FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

ANALYSIS AND APPROACHES SL

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WORKED SOLUTIONS

DYNAMIC LEARNING

HODDER EDUCATION

1 Core: Exponents and logarithms

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 1A

44 $\frac{4x^2+8x^3}{2x} = 2x + 4x^2$

45 $\frac{(2x^2y)^3}{8xy} = \frac{8x^6y^3}{8xy} = x^5y^2$

46 $(2ab^{-2})^{-3} = \left(\frac{2a}{b^2}\right)^{-3} = \left(\frac{b^2}{2a}\right)^3 = \frac{b^6}{8a^3}$

47 a $D = \frac{10^6}{n^2}$ so $n^2 = \frac{10^6}{D}$

Then $n = \sqrt{\frac{10^6}{d}} = \frac{1000}{\sqrt{D}}$

b If $n = 2$ then $D = \frac{10^6}{4} = 250\,000$

c If $D = 10^4$ then $n = \frac{10^3}{\sqrt{10^4}} = 10$ so \$10 million should be spent.

48 a When $n = 5$, $T_A = 1000 = k_A \times 5^3$ so $k_A = \frac{1000}{5^3} = 8$

When $n = 5$, $T_B = 1000 = k_B \times 5^2$ so $k_B = \frac{1000}{5^2} = 40$

b $\frac{T_A}{T_B} = \frac{k_A n^3}{k_B n^2} = \frac{k_A}{k_B} n$

c Method B will be faster at factorising a 10 digit number.

The answer to part **b** shows that the ratio of times is proportional to n. Since the times are equal for $n = 5$, it follows that method B is faster for $n > 5$ and method A is faster for $n < 5$.

49 $10 + 2 \times 2^x = 18$

$2^{x+1} = 8 = 2^3$

$x = 2$

50 $9^x = 3^{x+5}$

$3^{2x} = 3^{x+5}$

$2x = x + 5$ so $x = 5$

51 $5^{x+1} = 25 \times 5^{2x}$

$5^{x+1} = 5^2 \times 5^{2x} = 5^{2+2x}$

$x + 1 = 2 + 2x$ so $x = -1$

52 $8^x = 2 \times 4^{2x}$

$(2^3)^x = 2 \times (2^2)^{2x}$

$2^{3x} = 2^{4x+1}$

$3x = 4x + 1$ so $x = -1$

53 $25^{2x+4} = 125 \times 5^{x-1}$

$(5^2)^{2x+4} = 5^3 \times 5^{x-1}$

$5^{4x+8} = 5^{x+2}$

$4x + 8 = x + 2$

$3x = -6$ so $x = -2$

54 a $P = 0.8T$ and $R = 5P^2$

Then $R = 5 \times (0.8T)^2 = 5 \times 0.64T^2 = 3.2T^2$

b Rearranging: $T = \sqrt{\frac{R}{3.2}}$

When $R = 2 \times 10^5$ then $T = \sqrt{\frac{2\times10^5}{3.2}} = \sqrt{\frac{10^5}{1.6}} = \sqrt{\frac{10^6}{16}} = \frac{10^3}{4} = 250$ °K

55 a Time $t = \frac{\text{Distance}}{\text{Speed}} = \frac{3}{v}$

b Fuel used $= 0.5v^2 \times t = 0.5v^2 \times \frac{3}{v} = 1.5v$

c If it uses the full tank of fuel, $1.5v = 60$ so $v = 40$.

The maximum constant speed for the journey is 40 km h^{-1}

56

$$\begin{cases} 8^x 2^y = 1 & (1) \\ \dfrac{4^x}{2^y} = 32 & (2) \end{cases}$$

$(1): (2^3)^x \times 2^y = 1$

$2^{3x+y} = 2^0$

$3x + y = 0$ (3)

$(2): \frac{(2^2)^x}{2^y} = 32$

$2^{2x-y} = 2^5$

$2x - y = 5$ (4)

$(3) + (4)$: $5x = 5$ so $x = 1$ and then $y = -3$

57 $6^x = 81 \times 2^x$

$2^x \times 3^x = 3^4 \times 2^x$

$3^x = 3^4$ so $x = 4$

58 $32 + 2^{x-1} = 2^x$

$32 + 2^{x-1} = 2 \times 2^{x-1}$

$32 = 2^{x-1}$

$2^5 = 2^{x-1}$ so $x = 6$

59 $(x-2)^{x+5} = 1$

So either

(a) $x + 5 = 0$ or

(b) $x - 2 = 1$ or

(c) $x - 2 = -1$ and $x + 5$ is even

Then the solutions are (a) $x = -5$ or (b) $x = 3$ or (c) $x = 1$

60 $2^7 = 128$ and $5^3 = 125$

So $2^7 > 5^3$, so it follows that $(2^7)^{1000} > (5^3)^{1000}$

$2^{7000} > 5^{3000}$.

61 Any integer ending in 6 will have all positive integer powers also ending in 6, and any integer ending in 1 will have all positive integer powers also ending in 1.

It follows that $316^a + 631^b$ will terminate in 7 for any positive integers a and b.

Exercise 1B

14

$$\begin{aligned} a \times b &= (4 \times 10^6) \times (5 \times 10^{-3}) \\ &= (4 \times 5) \times (10^6 \times 10^{-3}) \\ &= 20 \times 10^3 \\ &= 2 \times 10^4 \end{aligned}$$

15

$$\begin{aligned} c \times d &= (1.4 \times 10^3) \times (5 \times 10^8) \\ &= (1.4 \times 5) \times (10^3 \times 10^8) \\ &= 7 \times 10^{11} \end{aligned}$$

16

$$\begin{aligned} \frac{a}{b} &= (4 \times 10^6) \div (5 \times 10^{-3}) \\ &= (4 \div 5) \times (10^6 \div 10^{-3}) \\ &= 0.8 \times 10^9 \\ &= 8 \times 10^8 \end{aligned}$$

17

$$\begin{aligned} c \times d &= (1.4 \times 10^3) \div (2 \times 10^8) \\ &= (1.4 \div 2) \times (10^3 \div 10^8) \\ &= 0.7 \times 10^{-5} \\ &= 7 \times 10^{-6} \end{aligned}$$

18

$$\begin{aligned} a - b &= (4.7 \times 10^6) - (7.1 \times 10^5) \\ &= (4.7 \times 10^6) - (0.71 \times 10^6) \\ &= 3.99 \times 10^6 \end{aligned}$$

19

$$\begin{aligned} d - c &= (4.2 \times 10^{14}) - (3.98 \times 10^{13}) \\ &= (4.2 \times 10^{14}) - (0.398 \times 10^{14}) \\ &= 3.802 \times 10^{14} \end{aligned}$$

20 a $p = 1.22 \times 10^8$

b

$$\begin{aligned} \frac{p}{q} &= (12.2 \times\ 10^7) \div (3.05 \times 10^5) \\ &= (12.2 \div 3.05) \times (10^7 \div 10^5) \\ &= 4 \times 10^2 \\ &= 400 \end{aligned}$$

c $\frac{p}{q} = 4 \times 10^2$

21 $\frac{6\times10^{23}}{10^{80}} = \ 6 \times 10^{-57}$

22 $\frac{12\ \mathrm{g}}{6.02\times10^{23}} \approx 1.99 \times 10^{-23}\ \mathrm{g}$

23 a $15 \times 10^{-15}\ \mathrm{m} = 1.5 \times 10^{-14}\ \mathrm{m}$

b

$$\begin{aligned} V &= \frac{1}{6}\pi(1.5 \times 10^{-14})^3 \\ &= \frac{1}{6}\pi \times 3.375 \times 10^{-42} \\ &= 1.77 \times 10^{-42}\ \mathrm{m}^3 \end{aligned}$$

24 a $(3.04 \times 10^{13}) \div (1.02 \times 10^{13}) = (3.04 \div 1.02) \times (10^{13} \div 10^{13}) = 2.98$

b $741 \times 10^6 = 7.41 \times 10^8$

c Europe:

$$\begin{aligned} \text{Population per m}^2 &= (7.41 \times 10^8) \div (1.02 \times 10^{13}) \\ &= (7.41 \div 1.02) \times (10^8 \div 10^{13}) \\ &= 7.26 \times 10^{-5} \end{aligned}$$

Africa:

$$\begin{aligned} \text{Population per m}^2 &= (1.2 \times 10^9) \div (3.04 \times 10^{13}) \\ &= (1.2 \div 3.04) \times (10^9 \div 10^{13}) \\ &= 0.395 \times 10^{-4} \\ &= 3.95 \times 10^{-5} \end{aligned}$$

Europe has more population per square metre.

25

$$\begin{aligned} c \times 10^d &= (3 \times 10^a) \times (5 \times 10^b) \\ &= (3 \times 5) \times (10^a \times 10^b) \\ &= 15 \times 10^{a+b} \\ &= 1.5 \times 10^{a+b+1} \end{aligned}$$

a $c = 1.5$

b $d = a + b + 1$

26

$$\begin{aligned} c \times 10^d &= (2 \times 10^a) \div (5 \times 10^b) \\ &= (2 \div 5) \times (10^a \div 10^b) \\ &= 0.4 \times 10^{a-b} \\ &= 4 \times 10^{a-b-1} \end{aligned}$$

a $c = 4$

b $d = a - b - 1$

27

$$\begin{aligned} c \times 10^r &= xy \\ &= (a \times 10^p) \times (b \times 10^q) \\ &= (a \times b) \times (10^p \times 10^q) \\ &= ab \times 10^{p+q} \end{aligned}$$

If $4 < a < b < 9$ then $16 < ab < 81$ so $1.6 < \frac{ab}{10} < 8.1$ which would be the required form for the value to be in standard form.

$$c \times 10^r = \left(\frac{ab}{10}\right) \times 10^{p+q+1}$$

Then $r = p + q + 1$

Exercise 1C

37

$$\begin{aligned} 1 + 2\log_{10} x &= 9 \\ 2\log_{10} x &= 8 \\ \log_{10} x &= 4 \\ x &= 10^4 = 10000 \end{aligned}$$

38

$$\begin{aligned} \log_2(3x + 4) &= 3 \\ 3x + 4 &= 10^3 = 1000 \\ x &= 332 \end{aligned}$$

39 $\ln(\mathrm{e}^a\, \mathrm{e}^b) = \ln(\mathrm{e}^{a+b}) = a + b$

40

$$\begin{aligned} 10^x &= 5 \\ x &= \log_{10}(5) = 0.699 \end{aligned}$$

41

$$3 \times 10^x = 20$$
$$10^x = \frac{20}{3}$$
$$x = \log_{10}\left(\frac{20}{3}\right) = 0.824$$

42

$$2 \times 10^x + 6 = 20$$
$$2 \times 10^x = 14$$
$$10^x = 7$$
$$x = \log_{10}(7) = 0.845$$

43

$$5 \times 20^x = 8 \times 2^x$$
$$5 \times (2 \times 10)^x = 8 \times 2^x$$
$$5 \times 2^x \times 10^x = 8 \times 2^x$$
$$5 \times 10^x = 8$$
$$10^x = \frac{8}{5} = 1.6$$
$$x = \log_{10} 1.6$$

44 a $\text{pH} = -\log(2.5 \times 10^{-8}) = 7.60$

b $1.9 = -\log[H^+]$

$[\text{H}^+] = 10^{-1.9} = 0.0126$ moles per litre.

45 a When $t = 0$, $R = 10 \times \text{e}^0 = 10$

b When $R = 5$, $5 = 10 \times \text{e}^{-0.1t}$

$$\frac{5}{10} = e^{-0.1t}$$
$$\frac{10}{5} = 2 = e^{0.1t}$$
$$0.1t = \ln 2$$
$$t = 10 \times \ln 2 = 6.93$$

46 a i When $t = 0$, $B = 1000 \times \text{e}^0 = 1000$

ii When $t = 2$, $B = 1000 \times \text{e}^{0.2} = 1221$

b When $B = 3000$, $3000 = 1000 \times \text{e}^{0.1t}$

$\text{e}^{0.1t} = 3$

$t = 10 \ln 3 \approx 11.0$ hours

47 Require $200 \times \text{e}^{0.1t} = 100 \times \text{e}^{0.25t}$

$\text{e}^{0.15t} = 2$

$t = \frac{1}{0.15} \ln 2 \approx 4.62$

48 a

$$\begin{aligned} L &= 10\log(10^{12} \times 5 \times 10^{-7}) \\ &= 10\log(5 \times 10^{5}) \\ &= 57.0 \text{ db} \end{aligned}$$

b

$$\begin{aligned} L &= 10\log(10^{12} \times 5 \times 10^{-6}) \\ &= 10\log(5 \times 10^{6}) \\ &= 67.0 \text{ db} \end{aligned}$$

c Increasing I by a factor of 10 Increases the noise level by 10 db

d $90 = 10\log(10^{12}I)$

$$\begin{aligned} \log(10^{12}I) &= 9 \\ 10^{12}I &= 10^{9} \\ I &= 10^{-3}\text{ W m}^{-2} \end{aligned}$$

49 a $20 = \mathrm{e}^{\ln 20}$

b

$$\begin{aligned} 20^{x} &= 7 \\ \left(\mathrm{e}^{\ln 20}\right)^{x} &= \mathrm{e}^{\ln 7} \\ \mathrm{e}^{x\ln 20} &= \mathrm{e}^{\ln 7} \\ x\ln 20 &= \ln 7 \\ x &= \frac{\ln 7}{\ln 20} \end{aligned}$$

50

$$\begin{cases} \log(xy) = 3 & (1) \\ \log\left(\frac{x}{y}\right) = 1 & (2) \end{cases}$$

$(1): xy = 10^{3}$ (3)

$(2): \frac{x}{y} = 10^{1}$ (4)

$(3) \times (4): x^{2} = 10^{4}$ so $x = \pm 10^{2} = \pm 100$

The solutions are: $x = 100, y = 10$ or $x = -100, y = -10$

51

Tip: You can solve this question very quickly if you know the rule of logarithms that

$log\ a - log\ b = log\left(\frac{a}{b}\right)$

Then $log(10x) - log(x) = log(10) = 1$.

Without this information, the question can still be solved, but it is more time-consuming.

One approach is shown below. If you generalise this, you can prove the rule of logarithms above.

Suppose $\log(10x) - \log x = c$

Then

$$\begin{aligned} 10^c &= 10^{\log(10x)-\log x} \\ &= 10^{\log(10x)} \div 10^{\log x} \\ &= 10x \div x \\ &= 10 \\ &= 10^1 \end{aligned}$$

So $c = 1$

Mixed Practice

1 **a**

$$\begin{aligned} \text{Perimeter} &= 2 \times (2680 \text{ cm} + 1970 \text{ cm}) \\ &= 9300 \text{ cm} \\ &= 9.3 \times 10^3 \text{ cm} \end{aligned}$$

b

$$\begin{aligned} \text{Area} &= (2680 \text{ cm} \times 1970 \text{ cm}) \\ &= 5\,279\,600 \text{ cm}^2 \\ &\approx 5\,280\,000 \text{ cm}^2 \end{aligned}$$

2

$$\begin{aligned} \frac{(3xy^2)^2}{(xy)^3} &= \frac{9x^2y^4}{x^3y^3} \\ &= \frac{9y}{x} \end{aligned}$$

3

$$\begin{aligned} (3x^2y^{-3})^{-2} &= \left(\frac{3x^2}{y^3}\right)^{-2} \\ &= \left(\frac{y^3}{3x^2}\right)^2 \\ &= \frac{y^6}{9x^4} \end{aligned}$$

4 **a** $k = PR = 4\,000\,000$

b When $R = 4$, $P = \frac{k}{4} = 1\,000\,000$

c $R = kP^{-1}$ so when $P = 250\,000$, $R = 16$

5

$$\begin{aligned} 8^x &= 2^{x+6} \\ (2^3)^x &= 2^x \times 2^6 \\ 2^{3x} &= 2^x \times 2^6 \\ 2^{2x} &= 2^6 \end{aligned}$$

$2x = 6$ so $x = 3$

6

$$\begin{aligned} ab &= (3\times 10^8)\times(4\times 10^4)\\ &= (3\times 4)\times(10^8\times 10^4)\\ &= 12\times 10^{12}\\ &= 1.2\times 10^{13} \end{aligned}$$

7

$$\begin{aligned} \frac{a}{b} &= (1\times 10^9)\div(5\times 10^{-4})\\ &= (1\div 5)\times(10^9\div 10^{-4})\\ &= 0.2\times 10^{13}\\ &= 2\times 10^{12} \end{aligned}$$

8

$$\begin{aligned} a-b &= (5\times 10^5)-(3\times 10^4)\\ &= (5\times 10^5)-(0.3\times 10^5)\\ &= 4.7\times 10^5 \end{aligned}$$

9

$$\begin{aligned} \text{Time} &= \frac{\text{distance}}{\text{speed}}\\ &= (1.5\times 10^{11}\text{ m})\div(3\times 10^8\text{ m s}^{-1})\\ &= (1.5\div 3)\times(10^{11}\div 10^8)\text{ s}\\ &= 0.5\times 10^3\text{ s}\\ &= 500\text{ s} \end{aligned}$$

10

$$\begin{aligned} \log(x+1) &= 2\\ x+1 &= 10^2 = 100\\ x &= 99 \end{aligned}$$

11

$$\begin{aligned} \ln(2x) &= 3\\ 2x &= \mathrm{e}^3\\ x &= 0.5\mathrm{e}^3 \end{aligned}$$

12 $\mathrm{e}^x = 2$

$x = \ln 2 = 0.693$ (to 3 s.f.)

13 $5\times 10^x = 17$

$$\begin{aligned} 10^x &= 3.4\\ x &= \log 3.4 = 0.531 \text{ (to 3 s.f.)} \end{aligned}$$

14 $5\mathrm{e}^x - 1 = y$

$$\begin{aligned} 5\mathrm{e}^x &= y+1\\ \mathrm{e}^x &= \frac{y+1}{5}\\ x &= \ln\left(\frac{y+1}{5}\right) \end{aligned}$$

15 **a** $2^m = 8 = 2^3$ so $m = 3$

$2^n = 16 = 2^4$ so $n = 4$

b

$$\begin{aligned} 8^{2x+1} &= 16^{2x-3} \\ (2^3)^{2x+1} &= (2^4)^{2x-3} \\ 2^{6x+3} &= 2^{8x-12} \\ 6x+3 &= 8x-12 \\ 2x &= 15 \\ x &= 7.5 \end{aligned}$$

16

$$\begin{aligned} (7\times 10^a)\times(4\times 10^b) &= c\times 10^d \\ (7\times 4)\times(10^a\times 10^b) &= c\times 10^d \\ 28\times 10^{a+b} &= c\times 10^d \\ 2.8\times 10^{a+b+1} &= c\times 10^d \end{aligned}$$

a $c = 2.8$

b $d = a + b + 1$

17

$$\begin{aligned} (6\times 10^a)\div(5\times 10^b) &= c\times 10^d \\ (6\div 5)\times(10^a\div 10^b) &= c\times 10^d \\ 1.2\times 10^{a-b} &= c\times 10^d \end{aligned}$$

a $c = 1.2$

b $d = a - b$

18 $\text{pH} = 6.1 + \log\left(\frac{[HCO_3^-]}{[H_2CO_3]}\right)$

Rearranging:

$$\frac{[HCO_3^-]}{[H_2CO_3]} = 10^{\text{pH}-6.1}$$

$$[H_2CO_3] = \frac{[HCO_3^-]}{10^{\text{pH}-6.1}}$$

When $\text{pH} = 7.35$ and $[HCO_3^-] = 0.579$, $[H_2CO_3] = 0.0326$

When $\text{pH} = 7.45$ and $[HCO_3^-] = 0.579$, $[H_2CO_3] = 0.0259$

So the range of carbonic acid concentrations is 0.0259 to 0.0326

19 $v = 1350(1 - e^{-0.007t})$

a When $t = 1$, $v = 1350\times(1 - e^{-0.007}) = 9.42\text{ m s}^{-1}$

b At $t = 600$, the model predicts $v = 1350\times(1 - e^{-4.2}) = 1330\text{ m s}^{-1}$

Yes, he would be expected to break the speed of sound.

Ordinarily, air resistance would prevent such high speeds being reached so soon during a free fall, but by jumping from the edge of the atmosphere, the effect of air resistance was reduced during the initial period of the fall.

20 a When $A = 1000$, $S = \log 1000 = 3$

b $A' = 10A$

$S' = \log 10A$

$S' = \log 10 + \log A$

$S' = 1 + S$

When A increases by a factor of 10, S increases by one unit.

c $A = 10^S$ so when $S = 9.5$, $A = 10^{9.5}$ μm $= 10^{3.5}$ m $= 3162$ m

21 $3 \times 20^x = 2^{x+1}$

$3 \times 20^x = 2 \times 2^x$

$10^x = \frac{2}{3}$

$x = \log\left(\frac{2}{3}\right) \approx -0.176$

22

$$\begin{cases} 9^x \times 3^y = 1 \\ \dfrac{4^x}{2^y} = 16 \end{cases}$$

$$\begin{cases} 3^{2x+y} = 3^0 \\ 2^{2x-y} = 2^4 \end{cases}$$

$$\begin{cases} 2x + y = 0 & (1) \\ 2x - y = 4 & (2) \end{cases}$$

$(1) + (2)$: $4x = 4$ so $x = 1, y = -2$

23

$$\begin{cases} \log(xy) = 0 \\ \log\left(\dfrac{x^2}{y}\right) = 3 \end{cases}$$

$$\begin{cases} xy = 1 & (1) \\ \dfrac{x^2}{y} = 10^3 & (2) \end{cases}$$

$(1) \times (2)$: $x^3 = 10^3$ so $x = 10, y = 0.1$

2 Core: Sequences

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 2A

21 $u_1 = 7, d = 11$

a $u_{20} = u_1 + 19d = 216$

b $S_n = \frac{n}{2}(2u_1 + d(n-1))$

$$S_{20} = \frac{20}{2}(2 \times 7 + 11 \times 19) = 2230$$

22 $u_1 = 3, u_2 = 7 = u_1 + d$

a $d = 4$

b $u_8 = u_1 + 7d = 31$

c $S_n = \frac{n}{2}(2u_1 + d(n-1))$

$$S_{15} = \frac{15}{2}(2 \times 3 + 4 \times 14) = 465$$

23 $d = 5, u_2 = 13 = u_1 + d$

a $u_1 = 8$

b $S_n = \frac{n}{2}(2u_1 + d(n-1))$

$$S_{10} = \frac{10}{2}(2 \times 8 + 5 \times 9) = 305$$

24 $u_1 = -8, u_{16} = 67 = u_1 + 15d$

a $15d = 67 - u_1 = 75$ so $d = 5$

b $u_{25} = u_1 + 24d = 112$

25 a 4% of £300 is £300 × 0.04 = £12

Simple interest: The same interest sum is earned each year.

At the end of the first year, Sam has £300 + 1 × £12 = £312.

b At the end of the tenth year, Sam has £300 + 10 × £12 = £420.

26 On his 21st birthday, the grandparents have paid in 20 supplementary £10 amounts.

The balance is £100 + 20 × £10 = £300

27 The nth stair is u_n cm off the ground.

Then $u_1 = 10$ and $d = 20$.

$$\begin{aligned} u_n &= u_1 + d(n-1) \\ 10 + 20(n-1) &= 270 \\ n - 1 &= \frac{260}{20} = 13 \\ n &= 14 \end{aligned}$$

28 $u_1 = 11, u_n = 75 = u_1 + d(n-1)$

a If $d = 8$ then $8(n-1) + 11 = 75$ so $n - 1 = \frac{64}{8} = 8; n = 9$

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{9}{2}(11 + 75) = 387$$

b If $d = 4$ then $4(n-1) + 11 = 75$ so $n - 1 = \frac{64}{4} = 16; n = 17$

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{17}{2}(11 + 75) = 731$$

Tip: Be careful not to assume that halving the difference for a sequence with a fixed start and end point either doubles the number of terms (it doubles $n - 1$ instead!) or doubles the total sum. Neither is valid.

29 $u_{10} = 26 = u_1 + 9d$ (1)

$u_{30} = 83 = u_1 + 29d$ (2)

$(2) - (1): 20d = 57$

$u_{50} = u_1 + 49d = u_{30} + 20d = 140$

Tip: Be alert to simple approaches. At no point in this problem did you need to calculate u_1 or d.

30 a $u_1 = 8, d = 3, u_n = 68 = u_1 + d(n-1)$

Then $3(n-1) = 60$ so,

$n - 1 = 20$

$n = 21$

b $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

$$S_{21} = \frac{21}{2}(2 \times 8 + 3 \times 20) = 798$$

31 a $u_7 = 35 = u_1 + 6d$ (1)

$u_{18} = 112 = u_1 + 17d$ (2)

$(2) - (1): 11d = 77$

So $d = 7$, and then $u_1 = u_7 - 6d = -7$

b $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

$$S_{18} = \frac{18}{2}(2 \times (-7) + 7 \times 17) = 945$$

32 **a** $u_{10} = 16 = u_1 + 9d \quad (1)$

$u_{30} = 156 = u_1 + 29d \quad (2)$

$(2) - (1){:}\ 20d = 140$

$u_{50} = u_1 + 49d = u_{30} + 20d = 296$

b $d = 7$ so $u_1 = 16 - 9d = -47$

$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$$

$$s_{20} = \frac{20}{2}(2 \times (-47) + 7 \times 19) = 390$$

33 Let $u_n = 5n - 3$

Then $u_1 = 2$ and $d = 5$

$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$$

$$S_{16} = \frac{16}{2}(2 \times 2 + 5 \times 15) = 632$$

34 There must be a common difference between the terms.

So:

$$\begin{aligned}(2x+1) - (3x+1) &= (4x-5) - (2x+1)\\ -x &= 2x - 6\\ 3x &= 6\\ x &= 2\end{aligned}$$

35 $u_5 = 2u_2, u_7 = 28$

Then $u_1 + 4d = 2(u_1 + d) \quad (1)$

and $u_1 + 6d = 28 \quad (2)$

$(1){:}\ u_1 = 2d$

Substituting into $(2){:}\ 8d = 28$ so $2d = 7$

$u_{11} = u_7 + 4d = 28 + 2 \times 7 = 42$

36 $u_1 + u_2 + u_3 = u_{10}, u_7 = 27$

$u_1 + (u_1 + d) + (u_1 + 2d) = u_1 + 9d \quad (1)$

$u_1 + 6d = 27 \quad (2)$

$(1){:}\ 3u_1 + 3d = u_1 + 9d$ so $2u_1 = 6d$

Substituting into $(2){:}\ 3u_1 = 27$ so $u_1 = 9$ and then $d = 3$

Then $u_{12} = u_1 + 11d = 42$

37 Arithmetic sequence with common difference $d = 4$

$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$$

$$S_{20} = \frac{20}{2}(2 \times 10 + 4 \times 19) = 960$$

38 **a** $u_1 = 7, d_u = 12$

If the sequence has N terms, $u_N = 139$

$u_N = u_1 + d_u(N-1) = 7 + 12(N-1) = 139$

$12N - 5 = 139$

$N = 12$

$S_N = \frac{N}{2}(u_1 + u_N) = \frac{12}{2}(7 + 139) = 876$

b $v_1 = 7, d_v = 6$

If the sequence has M terms, $v_M = 139$.

$$\begin{aligned} v_M &= v_1 + d_v(M-1) \\ &= 7 + 6(M-1) = 139 \end{aligned}$$

$6M + 1 = 139$

$M = 23$

$S_M = \frac{M}{2}(v_1 + v_M) = \frac{23}{2}(7 + 139) = 1679$

c $w_1 = 7, d_w = 6, N = 12$

$S_N = \frac{N}{2}\left(2w_1 + d_w(N-1)\right) = \frac{12}{2}(2 \times 7 + 6 \times 11) = 480$

39 Estimating $u_1 = 53, u_5 = 211 = u_1 + 4d$ so $d = \frac{158}{4} = 39.5$

Then $u_{10} = u_1 + 9d = 408.5$

40 Estimating $u_1 = 3, u_4 = 33 = u_1 + 3d$ so $d = \frac{30}{3} = 10$

Then $u_6 = u_1 + 5d = 53$

41 $u_1 = 1$

$u_2 = a = u_1 + d$ (1)

$u_3 = 3a + 5 = u_1 + 2d$ (2)

$2(1) - (2): u_1 = -a - 5$

So $a - 5 = 1$ and therefore $a = -6$ so $d = -7$

Then $u_4 = u_1 + 3d = -20$

Tip: Alternatively, use the fact that $u_n + 1 = 2u_n - u_{n-1}$ for any n in an arithmetic sequence.

So $u_3 = 2u_2 - u_1$ so $3a + 5 = 2a - 1$ from which $a = -6$

Then $u_4 = 2u_3 - u_2 = 5a + 10 = -20$ directly, without calculating d at all.

42 Let u_n be the number of minutes of screentime on day n.

$u_1 = 200$ and $d = -5$ in the arithmetic progression.

a He gives gets to zero minutes on day N where $u_N = 0$

$u_N = u_1 + d(N-1) = 200 - 5(N-1) = 0$ so $N = 41$

b

$$S_N = \frac{N}{2}\left(2u_1 + d(N-1)\right)$$
$$= \frac{41}{1}(2 \times 200 - 5 \times 41) = 7995 \text{ minutes total}$$

43 Let u_n be the number of steps taken on day n.

$u_1 = 1000, d = 500$ in an arithmetic sequence.

a Require least N such that $u_N = 10000$

$$u_N = u_1 + (N-1)d$$
$$10000 = 1000 + 500(N-1)$$
$$N = 1 + \frac{9000}{500} = 19$$

On the 19th day he takes 10 000 steps.

b Require M such that $S_M = 540\,000$

$$S_M = \frac{M}{2}\left(2u_1 + d(M-1)\right)$$
$$540000 = \frac{M}{2}\left(2 \times 1000 + 500(M-1)\right)$$
$$1080000 = 2000M + 500M^2 - 500M$$
$$2160 = M^2 + 3M$$
$$M^2 + 3M - 2160 = 0$$
$$(M+48)(M-45) = 0$$

$M = -48$ (reject) or $M = 45$

It takes 45 days.

44

Comment: Three methods are shown here; the first method takes the formula given for the sum and compares it to the standard general formula to find u_1 and d.

The second method is uses the fact that $u_1 = S_1$ and that the difference in sequential sums is a single term: $S_N - S_{N-1} = u_N$; from this we can find both u_1 and d indirectly, if needed.

The third uses again that $S_N - S_{N-1} = u_N$ to find the formula for the general term u_N. For this question, this is unnecessarily laborious but is potentially useful in more complicated situations, or when only this is required, as in question 25 below.

Method 1: Compare formulae to find parameter values

a $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right) \equiv 2n^2 + n$ for all n for this sequence

$$n(2u_1 - d + dn) \equiv 4n^2 + 2n$$
$$dn^2 + (2u_1 - d)n \equiv 4n^2 + 2n$$

Comparing coefficients, $d = 4$ and $2u_1 - d = 2$ so $2u_1 = 6$ and so $u_1 = 3$

Then $u_1 = 3, u_2 = 7$

b $u_{50} = u_1 + 49d = 3 + 49 \times 4 = 199$

Method 2: Use given formula directly

a $S_n = 2n^2 + n$ so $S_1 = 3$ and $S_2 = 10$.

$u_1 = S_1 = 3$

$u_2 = S_2 = S_1 = 7$

b

$$u_{50} = S_{50} - S_{49}$$
$$= (2 \times 50^2 + 50) - (2 \times 49^2 + 49)$$
$$= 5050 - 4851 = 199$$

Method 3: Use given sum formula to find the formula for the general term.

$S_n - S_{n-1} = u_n$

$$u_n = (2n^2 + n) - \left(2(n-1)^2 + (n-1)\right)$$
$$= 2n^2 + n - (2n^2 - 4n + 2 + n - 1)$$
$$= 4n - 1$$

a $u_1 = 4 \times 1 - 1 = 3$

$u_2 = 4 \times 2 - 1 = 7$

b $u_{50} = 4 \times 50 - 1 = 199$

45 $S_n - S_{n-1} = u_n$

$$u_n = (n^2 + 4n) - \left((n-1)^2 + 4(n-1)\right)$$
$$= n^2 + 4n - (n^2 - 2n + 1 + 4n - 4)$$
$$= 2n + 3$$

46 $u_9 = 5u_3$

Then $u_1 + 8d = 4(u_1 + 2d)$

$u_1 + 8d = 4u_1 + 8d$

$u_1 = 0$

47 $7 \times 15 = 105$

$7 \times 143 = 1001$

The multiples of 7 form an arithmetic sequence with common difference $d = 7$.

Let $u_1 = 105$, $d = 7$ and $n = 143 - 15 = 128$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

$$S_{128} = \frac{128}{2}(2 \times 105 + 7 \times 127) = 70336$$

48 Multiples of 5 (not including 0 which does not change the sum): $u_1 = 5, d = 5, u_{20} = 100$

$$S_{20} = \frac{20}{2}(2 \times 5 + 5 \times 19) = 1050$$

Multiples of 15 (not including 0): $v_1 = 15, d = 15, u_6 = 90$

$$S_6 = \frac{6}{2}(2 \times 15 + 15 \times 5) = 315$$

Then the sum of multiples of 5 which are not also multiples of 3 will be $1050 - 315 = 735$

49 $u_{10} = 3u_1$ and $S_{10} = 400$ cm

$S_{10} = \frac{10}{2}(u_1 + u_{10}) = 20u_1 = 400$ cm so $u_1 = 20$ cm

50 $u_1 = 4$

$u_2 = a = 4 + d$ (1)

$u_3 = b = 4 + 2d$ (2)

$u_4 = a - b = 4 + 3d$ (3)

$(3) - (2) - (1): -2b = -4$ so $b = 2$. Then $d = -1$ so $u_6 = u_1 + 5d = -1$

51 a

$$\begin{aligned} u_n - u_{n-1} &= (a + nd) - (a + (n-1)d) \\ &= d \end{aligned}$$

The difference between consecutive terms is shown to be constant d.

b

$$\begin{aligned} u_n - u_{n-1} &= (an^2 + bn) - (a(n-1)^2 + b(n-1)) \\ &= (an^2 + bn) - (an^2 - 2an + a + bn - b) \\ &= 2an + a - b \\ &= A + nD \end{aligned}$$

where $A = a - b$ and $D = 2a$

Using part **a**, this is the formula for an arithmetic sequence with constant difference $D = 2a$.

52 a The first 9 positive integers are single digits, for a total of 9 digits written

The next 10 integers (10 to 19) are double-digit numbers, so contain a total of 20 digits.

In all, she has written 29 digits.

b The values up to 99 will contain a total of

9 digits (single digit values 1 to 9)

180 digits (double digit values 10 to 99)

If she has written a total of 342 digits then the remaining 153 digits are from triple-digit numbers.

$153 \div 3 = 51$ so she has written 51 triple-digit values

100 is the first, so 150 is the 51st.

Alessia has written values up to 150.

Exercise 2B

16 a $u_1 = 128, r = 0.5$

$u_8 = u_1 \times r^7 = 1$

b $S_8 = u_1 \times \frac{1-r^8}{1-r} = 128 \times \frac{1-\frac{1}{256}}{1-\frac{1}{2}} = 255$

17 $u_1 = 3, u_2 = 6 = r \times u_1$

a $r = 2$

b $u_6 = u_1 \times r^5 = 96$

c $S_{10} = u_1 \times \frac{r^{10}-1}{r-1} = 3 \times \frac{1024-1}{2-1} = 3069$

18 $u_2 = 24 = u_1 \times r \qquad (1)$

$u_5 = 81 = u_1 \times r^4 \qquad (2)$

a $(2) \div (1): r^3 = \frac{81}{24} = \frac{27}{8} = \left(\frac{3}{2}\right)^3$

$r = 1.5$

b $u_7 = u^5 \times r^2 = 81 \times \frac{9}{4} = 182.25$

19

Tip: Rather than use n to count the number of days, it can be simpler to use n to count the number of complete periods of doubling. You then need less work for calculating terms of the sequence, but must take care when interpreting what day corresponds to what value of n.

As is always the case in problems like this, if the question does not define the terms of a sequence, you should give a definition in your answer, but are free to do this however seems most convenient.

Let u_n be the area of algal cover on day $8n - 7$ (the start of the nth 8-day period).

$u_1 = 15, r = 2$ in a geometric sequence.

The start of week 9 is day 57 which corresponds to $n = 8$

$u_8 = u_1 \times r^7 = 15 \times 2^7 = 1920 \text{ cm}^2$

20 Let u_n be the concentration on day $2n - 1$.

$u_1 = 1.2$ and $r = \frac{1}{2}$ for a geometric sequence.

After 12 days, $n = 6$.

$u_6 = u_1 \times r^5 = 1.2 \times 0.5^5 = 0.0375 \text{ mg ml}^{-1}$

21 $u_1 = 8, u_1 + u_2 = 12$ so $u_2 = 4 = u_1 \times r$

$r = 0.5$

$$S_5 = u_1 \times \frac{1 - r^5}{1 - r} = 8 \times \frac{1 - \frac{1}{32}}{1 - \frac{1}{2}} = 15.5$$

22 Let u_n be the time taken on the nth attempt

$u_1 = 5, r = 0.8$ in a geometric sequence.

$u_{10} = u_1 \times r^9 = 0.671$ seconds

23

Tip: As in several questions towards the end of this exercise, the question describes a situation and defining a sequence should be the first step of a formal answer.

Let u_n be the volume of water at the start of day n of the drought.

a $u_1 = 5000, r = 0.92$ in a geometric sequence

$u_6 = u_1 \times r^4 = 3580 \text{ m}^3$

b Require N such that $u_N < 2000$

Then $u_1 \times r^{N-1} < 2000$

$$r^{N-1} < \frac{2000}{u_1} = 0.4$$
$$(N-1)\log r < \log 0.4$$
$$-0.036(N-1) < -0.398$$
$$N - 1 > \frac{0.398}{0.036}$$
$$N > 11.99$$

At the start of day 12, the reservoir holds less than 2000 m^3, so it takes 11 days to use up 3000 m^3.

24 $u_5 = u_1 \times r^4$ (1)

$u_2 = u_1 \times r$ (2)

But $u_5 = 8u_2$ so $r^3 = 8$, from which $r = 2$

$\frac{S_8}{u_1} = \frac{r^8-1}{r-1} = \frac{256-1}{1} = 255$

25 Let u_n be the number of grains on the nth square; $\{u_n\}$ is a geometric sequence with $r = 2$.

a $u_{64} = u_1 \times r^{63} = 2^{63} = 9.22 \times 10^{18}$

b $S_{64} = \frac{(r^{64}-1)}{r-1} = 2^{64} - 1 = 1.84 \times 10^{19}$

c Mass of Sissa's reward: $s_{64} \times 0.1 \text{ g} = 1.84 \times 10^{18}$ g

This would take $1.84 \times 10^{18} \div (7.5 \times 10^{14}) \approx 2450$ years.

26 $r = \frac{u_n}{u_{n-1}}$

$\frac{y^3}{xy^2} = yx^{-1}$ and $\frac{x^{-1}y^4}{y^3} = x^{-1}y$ so the sequence is consistent, and no relation between x and y can be determined.

$u_n = u_1 \times r^{n-1}$ so the general term $u_n = xy^2 \times (x^{-1}y)^{n-1} = x^{2-n}y^{n+1}$

27 Geometric sequence with $u_1 = 3$ and $r = 2$.

$S_{10} = u_1\frac{(r^{10}-1)}{r-1} = 3 \times \frac{2^{10}-1}{2-1} = 3 \times 1023 = 3069$

28 $r = \frac{u_2}{u_1} = x$ and also $r = \frac{u_3}{u_2} = \frac{2x^2+x}{x} = 2x + 1$

So $x = 2x + 1$

$x = -1$

$u_{10} = u_1 \times r^9 = -1$

29 Sum of a geometric sequence with $u_1 = 3$ and $r = 3$.

$S_{10} = u_1\frac{r^{10}-1}{r-1} = 3 \times \frac{3^{10}-1}{3-1} = 88\,572$

30 Let u_n be the height reached on the nth bounce, in metres.

$u_1 = 0.6$ and $r = 0.8$ in a geometric sequence.

a $u_5 = u_1 \times r^4 = 0.246$ so the height of the 5th bounce is 0.246 m.

b From the top of the 1st bounce to the top of the 5th bounce would be distance

$$u_1 + 2(u_2 + u_3 + u_4) + u_5 = 0.6 + 2(0.48 + 0.384 + 0.3072) + 0.24576 \approx 3.19 \text{ m}$$

c The bounce height and value of r are only given to 1 significant figure, so even if a geometric model were precise for the loss of energy from a bouncing system, the accuracy of the output would be poor for high powers of r.

In any case, at the level of detail predicted by the model at the 20th bounce, we would expect that measurement error and imperfections in the ground surface and ball surface and composition would swamp the prediction.

Exercise 2C

16 $£800 \times 1.03^4 = £900.41$

17 $£10\,000 \times 1.057^7 = £14\,071$

18 4% annual rate is equivalent to $1 + \frac{0.04}{12} = 1.00333$ monthly multiplier

$\$8000 \times 1.00333^{30} = \$8\,839.90$

19 5.8% annual rate is equivalent to $1 + \frac{0.058}{12} = 1.00483$ monthly multiplier.

$€15\,000 \times 1.00483^{18} = €16\,360.02$

20 $£20\,000 \times 0.85^5 = £8870$ (to 3 s.f.)

21 a The return will be better when interest is applied monthly, due to the effect of compounding.

After 1 year at 6% compounded annually, the account stands at

$£1000 \times 1.06 = £1060$

After 1 year at 6% compounded monthly, the account stands at

$$£1000 \times \left(1 + \frac{0.06}{12}\right)^{12} = £1061.68$$

b After 10 years at 6% compounded annually, the account stands at

$£1000 \times 1.06^{10} = £1790.85$

After 10 years at 6% compounded monthly, the account stands at

$$£1000 \times \left(1 + \frac{0.06}{12}\right)^{120} = £1819.40$$

The difference is £28.55

22 20% depreciation and 2.5% interest rate: After the first year the value is

$£15\,000 \times 0.775 = £11\,625$

10% depreciation and 2.5% interest rate for 4 years: $£11\,625 \times 0.875^4 = £6814.36$

23

Year	Start-year value ($)	Depreciation expense ($)	End-year value($)
1	20 000	6 000	14 000
2	14 000	4 200	9 800
3	9 800	2 940	6 860
4	6 860	2 058	4 802
5	4 802	1 441	3361
6	3361	1 008	2 353
7	2 353	706	1 647
8	1 647	147	1 500

24 The annual real terms percentage change is $r \approx c - i = 6.2\% - 3.2\% = 3\%$

So over 5 years, the real terms percentage increase is $1.03^5 - 1 = 15.9\%$

25 12% annual interest: 1% monthly interest

Monthly interest of 1% gives $(1 + 0.01)^{12} - 1 = 12.7\%$ annual equivalent rate (AER)

Tip: Be careful to distinguish between 'annual interest rate' which is $12 \times$ monthly interest and "Annual Equivalent Rate" (AER), which is the effective rate over a year, after consideration of compounding.

AER is often published in financial offers because it allows easy comparison between products applying compound interest at different intervals.

For example:

Account *A* compounds monthly at an annual rate of 12% so would have a monthly rate of 1% for an AER of 12.68% (as in this question)

Account *B* compounds quarterly at an annual rate of 12.1% so would have a quarterly rate of 3.025% for an AER of 12.66%. Although account *B* has a higher annual rate, account *A* actually has a slightly higher AER.

26 a Real terms percentage increase $r = \frac{1+\frac{c}{100}}{1+\frac{i}{100}} - 1 = \frac{1.05}{1.025} - 1 = 2.44\%$

b Over the course of 5 years, this would produce a $1.0244^5 - 1 = 12.8\%$ increase in real terms.

27 a $250 \times (1 + 10^9) \approx 250 \times 10^9$ marks (250 billion marks)

b Using the exact formula for adjustment: $r = \frac{1+\frac{c}{100}}{1+\frac{i}{100}} = \frac{1.2}{1+10^9}$

So at year end, 2×10^6 marks would have a real terms value of

$2 \times 10^6 \times \left(\frac{1.2}{10^9}\right) = 2.4 \times 10^{-3} = 0.0024$ marks c $= \frac{1+\frac{c}{100}}{1+\frac{i}{100}} = \frac{1.15}{1+10^9}$

So at year end, 25×10^6 marks after application of interest would have a real terms value of $25 \times 10^6 \times \left(\frac{1.15}{10^9}\right) \approx 29 \times 10^{-3} = 0.029$ marks

Mixed Practice

1 **a** 4.5% nominal annual rate is equivalent to $1 + \frac{0.045}{12} = 1.00375$ monthly multiplier

$\$5000 \times 1.00375^{7\times 12} = \$6847.26$

b Current value $= 7000$, Future value $= 14\,000$, $n = 10$, payments/year $= 1$.

Calculator gives $I\% = 7.18$

Carla requires at least 7.18% annual rate to achieve her aim.

2 **a** **i** $\{d\}$ is an arithmetic sequence, with common difference -0.05

ii $\{b\}$ is a geometric sequence, with common ratio $\frac{3}{2}$

b **i** Common ratio is $\frac{-3}{-6} = \frac{1}{2}$

ii $e_1 = -6, r = \frac{1}{2}$

$e_{10} = e_1 \times r^9 = -\frac{6}{2^9} = -\frac{3}{256}$

3 **a** $d = u_9 - u_8 = 2$

b $u_8 = u_1 + 7d = u_1 + 14 = 10$ so $u_1 = -4$

c $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

So $S_{20} = \frac{20}{2}(2 \times (-4) + 2 \times 19) = 300$

4 **a** $u_1 = 2$ and $u_2 = 8 = u_1 \times r$ so $r = 4$

b $u_5 = u_1 \times r^4 = 2 \times 4^4 = 512$

c $S_n = u_1 \frac{(r^n - 1)}{r-1}$

$$S_8 = 2\frac{4^8 - 1}{4 - 1} = 43\,690$$

5 Let u_n be net profit in year n, in thousand dollars.

$u_1 = -100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 15$

Least N such that $u_N > 0$:

$$u_N = u_1 + (N-1)d > 0$$
$$-100 + 15(N-1) > 0$$
$$15(N-1) > 100$$
$$N > 1 + \frac{100}{15} = 7.6$$

The company is first profitable in the 8th year.

6 Let u_n be the value at the start of the nth year, in thousand dollars.

$u_1 = 25$ and $\{u\}$ follows an arithmetic progression with common difference $d = -1.5$

Least N such that $u_N \leqslant 10$:

$$\begin{aligned} u_N = u_1 + (N-1)d &\leqslant 10 \\ 25 - 1.5(N-1) &\leqslant 10 \\ 1.5(N-1) &\geqslant 15 \\ N &\geqslant 11 \end{aligned}$$

At the start of the 11th year, the car has value \$10 000 so it takes 10 years to fall to that value.

7 Let u_n be the height of the sunflower n weeks after being planted, in cm.

$u_1 = 20$ and $\{u\}$ follows a geometric progression with common ratio $r = 1.25$

a $u_5 = u_1 \times r^4 = 20 \times 1.25^4 = 48.8$ cm

b Least N such that $u_N > 100$:

$$\begin{aligned} u_N &= u_1 \times r^{N-1} \\ 20 \times 1.25^{N-1} &> 100 \\ 1.25^{N-1} &> 5 \end{aligned}$$

Tip: If you are comfortable using logarithms with base other than 10 or e then you can calculate the solution directly as $N - 1 > \log_{1.25} 5$. Otherwise, use a change of base method or simply let the calculator solve this directly using an equation solver or graph solver.

$$\begin{aligned} N - 1 &> 7.2 \\ N &> 8.2 \end{aligned}$$

It takes 9 weeks for the plant to exceed 100 cm.

8 **a** Current value $=$ 500, $n = 16$, payments/year $=$ 4, I% $= 3$.

Calculator gives Future value $=$ 563.50

After 4 years, Kunal has a balance of 563.50 euros.

b Current value $=$ 500, payments/year $=$ 4, I% $=$ 3, Future value $=$ 600

Calculator gives $n = 24.4$

It will take 25 quarters (six and a quarter years) for Kunal to earn 100 euros of interest.

9 Let u_n be the population at the end of year $2000 + n$

$u_{18} = 7.7 \times 10^9$ and $\{u\}$ follows a geometric progression with $r = 1.011$.

a $u_{22} = u_{18} \times r^4 = 8.04 \times 10^9$

According to the model, at the end of 2022 the world's population will be approximately 8.0 billion.

b Least N such that $u_N > 9 \times 10^9$:

$$\begin{aligned} u_N &= u_{18} \times r^{N-18} \\ 7.7 \times 1.011^{N-1} &> 9 \\ N &> 32.3 \end{aligned}$$

According to the model, the world's population will exceed 9 billion during 2033.

10 a i $u_2 = 30; u_5 = 90 = u_2 + 3d$ so $d = \frac{90-30}{3} = 20$

ii $u_1 = u_2 - d = 10$

b Let $\{v\}$ be the geometric sequence.

$v_1 = u_1 = 10; v_2 = u_2 = 30; v_3 = u_5 = 90$

So the common ratio $r = 3$

$v_7 = v_1 \times r^6 = 10 \times 3^6 = 7290$

11 a Substituting for n in the formula of S_n:

i $S_1 = 6 + 1 = 7$

ii $S_2 = 12 + 4 = 16$

b $u_2 = S_2 - S_1 = 16 - 7 = 9$

c $S_1 = u_1 = 7$ and $u_2 = 9$ so common difference $d = u_2 - u_1 = 2$

d $u_{10} = u_1 + 9d = 7 + 18 = 25$

e Least N such that $u_N > 1000$:

$$u_N = u_1 + d(N-1)$$
$$7 + 2(N-1) > 1000$$
$$N > 1 + \frac{993}{2} = 497.5$$

The first term which exceeds 1000 is u_{498}

f $S_n = 1512 = 6n + n^2$

Solving the quadratic: $n = 36$ (reject solution $n = -42$)

12 $u_1 = x$

$u_2 = u_1 + d = 2x + 4$ so $d = x + 4$ (1)

$u_3 = u_1 + 2d = 5x$ so $d = 2x$ (2)

$(1)\&(2)$: $x + 4 = 2x$ so $x = 4$

13 a The difference between consecutive terms in the sequence are

$d_1 = 10, d_2 = 12, d_3 = 9, d_4 = 9$.

Mean difference is 10 and all the observed differences are close to this value.

b $u_6 = u_1 + 5d$

Taking the value $d = 10$, this predicts $u_6 = 74$ audience members.

14 Sum of a geometric sequence with $u_1 = 2$ and $r = 2$.

$$S_{12} = u_1 \frac{r^{12}-1}{r-1} = 2 \times \frac{2^{12}-1}{2-1} = 8190$$

15 Let u_n be the number of widgets sold in the nth month.

$u_1 = 100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 20$.

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

Require least N such that $S_N \geqslant 4000$

$$\frac{N}{2}\left(200 + 20(N-1)\right) \geqslant 4000$$
$$10N^2 + 90N - 4000 \geqslant 0$$

From calculator, this has solution $N \geqslant 16$

It takes 16 months to sell a total of 4000 widgets.

16 $u_1 = a^2b^2$ and $r = \frac{u_2}{u_1} = \frac{a^4b}{a^2b^2} = a^2b^{-1}$

$$u_n = u_1 \times r^{n-1} = a^2b^2 \times \left(a^2b^{-1}\right)^{n-1} = a^2b^2 \times (a^{2n-2}b^{1-n})$$
$$u_n = a^{2n}b^{3-n}$$

17 Let u_n be the cost of the nth metre in thousand dollars.

$u_1 = 10$ and $\{u\}$ follows an arithmetic progression with common difference $d = 0.5$.

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$
$$S_{200} = 100(20 + 0.5 \times 199) = 11\,950$$

A 200 m tunnel will cost \$11.95 million.

18 Let u_n be the amount deposited on Elsa's $(n-1)$th birthday, in dollars.

$u_1 = 100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 50$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$
$$S_{19} = 9.5(200 + 50 \times 18) = \$10\,450$$

Tip: Note that this working used S_{19} because the standard formula for S_n gives $\sum_{r=1}^{n} u_r$. This requires defining u_n so that the first value is given as u_1. The 18th birthday gift was therefore u_{19}.

An alternative approach would be to define v_n as the amount deposited on the nth birthday, so $v_1 = 150$.

Then the total up to and including the 18th birthday gift would be

$$100 + \sum_{r=1}^{18} v_r = 100 + \frac{18}{2}(2 \times 150 + 50 \times 17) = 100 + 1\,350 = 1\,450$$

The answer is of course the same, and the method is entirely for the mathematician to choose; just make sure you define the terms in your sequence clearly, for both your own and your readers' benefit.

19 The annual real terms change multiplier is $1 + r\% \approx \frac{1+c\%}{1+i\%} = \frac{1.058}{1.0292} = 1.028$

So over 3 years, the real terms percentage increase is $1.028^3 - 1 = 8.63\%$

20 The annual real terms change multiplier is $1 + r\% \approx \frac{1+c\%}{1+i\%} = \frac{0.9}{1.02} = 0.882$

So after 4 years, the real terms value is $\$2000 \times 0.883^4 = \$1212.27$

21 Under scheme A, his balance after n years is $\$(1000 + 25n)$

Under scheme B, his balance after n years is $\$(1000 \times 1.02^n)$

By calculator, $A > B$ for $n < 22.7$

So, for the first 22 years, A is better than B.

22 **a** $a_1 = 6$

b **i** $a_2 = 8$

ii $a_3 = 10$

c $d = 2$

d **i** $a_n = 4 + 2n$

If the final (Nth) pumpkin has $a_N = 48$ (distance to the pumpkin and back to the start line) then $N = 22$.

ii $S_n = \frac{n}{2}\left(2a_1 + d(n-1)\right)$

$S_{22} = 11(12 + 2 \times 21) = 594$

Sirma ran a total of 594 m.

e $S_m = \frac{m}{2}\left(12 + 2(m-1)\right) = 940$

$m^2 + 5m = 940$

By calculator, $m = 28.3$

So Peter has completed 28 runs and has collected 28 pumpkins.

f $S_{28} = 14(12 + 2 \times 27) = 924$

The 29th pumpkin is $a_{29} = 62$ m away from the start line.

Peter has run $940 - 924 = 16$ m of the way through the 29th collection, so he has not reached the pumpkin and is still on the outward journey, 16 m away from the start line.

23 Let the arithmetic sequence be $\{u\}$ with common difference d and the geometric sequence be $\{v\}$ with common ratio r.

Then $u_n = a + d(n-1)$ and $v_n = v_1 \times r^{n-1}$

$u_7 = v_1 = a + 6d$ (1)

$u_3 = v_2 = a + 2d = v_1 r$ (2)

$u_1 = v_3 = a = v_1 r^2$ (3)

$(2) \div (1): r = \frac{a+2d}{a+6d}$ (4)

$(3) \div (2): r = \frac{a}{a+2d}$ (5)

Equating (4) and (5): $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$

$$(a+2d)^2 = a(a+6d)$$
$$a^2 + 4ad + 4d^2 = a^2 + 6ad$$
$$2ad = 4d^2$$

Then $a = 2d$ as required.

b $v_1 = u_7 = 3 = a + 6d = 4a$ so $a = \frac{3}{4}$ and $d = \frac{3}{8}$

Then $r = \frac{a}{a+2d} = \frac{1}{2}$

$$\sum_{r=1}^{n} u_r = \frac{n}{2}\big(2a + d(n-1)\big) = \frac{n}{2}\left(\frac{3}{4} + \frac{3}{8}(n-1)\right) = \frac{6n^2 + 9n}{16}$$

$$\sum_{r=1}^{n} v_r = v_1 \frac{(1-r^n)}{1-r} = 6\left(1 - \frac{1}{2^n}\right)$$

Require the least n such that $\frac{6n^2+9n}{16} \geqslant 200 + 6\left(1 - \frac{1}{2^n}\right)$

From the calculator, $n \geq 31.7$ so the least such n is 32.

24 a Under program A, she runs $a_m = 10 + 2(m-1)$ km on day m.

Under program B, she runs $b_n = 10 \times 1.15^{n-1}$ km on day n.

If $a_m \geqslant 42$ then $m \geqslant 17$; she first reaches 42 km on day 17.

If $b_n \geqslant 42$ then $n \geqslant 11.3$; she first reaches 42 km on day 12.

b) $\{a\}$ follows an arithmetic progression with $a_1 = 10$ and $d = 2$

$$\sum_{r=1}^{m} a_r = \frac{m}{2}\big(2a_1 + d(m-1)\big) = \frac{m}{2}\big(20 + 2(m-1)\big) = m^2 + 9m$$

$m^2 + 9m \geqslant 90$ for $m \geqslant 6$ so under program A she completes 90 km on day 6.

$\{b\}$ follows a geometric progression with $b_1 = 10$ and $r = 1.15$

$$\sum_{r=1}^{n} b_r = b_1 \frac{r^n - 1}{r - 1} = 10\frac{1.15^n - 1}{0.15}$$

$10\frac{1.15^n-1}{0.15} \geq 90$ for $m \geq 6.1$ so under program B she completes 90 km on day 7.

25 a Let u_n be the salary in year n, in thousand pounds.

$\{u\}$ follows an arithmetic progression with $u_1 = 25$ and common difference $d = 1.5$

$u_n = u_1 + d(n-1)$ so $u_{30} = 25 + 1.5 \times 29 = 68.5$

Final year salary is £68 500

b $S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$

$S_{30} = 15(50 + 1.5 \times 29) = £1\,402\,500$

c $\frac{68\,500}{1.015^{30}} = 43\,800$

In terms of the value at the beginning of the 30 year career, the final year salary has real value £43 800.

26 $u_{10} = u_1 + 9d$

$u_4 = u_1 + 3d$

If $u_{10} = 2u_4$ then $u_1 + 9d = 2u_1 + 6d$

Rearranging: $u_1 = 3d$ so $\frac{u_1}{d} = 3$

27 If the common difference of the sequence is k then

$b = a + k$

$c = a + 2k$

$d = a + 3k$

Then

$$2(b-c)^2 = 2(-k)^2 = 2k^2$$
$$\begin{aligned} bc - ad &= (a+k)(a+2k) - a(a+3k) \\ &= a^2 + 3ak + 2k^2 - (a^2 + 3ak) \\ &= 2k^2 \end{aligned}$$

Putting these together: $2(b-c)^2 = bc - ad$ as required.

3 Core: Functions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 3A

26 a $g(-2) = 4(-2) - 5$

$= -13$

b $4x - 5 = 7$ so $x = 3$

27

$$\frac{x-5}{3} = 12$$
$$x - 5 = 36$$
$$x = 41$$

28 a $v(1.5) = 3.8(1.5)$

$= 5.7 \text{ m s}^{-1}$

b It is unlikely that a car can accelerate uniformly at that level for 30 seconds (resistive forces that relate to speed would become significant before that time); the model would predict a speed of 114 m s^{-1} at 30 seconds, which is unrealistic for a car (over 400 km per hour!)

29 a Require that the denominator not equal zero: The largest domain of $f(x)$ is $x \neq 5$

b

$$f(2) = \frac{3}{(2-5)^2}$$
$$= \frac{3}{3^2}$$
$$= \frac{1}{3}$$

30 a

$$q\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4} - 2$$
$$= -\frac{5}{4}$$

b The function is in completed square form, so has minimum value -2.

The range is $q(x) \geqslant -2$

c $3x^2 - 2 = 46$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

31 a $N(5) = 2.3\mathrm{e}^{0.49} + 1.2 = 4.95$

The model predicts 4.95 billion smartphones in 5 years

b Market saturation (population that can afford a smartphone having sufficient, not buying additional without discarding/recycling one) might slow the exponential growth. Smartphones may get replaced by a newer technology within a 5 year period.

32 a $\mathrm{f}(x) = 1.3x$

b $\mathrm{f}^{-1}(x)$ would return the amount in pounds equal to $\$x$

33 a $\mathrm{f}^{-1}(6) = 1$

b $\mathrm{f}(x) = x + 2$ has solution $x = 2$

34 a Require that the square root has a non-negative argument.

$2x - 5 \geqslant 0$ so the largest possible domain is $x \geqslant \frac{5}{2} = 2.5$

b If the argument of the square root takes any value greater than or equal to zero then the range of the function is $\mathrm{f}(x) \geqslant 0$.

c $\sqrt{2x - 5} = 3$

$2x - 5 = 9$

$2x = 14$

$x = 7$

35 a At $t = 0$, $N = 150 - 90 \times 1 = 60$ fish

b

$N(15) = 150 - 90\mathrm{e}^{-1.5}$

$= 130$ fish (rounding to the nearest whole number)

c The model is continuous (it predicts non-integer values). Such a model is an acceptable approximation for populations numbering in the millions (such as microbial populations in a culture) but the inaccuracy becomes more relevant for a small population such as seen here.

36 a $\mathrm{f}(18) = \frac{18}{2} + 5$

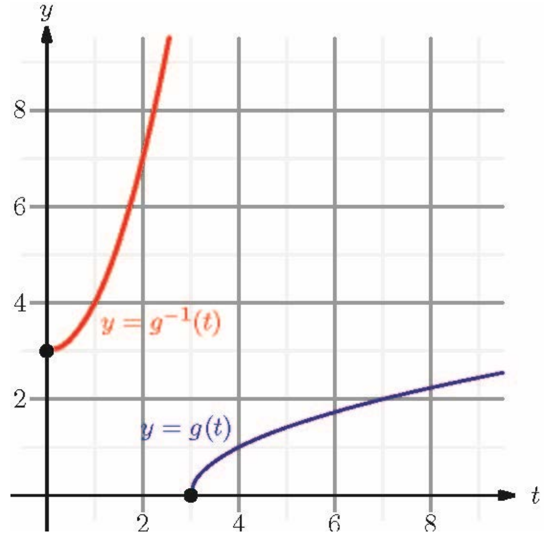
$= 14$

b $\frac{x}{2} + 5 = 7$

$\frac{x}{2} = 2$

$x = 4$

37 a The graph of the inverse function is the original graph after a reflection through $y = x$.



38 a Require that the logarithm takes a positive argument so the largest possible domain of g is $x > 0$.

b

$$\begin{aligned} g(81) &= \log_3 81 \\ &= \log_3 3^4 \\ &= 4 \end{aligned}$$

c $\log_3 x = -2$

$$\begin{aligned} x &= 3^{-2} \\ &= \frac{1}{9} \end{aligned}$$

39 Require that the logarithm has a positive argument.

$3x - 15 > 0$ so the largest domain of $n(x)$ is $x > 5$

40 a Require that the logarithm has a positive argument.

$7 - 3x > 0$ so the largest domain of $h(x)$ is $x < \frac{7}{3}$

b $\log(7 - 3x) = 2$

$$\begin{aligned} 7 - 3x &= 10^2 \\ 3x &= 7 - 100 \\ x &= -31 \end{aligned}$$

41 a $f(-3) = 10 - 3(-3)$

$= 19$

b The domain is $x \leqslant 2$ so the range is $f(x) \geqslant 10 - 3(2)$

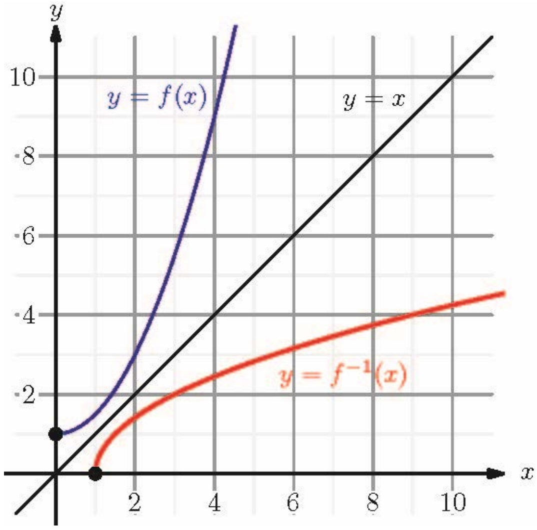
$f(x) \geqslant 4$

c The value 1 lies outside the range of the function.

42 a $f(4) = 9$

b $f^{-1}(4) \approx 2.5$

c The graph of the inverse function is the original graph after a reflection through $y = x$.



43 a

$$N(7) = 3e^{-2.8}$$
$$= 0.182$$

b $3e^{-0.4t} = 2.1$

$$t = -\frac{1}{0.4}\ln 0.7 = 0.892$$

44 a Require that the denominator is non-zero, so $4 - \sqrt{x-1} \neq 0$

$x - 1 \neq 16$

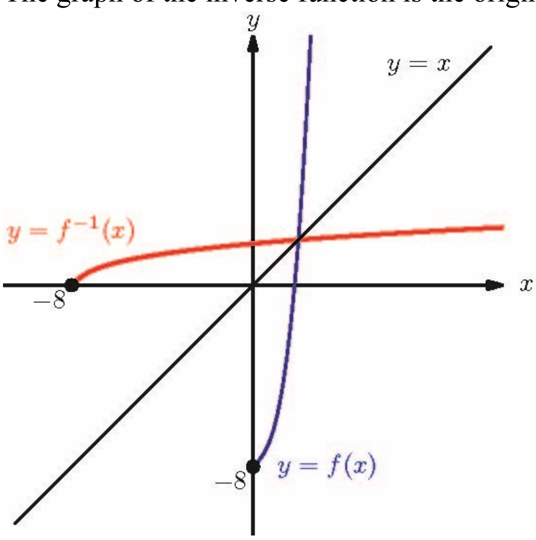
Also require that the argument of the square root is non-negative so $x - 1 \geqslant 0$

Then the largest domain is $x \geqslant 1, x \neq 17$

b From calculator (or by considering the two cases $4 - \sqrt{x-1} > 0$ and $4 - \sqrt{x-1} < 0$), the range is $f(x) \geqslant \frac{3}{4}$ or $f(x) < 0$

45 a Using GDC

b The graph of the inverse function is the original graph after a reflection through $y = x$.



c Either from GDC or by noting that where $f(x) = f^{-1}(x)$ the curves meet the line $y = x$ as well.

Thus $f(x) = x$ at the solution.

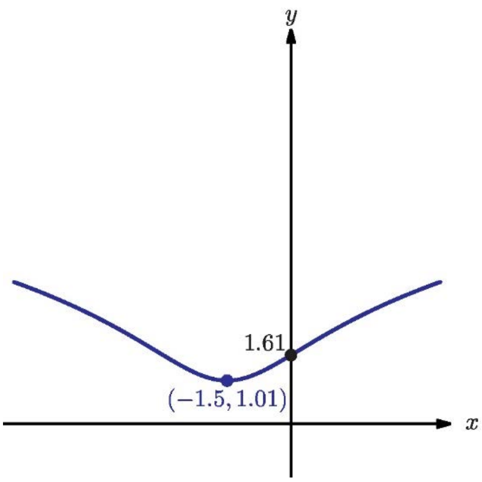
Then $x^3 + x - 8 = x$ so $x^3 = 8$, which gives solution $x = 2$.

Tip: It is tempting to hope that this always works to solve $f(x) = f^{-1}(x)$ but this is not the case for every function; you should always consider the graphs of f and f^{-1}. Consider the function $f(x) = 1 - x$. Since $f^{-1}(x) = 1 - x$ as well (the function is “self-inverse”), you can see that $f(x) = f^{-1}(x)$ for all values of x, even though the only solution on the line $y = x$ is $x = \frac{1}{2}$.

See if you can describe a condition on the graph of a function $h(x)$ for there to be solutions to $h(x) = h^{-1}(x)$ which do not lie on $y = x$.

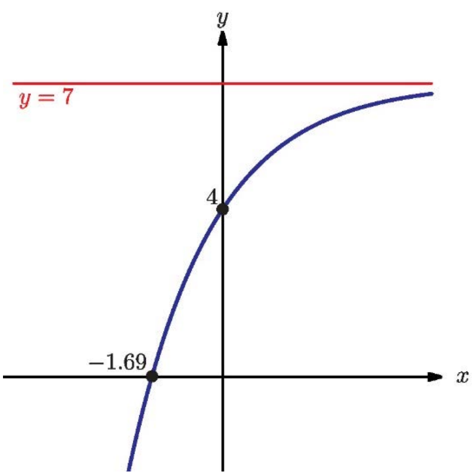
Exercise 3B

19



From GDC: Vertex is at $(-1.5, 1.01)$

20

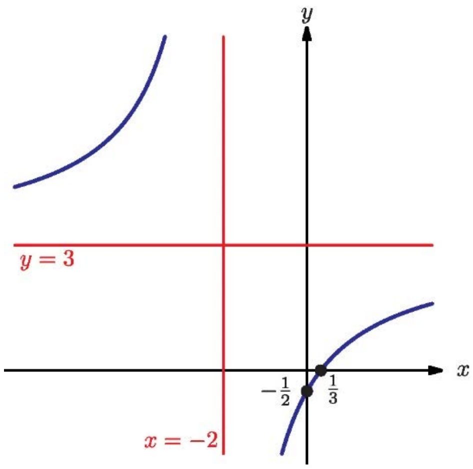


Asymptote $y = 7$; axis intercepts $(-1.69, 0)$ and $(0, 4)$

21 a Require denominator to be non-zero.

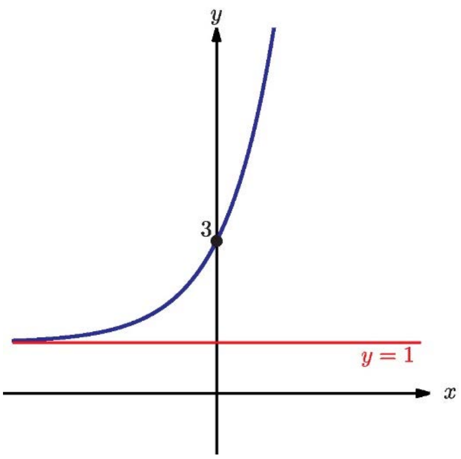
Largest domain is $x \neq 2$

b



Asymptotes are $y = 3$ and $x = -2$

22 For example, $y = 1 + 2e^x$



23 From GDC, the intersection of $y = 5 - x$ and $y = \frac{1}{2}e^x$ is $(1.84, 3.16)$

24 From GDC, maximum point is $(-1, 2.5)$

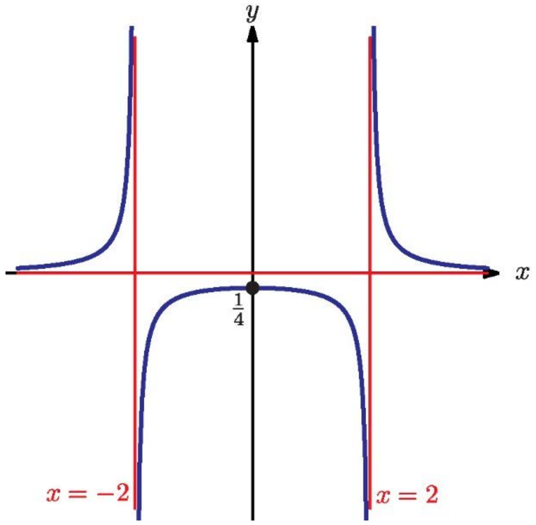
(Analytically, complete the square of the denominator and reason that since the denominator is always positive, the maximum will occur at the minimum value of the denominator, at $x = -1$.

$$y = \frac{10}{(x+1)^2 + 4}$$

25 From GDC, maximum value P occurs when $q = 235$

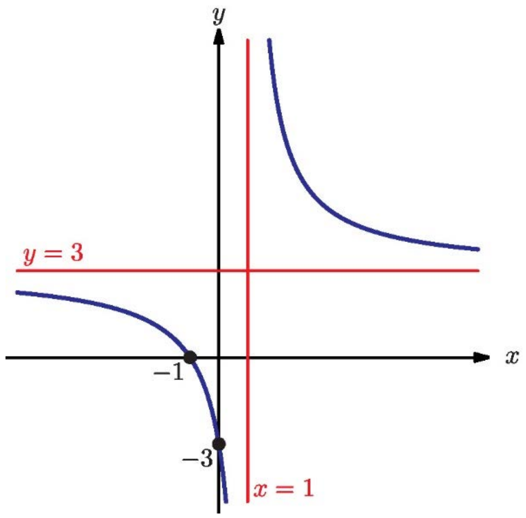
26 From GDC or completing the square to $y = \frac{1}{(x+1)^2+2}$, the line of symmetry is $x = -1$

27



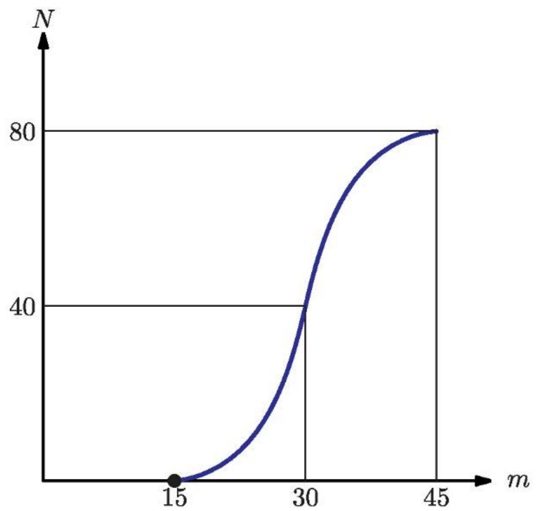
Vertical asymptotes are $x = -2$ and $x = 2$, horizontal asymptote is $y = 0$.

28



(Example: $y = \frac{3x+3}{x-1}$)

29



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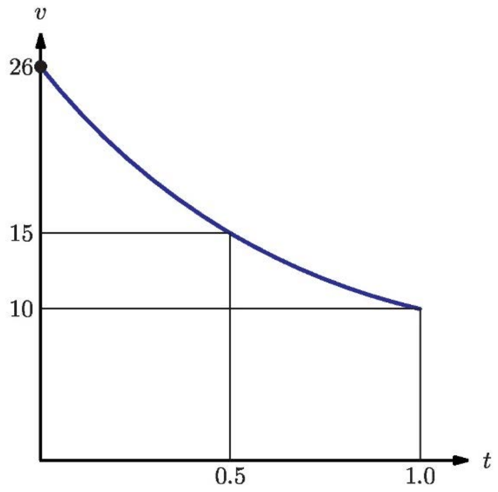
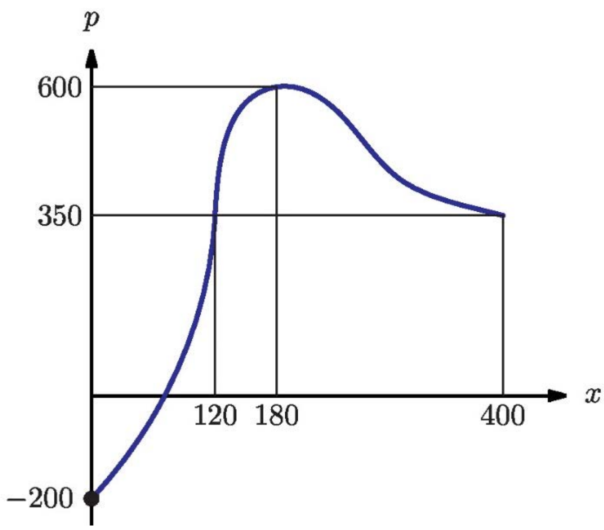


31



32 From GDC: $x = 0.755$

33 From GDC: $x = -2.20, -1.71$ or 1.91

34 From GDC: $x = -2.41, 0.414, 2$

35 From GDC: $x = \pm 1.41$

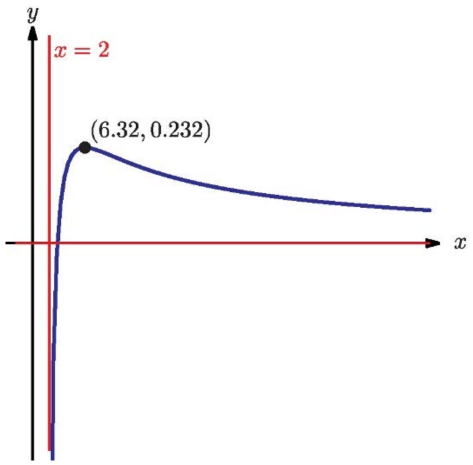
Solving exactly: $|4 - x^2| = x^2$

$$4 - x^2 = \pm x^2$$
$$4 = 0 \text{ (reject) or } 4 = 2x^2$$
$$x = \pm\sqrt{2} = \pm 1.41$$

36 From the calculator, maximum value is $f(2.5) = 0.920$

Tip: If you study calculus, you will learn to show algebraically that the maximum value occurs at $x = 2.5$)

37



38 From GDC: $x \geqslant 2$

Algebraically: $\ln|x-1| = |\ln(x-1)|$.

LHS equals a modulus function so must be non-negative. $\ln|x-1| \geqslant 0$ so $x \geqslant 2$

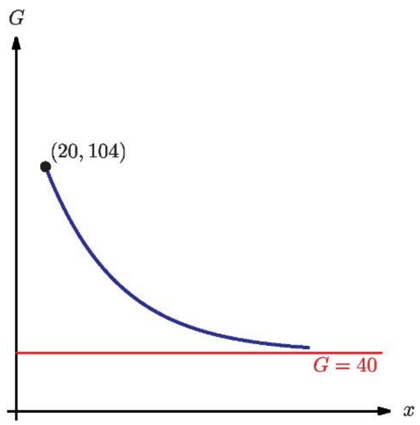
But for $x \geqslant 2$, $\ln|x-1| = \ln(x-1)$ and $\ln(x-1) \geqslant 0$ so $\ln|x-1| = \ln(x-1)$

So for $x \geqslant 2$, the two sides are always equivalent, with no further restriction.

Therefore the full solution is just $x \geqslant 2$.

Mixed Practice

1 **a**



b $G(45) = 78.6$

So the total cost is $45 \times G(45) = \$3\,538.09 \approx \$3\,540$

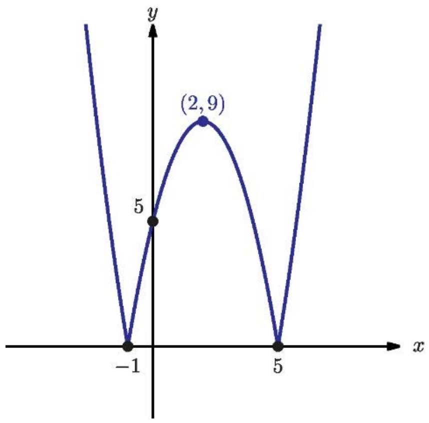
2 From GDC: $x = 1.86$ or 4.54

3 **a** Require that the argument of the square root is non-negative so require $x \geqslant -5$

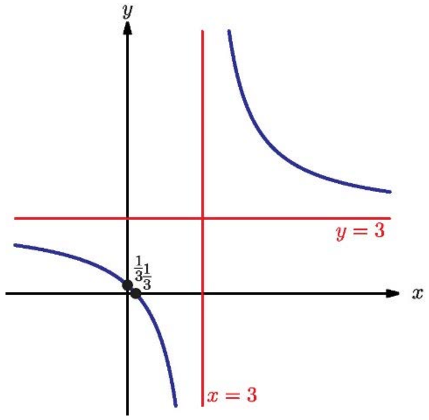
b From GDC, $x = -2.38$

4 From GDC, intersection points are $(-1.68, 0.399)$ and $(0.361, 2.87)$

5



6 a



b Domain: $x \neq 3$

Range: $f(x) \neq 3$

7 a $v(1.5) = 18 \times 1.5e^{-0.3} \approx 20.0 \text{ m s}^{-1}$

b From GDC: At $t = 0.630$ s and $t = 17.1$ s

c Maximum speed occurs at $t = 5$ s

8 a $v_1(0) = 8 - 6 = 2$

$v_2(0) = 2 + 0 - 0 = 2$

$v_1(0) = v_2(0)$ so the two runners have the same initial speed.

b From GDC, $v_1(t) = v_2(t)$ when $t = 1.30$ s

9 a $T(0) = 100$. Boiling point at sea level is 100 °C

b $T(1370) = 100 - 0.0034 \times 1370 = 95.3$ °C

c $70 = 100 - 0.0034h$ so $h = \frac{30}{0.0034} = 8\,820 \text{ m} = 8.82 \text{ km}$

10 a $f(x) = 1.8x + 32$

b $f^{-1}(x)$ gives the temperature in Celsius equivalent to x °F

11 a $p = f(1) = 3 - 5 = -2$

$3q - 5 = 7$ so $q = 4$

b **i** Require that the denominator is non-zero, so the denominator is $x \neq 2$

ii The horizontal asymptote is $y = 0$ so the range of the function is $g(x) > 0$

iii The vertical asymptote is $x = 2$

12 **a** $f(7) = 20$ and the function increases so the range is $f(x) \geqslant 20$

b $3x - 1 = 35$

$3x = 36$

$x = 12$

13 From GDC: $x = -5.24$ or 3.24

Algebraically:

$$2x - 1 = (4 - x)(4 + x)$$
$$= 16 - x^2 \text{ and } x \neq -4$$

$x^2 + 2x - 17 = 0$ and $x \neq -4$

$$x = -1 \pm 3\sqrt{2}$$

14 From GDC: $x = 1$ or 2.41

Algebraically:

$$x - 2 = \pm\frac{1}{x} \text{ and } x > 0 \text{ (since } \frac{1}{x} > 0)$$

$x^2 - 2x \pm 1 = 0$ and $x > 0$

$$(x - 1)^2 = 0 \text{ or } \left(x - 1 + \sqrt{2}\right)\left(x - 1 - \sqrt{2}\right) = 0 \text{ and } x > 0$$

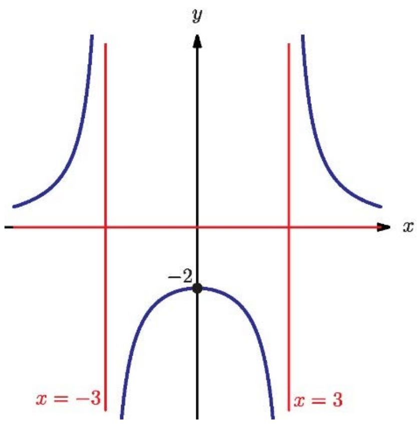
$x = 1$ or $x = 1 + \sqrt{2}$

15 Require denominator is non-zero so domain is $x \neq \pm 3$

From GDC (or symmetry), maximum point is at $x = 0$ so local max in the $-2 < x < 2$ interval is $h(0) = -2$.

For $x < -2$ and $x > 2$, $h(x) > 0$ with an asymptote of $y = 0$.

Hence the range is $h(x) \leqslant -2$ or $h(x) > 0$



16 From GDC, function has single stationary point at minimum $f(3.5) = -9.25$

The (unincluded) endpoints of the curve are $(0,3)$ and $(6,-3)$

The range is therefore $-9.25 \leqslant f(x) < 3$

17 a From GDC, the minimum is $f(\ln 0.8) = 4.89$

b $f^{-1}(x) = 2$ so $x = f(2) = 5e^2 - 8 \approx 28.9$

18 a Function has maximum at $f(0) = 8$.

The (unincluded) endpoints of the curve are $(-3,-19)$ and $(2,-4)$

The range is therefore $-19 < f(x) \leqslant 8$

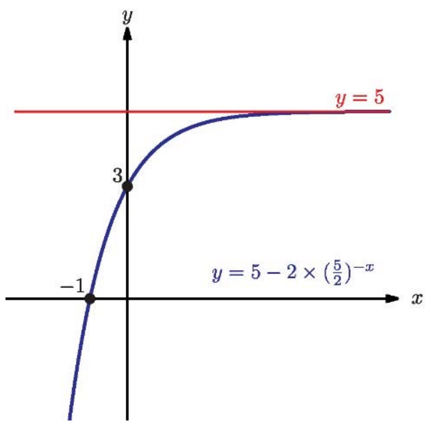
b $g(x) = 8 - 3x^2 = 5$

$3x^2 = 3$

$x = 1, x = -1$

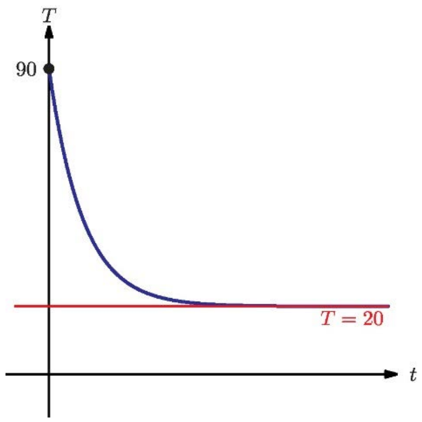
c -20 is outside the defined range of $g(x)$.

19



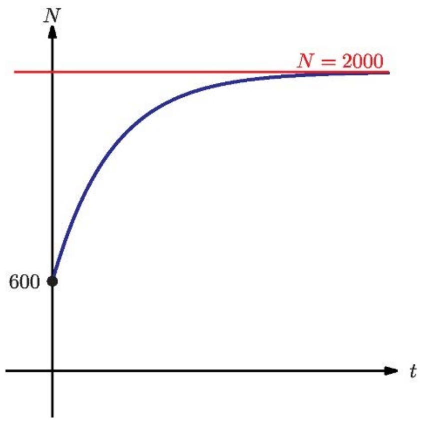
(Example: $y = 5 - 2 \times 0.4^x$)

20



(Example: $T = 20 + 70e^{-t}$)

21



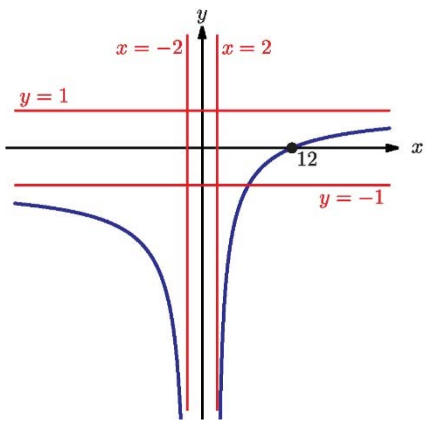
(Example: $N = 2000 - 1400e^{-t}$)

22 a From GDC, maximum is $v(1) = 1.10$ m s^{-1}

b The model predicts an insignificant speed at $t = 20$. Realistically in a physical system, the car would have halted before that time, when resistances become significant when compared to the model's predicted behaviour.

23 From the GDC, solutions are $x = -2.50, -1.51, 0.440$

24 a



b i x-intercept is at $(12,0)$

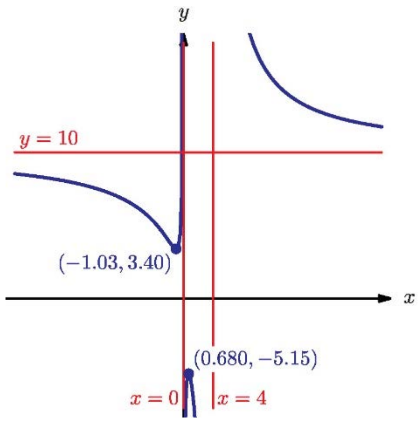
ii Asymptotes are $x = \pm 2$, $y = \pm 1$

25 From GDC, solutions are $x = 2.27$ or 4.47

26 From GDC, the minimum value is $f(1.16) = -1.34$

27 a Require the denominator is non-zero so the domain is $x \neq 0,4$

b



Range is $\mathrm{f}(x) \leqslant -5.15$ or $\mathrm{f}(x) \geqslant 3.40$

28 a Require denominator to be non-zero, so $x \neq \mathrm{e}^{-3}$

Also require argument of logarithm to be positive, so $x > 0$

So the largest possible domain of $g(x)$ is $x > 0, x \neq \mathrm{e}^{-3}$

b From GDC:

As $x \to 0$, $g(x) \to 0$ (from below)

As $x \to \mathrm{e}^{-3}$ from below, $g(x) \to -\infty$

As $x \to \mathrm{e}^{-3}$ from above, $g(x) \to \infty$

As $x \to \infty$, $g(x) \to \infty$

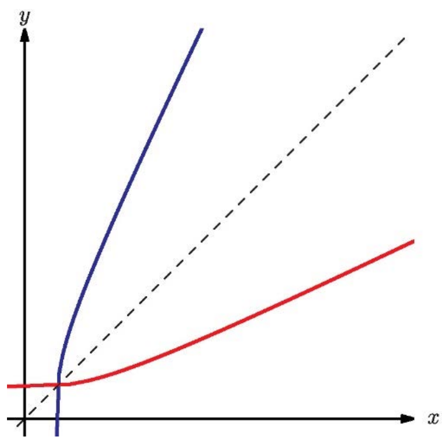
The curve has a single minimum at $f(0.135) = 0.271$

Tip: If you study further differentiation, see if you can show that the minimum has exact value $f(\mathrm{e}^{-2}) = 2\mathrm{e}^{-2}$. For this exercise though, the decimal approximation from your calculator is the faster approach, and is what the question requires.

29 a Require the argument of the logarithm to be positive, so the domain is $x > 2$

For $x > 2$, $g(x)$ has range $\mathbb{R}$.

b



c From GDC, $g(x) = g^{-1}(x)$ at $x = 2.12$

4 Core: Coordinate geometry

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 4A

34 a

$$\text{Gradient } m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-7)}{5 - 1} = \frac{7}{4}$$

b Perpendicular gradient $m = -\frac{1}{m_{AB}} = -\frac{4}{7}$

$$y - y_C = m(x - x_C)$$
$$y - 3 = -\frac{4}{7}(x - 8)$$
$$y = -\frac{4}{7}x + \frac{53}{7}$$

35 a Using the axis intercepts $(0,12)$ and $(9,0)$ to determine gradient:

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 12}{9 - 0} = -\frac{4}{3}$$

b $y - y_1 = m(x - x_1)$

$$y - 12 = -\frac{4}{3}(x - 0)$$

$4x + 3y = 36$

36 a Line $y = -\frac{5}{7}x + \frac{17}{7}$ has gradient $-\frac{5}{7}$

b When $y = 0$. $x = \frac{17}{5}$

37 a

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{7-(-3)} = \frac{1}{5}$$

b $y - y_1 = m(x - x_1)$

$$y - 1 = \frac{1}{5}(x - (-3))$$
$$5y - 5 = x + 3$$
$$x - 5y = -8$$

38 a Rearranging: $l_1: y = 21 - \frac{7}{2}x$

Gradient $m_1 = -\frac{7}{2}$

b Substituting $x = 8, y = -5$:

$7x + 2y = 56 - 4 = 52 \neq 42$

P does not lie on the line.

c Gradient $m_2 = -\frac{1}{m_1} = \frac{2}{7}$

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = \frac{2}{7}(x - 8)$$
$$y = \frac{2}{7}x - \frac{51}{7}$$

39 Rise $= 8\,800$

Tread $= 12\,000$ (half the cone diameter)

Gradient $m = \frac{\text{Rise}}{\text{Tread}} = \frac{8800}{12000} = 0.733$

40 a $Q: (8, 11)$

b $k = 8$

c QR is the vertical line $x = 8$

d Area $= \frac{1}{2}bh = \frac{1}{2} \times 8 \times 14 = 56$

41 a $y = -\frac{3}{2}x - 6$

b Gradient $m_2 = \frac{-2-1}{7-5} = -\frac{1}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = -\frac{1}{4}(x - 7)$$
$$4y + 8 = -x + 7$$
$$x + 4y = -1$$

c From GDC, the lines intersect at $(-4.6, 0.9)$

42 If the pivot point is the origin, then the end of the plank is at (x, y) where $\frac{y}{x} = 0.7$ and $y = 3.5$

So $x = \frac{y}{0.7} = 5$

Then the plank length is $\sqrt{5^2 + 3.5^2} = 6.10$ m

43 a $V = 0.1 + 0.5t$

b When $V = 5, 0.5t = 4.9$

$t = 9.8$

It would take 9.8 seconds to pop the balloon.

44 a Newtons per metre (N/m)

b $F = 0.3x$

If $x = 0.06$ then $F = 0.018$

Require 0.018 N

c A stiffer spring will require more force for the same stretch; k will be greater.

d $x = \frac{F}{k}$

So when $F = 0.14$ and $k = 0.3$, the stretch is $\frac{0.14}{0.3} = 0.467$ m $= 46.7$ cm

45 a $C_1 = 5 + 0.01m$

b $C_1(180) = 6.8$

The cost is \$6.80 per month

c New model: $C_2 = 0.02m$

Intersection: $0.02m = 5 + 0.01m$

$m = 500$

The first contract will be better if Joanna expects to talk more than 500 minutes per month (6 hours 20 minutes).

46 a For n items sold in a single month, the monthly profits are

$P_1 = 10n - 2000$

b $P = 1500 = 10n - 2000$

$n = 350$

c

$$\begin{aligned} P_2 &= 10n - (1200 + 2n) \\ &= 8n - 1200 \end{aligned}$$

d Intersection: $10n - 2000 = 8n - 1200$

$2n = 800$

$n = 400$

Beyond this point, the first model predicts greater profits (steeper slope/no marginal costs).

For $P = 1500$, which is lower than the intersection point of the two models, the company will need to sell fewer items under the second model.

47 a Gradient of AB is $m_{AB} = \frac{8-3}{3-(-4)} = \frac{5}{7}$

Gradient of CD is $m_{CD} = \frac{-11-(-1)}{-9-5} = \frac{-10}{-14} = \frac{5}{7}$

So lines AB and CD are parallel, since they have the same gradient

b However, AB and CD are different lengths, so the shape $ABCD$ is not a parallelogram.

48 Considering the path as the hypotenuse of a right-angled triangle, the rise must be 400 m and $\frac{\text{rise}}{\text{tread}} = 0.3$

So tread $= \frac{\text{rise}}{0.3} = 1333$ m

Then the length of the path is $\sqrt{400^2 + 1333^2} = 1390$ m

Exercise 4B

10 a $\left(\frac{-4+7}{2}, \frac{1+0}{2}, \frac{9+2}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}, \frac{11}{2}\right) = (1.5, 0.5, 5.5)$

b

$$\begin{aligned} AB &= \sqrt{(7-(-4))^2 + (0-1)^2 + (2-9)^2} \\ &= \sqrt{121 + 1 + 49} \\ &= \sqrt{171} \end{aligned}$$

11 $B: (b_x, b_y, b_z)$ where $(5, 1, -3) = \left(\frac{b_x+4}{2}, \frac{b_y-1}{2}, \frac{b_z+2}{2}\right)$

So B is $(6, 3, -8)$

12 $\left(\frac{-4+b}{2}, \frac{a+1}{2}, \frac{1+8}{2}\right) = (8, 2, c)$

$a = 3, b = 20, c = \frac{9}{2} = 4.5$

13 Midpoint M has coordinates $\left(\frac{3+2}{2}, \frac{-18-2}{2}, \frac{8+11}{2}\right) = (2.5, -10, 9.5)$

Distance $OM = \sqrt{2.5^2 + (-10)^2 + 9.5^2} = \sqrt{196.5}$

14 Distance travelled $d = \sqrt{14^2 + 3^2 + 6.7^2} = \sqrt{249.89}$

Average speed $= \frac{\text{total distance}}{\text{time taken}} = \frac{d}{3.5} = 4.52\text{ m s}^{-1}$

15 $\sqrt{k^2(1^2+2^2+5^2)}=30$

$k=\frac{30}{\sqrt{30}}=\sqrt{30}$ (selecting positive root for k)

16 $\sqrt{(k-1)^2+(k+1)^2+(-3k)^2}=\sqrt{46}$

$$\begin{aligned}11k^2+2&=46\\ k^2&=4\\ k&=\pm 2\end{aligned}$$

17 $\sqrt{a^2(2^2+1^2+5^2)}=2\sqrt{(-4)^2+1^2+7^2}$

$$a=2\frac{\sqrt{66}}{\sqrt{30}}=2.97$$

18 a $\left(\frac{3a+1+5-b}{2},\frac{2a+b+3}{2}\right)=(4,-5)$

$3a-b+6=8 \quad (1)$

$2a+b+3=-10 \quad (2)$

$(1)+(2): 5a+9=-2$ so $a=-2.2$

$(2): b=-13-2a=-8.6$

b So $P: (-5.6,-4.4)$ and $Q: (13.6,-5.6)$

$$\begin{aligned}\text{Gradient } m&=\frac{y_2-y_1}{x_2-x_1}\\ &=\frac{-5.6-(-4.4)}{13.6-(-5.6)}\\ &=\frac{-1.2}{19.2}\\ &=-\frac{1}{16}\end{aligned}$$

$$\begin{aligned}y-y_1&=m(x-x_1)\\ y+5&=-\frac{1}{16}(x-4)\end{aligned}$$

19 a Rearranging: $y=\frac{4}{7}x-5$

Gradient $m_1=\frac{4}{7}$

b Perpendicular gradient $m_2=-\frac{7}{4}$

$$\begin{aligned}y-y_1&=m(x-x_1)\\ y-2&=-\frac{7}{4}\big(x-(-4)\big)\\ 4y-8&=-7x-28\\ 7x+4y&=-20\end{aligned}$$

c $l_1: 4x-7y=35 \quad (1)$

$l_2: 7x+4y=-20 \quad (2)$

$4(1)+7(2): 65x=0$ so $x=0, y=-5$

$P: (0,-5)$

d Shortest distance will be perpendicular to l_1 and so equals NP.

$$NP = \sqrt{\left(0-(-4)\right)^2 + (-5-2)^2}$$
$$= \sqrt{65}$$
$$= 8.06$$

20 a 1.8 m

b Midpoint of BC is $G: (3, 1, 0)$

$$GE = \sqrt{(3-0.2)^2 + (1-1)^2 + (0-1.8)^2}$$
$$= \sqrt{11.08}$$
$$= 3.33 \text{ m}$$

21 a Midpoint $M: (3,7)$

b From A to M: 2 units right and 3 units down

So from M to B will be the same translation, but rotated 90°: 2 units down and 3 units left

$B: (6,5)$

Then M is the midpoint of BD, since it is the centre of the square

$D: (0,9)$

22 a Gradient $m_{AC} = \frac{8-1}{8-1} = 1$

So Gradient $m_{BD} = -1$

The centre of the rhombus is M, the midpoint of AC

M has coordinates $(4.5, 4.5)$

So line BD has equation $y - 4.5 = -(x - 4.5)$

$y = 9 - x$ or $y + x = 9$

b $m_{AB} = \frac{4}{3}$ so line AB has equation $y - 1 = \frac{4}{3}(x - 1)$

$$y = \frac{4}{3}x - \frac{1}{3}$$

B is the intersection of these lines. Substituting:

$$9 - x = \frac{4}{3}x - \frac{1}{3}$$
$$\frac{7}{3}x = \frac{28}{3}$$

$x = 4$ so $y = 5$. B has coordinates $(4,5)$

Since M is the midpoint of BD, it follows that D has coordinates $(5,4)$

c $AB = \sqrt{(4-1)^2 + (5-1)^2} = 5$

23 a Diagonal of the cuboid has length $\sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$ m

b Considering the net of the room, the shortest distance for the spider would be the diagonal of the rectangle formed by any two sides of the cuboid.

The options are:

Diagonal of a 3 by $(4+5)$: $\sqrt{3^2+9^2}=\sqrt{90}$

Diagonal of a 4 by $(3+5)$: $\sqrt{4^2+8^2}=\sqrt{80}$

Diagonal of a 5 by $(3+4)$: $\sqrt{5^2+7^2}=\sqrt{74}\approx 8.60$ m

Mixed Practice 4

1 **a** **i** $M: \left(\frac{4+0}{2}, \frac{1-5}{2}\right) = (2, -2)$

ii Gradient $m_{PQ} = \frac{-5-1}{0-4} = \frac{3}{2}$

iii Perpendicular gradient is $-\frac{2}{3}$

b Line has equation $y - y_1 = m(x - x_1)$

$$y + 2 = -\frac{2}{3}(x - 2)$$

Substituting $x = 0, y = k$: $k + 2 = \frac{4}{3}$ so $k = -\frac{2}{3}$

2 **a** $M: (2,3) = \left(\frac{s-2}{2}, \frac{8+t}{2}\right)$

So $s = 6, t = -2$

b Gradient of AB is $m_{AB} = \frac{t-8}{-2-s} = \frac{-10}{-8} = \frac{5}{4}$

So the perpendicular gradient is $m_{\perp} = -\frac{4}{5}$

Line has equation $y - y_1 = m(x - x_1)$

$$y - 3 = -\frac{4}{5}(x - 2)$$
$$5y - 15 = -4x + 8$$
$$4x + 5y = 23$$

3 **a** When $x = 6, y = 0$, $2y - 3x = 0 - 18 = -18 \neq 11$

Since $2y - 3x \neq 11$ at point A, it follows that A does not lie on L_1

b Rearranging: $y = \frac{3}{2}x + \frac{11}{2}$

Gradient of L_1 is $\frac{3}{2}$

c Perpendicular gradient $m_{\perp} = -\frac{2}{3}$

d Line has equation $y - y_1 = m(x - x_1)$

$$y - 0 = -\frac{2}{3}(x - 6)$$
$$y = -\frac{2}{3}x + 4$$
$$c = 4$$

4 a $m_{AB} = \frac{(10-6)}{-1-3} = -1$

b Perpendicular gradient $m_{\perp} = 1$

Line has equation $y - y_1 = m(x - x_1)$

$$y - 6 = 1(x - 3)$$
$$y = x + 3$$

c P has coordinates $(0,3)$ and Q has coordinates $(3,0)$

Area $OPQ = \frac{1}{2} \times 3 \times 3 = 4.5$

5 a $M: \left(\frac{-1+6}{2}, \frac{2-4}{2}\right) = (2.5, -1)$

b

$$PQ = \sqrt{\left(6-(-1)\right)^2 + (-4-2)^2}$$
$$= \sqrt{85}$$
$$= 9.22$$

c Gradient of PQ is $m_{PQ} = \frac{-4-2}{6-1} = -\frac{6}{7}$

Perpendicular gradient $m_{\perp} = \frac{7}{6}$

Line has equation $y - y_1 = m(x - x_1)$

$$y - (-1) = \frac{7}{6}(x - 2.5)$$
$$y = \frac{7}{6}x - \frac{47}{12}$$

6 a $M: \left(\frac{-1+6}{2}, \frac{2-4}{2}, \frac{5+3}{2}\right) = (2.5, -1, 4)$

b

$$PQ = \sqrt{\left(6-(-1)\right)^2 + (-4-2)^2 + (3-5)^2}$$
$$= \sqrt{89}$$
$$= 9.43$$

7 a When $x = 0$, $y = 3$ and when $y = 0$, $x = 6$ so P has coordinates $(0,3)$ and Q has coordinates $(6,0)$.

b $PQ = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$

c Substituting $y = x$ into the equation for l_1: $3x = 6$ so $x = 2$, and the intersection is $(2,2)$

8 a Rearranging the line equation: $y = -\frac{7}{4}x + \frac{d}{4}$

Gradient $m_{MN} = -\frac{7}{4}$

b Gradient $m_{MN} = \frac{k-(-5)}{-1-3} = -\frac{k+5}{4}$

Equating with the answer to part **a**: $k = 2$

c Line through $N(-1,2)$ with gradient $m_{MN} = -\frac{7}{4}$ has equation

$$(y-2) = -\frac{7}{4}(x-(-1))$$
$$y = -\frac{7}{4}x + \frac{1}{4}$$

Equating with the answer to part **a**: $d = 1$

9 **a** $m_{AB} = \frac{5-8}{2-(-3)} = -\frac{3}{5}$

$$m_{DC} = \frac{(6-9)}{1-(-4)} = -\frac{3}{5}$$

So $AB \parallel DC$

$$m_{AD} = \frac{9-8}{-4-(-3)} = -1$$
$$m_{BC} = \frac{6-5}{1-2} = -1$$

So $AD \parallel BC$

The quadrilateral has two pairs of parallel sides, so is a parallelogram

b Since $m_{AD} \times m_{AB} \neq -1$, the angle at A is not a right-angle, so the shape is not a rectangle.

10 If the three vertices are $A(-2,5)$, $B(1,3)$ and $C(5,9)$

$$m_{AB} = \frac{3-5}{1-(-2)} = -\frac{2}{3}$$
$$m_{AC} = \frac{9-5}{5-(-2)} = \frac{4}{7}$$
$$m_{BC} = \frac{9-3}{5-1} = \frac{3}{2}$$

Then $m_{AB} \times M_{BC} = -1$ so $AB \perp BC$ and therefore the triangle has a right angle at B.

11 $\sqrt{(-4)^2 + a^2 + (3a)^2} = \sqrt{416}$

$$16 + 10a^2 = 416$$
$$a^2 = 400$$
$$a = \pm 20$$

12 **a** Midpoint M: $\left(\frac{2-6}{2}, \frac{p+5}{2}, \frac{8+q}{2}\right) = (-2,3,-5)$

$p = 1, q = -18$

b $A: (2,1,8), B: (-6,5,-18)$

$$AB = \sqrt{(2-(-6))^2 + (1-5)^2 + (8-(-18))^2}$$
$$= \sqrt{756}$$
$$= 27.5$$

13 $m = \frac{6}{4} = 1.5$

14 Taking the equations simultaneously: $\frac{1}{2}x - 3 = 2 - \frac{2}{3}x$

$$\frac{7}{6}x = 5$$

$x = \frac{30}{7}$ so $y = -\frac{6}{7}$

$$P: \left(\frac{30}{7}, -\frac{6}{7}\right)$$

Distance $OP = \frac{1}{7}\sqrt{30^2 + 6^2} = \frac{6}{7}\sqrt{26} = 4.37$

15 a Gradient of AB $m_{AB} = \frac{0-3}{5-(-4)} = -\frac{1}{3}$

The perpendicular gradient is $m_{\perp AB} = 3$ so line $y = 3x$ is perpendicular to AB.

b Gradient of AC $m_{AC} = \frac{7-3}{4-(-4)} = \frac{1}{2}$

The perpendicular gradient is $m_{\perp AC} = -2$

Midpoint of AC $M = (0,5)$

Line equation $y - y_1 = m(x - x_1)$

$$y - 5 = -2(x - 0)$$

l_2 has equation $y = -2x + 5$

c Intersection of $y = 3x$ and $y = -2x + 5$:

$$3x = -2x + 5$$

$$5x = 5$$

$x = 1$ so S has coordinates $(1,3)$

$$SA = \sqrt{(1 - (-4))^2 + (3 - 3)^2} = 5$$

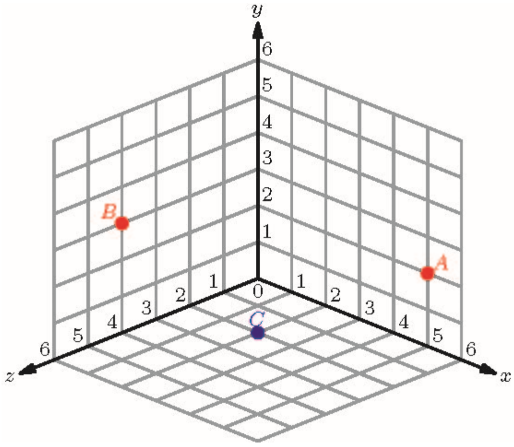
$$SB = \sqrt{(1 - 4)^2 + (3 - 7)^2} = 5$$

$$SC = \sqrt{(1 - 5)^2 + (3 - 0)^2} = 5$$

So point S is equidistant from the three points.

Tip: S is called the "circumcentre" of the triangle, and lies at the common intersection of all three side perpendicular bisectors. Because it is equidistant from all three vertices, a circle drawn with centre at the circumcentre through one of the vertices will also pass through the others as well, so that the triangle is inscribed exactly in a circle. You may like to investigate properties of the other major triangle "centre points" called orthocentre, centroid and incentre, and their relationships.

17 a



b $M: \left(\frac{5+0}{2}, \frac{2+3}{2}, \frac{0+4}{2}\right) = (2.5, 2.5, 2)$

c

$$AB = \sqrt{(0-5)^2 + (3-2)^2 + (4-0)^2}$$
$$= \sqrt{42}$$
$$= 6.48$$

18 a Gradient $m_{AB} = \frac{8-2}{5-(-7)} = \frac{1}{2}$

b $M: \left(\frac{5-7}{2}, \frac{2+8}{2}\right) = (-1,5)$

c Perpendicular gradient $m_1 = -2$

Equation of l_1:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -2(x - (-1))$$
$$2x + y = 3$$

d Substituting $x = 1, y = 1$ into the equation of l_1:

$2x + y = 2 + 1 = 3$ so N does lie on l_1

e Then the distance from N to line AB is the distance MN

$$MN = \sqrt{(1-(-1))^2 + (1-5)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5} = 4.47$$

19 a When $x = 0$, $y = -6$ so $A: (0,6)$ is the y-axis intercept

When $y = 0, x = 14$ so $B: (14,0)$ is the x-axis intercept

So triangle AOB has area $\frac{1}{2} \times 6 \times 14 = 42$

b

$$AB = \sqrt{6^2 + 14^2}$$
$$= \sqrt{232}$$
$$= 15.2$$

c Since the area can be calculated using any base side, and $AB = \sqrt{232}$, it follows that the distance from AB to the vertex O is $2 \times \frac{42}{\sqrt{232}} = 5.51$

20 When $y = 0$ on l_1 then $x = 10$ so $P: (10, 0)$

When $y = 0$ on l_2 then $x = \frac{9}{2}$ so $Q: \left(\frac{9}{2}, 0\right)$

Intersecting the lines: $l_1: x = 10 - 2y$ and $l_2: x = \frac{3}{2}y + \frac{9}{2}$

$$10 - 2y = \frac{3}{2}y + \frac{9}{2}$$
$$\frac{7}{2}y = \frac{11}{2}$$
$$y = \frac{11}{7}$$

So R has y-coordinate $\frac{11}{7}$ which is the altitude of triangle PQR , since the base lies along the x-axis.

Area $PQR = \frac{1}{2} \times \frac{11}{7} \times \frac{9}{2} = 4.32$

21 a Midpoint M of AC, which is the midpoint of the square, is $M: \left(\frac{8+2}{2}, \frac{3+1}{2}\right) = (5, 2)$

Gradient of AC is $m_{AC} = \frac{3-1}{2-8} = -\frac{1}{3}$

So the perpendicular gradient $m_{\perp AC} = m_{BD} = 3$

Then the equation of the other diagonal is $y - 2 = 3(x - 5)$

BD has equation $y = 3x - 13$

b From M to A is translation 3 right and 1 down, so from M to B will be a rotation 90° of this: 3 up and 1 right, so B is $(6, 5)$ and then since M is the midpoint of BD, D has coordinates $(4, -1)$

22 Midpoint of AC is $M_{AC} = \left(\frac{-3+9}{2}, \frac{2+(-2)}{2}\right) = (3, 0)$

Midpoint of $BD$is $M_{BD} = \left(\frac{4+2}{2}, \frac{3+(-3)}{2}\right) = (3, 0)$

So the two diagonals have a common midpoint.

Gradient of AC is $m_{AB} = \frac{-2-2}{9-3} = -\frac{1}{3}$

Gradient of BD is $m_{BD} = \frac{-3-3}{2-4} = 3$

$m_{AC} \times m_{BD} = -1$ so the two diagonals are perpendicular.

Since only a rhombus has bisecting diagonals which are perpendicular, it follows that $ABCD$ is a rhombus.

Tip: There are several alternatives here – pick a set of defining properties of a rhombus and show that they are true; an alternative would be to show that $AB \parallel CD$ and $AD \parallel BC$ (so that it is shown to be a parallelogram) and then also show that $AB = AD$.

As another option, ignore gradients altogether, and show that $AB = BC = CD = AD$

23 Gradient $0.15 = \frac{\text{Rise}}{\text{Tread}}$

So the horizontal distance is Tread $= \frac{\text{Rise}}{0.15} = \frac{20}{0.15} = 133.3$ m

Then the distance travelled (the hypotenuse of the triangle with vertical distance 20 and horizontal distance 133.3 is $\sqrt{20^2 + 133.3^2} = 135$ m

24 If the vertical distance of each section is 6 m and the gradient is 0.75 then the horizontal distance (when elevated) is $\frac{6}{0.75} = 8$ m.

The length of each section is therefore $\sqrt{6^2 + 8^2} = 10$ m

Then by elevating the bridge section, the ends each move 2 m from the midpoint of the bridge, for a total 4 m separation.

5 Core: Geometry and trigonometry

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 5A

16

$$\begin{aligned}\text{Surface Area} &= 4\pi r^2\\ &= 4\pi \times (7.5)^2\ \text{cm}^2\\ &= 225\pi\ \text{cm}^2\\ &= 707\ \text{cm}^2\end{aligned}$$

17

$$\begin{aligned}\text{Curved Surface Area} &= 2\pi r^2\\ &= 2\pi \times (3.2)^2\end{aligned}$$

$$\begin{aligned}\text{Circle Face Area} &= \pi r^2\\ &= \pi \times (3.2)^2\end{aligned}$$

$$\begin{aligned}\text{Total Surface Area} &= 3\pi \times (3.2)^2\\ &= 96.5\ \text{cm}^2\end{aligned}$$

18

$$\begin{aligned}\text{Base Area} &= \pi r^2\\ &= \pi \times (1.6)^2\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \frac{1}{3} \times \text{Base Area} \times \text{height}\\ &= \frac{1}{3} \times 2.56\pi \times 3.1\\ &= 8.31\ \text{m}^3\end{aligned}$$

19

$$\begin{aligned}\text{Volume} &= \frac{2}{3}\pi r^3\\ &= \frac{128}{3}\pi\\ &= 134\ \text{cm}^3\end{aligned}$$

20

$$\begin{aligned}\text{Volume} &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi \times (9.15)^3\\ &= 3\,210\ \text{cm}^3\end{aligned}$$

21 Converting all measurements to cm.

Cylinder:

$$\begin{aligned}\text{Curved Surface Area} &= 2\pi rl \\ &= 2 \times \pi \times 5 \times 100 \\ &= 1000\pi\end{aligned}$$

$$\begin{aligned}\text{Base Surface Area} &= \pi r^2 \\ &= \pi \times 5^2 \\ &= 25\pi\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \pi r^2 l \\ &= \pi \times 5^2 \times 100 \\ &= 2500\pi\end{aligned}$$

Cone:

$$\begin{aligned}\text{Curved Surface Area} &= \pi r\sqrt{r^2 + h^2} \\ &= \pi \times 5 \times \sqrt{125} \\ &= 25\pi\sqrt{5}\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 5^2 \times 10 \\ &= \frac{250}{3}\pi\end{aligned}$$

$$\begin{aligned}\text{Total Surface Area} &= \pi\left(1000 + 25 + 25\sqrt{5}\right) \\ &= 3400\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Total Volume} &= \pi\left(2500 + \frac{250}{3}\right) \\ &= 8120\ \text{cm}^3\end{aligned}$$

22 Cylinder height is 59 cm

$$\begin{aligned}\text{Cylinder Volume} &= \pi r^2 h \\ &= \pi \times 14^2 \times 59 \\ &= 11\,564\pi\end{aligned}$$

$$\begin{aligned}\text{Hemisphere Volume} &= \frac{2}{3}\pi r^3 \\ &= \frac{5\,488}{3}\pi\end{aligned}$$

$$\begin{aligned}\text{Total Volume} &= \left(11\,564 + \frac{5\,488}{3}\right)\pi \\ &= 42\,100\ \text{cm}^3 = 4.21 \times 10^4\ \text{cm}^3\end{aligned}$$

23 a $r = \sqrt{17^2 - 12^2} = 12.0$ cm

b

$$\text{Curved Surface Area} = \pi r l$$
$$= \pi \times 12.0 \times 17$$
$$= 643$$

$$\text{Base Surface Area} = \pi r^2$$
$$= 145\pi$$
$$= 456$$

$$\text{Total Surface Area} = 643 + 456$$
$$= 1\,100 \text{ cm}^2$$

24 a $V = \pi r^2 h$

$503.7 = \pi r^2 \times 12$

$$r = \sqrt{\frac{503.7}{12\pi}}$$
$$= 3.61 \text{ cm}$$

b $3 \times 12.3 = 36.9$ cm

Tip: Remember that a tapered solid has one third the volume of a prism the same height and base area, so a tapered solid with the same volume and base area would be three times the height.

25 a

$$V = \frac{1}{3}\pi r^2 h$$
$$= 192 \text{ cm}^3$$

b $V_{sphere} = \frac{4}{3}\pi r^3 = 192$

$$r = \sqrt[3]{192 \times \frac{3}{4\pi}}$$
$$= 3.58 \text{ cm}$$

26 a Height of cone $h_{\text{Cone}} = 35 - 23 = 12$ cm

Radius $r = 9$ cm

$$\text{Slant height } l = \sqrt{9^2 + 12^2}$$
$$= 15 \text{ cm}$$

b Curved SA of cone $S_{\text{Cone}} = \pi \times 9 \times 15 = 135\pi \text{ cm}^2$

Side SA of cylinder $S_{Cyl} = 2\pi \times 9 \times 23 = 414\pi \text{ cm}^2$

Base area of cylinder $S_{\text{Base}} = \pi \times 9^2 = 81\pi \text{ cm}^2$

Total SA $= 630\pi \text{ cm}^2 = 1980 \text{ cm}^2$

27 a A side face is isosceles with base 12 cm and perpendicular distance $\sqrt{15^2 - 6^2} = 3\sqrt{21}$ cm

Side face area $= \frac{1}{2} \times 12 \times 3\sqrt{21} = 82.5 \text{ cm}^2$

b Base area $= 12 \times 12 = 144\ \text{cm}^2$

Total surface area $= 4 \times 18\sqrt{21} + 144 = 474\ \text{cm}^2$

c Diagonal of the base has length $12\sqrt{2}$ cm so half the diagonal is $6\sqrt{2}$ cm

Pyramid height $h = \sqrt{15^2 - (6\sqrt{2})^2} = \sqrt{153}$ cm

$$\begin{aligned} V &= \frac{1}{3} \times 144 \times \sqrt{153} \\ &= 594\ \text{cm}^3 \end{aligned}$$

28 For the complete cone:

$$V = \frac{1}{3}\pi \times 2^2 \times 6 = 8\pi\ \text{cm}^3$$

After the hole is bored, the volume removed is $\frac{2}{3}\pi \times 1^3 = \frac{2}{3}\pi$

So the end volume is $\left(8 - \frac{2}{3}\right)\pi = 22\ \text{cm}^3$

Slant length $l = \sqrt{2^2 + 6^2} = \sqrt{40}$ cm

Cone curved SA $= \pi \times 2 \times \sqrt{40}$

Hemisphere curved SA $= 2\pi \times 1^2 = 2\pi\ \text{cm}^2$

Base ring area $= \pi(2^2 - 1^2) = 3\pi\ \text{cm}^2$

$$\begin{aligned} \text{Total SA} &= (5 + 2\sqrt{40})\pi\ \text{cm}^2 \\ &= 55.4\ \text{cm}^2 \end{aligned}$$

29

$$\begin{aligned} \text{Total volume} &= \frac{2}{3}\pi(8^3 + 10^3) \\ &= 3170\ \text{mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Curved area} &= 2\pi(8^2 + 10^2) \\ &= 328\pi\ \text{mm}^2 \end{aligned}$$

Ring area $= \pi(10^2 - 8^2) = 36\pi\ \text{mm}^2$

$$\begin{aligned} \text{Total SA} &= 364\pi\ \text{mm}^2 \\ &= 1140\ \text{mm}^2 \end{aligned}$$

30 Main cone:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 8^2 \times 30 \\ &= 640\pi\ \text{mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Curved SA} &= \pi r l \\ &= \pi \times 8 \times \sqrt{8^2 + 30^2} \\ &= 248.4\pi\ \text{mm}^2 \end{aligned}$$

Removed cone:

$$r = 8 \times \frac{12}{30} = 3.2 \text{ mm}$$

$$\begin{aligned} V &= \frac{1}{3}\pi \times 3.2^2 \times 12 \\ &= 40.96\pi \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Curved SA} &= \pi \times 3.2 \times \sqrt{3.2^2 + 12^2} \\ &= 39.7\pi \text{ mm}^2 \end{aligned}$$

$$\begin{aligned} \text{Frustum } V &= 640\pi - 40.96\pi \\ &= 1880 \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Frustum SA} &= (248.4\pi - 39.7\pi) + \pi(8^2 + 3.2^2) \\ &= 889 \text{ mm}^2 \end{aligned}$$

31 a Let x be the length of one side of the base, so that the base area is x^2

$$\begin{aligned} V &= \frac{1}{3}x^2h \\ x^2 &= \frac{3V}{h} = \frac{3 \times 1352}{24} = 169 \\ x &= 13 \text{ cm} \end{aligned}$$

b Let l be the altitude of one of the isosceles triangle faces.

$$\begin{aligned} l &= \sqrt{h^2 + 0.25x^2} \\ &= \sqrt{24^2 + 6.5^2} \\ &= 24.86 \text{ cm} \end{aligned}$$

Then the area of one of the triangles A is given by

$$A = \frac{1}{2}lx = 162 \text{ cm}^2$$

So the total surface area is $4A + x^2 = 815 \text{ cm}^2$

32 $V = \frac{4}{3}\pi r^3 = 354$

$$r = \sqrt[3]{\frac{3}{4} \times \frac{354}{\pi}} = 4.39$$

$$SA = 4\pi r^2 = \frac{3V}{r} = 242 \text{ m}^2$$

33 a Cylinder:

$$\begin{aligned} \text{Curved SA} &= 2\pi rh \\ &= 2\pi \times 2 \times 8 \\ &= 32\pi \text{ mm}^2 \\ V &= \pi r^2 h \\ &= \pi \times 2^2 \times 8 \\ &= 32\pi \text{ mm}^2 \end{aligned}$$

Hemispheres:

$$\begin{aligned}\text{Curved SA} &= 4\pi r^2\\ &= 4\pi \times 2^2\\ &= 16\pi \text{ mm}^2\\ V &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi \times 2^3\\ &= \frac{32}{3}\pi \text{ mm}^3\end{aligned}$$

Total:

$$\text{SA} = 48\pi \text{ mm}^2 = 151 \text{ mm}^2$$

$$V = \frac{128}{3}\pi \text{ mm}^3 = 134 \text{ mm}^3$$

b Require $r = 1.8$ mm and $V = 0.9 \times 134 = 121 \text{ mm}^3$

$$V = \pi r^2\left(h + \frac{4r}{3}\right)$$

Rearranging:

$$\begin{aligned}h &= \frac{V}{\pi r^2} - \frac{4r}{3}\\ &= 9.45 \text{ mm}\end{aligned}$$

So the total length of the new tablet is $9.45 + 2(1.8) = 13.1$ mm

34 a Converting all lengths to cm for consistency of calculation:

Cylinder height $h_c = 180$

Spike height $h_s = 10$

$$\begin{aligned}\text{Cylinder: } V &= \pi r^2 h_c\\ &= \pi \times 4^2 \times 180\\ &= 2880\pi \text{ cm}^3\\ \text{Spike: } V &= \frac{1}{3}\pi r^2 h_s\\ &= \frac{1}{3}\pi \times 4^2 \times 10\\ &= \frac{160\pi}{3} \text{ cm}^3\\ \text{Hemisphere: } V &= \frac{2}{3}\pi r^3\\ &= \frac{2}{3}\pi \times 4^3\\ &= \frac{128\pi}{3} \text{ cm}^3\\ \text{Total: } V &= 2976\pi \text{ cm}^3\\ &= 9349 \text{ cm}^3\end{aligned}$$

For a thousand posts, the volume of metal required would be $9\,349\,000 \text{ cm}^3 = 9.35 \text{ m}^3$

b Paint volume needed could be calculated by multiplying the surface area of the entire shape by 0.44 mm:

Unpainted Spike: slope length $l = \sqrt{10^2 + 4^2} = 10.8$ cm

So unpainted spike SA $= \pi r l = 4\pi \times 10.8 = 135.3$ cm^2

Unpainted cylinder SA $= 2\pi r h = 2\pi \times 4 \times 180 = 4523.9$ cm^2

Unpainted hemisphere SA $= 2\pi r^2 = 2\pi \times 4^2 = 100.5$ cm^2

Total unpainted SA $= 4759.8$ cm^2

Then approximate volume of paint is $4759.8 \times 0.04 = 190$ cm^3

This method is imprecise as it takes no account of how the paint accommodates to angles surfaces, but given the shape concerned, this is immaterial.

Alternatively, we can estimate the volume of paint as the increase in volume when the original calculations use $r = 4.04$ instead of 4 and spike length 10.04 instead of 10.

Total shape volume, from part **a**, is given by

$$V_{unpainted} = \pi r^2 h_c + \frac{1}{3}\pi r^2 h_s + \frac{2}{3}\pi r^3 = 9349 \text{ cm}^3$$

Using $r = 4.04$, $h_c = 180$ and $h_s = 10.04$, this gives

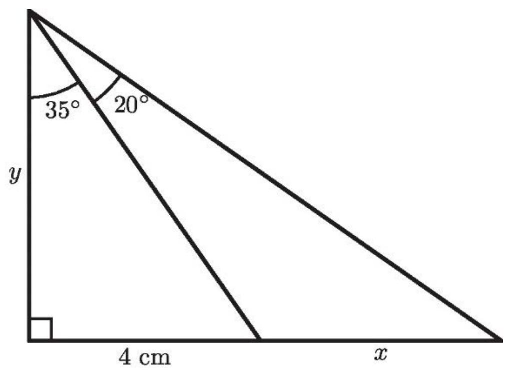
$V_{painted} = 9539.4$

The difference is accounted for by the paint, so the approximate volume of paint is 190 cm^3

For 1000 posts, the total paint needed is 190000 cm^3 $= 0.190$ m^3

Exercise 5B

34

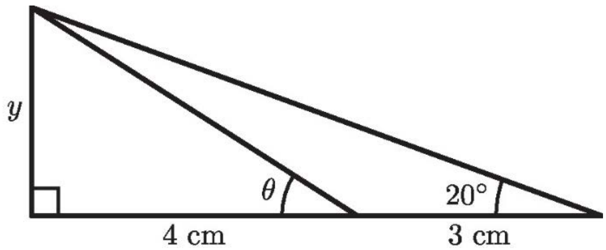


$$y = \frac{4}{\tan 35°}$$

$$x + 4 = y \tan 55°$$

$$x = \frac{4 \tan 55°}{\tan 35°} - 4$$
$$= 4.16 \text{ cm}$$

35



$y = 7\tan 20°$

$\theta = \tan^{-1}\left(\frac{y}{4}\right)$

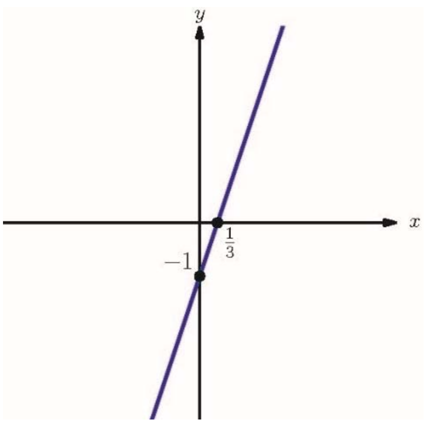
$= \tan^{-1}\left(\frac{7\tan 20°}{4}\right)$

$= 32.5°$

36 a $(0,3)$ and $(-6,0)$

b $\tan^{-1}\left(\frac{3}{6}\right) = 26.6°$

37 a



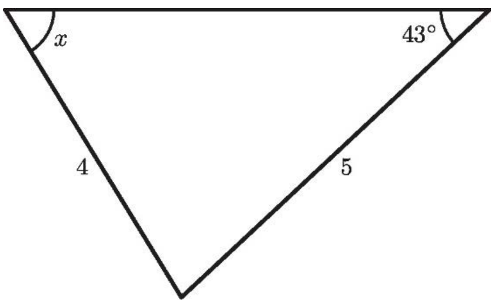
b $\tan^{-1}(3) = 71.6°$

38 Cosine Rule:

$$a^2 = 11^2 + 12^2 - 2(11)(12)\cos 35°$$
$$= 48.7$$

$$a = \sqrt{48.7} = 6.98 \text{ cm}$$

39



Sine Rule:

$$\frac{\sin x}{5} = \frac{\sin 43°}{4}$$

$$x = \sin^{-1}\left(\frac{5\sin 43°}{4}\right)$$
$$= 58.5°$$

Third angle $= 180° - 58.5° - 43° = 78.5°$

40 Cosine Rule:

$$C\hat{A}B = \cos^{-1}\left(\frac{9^2 + 16^2 - 18^2}{2(9)(16)}\right)$$
$$= 87.4°$$

41 a Cosine Rule:

$$\hat{C} = \cos^{-1}\left(\frac{10 + 20^2 - 11^2}{2(10)(20)}\right)$$
$$= 18.6°$$

b Sine Rule for area:

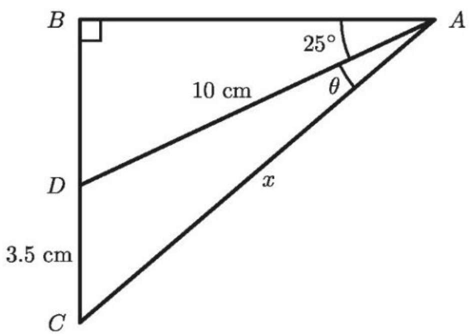
$$\text{Area} = \frac{1}{2}(10)(20)\sin 18.6°$$
$$= 32.0$$

42 Sine Rule for area:

$$\text{Area} = \frac{1}{2}(AB)(27)\sin 70° = 241$$

$$AB = \frac{482}{27\sin 70°} = 19.0$$

43



$$BD = 10\sin 25° = 4.226\ldots$$

$$AB = 10\cos 25° = 9.063\ldots$$

$$x = \sqrt{(BA)^2 + (BC)^2}$$
$$= \sqrt{9.06\ldots^2 + 7.72\ldots^2}$$
$$= 11.8\text{ cm}$$

$$\theta = \cos^{-1}\left(\frac{AB}{x}\right) - 25$$
$$= 15.5°$$

44 y-intercept is 8, crosses x-axis at 10

Angle with the x-axis is $\tan^{-1}\left(\frac{8}{10}\right) = 38.7°$

45 a Intersection: $2x - 8 = \frac{1}{4}x - 1$

$$\frac{7}{4}x = 7$$
$$x = 4$$

The lines intersect at $(4,0)$

b Angle between the lines:

Angle of $y = 2x - 8$ is $\tan^{-1}(2) = 63.4°$

Angle of $y = \frac{1}{4}x - 1$ is $\tan^{-1}\left(\frac{1}{4}\right) = 14.0°$

Angle between the lines is therefore $63.4 - 14.0 = 49.4°$

46 Line $2x - 5y = 7$ has gradient $\frac{2}{5}$ and angle to the horizontal $\tan^{-1}\left(\frac{2}{5}\right) = 21.8°$

Line $4x + y = 8$ has gradient -4 and angle to the horizontal $\tan^{-1}(-4) = -76.0°$

Angle between the lines is therefore $21.8 — 76.0 = 97.8°$

Acute angle is $82.2°$

47 Sine Rule:

$$\frac{b}{\sin 60°} = \frac{12}{\sin 40°}$$
$$b = \frac{12 \sin 60°}{\sin 40°} = 16.2 \text{ cm}$$

48 Cosine Rule:

$$a = \sqrt{5^2 + 8^2 - 2(5)(8)\cos 45°}$$
$$= 5.69 \text{ cm}$$

49 Cosine Rule:

$$A = \cos^{-1}\left(\frac{6^2 + 8^2 - 4^2}{2(6)(8)}\right)$$
$$= 29.0°$$

50 Sine Rule:

$$\frac{\sin Y}{8} = \frac{\sin 66°}{10}$$
$$y = \sin^{-1}\left(\frac{8 \sin 66°}{10}\right)$$
$$= 47.0°$$

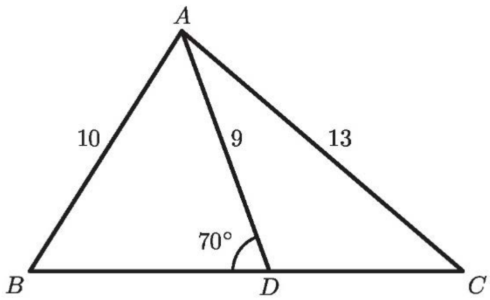
52 $A = 180 - 32 - 64 = 84°$

Sine Rule:

$$\frac{a}{\sin 84°} = \frac{3}{\sin 32°}$$

$$a = \frac{3\sin 84°}{\sin 32°}$$
$$= 5.63 \text{ cm}$$

53



Sine Rule in ABD

$$A\hat{B}D = \sin^{-1}\left(\frac{9\sin 70°}{10}\right) = 57.7°$$

Sine Rule in ACD

$$A\hat{C}D = \sin^{-1}\left(\frac{9\sin 110°}{13}\right) = 40.6°$$

Then $B\hat{A}C = 180 - 57.7 - 40.6 = 81.7°$

Cosine Rule in ABC

$$BC = \sqrt{10^2 + 13^2 - 2(10)(13)\cos 81.7°}$$
$$= 15.2$$

54 Sine Rule for area:

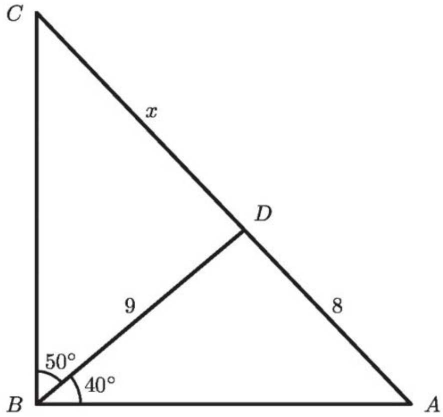
$$\text{Area} = \frac{1}{2}(6)(11)\sin\theta = 26$$

$$\theta = \sin^{-1}\left(\frac{52}{66}\right) = 52.0°$$

Cosine Rule:

$$AB = \sqrt{6^2 + 11^2 - 2(6)(11)\cos\theta}$$
$$= 8.70$$

55



Sine Rule in ABD:

$$\hat{B} = \sin^{-1}\left(\frac{9\sin 40°}{8}\right) = 46.3°$$

$$\hat{C} = 90 - 46.3 = 43.7°$$

Sine Rule in ACD:

$$x = \frac{9\sin 50°}{\sin \hat{C}} = 9.98$$

56 Let the base length be b.

Then $h = b\tan 40°$ and $d + h = b\tan 50°$

$$h = b\tan 50° - d$$
$$= \frac{h}{\tan 40°}\tan 50° - d$$

Rearranging:

$$h\left(\frac{\tan 50°}{\tan 40°} - 1\right) = d$$

$$h = \frac{d\tan 40°}{\tan 50° - \tan 40°}$$

57 **a** $x = \frac{h}{\tan 30°}, y = \frac{4-h}{\tan 10°}$

b Since $x + y = 8$

$$h\left(\frac{1}{\tan 30°} - \frac{1}{\tan 10°}\right) + \frac{4}{\tan 10°} = 8$$

$$h = \frac{8 - \frac{4}{\tan 10°}}{\left(\frac{1}{\tan 30°} - \frac{1}{\tan 10°}\right)} = 3.73$$

Exercise 5C

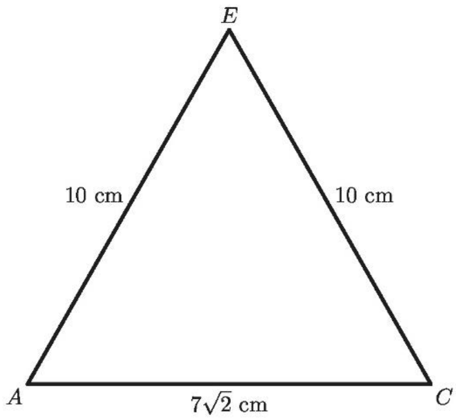
20 $h = 40 \tan 55° = 57.1$ m

21 a $HB = \sqrt{5^2 + 12^2 + 9^2} = 15.8$

b $BD = \sqrt{12^2 + 9^2} = 15$

$$H\hat{B}D = \cos^{-1}\left(\frac{BD}{HB}\right) = 18.4°$$

22 a



b Let X be the centre of the base, midpoint of AC.

$$AX = \frac{7\sqrt{2}}{2}$$

The pyramid height $EX = \sqrt{EA^2 - AX^2} = \sqrt{100 - 24.5} = 8.69$ cm

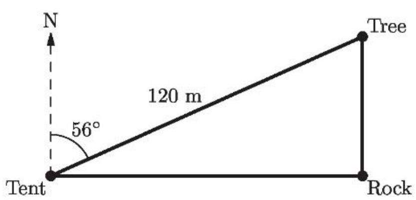
c Cosine Rule:

$$A\hat{E}C = \cos^{-1}\left(\frac{10^2 + 10^2 - (7\sqrt{2})^2}{2(10)(10)}\right) = 59.3°$$

23 Radius = 6

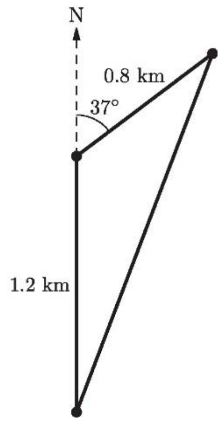
$$\theta = \tan^{-1}\left(\frac{9}{6}\right) = 56.3°$$

24 a



b $RT = 120 \cos 56° = 67.1$ m

25 a



b Distance east of the port is $0.8 \sin 37° = 0.481$ km

Distance north of the port is $1.2 + 0.8 \cos 37° = 1.84$ km

Total distance from port is $\sqrt{1.2^2 + 0.481^2} = 1.90$ km

26 $BT = 1.6 + 6.5 \tan 62° = 13.8$ m

27 $d = 9 \tan 78° = 42.3$ m

28 If the initial distance from the statue is x and the height of the statue is h

$x \tan 17.7° = h \qquad (1)$

$(x+5) \tan 12.0° = h \qquad (2)$

$$(1): x = \frac{h}{\tan 17.7°}$$

$$(2): h\left(1 - \frac{\tan 12.0°}{\tan 17.7°}\right) = 5 \tan 12.0°$$

$$h = \frac{5 \tan 12.0°}{\left(1 - \frac{\tan 12.0°}{\tan 17.7°}\right)} = 3.18 \text{ m}$$

29 If the height of the lighthouse is h and the distance of the first buoy from the lighthouse is x then

$h = x \tan 42.5° \qquad (1)$

$h = \sqrt{x^2 + 18^2} \tan 41.3° \qquad (2)$

$$(1): x = \frac{h}{\tan 42.5°}$$

$$(2): \left(\frac{h}{\tan 41.3°}\right)^2 = \left(\frac{h}{\tan 42.5°}\right)^2 + 324$$

$$h^2 = \frac{324}{\frac{1}{\tan^2 41.3°} - \frac{1}{\tan^2 42.5°}}$$

$$h = \sqrt{\frac{324}{\frac{1}{\tan^2 41.3°} - \frac{1}{\tan^2 42.5°}}} = 55.6 \approx 56 \text{ m}$$

30 a $AG = \sqrt{7^2 + 4^2 + 9^2} = 12.1$ cm

b $AC = \sqrt{4^2 + 9^2} = 9.85$ cm

$C\hat{A}G = \cos^{-1}\left(\frac{9.85}{12.1}\right) = 35.4°$

c $B\hat{A}G = \cos^{-1}\left(\frac{9}{12.1}\right) = 41.9°$

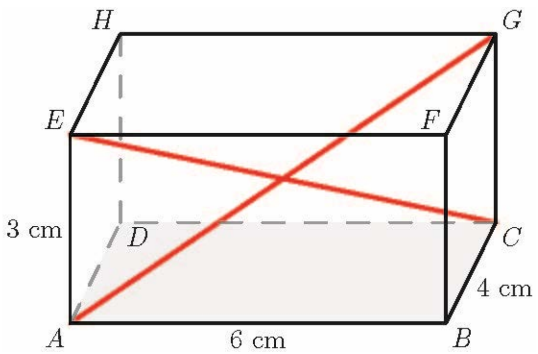
31 a

$$AG = \sqrt{AB^2 + AD^2 + AE^2}$$
$$AC^2 = AB^2 + AD^2 = 13^2$$
$$AF^2 = AB^2 + AE^2 = 7^2$$
$$CF^2 = AH^2 = AD^2 + AE^2 = 11^2$$
$$AB^2 + AD^2 + AE^2 = \frac{1}{2}(AC^2 + AF^2 + CF^2) = \frac{339}{2}$$
$$AG = \sqrt{\frac{339}{2}} = 13.02 \text{ m}$$

b

$$C\hat{A}G = \cos^{-1}\left(\frac{AC}{AG}\right)$$
$$= \cos^{-1}\left(\frac{13\sqrt{2}}{\sqrt{339}}\right)$$
$$= 3.11°$$

32

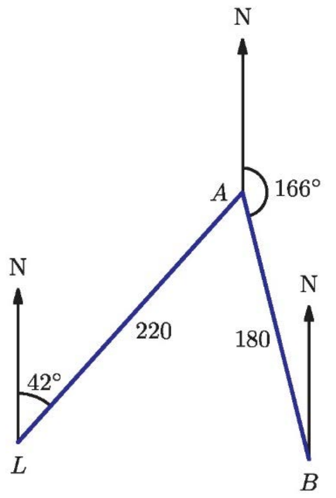


a $AG = CE = \sqrt{3^2 + 6^2 + 4^2} = 7.81$ cm

b The two lines cross in the centre of the cuboid at point X, so CGX is an isosceles, with sides 3.91 and base 3.

$$C\hat{X}G = \cos^{-1}\left(\frac{3.91^2 + 3.91^2 - 3^2}{2(3.91)(3.91)}\right)$$
$$= 45.2°$$

33



The dog runs from lighthouse L to A and then from A to B.

$$L\hat{A}B = 42° + (180 - 166)° = 56°$$

Cosine Rule in LAB:

$$LB = \sqrt{220^2 + 180^2 - 2(220)(180)\cos 56°} = 191 \text{ m}$$

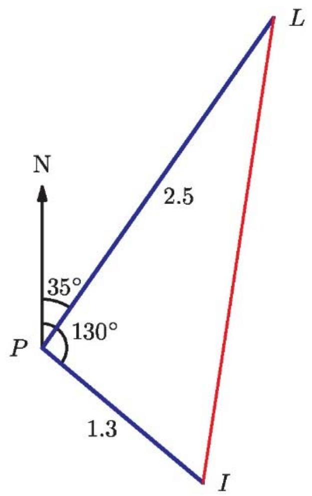
Sine Rule:

$$A\hat{L}B = \sin^{-1}\left(\frac{180 \sin 56°}{191}\right) = 51.3°$$

The bearing from L to B is $51.3 + 42 = 93.3°$

So the bearing for the return journey from B to L is $273°$

34 If the lighthouse is at L, the port at P and the island at I then



$$L\hat{P}I = 130 - 35 = 95°$$

Cosine Rule:

$$LI = \sqrt{2.5^2 + 1.3^2 - 2(2.5)(1.3)\cos 95°} = 2.92 \text{ km}$$

Worked solutions

Sine Rule:

$$L\hat{I}P = \sin^{-1}\left(\frac{2.5\sin 95°}{2.92}\right) = 58.64°$$

The bearing from I to L is $(58.64 + 130 - 180) = 8.64°$

35 If the apex is at A, centre of base at X and the centre of one side Q

$$QX = 115 \text{ m}$$

$$A\hat{Q}X = 42°$$

So height of pyramid $AX = 115\tan 42° = 104$ m

36 **a** $BT = 19.5\tan 26° = 9.51$ m

b $d = \frac{BT}{\tan 41°} = 10.9$ m

c Cosine Rule:

$$R\hat{B}M = \cos^{-1}\left(\frac{19.5^2 + 10.9^2 - 14.7^2}{2(19.5)(10.9)}\right) = 48.3 \approx 48°$$

37 Let the distance of the base of the painting from the floor be x and the height of the painting be h.

$$x = 2.4\tan 55°$$

$$x + h = 2.4\tan 72°$$

So $h = 2.4(\tan 72° - \tan 55°) = 3.96$ m

38 **a** Volume of a square based pyramid is given by $V = \frac{1}{3}b^2h$ where $b$is the length of the sides of the base and h is the pyramid's vertical height.

Therefore, the volume of the Louvre pyramid is $V = \frac{1}{3}(34^2)(21.6) = 8323.2 \ldots m^3$.

We require one unit per $1000m^3$

Hence, 9 units are required.

b Note that we only have to consider the heat loss through the four glass sides exposed to the outside air.

The surface area of the pyramid through which heat is lost is given by

$$SA = 2a\sqrt{\frac{a^2}{4} + h^2} = 2(34)\sqrt{\frac{34^2}{4} + 21.6^2} = 1869.14 \ldots \text{m}^2$$

And the power required to offset this heat loss is simply
$192 \times 1869.14 \ldots = 359000 \text{ W} = 359 \text{ KW}$

c Consider the triangle formed by the base of the pyramid, the vertical to the pyramid's peak, and the side of the pyramid.

We can apply trig. to this triangle to find the elevation, θ.

$$\tan\theta = \frac{21.6}{0.5 \times 34}$$

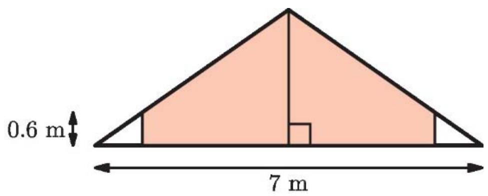
Hence, the elevation is 51.9 and therefore scaffolding is required.

39 **a** We have a maximum height h when $h = \frac{7}{2}\tan 35° = 2.45$ m

b The area of the face of the prism is given by $A = \frac{1}{2}(7)(2.45) = 8.58\ \text{m}^2$

And so, the volume of the roof space is $V = \frac{1}{2}(7)(2.45)(5) = 42.9\ \text{m}^3$

c



Using proportion: If the maximum height is 2.45 m then the proportion of the width which is over 0.6 m is $1 - \left(\frac{0.6}{2.45}\right) = 75.5\%$

d The non-usable volume will be contained in the two identical triangular prisms at the edge of the roof. Each has height 0.6 m and width $(1 - 0.755) \times 3.5 = 0.857$ m so has cross-sectional area $0.6 \times 0.857 = 0.514\ \text{m}^2$

The proportion of the volume that is unusable is therefore $\frac{0.514}{8.58} = 5.99\%$

The proportion of the volume that is usable is 94.0%

(Alternatively, it can be seen that the non-usable triangles together form an isosceles similar to the overall triangle; since scale factor from whole triangle to unused part is 24.5%, it follows that the proportion of the cross-sectional area not used, and hence the proportion of the prism volume not used, will be $(24.5\%)^2 = 5.99\%$. Again, the proportion of the volume that is usable is 94.0%.)

40 $BE = \sqrt{7^2 + 9^2} = \sqrt{130}$

$BG = \sqrt{4^2 + 7^2} = \sqrt{65}$

$EG = \sqrt{4^2 + 9^2} = \sqrt{97}$

Cosine Rule:

$$B\hat{E}G = \cos^{-1}\left(\frac{130 + 97 - 65}{2\sqrt{130}\sqrt{97}}\right) = 43.8°$$

Sine Rule for area:

$$\text{Area } BEG = \frac{1}{2}\sqrt{130}\sqrt{97}\sin 43.8° = 38.9\ \text{cm}^2$$

41 ABV is isosceles with equal sides 23 cm and base 20 cm. Its altitude $MV = \sqrt{23^2 - 10^2} = 20.7$ cm

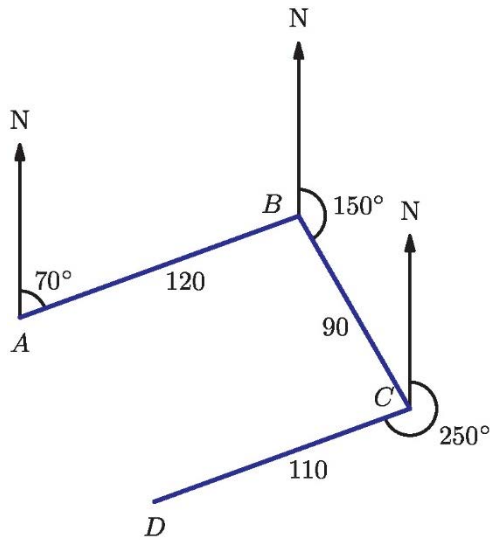
Since the pyramid has a square base, $NV = MV$ and MNV is isosceles.

$MN = \sqrt{200}$ cm

Cosine Rule:

$$M\hat{V}N = \cos^{-1}\left(\frac{20.7^2 + 20.7^2 - 200}{2(20.7)(20.7)}\right) = 39.9°$$

42



If Amy starts at A, then travels to B, C and finally D:

$$A\hat{B}C = 70 + 180 - 150 = 100°$$

Cosine Rule in ABC:

$$AC = \sqrt{AB^2 + BC^2 - 2(AB)(BC)\cos A\hat{B}C}$$
$$= \sqrt{120^2 + 90^2 - 2(120)(90)\cos 100°}$$
$$= 162 \text{ m}$$

Sine Rule in ABC:

$$A\hat{C}B = \sin^{-1}\left(\frac{120 \sin 100°}{162}\right) = 46.8°$$

Then $A\hat{C}D = 360 - 250 - 30 - 46.8 = 33.2°$

Cosine Rule in ACD:

$$AD = \sqrt{110^2 + 162^2 - 2(110)(162)\cos 33.2°} = 92.3 \text{ m}$$

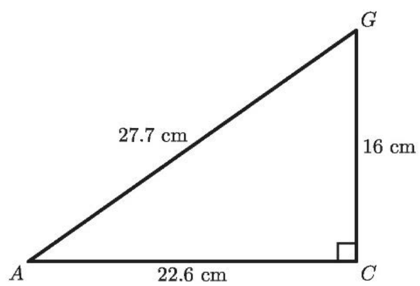
Mixed Practice

1 $h = 50 \tan 35° = 35.0$ m

2 **a** $AC = 16\sqrt{2} = 22.6$ cm

$AG = 16\sqrt{3} = 27.7$ cm

b



c $C\hat{A}G = \tan^{-1}\left(\frac{16}{16\sqrt{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3°$

3 a $AC = 23\sqrt{2} = 32.5$ cm

b If X is the centre of the base, then $AX = 0.5AC = 16.3$ cm

$XE = 16.3 \tan 56° = 24.1$ cm

c $AE = \frac{16.3}{\cos 56°} = 29.1$ cm

4 $\theta = 2\tan^{-1}\left(\frac{5}{12}\right) = 45.2°$

5 a $(0,6)$

b Gradient $= \frac{2-5}{8-2} = -\frac{1}{2} = -0.5$

c $\theta = \tan^{-1}(0.5) = 26.6°$

6 a Sine Rule in ABC:

$$AC = \frac{10 \sin 100°}{\sin 50°} = 12.9 \text{ cm}$$

b Cosine Rule in ADC:

$$A\hat{D}C = \cos^{-1}\left(\frac{7^2 + 12^2 - 12.9^2}{2(7)(12)}\right) = 80.5°$$

7 Sine Rule:

$$b = \frac{10 \sin 70°}{\sin 50°} = 12.3 \text{ cm}$$

8 Cosine Rule:

$$a = \sqrt{8^2 + 10^2 - 2(8)(10)\cos 15°} = 3.07 \text{ cm}$$

9 Cosine Rule:

$$A = \cos^{-1}\left(\frac{5^2 + 7^2 - 3^2}{2(5)(7)}\right) = 21.8°$$

10 Sine Rule:

$$Y = \sin^{-1}\left(\frac{12 \sin 42°}{15}\right) = 32.4°$$

11 Sine Rule:

$$Q = \sin^{-1}\left(\frac{4 \sin 120°}{9}\right) = 22.6°$$

So $R = 180 - 120 - 22.6 = 37.4°$

12 $A = 180 - 32 - 72 = 76°$

Sine Rule:

$$a = \frac{10 \sin 76°}{\sin 32°} = 18.3 \text{ cm}$$

13 If the initial distance from the base of the tower is x then

$x \tan 47.7° = h$ (1)

$(x+20)\tan 38.2° = h$ (2)

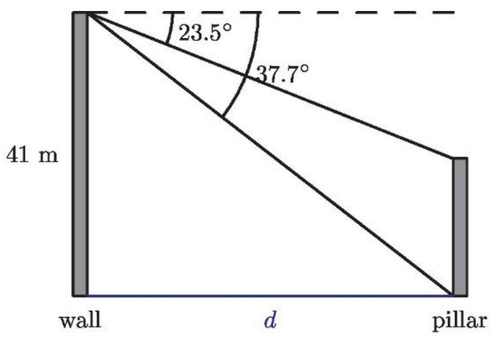
$$(1): x = \frac{h}{\tan 47.7°}$$

Substituting into (2):

$$h\left(1 - \frac{\tan 38.2°}{\tan 47.7°}\right) = 20 \tan 38.2°$$

$$h = \frac{20 \tan 38.2°}{\left(1 - \frac{\tan 38.2°}{\tan 47.7°}\right)} = 55.4 \text{ m}$$

14



The distance between wall and pillar is d

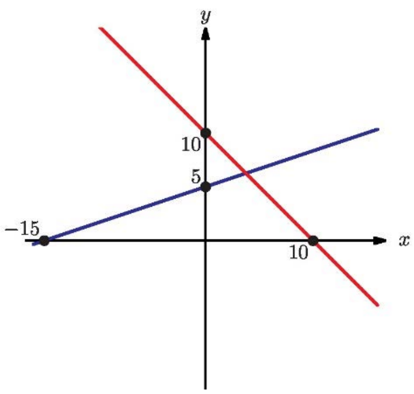
$$d = \frac{41}{\tan(37.7°)} = 53.0 \text{ m}$$

The height of the pillar is h so the difference in height between wall and pillar is $d - h$

$$d - h = 53.0 \tan 23.5° = 23.1 \text{ m}$$

So the height of the pillar is $41 - 23.1 = 17.9 \approx 18$ m

15 a



b $y_1 = 10 - x, y_2 = \frac{1}{3}x + 5$

Intersection where $10 - x = \frac{1}{3}x + 5$

$$\frac{4}{3}x = 5$$
$$x = \frac{15}{4} = 3.75$$

Point is $(3.75, 6.25)$

c Angle between y_1 and the x-axis is $\tan^{-1}\left(\frac{1}{3}\right) = 18.4°$

Angle between y_2 and the x-axis is $\tan^{-1}(-1) = -45°$

Angle between the two lines is $18.4 — 45 = 63.4°$

16 If the apex of the pyramid is E, and the base is $ABCD$ with centre X

$$AE = 26 \text{ cm}$$
$$A\hat{E}X = 35° \text{ and } A\hat{X}E = 90°$$

$$AX = 26 \tan 35° = 18.2 \text{ cm}$$
$$AB = 18.2\sqrt{2} = 25.7 \text{ cm}$$

$$\text{Vol} = \frac{1}{3}\text{base} \times \text{height}$$
$$= \frac{1}{3}(25.7^2)(26) = 5740 \text{ cm}^3$$

17 a $AM = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$

$$A\hat{M}E = \tan^{-1}\left(\frac{15}{\sqrt{45}}\right) = \tan^{-1}(\sqrt{5}) = 65.9°$$

b HME is isosceles with sides $EM = \sqrt{15^2 + 45} = \sqrt{270}$ and base 6

Cosine Rule:

$$H\hat{M}E = \cos^{-1}\left(\frac{270 + 270 - 6^2}{2\sqrt{270}\sqrt{270}}\right) = \cos^{-1}\left(\frac{504}{540}\right) = 21.0°$$

18 Cosine Rule:

$$(x + 4)^2 = x^2 + (2x)^2 - 2(x)(2x)\cos 60°$$
$$x^2 + 8x + 16 = x^2 + 4x^2 - 2x^2$$
$$2x^2 - 8x - 16 = 0$$
$$x^2 - 4x - 8 = 0$$
$$x = 2 \pm \sqrt{12}$$

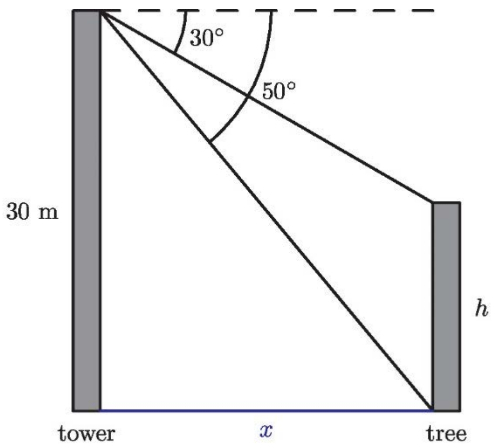
In context, $x > 0$ so the only solution is $x = 2 + \sqrt{12} = 2 + 2\sqrt{3} = 5.46$

19 Sine Rule for area:

$$84 = \frac{1}{2}x(x-5)\sin 150°$$
$$84 = \frac{1}{4}x(x-5)$$
$$x^2 - 5x - 336 = 0$$
$$(x-21)(x+16) = 0$$

In context, $x > 5$ so the only solution is $x = 21$

20



Let x be the distance between the tree and the tower on the ground.

$x = 30\tan 40° = 25.2$ m

Let h be the height of the tree.

$h = x\tan 35° = 17.6$ m

21 a $V_{ball} = \frac{4}{3}\pi(3.15)^3 = 131\text{ cm}^3$

b $V_{tube} = \pi(3.2)^2 \times 26 = 836\text{ cm}^3$

$V_{cube} - 4V_{ball} = 313\text{ cm}^3$

22 a $AB = \sqrt{9.5^2 - 8^2} = \sqrt{26.25} = 5.12$ cm

b $EF = AD = AB$

$FM = \frac{1}{2}EF = 2.56$ cm

$BM = \sqrt{BF^2 + FM^2} = \sqrt{96.8125} = 9.84$ cm

c $AM = \sqrt{AF^2 + FM^2} = \sqrt{70.5625} = 8.40$ cm

$A\hat{M}B = \cos^{-1}\left(\frac{AM}{BM}\right) = \cos^{-1}\left(\frac{8.40}{9.84}\right) = 31.4°$

23 Part A

a $XM = 5$ cm

b $VM = \sqrt{XM^2 + VX^2} = \sqrt{5^2 + 8^2} = \sqrt{89} = 9.43$ cm

c $V\hat{M}X = \cos^{-1}\left(\frac{XM}{VM}\right) = \cos^{-1}\left(\frac{5}{\sqrt{89}}\right) = 58.0°$

Part B

a Cosine Rule:

$$d = \sqrt{290^2 + 550^2 - 2(290)(550)\cos 115°} = 722 \text{ m} \approx 720 \text{ m}$$

b Sine Rule for area:

$$\text{Area} = \frac{1}{2}(290)(550)\sin 115° = 72\,300 \text{ m}^2$$

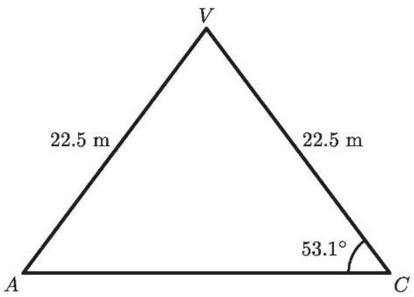
c Sine Rule

$$A\hat{B}C = \sin^{-1}\left(\frac{180 \sin 53°}{230}\right) = 38.7°$$

Then $A\hat{C}B = 180 - 53 - 38.7 = 88.3°$

24 a i 22.5 m

ii



b If the centre of $ABCD$ is X then the height of the pyramid is VX.

$$V\hat{X}C = 90°$$

$$VX = 22.5 \sin 53.1° = 18.0 \text{ m}$$

c $CX = 22.5 \cos 53.1° = 13.5$ m

$AC = 2CX = 27$ m

d $BC = \frac{AC}{\sqrt{2}} = 19.1$ m

e $AP = 126 - 18.0 = 108$ m

Volume of cuboid $= AP \times AB^2 = 108 \times 19.1^2 = 39\,421 \text{ m}^3$

Volume of pyramid $= \frac{1}{3}AB^2 \times VX = \frac{1}{3} \times 19.1^2 \times 18.0 = 2187 \text{ m}^3$

Total volume $= 39\,421 + 2\,187 \approx 41\,600 \text{ m}^3$

f 90% volume $= 41\,600 \times 0.9 = 37\,500 \text{ m}^3$

Mass of $37\,500 \text{ m}^3$ air $= 37\,500 \times 1.2 = 44\,900$ kg

25

Tip: If you know how to find areas using vectors this is a much easier problem to approach.

If we just use the techniques of this chapter and some basic linear algebra, we can find the intersection points, two side lengths and then find the angle between two lines, then apply the sine rule for area.

Intersection points of $y_1 = 8 - x, y_2 = 2x - 10$ and $y_3 = 12.5 - 5.5x$

y_3 & y_2:

$$12.5 - 5.5x = 2x - 10$$
$$7.5x = 22.5$$

$x = 3$ so intersection is at $A(3, -4)$

y_1 & y_3:

$$8 - x = 12.5 - 5.5x$$
$$4.5x = 4.5$$

$x = 1$ so intersection is at $B(1,7)$

y_1 & y_2:

$$8 - x = 2x - 10$$
$$3x = 18$$

$x = 6$ so intersection is at $C(6,2)$

$$AB = \sqrt{2^2 + 11^2} = 5\sqrt{5}$$

$$AC = \sqrt{3^2 + 6^2} = 3\sqrt{5}$$

Angle made by y_2 and the x-axis is $\tan^{-1}(2) = 63.4°$

Angle made by y_3 and the x-axis is $\tan^{-1}(-5.5) = -79.7°$

Angle between y_2 and y_3 is $63.4 - (-79.7) = 143° = B\hat{A}C$

$$\begin{aligned}\text{Area } ABC &= \frac{1}{2}(AB)(AC)\sin B\hat{A}C \\ &= \frac{1}{2} \times 5\sqrt{5} \times 3\sqrt{5} \sin 143° \\ &= \frac{75}{2} \times 0.6 \\ &= 22.5\end{aligned}$$

6 Core: Statistics

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 6A

5 **a** All households in Germany

b Convenience sampling

c Households in the city may have fewer pets than in the countryside

There may be a link between pet ownership and having a child at Anke's school, such as affluence or catchment area of the school (different regions of a city have different sizes of garden or access to parks, for example).

There may even be a link between pet ownership rates and having children in the household (and since she is enquiring only from households with children, this would introduce a bias)

6 **a** The number or proportion of pupils in each year group.

b Quota sampling

c **i** Keep – this is potentially a valid result, even if it is an outlier.

ii Discard – this cannot be a true result (misrecorded, question misunderstood or answer deliberately mischievous)

7 **a** Convenience sampling

b All residents of his village

c His sample is tied to bus-stop users and may not be representative of the village as a whole, since preferred mode of transportation is relevant to environmental attitudes.

d He would need access to all residents, and a means of requiring responses from the randomly selected individuals.

8 The observation is not necessarily correct; this feature of the sample could have occurred by chance, even with correct sampling technique.

9 **a** Continuous

b Testing the population would require running all the lightbulbs under test conditions until they failed, which would mean none would be sold.

c Listing the serial numbers in order, select every twentieth bulb.

10 **a** Quota sampling

b A stratified sample would be more representative of the scarves sold; this might be significant if colour selection were linked to gender of customer.

c Applying the percentages to the sample size 40:

30% of 40: 12 red scarves

30% of 40: 12 green scarves

25% of 40: 10 blue scarves

15% of 40: 6 white scarves

11 a Quota sampling

b The sample will be more representative of the population she believes to be in the park.

c It would be difficult to compile a list of all the animals in the park and then specifically catch and test those animals predetermined for the sample.

12 a Continuous

b The basketball team are unlikely to be representative of the students as a whole, since height is a relevant characteristic associated with membership of a basketball team

c Systematic sampling

d To be a random sample, every sample of the same size must have equal probability of being taken. However, with the sampling described, it would not be possible to select two people who are adjacent on the list; in fact, for a 'one every ten names' system, there would only be 10 possible lists, determined entirely by which student in the list Shakir began his sample at.

13 The proportions in the table are

Cat 27%, Dog 43% and Fish 30%.

Applying these proportions to the sample size, we would take 5 cats, 9 dogs and 6 fish

14 a The total number in the table is 200, so applying the same proportions to a total of 20 gives

Gender/age	12	13	14
Boys	4	5	5
Girls	0	4	2

b His sample might be representative of his school but there is no basis to believe it to be representative of the country.

(For example, if he is correct about gender and age being relevant, the fact that his sample proportions will not reflect that of the country, where 12 year old girls exist, will be a problem!)

15 a Discrete

b Not reasonably

c Convenience sampling

d Different species or genetic family lines might be clustered in different parts of the field, and if she were generalising to the country as a whole, the field itself might not be representative.

16 a **i** Possible, if her sample is representative.

ii Not necessarily; the school may not reflect the distribution seen nationally

iii Possible; the school might follow the national pattern and she happened to select taller students by chance.

b **i** and **ii** would have the same answer.

iii would look less likely, since her sample is now larger and less likely to be extreme by random chance.

c -32 is clearly an error, since negative height is impossible. Check the original data and if the value is clearly recorded as -32, discard it.

155 cm is very tall for a primary school student, but is theoretically possible, so should be checked and then if truly recorded as 155, the value should be retained.

17a You would expect 20 to have taken the 'say yes' and 20 to have taken 'say no', so of the subjects who had a free choice, 4 of 20 answered 'yes'. This implies a 20% 'yes' answer from the sample who had a free choice.

b **i** Out of 120 students, you would expect 20 to be answering statement 1 and 100 to be answering statement 2.

Assuming they respond honestly to the statements as instructed, we would expect 4 cheaters to be answering statement 1 and 20 cheaters to be answering statement 2.

Then $4 + 80 = 84$ would answer 'True'

ii Let the proportion who have cheated be p.

Then the number saying 'True' would equal $20p + 100(1 - p) = 48$

Rearranging:

$$\begin{aligned} 100 - 80p &= 48 \\ 80p &= 52 \\ p &= 65\% \end{aligned}$$

18 a Of the recaptured fish, $\frac{20}{40\,000}$ were labelled.

If this is a representative sample, then the proportion of labelled fish in the Sea would be

$$\frac{20}{40\,000} = \frac{50\,000}{100\,000\,000}$$

So we would estimate 100 million cod in the North Sea

b We would have to assume that all 50 000 labelled fish were still available when the sample was taken – that is that none had died (natural death or removed by predation or fishing)

We would also have to assume that there was thorough mixing of the cod population so that the sample was representative of the cod stocks as a whole. Given fish behaviour typically involves grouping in schools, this is questionable, unless the sampling was very extensive and avoided sampling largely from only a few groups.

Exercise 6B

19 **a** Sum of frequencies is 48

b $8 + 11 = 19$

c From GDC: $\bar{x} = 6.17$ min, $\sigma = 1.31$ min

20 **a** $12 + 19 = 31$

b $1.4 \leqslant m < 1.6$

c 1.34 kg

21 **a** Sum of frequencies $= 28$

b $15.5 \leqslant t < 17.5$

c From GDC: $\bar{t} = 17.3$ sec

This is an estimate because the actual times are not known, so the data are assumed to lie at the midpoints of their groups.

22 From GDC:

a Median $= 4.25$ min

b IQR$= 1.5$

c The second artist has songs which are longer on average and more consistent (less spread).

23 From GDC:

a Median $= 4$

b IQR $= 3$

c $Q_1 = 3$ and IQR $= 3$ so any value below $3 - 1.5(3) = -1.5$ mins would be an outlier, but there are no such values.

$Q_3 = 6$ and IQR $= 3$ so any value above $6 + 1.5(3) = 10.5$ is an outlier. 11 is an outlier.

24 **a** Sum of frequencies $= 67$

b 23 cm

c From GDC: mean length $= 24.7$ cm

d

$\mathbf{20.5 \leqslant l < 23.5}$	$\mathbf{23.5 \leqslant l < 26.5}$	$\mathbf{26.5 \leqslant l < 29.5}$
23	30	14

e $30 + 14 = 44$

f No. The modal group is $23.5 \leqslant l < 26.5$ but the modal value is 23.

25 a Total value is $10 + 6a = 17 \times 5$

$$\begin{aligned} 10 + 6a &= 85 \\ 6a &= 75 \\ a &= 12.5 \end{aligned}$$

b From GDC, $\sigma = 13.4$

26 a Total value is $12x - 3 = 10.5 \times 6$

$$\begin{aligned} 12x - 3 &= 63 \\ 12x &= 66 \\ x &= 5.5 \end{aligned}$$

b $\text{var} = 30.9$

27 Total mark for first group: $67.5 \times 12 = 810$

Total mark for the second group: $59.3 \times 10 = 593$

Total marks over both groups: $810 + 593 = 1403$

Mean mark over both groups: $\frac{1403}{22} = 63.8$

28 Total marks required for average 60 marks: $5 \times 60 = 300$

Total marks in the first four papers: 238

She needs $300 - 238 = 62$ marks in the final paper.

29 a Mean $= \frac{2\times5+5\times8+7\times13+14a+10(a+2)}{5+8+13+14+10} = \frac{161+24a}{50}$

b $\frac{161+24a}{50} = 8.02$

$$\begin{aligned} 161 + 24a &= 401 \\ 24a &= 240 \\ a &= 10 \end{aligned}$$

30 a

$$\begin{aligned} \text{Mean shoe size} &= \frac{4.5 \times 4 + 5 \times 12 + 6.5 \times 8 + 7 \times \ 4 + 2x}{4 + 12 + 8 + 4 + 2} \\ &= \frac{158 + 2x}{30} \\ &= 5.9 \end{aligned}$$

$$\begin{aligned} 158 + 2x &= 177 \\ 2x &= 19 \\ x &= 9.5 \end{aligned}$$

b $n = 30$ so the median is midway between the 15th and 16th value, Q_1 is the 8th value and Q_3 is the 23rd value.

$Q_1 = 5, Q_2 = 5, Q_3 = 6.5$

c So $\text{IQR} = 1.5$

Values above 1.5IQR above Q_3 would be considered outliers; $6.5 + 1.5\text{IQR} = 8.75$ so $x = 9.5$ is an outlier value.

31 **a** From GDC: $\bar{x} = £440$, $\sigma = £268$

b Median would seem better, since there is one value substantially larger than the others which distorts the mean.

c Both will scale with the conversion:

$\bar{x} = \$(440 \times 1.31) = \$576$

$\sigma = \$(268 \times 1.31) = \$351$

32 **a** From GDC: $\bar{x} = 4$, $\sigma = 2.94$

b The new data values are the original values x under transformation $f(x) = 3x + 2000$

The mean will rise to $f(\bar{x})$ and the standard deviation will rise to 3σ

The new mean is $f(4) = 2012$

The new standard deviation is $3 \times 2.94 = 8.83$

33 The mean is $\frac{5}{9}51 - \frac{160}{9} = 10.6\ °\text{C}$

The standard deviation is $\frac{5}{9}3.6 = 2\ °\text{C}$

34 The mean is $5.7 \times 1.61 = 9.18\ \text{km}$

The variance is $4.5 \times 1.61^2 = 11.9\ \text{km}^2$

35 New median is $-3 \times 25 = -75$

New IQR is $|-3 \times 14| = 42$

36 **a** $n = 11$ so the median is the 6th mark, Q_1 is the 3rd mark and Q_3 is the 9th mark, in an ordered list.

Since $m > 46$, that means $Q_2 = 42, Q_1 = 37$ and $Q_3 = 45$

Median is 42 and IQR is 8

b Outliers exist for values above $Q_3 + 1.5\text{IQR}$, which equals $45 + 12 = 57$.

So any value of m greater than 57 would represent an outlier.

The smallest such m is 58.

37 Suppose each test is scored out of K marks.

Total marks for the first four tests: $0.68K \times 4 = 2.72K$

Total marks required for all six tests: $0.7K \times 6 = 4.2K$

Total required in the final tests: $4.2K - 2.72K = 1.48K$

Average required in each of the final two tests: $\frac{1.48K}{2K} = 74\%$

38 **a**

$$\text{Mean grade} = \frac{3 \times 1 + 4 \times 8 + 5 \times 15 + 6p + 7 \times 4}{1 + 9 + 14 + p + 4}$$

$$= \frac{138 + 6p}{28 + p} = 5.25$$

$$138 + 6p = 5.25(28 + p)$$

$$138 + 6p = 147 + 5.25p$$

$$0.75p = 9$$
$$p = 12$$

b From GDC: With $p = 12$, $\sigma = 0.968$

39 Total frequency:

$$5 + 10 + 13 + 11 + p + q = 50$$
$$p + q = 11 \qquad (1)$$

Mean:

$$\frac{3 \times 5 + 4 \times 10 + 5 \times 13 + 6 \times 11 + 7p + 8q}{50} = 5.34$$
$$186 + 7p + 8q = 267$$
$$7p + 8q = 81 \qquad (2)$$
$$(2) - 7(1): q = 4$$

Then $(1): p = 7$.

40 Median is the central value: $b = 26$ (1)

Range is 11 so $a = c - 11$ (2)

Mean is 25 so $(a + b + c) = 3 \times 25 = 75$ (3)

Substituting (1) and (2) into (3):

$$2c + 15 = 75$$
$$c = 30$$

41 Median is the central value, so $a, b \geqslant 3$ since either being less than 3 would decrease the median.

Mean is 4 so $a + b + 6 = 4 \times 5 = 20$

Then $a + b = 14$

The largest possible range occurs when one equals 3 (the lowest possible) and the other is 11.

Largest possible range is therefore 10.

42 The median of 8 values is midway between the 4th and 5th values.

Mean is 7 so $1 + 3 + 4 + 10 + 10 + 16 + x + y = 8 \times 7 = 56$

Then $x + y = 12$

The values are all integers.

We could choose $x = y = 6$, which would give a median of 6 and $x + y = 12$ as required; however, this would contradict the requirement that $x < y$.

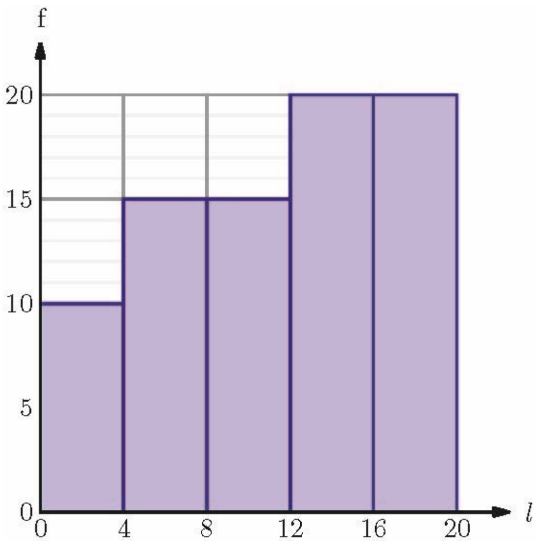
Choosing $x = 5, y = 7$ would again maintain a median of 6 and also satisfy the condition $x + y = 12$. As would the case $x = 4, y = 8$.

At $x = 3, y = 9$, x becomes the (2nd or) 3rd value, with 4 being the 4th and y the 5th, hence the median will no longer be 6.

The only possible values for (x, y) are $(5, 7)$ and $(4, 8)$.

Exercise 6C

10 a

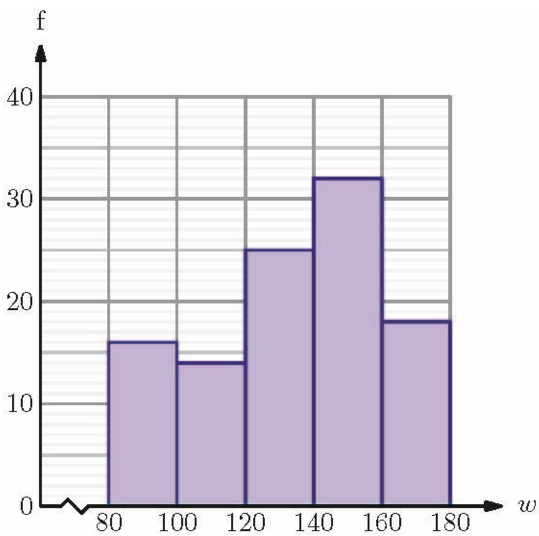


b Linear estimate: one quarter of the $12 < l \leqslant 16$ group and all the $16 < l \leqslant 20$ group:

$5 + 20 = 25$

11 a 105

b

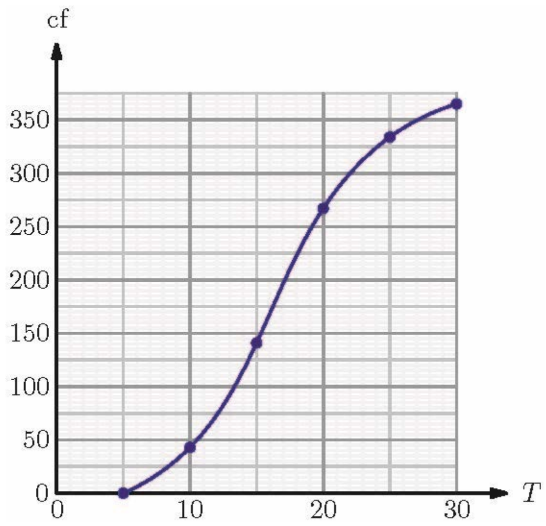


c Linear estimate: half of the $100 \leqslant m < 120$ group , all the $120 \leqslant m < 140$ group and half of the $140 \leqslant m < 160$ group:

$7 + 25 + 16 = 48$

This is $\frac{48}{105} \times 100\% = 46\%$ of the apples sampled.

12 a



b i 17 °C

ii $Q_1 \approx 13$ °C, $Q_3 \approx 20$ °C so IQR ≈ 7 °C

13 a 45

b Linear approximation: $\frac{3}{5}$ of the group $20 \leqslant t < 25$ and all the group $25 \leqslant t < 30$

$6 + 4 = 10$ pupils.

This represents $\frac{10}{45} \times 100\% = 22\%$

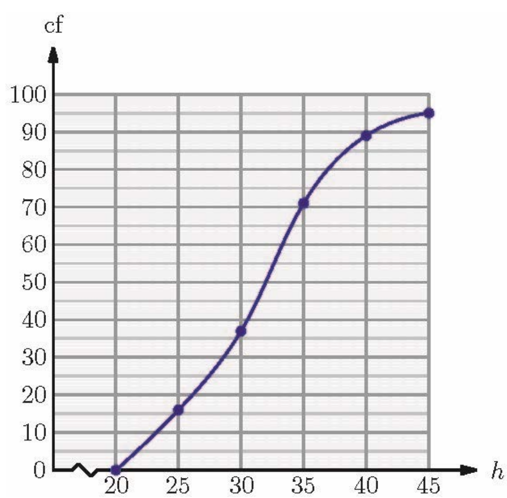
c

Time (min)	$5 \leqslant t < 10$	$10 \leqslant t < 15$	$15 \leqslant t < 20$	$20 \leqslant t < 25$	$25 \leqslant t < 30$
Freq	7	9	15	10	4

d Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{7 \times 7.5 + 9 \times 12.5 + 15 \times 17.5 + 10 \times 22.5 + 4 \times 27.5}{45} = 16.9 \text{ min}$$

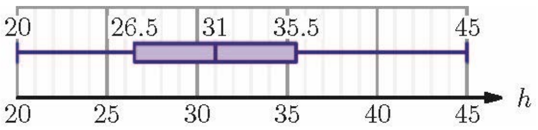
14 a



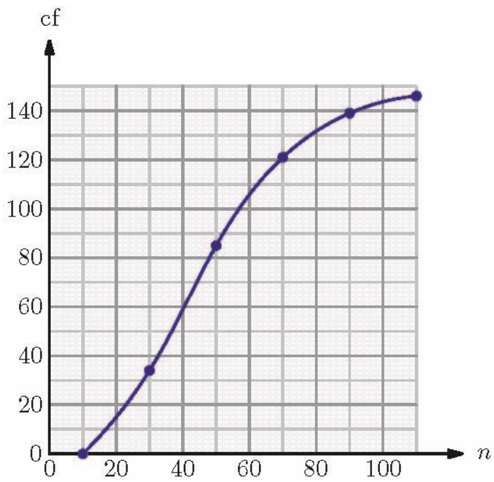
b Median ≈ 31 cm

$Q_1 \approx 26.5$ cm, $Q_3 \approx 35.5$ cm so IQR ≈ 9 cm

c



15 a

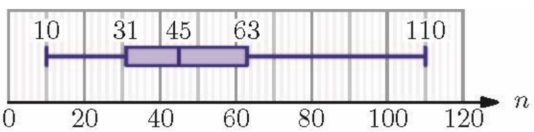


b 40

c Median ≈ 45

$Q_1 \approx 31,\ Q_3 \approx 63$ so IQR ≈ 32

d



e The distribution of number of candidates for Mathematics SL has a greater central value but is less widely spread than the distribution of number of candidates for History SL.

16 Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{35 \times 145 + 60 \times 155 + 55 \times 165 + 20 \times 175}{35 + 60 + 55 + 20} = 159 \text{ cm}$$

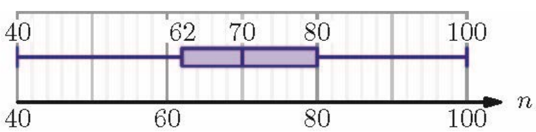
17 a 160

b Number scoring at least 55: approximately 270 so percentage passing is approximately 90%.

c 60th centile is the score exceeded by 120 students: 75 marks

d Median ≈ 72

$Q_1 \approx 62,\ Q_3 \approx 80$ so IQR ≈ 18



e The first school has a lower central score but is less widely spread (more consistent) in the scores achieved by its students, and has fewer failing the examination.

18 From GDC:

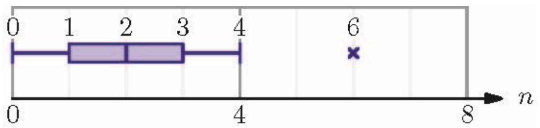
a Median, $Q_2 = 2$

$Q_1 = 1, Q_3 = 3$

b IQR = 2

c $Q_3 = 3$ and IQR $= 2$ so any value above $3 + 1.5(2) = 6$ would be an outlier, so 7 is an outlier.

d



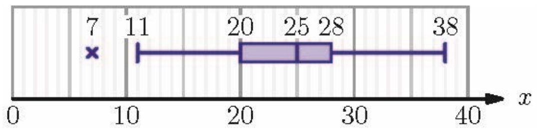
19 a IQR = 8

Outliers are values greater than 1.5IQR from the central 50% of the data.

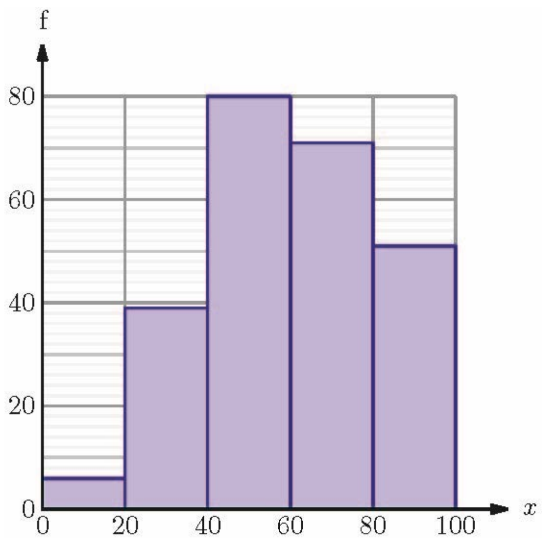
$Q_1 - 1.5 \times 8 = 8$ so 7 is an outlier.

$Q_3 + 1.5 \times 8 = 40$ so there are no outliers at the high end of the data.

b



20 a



b Using midpoints of groups to estimate the mean, and assuming the lowest possible mark is 0:

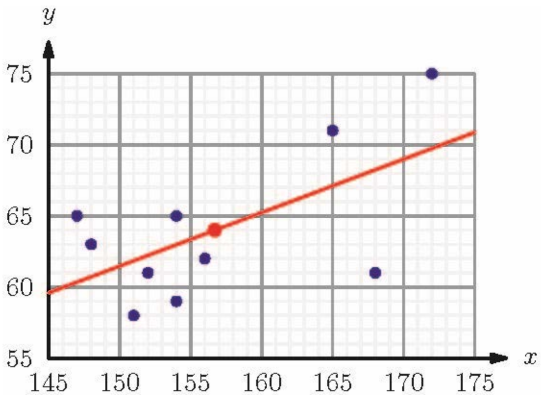
Midpoint	10	30	50	70	90
Frequency	6	39	80	71	51

$$\text{Mean} \approx \frac{6 \times 10 + 39 \times 20 + 80 \times 50 + 71 \times 70 + 51 \times 90}{247} = 59.9$$

21 A2, B3, C1

Exercise 6D

14 a, d



b The scatter growth shows a weak positive correlation

c Mean height = 156.7 cm

Mean arm length = 64 cm

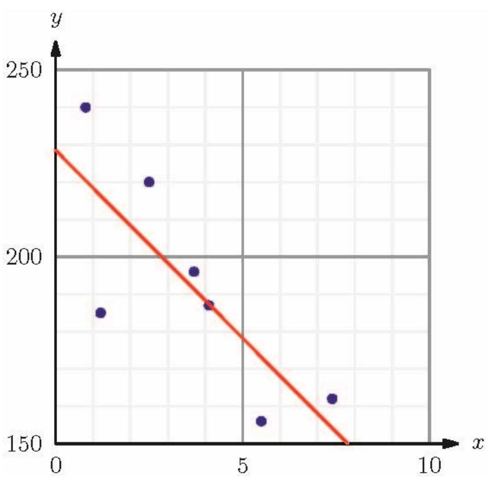
e 61.5 cm

f i With reservations, since the data is from 15-year olds, but it reasonable to predict arm length for a 16-year old from this data.

ii 192 cm is outside the range of the data taken, and extrapolating beyond the data would be unreliable.

iii The data is from adolescents and it would not be suitable to use the best fit line to estimate the arm length for a 72-year old.

15 a, d



b There is a fairly strong negative correlation between distance from the nearest train station and average house price; as distance increases, average house price tends to fall.

c Mean distance $= 3.6$ km

Mean price $= \$192\ 000$

e Using the line to estimate: A village 6.7 km from its nearest train station would be predicted to have an average house price approximately $= \$161\ 000$

16 A) Strong positive correlation: Graph 1

B) Weak negative correlation: Graph 2

C) Strong negative correlation: Graph 3

D) Weak positive correlation: Graph 4

17 a $r = 0.688$

b From GDC: $y = 0.418x + 18.1$

c The model predicts a second test score of 46.5

d The statement is not appropriate, as it infers causation from the correlation. There is no reason to believe that math score drives chemistry score (or vice versa), the correlation merely indicates that they are linked.

18 a $r = 0.745$

b Greater spending on advertising tends to yield a greater profit.

c From GDC: $y = 10.8x + 188$

d i $\$1270$

ii $\$2350$

e The estimate for $100 is within the values of the data, and is therefore more reliable than the estimate for $\$200$, for which we have to extrapolate from the graph.

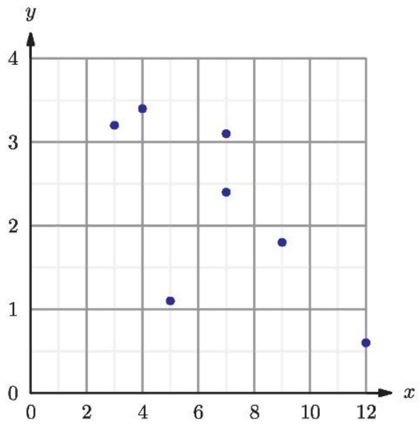
19 a $r = -0.0619$

b $y = 51.6 - 0.0370x$

c It would not be reasonable to use a mark in History to predict marks in a French test. The correlation is very low, far lower than the critical value, so the data does not indicate a significant link between the two data sets.

d As for part **c**, the correlation value is so low that no link is established between the two data sets. Neither is seen to predict the other, so a French test result could also not reliably be used to predict score in a History test.

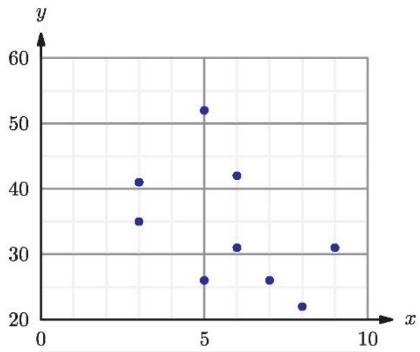
20 **a**



b $r = -0.695$

c The calculated correlation coefficient has an absolute value greater than the critical value, so we conclude that there is statistically significant negative correlation between age and value; as age increases, value tends to decrease.

21 **a**



b Weak, negative correlation

c $r = -0.480$

d The absolute value of the correlation coefficient is less than the critical value. The data does not show a significant correlation.

22 **a** There is a moderate positive correlation between head circumference and arm length in the sample.

b This head circumference is within the values of the data set and there is a moderate correlation, so the estimate can be considered reliable.

c Substituting $x = 51.7$ into the regression line equation: $y = 40.8$.

An arm length of 40.8 cm is predicted.

23 **a** $r = 0.828$

There is a moderate positive correlation between time spent practising and test mark.

b $m = 0.631t + 5.30$

c The intercept value suggests that if no time is spent revising, Theo would expect to score 5.3 on a test.

The gradient suggests that for every minute spent studying, he increases his expected test score by 0.631 marks.

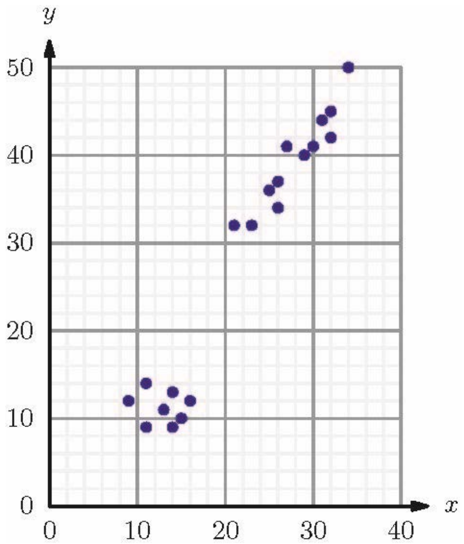
24 **a** The moderate positive correlation suggests that as advertising budget increases, so too does profit.

b No. The correlation shows a linkage, it does not show causation.

c **i** The slope 3.25 suggests that for every thousand Euros spent in advertising, the profits are expected to rise by 3 250 Euros, within the interval of the advertising budget investigated.

ii The intercept 138 suggests that with no advertising at all, there would be a profit of 138 000 Euros.

25 **a**



b The two regions represent sales in cold weather (perhaps winter) and sales in warm weather (summer).

c For the winter temperatures, there appears to be no significant correlation between temperature and sales.

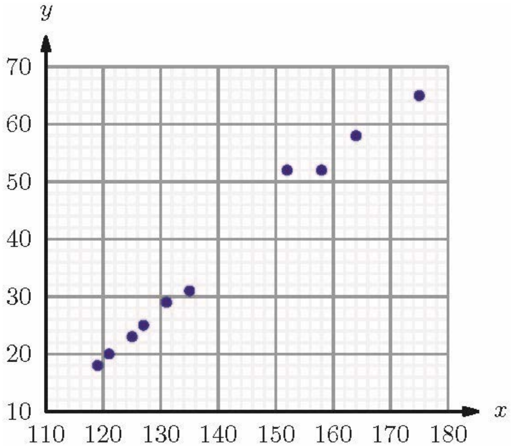
For the summer temperatures, there appears to be a strong positive correlation between temperature and sales.

d Excluding the lower population (temperatures below 20 °C), the regression line is calculated as

$$y = 2.97 + 1.30x$$

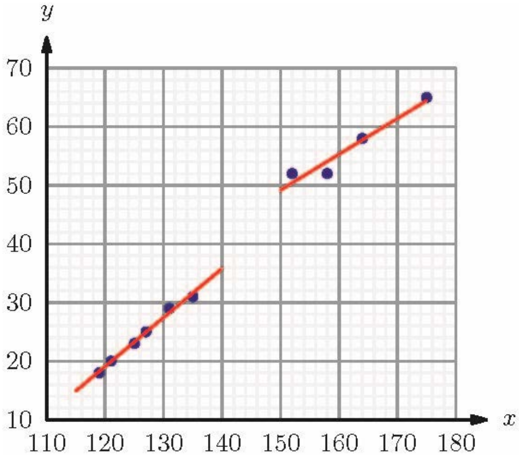
This predicts that for a temperature of 28 °C, the sales would be approximately 39.5.

26 a



b The children and the adults have height to mass relationships, but each subpopulation shows a (different) positive linear correlation.

c



children $(126, 24.3)$, adults $(162, 56.8)$

d The regression line for the childrens' data (heights less than 140 cm) in the data has equation

$$m = 0.835h - 81.1$$

This predicts a mass of 19.0 kg for a child with height 120 cm.

27 a For x:

$$Q_1 = 12, Q_3 = 24.$$

For y,

$$Q_1 = 11, Q_3 = 19$$

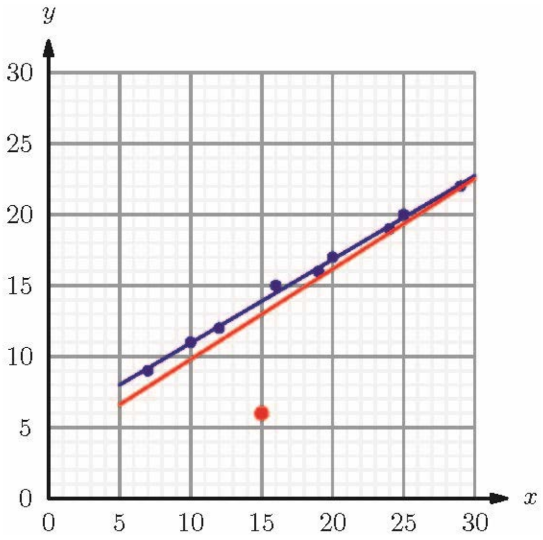
b Outliers are values more than 1.5IQR above Q_3 or below Q_1.

For x, IQR $= 12$ and all the data values lie within $\big(12 - 1.5(12), 24 + 1.5(12)\big) = (-6,42)$

For y, IQR $= 8$ and all the data values lie within $\big(11 - 1.5(8), 19 + 1.5(8)\big) = (-1,31)$

There are no outliers in either data set.

c, d, e



Mixed Practice

1 **a** **i** Systematic sampling

ii Students might have a weekly borrowing pattern, so would either be sampled every time or not at all.

Even without such a regular pattern, some days of the week might be busier than others, and picking the same day every week would then not be representative.

b **i** Each possible ten day selection has an equal chance of being picked for the sample.

ii The sample is more likely to be representative of all the days in the investigation.

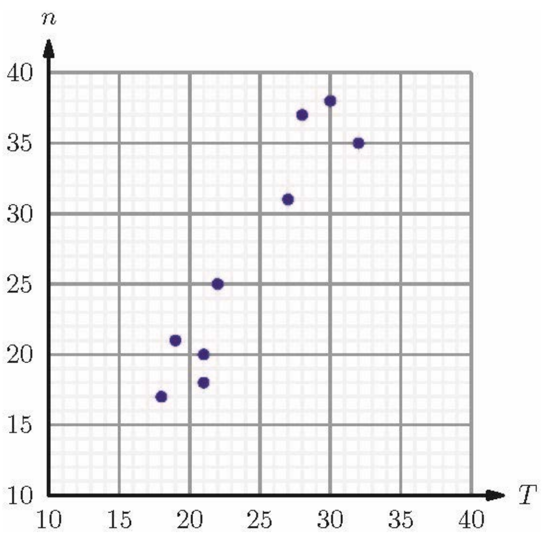
c From GDC:

i Range $= 11$

ii $\bar{x} = 17.4$

iii $\sigma = 3.17$

2 **a**



b There is a strong positive correlation; as temperature increases, sale of cold drinks also tends to increase.

c From GDC: $n = 1.56T - 10.9$

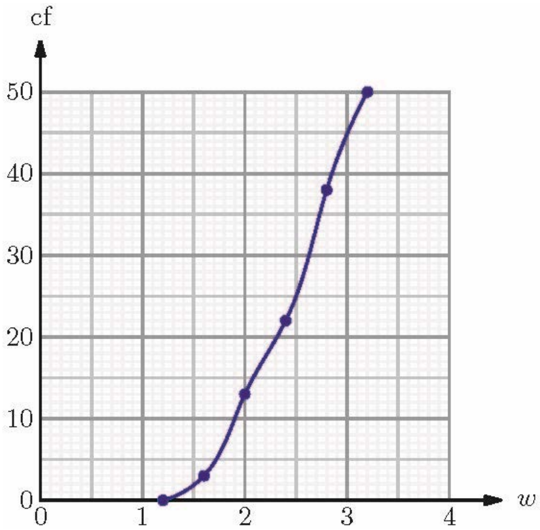
d 26 °C is within the range of data values for a high correlation line, so we can confidently estimate using the regression equation.

When $T = 26, n \approx 30.0$

3 **a** Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{4 \times 1.4 + 10 \times 1.8 + 8 \times 2.2 + 16 \times 2.6 + 12 \times 3.0}{50} = 2.38 \text{ kg}$$

b

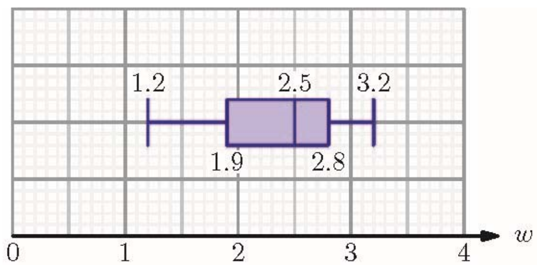


c From the graph:

Median $= 2.5$ kg

$Q_1 = 1.9$ kg, $Q_3 = 2.8$ kg so IQR $= 0.9$ kg

d



4 **a** Discrete

b Mode $= 0$

c From GDC:

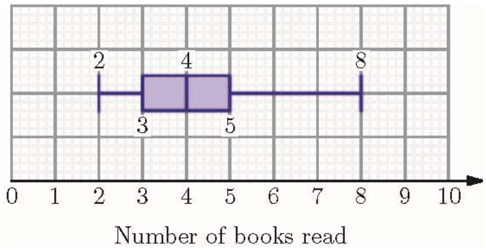
i $\bar{x} = 1.47$ passengers

ii Median $= 1.5$ passengers

iii $\sigma = 1.25$ passengers

5 **a** Median = 4

b



c $40 \times 25\% = 10$ students

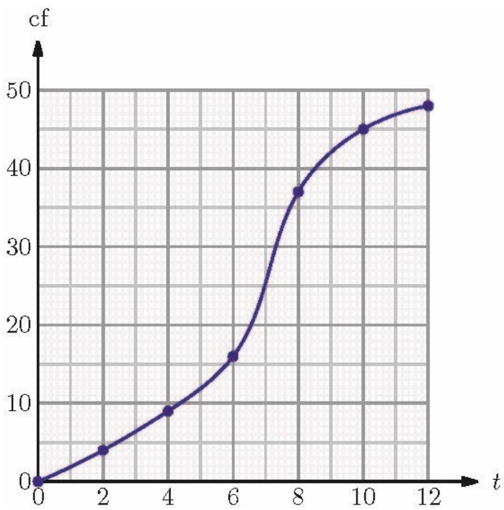
6 **a** From GD: $r = 0.996$

b From GDC: $y = 3.15x - 15.4$

c High correlation and value within the data range allows confident approximation from the regression equation.

When $x = 26, y \approx 66.5$

7 **a**



b From the graph:

i Median = 6.9 mins

ii $Q_1 = 5.0$ mins, $Q_3 = 7.9$ mins so IQR = 1.9 mins

iii 90th centile is at $cf = 43.2$, which corresponds to approximately 9.3 mins

c

Time	Freq
$0 \leqslant t < 2$	4
$2 \leqslant t < 4$	5
$4 \leqslant t < 6$	7
$6 \leqslant t < 8$	11
$8 \leqslant t < 10$	8
$10 \leqslant t < 12$	3

d Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{4 \times 1 + 5 \times 3 + 7 \times 5 + 21 \times 7 + 8 \times 9 + 3 \times 11}{48} = 6.375 \text{ mins}$$

8 **a** From GDC:

Median $= 46$

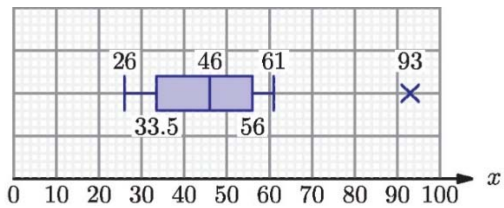
$Q_1 = 33.5, Q_3 = 56$ so IQR $= 22.5$

b Outliers are values more than 1.5IQR above Q_3 or below Q_1.

The lower boundary for outliers is $33.5 - 1.5(22.5) = -0.25$ so there are no outliers at the lower end of the data.

The upper boundary for outliers is $56 + 1.5(22.5) = 89.75$ so 93 is an outlier.

c



9 **a** From GDC:

Mean $= 121$ cm

Var $= 22.9 \text{ cm}^2$

b Adding a constant changes the mean but not the variance.

For the new data,

Mean $= 156$ cm

Var $= 22.9 \text{ cm}^2$

10 Let X be the distance travelled, in kilometres.

$\bar{x} = 11.6, \sigma_x = 12.5$

Let Y be the cost of his travels, in \$

$Y = 15 + 3.45X$

Then $\bar{y} = 15 + 3.45\bar{x} = \$55.02$

And $\sigma_y = 3.45\sigma_x = \$43.13$

11

$$\text{Mean} = \frac{5 \times 0 + 6 \times 1 + 8 \times 2 + 3x}{19 + x}$$

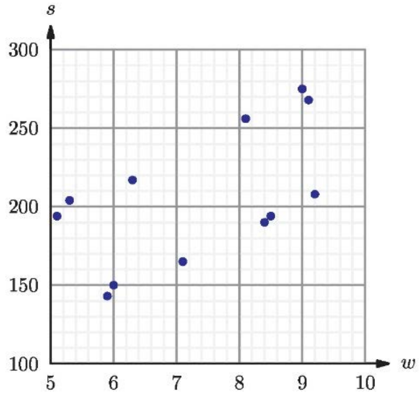
$$= \frac{22 + 3x}{19 + x} = 1.6$$

$22 + 3x = 30.4 + 1.6x$

$1.4x = 8.4$

$x = 6$

12 a



b From GDC: $w = 19.1s + 99.0$

c From GDC: $r = 0.994$

This indicates a strong positive correlation between shell length and mass; as shell length increases, mass reliably increases as well.

d 8 cm is within the range of data values, and the correlation is high, so an estimate from the regression equation is reliable.

The model predicts a mass of 252 g

e The model would predict a mass of between 137 g and 175 g.

This prediction is not reliable, since these masses are outside the data gathered, and cannot reliably be estimated by extrapolating the linear model for adult crabs.

13 a i A positive y-intercept of 2 could be interpreted as a fixed increase in performance of 2 miles for all athletes, irrespective of their previous fitness levels.

ii The positive gradient of 1.2 could be interpreted as a variable 20% increase in performance levels for all athletes.

b The new mean would be $1.2(8) + 2 = 11.6$ miles

c i The correlation is unchanged by a linear transformation on the data. It remains 0.84

ii The equation of the regression line becomes

$$\begin{aligned} Y &= 1.6y \\ &= 1.6(1.2x + 2) \\ &= 1.6\left(\frac{1.2}{1.6}X + 2\right) \\ &= 1.2X + 3.2 \end{aligned}$$

Since both axes undergo the same linear transformation, the gradient remains unchanged but the intercept is adjusted (what was an intercept of 2 miles is now 3.2 km).

7 Core: Probability

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 7A

16 **a** Estimate $\text{P(negative side effect)} = \frac{26}{350} = \frac{13}{175} \approx 0.0743$

b $\text{Expected number with side effects} = 900 \times \frac{13}{175} = 66.9$

17 $\text{P}(2) = \frac{2}{8} = \frac{1}{4}$

$\text{Expected number of twos after 30 rolls} = 30 \times \frac{1}{4} = 7.5$

18 $\text{P(not a diamond)} = \frac{3}{4}$

$\text{Expected number not diamonds} = 20 \times \frac{3}{4} = 15$

19 $\text{P}(1) = \frac{1}{n}$

So if $400 \times \text{P}(1) = 50$ then $n = 8$

20 $\text{P(Odd)} + \text{P(Even)} = 1$ because 'Odd' and 'Even' are complementary events.

$\text{P(Odd)} = 3\text{P(Even)}$ so $4\text{P(Even)} = 1$

$\text{P(Even)} = \frac{1}{4}, \text{P(Odd)} = \frac{3}{4}$

21 $\text{P(accident)} = \frac{124}{2000} = 0.062$

If the policy price is $\$A$, the company wants the expected payout to be $0.8\$A$ in order to make a 20% profit on policies sold.

The expected payout per policy is $0.062 \times \$15\,000 = \$930$

$0.8A = 930$ so $A = \$1162.50$

22 **a** $\frac{0.6}{0.4} = 1.5$

b $\frac{p}{1-p} = 4$

$p = 4 - 4p$

$5p = 4$

$p = 0.8$

23 Let r be the number of red balls.

Then there are $3r$ green balls and $12r$ blue balls, for a total of $16r$ balls.

$$\text{P(Red ball)} = \frac{r}{16r} = \frac{1}{16}$$

24 The area of the square is 1 and the area of the circle, which has radius $\frac{1}{2}$, is $\frac{\pi}{4}$

The probability of any point falling into the circle is therefore $\frac{\pi}{4}$.

The relative frequency in the simulation gives $\frac{78}{100} \approx \frac{\pi}{4}$

So $\pi \approx \frac{78}{25} = 3.12$

Exercise 7B

40 a

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second die	**1**	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

b $\text{P}(X \geqslant 20) = \frac{8}{36} = \frac{4}{18}$

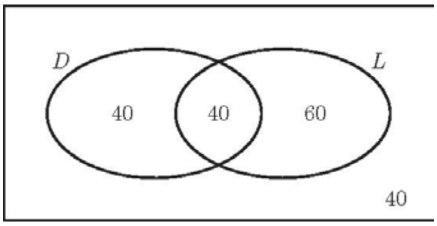
Expected number of twenties in 180 trials $= 180 \times \frac{4}{18} = 40$

41

		First die			
		1	**2**	**3**	**4**
Second die	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8
	5	6	7	8	9
	6	7	8	9	10
	7	8	9	10	11
	8	9	10	11	12

$$\text{P}(X < 10) = \frac{26}{32} = \frac{13}{16}$$

42 a



b $P(D' \cap L') = \frac{40}{180} = \frac{2}{9}$

c $P(L|D') = \frac{40}{80} = \frac{1}{2}$

43 $P(G, Y) = \frac{12}{30} \times \frac{18}{29} = \frac{36}{145}$

$P(Y, G) = \frac{18}{30} \times \frac{12}{29} = \frac{36}{145}$

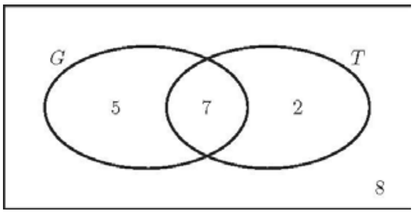
So $P(\text{different colours}) = \frac{72}{145} \approx 0.496$

44 a i $P(\text{late}|\text{rain}) = \frac{5}{40} = \frac{1}{8}$

ii $P(\text{late}|\text{not rain}) = \frac{6}{48} = \frac{1}{8}$

b Since $P(\text{late}|\text{rain}) = P(\text{late}|\text{not rain})$, rain and late are independent events.

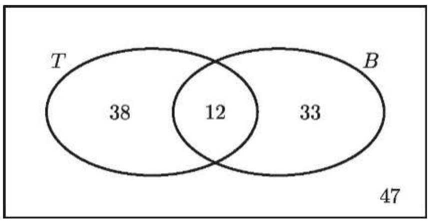
45 a



b $P(G) = \frac{7}{22}$

c $P(T \mid G) = \frac{5}{7}$

46



a $130 - (50 + 45 - 12) = 47$

b $\frac{83}{130}$

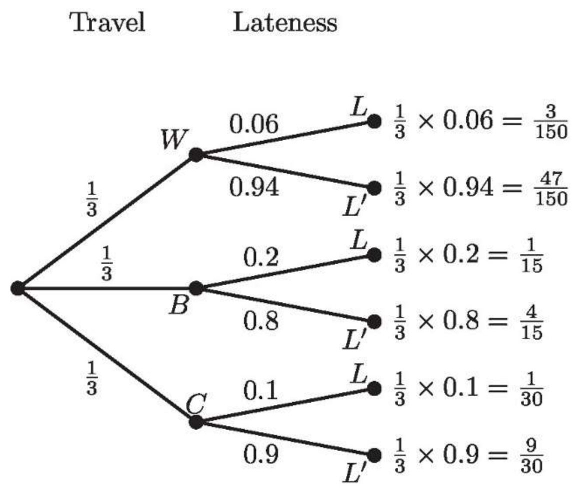
c $\frac{33}{38+33} = \frac{33}{71}$

47 There are 4 teams higher in the league and 13 teams lower in the league.

$$\text{P(win)} = \frac{4}{17} \times 20\% + \frac{13}{17} \times 70\% = 58.2\%$$

48 $\text{P(working)} = \frac{1}{2} \times 0.8 + \frac{1}{2} \times 0.7 = 0.75$

49



a $\text{P}(\text{B} \cap \text{L}) = \frac{1}{15}$

b $\text{P}(\text{L}') = \frac{47}{150} + \frac{4}{15} + \frac{9}{30} = \frac{132}{150} = \frac{22}{25} = 0.88$

50 There are $6^3 = 216$ possible throws (using ordered dice)

Means of throwing a total of 5:

1,1,3	1,2,2	1,3,1	2,1,2	2,2,1	3,1,1

Total of 6 ways, so the probability of a score 5 is $\frac{6}{216} = \frac{1}{36}$

51 a i $\text{P(blue)} = \frac{77}{77+59+51} = \frac{77}{187} = \frac{7}{17}$

ii $\text{P(blond)} = \frac{79}{187}$

iii $\text{P}(\text{blue} \cap \text{blond}) = \frac{26}{187}$

52

$$\begin{aligned} \text{P}(B) &= \text{P}(A \cup B) - \big(\text{P}(A) - \text{P}(A \cap B)\big) \\ &= 0.9 - (0.6 - 0.2) \\ &= 0.5 \end{aligned}$$

53

$$\begin{aligned} \text{P}(A \cap B) &= \big(\text{P}(A) + \text{P}(B)\big) - \text{P}(A \cup B) \\ &= (0.7 + 0.7) - 0.9 \\ &= 0.5 \end{aligned}$$

54 a

$$\begin{aligned}P(A \cap B) &= P(A) \times P(B|A)\\ &= \frac{2}{5} \times \frac{1}{2}\\ &= \frac{1}{5}\end{aligned}$$

b

$$\begin{aligned}P(B) &= P(A \cup B) - \big(P(A) - P(A \cap B)\big)\\ &= \frac{3}{4} - \left(\frac{2}{5} - \frac{1}{5}\right)\\ &= \frac{11}{20}\end{aligned}$$

55 a $P(R_2|R_1) = \frac{39}{69} = \frac{13}{23}$

b

$$\begin{aligned}P(R_1 \cap R_2) &= \frac{40}{70} \times \frac{39}{69}\\ &= \frac{52}{161}\end{aligned}$$

c

$$\begin{aligned}P(G_1 \cap G_2) &= \frac{30}{70} \times \frac{29}{69}\\ &= \frac{29}{161}\end{aligned}$$

Then the probability of same colours is $\frac{52+29}{161} = \frac{81}{161} > \frac{1}{2}$

It is more likely to get the same colours than to get different colours.

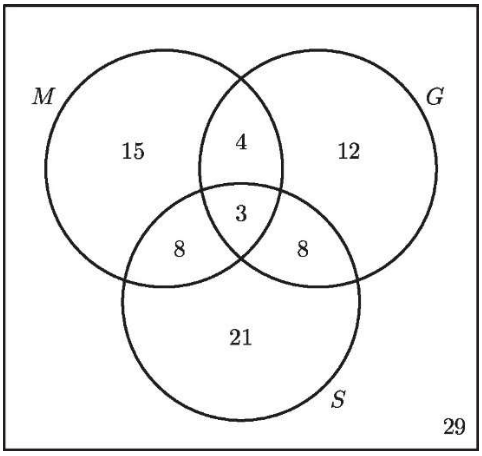
56 a Let R_n be the event of rain on day n

$$\begin{aligned}P(R_1 \cup R_2) &= 1 - P(R_1' \cap R_2')\\ &= 1 - (0.88^2)\\ &= 0.2256\end{aligned}$$

b

$$\begin{aligned}P(R_1 \cap R_2 \cap R_3) &= 0.12^3\\ &= 0.00173\end{aligned}$$

57 a

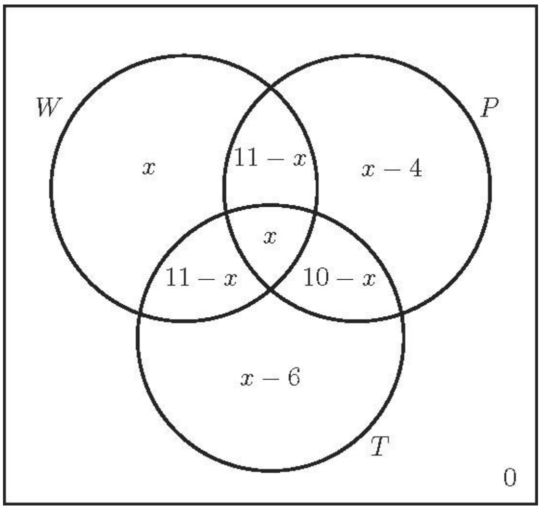


b $P(S \cap G' \cap M') = \frac{21}{100} = 0.21$

c

$$P(M|S) = \frac{n(M \cap S)}{n(S)}$$
$$= \frac{11}{40}$$

58 a



b The total in all the regions of the diagram must equal 30.

$$x + (11 - x) + (11 - x) + x + (x - 4) + (10 - x) + (x - 6) = 30$$
$$x + 22 = 30$$

c $x = 8$

$$P(W \cap P \cap T) = \frac{8}{30} = \frac{4}{15}$$

d

$$P(P|T') = \frac{n(P \cap T')}{n(T')}$$
$$= \frac{7}{15}$$

e

$$\begin{aligned} P(W \cap T|2 \text{ items}) &= \frac{n(W \cap T \cap P')}{n(2 \text{ items})} \\ &= \frac{3}{8} \end{aligned}$$

59 a There are 3 options for older and younger: (B, G), (G, B) and (B, B). In the absence of other information, each is equally likely.

$$P(B, B) = \frac{1}{3}$$

b There are now only two options for older and younger: BG or BB. In the absence of other information, each is equally likely.

$$P(B, B) = \frac{1}{2}$$

Mixed Practice

1 $650 \times \frac{128}{200} = 416$ participants

2

	boy	girl	total
apples	16	21	37
bananas	32	14	46
strawberries	11	21	32
total	59	56	115

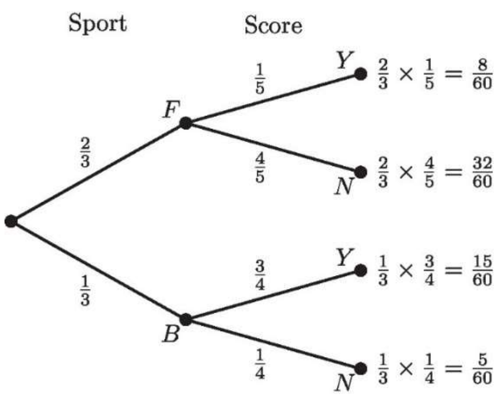
a $P(\text{girl}) = \frac{56}{115}$

b $P(\text{girl} \cap \text{apples}) = \frac{21}{115}$

c $P(\text{banana}|\text{boy}) = \frac{32}{59}$

d $P(\text{girl}|\text{strawberry}) = \frac{21}{32}$

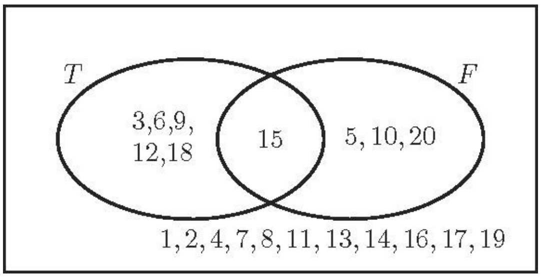
3



a $P(F \cap Y) = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

b $P(N) = \frac{32}{60} + \frac{5}{60} = \frac{37}{60}$

4 **a**



b **i** $P(F) = \frac{4}{20} = \frac{1}{5}$

ii $P(F|T') = \frac{3}{14}$

5 **a**

$$\begin{aligned} P(F \cap G) &= P(F) + P(G) - P(F \cup G) \\ &= 0.4 + 0.6 - (1 - 0.2) \\ &= 0.2 \end{aligned}$$

b

$$\begin{aligned} P(G|F') &= \frac{P(G \cap F')}{P(F')} \\ &= \frac{0.4}{0.6} \\ &= \frac{2}{3} \end{aligned}$$

6 **a**

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.6 + 0.3 - 0.72 \\ &= 0.18 \end{aligned}$$

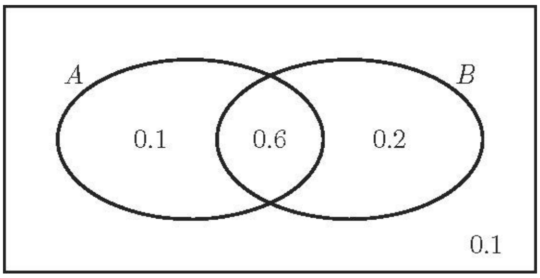
b Since $P(A \cap B) = P(A) \times P(B)$, A and B are independent events.

7 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are independent then $P(A \cap B) = P(A) \times P(B)$

$$\begin{aligned} \text{So } P(A \cup B) &= P(A) + P(B) - P(A) \times P(B) \\ &= 0.6 + 0.8 - 0.6 \times 0.8 \\ &= 1.4 - 0.48 \\ &= 0.92 \end{aligned}$$

8 **a**



b $P(A|B) = \frac{0.6}{0.8} = \frac{3}{4} = 0.75$

c $P(B|A') = \frac{0.2}{0.3} = \frac{2}{3} = 0.667$

9 **a** **i**

$$\begin{aligned} P(1) &= P(1|H) \times P(H) + P(1|T) \times P(T) \\ &= \frac{1}{6} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3} \\ &= \frac{1}{18} + \frac{3}{18} \\ &= \frac{2}{9} \end{aligned}$$

ii

$$\begin{aligned} P(6) &= P(6|H) \times P(H) + P(6|T) \times P(T) \\ &= \frac{1}{6} \times \frac{1}{3} + 0 \times \frac{2}{3} \\ &= \frac{1}{18} \end{aligned}$$

b

$$\begin{aligned} P(3 \cup 6) &= P(3 \cup 6|H) \times P(H) + P(3 \cup 6|T) \times P(T) \\ &= \frac{1}{3} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3} \\ &= \frac{2}{18} + \frac{3}{18} \\ &= \frac{5}{18} \end{aligned}$$

10 Let S be the event that the number is a multiple of 7 and N be the event that the number is a multiple of 9.

a $\frac{1000}{7} = 142.9$

$$\begin{aligned} P(S) &= \frac{143}{1000} \\ &= 0.143 \end{aligned}$$

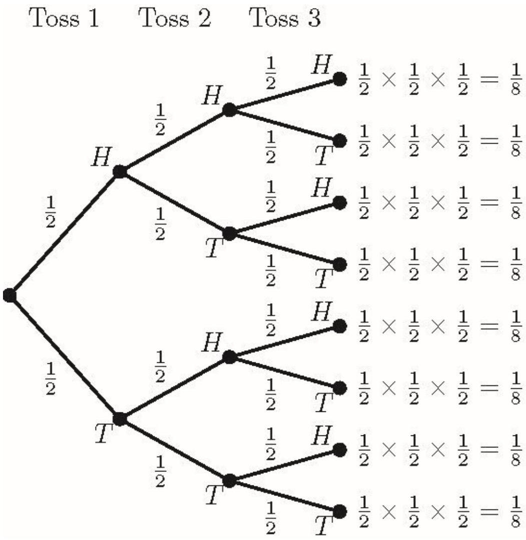
b $\frac{1000}{9} = 111.19$

$$\begin{aligned} P(N) &= \frac{111}{1000} \\ &= 0.111 \end{aligned}$$

c $\frac{1000}{63} = 15.9$

$$\begin{aligned} P(S \cup N) &= P(S) + P(N) - P(S \cap N) \\ &= 0.142 + 0.111 - 0.015 \\ &= 0.238 \end{aligned}$$

11 a



b $P(T, T, T) = \frac{1}{8}$

c $P(\text{at least one } H) = 1 - P(T, T, T) = \frac{7}{8}$

d $P(2H + 1T) = P(H, H, T) + P(H, T, H) + P(T, H, H) = \frac{3}{8}$

12 For Asher:

$$\begin{aligned} P(WB \cup BW) &= \frac{8}{14} \times \frac{6}{13} + \frac{6}{14} \times \frac{8}{13} \\ &= \frac{48}{91} \end{aligned}$$

For Elsa:

$$\begin{aligned} P(WB \cup BW) &= \frac{8}{14} \times \frac{6}{14} + \frac{6}{14} \times \frac{8}{14} \\ &= \frac{48}{98} = \frac{24}{49} \end{aligned}$$

Asher has a higher chance of selecting one of each colour.

13

$$\begin{aligned} P(RR \cup WW \cup BB) &= \frac{8}{19} \times \frac{7}{18} + \frac{6}{19} \times \frac{5}{18} + \frac{5}{19} \times \frac{4}{18} \\ &= \frac{106}{342} \\ &= \frac{53}{171} \\ &\approx 0.310 \end{aligned}$$

14 a

$$P(R_2 | B_1) = \frac{8}{23}$$

b

$$\begin{aligned}P(R_1 \cup R_2) &= P(R_1) + P(R_2) - P(R_1 \cap R_2)\\ &= \frac{8}{24} + \frac{8}{24} - \frac{8}{24} \times \frac{7}{23}\\ &= \frac{13}{23}\end{aligned}$$

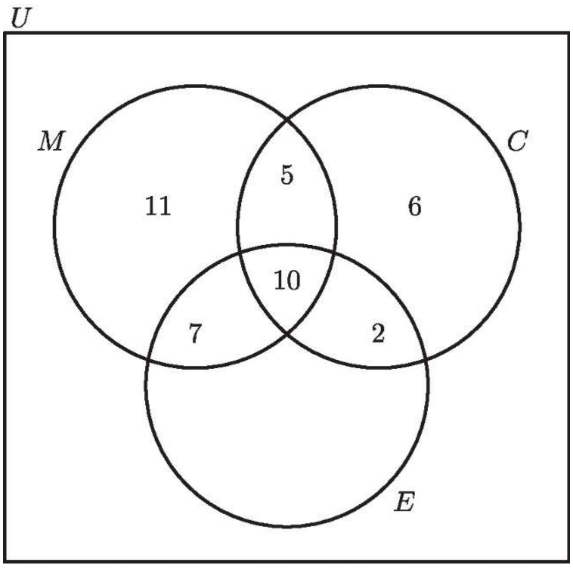
15 a $P(R) = \frac{3}{12} = \frac{1}{4}$

b $P(G_1B_2) = \frac{2}{12} \times \frac{7}{11} = \frac{7}{66}$

c

$$\begin{aligned}P(G_1G_2 \cup R_1R_2 \cup B_1B_2) &= \frac{2}{12} \times \frac{1}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{7}{12} \times \frac{6}{11}\\ &= \frac{25}{66}\end{aligned}$$

16 a



b 16

c i $22 - 19 = 3$

ii Total in the diagram studying at least one of the subjects:
$11 + 5 + 6 + 7 + 10 + 2 + 3 = 44$

Total who study none of these subjects: $100 - 44 = 56$

d i $P(E) = \frac{22}{100} = \frac{11}{50} = 0.22$

ii $P(M \cap C \cap E') = \frac{5}{100} = \frac{1}{20} = 0.05$

iii $P(M' \cap E') = \frac{62}{100} = \frac{31}{50} = 0.62$

iv

$$P(M'|E') = \frac{P(M' \cap E')}{P(E')} = \frac{0.62}{0.78} = \frac{31}{39}$$

17 $P(R_1R_2) = \frac{10}{10+n} \times \frac{9}{9+n}$

$$P(Y_1Y_2) = \frac{n}{10+n} \times \frac{n-1}{9+n}$$

$$P(\text{Same colour}) = P(R_1R_2) + P(Y_1Y_2) = \frac{90 + n^2 - n}{(10+n)(9+n)} = \frac{1}{2}$$

Rearranging:

$$2(90 + n^2 - n) = (10+n)(9+n)$$
$$180 + 2n^2 - 2n = 90 + 19n + n^2$$
$$n^2 - 21n + 90 = 0$$

b Factorising:

$$(n-15)(n-6) = 0$$

$n = 15$ or $n = 6$

18 a $P(F \cup S) = 100\%$ since all students have to learn at least one of the languages.

$$P(F \cap S) = P(F) + P(S) - P(F \cup S) = 40\% + 75\% - 100\% = 15\%$$

b

$$P(S \cap F') = P(S) - P(F \cap S) = 75\% - 15\% = 60\%$$

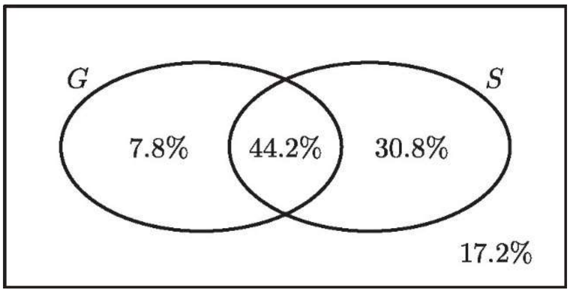
c i

$$P(G \cap S) = P(S|G) \times P(G) = 85\% \times 52\% = 0.442 = 44.2\%$$

ii $P(G \cap S) = 44.2\%$ and $P(G) \times P(S) = 52\% \times 75\% = 39\%$

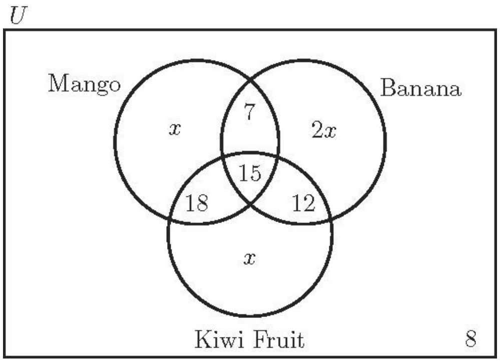
Since $P(G \cap S) \neq P(G) \times P(S)$, the two events are not independent.

d



$$\mathrm{P}(S|G') = \frac{\mathrm{P}(S \cap G')}{\mathrm{P}(G')} = \frac{30.8\%}{48\%} = 0.642 = 64.2\%$$

19 a and b



c $x + 7 + 2x + 18 + 15 + 12 + x + 8 = 100$

$4x + 60 = 100$

$x = 10$

d i $n(\text{Mango}) = 50$

ii $n(\text{Mango} \cup \text{Banana}) = 82$

e i $\mathrm{P}(M' \cap B' \cap K') = \frac{8}{100} = \frac{2}{25} = 0.08$

ii $\mathrm{P}(M' \cap B \cap K) + \mathrm{P}(M \cap B' \cap K) + \mathrm{P}(M \cap B \cap K') = \frac{37}{100} = 0.37$

iii $\mathrm{P}(M \cap B \cap K|M \cap B) = \frac{15}{22} = 0.682$

f $\mathrm{P}(\text{both dislike all}) = \frac{8}{100} \times \frac{7}{99} = \frac{14}{2475} = 0.566\%$

8 Core: Probability distributions

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Tip: All the values calculated here are given exactly or to 3 significant figures.

Wherever a previously calculated value has to be used in a subsequent part of the question, the calculated value has been retained in the calculator and reused. If you consistently get a slightly different answer, you may be using rounded values in subsequent calculations, and are experiencing the effects of cumulative rounding errors.

As a rule of thumb, if your final answer is to be accurate to 3 s.f., you need to keep at least 4 and preferably 5 s.f. for all preliminary calculated values, or store results in your calculator for future use in the same question.

Exercise 8A

13 a Require $\sum \mathrm{P}(X = x) = 1$

$$0.2 + 0.1 + 0.3 + k = 1$$
$$k = 0.4$$

b $\mathrm{P}(X \geq 3) = 0.3 + k = 0.7$

c

$$\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$$
$$= (1 \times 0.2) + (2 \times 0.1) + (3 \times 0.3) + 4k$$
$$= 2.9$$

14 a Require $\sum \mathrm{P}(Y = y) = 1$

$$0.1 + 0.3 + k + 2k = 1$$
$$3k = 0.6$$
$$k = 0.2$$

b $\mathrm{P}(Y < 6) = 0.1 + 0.3 = 0.4$

c

$$\mathrm{E}(Y) = \sum y\,\mathrm{P}(Y = y)$$
$$= (1 \times 0.1) + (3 \times 0.3) + 6k + 10(2k)$$
$$= 6.2$$

15 a $\mathrm{P(R, R)} = \frac{8}{14} \times \frac{7}{13} = \frac{4}{13}$

$\mathrm{P(R, Y)} = \frac{8}{14} \times \frac{6}{13} = \frac{24}{91}$

$\mathrm{P(Y, R)} = \frac{6}{14} \times \frac{8}{13} = \frac{24}{91}$

$\mathrm{P(Y, Y)} = \frac{6}{14} \times \frac{5}{13} = \frac{15}{91}$

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{4}{13}$	$\frac{48}{91}$	$\frac{15}{91}$

b

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= 0 + \left(1 \times \frac{48}{91}\right) + \left(2 \times \frac{15}{91}\right)\\ &= \frac{78}{91} \approx 0.857\end{aligned}$$

16 a $\mathrm{P(H, H)} = \frac{13}{52} \times \frac{12}{51} = \frac{1}{17}$

b $\mathrm{P(H, H')} = \frac{13}{52} \times \frac{39}{51} = \frac{13}{68}$

$\mathrm{P(H', H)} = \frac{39}{52} \times \frac{13}{51} = \frac{13}{68}$

$\mathrm{P(H', H')} = \frac{39}{52} \times \frac{38}{51} = \frac{19}{34}$

h	0	1	2
$\mathrm{P}(H = h)$	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{1}{17}$

c

$$\begin{aligned}\mathrm{E}(H) &= \sum h\,\mathrm{P}(H = h)\\ &= 0 + \left(1 \times \frac{13}{34}\right) + \left(2 \times \frac{1}{17}\right)\\ &= \frac{17}{34}\\ &= 0.5\end{aligned}$$

17 Two outcomes:

Heads: Olivia has a loss of £2

Tails: Olivia has a gain of £3

Olivia's expected gain is $\frac{1}{2} \times (-£2) + \frac{1}{2} \times £3 = 50$ pence

Since Olivia has an expectation greater than zero, the game is biased in her favour, so is not fair.

18 Two outcomes:

Diamond: Shinji has a loss of £3

Not diamonds: Shinji has a gain of £n

Shinji's expected gain is $\frac{1}{4} \times (-£3) + \frac{3}{4} \times £n = (75n - 75)$ pence

The game is 'fair' (Shinji's expected gain is zero) if $n = 1$

(This is a zero-sum game, so if fair for Shinji then it is also fair for Maria)

Worked solutions

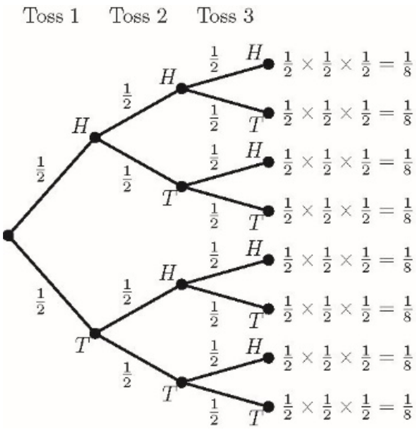
19 Let X be the number or tails and therefore the number of dollars paid out.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= 0 + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) \\ &= \frac{12}{8} \\ &= 1.5 \end{aligned}$$

So the stall should charge \$1.50 to make the game fair.

20 a



b

Let X be the number of heads.

$$P(X = 2) = \frac{3}{8}$$

c

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= 0 + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) \\ &= \frac{12}{8} \\ &= 1.5 \end{aligned}$$

21 a

		First die			
		1	**2**	**3**	**4**
Second Die	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

x	2	3	4	5	6	7	8
$\mathrm{P}(X = x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

d

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= \left(2 \times \frac{1}{16}\right) + \left(3 \times \frac{1}{8}\right) + \left(4 \times \frac{3}{16}\right) + \left(5 \times \frac{1}{4}\right) + \left(6 \times \frac{3}{16}\right) + \left(7 \times \frac{1}{8}\right) + \left(8 \times \frac{1}{16}\right)\\ &= 5\end{aligned}$$

22 a $\mathrm{P}(X = 3) = \frac{1}{15}(6) = \frac{2}{5} = 0.4$

b

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= \left(1 \times \frac{4}{15}\right) + \left(2 \times \frac{5}{15}\right) + \left(3 \times \frac{6}{15}\right)\\ &= \frac{32}{15} = 2.13\end{aligned}$$

23 a Require $\sum \mathrm{P}(y = y) = 1$

$$\begin{aligned}k(3 + 4 + 5) &= 1\\ k &= \frac{1}{12}\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(Y \geq 5) &= \mathrm{P}(Y = 5) + \mathrm{P}(Y = 6)\\ &= 4k + 5k\\ &= \frac{3}{4} = 0.75\end{aligned}$$

c

$$\begin{aligned}\mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y)\\ &= \left(4 \times \frac{3}{12}\right) + \left(5 \times \frac{4}{12}\right) + \left(6 \times \frac{5}{12}\right)\\ &= \frac{62}{12} = 5.17\end{aligned}$$

24 a Require $\sum \mathrm{P}(X = x) = 1$

$$c\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = 1$$
$$\frac{25}{12}c = 1$$
$$c = \frac{12}{25} = 0.48$$

b

x	1	2	3	4
$\mathbf{P}(X = x)$	$\frac{12}{25}$	$\frac{12}{50}$	$\frac{12}{75}$	$\frac{12}{100}$

$$\mathrm{P}(X \leq 3) = \frac{88}{100}$$

So $\mathrm{P}(X = 2 | X \leq 3) = \dfrac{\left(\frac{12}{50}\right)}{\left(\frac{88}{100}\right)} = \frac{24}{88} = \frac{3}{11}$

c

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= \left(1 \times \frac{12}{25}\right) + \left(2 \times \frac{12}{50}\right) + \left(3 \times \frac{12}{75}\right) + \left(4 \times \frac{12}{100}\right)\\ &= 1.92\end{aligned}$$

25 Require $\sum \mathrm{P}(X = x) = 1$

$a + 0.2 + 0.3 + b = 1$

$$a + b = 0.5 \qquad (1)$$

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= (1 \times a) + (2 \times 0.2) + (3 \times 0.3) + (4 \times b)\\ &= a + 4b + 1.3\\ &= 2.4\end{aligned}$$

$$a + 4b = 1.1 \qquad (2)$$

$(2) - (1): 3b = 0.6$

So $b = 0.2, a = 0.3$

Exercise 8B

19 a $X \sim \mathrm{B}\left(10, \frac{1}{6}\right)$

b From GDC: $\mathrm{P}(X = 2) = 0.291$

c From GDC:

$$\begin{aligned}\mathrm{P}(X > 2) &= 1 - \mathrm{P}(X \leqslant 2)\\ &= 1 - 0.775\\ &= 0.225\end{aligned}$$

20 a Let X be the number of times scored in 12 shots.

$X \sim (12, 0.85)$

From GDC: $P(X = 10) = 0.292$

b From GDC

$$P(X \geqslant 10) = 1 - P(X \leqslant 9) \\ = 1 - 0.264 \\ = 0.736$$

c $E(X) = 12 \times 0.85 = 10.2$

21 a From GDC: $P(Y = 11) = 0.160$

b From GDC

$$P(X > 9) = 1 - P(X \leqslant 9) \\ = 1 - 0.128 \\ = 0.872$$

c From GDC

$$P(7 \leqslant X < 10) = P(X \leqslant 9) - P(X \leqslant 6) \\ = 0.128 - 0.006 \\ = 0.121$$

d $\text{Var}(X) = 20 \times 0.6 \times 0.4 = 4.8$

22 Let X be the number of correct answers. $X \sim B(25,0.2)$

a

$$P(X < 10) = P(X \leqslant 9) \\ = 0.983$$

b $E(X) = 25 \times 0.2 = 5$

c

$$P(X > 5) = 1 - P(X \leqslant 5) \\ = 1 - 0.617 \\ = 0.383$$

23 a Assume independent events and constant probability:

Each employee gets a cold with the same probability.

One employee having a cold has no impact on the probability of another employee catching a cold.

b For an infectious condition, independence is questionable; if one employee has a cold, one would imagine the probability of others getting a cold would be significantly increased.

c Let X be the number of employees suffering from a cold. $X \sim B(80,0.012)$.

From GDC: $P(X = 3) = 0.0560$

d

$$P(X > 3) = 1 - P(X \leqslant 3) \\ = 1 - 0.9841 \\ = 0.0159$$

24 Let X be the number of sixes rolled in 20 rolls. $X \sim \mathrm{B}\left(20, \frac{1}{6}\right)$.

$$\mathrm{E}(X) = \frac{20}{6} = 3.33$$

$$\begin{aligned} \mathrm{P}(X > 3.33) &= 1 - \mathrm{P}(X \leqslant 3) \\ &= 1 - 0.567 \\ &= 0.433 \end{aligned}$$

25 a Let X be the number of eggs broken in one box. $X \sim \mathrm{B}(6, 0.06)$.

$$\begin{aligned} \mathrm{P}(X \geqslant 1) &= 1 - \mathrm{P}(X = 0) \\ &= 1 - 0.690 \\ &= 0.310 \end{aligned}$$

b Let Y be the number of boxes out of ten which will be returned. $Y \sim \mathrm{B}(10, 0.310)$.

Tip: Always clearly define your variables and use different letters for different variables. This is very important for making your working clear in questions like this where one variable has parameters calculated from the distribution of another.

$$\begin{aligned} \mathrm{P}(Y \geqslant 1) &= 1 - \mathrm{P}(Y = 0) \\ &= 1 - 0.0245 \\ &= 0.976 \end{aligned}$$

c

$$\begin{aligned} \mathrm{P}(Y > 2) &= 1 - \mathrm{P}(Y \leqslant 2) \\ &= 1 - 0.3565\ldots \\ &= 0.643 \end{aligned}$$

26 Let X be the number of hits out of 10 shots. $X \sim \mathrm{B}(10, 0.7)$.

Assume constant probability and independence between shots.

a

$$\begin{aligned} \mathrm{P}(X \geqslant 7) &= 1 - \mathrm{P}(X \leqslant 6) \\ &= 1 - 0.350 \\ &= 0.650 \end{aligned}$$

b Let Y be the number of rounds in which the archer hits at least seven times. $Y \sim \mathrm{B}(5, 0.650)$.

$$\begin{aligned} \mathrm{P}(Y \geqslant 3) &= 1 - \mathrm{P}(Y \leqslant 2) \\ &= 1 - 0.235 \\ &= 0.765 \end{aligned}$$

27 Let X be the number of sixes rolled in ten rolls. $X \sim \mathrm{B}(10, p)$.

Empirical data suggests that $\mathrm{E}(X) = 2.7 = 10p$ so estimate $p = 0.27$

$$X \sim \mathrm{B}(10, 0.27)$$

From GDC:

$$\begin{aligned} \mathrm{P}(X > 4) &= 1 - \mathrm{P}(X \leq 4) \\ &= 1 - 0.896 \\ &= 0.104 \end{aligned}$$

28 $\mathrm{E}(X) = np = 36 \quad (1)$

$\mathrm{Var}(X) = np(1-p) = 3^2 = 9 \quad (2)$

$(2) \div (1): (1-p) = \frac{9}{36} = \frac{1}{4}$

Then $p = \frac{3}{4}$ and $n = 48$

$X \sim \mathrm{B}(48, 0.75)$

From GDC: $\mathrm{P}(X = 36) = 0.132$

29 a Let X be the number of defective components in a pack of ten. $X \sim \mathrm{B}(10, 0.003)$

$$\begin{aligned}\mathrm{P}(X \geq 1) &= 1 - \mathrm{P}(X = 0)\\ &= 1 - 0.9704\\ &= 0.0296\end{aligned}$$

b A batch is rejected if both of two selected packs contain at least one defective component.

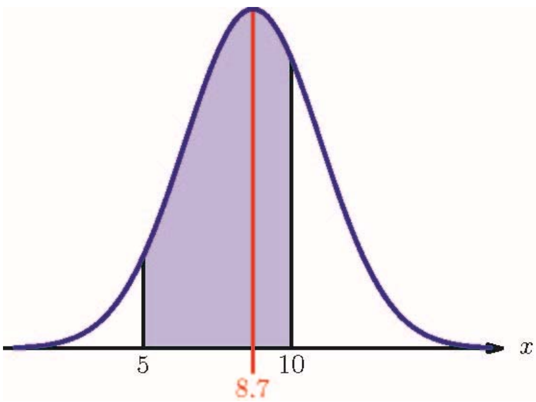
The probability of rejecting a batch is therefore $0.0296^2 = 0.000876$

c In a manufacturing situation, defective components may arise due to a faulty manufacturing machine or lower quality materials, either of which might be expected to affect multiple components.

Exercise 8C

13 No; the data suggest the distribution is not symmetrical

14 a



b Let X be tree height in metres. $X \sim \mathrm{N}(8.7, 2.3^2)$.

From GDC: $\mathrm{P}(5 < X < 10) = 0.660$

15 Let X be phone battery life in hours. $X \sim \mathrm{N}(56, 8^2)$.

a From GDC: $\mathrm{P}(50 < X < 60) = 0.465$

b From GDC: $\mathrm{P}(X > 72) = 0.0228$

16 Let X be tree height in metres. $X \sim \mathrm{N}(17.2, 6.3^2)$.

a From GDC: $\mathrm{P}(15 < X < 20) = 0.308$

b From GDC: $\mathrm{P}(X > 20) = 0.328$

17 Let X be the time Charlotte runs a 400 m race, in seconds. $X \sim N(62.3, 4.5^2)$.

a From GDC: $\mathrm{P}(X > 65) = 0.274$

b Let Y be the number of races out of 38 in which she ran over 65 seconds.

$$Y \sim \mathrm{B}(38, 0.274)$$
$$\mathrm{E}(Y) = 38 \times 0.274 = 10.4$$

c From GDC: $\mathrm{P}(X < 59.7) = 0.282$

18 Let X be the time (in minutes) taken to complete a puzzle. $X \sim \mathrm{N}(4.5, 1.5^2)$.

a From GDC: $\mathrm{P}(X > 7) = 0.0478$

b Require x such that $\mathrm{P}(X < x) = 0.9$

From GDC: $x = 6.42$ minutes

19 Let X be the time (in hours) of screen time. $X \sim \mathrm{N}(4.2, 1.3^2)$.

a From GDC: $\mathrm{P}(X > 6) = 0.0831$

b Require T such that $\mathrm{P}(X < T) = 0.95$

From GDC: $T = 6.34$ hours

c From GDC: $\mathrm{P}(X < 3) = 0.178$

Let Y be the number of teenagers in a group of 350 who get less than 3 hours of screen time per day.

$$Y \sim \mathrm{B}(350, 0.178)$$
$$\mathrm{E}(Y) = 350 \times 0.178 = 62.3$$

20 Let X be the distance (in metres) achieved in long jump for this group. $X \sim \mathrm{N}(5.2,0.6)$.

a From GDC: $\mathrm{P}(X > 6) = 0.151$

Let Y be the number of competitors out of 30 who jump further than 6 m.

$\mathrm{E}(Y) = 30 \times 0.151 = 4.53$

b Require x such that $\mathrm{P}(X < x) = 0.95$

From GDC: $x = 6.47$ m

21 Let T be the time (in seconds) taken to run 100 m for this group. $T \sim \mathrm{N}(14.3, 2.2)$.

Require t such that $\mathrm{P}\left(t < t_q\right) = 0.15$ where t_q is the greatest possible qualifying time.

Now, when $P\left(z < Z_q\right) = 0.15$, we have that $Z_q = 1.03624$.

I.e. $Z_q = \frac{14.3 - t_q}{\sqrt{2.2}} = 1.03624$

Hence, $t_q = 12.763 = 12.8$ s

22 $X \sim \mathrm{N}(12, 5)$

a Require x such that $\mathrm{P}(X < x) = 0.75$

From GDC: $x = 15.4$ s

b Require x_U such that $\mathrm{P}(X < x_U) = 0.75$ and x_L such that $\mathrm{P}(X < x_L) = 0.25$

From GDC: $x_U = 15.4$ s and $x_L = 8.63$ s

So $\mathrm{IQR} = x_U - x_L = 6.74$ s

23 $X \sim \mathrm{N}(3.6, 1.2)$

Require x such that $\mathrm{P}(X < x) = 0.8$

From GDC: $x = 4.61$

24 $X \sim \mathrm{N}(17, 3.2^2)$

From GDC: $\mathrm{P}(X > 15) = 0.734$

Require k such that $\mathrm{P}(X > k) = 0.734 - 0.62$

From GDC: $k = 20.9$

25 $Y \sim \mathrm{N}(13.2, 5.1^2)$

From GDC: $\mathrm{P}(X < 17.3) = 0.789$

Require c such that $\mathrm{P}(X < c) = 0.789 - 0.14$

From GDC: $k = 15.2$

26 For a Normal distribution to apply, require a symmetrical unimodal distribution with approximately 95% of the distribution within 2 standard deviations from the mean.

The test data show that the minimum mark (0%) is only 1.75 standard deviations below the mean, which would mean the distribution cannot have that Normal bell-shape.

(An alternative way of looking at it would be that if the distribution were $X \sim \mathrm{N}(35, 20^2)$ then there would be a predicted probability of $\mathrm{P}(X < 0) = 0.04 = 4\%$ that a member of the population scores less than 0%.)

27 **a** The distribution appears symmetrical, with the central 50% occupying an interval equivalent to between half and two thirds of each of the 25% tails of the distribution.

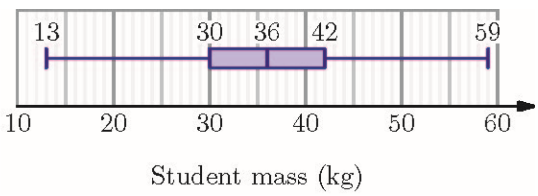
b Let X be the weights of children; $X \sim N(36, 8.5^2)$

Require x_U such that $\mathrm{P}(X < x_U) = 0.75$ and x_L such that $\mathrm{P}(X < x_L) = 0.25$

From GDC: $x_U = 41.7$ and $x_L = 30.3$

This gives $\mathrm{IQR} = 11.7$.

Given there are no outliers, we understand the minimum to be no less than $x_L - 1.5 \times \mathrm{IQR} = 12.8$ and the maximum to be no greater than $x_U + 1.5\mathrm{IQR} = 59.2$



Tip: If you are only considering the Normal model aspect of the question, you might consider that the correct upper and lower boundaries would be more like $\mu \pm 3\sigma$ but then – by the definition of outliers in a box plot – the minimum and maximum values you plot for the of the population would be outliers, which the question explicitly excludes.

28 Let X be the sprinter's reaction time, in seconds. $X \sim \mathrm{N}(0.2, 0.1^2)$.

a From GDC: $\mathrm{P}(X < 0) = 0.0228$.

If the reaction time is recorded as negative, that means the sprinter anticipated the start signal (left the blocks before the signal was given).

b From GDC: $P(X < 0.1) = 0.159$.

c Let Y be the number of races out of ten in which the sprinter gets a false start.

$$Y \sim B(10, 0.159)$$

$$\begin{aligned} P(Y > 1) &= 1 - P(Y \leqslant 1) \\ &= 1 - 0.513 \\ &= 0.488 \end{aligned}$$

d It is assumed that the sprinter's probability of a false start is constant and that false starts are independent, so that Y can be modelled by a binomial distribution. Since repeat offences might lead to disqualification or being barred from future events, it is likely that after one false start, the sprinter would adjust behaviour; this might be expected to equate to raising the mean reaction time above 0.2 s, for example, and would mean that the set of 10 races would not have a constant probability of a false start, and would not be independent of each other.

29 Let X be the mass of an egg. $X \sim N(60, 5^2)$.

$P(X < 53) = 0.0808$

$P(53 < X < 63) = 0.645$

$P(X > 63) = 0.274$

Let Y be the income from an egg, in cents

y	0	12	16
$P(Y = y)$	0.0808	0.645	0.274

$$\begin{aligned} E(Y) &= \sum y\, P(Y = y) \\ &= (0 \times 0.0808) + (12 \times 0.645) + (16 \times 0.274) \\ &= 12.1 \end{aligned}$$

So the expected value of each egg is 12.1 ¢

Then the expected value of 6000 eggs is 6000×12.1 ¢ $= \$728$ (to 3 s.f.)

Mixed Practice

1 **a** Require $\sum P(X = x) = 1$

$$\begin{aligned} 0.2 + 0.2 + 0.1 + k &= 1 \\ k &= 0.5 \end{aligned}$$

b $P(X \geqslant 3) = 0.1 + k = 0.6$

c

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= (1 \times 0.2) + (2 \times 0.2) + (3 \times 0.1) + 4k \\ &= 2.9 \end{aligned}$$

2 Let X be the number of sixes rolled in twelve rolls. $X \sim B\left(12, \frac{1}{6}\right)$.

a From GDC:

$P(X = 2) = 0.296$

b From GDC:

$$\begin{aligned}P(X > 2) &= 1 - P(X \leqslant 2)\\ &= 1 - 0.677\\ &= 0.323\end{aligned}$$

3 Let X be film length, in minutes. $X \sim N(96, 12^2)$.

a From GDC: $P(100 < X < 120) = 0.347$

b From GDC: $P(X > 105) = 0.227$

4 Let X be a score on the test. $X \sim N$.

Require x such that $P(X < x) = 0.985$

From GDC: $x = 215$

5 Let X be the number of defective plates in a random sample of twenty. $X \sim B(20, 0.021)$

From GDC

$$\begin{aligned}P(X \geqslant 1) &= 1 - P(X = 0)\\ &= 1 - 0.654\\ &= 0.346\end{aligned}$$

6 **a**

x	\$1	−\$0.50	−\$1.50
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

b Expected value of each play for Alessia is

$$\begin{aligned}E(X) &= \sum x\, P(X = x)\\ &= \left(1 \times \frac{1}{2}\right) + \left(-0.5 \times \frac{1}{3}\right) + \left(-1.5 \times \frac{1}{6}\right)\\ &= -\frac{1}{12}\end{aligned}$$

The game is not fair; Alessia expects to lose $\$\frac{1}{12}$ on each play.

7 Let X be the profit a player makes on the game.

x	-2	0	3	$N - 3$
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$

For the game to be fair, require $E(X) = 0$

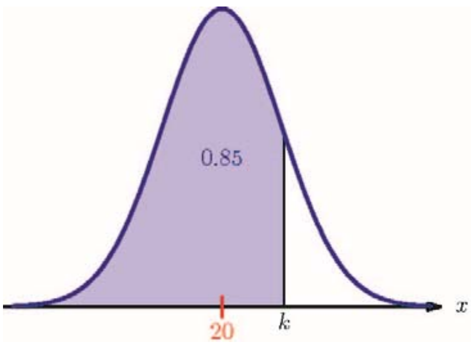
$$\begin{aligned}E(X) &= \sum x\, P(X = x)\\ &= \left(-2 \times \frac{1}{2}\right) + \left(0 \times \frac{1}{5}\right) + \left(3 \times \frac{1}{5}\right) + \frac{N-3}{10}\\ &= \frac{N-3}{10} - \frac{2}{5}\end{aligned}$$

Therefore $N - 3 = 4$ from which $N = 7$

8 $X \sim \mathrm{N}(20, 3^2)$

a From GDC: $\mathrm{P}(X \leqslant 24.5) = 0.933$

b **i**



ii From GDC: $k = 23.1$

9 $X \sim \mathrm{B}(30, p)$

a $\mathrm{E}(X) = 10 = 30P$ so $P = \frac{1}{3}$

b From GDC: $\mathrm{P}(X = 10) = 0.153$

c From GDC:

$$\begin{aligned}\mathrm{P}(X \geq 15) &= 1 - \mathrm{P}(X \leq 14)\\ &= 1 - 0.9565\\ &= 0.0435\end{aligned}$$

10 a Let X be the mass of an apple. $X \sim \mathrm{N}(110, 12.2^2)$.

From GDC: $\mathrm{P}(X < 100) = 0.206$

b Let Y be the number of apples in a bag of six with mass less than 100 g.

Assuming independence and constant probability from part **a**:

From GDC:

$$\begin{aligned}\mathrm{P}(Y > 15) &= 1 - \mathrm{P}(Y \leqslant 1)\\ &= 1 - 0.640\\ &= 0.360\end{aligned}$$

11 a Let X be the mass of an egg. $X \sim \mathrm{N}(63, 6.8^2)$.

$\mathrm{P}(X > 73) = 0.0707$

Let Y_6 be the number of very large eggs in a box of six eggs.

Assuming independence and a constant probability, $Y_6 \sim \mathrm{B}(6, 0.0707)$.

$$\begin{aligned}\mathrm{P}(Y \geqslant 1) &= 1 - \mathrm{P}(Y = 0)\\ &= 1 - 0.644\\ &= 0.356\end{aligned}$$

b Let Y_{12} be the number of very large eggs in a box of twelve. Assuming the same independence and constant probability of a very large egg: $Y_{12} \sim \mathrm{B}(12, 0.0707)$.

From GDC: $\mathrm{P}(Y_6 = 1) = 0.294$

From GDC: $\mathrm{P}(Y_{12} = 2) = 0.158$

It is more likely that a box of six contains a single very large egg.

12 Let X be jump length in metres. $X \sim \mathrm{N}(7.35, 0.8^2)$.

a From GDC: $\mathrm{P}(X > 7.65) = 0.354$

b Let Y be the number of jumps in three attempts which exceed 7.65 m. $Y \sim \mathrm{B}(3, 0.354)$.

From GDC:

$$\begin{aligned}\mathrm{P}(Y \geqslant 1) &= 1 - \mathrm{P}(Y = 0)\\ &= 1 - 0.2695\ldots\\ &= 0.730\end{aligned}$$

13 Let X be the 100 m time of a club runner in seconds. $X \sim \mathrm{N}(15.2, 1.6^2)$

From GDC: $\mathrm{P}(X < 13.8) = 0.191$

Let Y be the number of racers (out of the other seven) who have a time less than 13.8 s.

$Y \sim \mathrm{B}(7, 0.191)$

Heidi wins the race when $Y = 0$.

From GDC: $\mathrm{P}(Y = 0) = 0.227$

14 Let X be the amount of paracetamol in a tablet, in milligrams. $X \sim \mathrm{N}(500, 80^2)$.

From GDC: $\mathrm{P}(X < 380) = 0.0668$

Let Y be the number of participants in a trial of twenty five who get a dose less than 380 mg.

$Y \sim \mathrm{B}(25, 0.0668)$

From GDC:

$$\begin{aligned}\mathrm{P}(Y > 2) &= 1 - \mathrm{P}(Y \leq 0)\\ &= 1 - 0.768\\ &= 0.232\end{aligned}$$

15 a

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= (1 \times 0.15) + (2 \times 0.25) + (3 \times 0.08) + (4 \times 0.17) + (5 \times 0.15) + (6 \times 0.20)\\ &= 3.52\end{aligned}$$

b Rolls are independent, so

$\mathrm{P}(5,6) = 0.15 \times 0.2 = 0.03$

$\mathrm{P}(6,5) = 0.2 \times 0.15 = 0.03$

So the probability of rolling a five and a six (in either order) is 0.06.

c Let X be the number of times in ten rolls that a one is rolled. $X \sim \mathrm{B}(10, 0.15)$.

From GDC:

$$\begin{aligned}\mathrm{P}(X \geqslant 2) &= 1 - \mathrm{P}(Y \leqslant 1)\\ &= 1 - 0.544\\ &= 0.456\end{aligned}$$

16 Require $\sum \mathrm{P}(X = x) = 1$

$$\frac{1}{3} + \frac{1}{4} + a + b = 1$$

$$b = \frac{7}{12} - a$$

Let X be the gain in a single game.

x	-2	-1	0	1
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{7}{12} - a$

$$\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$$

$$= \left(-2 \times \frac{1}{3}\right) + \left(-1 \times \frac{1}{4}\right) + \frac{7}{12} - a$$

$$= -\frac{4}{12} - a$$

Expected gain per game is $-\frac{1}{2}$

So $-\frac{1}{3} - a = -\frac{1}{2}$

$$a = \frac{1}{6}$$

17 Let X be the height of a pupil at the school in cm. $X \sim \mathrm{N}(148, 8^2)$.

a Require x_U such that $\mathrm{P}(X < x_U) = 0.75$ and x_L such that $\mathrm{P}(X < x_L) = 0.25$

From GDC: $x_U = 153.4$ cm and $x_L = 142.6$ cm

So IQR $= 153.4 - 142.6 = 10.8$ cm

b Outliers are more than 1.5 IQR outside the quartile marks.

From GDC:

$\mathrm{P}(X < x_L - 1.5 \times 10.8) = 0.00349$

By symmetry, there is the same probability of a value above $x_U + 1.5$ IQR

Total probability of outlier: 0.00698 or 0.698% of the pupils.

18 For a Normal distribution to apply, require a symmetrical unimodal distribution with approximately 95% of the distribution within 2 standard deviations from the mean.

The context shows that the minimum time (0 minutes, which is clearly unrealistic anyway) is only 2 standard deviations below the mean, which would mean the distribution cannot have that Normal bell-shape.

(An alternative way of looking at it would be that if the distribution were $X \sim \mathrm{N}(10, 5^2)$ then there would be a predicted probability of $\mathrm{P}(X < 0) = 0.0228$ that a member of the population finishes the test before starting it!)

19 Let X be the volume of water in a bottle (in millilitres). $X \sim \text{N}(330, 5^2)$.

Require x such that $\text{P}(X < x) = 0.025$.

From GDC: $x = 320.2$

The bottle should be labelled 320 ml.

20

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second Die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

a $\text{P}(X = 5) = \frac{4}{36} = \frac{1}{9}$

b Let Y be the number of prizes won in eight attempts. $Y \sim \text{B}\left(8, \frac{1}{9}\right)$

$\text{P}(\text{Y} = 3) = 0.0426$

21 a Let X be the time to complete a task. $X \sim \text{N}(20, 1.25^2)$.

From GDC: $\text{P}(\text{X} < 21.8) = 0.925$

b Require x such that $\text{P}(X < x) = 0.925 - 0.3 = 0.625$

From GDC: $x = 20.4$

22 a $W \sim \text{N}(1000, 4^2)$

From GDC: $\text{P}(990 < W < 1004) = 0.835$

b Require k such that $\text{P}(W < k) = 0.95$

From GDC: $k = 1006.58$

Tip: Given the context of the question, you clearly don't want to give an answer to 3s.f. here! Quick judgement suggests that the values should be 3 s.f. detail for their difference from 1000 g

c Interval symmetrical about the distribution mean, so we require 5% above the interval and 5% below the interval. Then $a = k - 1000 = 6.58$

23 $X \sim \text{B}\left(5, \frac{1}{5}\right)$

a i $\text{E}(X) = 5 \times \frac{1}{5} = 1$

ii From GDC:

$$\begin{aligned}\text{P}(X \geq 3) &= 1 - \text{P}(X \leq 2)\\ &= 1 - 0.9421\\ &= 0.0579\end{aligned}$$

b **i** Require $\sum \mathrm{P}(Y = y) = 1$

$$0.67 + 0.05 + (a + 2b) + (a - b) + (2a + b) + 0.04 = 1$$
$$4a + 2b = 0.24$$

ii

$$\begin{aligned}\mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y)\\ &= (0 \times 0.67) + (1 \times 0.05) + 2(a + 2b) + 3(a - b) + 4(2a + b)\\ &\quad + (5 \times 0.04)\\ &= 0.25 + 13a + 5b\\ &= 1\end{aligned}$$

Simultaneous equations:

$4a + 2b = 0.24$ (1)

$13a + 5b = 0.75$ (2)

$2(2) - 5(1)$: $6a = 0.3$

So $a = 0.05$ and $b = 0.02$

c $\mathrm{P}(Y \geqslant 3) = (a - b) + (2a + b) + 0.04 = 0.19 > 0.0579$

Bill is more likely to pass the test.

24 Let X be the distance thrown in metres. $X \sim \mathrm{N}(40, 5^2)$.

By symmetry, $\mathrm{P}(X < 40) = 0.5$

From GDC: $\mathrm{P}(40 < X < 46) = 0.385$

Then $\mathrm{P}(X > 46) = 0.115$

Let Y be the number of points achieved in a single throw.

y	0	1	4
$\mathrm{P}(Y = y)$	0.5	0.385	0.115

a **i**

$$\begin{aligned}\mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y)\\ &= 0 + (1 \times 0.385) + (4 \times 0.115)\\ &= 0.845 \text{ points}\end{aligned}$$

ii If she throws twice, the expected value is $2 \times 0.845 = 1.69$ points.

b It is assumed that the probabilities stay the same between throws, so that each attempt follows the same distribution and that attempts are independent. However, Josie may be expected to improve as she warms up, or perhaps get worse in the later attempts as she tires. Success may lift her ability to achieve, and failure to score may make her tense up and perform worse. At any rate, we may anticipate that subsequent throws do not follow the same normal distribution, and depend on previous throws.

25 Let X be the number of sixes rolled in n throws. $X \sim \mathrm{B}\left(n, \frac{1}{6}\right)$.

Then $\mathrm{P}(X = 0) = (1 - p)^n$

$$\left(\frac{5}{6}\right)^n = 0.194$$

From GDC: $n = 9$

26 Let X be the number of heads when a fair coin is tossed n times. $X \sim (n, 0.5)$.

Require n such that $\mathrm{P}(X = 0) < 0.001$

$\mathrm{P}(X = 0) = 0.5^n$

So $0.5^n < 0.001$

Then $2^n > 1000$

The least such n is 10, since $2^{10} = 1024$

27 Let X_{10} be the number of tails from ten tosses of the biased coin. $X_{10} \sim \mathrm{B}(10, 0.57)$.

a From GDC:

$$\begin{aligned}\mathrm{P}(X_{10} \geq 4) &= 1 - \mathrm{P}(X_{10} \leq 3)\\ &= 1 - 0.0806\\ &= 0.919\end{aligned}$$

b For the fourth tail on the tenth toss, require three tails in the first nine tosses and then a tail.

Let X_9 be the number of tails from nine tosses. $X_9 \sim \mathrm{B}(9, 0.57)$

From GDC: $\mathrm{P}(X_9 = 3) = 0.0983$

So the required probability is $0.0983 \times 0.57 = 0.0561$

28

Tip: This may not look immediately like anything you have studied so far. Try to reframe the question in terms of something you do know. Many probability problems can be considered from different perspectives, to make them solvable by known methods.

For each day of the year, the number of people with that birthday will follow a $X \sim \mathrm{B}\left(100, \frac{1}{365}\right)$ distribution, assuming independence between the people.

The probability for each day that no person has a birthday that day is $\mathrm{P}(X = 0) = 0.760$

Although in this scenario, days are not truly independent (in that, each time you find a day with no birthdays, the probability of birthdays on subsequent assessed days changes slightly) the number of possible days is large enough to disregard this issue.

The total number of days expected to have no birthdays is therefore $365 \times 0.760 = 277$

29 a Assuming the responses from invited guests to be independent:

Let X be the number of invited guests who attend when he invites 5: $X \sim \mathrm{B}(5, 0.5)$

$\boldsymbol{x}$	0	1	2	3	4	5
$\mathbf{P}(\boldsymbol{X} = \boldsymbol{x})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Let Y be the profit made, in pounds:

$\boldsymbol{x}$	0	1	2	3	4	5
$\boldsymbol{y}$	0	50	100	150	200	100
$\mathbf{P}(\boldsymbol{Y} = \boldsymbol{y})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

$$\mathrm{E}(Y) = \sum y\,\mathrm{P}(Y = y)$$
$$= \left(0 \times \frac{1}{32}\right) + \left(50 \times \frac{5}{32}\right) + \left(100 \times \frac{10}{32}\right) + \left(150 \times \frac{10}{32}\right) + \left(200 \times \frac{5}{32}\right) + \left(100 \times \frac{1}{32}\right)$$
$$= 120.3$$

Expected profit from 5 invitations is £120

b For four invitations sent out:

x	0	1	2	3	4	5
y	0	50	100	150	200	100
$\mathbf{P}(Y = y)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	0

$\mathrm{E}(Y) = 100$

For six invitations sent out:

x	0	1	2	3	4	5	6
y	0	50	100	150	200	100	0
$\mathbf{P}(Y = y)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

$\mathrm{E}(Y) = 131.25$

For seven invitations sent out:

x	0	1	2	3	4	5	6	7
y	0	50	100	150	200	100	0	-100
$\mathbf{P}(Y = y)$	$\frac{1}{128}$	$\frac{7}{128}$	$\frac{21}{128}$	$\frac{35}{128}$	$\frac{35}{128}$	$\frac{21}{128}$	$\frac{7}{128}$	$\frac{1}{128}$

$\mathrm{E}(Y) = 130.5$

On the basis solely of maximum profit for a single evening, the best number of invitations to send is 6. Inviting 8 or more people leads to an expected number of guests of 4 or more, so expected profit will decrease, since each additional attendee has a negative profit effect.

In context, getting a reputation for turning invited guests away may not – in the long term – be good for business, and might start to reduce the proportion of people who attend.

9 Core: Differentiation

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 9A

27 a $\frac{\mathrm{d}V}{\mathrm{d}t} = V + 1$

b When $V = 4$, the equation in part **a** gives $\frac{\mathrm{d}V}{\mathrm{d}t} = 5$.

28

x	$\frac{\sin x^2}{\left(\frac{\pi x}{180}\right)^2}$
10	32.33
5	55.49
1	57.29
0.1	57.30

The limit appears to be 57.3.

29

x	$\frac{\ln x}{x-1}$
0.5	1.38629
0.9	1.05361
0.99	1.00503
0.999	1.00050

The limit appears to equal 1

30 a Gradient $= \frac{\text{change in } x}{\text{change in } y} = \frac{x^2-1}{x-1} = x + 1$ for $x \neq 1$

b As the value of x tends towards 1, the gradient of the chord tends towards 2. The significance of this is that the gradient of the tangent at $x = 1$ must be 2.

31

x	$\frac{\ln x}{x-1}$
10	2.6491
100	2.9563
1000	2.9955
10000	2.9996

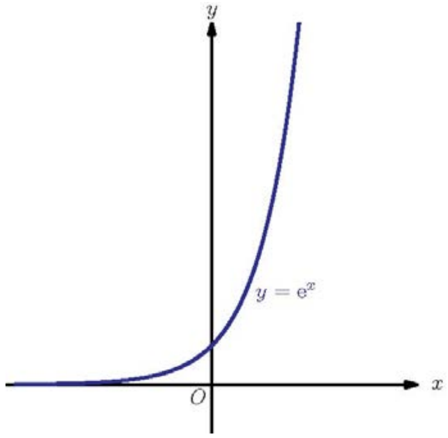
The limit appears to be 3 as x tends to ∞.

32 $\frac{dx}{dy} = 3x$ so $\frac{dy}{dx} = \frac{1}{3x}$

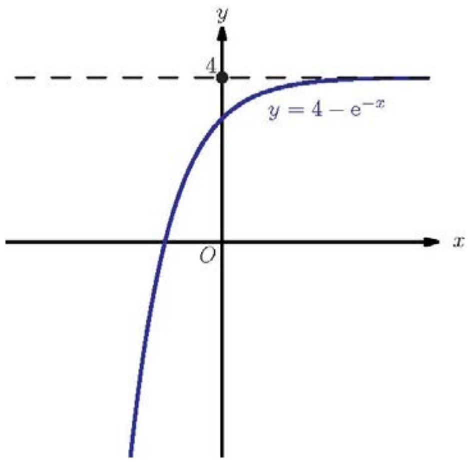
When $x = 2$, the gradient is $\frac{1}{6}$.

Exercise 9B

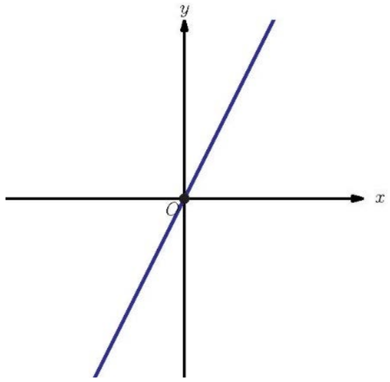
19 For example, $y = e^x$, which has derivative $\frac{dy}{dx} = e^x$



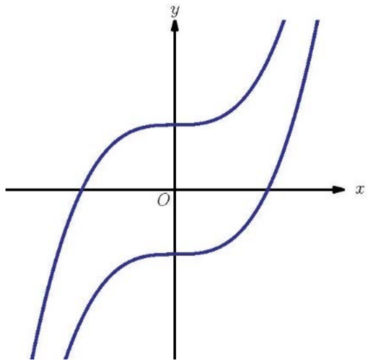
20 For example, $y = 4 - e^{-x}$, which increases towards $y = 4$ as x tends to ∞, and has derivative $\frac{dy}{dx} = e^{-x}$, which decreases towards $\frac{dy}{dx} = 0$ as x tends to ∞.



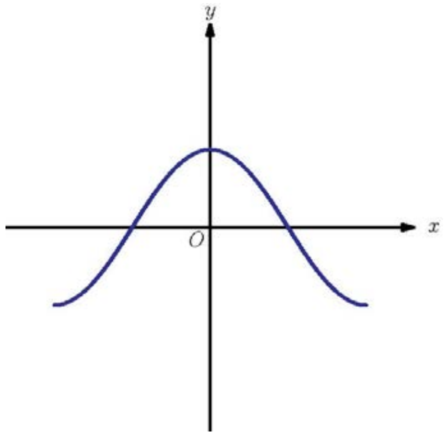
21 a The curve decreases (negative gradient) to a minimum at $x = 0$ and then rises (positive gradient) so the derivative graph should pass through the origin, from negative to positive.



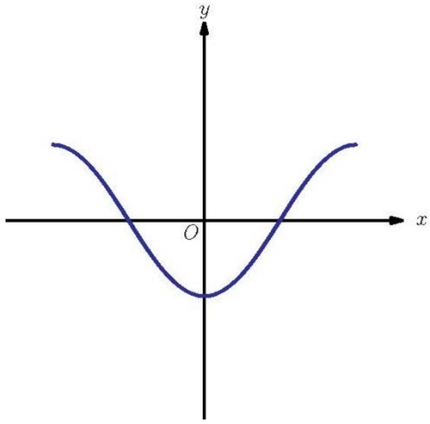
b The gradient curve is non-negative throughout: it decreases to zero at $x = 0$ and then rises again, so there is a stationary point of inflexion at $x = 0$ in the graph of $f(x)$, with the rest of the graph increasing; the gradient is steeper further from $x = 0$.



22 a The curve decreases (negative gradient) to a minimum (zero gradient), rises (positive gradient) to a maximum (zero gradient) and decreases again (negative gradient).

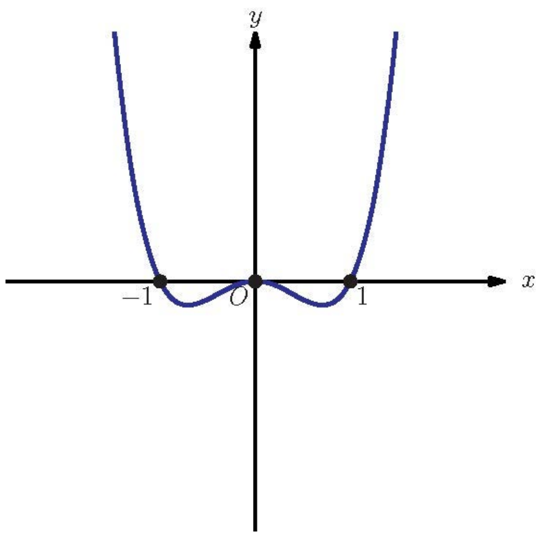


b The gradient curve begins at zero, is increasingly negative then returns to zero at the origin, continues to increase and then falls back to zero, so there is a stationary point at $x = 0$ on the graph of $\mathrm{f}(x)$ and also at the two ends of the curve:



c The gradient describes the shape of a curve but not its position vertically; any vertical translation of the graph in part **b** would have the same gradient curve.

23 a $y = x^4 - x^2$



b $f(x) > 0$ for $x < -1$ and for $x > 1$

c The graph is decreasing for $x < -\frac{\sqrt{2}}{2} = -0.707$ and for $0 < x < \frac{\sqrt{2}}{2} = 0.707$

This can either be found using technology or by differentiating the curve and finding stationary points, using the methods from Section C.

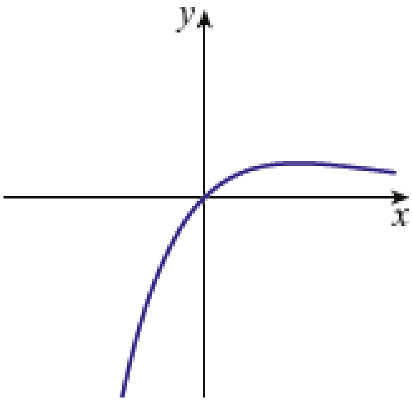
$f(x) = x^4 - x^2$ so $f'(x) = 4x^3 - 2x$

When $f'(x) = 0$, $2x(2x^2 - 1) = 0$ so turning points are at $x = 0, \pm\frac{\sqrt{2}}{2}$.

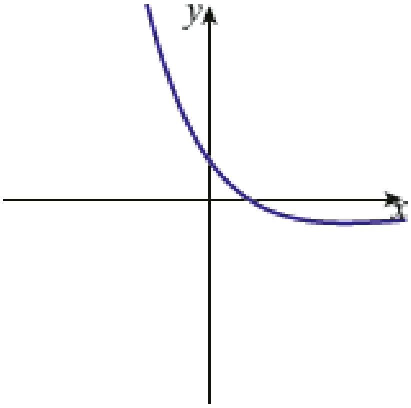
Inspecting the curve in part **a** shows clearly which regions are decreasing.

24 A increases to a stationary point at positive x then has a shallow negative gradient; its derivative graph is shown in E.

A

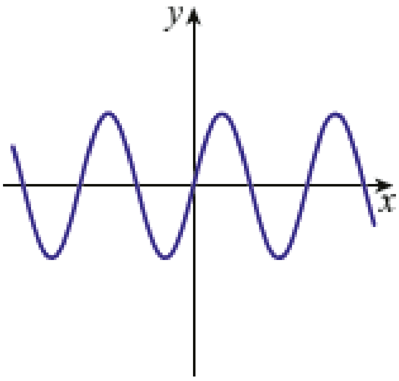


E

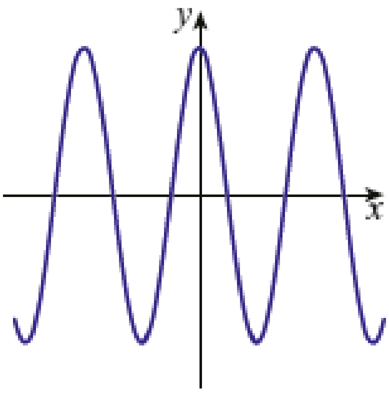


B oscillates regularly with a positive gradient at $x = 0$; its gradient is shown in D.

B

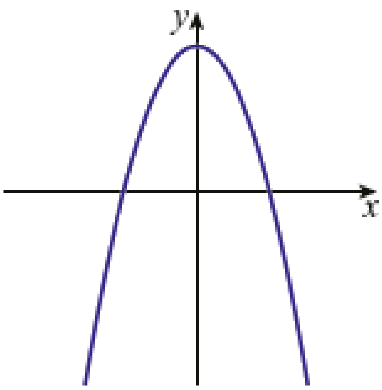


D

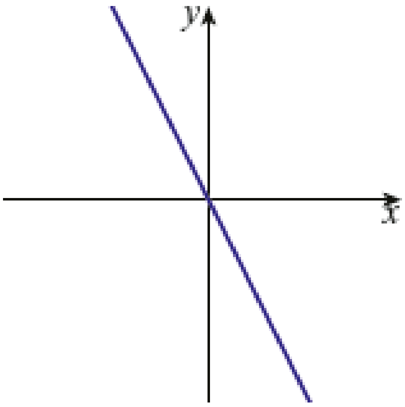


F increases to a stationary point at $x = 0$ and then decreases; its gradient is shown in C

F

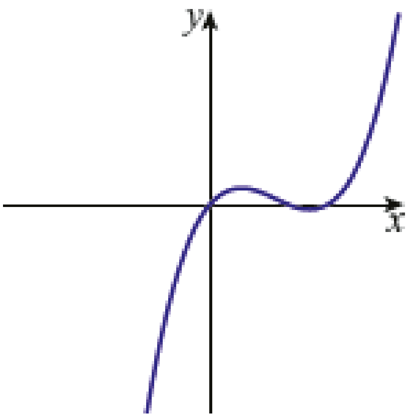


C

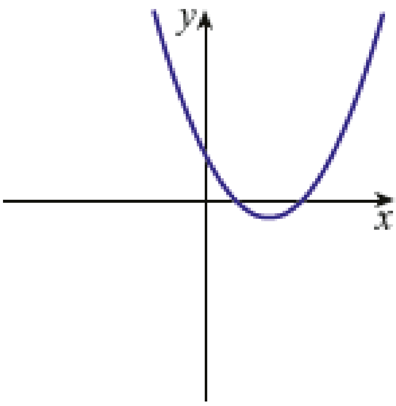


25 A increases through the origin to a maximum point for some $x > 0$, falls to a minimum and then increases with increasing gradient. Its derivative graph is given in C.

A

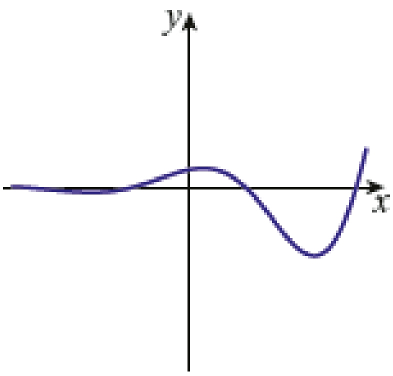


C

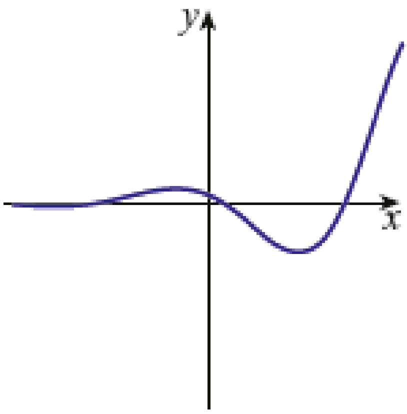


D has a minimum for some $x < 0$, then rises to a maximum at some $x > 0$ and falls to a further minimum value before rising steeply. Its derivative graph is given in E.

D

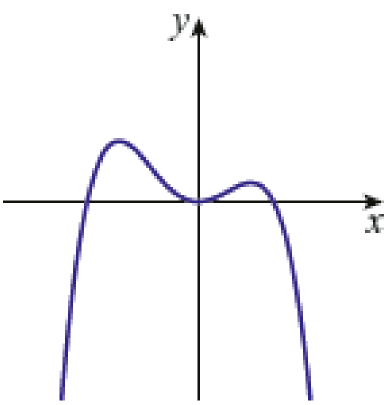


E

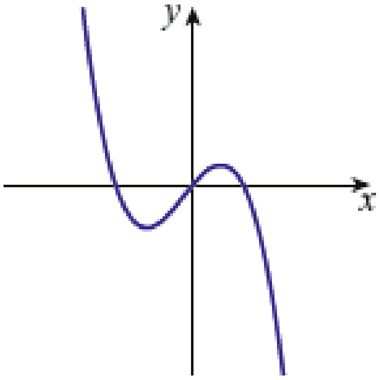


F increases to a maximum for some $x < 0$, falls to a minimum at $x = 0$ and rises to a further maximum for some $x > 0$ before falling increasingly steeply. Its derivative graph is given in B.

F



B



Exercise 9C

31 $d = 6t - 4t^{-1}$

$$\frac{\mathrm{d}d}{\mathrm{d}t} = 6 + 4t^{-2} = 6 + \frac{4}{t^2}$$

32 $q = m + 2m^{-1}$

$$\frac{\mathrm{d}q}{\mathrm{d}m} = 1 - 2m^{-2} = 1 - \frac{2}{m^2}$$

33 $E = \frac{3}{2}kT$

$$\frac{\mathrm{d}E}{\mathrm{d}T} = \frac{3}{2}k$$

34 Let $\mathrm{f}(x) = x^2 - x$

Then $\mathrm{f}'(x) = 2x - 1$

$\mathrm{f}(x)$ is increasing when $\mathrm{f}'(x) > 0$

$2x - 1 > 0$ so $x > \frac{1}{2}$

35 Let $\mathrm{f}(x) = x^2 + bx + c$

Then $\mathrm{f}'(x) = 2x + b$

$\mathrm{f}(x)$ is increasing when $\mathrm{f}'(x) > 0$

$2x + b > 0$ so $x > -\frac{b}{2}$

36 $x + y = 8$ so $y = 8 - x$

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$

37 $x^3 + y = x$ so $y = x - x^3$

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 3x^2$

38 $V = kr^{-1}$

a Force $= \frac{\mathrm{d}V}{\mathrm{d}r} = -kr^{-2}$

b Rearranging the original formula, $r^{-1} = \frac{V}{k}$. Substituting into the answer from part **a:**

Force $= -k\left(\frac{V}{k}\right)^2 = -\frac{V^2}{k}$

c Let the distance to Alpha be d; then the distance to Omega is $2d$.

Using the formula from part **a**,

$$\frac{\text{Force on Alpha}}{\text{Force on Omega}} = \frac{-kd^{-2}}{-k(2d)^{-2}} = 4$$

39 a $A = 2 + qL + qL^2$ so $\frac{\mathrm{d}A}{\mathrm{d}L} = q + 2qL$

b In the context, one would expect that the reading age rating of a book would increase with longer sentences. Since the function is only defined for positive values of L, the function is increasing as long as $q > 0$.

40 $\mathrm{f}(x) = 4x^3 + 7x - 2$

So $\mathrm{f}'(x) = 12x^2 + 7 > 0$ for all x, since square numbers can never be negative.

Since $\mathrm{f}'(x) > 0$ for all x, the function $f(x)$ is increasing for all x.

Exercise 9D

28 a $y = x^4 - x$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 1$

b When $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 4(0)^3 - 1 = -1$

c Then the gradient of the normal at $x = 0$ is 1.

$y(0) = 0$ so the normal passes through the origin and has gradient 1 so has equation $y = x$.

29 a $f(x) = x^3 + x^{-1}$ so $f'(x) = 3x^2 - x^{-2}$

b $f'(1) = 3 - 1 = 2$ so the gradient of the tangent at $x = 1$ is 2.

c $f(1) = 1 + 1 = 2$

The tangent passes through $(1,2)$ and has gradient 2 so has equation $y - 2 = 2(x - 1)$ which rearranges to $y = 2x$, and so the tangent passes through the origin.

30 Let $f(x) = x\sqrt{x+1}$.

From the GDC: $f'(3) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{x+1}\right)\Big|_{x=3} = \frac{11}{4}$

$f(3) = 6$ so the tangent line passes through $(3,6)$ with gradient $\frac{11}{4}$.

The tangent equation is $y - 6 = \frac{11}{4}(x - 3)$ which rearranges to $y = \frac{11}{4}x - \frac{9}{4}$.

31 Let $f(x) = \frac{1}{x+4}$.

From the GDC: $f'(-2) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x+4}\right)\Big|_{x=-2} = -\frac{1}{4}$

$f(-2) = \frac{1}{2}$ so the tangent line passes through $\left(-2, \frac{1}{2}\right)$ with gradient $-\frac{1}{4}$.

Then the normal line passes through $\left(-2, \frac{1}{2}\right)$ with gradient 4.

The normal equation is $y - \frac{1}{2} = 4(x + 2)$ which rearranges to $y = 4x - \frac{17}{2}$.

32 Let $f(x) = x^2\mathrm{e}^x$.

From the GDC: $f'(0) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x)\Big|_{x=0} = 0$

$f(0) = 0$ so the tangent line passes through the origin with gradient 0.

The normal is therefore the y-axis, with equation $x = 0$.

33 $y = x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

The gradient at $x = 1$ is 2, and the curve passes through $(1,1)$.

The normal has gradient $-\frac{1}{2}$ and passes through $(1,1)$ so has equation $y - 1 = -\frac{1}{2}(x - 1)$ which rearranges to $y = \frac{3}{2} - \frac{1}{2}x$.

Substituting back into the original curve equation:

$$x^2 = \frac{3}{2} - \frac{1}{2}x$$
$$2x^2 + x - 3 = 0$$

One intersection is already known: $(1,1)$, so $(x - 1)$ is a factor of the quadratic.

$$(x - 1)(2x + 3) = 0$$

The other intersection is at $\left(-\frac{3}{2}, \frac{9}{4}\right)$.

34 $y = x^{-1}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2}$.

The gradient at $x = 2$ is $-\frac{1}{4}$, and the curve passes through $\left(2, \frac{1}{2}\right)$.

The normal has gradient 4 and passes through $\left(2, \frac{1}{2}\right)$ so has equation $y - \frac{1}{2} = 4(x - 2)$ which rearranges to $y = 4x - \frac{15}{2}$.

Substituting back into the original curve equation:

$$\frac{1}{x} = 4x - \frac{15}{2}$$
$$8x^2 - 15 - 2 = 0$$

One intersection is already known: $\left(2, \frac{1}{2}\right)$, so $(x - 2)$ is a factor of the quadratic.

$$(x - 2)(8x + 1) = 0$$

The other intersection is at $\left(-\frac{1}{8}, -8\right)$.

35 $y = x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

Require the tangent to have gradient -2 so $2x = -2$ from which $x = -1$.

At $x = -1$, $y = 1$ so the tangent passes through $(-1, 1)$ and has gradient -2.

The equation is $y - 1 = -2(x + 1)$ which rearranges to $y = -2x - 1$ or $y + 2x = -1$.

36 $y = x^2 + 2x$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2$.

Require the normal to have gradient $\frac{1}{4}$ so the tangent has gradient -4 so $2x + 2 = -4$ from which $x = -3$.

At $x = -3$, $y = 3$ so the normal passes through $(-3, 3)$ and has gradient $\frac{1}{4}$.

The equation is $y - 3 = \frac{1}{4}(x + 3)$ which rearranges to $y = \frac{1}{4}x + \frac{15}{4}$.

37 $y = x^3$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

Require the tangent to have gradient 3 so $3x^2 = 3$ from which $x = \pm 1$.

At $x = -1$, $y = -1$ so the normal passes through $(-1, -1)$ and has gradient $-\frac{1}{3}$.

The equation is $y+1=-\frac{1}{3}(x+1)$ which rearranges to $y=-\frac{1}{3}x-\frac{4}{3}$.

At $x=1$, $y=1$ so the normal passes through $(1,1)$ and has gradient $-\frac{1}{3}$.

The equation is $y-1=-\frac{1}{3}(x-1)$ which rearranges to $y=-\frac{1}{3}x+\frac{4}{3}$.

The two normals have equations $y=-\frac{1}{3}x\pm\frac{4}{3}$.

38 $y=x^3-x$ so $\frac{dy}{dx}=3x^2-1$.

Require the tangent to have gradient 11 so $3x^2-1=11$ from which $x=\pm2$.

At $x=-2$, $y=-6$ so one such tangent passes through $(-2,-6)$.

At $x=2$, $y=6$ so the other such tangent passes through $(2,6)$.

39 $y=x^2+4x+1$ so $\frac{dy}{dx}=2x+4$.

Require the tangent to have gradient equal to the y-coordinate.

Substituting: $2x+4=x^2+4x+1$

$$x^2+2x-3=0$$
$$(x+3)(x-1)=0$$

So $x=-3$ or 1

When $x=-3$, $y=-2$ and when $x=1$, $y=6$ so the points are $(-3,-2)$ and $(1,6)$.

40 $y=x^{-1}$ so $\frac{dy}{dx}=-x^{-2}$.

Let point P on the curve have x-coordinate p so P is $\left(p,\frac{1}{p}\right)$.

Then the tangent at P has gradient $-\frac{1}{p^2}$, so the tangent equation is $y-\frac{1}{p}=-\frac{1}{p^2}(x-p)$.

Given this passes through $(4,0)$, substitute $x=4, y=0$ into the tangent equation to find p:

$$-\frac{1}{p}=-\frac{1}{p^2}(4-p)$$
$$\frac{4}{p^2}=\frac{2}{p}$$
$$p=2$$

So point P has coordinates $\left(2,\frac{1}{2}\right)$.

Tip: When faced with a question of this sort, where you have a condition you cannot immediately apply (in this case that the tangent line passes through $(4,0)$), it is almost always best to assign an unknown value to the independent variable and calculate everything in terms of that unknown. You will end with an equation into which you can substitute your condition and then solve for the unknown.

41 $y=4x^{-1}$ so $\frac{dy}{dx}=-4x^{-2}$.

Let point P on the curve have x-coordinate p so P is $\left(p,\frac{4}{p}\right)$.

Then the tangent at P has gradient $-\frac{4}{p^2}$, so the tangent equation is $y-\frac{4}{p}=-\frac{4}{p^2}(x-p)$.

Given this passes through $(1,3)$, substitute $x = 1, y = 3$ into the tangent equation to find p:

$$3 - \frac{4}{p} = -\frac{4}{p^2}(1 - p)$$
$$\frac{4}{p^2} - \frac{8}{p} + 3 = 0$$
$$3p^2 - 8p + 4 = 0$$
$$(3p - 2)(p - 2) = 0$$

So $p = \frac{2}{3}$ or 2.

42 $y = x^2$ so $\frac{dy}{dx} = 2x$.

Let point P on the curve have x-coordinate p so P is (p, p^2).

Then the tangent at P has gradient $2p$, so the tangent equation is $y - p^2 = 2p(x - p)$.

Given this passes through $(2,3)$, substitute $x = 2, y = 3$ into the tangent equation to find p:

$$3 - p^2 = 2p(2 - p)$$
$$p^2 - 4p + 3 = 0$$
$$(p - 1)(p - 3) = 0$$

$p = 1$ or 3 so the coordinates of P are $(1,1)$ or $(3,9)$.

43 $y = x^{-1}$ so $\frac{dy}{dx} = -x^{-2}$.

Let A be the point with coordinates $\left(a, \frac{1}{a}\right)$.

Then the tangent at A has gradient $-\frac{1}{a^2}$, so the tangent equation is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$.

This tangent intersects the x-axis when $y = 0$: $-\frac{1}{a} = \frac{1}{a} - \frac{1}{a^2}x$ so $x = 2a$.

Then Q has coordinates $(2a, 0)$.

The tangent intersects the y-axis when $x = 0$: $y - \frac{1}{a} = \frac{1}{a}$ so $y = \frac{2}{a}$.

Then P has coordinates $\left(0, \frac{2}{a}\right)$.

The area of right-angled triangle OPQ is $\frac{1}{2} \times OP \times OQ = \frac{1}{2} \times \frac{2}{a} \times 2a = 2$, which is independent of a as required.

44 $y = ax^{-1}$ so $\frac{dy}{dx} = -ax^{-2}$.

Point P has coordinates $\left(2, \frac{a}{2}\right)$, so the gradient at P is $-\frac{a}{4}$.

The normal at P therefore has gradient $\frac{4}{a}$ which must equal $\frac{1}{5}$.

The gradient of line $y = \frac{1}{5}x - 2$.

$\frac{4}{a} = \frac{1}{5}$ so $a = 20$.

45 $y = x^3 + x + 1$ so $\frac{dy}{dx} = 3x^2 + 1$.

Let point P have x-coordinate k so P is given by $(k, k^3 + k + 1)$.

Then the tangent at P has gradient $3k^2 + 1$ so the equation of the tangent is

$$y - (k^3 + k + 1) = (3k^2 + 1)(x - k)$$

To find where this tangent intersects the curve, substitute back into the original equation:

$$(3k^2 + 1)(x - k) + k^3 + k + 1 = x^3 + x + 1$$

By construction, this equation must have solution $x = k$ as a repeated root (the tangent touches the curve at $x = k$), so $(x - k)^2$ is a factor of the equation.

$$-2k^3 + 3k^2x + x + 1 = x^3 + x + 1$$
$$x^3 - 3k^2x + 2k^3 = 0$$

By construction, this equation must have solution $x = k$ as a repeated root (the tangent touches the curve at $x = k$), so $(x - k)^2$ is a factor of the equation.

$$(x - k)(x^2 + kx - 2k^2) = 0$$
$$(x - k)(x - k)(x + 2k) = 0$$

So the tangent touches the curve at $x = k$ and intersects it at $x = -2k$.

Mixed Practice

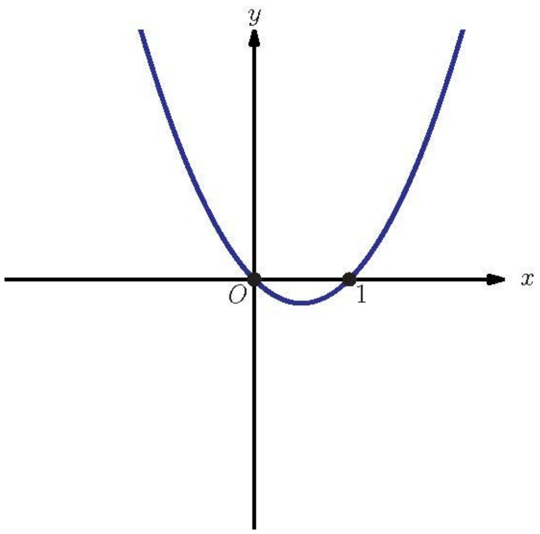
1 **a** $y = 4x^2 - x$

$$\frac{dy}{dx} = 8x - 1$$

b Require $8x - 1 = 15$ so $x = 2$.

When $x = 2, y = 14$ so the point is $(2, 14)$.

2 **a** $f(x) = x^2 - x$

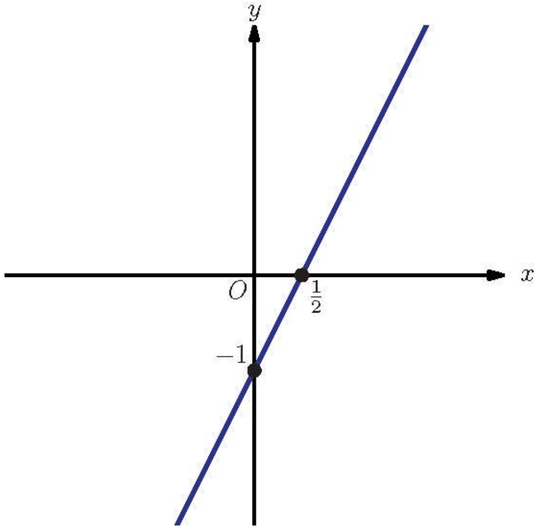


b $f(x) = x^2 - x$ so $f'(x) = 2x - 1$

The curve is increasing when $f'(x) > 0$ so $2x - 1 > 0$

The curve is increasing for $x > \frac{1}{2}$.

c $f'(x) = 2x - 1$



3 **a** $f(x) = 2x^3 + 5x^2 + 4x + 3$

So $f'(x) = 6x^2 + 10x + 4$

b Then $f'(-1) = 6 - 10 + 4 = 0$

c The tangent is horizontal at $(-1,2)$ so has equation $y = 2$.

4

x	$\frac{\ln(1+x) - x}{x^2}$
1	−0.38605
0.5	−0.37814
0.1	−0.46898
0.01	−0.49669
0.0001	−0.49997

The limit appears to equal -0.5.

5 $y = x^3 - 4$ so $\frac{dy}{dx} = 3x^2$.

When $y = 23$, $x^3 = 27$ so $x = 3$.

At $(3, 23)$, the gradient is $3(3^2) = 27$ so the tangent has equation $y - 23 = 27(x - 3)$.

This rearranges to $y = 27x - 58$.

6 **a** $V = 50 + 12t + 5t^2$ so $\frac{dV}{dt} = 12 + 10t$.

b $V(6) = 302, \frac{dV}{dt}(6) = 72$

After 6 minutes, there is 302 m^3 of water in the tank, and the volume of water in the tank is increasing at a rate of 72 m^3 per minute.

c $\frac{dV}{dt}(10) = 112$, so the volume is increasing faster after 10 minutes than after 6 minutes.

7 **a** $A = 2t - t^2$ so $\frac{dA}{dt} = 2 - 2t$

b **i** $A(5) = 0.75$

ii $\frac{dA}{dt}(0.5) = 1$

c A is increasing when $\frac{dA}{dt} > 0$

$2 - 2t > 0$ for $t < 1$ so the accuracy is increasing for $0 < t < 1$.

Tip: Be careful to check the context and question wording for the domain of the function; here A is only defined for $0 < t < 2$ so it would be incorrect to say that accuracy is increasing for $t < 1$.

8 $f(x) = \frac{1}{2}x^2 - 8x^{-1}, x \neq 0$

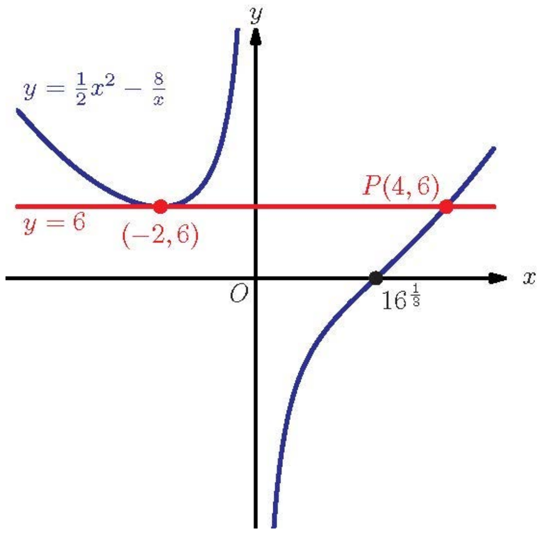
a $f(-2) = 6$

b $f'(x) = x + 8x^{-2}$

c $f'(-2) = -2 + 2 = 0$

d T is a line passing through $(-2,6)$ with gradient 0 so has equation $y = 6$.

e, f



g From GDC: the tangent intersects the graph again at $P(4, 6)$.

9 $y = 2x^3 - 8x + 3$ so $\frac{dy}{dx} = 6x^2 - 8$.

Require that the gradient is -2 so $6x^2 - 8 = -2$.

$6x^2 = 6$ so $x = \pm 1$.

When $x = -1, y = 9$ and when $x = 1, y = -3$ so the two points are $(-1,9)$ and $(1,-3)$.

10 **a** $y = x^3 - 6x^2$ so $\frac{dy}{dx} = 3x^2 - 12x$.

Gradient is zero where $3x^2 - 12x = 0$.

$3x(x - 4) = 0$ so $x = 0$ or $x = 4$.

When $x = 0, y = 0$ and when $x = 4, y = -32$ so the points are $(0,0)$ and $(4,-32)$.

b The line passing through the origin and $(4, -32)$ is $y = -8x$.

11 a $y = 3x^2 + 6x$ so $\frac{dy}{dx} = 6x + 6$.

$y = 3x(x + 2)$ has roots $x = 0$ and $x = -2$.

When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 6$ so the tangent has equation $y = 6x$.

When $x = -2$, $y = 0$ and $\frac{dy}{dx} = -6$ so the tangent has the equation:

$y = -6(x + 2) = -6x - 12$

b Substituting to find the intersection:

$6x = -6x - 12$ so $12x = -12$, and then $x = -1$. The two lines intersect at $(-1, -6)$.

12 $y = 2x^2 + c$ so $\frac{dy}{dx} = 4x$.

If the gradient at $(p, 5)$ is -8 then $4p = -8$ so $p = -2$.

Substituting $(-2,5)$ into the curve equation: $5 = 8 + c$ so $c = -3$.

13 a $P(t) = 15t^2 - t^3$ so $\frac{dP}{dt} = 30t - 3t^2$.

b $\frac{dP}{dt}(6) = 72$ and $\frac{dP}{dt}(12) = -72$

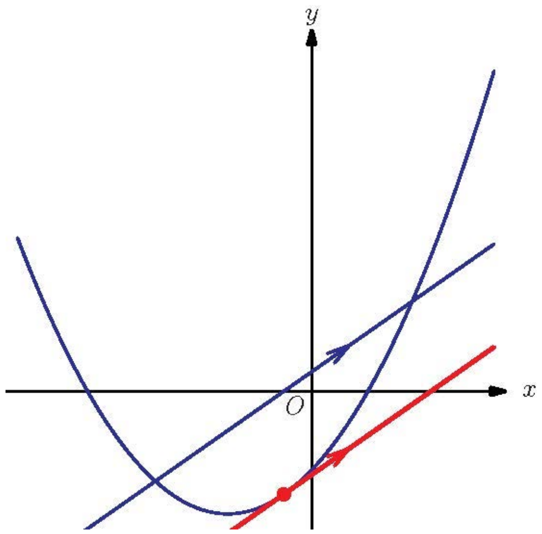
c The profit is increasing by \$72 per month after 6 months, but is falling at a rate of \$72 per month after 12 months.

14 a i $f(x) = 2x + 1$ so $f'(x) = 2$.

ii $g(x) = x^2 + 3x - 4$ so $g'(x) = 2x + 3$.

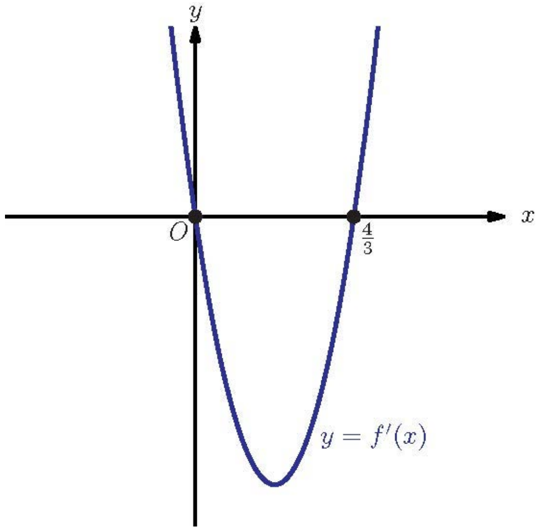
b If the two graphs have the same gradient for a value of x then $2x + 3 = 2$ so $x = -\frac{1}{2}$.

c The tangent line is parallel to the line $y = f(x)$.



15 a i The graph is decreasing for $0 < x < \frac{4}{3}$

ii $f(x)$ has stationary points at $x = 0$ and $x = \frac{4}{3}$, so these are the roots fo $f'(x)$. The graph is increasing for $x < 0$ and $x > \frac{4}{3}$, so $f'(x)$ is positive in these regions, and negative in $0 < x < \frac{4}{3}$.

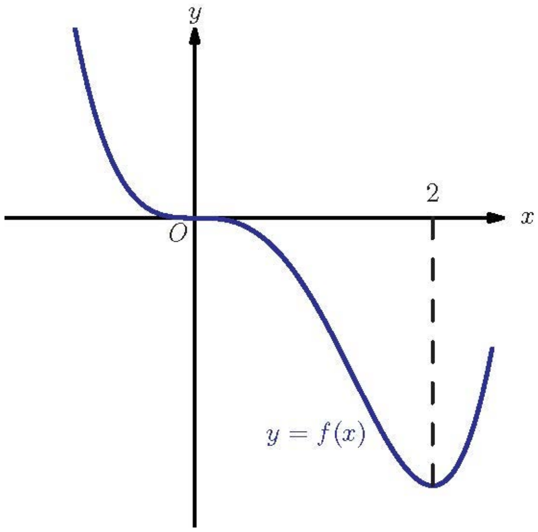


b i $f(x)$ is decreasing when $f'(x) < 0$ which is for $x < 0$ and $0 < x < 2$.

ii Stationary points occur when $f'(x) = 0$: at $x = 0$ and $x = 2$.

$x = 0$ will be a horizontal inflexion point, with negative gradient either side.

$x = 2$ will be a minimum point, with the gradient changing from negative to positive.



16 a Line connecting $(-1, -2)$ and $(1, 4)$: Gradient $= \frac{4-(-2)}{1-(-1)} = 3$.

b $f(x) = x^2 - x + 2$ so $f'(x) = 2x - 1$.

c Require that $2x - 1 = 3$ so $x = 2$.

$f(2) = 4$ so at point $(2,4)$, the curve $y = f(x)$ is parallel to line l.

d Require that $2x - 1 = -\frac{1}{3}$ so $x = \frac{1}{3}$.

$f\left(\frac{1}{3}\right) = \frac{16}{9}$ so at point $\left(\frac{1}{3}, \frac{16}{9}\right)$, the curve $y = f(x)$ is perpendicular to line l.

e $f(3) = 5$ so the gradient at $(3,8)$ is 5.

The tangent has equation $y - 8 = 5(x - 3)$ which rearranges to $y = 5x - 7$.

f The vertex occurs when $f'(x) = 0$.

$2x - 1 = 0$ so $x = \frac{1}{2}$.

$f\left(\frac{1}{2}\right) = \frac{7}{4}$ so the vertex is at $\left(\frac{1}{2}, \frac{7}{4}\right)$, and the gradient is zero at this point.

17 a $f(x) = x^2 + x - 5$ so $f'(x) = 2x + 1$.

b Substituting: $2x + 1 = x^2 + x - 5$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

18 $y = ax^2 + bx$ so $\frac{dy}{dx} = 2ax + b$.

Substituting $x = 2, y = -2$ into the curve equation:

$-2 = 4a + 2b$ (1)

Substituting $x = 2, \frac{dy}{dx} = 3$ into the gradient equation:

$3 = 4a + b$ (2)

$(1) - (2): -5 = b$

So $b = -5, a = 2$

19 $y = ax^2 + bx$ so $\frac{dy}{dx} = 2ax + b$.

Substituting $x = 1, y = 5$ into the curve equation:

$5 = a + b$ (1)

If the normal gradient is $\frac{1}{3}$ then the curve gradient is -3.

Substituting $x = 1, \frac{dy}{dx} = -3$ into the gradient equation:

$-3 = 2a + b$ (2)

$(2) - (1): -8 = a$

So $a = -8, b = 13$

20 $y = 5x^2 - 4$ so $\frac{dy}{dx} = 10x$.

When $x = 1$, $y = 1$ and the gradient is 10.

The equation of the tangent at $(1, 1)$ is $y - 1 = 10(x - 1)$ which rearranges to $y = 10x - 9$

When $x = 2$, $y = 16$ and the gradient is 20.

The equation of the tangent at $(2,16)$ is $y - 16 = 20(x - 2)$ which rearranges to $y = 20x - 24$

At the intersection of these lines, $10x - 9 = 20x - 24$

$10x = 15$ so $x = \frac{3}{2}, y = 6$. The two tangents intersect at $\left(\frac{3}{2}, 6\right)$

21 $f(x) = \frac{\ln(4x)}{x}$

From the GDC: $f'(0.25) = 16$ so the tangent at $P(0.25,0)$ is $y = 16\left(x - \frac{1}{4}\right) = 16x - 4$

This is perpendicular to the tangent at Q so the tangent at Q has gradient $-\frac{1}{16}$.

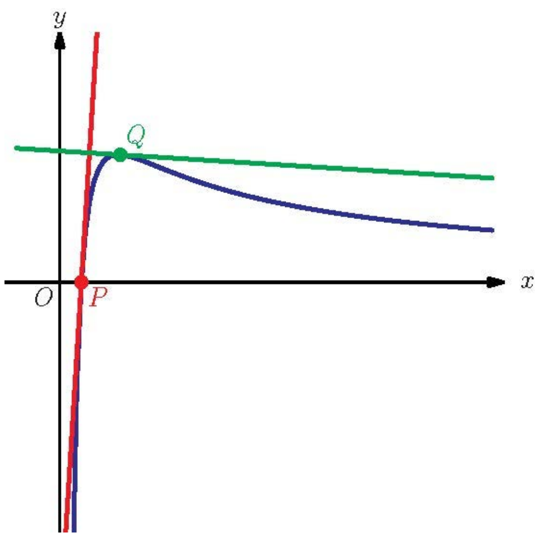
The graph of $f(x)$ increases to a maximum at $x = 0.68$ and then decreases, with gradient decreasing towards zero.

From the GDC: $f'(5) < -\frac{1}{16}$ so the only point within the domain where the tangent is perpendicular to the tangent at $x = \frac{1}{4}$ must be close to the stationary point.

x	$-\frac{1}{f'(x)}$
0.8	3.92
0.75	5.70
0.7	16.54
0.71	11.50
0.705	13.53
0.703	14.58
0.702	15.17
0.701	15.83
0.7005	16.17

The x-coordinate of Q must lie between 0.7 and 0.7005, so to 3DP, the x-coordinate is 0.7.

Q has coordinates $(0.7,1.47)$.



10 Core: Integration

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 10A

23

$$\int \frac{4}{3t^2} - \frac{2}{t^5}\,\mathrm{d}t = \int \frac{4}{3}t^{-2} - 2t^{-5}\,\mathrm{d}t$$
$$= -\frac{4}{3}t^{-1} + \frac{2}{4}t^{-4} + c$$
$$= -\frac{4}{3}t^{-1} + \frac{1}{2}t^{-4} + c$$

24

$$y = \int 3x^2 - 4\ \mathrm{d}x$$
$$= \frac{3x^3}{3} - \frac{4x}{1} + c$$
$$= x^3 - 4x + c$$

When $x = 1, y = 4$ so,

$$4 = 1 - 4 + c$$

So, $c = 7$ and then $y = x^3 - 4x + 7$

25

$$y = \int 4x^{-2} - 3x^2\ \mathrm{d}x$$
$$= \frac{4x^{-1}}{-1} - \frac{3x^3}{3} + c$$
$$= -\frac{4}{x} - x^3 + c$$

When $x = 2, y = 0$ so,

$$0 = -\frac{4}{2} - 2^3 + c$$

So, $c = 10$ and then $y = -\frac{4}{x} - x^3 + 10$

26

$$\int (3x - 2)(x^2 + 1)\,\mathrm{d}x = \int 3x^3 - 2x^2 + 3x - 2\ \mathrm{d}x$$
$$= \frac{3x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2} - 2x + c$$

27

$$\int z^2\left(z+\frac{1}{z}\right)\mathrm{d}z = \int z^3 + z\,\mathrm{d}z$$
$$= \frac{z^4}{4} + \frac{z^2}{2} + c$$

28

$$\int \frac{x^5 - 2x}{3x^3}\mathrm{d}x = \int \frac{x^2}{3} - \frac{2x^{-2}}{3}\,\mathrm{d}x$$
$$= \frac{x^3}{9} + \frac{2x^{-1}}{3} + c$$
$$= \frac{x^3}{9} + \frac{2}{3x} + c$$

29 a When $t = 2, \frac{\mathrm{d}m}{\mathrm{d}t} = 0.5$ so $0.5 = 2k + 0.1$

$k = 0.2$

b Take $m(0) = 0$ since initially the mass is negligible.

$$m = \int kt + 0.1\ \mathrm{d}t$$
$$= \frac{1}{2}kt^2 + 0.1t + c$$

Substituting $k = 0.2$ and $m(0) = 0$ so $c = 0$:

$$m(t) = 0.1t^2 + 0.1t$$

Then $m(5) = 0.1 \times 25 + 0.1 \times 5 = 3$ kg

30 a Let the volume of water in the bath be given by V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{80}{t^2}$$

$V(1) = 0$:

$$V = \int 80t^{-2}\ \mathrm{d}t$$
$$= -80t^{-1} + c$$

$V(1) = 0$:

$$0 = -80 + c \Rightarrow c = 80$$
$$V = 80\left(1 - \frac{1}{t}\right)$$

When $t = 2, V = 40$ l

b When $V = 60, 60 = 80\left(1 - \frac{1}{t}\right)$

$$\frac{1}{t} = \frac{1}{4}$$

$t = 4$; after 4 minutes, the bath holds 60 l.

c A graph of the curve shows that it has an asymptote at $V = 80$ and it will never exceed this value, so the bath will never overflow.

Exercise 10B

5 From GDC:

$$\int_2^5 2x + \frac{1}{x^2} \, dx = 21.3 \text{ (to 3 s.f.)}$$

6 From GDC:

$$\int_1^2 (x-1)^3 \, dx = 0.25$$

7 From GDC:

$$\int_1^4 x^2 + 3 \, dx = 30$$

8 From GDC:

$$\int_{0.5}^2 4 - x^{-2} \, dx = 4.5$$

9 From GDC plot, intercepts between x-axis and the graph are at $(2,0)$ and $(6,0)$

Then from GDC, the area is

$$\left| \int_2^6 -x^2 + 8x - 12 \, dx \right| = \frac{32}{3}$$

10 From GDC:

$$\int_0^3 9 - x^2 \, dx = 18$$

11 a From GDC, the two graphs intersect at the origin and at $(2.5, 6.25)$

b Shaded area $= \int_0^{2.5} (5x - x^2) - \frac{5x}{2} \, dx = 2.60$

12 When $t = 2$, $5 + kt^2 = 9$

$5 + 4k = 9$ so $k = 1$

Total filtered in the first two minutes is

$$\int_0^2 5 + t^2 \, dt = \frac{38}{3} \text{ l}$$

13 Let $p(t)$ be the amount of paint sprayed in grams per second at time t seconds.

$p = kt$ where k is the constant of proportionality in the system.

$p(10) = 20$ so $k = 2$

$$\int_0^{60} p \, dt = \int_0^{60} 2t \, dt$$
$$= 3600 \text{ g}$$

Worked solutions

14 a Let $v(t)$ be the rate of sand falling at time t seconds.

$$v = 100t^{-3}$$

Amount of sand after 5 seconds is given by

$$\int_0^5 v \, dt = 10 + \int_1^5 100t^{-3} \, dt$$
$$= 58 \text{ g}$$

b From the calculator, the amount which will eventually fall is

$$10 + \int_1^\infty 100t^{-3} \, dt = 60 \text{ g}$$

c The model suggests that it takes infinitely long for all 60 g to fall through the timer, and yet is always falling; this in turn suggests that sand is infinitely divisible.

The model also suggests that at the start, the rate at which the sand falls is infinite.

15 $f'(x) = x^2$ so

$$f(x) = \int x^2 \, dx$$
$$= \frac{1}{3}x^3 + c$$

Given $f(0) = 4$, $c = 4$.

$$\int_0^3 f(x) \, dx = \int_0^3 \frac{1}{3}x^3 + 4 \, dx$$
$$= 18.75$$

16 $\int f(x) \, dx = 4(x^3 + x^{-2} + c)$

Then

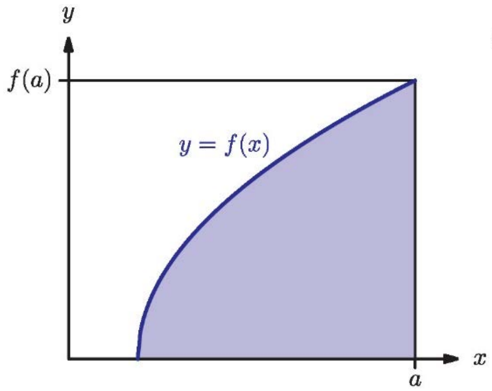
$$f(x) = \frac{d}{dx}(4(x^3 + x^{-2} + c)$$
$$= 12x^2 - 8x^{-3}$$

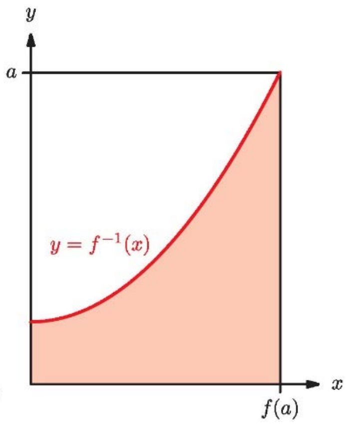
17 This is the same as the area enclosed by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$, by symmetry.

$$\int_0^4 \sqrt{x} \, dx = \int_0^4 x^{\frac{1}{2}} \, dx$$
$$= \frac{16}{3}$$

18 Since the graph of $f^{-1}(x)$ is the reflection of $f(x)$ in the line $y = x$,

$$\int_0^a \text{f}(x) \, dx + \int_0^{f(a)} f^{-1}(x) \, dx = af(a)$$





where $af(a)$ is the area of the rectangle bordering the limits of the integral area, which would have vertices at the origin, $(0, f(a))$, $(a, 0)$ and $(a, f(a))$.

Then

$$\int_0^{f(a)} f^{-1}(x)\ dx = af(a) - A$$

Mixed Practice 10

1 $\int x^3 - 3x^{-2}\ dx = \frac{x^4}{4} + 3x^{-1} + c$

2 $\int 4x^2 - 3x + 5\ dx = \frac{4}{3}x^3 - \frac{3}{2}x^2 + 5x + c$

3 $\int_1^5 2x^{-4}\ dx = 0.661$

4 $y = \int 3x^2 - 8x\ dx = x^3 - 4x^2 + c$

$y(1) = 3 = 1 - 4(1) + c$

$c = 6$

$y = x^3 - 4x^2 + 6$

5 $I = \int_2^a 2 - \frac{8}{x^2} dx = 9$

From GDC:

When $a = 5, I = 3.6$

When $a = 7, I = 7.14$

When $a = 8, I = 9$

$a = 8$

Tip: If you learn further integration methods, you will find how to calculate this algebraically.

6 $I = \int_1^b 9x - x^2 - 8\,dx = 42.7$

From GDC:

When $b = 5, I = 34.7$

When $b = 6, I = 45.8$

When $b = 5.5, I = 40.5$

When $b = 5.7, I = 42.71$

When $b = 5.65, I = 42.2$

So the value of b for which $I = 42.7$ is between 5.65 and 5.7, so to 1 decimal place, $b = 5.7$

7 **a** From calculator, solutions to $-x^3 + 9x^2 - 24x + 20 = 0$ are $x = 2, 5$

So P has coordinates $(2, 0)$ and Q has coordinates $(5, 0)$

b From calculator,

$$\int_2^2 -x^3 + 9x^2 - 24x + 20\,\mathrm{d}x = 6.75$$

8 **a** $y = 0.2x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.4x$

On the curve, at $x = 4$, $y = 0.2 \times 4^2 = 3.2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1.6$

So the tangent has gradient 1.6 and passes through $(4, 3.2)$

Tangent equation is $y - 3.2 = 1.6(x - 4)$

$$y = 1.6x - 3.2$$

b Substituting $x = 2$ into the tangent equation gives $y = 3.2 - 3.2 = 0$ so the x-intercept is at $(2, 0)$.

c

$$\text{Shaded area} = \int_0^4 0.2x^2\,\mathrm{d}x - \int_2^4 1.6x - 3.2\,\mathrm{d}x$$
$$= 1.07$$

9 **a** $y = x^{-2}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-3}$

When $x = 1$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

Then the normal at $(1, 1)$ has gradient $\frac{1}{2}$.

Normal has equation $y - 1 = \frac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + \frac{1}{2}$$

When $x = -1$, $y = -\frac{1}{2} + \frac{1}{2} = 0$ so the x-axis intercept of the normal is at $(-1, 0)$.

b

$$\text{Shaded area} = \int_{-1}^1 \frac{1}{2}x + \frac{1}{2}\,\mathrm{d}x - \int_1^2 x^{-2}\,\mathrm{d}x$$
$$= 1.5$$

10 If the gradient of the normal is always x^2 then $f'(x) = -\frac{1}{x^2} = -x^{-2}$

$$f(x) = \int -x^{-2}\ \mathrm{d}x$$
$$= x^{-1} + c$$

$f(1) = 2 = 1 + c$ so $c = 1$

$$y = 1 + \frac{1}{x}$$

Then $f(2) = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

11 Let $V(t)$ be the volume of water in the container at time t, measured in cm^3.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20t$$

$$V = \int 20t\ \mathrm{d}t$$
$$= 10t^2 + c$$

If $V(0) = 0$ then $0 = 0 + c$ so $c = 0$

$$V(t) = 10t^2$$

Then $V(10) = 1000\ \mathrm{cm}^3$

12 Let $M(t)$ be the mass of the puppy at A months, measured in kg.

$$M(6) = 2.3$$

Then $\frac{dM}{dA} = \frac{A}{20} + c$

When $A = 10$, $\frac{\mathrm{d}M}{\mathrm{d}A} = 1.5 = \frac{1}{2} + c$ so $c = 1$

$$\frac{dM}{dA} = \frac{A}{20} + 1$$

Then for $A \geqslant 6$,

$$M(A) = M(6) + \int_6^{18} \frac{A}{20} + 1\ \mathrm{d}A$$
$$= 21.5\ \mathrm{kg}$$

13 $f'(x) = 3x^2 + k$

$$f(x) = \int 3x^2 + k\ \mathrm{d}x$$
$$= x^3 + kx + c$$

Substituting:

$f(1) = 13 = 1 + k + c$ (1)

$f(2) = 24 = 8 + 2k + c$ (2)

(2) − (1): $k + 7 = 11$

So $k = 4$ and therefore $c = 8$

Then $f(3) = 3^3 + 4 \times 3 + 8 = 47$

14 a The measurement is of temperature increase in the water, but this is assumed equal to the energy output of the nut – that is, there is an assumption that no energy is lost from the system into the surroundings.

b Let $E(t)$ be the amount of energy (in calories) absorbed by the water by time t seconds.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{k}{t^2} \text{ for } t > 1$$

$E(1) = 10$ and $E(2) = 85$

$$\begin{aligned} E(2) &= E(1) + \int_1^2 kt^{-2}\, \mathrm{d}t \\ &= 10 + 0.5k \\ &= 85 \end{aligned}$$

Then $0.5k = 75$ so $k = 150$

Total energy after an indefinite burn is

$$\begin{aligned} E(1) + \int_1^\infty kt^{-2}\, \mathrm{d}t &= 10 + k \\ &= 160 \text{ calories} \end{aligned}$$

15 $\frac{\mathrm{d}y}{\mathrm{d}x} = kx^2$

$$\begin{aligned} y &= \int kx^2\, \mathrm{d}x \\ &= \frac{k}{3}x^3 + c \end{aligned}$$

Substituting $x = 0, y = 3$ and $x = 1, y = \frac{14}{3}$:

$3 = 0k + c$ so $c = 3$

$\frac{14}{3} = \frac{1}{3}k + 3$ so $k = 5$

Then $y = \frac{5}{3}x^3 + 3$

16 $f'(x) = 3x - x^2$

$$\begin{aligned} f(x) &= \int 3x - x^2\, \mathrm{d}x \\ &= \frac{3}{2}x^2 - \frac{1}{3}x^3 + c \end{aligned}$$

Substituting $x = 4, y = 0$:

$0 = \frac{3}{2}(16) - \frac{1}{3}(64) + c$ so $c = \frac{64}{3} - 24 = -\frac{8}{3}$

$$\begin{aligned} \int_0^4 f(x)\, \mathrm{d}x &= \int_0^4 \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{8}{3}\, \mathrm{d}x \\ &= 0 \end{aligned}$$

17

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\ \mathrm{d}x\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 2x^{-1} + c)$$
$$= 6x + 2x^{-2}$$

18 a $y = x^2$ so when $y = 9$ for $x > 0$, $x = 3$.

B has coordinates $(3, 0)$.

b Shaded area is the rectangle area less the area under the curve.

$$\text{Shaded area} = 27 - \int_0^3 x^2\ \mathrm{d}x$$
$$= 18$$

Core Review Exercise

1 **a**

$$u_1 = 7$$
$$u_3 = 15 = u_1 + 2d$$
$$2d = 8$$
$$d = 4$$

b $u_{20} = u_1 + 19d = 83$

c

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$S_{20} = \frac{20}{2}(7 + 83) = 900$$

2 **a** $y = \frac{2x-12}{3} = \frac{2}{3}x - 4$

Gradient of L is $\frac{2}{3}$

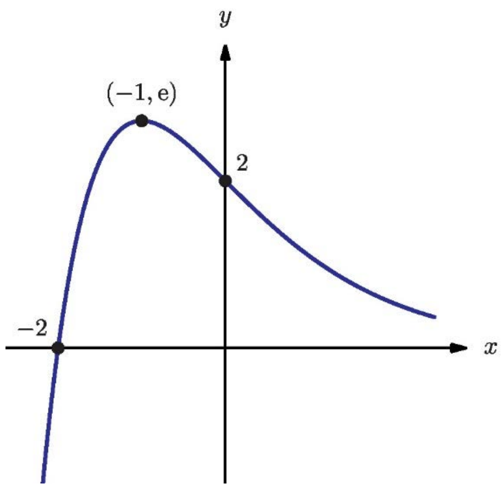
b $k = y(9) = \frac{2}{3}(9) - 4 = 2$

c M has gradient $-\frac{3}{2}$, passes through $(9, 2)$.

M has equation $y - 2 = -\frac{3}{2}(x - 9)$

$$2y - 4 = -3x + 27$$
$$3x + 2y = 31$$

3 **a** From GDC:



b Horizontal asymptote $y = 0$

c $f(x) \leqslant e$

4 **a** $a = \frac{5}{2} = 2.5$

The domain is $x \geqslant 2.5$

b Range is $\mathrm{f}(x) \geqslant 0$

c

$$\mathrm{f}(x) = \sqrt{x - \frac{5}{2}}$$

$$\mathrm{f}^{-1}(x) = x^2 + \frac{5}{2}$$

$$\mathrm{f}^{-1}(4) = \frac{37}{2} = 18.5$$

5 **a** Using cosine rule:

$$\cos A\hat{C}B = \frac{AC^2 + BC^2 - AB^2}{2(AC)(BC)} = 0.778$$

$$A\hat{C}B = 38.9°$$

b

$$\text{Area} = \frac{1}{2}(AC)(CB)\sin A\hat{C}B$$

$$= 95.5 \text{ cm}^2$$

6 $X \sim \mathrm{N}(26, 20.25)$

a $\sigma = \sqrt{20.25} = 4.5$

b From GDC, $\mathrm{P}(21.0 < x < 25.3) = 0.305$

c From GDC, $a = 28.2$

7 **a** $y = (2 - x)(2 + x)\mathrm{e}^{-x}$ has roots at $A(-2,0)$ and $B(2,0)$

b From GDC:

$$\text{Shaded area} = \int_{-2}^{2} (4 - x^2)\mathrm{e}^{-x} \, \mathrm{d}x = 15.6$$

8 **a** From GDC: Median is 4.5

b Lower quartile: 3

Upper quartile: 5

$$\text{IQR} = 5 - 3 = 2$$

c $n = 10$

Number of students scoring at least 4: 7

$$\mathrm{P}(X \geqslant 4) = \frac{7}{10}$$

Worked solutions

9 **a**

$$\text{Volume} = \frac{1}{3}(\text{Base area}) \times \text{height}$$
$$= \frac{1}{3} \times 3.2^2 \times 2.8$$
$$= 9.56 \text{ cm}^3$$

b Mass $= 9.3 \times 9.56 = 88.9$ g

c Considering triangle ABC:

$$AC^2 = AB^2 + BC^2 = 2 \times 3.2^2$$

So $AO^2 = \frac{3.2^2}{2}$

Considering triangle AOV:

$$AV = \sqrt{AO^2 + OV^2}$$
$$= \sqrt{\frac{3.2^2}{2} + 2.8^2}$$
$$= 3.6 \text{ cm}$$

d Using cosine rule in triangle BVC:

$$\cos B\hat{V}C = \frac{BV^2 + VC^2 - BC^2}{2(BV)(VC)} = 0.605$$

$$B\hat{V}C = 52.8°$$

e Each triangular side has area

$$\triangle \text{ Area} = \frac{1}{2}(BV)(VC)\sin\left(B\hat{V}C\right) = 5.16 \text{ cm}^2$$

$$\text{Base area} = 3.2^2 = 10.2 \text{ cm}^2$$

$$\text{Total surface area} = 4(5.16) + 10.2 = 30.9 \text{ cm}^2$$

10 **a** Root at root of numerator: $x = -\frac{3}{2}$

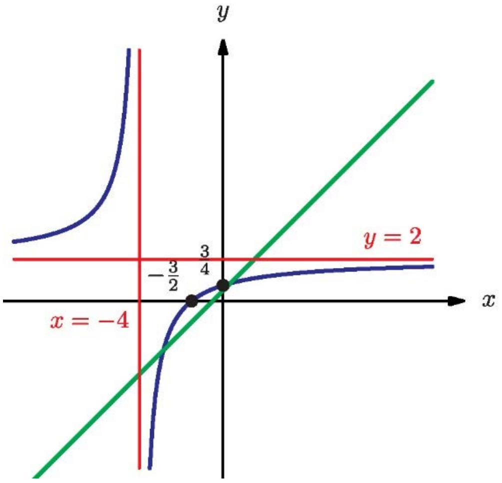
y-intercept at $x = 0$: $\left(0, \frac{3}{4}\right)$

Vertical asymptote at root of denominator: $x = -4$

Horizontal asymptote is value as $x \to \pm\infty$: $y = 2$

b $x = -4$

c



d $(-2.85078, -2.35078)$ or $(0.35078, 0.85078)$

e 1

f L has gradient -1 and passes through $(-2, -3)$

L has equation $y + 3 = -(x + 2)$

$$y = -x - 5$$

11

$$y(-2) = -2 + 3(-2)^2 = 10$$
$$\frac{dy}{dx} = 1 + 6x$$
$$\frac{dy}{dx}(-2) = -11$$

Normal at $(-2,10)$ has gradient $\frac{1}{11}$

Normal has equation $y - 10 = \frac{1}{11}(x + 2)$

$$11y - 110 = x + 2$$
$$x - 11y + 112 = 0$$

12 a $\log_{10} x = 3$ so $x = 10^3 = 1000$

b $\log_{10} 0.01 = -2$

13 a The question has been corrected to: Expand and simplify $(2x^2)^3(x - 3x^{-5})$

$(2x^2)^3(x - 3x^{-5}) = 8x^6(x - 3x^{-5}) = 8x^7 - 24x^{-2}$

b $\frac{d}{dx}(2x^2)^3(x - 3x^{-5}) = 56x^6 + 48x^{-3}$

14 a $u_4 = 18r^3$

b $S_{15} = \frac{18(1-r^{15})}{1-r}$

c $\frac{18(1-r^{15})}{1-r} = 26.28$

From GDC: $r = -1.05$ or 0.315

Try computing the sum using r=-1.05, and you will find that the only valid solution is r=0.315

15 Assuming monthly compounded interest

Using GDC:

$n = 18$

$I\% = ?$

$PV = 200$

$PMT = 0$

$FV = -211$

$P/Y = 12$

$C/Y = 12$

Solving for $I\%$: $I\% = 3.57$

He needs an annual interest rate of 3.6% (to 1 d.p.)

16 Require that

$$\sum \mathrm{P}(X = x) = 1$$

$$k\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) = \frac{205}{144}$$

$$k = \frac{144}{205}$$

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= k\left(\frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \frac{4}{4^2}\right)\\ &= \frac{25}{12}k\\ &= \frac{300}{205}\\ &= \frac{60}{41}\end{aligned}$$

17 Let X be the number of brown eggs in a box.

$X \sim \mathrm{B}(240, 0.05)$

a $\mathrm{E}(X) = 240 \times 0.05 = 12$

b $\mathrm{P}(X = 15) = 0.0733$

c

$$\begin{aligned}\mathrm{P}(X \geqslant 10) &= 1 - \mathrm{P}(X \leqslant 9)\\ &= 1 - 0.236\\ &= 0.764\end{aligned}$$

18 **a** Gradient $CD = \frac{1-(-1)}{-2-(-1)} = -2$

b Gradient $AD = \frac{1-(-1)}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$

Since $-2\left(\frac{1}{2}\right) = -1$, the two lines are perpendicular

c CD has gradient -2 and passes through $D(-1,-1)$

CD has equation $y+1=-2(x+1)$

$y=-2x-3 \qquad (1)$

d AB has equation $x+3y=6 \qquad (2)$

Finding the intersection point:

$(1)-2(2):\ -5y=-15$

$y=3$ so $x=6-3y=-3$

The intersection point is $E(-3,3)$

e $AD=\sqrt{\left(3-(-1)\right)^2+\left(1-(-1)\right)^2}=\sqrt{20}$

f Since $AD \perp ED$:

Area $ADE=\frac{1}{2}(AD)(ED)=\frac{1}{2}\sqrt{20}\sqrt{20}=10$

19 a i

$$\begin{aligned}\text{Volume} &= \pi r^2 h\\ &= \pi(3.25)^2\times 39\\ &= 1294.14\ \text{cm}^2\end{aligned}$$

ii The diameter is 6.5 cm so the maximum number of balls is the rounded down value of $\frac{39}{6.5}=6$

iii I Total volume occupied by the balls:

$$\begin{aligned}\text{Ball volume} &= 6\times\frac{4}{3}\pi r^3\\ &= 8\pi(3.25)^3\\ &= 862.76\ \text{cm}^3\end{aligned}$$

Then the volume of air is $1294.14-862.76=431.38\ \text{cm}^3$

II This is equal to $431.38\times10^{-6}=4.31\times10^{-4}\ \text{m}^3$

b i I $B\hat{T}L+B\hat{L}T+T\hat{B}L=180°$ so $B\hat{T}L=180-80-26.5=73.5°$

II Using the sine rule:

$$\begin{aligned}BT &= \frac{BL}{\sin B\hat{T}L}\times\sin B\hat{L}T\\ &= 55.8\ \text{m}\end{aligned}$$

ii $TG=BT\sin 80°=55.0\ \text{m}$

iii Using cosine rule in triangle BTM:

$$\begin{aligned}TM^2 &= BM^2+BT^2-2(BM)(BT)\cos T\hat{B}M\\ &= 200^2+55.8^2-2(200)(55.8)\cos 100°\\ &= 46997\end{aligned}$$

$TM=217\ \text{m}$

20 a **i** $r = 0.985$

ii This is a strong positive correlation; as number of bicycles increases, production cost reliably increases in a mostly linear pattern.

b From GDC:

$y = 260x + 699$

c Using the regression equation, $\hat{y}(13) = \$4\,079$

d The total income for those 13 bicycles is $13 \times 304 = \$3\,952$ which is less than the estimated cost of production given in part **c**.

e **i** $\text{Sale price}(x) = 304x$

ii $\text{Profit} = 304x - (260x + 699) = 44x - 699$

iii For positive profit, require $44x > 699$

$x > 15.8$

For positive profit, the factory must produce at least 16 bicycles per day.

21 Let X be the time taken (in minutes) to complete a test paper.

$X \sim \text{N}(52, 7^2)$

a From GDC: $\text{P}(X < 45) = 0.159$

b Let Y be the number of students, in a group of 20, who complete the test in less than 45 minutes.

$Y \sim \text{B}(20, 0.159)$

i $\text{P}(Y = 1) = 0.119$

ii

$$\begin{aligned}\text{P}(Y > 3) &= 1 - \text{P}(Y \leq 3)\\ &= 1 - 0.606\\ &= 0.394\end{aligned}$$

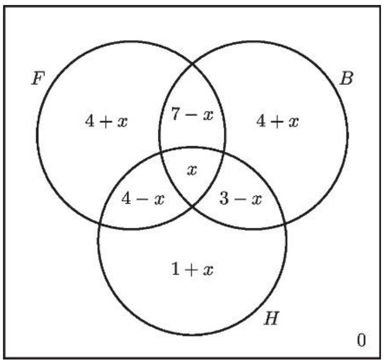
22 If x study all three subjects (and every student studies at least one):

$7 - x$ study French and Biology but not History

$4 - x$ study French and History but not Biology

Then the number studying French only must be $15 - (7 - x + 4 - x + x) = 4 + x$

Working similarly:



a The total number of students displayed in the diagram is

$$4 + x + 7 - x + 4 - x + x + 4 + x + 3 - x + 1 + x = 23 + x = 26$$

So $x = 3$

b $\mathrm{P}(F \cap B' \cap H') = \frac{7}{26}$

c $\mathrm{P}(B'|F) = \frac{\mathrm{P}(B' \cap F)}{\mathrm{P}(F)} = \frac{8}{15}$

d A total of 8 students study History, so 18 do not.

The probability that neither of two randomly selected students study History is

$$P(H', H') = \frac{18}{26} \times \frac{17}{25} = \frac{153}{325}$$

Then the probability that at least one studies history is the complement of this:

$$\mathrm{P}(\text{at least one } H) = 1 - \frac{153}{325} = \frac{172}{325}$$

11 Analysis and approaches: Proof

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 11A

1 Expanding and comparing coefficients: $x^2 + 8x + 23 \equiv x^2 + 2ax + a^2 + b$

x^2: $1 = 1$

x^1: $8 = 2a$ so $a = 4$

x^0: $23 = a^2 + b = 16 + b$ so $b = 7$

$a = 4; b = 7$

2 Expanding and comparing coefficients: $x^2 - 12x - 1 \equiv x^2 - 2ax + a^2 - b$

x^2: $1 = 1$

x^1: $-12 = -2a$ so $a = 6$

x^0: $-1 = a^2 - b = 36 + b$ so $b = 37$

$a = 6; b = 37$

3 Expanding: $4x^2 - 25 \equiv px^2 - q^2$

Comparing coefficients:

x^2: $4 = p$ so $p = 4$

x^1: $0 = 0$

x^0: $-25 = -q^2$ so $q = \pm 5$

$p = 4, q = \pm 5$

4 Expanding the left side:

$$\begin{aligned}(n-1)^2 + n^2 + (n+1)^2 &\equiv (n^2 - 2n + 1) + n^2 + (n^2 + 2n + 1)\\ &\equiv n^2(1+1+1) + n(-2+2) + (1+1)\\ &\equiv 3n^2 + 2\end{aligned}$$

5

$$\begin{aligned}\frac{3x-2}{3} - \frac{2x-3}{2} &\equiv \frac{2(3x-2)}{6} - \frac{3(2x-3)}{6}\\ &\equiv \frac{(6x-4)-(6x-9)}{6}\\ &\equiv \frac{5}{6}\end{aligned}$$

6

$$\frac{7x+6}{12}-\frac{3x+5}{10}\equiv\frac{5(7x+6)}{60}-\frac{6(3x+5)}{60}$$
$$\equiv\frac{(35x+30)-(18x+30)}{60}$$
$$\equiv\frac{17x}{60}$$

7 **a** Show that $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

Consider the LHS:

$$\frac{1}{2}+\frac{1}{4}=1\times\frac{1}{2}+\frac{1}{4}=\frac{2}{2}\times\frac{1}{2}+\frac{1}{4}$$
$$=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$$

Which is equal to the RHS as required.

b

$$\frac{1}{n}+\frac{1}{2n}\equiv\frac{2}{2n}+\frac{1}{2n}$$
$$\equiv\frac{3}{2n}$$

In the instance $n=1$, this gives $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

8 **a** Is an identity, true for all values of x. (Difference of Two Squares)

b Is not an identity; for example, when $x=1$ the statement is false. In fact, the equation can be solved to show it is only valid when $x=0$.

c Is an identity, true for all values of x.

9 Expanding:

$$x^3-5x^2+3x+9\equiv x^3+x^2(a-2b)+x(b^2-2ab)+ab^2$$

Comparing coefficients:

x^3: $1=1$

x^2: $-5=a-2b$ (1)

x^1: $3=b^2-2ab$ (2)

x^0: $9=ab^2$ (3)

(1): $a=2b-5$

Substituting into (2): $3=b^2-2b(2b-5)=-3b^2+10b$

$3b^2-10b+3=0$

$(3b-1)(b-3)=0$

$b=\frac{1}{3}$ or 3, from which (1) gives $a=-\frac{13}{3}$ and $b=\frac{1}{3}$ or $a=1, b=3$

Checking in (3), only the second solution is valid.

$a=1; b=3$

10 Expanding:

$$x^3 - 2x^2 + 5x - 10 \equiv x^3 - qx^2 + px - pq$$

Comparing coefficients:

x^3: $1 = 1$

x^2: $-2 = -q$ so $q = 2$

x^1: $5 = p$

x^0: $-10 = -pq$ which is consistent with the above

$p = 5; q = 2$

11

$$\frac{1}{x-2} - \frac{1}{x+3} \equiv \frac{x+3}{(x-2)(x+3)} - \frac{x-2}{(x-2)(x+3)}$$
$$\equiv \frac{(x+3)-(x-2)}{x^2+x-6}$$
$$\equiv \frac{5}{x^2+x-6}$$

12 a

$$\frac{1}{n-1} - \frac{1}{n+1} \equiv \frac{n+1}{(n-1)(n+1)} - \frac{n-1}{(n-1)(n+1)}$$
$$\equiv \frac{(n+1)-(n-1)}{(n-1)(n+1)}$$
$$\equiv \frac{2}{n^2-1}$$

Taking $n = 4$ this gives $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

b As shown above.

13 a If $x = 2, y = 1$ then

LHS: $x^3 - y^3 = 8 - 1 = 7$

RHS: $(x-y)(x^2+y^2) = (2-1)(4+1) = 5$

Since $7 \neq 5$, the equation is shown not be an identity.

b Expanding and comparing:

$$x^3 - y^3 = x^3 + x^2y(a-1) + xy^2(1-a) - y^3$$

x^3: $1 = 1$

x^2y: $a - 1 = 0$ so $a = 1$

xy^2: $1 - a = 0$ is consistent with this

y^3: $-1 = -1$

So if $a = 1$, the equivalence is valid.

14 Any value k which can be expressed as $k = 2m$ for some $m \in \mathbb{Z}$ is an integer

$$\begin{aligned}(2n+1)^2 - (2n+1) &\equiv (4n^2 + 4n + 1) - (2n + 1)\\ &\equiv 4n^2 + 2n\\ &\equiv 2(2n^2 + n)\end{aligned}$$

Since $2n^2 + n$ is an integer, for integer n, it follows that $(2n+1)^2 - (2n+1)$ must be even.

15 Any value k which can be expressed as $k = 8m$ for some $m \in \mathbb{Z}$ is a multiple of 8.

$$\begin{aligned}(2n+1)^2 - (2n-1)^2 &\equiv (4n^2 + 4n + 1) - (4n^2 - 4n + 1)\\ &\equiv 8n\end{aligned}$$

So for integer any n, $(2n+1)^2 - (2n-1)^2$ is a multiple of 8.

16 Two consecutive integers can be expressed as n and $n + 1$.

The difference in squares between the two values is given by

$$(n+1)^2 - n^2 \equiv n^2 + 2n + 1 - n^2 \equiv 2n + 1$$

Since any value given by $2n + 1$ for an integer n is by definition an odd number, this shows that the difference between consecutive squares is always an odd number.

17 Two consecutive odd numbers can be expressed as $2n - 1$ and $2n + 1$.

The sum of the squares of two such values is given by

$$\begin{aligned}(2n-1)^2 + (2n+1)^2 &\equiv (4n^2 - 4n + 1) + (4n^2 + 4n + 1)\\ &\equiv 8n^2 + 2\\ &\equiv 4(2n^2) + 2\end{aligned}$$

Since this value cannot be expressed in the form $4m$ for some integer m (there is always a remainder of 2), it is proved that the sum of squares of two consecutive odd numbers is even but never a multiple of 4.

Mixed Practice

1 **a** Is not an identity; for example, if $x = 1$ then it is not valid, since $3^3 = 27 \neq 9 = 1^3 + 8$

b Is an identity, as an example of the identity $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ with $y = 4$.

c Is an identity, true for all values of x. (Difference of Two Squares)

2 **a** If $x = 1$ then

LHS: $x^2 + 5 = 1^2 + 5 = 6$

RHS: $(x+1)(x+5) = 2 \times 6 = 12$

Since $6 \neq 12$, the equation is shown not be an identity.

b Expanding and comparing coefficients:

$$x^2 + kx + 5 \equiv x^2 + 6x + 5$$

The identity is valid for $k = 6$.

3 **a** If $x = 1$ then

LHS: $x^2 + 4x + 9 = 1^2 + 4 \times 1 + 9 = 14$

RHS: $(x + 3)^2 = (1 + 3)^2 = 16$

Since $4 \neq 16$, the equation is shown not be an identity.

b Expanding and comparing coefficients:

$x^2 + 4x + 9 \equiv x^2 + 2px + p^2 + q$

x^2: $1 = 1$

x^1: $4 = 2p$ so $p = 2$

x^0: $9 = p^2 + q = 4 + q$ so $q = 5$

The identity is valid for $p = 2, q = 5$.

4 Expanding and comparing coefficients: $x^2 - 8x \equiv x^2 - 2ax + a^2 - b$

x^2: $1 = 1$

x^1: $-8 = -2a$ so $a = 4$

x^0: $0 = a^2 - b = 16 - b$ so $b = 16$

$a = 4; b = 16$

5

$$\frac{x-2}{2} - \frac{x-3}{3} \equiv \frac{3(x-2)}{6} - \frac{2(x-3)}{6}$$
$$\equiv \frac{(3x-6)-(2x-6)}{6}$$
$$\equiv \frac{x}{6}$$

6 Expanding and comparing coefficients: $Ax + 2A + 2Bx - B \equiv 5x$

x^1: $A + 2B = 5$ (1)

x^0: $2A - B = 0$ (2)

$2(1) - (2)$: $5B = 10$ so $B = 2; A = 1$

7 Any value k which can be expressed as $k = 12m$ for some $m \in \mathbb{Z}$ is a multiple of 12.

$$(2n+3)^2 - (2n-3)^2 \equiv (4n^2 + 12n + 9) - (4n^2 - 12n + 9)$$
$$\equiv 24n$$
$$\equiv 12(2n)$$

So for any integer value n, $(2n + 3)^2 - (2n - 3)^2$ is a multiple of 12.

8 Consider $m, n \in \mathbb{Z}$

Then $(2m + 1)$ and $(2n + 1)$ are both odd numbers.

Without loss of generality, let the difference between these two odd numbers be given by $d \equiv (2m + 1) - (2n + 1)$

$d \equiv 2m + 1 - 2n - 1 \equiv 2m - 2n \equiv 2(m - n)$

Hence d is divisible by 2 for all values of m, n i.e. the difference is even.

9 Two consecutive integers can be expressed as n and $n + 1$.

Then the product of the two values added to the larger is given by $n(n+1) + (n+1) = (n+1)^2$ which is a square number.

10

$$\begin{aligned}\frac{x}{(x-2)^2} - \frac{1}{x-2} &\equiv \frac{x}{(x-2)^2} - \frac{x-2}{(x-2)^2} \\ &\equiv \frac{x}{(x-2)^2} - \frac{x-2}{(x-2)^2} \\ &\equiv \frac{2}{(x-2)^2}\end{aligned}$$

11 Any value k which can be expressed as $k = 12m$ for some $m \in \mathbb{Z}$ is a multiple of 12.

Three consecutive odd numbers can be expressed as $2n - 1,\ 2n + 1$ and $2n + 3$.

The sum of squares of these values is then given by

$$\begin{aligned}&(2n-1)^2 + (2n+1)^2 + (2n+3)^2 \\ &\equiv (4n^2 - 4n + 1) + (4n^2 + 4n + 1) + (4n^2 + 12n + 9) \\ &\equiv 12n^2 + 12n + 11 \\ &\equiv 12(n^2 + n) + 11\end{aligned}$$

Since $n^2 + n$ is an integer, it follows that $12(n^2 + n) + 11$ is 11 more than a multiple of 12.

12 **a** Expanding and comparing coefficients:

$x^3 - 8 = x^3 + x^2(a - 2) + x(b - 2a) - 2b$

x^3: $1 = 1$

x^2: $0 = a - 2$ so $a = 2$

x^1: $0 = b - 2a = b - 4$ so $b = 4$

x^0: $-8 = -2b$ is consistent with the above.

$a = 2; b = 4$

b By part a, $n^3 - 8 \equiv (n - 2)(n^2 + 2n + 4)$

If n is an integer greater than 3 then $n - 2 > 2$ and $n^2 + 2n + 4 > 28$ so $n^3 - 8$ is the product of two values greater than or equal to 2.

Any integer factorisation of a prime number must be 1 and the prime number.

Therefore $n^3 - 8$ cannot be prime.

12 Analysis and approaches: Exponents and logarithms

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 12A

14

$$\begin{aligned} 8^{-\frac{4}{3}} &= (2^3)^{-\frac{4}{3}} \\ &= 2^{-4} \\ &= \frac{1}{2^4} \\ &= \frac{1}{16} \end{aligned}$$

15

$$\begin{aligned} \left(\frac{1}{4}\right)^{-\frac{1}{2}} &= (2^{-2})^{-\frac{1}{2}} \\ &= 2^1 \\ &= 2 \end{aligned}$$

16

$$\begin{aligned} \left(\frac{4}{9}\right)^{\frac{3}{2}} &= \left(\left(\frac{2}{3}\right)^2\right)^{\frac{3}{2}} \\ &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27} \end{aligned}$$

17

$$\begin{aligned} \sqrt[3]{x^2} \times \sqrt[4]{x} &= x^{\frac{2}{3}} \times x^{\frac{1}{4}} \\ &= x^{\frac{8}{12}+\frac{3}{12}} \\ &= x^{\frac{11}{12}} \end{aligned}$$

18 $\frac{3}{\sqrt[3]{x}} + 2\sqrt{x} = 3x^{-\frac{1}{3}} + 2x^{\frac{1}{2}}$

19 $\frac{1}{3\sqrt{x^3}} = \frac{1}{3}x^{-\frac{3}{2}}$

20 $x^{\frac{3}{2}} = 2^{-3}$

$$x = (2^{-3})^{\frac{2}{3}} = 2^{-2} = \frac{1}{4}$$

21 $x^{-\frac{1}{2}} = \frac{2}{5}$

$$x = \left(\frac{2}{5}\right)^{-2} = \left(\frac{5}{2}\right)^{2} = \frac{25}{4}$$

22

$$\frac{x\sqrt{x}}{\sqrt[3]{x}} = \frac{x^{\frac{3}{2}}}{x^{\frac{1}{3}}}$$
$$= x^{\frac{3}{2}-\frac{1}{3}}$$
$$= x^{\frac{7}{6}}$$

23

$$\frac{x}{x^2\sqrt{x}} = \frac{x^1}{x^{\frac{5}{2}}}$$
$$= x^{1-\frac{5}{2}}$$
$$= x^{-\frac{3}{2}}$$

24

$$\left(x \times \sqrt[3]{x}\right)^2 = \left(x^{\frac{4}{3}}\right)^2$$
$$= x^{\frac{8}{3}}$$

25

$$\left(\frac{1}{2\sqrt{x}}\right)^3 = \left(\frac{1}{2}x^{-\frac{1}{2}}\right)^3$$
$$= \frac{1}{8}x^{-\frac{3}{2}}$$

26

$$\frac{x^2 + \sqrt{x}}{x\sqrt{x}} = \frac{x^2 + x^{\frac{1}{2}}}{x^{\frac{3}{2}}}$$
$$= x^{\frac{1}{2}} + x^{-1}$$

27

$$\frac{\left(x + \sqrt{x}\right)\left(x - \sqrt{x}\right)}{\sqrt{x}} = \frac{x^2 - x}{x^{\frac{1}{2}}}$$
$$= x^{\frac{3}{2}} - x^{\frac{1}{2}}$$

28 $\frac{1}{3x\sqrt{x}} = \frac{1}{3}x^{-\frac{3}{2}}$

29 $\frac{x^2+3\sqrt{x}}{2x} = \frac{1}{2}x^1 + \frac{3}{2}x^{-\frac{1}{2}}$

30 $y = 2x^{\frac{2}{3}}$

$y^4 = \left(2x^{\frac{2}{3}}\right)^4 = 16x^{\frac{8}{3}}$

31 $y = 3^3 \times x^{\frac{1}{2}}$

$\sqrt[3]{y} = 3x^{\frac{1}{6}}$

32 $x^{\frac{1}{2}} = y^{\frac{1}{3}}$

$y = \left(x^{\frac{1}{2}}\right)^3 = x^{\frac{3}{2}}$

33 $y = \frac{2}{3}x^{-\frac{1}{2}}$

$$y^3 = \left(\frac{2}{3}\right)^3 \times \left(x^{-\frac{1}{2}}\right)^3$$
$$= \frac{8}{27}x^{-\frac{3}{2}}$$

34 $y = x^{\frac{3}{2}}$

$x = y^{\frac{2}{3}}$

35 $y^{\frac{2}{3}} = 3^2$

$y^2 = 3^6$

$y = \pm 3^3 = \pm 27$

36 $x^{\frac{1}{2}} = 2x^{\frac{1}{3}}$

$x^{\frac{1}{2}-\frac{1}{3}} = 2$

$x^{\frac{1}{6}} = 2$

$x = 2^6 = 64$

Exercise 12B

34 a

$$\begin{aligned}\log_3(ab^4) &= \log_3 a + \log_3(b^4)\\ &= \log_3 a + 4\log_3 b\\ &= x + 4y\end{aligned}$$

b

$$\begin{aligned}\log_3\left(\frac{a^2b}{c^5}\right) &= \log_3(a^2) + \log_3 b - \log_3(c^5)\\ &= 2\log_3 a + \log_3 b - 5\log_3 c\\ &= 2x + y - 5z\end{aligned}$$

c

$$\begin{aligned}\log_3(27a^2b^3) &= \log_3 27 + \log_3(a^2) + \log_3(b^3)\\ &= 3 + 2\log_3 a + 3\log_3 b\\ &= 3 + 2x + 3y\end{aligned}$$

35 a

$$\begin{aligned}\log_5(25\sqrt{a}) &= \log_5 25 + \log_5\left(a^{\frac{1}{2}}\right)\\ &= 2 + \frac{1}{2}\log_5 a\\ &= 2 + \frac{1}{2}x\end{aligned}$$

b

$$\begin{aligned}\log_5\left(\frac{b}{5c^5}\right) &= \log_5 b - \log_5 5 - \log_5(c^5)\\ &= \log_5 b - 1 - 5\log_5 c\\ &= y - 1 - 5z\end{aligned}$$

36

$$\begin{aligned}2\ln a + 6\ln b &= \ln(a^2) + \ln(b^6)\\ &= \ln(a^2b^6)\end{aligned}$$

37

$$\begin{aligned}\frac{1}{3}\ln x - \frac{1}{2}\ln y &= \ln(\sqrt[3]{x}) - \ln(\sqrt{y})\\ &= \ln\left(\frac{\sqrt[3]{x}}{\sqrt{y}}\right)\end{aligned}$$

38 a

$$\begin{aligned}\log_2\left(\frac{1}{\sqrt{2}}\right) &= \log_2\left(2^{-\frac{1}{2}}\right)\\ &= -\frac{1}{2}\end{aligned}$$

b $\log_x 27 = -3$

$$x^{-3} = 27 = 3^3$$

$$x = \frac{1}{3}$$

39 $\log_2(x+3) = 3$

$$x + 3 = 2^3$$

$$x = 8 - 3 = 5$$

40 $\log_3(2x-3) = 4$

$$2x - 3 = 3^4$$

$$x = \frac{3^4 + 3}{2} = 42$$

41 a $5^x = 10$

$x = \log_5 10 = 1.43$

b $2 \times 3^x = 14$

$3^x = 7$

$x = \log_3 7 = 1.77$

42 $1.1^x = \frac{20}{3}$

$x = \log_{1.1}\left(\frac{20}{3}\right) = 19.9$

43 $\log_x(2^5) = 5$

$2^5 = x^5$

$x = 2$

44 $\log_x(2^6) = 3$

$2^6 = x^3$

$x = 2^2 = 4$

45 a $5 \times 6^x = 12 \times 3^x$

$2^x = \frac{12}{5} = 2.4$

b $x = \log_2(2.4) = 1.26$

46 a

$$\log_x e = \frac{\ln e}{\ln x}$$
$$= \frac{1}{\ln x}$$

b $16 \ln x = \frac{1}{\ln x}$

$$(\ln x)^2 = \frac{1}{16}$$
$$\ln x = \pm\frac{1}{4}$$
$$x = e^{\frac{1}{4}} \text{ or } e^{-\frac{1}{4}}$$

47

$$(2^3)^{3x+1} = (2^2)^{x-3}$$
$$2^{9x+3} = 2^{2x-6}$$
$$9x + 3 = 2x - 6$$
$$7x = -9$$
$$x = -\frac{9}{7}$$

48 $5^{2x+3} = 9^{x-5}$

Taking natural logarithms:

$$\begin{aligned} \ln(5^{2x+3}) &= \ln(9^{x-5}) = \ln(3^{2x-10}) \\ (2x+3)\ln 5 &= (2x-10)\ln 3 \\ x(2\ln 5 - 2\ln 3) &= -10\ln 3 - 3\ln 5 \\ x &= \frac{10\ln 3 + 3\ln 5}{2\ln 3 - 2\ln 5} \end{aligned}$$

Note that $\ln 9 = \ln 3^2 = 2\ln 3$

49 **a** When $t = 0$, $R = 10$

b When $R = 5, 0.9^t = 0.5$

$t = \log_{0.9} 0.5 = 6.58$ days

50 **a** **i** When $t = 0$, $B = 1000$

ii When $t = 2$, $B = 1000 \times 1.1^2 = 1210$

b When $B = 2000$, $1.1^t = 2$

$t = \log_{1.1} 2 = 7.27$ hours

51 Require $200 \times 1.1^t = 100 \times 1.2^t$

$$\left(\frac{1.2}{1.1}\right)^t = 2$$

$t = \log_{\frac{1.2}{1.2}} 2 = 7.97$ years

It will take approximately 8 years for the populations to equal.

52 $\log_4 x = 6\log_x 8$

$$\frac{\ln x}{\ln 4} = \frac{6\ln 8}{\ln x}$$

$$(\ln x)^2 = \ln 4 \times 6\ln 8 = 2\ln 2 \times 18\ln 2 = (6\ln 2)^2 = (\ln 64)^2$$

$\ln x = \pm\ln 64$

$x = 64$ or $\frac{1}{64}$

53 $2^{5-3x} = 3^{2x-1}$

Taking logarithms:

$$\begin{aligned} (5-3x)\ln 2 &= (2x-1)\ln 3 \\ x(2\ln 3 + 3\ln 2) &= 5\ln 2 + \ln 3 \\ x(\ln 9 + \ln 8) &= \ln 32 + \ln 3 \\ x\ln 72 &= \ln 96 \\ x &= \frac{\ln 96}{\ln 72} \end{aligned}$$

54 Let X be the density of transistors. $X = k \times 2^{0.5t}$ for some constant k.

Require $2^{0.5t} = 10$

$$0.5t = \log_2 10$$

$$t = 2\log_2 10 \text{ years}$$

55 **a** $\log_a(x^2) = b$

Then $x^2 = a^b$

So $x = \pm a^{0.5b}$

$x_1 = a^{0.5b}$, $x_2 = -a^{0.5b}$ so $x_1x_2 = -a^b$

b $(\log_a x)^2 = b$

Then $\log_a x = \pm\sqrt{b}$

$$x = a^{\pm\sqrt{b}}$$

$x_1 = a^{\sqrt{b}}$, $x_2 = a^{-\sqrt{b}}$ so $x_1x_2 = 1$

Mixed Practice

1 **a**

$$\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{2^2}{3^2}\right)^{-\frac{3}{2}}$$

$$= \left(\frac{2}{3}\right)^{-3}$$

$$= \left(\frac{3}{2}\right)^{3}$$

$$= \frac{27}{8}$$

b $\log_2\left(\frac{1}{8}\right) = \log_2(2^{-3}) = -3$

2 $\ln 4 + 2\ln 3 = \ln 4 + \ln 9 = \ln 36$

3 $3^{x-2} = 27 = 3^3$

$$x = 5$$

4 $1.05^x = 2$

$$x = \log_{1.05} 2 = 14.2$$

5 $100^{x+1} = 10^{3x}$

$10^{2x+2} = 10^{3x}$

$2x + 2 = 3x$

$$x = 2$$

6 $\log_3(5x+1) = 2$

$$5x + 1 = 3^2$$

$$x = \frac{9}{5} = 1.6$$

7

$$\begin{aligned} 3\ln x + 2 &= 2(\ln x - 1) \\ \ln x^3 + 2 &= \ln x^2 - 2 \\ \ln x^3 - \ln x^2 &= -4 \\ \ln x &= -4 \\ x &= \mathrm{e}^{-4} \end{aligned}$$

8 **a**

$$\begin{aligned} \log_3(x^2 y) &= \log_3 x^2 + \log_3 y \\ &= 2\log_3 x + \log_3 y \\ &= 2a + b \end{aligned}$$

b

$$\begin{aligned} \log_3\left(\frac{x}{yz^3}\right) &= \log_3 x - \log_3 y - \log_3 z^3 \\ &= \log_3 x - \log_3 y - 3\log_3 z \\ &= a - b - 3c \end{aligned}$$

c

$$\begin{aligned} \log_3\left(\sqrt{zx^3}\right) &= \log_3\left(z^{\frac{1}{2}}\right) + \log_3\left(x^{\frac{3}{2}}\right) \\ &= \frac{1}{2}\log_3 z + \frac{3}{2}\log_3 x \\ &= \frac{c + 3a}{2} \end{aligned}$$

9 **a** $\log_6(6^2) = 2$

b $\log_6 4 + \log_6 9 = \log_6 36 = 2$

c $\log_6 2 - \log_6 12 = \log_6\left(\frac{1}{6}\right) = -1$

10 **a**

$$\begin{aligned} \log_2\left(\frac{a}{\sqrt{b}}\right) &= \log_2 a + \log_2\left(b^{-\frac{1}{2}}\right) \\ &= \log_2 a - \frac{1}{2}\log_2 b \\ &= x - \frac{1}{2}y \end{aligned}$$

b

$$\begin{aligned} \log_2\left(\frac{a^2}{8b^3}\right) &= \log_2(a^2) - \log_2 8 - \log_2(b^3) \\ &= 2\log_2 a - 3 - 3\log_2 b \\ &= 2x - 3 - 3y \end{aligned}$$

11 **a** $\ln 10 = \ln(2 \times 5) = \ln 2 + \ln 5 = x + y$

b $\ln 50 = \ln(2 \times 5^2) = \ln 2 + 2\ln 5 = x + 2y$

c $\ln 0.08 = \ln\left(\frac{2}{25}\right) = \ln 2 - 2\ln 5 = x - 2y$

12

$$\begin{aligned} 3 + 2\log 5 - 2\log 2 &= \log 10^3 + \log 5^2 - \log 2^2 \\ &= \log\left(\frac{10^3 \times 5^2}{2^2}\right) \\ &= \log 6250 \end{aligned}$$

13

$$\begin{aligned} 3\ln x + \ln 8 &= 5 \\ \ln(x^3) + \ln 8 &= \ln(\mathrm{e}^5) \\ \ln(x^3) &= \ln\left(\frac{\mathrm{e}^5}{8}\right) \\ x^3 &= \frac{1}{8}\mathrm{e}^5 \\ x &= \frac{1}{2}\mathrm{e}^{\frac{5}{3}} \end{aligned}$$

14 Using change of base:

$$\begin{aligned} \frac{\log x}{\log 3} &= \frac{4\log 3}{\log x} \\ (\log x)^2 &= 4(\log 3)^2 \\ \log x &= \pm 2\log 3 = \log(3^{\pm 2}) \\ x &= 9 \text{ or } \frac{1}{9} \end{aligned}$$

15

$$\begin{aligned} 4^{3x+5} &= 8^{x-1} \\ 2^{6x+10} &= 2^{3x-3} \\ 6x + 10 &= 3x - 3 \\ x &= -\frac{13}{3} \end{aligned}$$

16 $\log_2 x + \log_3 y = 5$ (1)

$\log_2 x - 2\log_3 y = -1$ (2)

$(1) - (2)$: $3\log_3 y = 6$

$\log_3 y = 2$

(1): $\log_2 x = 5 - \log_3 y = 3$

$y = 3^2 = 9$

$x = 2^3 = 8$

17 $\log_x 8 = 6$

$x^6 = 8$

$x = \sqrt{2}$

18 $3^{2x} = 2e^x$

Since $3 = e^{\ln 3}$ and $2 = e^{\ln 2}$

$$e^{2x\ln 3} = e^{x+\ln 2}$$
$$2x\ln 3 = x + \ln 2$$
$$x(2\ln 3 - 1) = \ln 2$$
$$x = \frac{\ln 2}{2\ln 3 - 1}$$

19 $5^{2x+1} = 7^{x-3}$

Taking logarithms:

$$(2x+1)\log 5 = (x-3)\log 7$$
$$x(2\log 5 - \log 7) = -3\log 7 - \log 5$$
$$x(\log 7 - 2\log 5) = 3\log 7 + \log 5$$
$$x = \frac{(3\log 7 + \log 5)}{\log 7 - 2\log 5}$$

20

$$12^{2x} = 4 \times 3^{x+1}$$
$$2^{4x} \times 3^{2x} = 2^2 \times 3^{x+1}$$
$$2^{4x-2} = 3^{1-x}$$

Taking logarithms:

$$(4x-2)\log 2 = (1-x)\log 3$$
$$x(4\log 2 + \log 3) = \log 3 + 2\log 2$$
$$x\log(2^4 \times 3) = \log(3 \times 2^2)$$
$$x = \frac{\log 12}{\log 48}$$

21 **a** At $t = 0$, $N = 150$

b At $t = 3$, $N = 150 \times e^{3.12} = 3397$

c When $N = 1000$,

$$e^{1.04t} = \frac{1000}{150}$$
$$t = \frac{1}{1.04}\ln\left(\frac{1000}{150}\right) = 1.82 \text{ hours}$$

22 $u_1 = 15, r = 1.2$

$$u_n = u_1 \times r^{n-1} = 231$$
$$1.2^{n-1} = \frac{231}{15}$$
$$(n-1)\ln 1.2 = \ln\left(\frac{231}{15}\right)$$
$$n = 1 + \frac{\ln\left(\frac{231}{15}\right)}{\ln 1.2} = 16$$

231 is the 16th term.

23 **a** $\log_2 40 - \log_2 5 = \log_2\left(\frac{40}{5}\right) = \log_2 8 = 3$

b $8^{\log_2 5} = (2^3)^{\log_2 5} = \left(2^{\log_2 5}\right)^3 = 5^3 = 125$

24 a $\log_2 32 = \log_2(2^5) = 5$

b $\log_2\left(\frac{32^x}{8^y}\right) = \log_2(2^{5x}) - \log_2(2^{3y}) = 5x - 3y$

$p = 5, q = -3$

25 a $\log_3 p^2 = 2\log_3 p = 12$

b $\log_3\left(\frac{p}{q}\right) = \log_3 p - \log_3 q = -1$

c $\log_3(9p) = \log_3 9 + \log_3 p = 2 + 6 = 8$

26

$$\begin{aligned} 8^{x-1} &= 6^{3x} \\ (2^3)^{x-1} &= (2^x \times 3^x)^3 \\ 2^{-3} &= 3^{3x} \\ 2^{-1} &= 3^x \\ -\ln 2 &= x\ln 3 \\ x &= -\frac{\ln 2}{\ln 3} \end{aligned}$$

27

$$\begin{aligned} 3^{x+1} &= 3^x + 18 \\ 3 \times 3^x &= 3^x + 18 \\ 2 \times 3^x &= 18 \\ 3^x &= 9 \\ x &= 2 \end{aligned}$$

28 $\log_b \sqrt{a} = \frac{\ln\sqrt{a}}{\ln b} = \frac{1}{2}\frac{\ln a}{\ln b} = \frac{x}{2y}$

29 a

$$\begin{aligned} \log_b\left(\frac{a^2}{b^3}\right) &= \log_b(a^2) - \log_b(b^3) \\ &= 2\log_b a - 3 \\ &= 2x - 3 \end{aligned}$$

b

$$\begin{aligned} \log_a b^2 &= \frac{\log_b b^2}{\log_b a} \\ &= \frac{2}{x} \end{aligned}$$

30

$$\begin{aligned} \ln x + \ln x^2 + \ln x^3 + \cdots + \ln x^{20} &= (1 + 2 + 3 + \cdots + 20)\ln x \\ &= 210\ln x \end{aligned}$$

Using the formula $\sum_1^n a = \frac{n(n+1)}{2}$

31

$$\begin{aligned} \log_3\left(\frac{1}{3}\right) + \log_3\left(\frac{3}{5}\right) + \log_3\left(\frac{5}{7}\right) + \cdots + \log_3\left(\frac{79}{81}\right) &= \log_3\left(\frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \ldots \times \frac{79}{81}\right) \\ &= \log_3\left(\frac{1}{81}\right) \\ &= -4 \end{aligned}$$

(cancelling fractions throughout the bracketed expression)

13 Analysis and approaches: Exponents and logarithms

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Tip: It is standard to use r as the common ratio of a geometric sequence. For this reason, sums which have used r as the dummy variable, such as

$$\sum_{r=0}^{\infty} \frac{1}{2^r}$$

have been recast throughout this exercise with k as dummy variable instead. Be ready to do an equivalent in any question you encounter; you are strongly advised to avoid using the same letter for two different purposes within any question

Exercise 13A

11 $u_1 = 3, r = \frac{1}{4}$

$$S_\infty = \frac{u_1}{1-r} = \frac{3}{1-\frac{1}{4}} = 4$$

12 $u_1 = 5, r = -\frac{1}{4}$

$$S_\infty = \frac{u_1}{1-r} = \frac{5}{1+\frac{1}{4}} = 4$$

13 Geometric series:

$$u_1 = 2, r = -\frac{1}{3}$$

$$S_\infty = \frac{u_1}{1-r} = \frac{2}{1+\frac{1}{3}} = \frac{3}{2}$$

14 $u_1 = 8$

$$S_\infty = \frac{u_1}{1-r} = 6$$

$$8 = 6(1-r)$$

$$r = -\frac{1}{3}$$

15 $u_1 = 3$

$$S_\infty = \frac{u_1}{1-r} = 4$$
$$3 = 4(1-r)$$
$$r = \frac{1}{4}$$

16 $r = \frac{1}{3}$

$$S_\infty = \frac{u_1}{1-r} = 3$$
$$3\left(1 - \frac{1}{3}\right) = u_1 = 2$$

Then $u_1 = 2, u_2 = \frac{2}{3}$ and $u_3 = \frac{2}{9}$

17 $u_2 = u_1 r = 2$

$$S_\infty = \frac{u_1}{1-r} = 9$$
$$u_1 = 9(1-r) = \frac{2}{r}$$
$$(9-9r)r = 2$$
$$9r^2 - 9r + 2 = 0$$
$$r = \frac{9 \pm \sqrt{81-72}}{18} = \frac{2}{3} \text{ or } \frac{1}{3}$$

18 **a** Changing the dummy variable to k and reserving r for use as the common ratio:

$$\sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k$$

$$k = 0: u_1 = 1$$
$$k = 1: u_2 = \frac{2}{5}$$
$$k = 2: u_3 = \frac{4}{25}$$

b $S_\infty = \frac{u_1}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

19 Geometric series with $u_1 = 2$ and common ratio $r = \frac{1}{3}$

$$\sum_{k=0}^{\infty} \frac{2}{3^k} = S_\infty = \frac{u_1}{1-r} = \frac{2}{1-\frac{1}{3}} = 3$$

20 **a** Geometric series with $u_1 = 3$ and $r = -\frac{x}{9}$.

Convergence criterion: $|r| < 1$ so $\left|-\frac{x}{9}\right| < 1$

$$|x| < 9$$

b $r = \frac{2}{9}$

$S_\infty = \frac{u_1}{1-r} = \frac{3}{1-\frac{2}{9}} = \frac{27}{7}$

Worked solutions

21 a $u_1 = 5, r = (x-3)$

Convergence criterion: $|r| < 1$

So $|x-3| < 1$

$2 < x < 4$

b $S_\infty = \frac{u_1}{1-r} = \frac{5}{1-(x-3)} = \frac{5}{4-x}$

22 a $u_1 = 2, r = 2x$

Convergence criterion: $|r| < 1$

$|2x| < 1$ so $|x| < \frac{1}{2}$

b $S_\infty = \frac{u_1}{1-r} = \frac{2}{1-2x}$

23

$$u_2 = u_1 r = -\frac{6}{5}$$

$$S_\infty = \frac{u_1}{1-r} = 5$$

$$u_1 = 5(1-r) = -\frac{6}{5r}$$

$$5r(5-5r) = -6$$

$$25r^2 - 25r - 6 = 0$$

$$r = \frac{25 \pm \sqrt{625+600}}{50} = 1.2 \text{ or } -0.2$$

The series converges so $r = 1.2$ is not a valid solution, since $|r| < 1$ for convergence.

$$r = -0.2 \text{ so } u_1 = 5(1-r) = 6$$

24 a Geometric series: $u_1 = x, r = 4x^2$

For convergence, $|r| < 1$ so $4x^2 < 1$

$$|x| < \frac{1}{2}$$

Since $x > 0$ from the question, the range of possible values is $0 < x < \frac{1}{2}$

b $S_\infty = \frac{u_1}{1-r} = \frac{x}{1-4x^2}$

25 a $\sum_{k=0}^{\infty} \frac{x^{k+1}}{2^k} = \sum_{k=0}^{\infty} x\left(\frac{x}{2}\right)^k = x\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k$

The series converges to 3

$$x \times \frac{1}{1-\frac{x}{2}} = 3$$

Rearranging:

$$x = 3 - \frac{3x}{2}$$
$$\frac{5x}{2} = 3$$
$$x = \frac{6}{5}$$

b The series only converges if $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$.

26

$$\frac{u_1}{1-r} = 27$$
$$u_1 + u_2 + u_3 = 19 = u_1(1 + r + r^2) \qquad (1)$$

Rearranging the first equation:

$$u_1 = 27(1 - r)$$

Substituting into (1):

$$27(1 + r + r^2)(1 - r) = 19$$
$$1 - r^3 = \frac{19}{27}$$
$$r^3 = \frac{8}{27}$$
$$r = \frac{2}{3}$$

Then $u_1 = 27\left(1 - \frac{2}{3}\right) = 9$

Exercise 13B

26

$$(10 - 3x)^4 = 10^4 + 4(10)^3(-3x)^1 + 6(10)^2(-3x)^2 + 4(10)^1(-3x)^3 + (-3x)^4$$
$$= 10\,000 - 12\,000x + 5400x^2 - 1080x^3 + 81x^4$$

27

$$(2x - 1)^5 = (2x)^5 + 5(2x)^4(-1) + 10(2x)^3(-1)^2 + 10(2x)^2(-1)^3 + 5(2x)^1(-1)^4 + (-1)^5$$
$$= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$$

28 General term: ${}^7\mathrm{C}_r(5)^r(3x)^{7-r}$

Require $7 - r = 5$ so $r = 2$

Term is

$${}^7\mathrm{C}_2(5)^2(3x)^5 = 21(5)^2(3x)^5 = 127575x^5$$

Coefficient is 127 575

29 General term: $^{12}\mathrm{C}_r(3x)^r(-2)^{12-r}$

Require $r = 8$

Term is

$$^{12}\mathrm{C}_8(3x)^8(-2)^4 = 495(3x)^8(-2)^4 = 51963120x^8$$

Coefficient is 51 963 120

30

$$^{n}\mathrm{C}_4 = \frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4} = 495$$

$$n(n-1)(n-2)(n-3) = 495\times24 = 11880$$

Using GDC to solve the quartic: $n = 12$

31 a $(2+3x)^{10} = (2)^{10} + 10(2)^9(3x) + 45(2)^8(3x)^2 + \cdots + (3x)^{10}$

First three terms: $1024, 15360x, 103680x^2$

b Substituting $x = 0.001$:

$$\begin{aligned}(2+3x)^{10} &= 2.003^{10}\\ &\approx 1024 + 15.360x + 0.10368\\ &\approx 1039\end{aligned}$$

32 a

$$\begin{aligned}(3-2x)^9 &= (3)^9 + 9(3)^8(-2x) + 36(3)^7(-2x)^2 + 84(3)^6(-2x)^3 + \cdots\\ &= 19\,683 - 118\,098x + 314\,928x^2 - 489\,888x^3 + \cdots\end{aligned}$$

b Substituting $x = 0.01$:

$$\begin{aligned}(3-2x)^9 &= 2.98^9\\ &\approx 19683 - 1180.98 + 31.4928 - 0.489888\\ &\approx 18\,533.0\end{aligned}$$

33 a

$$\begin{aligned}(2+x)^7 &= (2)^7 + 7(2)^6(x) + 21(2)^5(x)^2 + \cdots\\ &= 128 + 448x + 672x^2 + \cdots\end{aligned}$$

b

$$\begin{aligned}(2+x)^7(3-x) &= (128 + 448x + 672x^2 + \cdots)(3-x)\\ &= 3\times128 + x(3\times448 - 128) + x^2(3\times672 - 448) + \cdots\\ &= 384 + 1216x + 1568x^2 + \cdots\end{aligned}$$

34 a

$$\begin{aligned}(2-3x)^5 &= (2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + \cdots\\ &= 32 - 240x + 720x^2 + \cdots\end{aligned}$$

b x^2 term in the expansion of $(2-3x)^5(1+2x)$ is $x^2(720 - 2\times240) = 240x^2$
So coefficient is 240

35

$$(2-x)^5 = (2)^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + \cdots$$
$$= 32 - 80x + 80x^2 + \cdots$$

$$(1+2x)^6 = 1 + 6(2x) + 15(2x)^2 + \cdots$$
$$= 1 + 12x + 60x^2 + \cdots$$

x^2 term in the expansion of $(2-x)^5(1+2x)^6$ is x^2

$$(32 \times 60 - 80 \times 12 + 80 \times 1) = 1040x^2$$

Coefficient is 1040.

36 a

$$(2+x)^4 = (2)^4 + 4(2)^3(x)^1 + 6(2)^2(x)^2 + 4(2)^1(x)^3 + (x)^4$$
$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

Substituting $-x$ instead of x gives

$$(2-x)^4 = 16 - 32x + 24x^2 - 8x^3 + x^4$$

Therefore $(2+x)^4 - (2-x)^4 = 16x^3 + 64x$

b When $x = 0.01$, this gives

$$2.01^4 - 1.99^4 = 0.64 + 0.000016 = 0.640016$$

37 General term is ${}^{n}\mathrm{C}_{r}(x)^r(2)^{n-r}$

If the index of x is $r = n-3$ then the term is ${}^{n}\mathrm{C}_{n-3}(x)^{n-3}(2)^3$

$${}^{n}\mathrm{C}_{n-3} \times 2^3 = \frac{8\big(n \times (n-1) \times (n-2)\big)}{1 \times 2 \times 3} = 1760$$
$$n(n-1)(n-2) = 1320$$

Using calculator to solve this cubic: $n = 12$

38 Term in x^4 has coefficient ${}^{n}\mathrm{C}_{4}(3)^{n-4} = \frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4} = 153090$

$$n(n-1)(n-2)(n-3) \times 3^n = 297\,606\,960$$

Using calculator to solve this equation: $n = 10$

39

$$(x + 2x^{-1})^4 = x^4 + 4(x)^3(2x^{-1})^1 + 6(x)^2(2x^{-1})^2 + 4(x)^1(2x^{-1})^3 + (2x^{-1})^4$$
$$= x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}$$

40

$$(x^2 + 3x)^5 = (x^2)^5 + 5(x^2)^4(3x)^1 + 10(x^2)^3(3x)^2 + 10(x^2)^2(3x)^3 + 5(x^2)^1(3x)^4 + (3x)^5$$
$$= x^{10} + 15x^9 + 90x^8 + 270x^7 + 405x^6 + 243x^5$$

41 General term is ${}^{15}\mathrm{C}_{r}(x^2)^r(3x)^{15-r} = {}^{15}\mathrm{C}_{r}x^{15+r}(3)^{n-r}$

Require that $15 + r = 27$ so $r = 12$

Coefficient is then ${}^{15}\mathrm{C}_{12} \times 3^3 = 12\,285$

42 General term is ${}^{10}\mathrm{C}_r(2x^2)^r(-3x^{-1})^{10-r} = {}^{10}\mathrm{C}_r x^{3r-10} \times 2^r \times (-3)^{10-r}$

Require that $3r - 10 = 5$ so $r = 5$

Coefficient is then ${}^{10}\mathrm{C}_5 \times 2^5 \times (-3)^5 = -1\,959\,552$

43

Tip: Always be alert for a shortcut – notice the difference of two squares in the question to answer without any need for binomial expansion!

$$(x-1)^7(x+1)^7 = (x^2-1)^7$$

All powers of x in the expansion must be even, so the coefficient of x^9 is zero.

Mixed Practice

1

$$\begin{aligned}(2+x)^4 &= (2)^4 + 4(2)^3(x)^1 + 6(2)^2(x)^2 + 4(2)^1(x)^3 + x^4 \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

2 $u_1 = \frac{1}{3}, r = \frac{1}{3}$

$$S_\infty = \frac{u_1}{1-r} = \frac{1}{2}$$

3 $u_1 = 7, r = -\frac{5}{9}$

$$S_\infty = \frac{u_1}{1-r} = \frac{9}{2}$$

4 $r = \frac{3}{4}$

$$S_\infty = \frac{u_1}{1-r} = 12$$

$$u_1 = 12 \times \frac{1}{4} = 3$$

5 General term in the expansion of $(1+x)^n$ is ${}^n\mathrm{C}_r(x)^r(1)^{n-r}$

Require that $r = 6$

Coefficient is then ${}^n\mathrm{C}_6 = 3003 = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\times2\times3\times4\times5\times6}$

$$n(n-1)(n-2)(n-3)(n-4)(n-5) = 2\,162\,160$$

Solving this polynomial with calculator: $n = 14$

6 **a** General term in the expansion of $(3x-2)^{12}$ is ${}^{12}\mathrm{C}_r(3x)^r(-2)^{n-r}$

Require that $r = 5$ for the x^5 term.

Coefficient is then ${}^{12}\mathrm{C}_5(3)^5(-2)^7$

$$p = 5, q = 7, r = 5$$

Note that $(3x-2) \equiv (-2+3x)$

Therefore, we may also write the general term as

$${}^{12}C_r(-2)^r(3x)^{n-r}$$

For the term in x^5 we then have $r = 7$ $(p = 5, q = 7)$.

Indeed, ${}^{12}C_5 = {}^{12}C_7$;

and, ${}^{n}C_r \equiv {}^{n}C_{n-r}$ in general.

b Coefficient is ${}^{12}\mathrm{C}_5(3)^5(-2)^7 = -24\,634\,368$

7 a There will be eleven terms (x^0 to x^{10})

b General term in the expansion of $(x+3)^{10}$ is ${}^{10}\mathrm{C}_r(x)^r(3)^{n-r}$

Require that $r = 3$ for the x^3 term.

Term is then ${}^{10}\mathrm{C}_3(3)^7x^3 = 120 \times 2187x^3 = 262\,440x^3$

8 a

$$\begin{aligned}(1+2x)^{10} &= (1)^{10} + 10(1)^9(2x)^1 + 45(1)^8(2x)^2 + \cdots \\ &= 1 + 20x + 180x^2 + \cdots\end{aligned}$$

b When $x = 0.001$, this approximation gives

$$\begin{aligned}1.002^{10} &\approx 1 + 0.02 + 0.00018 \\ &\approx 1.02018\end{aligned}$$

9 a $u_1 = 2 - 3x = r$

Convergence when $|r| < 1$

So $|2 - 3x| < 1$

$$\begin{aligned}1 < |3x| < 3 \\ \frac{1}{3} < x < 1\end{aligned}$$

b

$$\begin{aligned}S_\infty &= \frac{u_1}{1-r} \\ &= \frac{2-3x}{3x-1}\end{aligned}$$

If $S_\infty = \frac{1}{2}$ then

$$\begin{aligned}3x - 1 &= 2(2-3x) \\ 9x &= 5 \\ x &= \frac{5}{9}\end{aligned}$$

c If $S_\infty = -\frac{2}{3}$ then $\frac{2-3x}{1-(2-3x)} = -\frac{2}{3}$

$$\begin{aligned}3(2-3x) &= -2(3x-1) \\ 6 - 9x &= 2 - 6x \\ 3x &= 4 \\ x &= \frac{4}{3}\end{aligned}$$

But this is a value outside the convergence requirement; for $x = \frac{4}{3}$, the series would be a sum of powers of (-2), which would not converge to $-\frac{2}{3}$.

10

$$S_\infty = \frac{u_1}{1-r} = 3u_1$$
$$1 - r = \frac{1}{3}$$
$$r = \frac{2}{3}$$

11 General term in the expansion of $(2-x)^{10}$ is ${}^{10}\mathrm{C}_r(2)^r(-x)^{n-r}$

Require that $r = 6$ for the x^4 term.

Coefficient is then ${}^{10}\mathrm{C}_6(2)^6(-1)^4 = 210 \times 64 = 13\,440$

12 a Expansion of $(3+x)^n$ is $3^n + n(3)^{n-1}(x) + \cdots$

Comparing first term: $3^n = 81$ so $n = 4$

b Then the second term gives $k = 4 \times 3^3 = 108$

13 a ${}^n\mathrm{C}_1 = n$

b i Expansion of $(x+2)^n$ is

$$x^n + n(x)^{n-1}(2)^1 + \frac{n(n-1)}{2}(x)^{n-2}(2)^2 + \cdots = x^n + 18x^{n-1} + bx^{n-2} + \cdots$$

Comparing terms:

x^n: $1 = 1$

x^{n-1}: $2n = 18$ so $n = 9$

ii x^{n-2}: $\frac{9\times 8}{2} \times 4 = b$

$b = 144$

14 General term in the expansion of $(2x+p)^6$ is ${}^6\mathrm{C}_r(2x)^r(p)^{n-r}$

Require that $r = 4$ for the x^4 term.

Term is then ${}^6\mathrm{C}_4(2x)^4(p)^2 = 60x^4$

$$15 \times 2^4 p^2 = 60$$
$$p^2 = \frac{1}{4}$$
$$p = \pm\frac{1}{2}$$

15 Geometric sequence with first term u_1 and common ratio r:

$u_1 + u_2 + u_3 = u_1(1 + r + r^2) = 62.755 \quad (1)$

$$S_\infty = \frac{u_1}{1-r} = 440$$
$$u_1 = 440(1-r)$$

Substituting into (1):

$$440(1-r^3) = 62.755$$
$$r^3 = 1 - \frac{62.755}{440} = 0.857375$$
$$r = 0.95 = \frac{19}{20}$$

16

$$\sum_{k=0}^{\infty} \frac{3^k + 4^k}{5^k} = \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k + \sum_{r=0}^{\infty} \left(\frac{4}{5}\right)^k$$
$$= \sum_{k=0}^{\infty} 0.6^k + \sum_{k=0}^{\infty} 0.8^k$$
$$= \frac{1}{1-0.6} + \frac{1}{1-0.8}$$
$$= \frac{15}{2}$$

17 a $\sum_{k=0}^{\infty}(\mathrm{e}^{-x})^k$ is a geometric series with common ratio $r = \mathrm{e}^{-x}$

Convergence criterion is $|r| < 1$

Since $\mathrm{e}^{-x} < 1$ for all positive x, the series will converge for all positive x.

b $S_\infty = \frac{u_1}{1-r} = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} = \frac{1}{\mathrm{e}^x - 1}$

c

$$\frac{1}{\mathrm{e}^x - 1} = 2$$
$$\mathrm{e}^x = \frac{3}{2}$$
$$x = \ln\left(\frac{3}{2}\right)$$

18 Geometric series with first term u_1 and common ratio r.

$$u_1 + u_2 + u_3 = u_1(1 + r + r^2) = 26 \qquad (1)$$

$$S_\infty = \frac{u_1}{1-r} = 27$$
$$u_1 = 27(1-r)$$

Substituting into (1):

$$27(1-r^3) = 26$$
$$r^3 = 1 - \frac{26}{27} = \frac{1}{27}$$
$$r = \frac{1}{3}$$

19 Expansion of $\left(1+\frac{2}{3}x\right)^n$ is $1+n\left(\frac{2}{3}x\right)^1+\cdots$

$$\begin{aligned}\left(1+\frac{2}{3}x\right)^n(3+nx)^2 &= \left(1+\frac{2n}{3}x+\cdots\right)\left(9+6nx+n^{2x^2}\right)\\ &= 9+x\left(9\times\frac{2n}{3}+6n\right)+\cdots\\ &= 9+84x+\cdots\end{aligned}$$

Comparing coefficient of x:

$$\begin{aligned}12n &= 84\\ n &= 7\end{aligned}$$

20 $(x-1)^3 = x^3-3x^2+3x-1$

$$\begin{aligned}(x^{-1}+2x)^6 &= x^{-6}(1+2x^2)^6\\ &= x^{-6}(1+6(2x^2)^1+15(2x^2)^2+20(2x^2)^3+\cdots)\\ &= x^{-6}+12x^{-4}+60x^{-2}+160+\cdots\end{aligned}$$

The coefficient of x^{-2} is $\left(12\times(-3)+60\times(-1)\right)=-96$

21 a i $u_n = a+(n-1)d$

So, $v_n = 2^{u_n} = 2^a\times 2^{d(n-1)}$

Then $\frac{v_{n+1}}{v_n} = \frac{2^a\times 2^{dn}}{2^a\times 2^{d(n-1)}} = 2^d$

ii $v_1 = 2^a$

iii $v_n = 2^a\left(2^d\right)^{n-1} = 2^{a+d(n-1)}$

b i $\{v_n\}$ is a geometric sequence with common ratio $r=2^d$ and first term $v_1=2^a$, from part **a**.

$$S_n = \frac{v_1(1-r^n)}{1-r} = 2^a\times\frac{1-2^{dn}}{1-2^d} = \frac{2^a(2^{nd}-1)}{2^d-1}$$

ii Convergence criterion: $|r|<1$

$|2^d|<1$

Require that $d<0$

iii $S_\infty = \frac{v_1}{1-r} = \frac{2^a}{1-2^d}$

iv

$$\begin{aligned}\frac{2^a}{1-2^d} &= 2^{a+1}\\ 1-2^d &= \frac{1}{2}\\ 2^d &= \frac{1}{2}\\ d &= -1\end{aligned}$$

c $w_n = pq^{n-1}$

$z_n = \ln(pq^{n-1}) = \ln p + (n-1)\ln q$

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} (\ln p + (i-1)\ln q)$$

$$= n\ln p + \ln q \sum_{i=1}^{n} (i-1)$$

$$= n\ln p + \frac{n(n-1)}{2}\ln q$$

$$= \ln\left[p^n q^{\frac{n(n-1)}{2}}\right]$$

$k = p^n q^{\frac{n(n-1)}{2}}$

Worked solutions

14 Analysis and approaches: Functions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 14A

15 a

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= 3g(x) - 1 \\ &= 3(4 - 3x) - 1 \\ &= 11 - 9x\end{aligned}$$

b

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= 4 - 3f(x) \\ &= 4 - 3(3x - 1) \\ &= 7 - 9x = 4\end{aligned}$$

So $9x = 3$ from which $x = \frac{1}{3}$

16 a

$$\begin{aligned}f(f(x)) &= (f(x))^2 + 1 \\ &= (x^2 + 1)^2 + 1 \\ &= x^4 + 2x^2 + 2\end{aligned}$$

b

$$\begin{aligned}f(g(x)) &= (g(x))^2 + 1 \\ &= (x - 1)^2 + 1 \\ &= x^2 - 2x + 2\end{aligned}$$

$$\begin{aligned}g(f(x)) &= f(x) - 1 \\ &= x^2\end{aligned}$$

If $f(g(x)) = g(f(x))$ then $x^2 - 2x + 2 = x^2$ so $x = 1$

17

$$\begin{aligned}f(f(x)) &= 2(f(x))^3 \\ &= 2(2x^3)^3 \\ &= 16x^9\end{aligned}$$

18 Domain of $\mathrm{f}(x)$ is $x \geqslant 4$ so require range of $\mathrm{g}(x)$ restricted to $\mathrm{g}(x) \geqslant 4$

$\mathrm{g}(x) = 3x + 10 \geqslant 4$ so $x \geqslant -2$

The largest possible domain for $(\mathrm{f} \circ \mathrm{g})(x)$ is $x \geqslant -2$.

b

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \sqrt{3x + 10 - 4} \\ &= \sqrt{3x + 6}\end{aligned}$$

If $\sqrt{3x+6} = 5$ then $3x + 6 = 25$ so $x = \frac{19}{3}$

19 a $\mathrm{fg}(8) = \ln(8 - 5) = \ln 3$

$\mathrm{gf}(8) = \ln 8 - 5$

b $\mathrm{fg}(x) = \ln(x - 5) = 8$

So $x = \mathrm{e}^8 + 5$

20 a i

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(3) &= \mathrm{f}(\mathrm{g}(3)) \\ &= \mathrm{f}(4) \\ &= 2\end{aligned}$$

ii

$$\begin{aligned}(\mathrm{g} \circ \mathrm{f})(4) &= \mathrm{g}(\mathrm{f}(4)) \\ &= \mathrm{g}(2) \\ &= 1\end{aligned}$$

b $\mathrm{f}(\mathrm{g}(x)) = 1 = \mathrm{f}(5)$ so $\mathrm{g}(x) = 5$

Then $x = 4$.

21 a i

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(3) &= \mathrm{f}(\mathrm{g}(3)) \\ &= \mathrm{f}(7) \\ &= 1\end{aligned}$$

ii

$$\begin{aligned}(\mathrm{g} \circ \mathrm{f})(9) &= \mathrm{g}(\mathrm{f}(9)) \\ &= \mathrm{g}(9) \\ &= 1\end{aligned}$$

b $\mathrm{f}(\mathrm{g}(x)) = 5 = \mathrm{f}(5)$ so $\mathrm{g}(x) = 5$

Then $x = 1$.

22 a Domain of $\mathrm{f}(x)$ is $x > 0$ so require range of $\mathrm{g}(x)$ restricted to $\mathrm{g}(x) > 0$

$\mathrm{g}(x) = x - 3 > 0$ so require $x > 3$

The largest possible domain for $(\mathrm{f} \circ \mathrm{g})(x)$ is $x > 3$.

b $(\mathrm{f} \circ \mathrm{g})(x) = \ln(x - 3) = 1$ so $x = \mathrm{e} + 3$

c $(\mathrm{g} \circ \mathrm{f})(x) = \ln x - 3 = 1$ so $x = \mathrm{e}^4$

d

$$\ln(x-3) = \ln x - 3$$
$$\ln x - \ln(x-3) = 3$$
$$\ln\left(\frac{x}{x-3}\right) = 3$$
$$\frac{x}{x-3} = e^3$$
$$x = e^3(x-3)$$
$$x(1-e^3) = -3e^3$$
$$x = \frac{3e^3}{e^3-1}$$

23 a

$$(f \circ g)(x) = f\left(\frac{1}{x-3}\right)$$
$$= \frac{1}{\frac{1}{x-3}+2}$$

Multiplying numerator and denominator by $(x-3)$:

$$(f \circ g)(x) = \frac{x-3}{1+2(x-3)}$$
$$= \frac{x-3}{2x-5}$$

b Domain of $f(x)$ is $x \neq -2$ so require range of $g(x)$ restricted to $g(x) \neq -2$

$g(x) = -2$ when $x = \frac{5}{2}$ so require $x \neq \frac{5}{2}$

Largest possible domain of $g(x)$ is $x \neq 3$

The largest possible domain for $(f \circ g)(x)$ is $x \neq 2.5, 3$.

c

$$\frac{x-3}{2x-5} = 2$$
$$x-3 = 2(2x-5)$$
$$3x = 7$$
$$x = \frac{7}{3}$$

24

$$(g \circ f)(x) = 2\left(\frac{4}{x}\right)^2 = \frac{32}{x^2}$$
$$(f \circ g)(x) = \frac{4}{2x^2} = \frac{2}{x^2}$$

So $(g \circ f)(x) = 16(f \circ g)(x)$

$$k = 16$$

25 a Largest possible domain of $f(x)$ is $x \neq -\frac{2}{3}$

$f(x) = -\frac{2}{3}$ when $x = -\frac{7}{6}$

The largest possible domain for $(f \circ f)(x)$ is $x \neq -\frac{2}{3}, -\frac{7}{6}$

b Range of $(\mathrm{f} \circ \mathrm{f})(x)$ for this largest domain is $(\mathrm{f} \circ \mathrm{f})(x) \neq 0, \frac{1}{2}$

c $(\mathrm{f} \circ \mathrm{f})(x) = 1$ so (using GDC) $x = -\frac{5}{3}$

Exercise 14B

Tip: When finding an inverse, always remember to check the range of the original, since that will be the domain of your inverse, and should be stated with your answer for a complete definition of the function.

18 Let $y = \mathrm{f}(x) = \frac{4}{3-x}$ (range $\mathrm{f}(x) \neq 0$)

Then $3 - x = \frac{4}{y}$ so $\mathrm{f}^{-1}(y) = x = 3 - \frac{4}{y} = \frac{3y-4}{y}$ (domain $y \neq 0$)

a $\mathrm{f}^{-1}(3) = \frac{5}{3}$

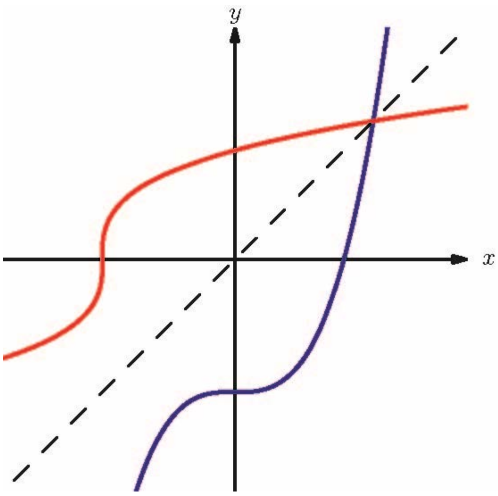
b $\mathrm{f}^{-1}(x) = \frac{3x-4}{x}$ (domain $x \neq 0$)

19 Let $y = \mathrm{f}(x) = 3\mathrm{e}^{5x}$ (range $\mathrm{f}(x) > 0$)

Then $\mathrm{f}^{-1}(y) = x = \frac{1}{5}\ln\left(\frac{y}{3}\right)$

Changing variables: $\mathrm{f}^{-1}(x) = \frac{1}{3}\ln\left(\frac{x}{3}\right)$, domain $x > 0$

20 a

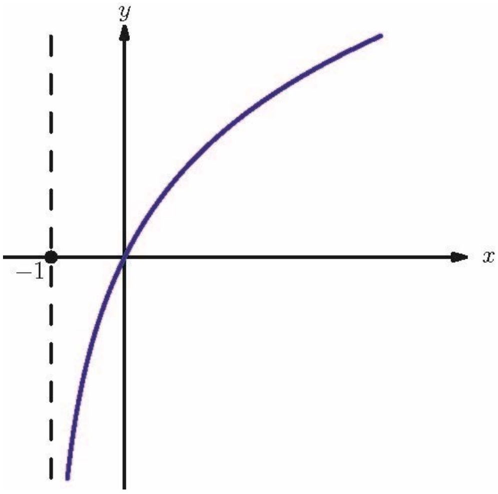


b Let $y = \mathrm{f}(x) = \frac{1}{5}x^3 - 3$ (range $\mathrm{f}(x) \in \mathbb{R}$)

Then $\mathrm{f}^{-1}(y) = x = \sqrt[3]{5(y+3)}$

Changing variables: $\mathrm{f}^{-1}(x) = \sqrt[3]{5x+15}$

21 a



b Let $x = \mathrm{f}(y) = e^{\frac{1}{2}y} - 1$ (range $\mathrm{f}(y) > -1$)

Then $x + 1 = e^{\frac{1}{2}y}$

$$\ln(x+1) = \frac{1}{2}y$$
$$y = 2\ln(x+1)$$
$$\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\big(f(y)\big) = y = 2\ln(x+1)$$

The domain of $\mathrm{f}^{-1}(x)$ is the range of $\mathrm{f}(x)$, in this instance $\{-1 < x\}$.

22 a Let $x = \mathrm{f}(y) = \frac{y^2+1}{y^2-4}$ (domain $y > 2$ so range $f(y) > 1$)

Rearranging:

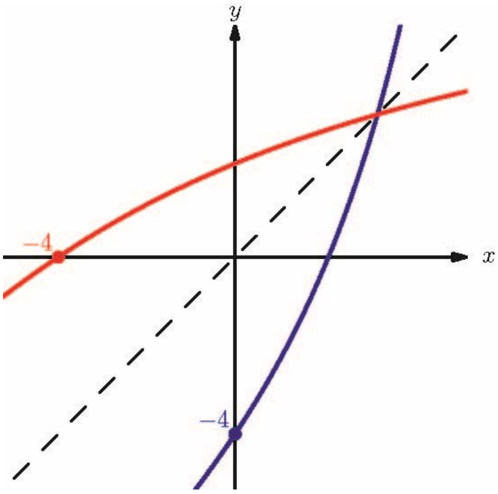
$$x = \frac{y^2+1}{y^2-4}$$
$$y^2x - 4x = y^2 + 1$$
$$y^2x - y^2 = 4x + 1$$
$$y^2(x-1) = 4x + 1$$
$$y^2 = \frac{4x+1}{x-1} \Leftrightarrow y = \sqrt{\frac{4x+1}{x-1}} \qquad (\text{as } 2 < y)$$
$$\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\big(\mathrm{f}(y)\big) = y = \sqrt{\frac{4x+1}{x-1}} \qquad \{x: 1 < x\}$$

b Range of $\mathrm{f}^{-1}(x)$ is the domain of $\mathrm{f}(x)$: $\mathrm{f}^{-1}(x) > 2$

23 $(\mathrm{g}^{-1} \circ \mathrm{f}^{-1})(x) = 4$ so

$$x = (\mathrm{f} \circ \mathrm{g})(4)$$
$$= \mathrm{f}\big(\mathrm{g}(4)\big)$$
$$= \frac{1}{4 + \sqrt{2 \times 4 + 1}}$$
$$= \frac{1}{7}$$

24 a



b From GDC, the intersection of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ is also the intersection with $y = x$.

$$\mathrm{e}^{\frac{x}{2}} + x - 5 = x$$
$$\mathrm{e}^{\frac{x}{2}} = 5$$
$$x = 2\ln 5$$

25 a Since $\mathrm{f}(x) = \mathrm{f}(-x)$, $\mathrm{f}(x)$ is one-to-one for $x \leqslant 0$ so the largest possible value is $a = 0$

b $\mathrm{f}(x) = x^2 + 3$ for $x \leqslant 0$ has range $\mathrm{f}(x) \geqslant 3$

Let $y = \mathrm{f}(x)$ so $\mathrm{f}^{-1}(y) = x = -\sqrt{y-3}$ for $y \geqslant 3$

Changing variables: $\mathrm{f}^{-1}(x) = -\sqrt{x-3}$ for $x \geqslant 3$

26 a Since $\mathrm{g}(5 + x) = \mathrm{g}(5 - x)$, $\mathrm{g}(x)$ is one-to-one for $x \leqslant 5$ so the largest possible value is $k = 5$.

b $\mathrm{g}(x) = 9(x-5)^2$ for $x \leqslant 5$ has range $\mathrm{g}(x) \geqslant 0$

Let $y = \mathrm{g}(x)$ so $\mathrm{g}^{-1}(y) = x = 5 - \frac{\sqrt{y}}{3}$ for $y \geqslant 0$

Changing variables: $\mathrm{g}^{-1}(x) = 5 - \frac{\sqrt{x}}{3}$ for $x \geqslant 0$.

27 $\mathrm{f}(x) = \frac{(ax+3)}{x-4}$ has domain $x \neq 4$

If $\mathrm{f}(x) \equiv \mathrm{f}^{-1}(x)$ then $(\mathrm{f}\circ\mathrm{f})(x) \equiv x$.

$$(\mathrm{f}\circ\mathrm{f})(x) = \frac{a\left(\frac{ax+3}{x-4}\right) + 3}{\frac{ax+3}{x-4} - 4}$$

Multiplying numerator and denominator by $(x-4)$ to simplify:

$$(\mathrm{f}\circ\mathrm{f})(x) = \frac{a(ax+3) + 3(x-4)}{(ax+3) - 4(x-4)}$$
$$= \frac{x(a^2+3) + 3a - 12}{(a-4)x + 19} \equiv x$$

$$x(a^2+3) + 3a - 12 \equiv x^2(a-4) + 19x$$

Comparing coefficients: If $a = 4$ then the x^2 term disappears, as is necessary, and the equivalence is met:

$$19x + 0 \equiv 0x^2 + 19x$$

So for $(x) \equiv \mathrm{f}^{-1}(x)$, $a = 4$

Tip: An alternative solution involves finding $\mathrm{f}^{-1}(x)$ in terms of a and then requiring the two function expressions to be equivalent

Mixed Practice

1 **a** Let $y = \mathrm{f}(x) = 3x - 1$

Then $y + 1 = 3x$

So $x = \mathrm{f}^{-1}(y) = \frac{1}{3}(y + 1)$

Changing variables: $\mathrm{f}^{-1}(x) = \frac{(x+1)}{3}$

b

$$\begin{aligned}(\mathrm{f} \circ \mathrm{f}^{-1})(x) &= \mathrm{f}\left(\mathrm{f}^{-1}(x)\right) \\ &= \mathrm{f}\left(\frac{1}{3}(x+1)\right) \\ &= 3\left(\frac{1}{3}(x+1)\right) - 1 \\ &= x\end{aligned}$$

2 **a** Square root must take non-negative arguments, so the greatest possible domain of $\mathrm{h}(x)$ is $x \leqslant 5$.

$a = 5$.

b

$$\begin{aligned}\sqrt{5 - x} &= 3 \\ 5 - x &= 9 \\ x &= -4\end{aligned}$$

3 $\mathrm{e}^{3x} = 4$ so $x = \frac{1}{3}\ln 4 = 0.462$

4 **a**

$$\begin{aligned}(\mathrm{f} \circ \mathrm{f})(2) &= \mathrm{f}\left(\mathrm{f}(2)\right) \\ &= \mathrm{f}(0) \\ &= 3\end{aligned}$$

b $\mathrm{f}^{-1}(4) = 1$

5

$$\begin{aligned}(\mathrm{g} \circ \mathrm{f})(x) &= \mathrm{g}\left(\mathrm{f}(x)\right) \\ &= \mathrm{e}^{2(x+3)} = 1\end{aligned}$$

$$2(x + 3) = \ln 1 = 0$$

$$x = -36.$$

6 **a** $\ln x$ takes only positive arguments for range in $\mathbb{R}$ so the domain of $g(x)$ is restricted to $x > 2$

b Let $y = \mathrm{g}(x) = 3\ln(x-2)$ with domain $x > 2$ and range $\mathbb{R}$

$$\frac{y}{3} = \ln(x-2)$$

$$x = \mathrm{g}^{-1}(y) = 2 + \mathrm{e}^{\frac{y}{3}} \text{ with domain}$$

Changing variables: $\mathrm{g}^{-1}(x) = 2 + \mathrm{e}^{\frac{x}{3}}$ with domain $\mathbb{R}$ and range $\mathrm{g}^{-1}(x) > 2$.

7 By inspection, $\mathrm{g}^{-1}(x) = \sqrt[3]{x}$ and $\mathrm{f}^{-1}(x) = \frac{x-1}{3}$.

$$\begin{aligned}(\mathrm{f}\circ \mathrm{g})^{-1}(x) &= (\mathrm{g}^{-1}\circ \mathrm{f}^{-1})(x)\\ &= \sqrt[3]{\frac{x-1}{3}}\end{aligned}$$

8 **a**

$$\begin{aligned}(\mathrm{f}\circ \mathrm{g})(x) &= \mathrm{f}\big(\mathrm{g}(x)\big)\\ &= 2(x^3) + 3\\ &= 2x^3 + 3\end{aligned}$$

b $2x^3 + 3 = 0$ so $x = \sqrt[3]{-\frac{3}{2}} = -1.14$ (to 3 s.f.)

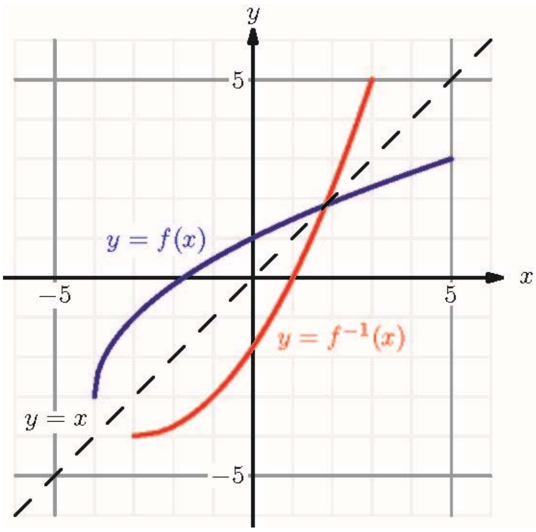
9 **a** From the graph:

i $\mathrm{f}(-3) = -1$

ii $\mathrm{f}^{-1}(1) = 0$

b The domain of f^{-1} is the range of f: $[-3,3]$

c The graph of f^{-1} is the graph of f under a reflection through the line $y = x$



10 a

$$\begin{aligned}(f \circ g)(x) &= f\big(g(x)\big)\\ &= \left(\frac{1}{x-1}\right) - 2\\ &= \frac{1-2(x-1)}{x-1}\\ &= \frac{3-2x}{x-1}\end{aligned}$$

The domain of $g(x)$ is $x \neq 1$ and the domain of $f(x)$ is unrestricted so the domain of $(f \circ g)(x)$ is $x \neq 1$.

b Let $y = (f \circ g)(x) = \frac{1}{x-1} - 2$

Then $y + 2 = \frac{1}{x-1}$

Taking reciprocals on both sides: $x - 1 = \frac{1}{y+2}$

Then $x = (f \circ g)^{-1}(y) = 1 + \frac{1}{y+2}$

Changing variables: $(f \circ g)^{-1}(x) = 1 + \frac{1}{x+2}$

By inspection, $f^{-1}(x) = x + 2$ and $g^{-1}(x) = 1 + \frac{1}{x}$

$$\begin{aligned}(g^{-1} \circ f^{-1})(x) &= g^{-1}\big(f^{-1}(x)\big)\\ &= 1 + \frac{1}{x+2}\end{aligned}$$

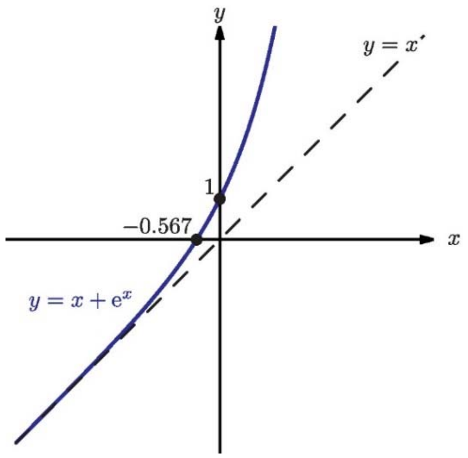
This demonstrates that $(f \circ g)^{-1}(x) \equiv (g^{-1} \circ f^{-1})(x)$

11 Let $y = f(x) = \frac{3x-1}{x+4}$, which has domain $x \neq -4$ and range $f(x) \neq 3$

$$\begin{aligned}y(x+4) &= 3x - 1\\ xy - 3x &= -4y - 1\\ x(3-y) &= 4y + 1\\ x = f^{-1}(y) &= \frac{4y+1}{3-y}\end{aligned}$$

Changing variables: $f^{-1}(x) = \frac{4x+1}{3-x}$ with domain $x \neq 3$ and range $f^{-1}(x) \neq -4$

12 a



The graph shows that $g(x)$ is one-to-one (for no value k are there two distinct values x_1 and x_2 such that $g(x_1) = g(x_2) = k$; this is seen in the graph as the quality that any horizontal line crosses the curve in at most one place). Therefore $g^{-1}(x)$ exists. Since the range of $g(x)$ is $\mathbb{R}$, the inverse function has domain $\mathbb{R}$.

b $g^{-1}(x) = 2$ so $x = g(2) = 2 + e^2 = 9.39$

13 a

$$
\begin{aligned}
(g \circ f)\left(\frac{1}{4}\right) &= g\left(f\left(\frac{1}{4}\right)\right) \\
&= g\left(\frac{1}{2}\right) \\
&= 3
\end{aligned}
$$

b $(f^{-1} \circ g)(x) = \frac{1}{3}$ so

$$
\begin{aligned}
x &= g^{-1}\left(f\left(\frac{1}{3}\right)\right) \\
&= \log_9\left(3^{-\frac{1}{2}}\right) \\
&= \log_9\left(9^{-\left(\frac{1}{4}\right)}\right) \\
&= -\frac{1}{4}
\end{aligned}
$$

14 a $h(x) = \ln(x-2)$ for $x > 2$ so $h^{-1}(x) = 2 + e^x$ with range $h^{-1}(x) > 2$

b $(g \circ h)(x) = e^{\ln(x-2)} = x - 2$

15 a The graph is symmetrical about $x = -\frac{1}{2}$ so the graph is one-to-one for $x \leqslant -\frac{1}{2}$

$$a = -\frac{1}{2}$$

b Let $y = f(x) = (2x+1)^2$

Then $x = f^{-1}(y) = \frac{-\sqrt{y}-1}{2}$ (selecting the negative root since the domain of f is to the negative side of $x = -\frac{1}{2}$)

Changing variables: $f^{-1}(x) = \frac{-\sqrt{x}-1}{2}$

16 a $\mathrm{f}(5) = 5^2 + 3 = 28$

b $\mathrm{gf}(x) = 12 - (x^2 + 3) = 9 - x^2$

c **i** The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections through the line $y = x$.

ii Let $y = \mathrm{f}(x) = x^2 + 3$

Rearranging gives $x = \mathrm{f}^{-1}(y) = \sqrt{y - 3}$

Changing variables gives $\mathrm{f}^{-1}(x) = \sqrt{x - 3}$

iii The domain of $\mathrm{f}(x)$ is $x \geqslant 1$ and so the range of $\mathrm{f}^{-1}(x)$ is $\mathrm{f}^{-1}(x) \geqslant 1$

17 a **i** $\mathrm{f}(7) = 3(7) + 1 = 22$

ii

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \frac{3(x+4)}{x-1} + 1 \\ &= \frac{(3x+12) + (x-1)}{x-1} \\ &= \frac{4x+11}{x-1}\end{aligned}$$

iii $(\mathrm{f} \circ \mathrm{f})(x) = 3(3x + 1) + 1 = 9x + 4$

b The domain of f is ℝ and f is a linear transformation, so f has range ℝ.

Since the domain of g requires $x \neq 1$, $(\mathrm{g} \circ \mathrm{f})(x)$ is not well-defined across the whole domain of f.

c **i**

$$\begin{aligned}(\mathrm{g} \circ \mathrm{g})(x) &= \frac{\frac{x+4}{x-1} + 4}{\frac{x+4}{x-1} - 1} \\ &= \frac{x + 4 + 4(x-1)}{x + 4 - (x-1)} \\ &= \frac{5x}{5} \\ &= x\end{aligned}$$

Since $(\mathrm{g} \circ \mathrm{g})(x) \equiv x$, it follows that $\mathrm{g}(x) \equiv \mathrm{g}^{-1}(x)$

ii Since $\mathrm{g}(x)$ is self-inverse, the range and domain are the same, so the range of $\mathrm{g}(x)$ is $\mathrm{g}(x) \neq 1$.

18 a $\mathrm{f}(x) = \sqrt{x - 5}$ so $\mathrm{f}^{-1}(x) = x^2 + 5$

$\mathrm{f}^{-1}(2) = 9$

b

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g}^{-1})(3) &= \mathrm{f}(\mathrm{g}^{-1}(3)) \\ &= \mathrm{f}(30) \\ &= \sqrt{30 - 5} \\ &= 5\end{aligned}$$

19 a Let $y = \mathrm{f}(x) = 3x - 2$

Then $\mathrm{f}^{-1}(y) = x = \frac{y+2}{3}$

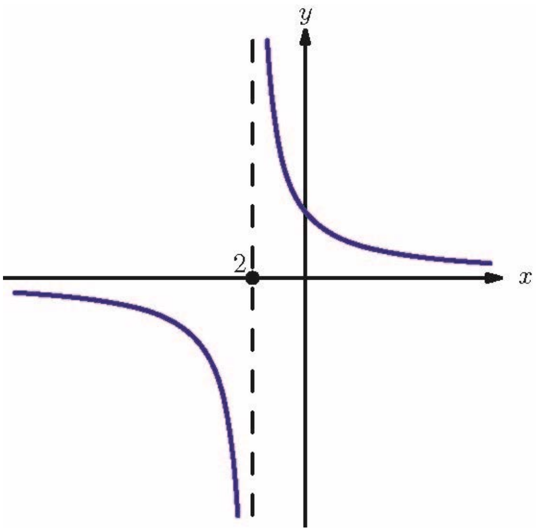
Changing variables: $\mathrm{f}^{-1}(x) = \frac{x+2}{3}$

b

$$(\mathrm{g} \circ \mathrm{f}^{-1})(x) = \mathrm{g}(\mathrm{f}^{-1}(x))$$
$$= \frac{5}{3\mathrm{f}^{-1}(x)}$$
$$= \frac{5}{x+2}$$

c i When $x = 0$, $h(x) = \frac{5}{2} = 2.5$

ii



d i When $\mathrm{h}^{-1}(x) = 0$, $x = \mathrm{h}(0) = \frac{5}{2} = 2.5$

ii The horizontal asymptote of $y = \mathrm{h}(x)$ is $y = 0$ so the vertical asymptote of $y = \mathrm{h}^{-1}(x)$ is $x = 0$

e $\mathrm{h}^{-1}(a) = 3$ so $a = \mathrm{h}(3) = \frac{5}{3+2} = 1$

20

$$(\mathrm{f} \circ \mathrm{g})(x) = \mathrm{e}^{2(\ln(x-2))}$$
$$= \mathrm{e}^{\ln((x-2)^2)}$$
$$= (x-2)^2$$

Let $y = (\mathrm{f} \circ \mathrm{g})(x) = (x-2)^2$

Then $x = (\mathrm{f} \circ \mathrm{g})^{-1}(y) = 2 + \sqrt{y}$

Changing variables: $(\mathrm{f} \circ \mathrm{g})^{-1}(x) = 2 + \sqrt{x}$

$\mathrm{g}^{-1}(x) = 2 + e^x$ and $\mathrm{f}^{-1}(x) = \frac{1}{2}\ln x = \ln(\sqrt{x})$

$\mathrm{g}^{-1}(\mathrm{f}^{-1}(x)) = 2 + \mathrm{e}^{\ln\sqrt{x}} = 2 + \sqrt{x}$

So $(\mathrm{f} \circ \mathrm{g})^{-1}(x) \equiv \mathrm{g}^{-1}(\mathrm{f}^{-1}(x))$

21 $f^{-1}(g^{-1}(x)) = 9$ so,

$$\begin{aligned} x &= g(f(9)) \\ &= \frac{1}{1+\sqrt{9}} + 7 \\ &= \frac{29}{4} \end{aligned}$$

22 a $f(x) = 2 + x - x^3$ has local maximum at $x = 0.577$ (GDC) so $f(x)$ is one-to-one for $x \geqslant 0.577$

b The graph of $y = f(x)$ is the reflection of the graph of $y = f^{-1}(x)$ in the line $y = x$.

c By **b**, the intersection of $y = f(x)$ and $y = f^{-1}(x)$ lies on $y = x$ so $f(x) = x$

Substituting: $2 + x - x^3 = x$

$$x = \sqrt[3]{2}$$

23 a i

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= \frac{1}{2x+3} \end{aligned}$$

Domain of $g(x)$ is $x \neq 0$ so require $f(x) \neq 0$: $x \neq -\frac{3}{2}$

ii

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= \frac{2}{x} + 3 \end{aligned}$$

Domain of $f(x)$ is $\mathbb{R}$ so the only restriction is the domain of $g(x)$: $x \neq 0$.

b If $f(x) = (g^{-1} \circ f \circ g)(x)$ then $(g \circ f)(x) = (f \circ g)(x)$

$$\frac{1}{2x+3} = \frac{2}{x} + 3 = \frac{2+3x}{x}$$

Rearranging: $x = (2x+3)(2+3x)$

$$\begin{aligned} &6x^2 + 12x + 6 = 0 \\ &x^2 + 2x + 1 = 0 \\ &(x+1)^2 = 0 \\ &x = -1 \end{aligned}$$

Substituting into $y = f(x)$ gives point of intersection $(-1, 1)$.

15 Analysis and approaches: Quadratics

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 15A

38 a y-intercept $c = -24$

b $y = (x-8)(x+3)$

Roots at $a = -3$ and $b = 8$

39 a Roots at -2 and 3 so $p = 2, q = 3$

b y-intercept $-apq = 18$ so $a = -3$

c

$$\begin{aligned} y &= -3(x+2)(x-3) \\ &= -3x^2 + 3x + 18 \end{aligned}$$

40 a

$$\begin{aligned} x^2 - 5x + 1 &= \left(x - \frac{5}{2}\right)^2 + 1 - \left(\frac{5}{2}\right)^2 \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{21}{4} \end{aligned}$$

b Line of symmetry is $x = \frac{5}{2}$

41 a y-intercept is at $(0,23)$

b

$$\begin{aligned} y &= 2x^2 + 12x + 23 \\ &= 2(x^2 + 6x) + 23 \\ &= 2[(x+3)^2 - 9] + 23 \\ &= 2(x+3)^2 + 5 \end{aligned}$$

c Vertex is at $(-3, 5)$.

42 a Vertex is at $(h, k) = (2,7)$

So $h = 2, k = 7$

b y-intercept $-13 = ah^2 + k = 4a + 7$ so $a = -5$

c

$$\begin{aligned} y &= -5(x-2)^2 + 7 \\ &= -5x^2 + 20x - 13 \end{aligned}$$

43 a

$$\begin{aligned} 3x^2 + 6x - 2 &= 3(x^2 + 2x) - 2 \\ &= 3[(x+1)^2 - 1] - 2 \\ &= 3(x+1)^2 - 5 \end{aligned}$$

b Minimum value is -5

44 a

$$\begin{aligned} 5x^2 - 10x + 3 &= 5(x^2 - 2x) + 3 \\ &= 5[(x-1)^2 - 1] + 3 \\ &= 5(x-1)^2 - 2 \end{aligned}$$

b Positive quadratic with minimum at $(1, -2)$

Range is $f(x) \geqslant -2$

Exercise 15B

29 a $x^2 - x - 12 = (x-4)(x+3)$

b $(x-4)(x+3) = 0$

So $x - 4 = 0$ or $x + 3 = 0$

$x = 4$ or -3

30 a

$$\begin{aligned} x^2 - 6x - 2 &= (x-3)^2 - 9 - 2 \\ &= (x-3)^2 - 11 \end{aligned}$$

b

$$\begin{aligned} (x-3)^2 - 11 &= 0 \\ x &= 3 \pm \sqrt{11} \end{aligned}$$

31 a $x^2 + 10x = (x+5)^2 - 25$

b

$$\begin{aligned} (x+5)^2 - 25 &= 7 \\ x &= -5 \pm \sqrt{32} \\ &= -5 \pm 4\sqrt{2} \end{aligned}$$

32

$$\begin{aligned} x^2 - 6x + 5 &< 0 \\ (x-1)(x-5) &< 0 \end{aligned}$$

Positive quadratic is less than zero between the roots

$1 < x < 5$

33

$$\begin{aligned} x^2 - b^2 &\geq 0 \\ (x+b)(x-b) &\geq 0 \end{aligned}$$

Positive quadratic is greater than zero outside the roots

$x \leqslant -b$ or $x \geqslant b$

34

$$x^2 - 4x - 21 = 0$$
$$(x - 7)(x + 3) = 0$$

$x = 7$ or $x = -3$

35 $A = x^2$ and $P = 4x$

Require $x^2 > 4x$ and (in context) $x > 0$

Since x is positive, dividing by x will not change the solution set.

$x > 4$.

Tip: Ordinarily, cancelling an equation by an expression in x can only be done if you also allow the possibility that the expression equals zero (so cancelling a factor of $(x - a)$ on both sides of an equation should only be done if you note the possible solution $x = a$). In this case, by stating $x > 0$ we can then cancel the factor x and avoid solving the quadratic inequality altogether.

36

$$3x^2 - 4x - 1 = 0$$
$$3\left(x^2 - \frac{4}{3}x\right) - 1 = 0$$
$$3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9}\right) - 1 = 0$$
$$3\left(x - \frac{2}{3}\right)^2 = \frac{7}{3}$$
$$x = \frac{2 \pm \sqrt{7}}{3}$$

37

$$x^2 - 2xp + p^2 > q^2$$
$$(x - p)^2 - q^2 > 0$$
$$(x - p - q)(x - p + q) > 0$$

Positive quadratic is greater than zero outside the roots

$x < p - q$ or $x > p + q$

38 a The two numbers are x and $x + 3$.

If their product is 40 then $x(x + 3) = 40$

$$x^2 + 3x - 40 = 0$$

b $(x + 8)(x - 5) = 0$

$x = -8$ (and the larger number is -5) or $x = 5$ (and the larger number is 8).

39 a If the width is x then the length is $x + 5$

$$\text{Area} = x(x + 5) = 24$$
$$x^2 + 5x - 24 = 0$$

b $(x + 8)(x - 3) = 0$

$x = -8$ (reject due to context) or $x = 3$

The perimeter is $2x + 2(x + 5) = 22$ cm

40 Substituting: $x + 2 = x^2 - 4$

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

$x = 3$ or -2

Coordinates of intersection are $(-2, 0)$ and $(3, 5)$

41

$$x^2 - 6x + 9 \le 0$$
$$(x-3)^2 \le 0$$

A square is only non-positive when it is zero. The inequality has solution $x = 3$.

42 a

$$10t - 5t^2 = 0$$
$$5t(2-t) = 0$$

The ball is at ground level at $t = 0$ (when it is hit) and $t = 2$ (when it lands).

The ball lands 2 seconds after being hit.

b

$$\begin{aligned} h &= -5(t^2 - 2t) \\ &= -5((t-1)^2 - 1) \\ &= -5(t-1)^2 + 5 \end{aligned}$$

Maximum height is 5 m at $t = 1$ second

c

$$\begin{aligned} h &> 1 \\ -5(t-1)^2 + 5 &> 1 \\ -5(t-1)^2 &> -4 \\ (t-1)^2 &< \frac{4}{5} \end{aligned}$$

Roots of $(t-1)^2 = \frac{4}{5}$ are $t = 1 \pm \frac{2}{\sqrt{5}}$

Positive quadratic is less than zero between the roots

The ball is > 1 m off the ground for $1 - \frac{2}{\sqrt{5}} < t < 1 + \frac{2}{\sqrt{5}}$,

so for $\frac{4}{\sqrt{5}}$ seconds $\approx \frac{4\sqrt{5}}{5} \approx 1.79$ seconds.

43 Substituting:

$$\begin{aligned} x^2 + (x-3)^2 &= 16 \\ 2x^2 - 6x + 9 - 16 &= 0 \\ 2(x^2 - 3x) - 7 &= 0 \\ 2\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right) - 7 &= 0 \\ 2\left(x - \frac{3}{2}\right)^2 - \frac{23}{2} &= 0 \\ x &= \frac{3}{2} \pm \frac{\sqrt{23}}{2} \end{aligned}$$

Coordinates are therefore $\left(\frac{3+\sqrt{23}}{2}, \frac{-3+\sqrt{23}}{2}\right), \left(\frac{3-\sqrt{23}}{2}, \frac{-3-\sqrt{23}}{2}\right)$.

44

$$\begin{aligned}\frac{6}{x-1} &= 3x+4\\(3x+4)(x-1) &= 6\\3x^2+x-10 &= 0\\(3x-5)(x+2) &= 0\\x &= \frac{5}{3} \text{ or } -2\end{aligned}$$

45 $4 < x^2$ and $x^2 < 9$, so

$(x < -2$ or $x > 2)$ and $(-3 < x < 3)$

The coincident regions of the two requirements are $-3 < x < -2$ or $2 < x < 3$

46 $x^2 - 3x \leqslant 0$ and $x^2 - 5x + 4 < 0$

$x(x-3) \leqslant 0$ and $(x-1)(x-4) < 0$

Positive quadratics are less than zero between the roots.

$0 \leqslant x \leqslant 3$ and $1 < x < 4$

The coincident region of the two requirements is $1 < x \leqslant 3$

47

$$\begin{aligned}y^2 - 3xy + 2x^2 &= 0\\(y-x)(y-2x) &= 0\\y &= x \text{ or } 2x\end{aligned}$$

48

$$\begin{aligned}x^2 - x - 2kx + k + k^2 &= 0\\(x-k)^2 - (x-k) &= 0\\(x-k)(x-k-1) &= 0\\x &= k \text{ or } k+1\end{aligned}$$

49

$$\begin{aligned}x^2 + y^2 - 2y + 1 &\leq 0\\(y-1)^2 &\leq -x^2\end{aligned}$$

Since $-x^2 \leq 0$ and $(y-1)^2 \geq 0$, the only possible solution is at $x = 0, y = 1$.

Exercise 15C

10 **a** Repeated root when $\Delta = 0$

$$\begin{aligned}8^2 - 4(3)(k) &= 0\\12k &= 64\\k &= \frac{16}{3}\end{aligned}$$

b $x = -\frac{8}{2(3)} = -\frac{4}{3}$

11 Repeated root when $\Delta = 0$

$$k^2 - 4(5)(20) = 0$$
$$k^2 = 400$$
$$k = \pm 20$$

12 Distinct real roots when $\Delta > 0$

$$(-3)^2 - 4(k)(2) > 0$$
$$9 - 8k > 0$$
$$k < \frac{9}{8}$$

13 No real roots when $\Delta < 0$

$$(-5)^2 - 4(3)(2k) < 0$$
$$25 - 24k < 0$$
$$k > \frac{25}{24}$$

14 **a** Repeated root when $\Delta = 0$

$$k^2 - 4(2)(k-2) = 0$$
$$k^2 - 8k + 16 = 0$$

b

$$(k-4)^2 = 0$$
$$k = 4$$

15 **a** Repeated root when $\Delta = 0$

$$(k+3)^2 - 4(k)(-1) = 0$$
$$k^2 + 10k + 9 = 0$$

b

$$(k+1)(k+9) = 0$$

$$k = -1 \text{ or } -9$$

16 Distinct real roots when $\Delta > 0$

$$k^2 - 4(2)(2) > 0$$
$$k^2 - 16 > 0$$

$$k < -4 \text{ or } k > 4$$

17 At least one real root when $\Delta \geqslant 0$

$$a^2 - 4(a)(3) \geqslant 0$$
$$a^2 - 12a \geqslant 0$$
$$a(a-12) \geqslant 0$$

Positive quadratic is greater than zero outside the roots

$$a \leqslant 0 \text{ or } a \geqslant 12$$

18 Repeated root when $\Delta = 0$

$$a^2 - 4(1)(9) = 0$$
$$a^2 = 36$$
$$a = \pm 6$$

19 No real roots when $\Delta < 0$

$$(-3)^2 - 4(2)(c) < 0$$
$$9 - 8c < 0$$
$$c > \frac{9}{8}$$

20 No real roots when $\Delta < 0$

$$b^2 - 4(1)(2b) < 0$$
$$b^2 - 8b < 0$$
$$b(b - 8) < 0$$

Positive quadratic is less than zero between the roots

$$0 < b < 8$$

21 Distinct real roots when $\Delta > 0$

$$(a + 1)^2 - 4(3)(4) > 0$$
$$(a + 1)^2 > 48$$

$$a + 1 < -\sqrt{48} \text{ or } a + 1 > \sqrt{48}$$

$$a < -1 - 4\sqrt{3} \text{ or } a > -1 + 4\sqrt{3}$$

22 No real roots when $\Delta < 0$

$$(3 - k)^2 - 4(k)(k) < 0$$
$$(3 - k)^2 - (2k)^2 < 0$$
$$(3 - 3k)(3 + k) < 0$$

Negative quadratic is less than zero outside the roots

$$k < -3 \text{ or } k > 1$$

23 $x = 0$ is not a solution, so we can multiply by x.

$$x^2 - 3x + a = 0$$

Real roots when $\Delta \geqslant 0$

$$(-3)^2 - 4(1)(a) \geqslant 0$$
$$a \leqslant \frac{9}{4}$$

24 For the graph to be always positive, the quadratic must be positive with no real roots.

No real roots when $\Delta < 0$

$$a^2 - 4(1)(4) < 0$$
$$a^2 - 16 < 0$$
$$(a + 4)(a - 4) < 0$$

Positive quadratic is less than zero between the roots

$$-4 < a < 4$$

25 For the graph to lie entirely below the x-axis, the quadratic must be negative with no real roots.

Negative quadratic: $a < 0$ (1)

No real roots when $\Delta < 0$

$$\begin{aligned} 4^2 - 4(a)(a-3) &< 0 \\ 4a^2 - 12a - 16 &> 0 \\ 4(a^2 - 3a - 4) &< 0 \\ (a+1)(a-4) &< 0 \end{aligned}$$

Positive quadratic is less than zero between the roots

$-1 < a < 4$ (2)

Taking these two conditions together $-1 < a < 0$

26

Tip: This could be answered using calculus and gradients, but it is elegant to solve by considering the number of intersection points.

If the line is tangent to the curve then there will be a single (repeated) point of intersection.

$$\begin{aligned} x + k &= x^2 + 7 \\ x^2 - x + 7 - k &= 0 \end{aligned}$$

Single repeated root when $\Delta = 0$

$$\begin{aligned} (-1)^2 - 4(1)(7-k) &= 0 \\ 7 - k &= \frac{1}{4} \\ k &= \frac{27}{4} \end{aligned}$$

27 Intersection occurs when $x^2 + (2x+k)^2 = 5$

$$5x^2 + 4kx + k^2 - 5 = 0$$

Real roots when $\Delta \geqslant 0$. When $\Delta = 0$ the line would be tangent to the circle (non properly intersecting), so restrict to $\Delta > 0$ for intersections.

$$\begin{aligned} (4k)^2 - 4(5)(k^2 - 5) &> 0 \\ -4k^2 + 100 &> 0 \\ k^2 &< 25 \\ -5 < k &< 5 \end{aligned}$$

28 Distinct real roots when $\Delta > 0$

$$\Delta = (a+2)^2 - 4(3)(-2) = (a+2)^2 + 24$$

This must be greater than zero for all real a, and hence there must always be two real roots to the original quadratic.

29 For $x^2 + bx + 9 = 0$ to have no solutions, the quadratic $y = x^2 + bx + 9$ must lie entirely above the x-axis; that is, it has no real roots. Let the discriminant of this quadratic be Δ_1.

No real roots when $\Delta_1 < 0$

$$\Delta_1 = b^2 - 4(1)(9) = b^2 - 36 < 0$$

Therefore $-6 < b < 6$

Let the discriminant of $y = x^2 + 9x + b$ be Δ_2

$$\Delta_2 = (9)^2 - 4(1)(b) = 81 - 4b$$

Given the values of b found above,

$$57 < \Delta_2 < 105$$

Therefore $\Delta_2 > 0$ and hence the second quadratic always has two distinct real roots.

Mixed Practice

1 **a** $x^2 + 7x - 18 = (x + 9)(x - 2)$

b $(-9,0)$ and $(2,0)$

2

$$\begin{aligned} 4x^2 - 9 &= 0 \\ (2x + 3)(2x - 3) &= 0 \\ x &= \pm\frac{2}{3} \end{aligned}$$

Coordinates of the roots are $\left(\pm\frac{2}{3}, 0\right)$.

3 **a**

$$\begin{aligned} -2x^2 + 8x - 3 &= -2(x^2 - 4x) - 3 \\ &= -2((x - 2)^2 - 4) - 3 \\ &= -2(x - 2)^2 + 5 \end{aligned}$$

b Vertex is $(2, 5)$.

4 **a** Positive quadratic with negative y-intercept: Graph 3.

b Positive quadratic with positive y-intercept: Graph 1.

c Negative quadratic: Graph 2.

5 **a** Vertex is at $(3, -2)$ so $a = 3, b = 2$

b

$$\begin{aligned} (x - 3)^2 &= 2 \\ x &= 3 \pm \sqrt{2} \end{aligned}$$

Coordinates of the roots are $(3 \pm \sqrt{2}, 0)$

6 **a** $y = (x - 5)(x + 3)$

Roots are at $(-3, 0)$ and $(5, 0)$

b $y = (x - 1)^2 - 16$

Line of symmetry is $x = 1$

c Minimum value is -16, the y-coordinate of the vertex.

7 **a**

$$\begin{aligned} 2x^2 - x - 5 &= 0 \\ \Delta &= (-1)^2 - 4(2)(-5) = 41 \end{aligned}$$

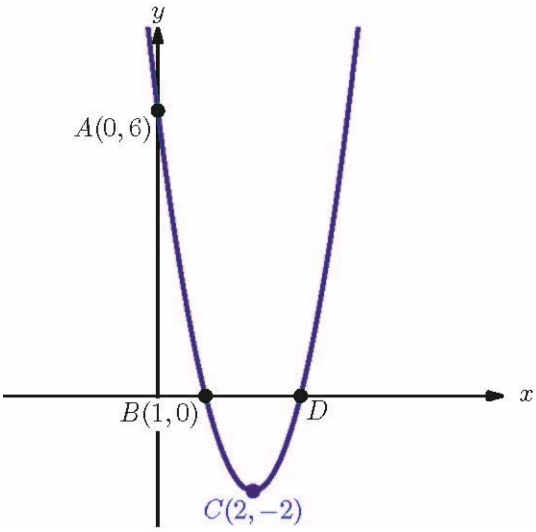
b $\Delta > 0$ so there are two distinct real roots.

8 No real roots when $\Delta < 0$

$$4^2 - 4(k)(5) < 0$$
$$k > \frac{4}{5}$$

9 a Axis of symmetry is vertical through the vertex: $x = 2$

b



c By symmetry, if one root is 1 unit to the left of the axis, the other must be one unit to the right, so D has x-coordinate 3.

10 a

$$\begin{aligned}3x^2 - 6x + 10 &= 3(x^2 - 2x) + 10\\ &= 3((x-1)^2 - 1) + 10\\ &= 3(x-1)^2 + 7\end{aligned}$$

b Positive quadratic with vertex at $(1, 7)$ has range $f(x) \geqslant 7$

11 a

$$\begin{aligned}-x^2 - 4x + 3 &= -(x^2 + 4x) + 3\\ &= -((x+2)^2 - 4) + 3\\ &= -(x+2)^2 + 7\end{aligned}$$

b Negative quadratic with vertex at $(-2, 7)$ has range $f(x) \leqslant 7$

12 Equal roots when $\Delta = 0$

$$\begin{aligned}\left(-(k+2)\right)^2 - 4(2)(3) &= 0\\ (k+2)^2 &= 24\\ k &= -2 \pm \sqrt{24} = -2 \pm 2\sqrt{6}\end{aligned}$$

13 a Let the length be y.

Then $2x + 2y = 12$ so $y = 6 - x$

Then the area is given by $A = xy = x(6 - x) = 6x - x^2$

b

$$\begin{aligned} A &= -(x^2 - 6x) \\ &= -((x-3)^2 - 9) \\ &= 9 - (x-3)^2 \end{aligned}$$

So the maximum area is 9 cm^2

14 At least one real root when $\Delta \geqslant 0$

$$\begin{aligned} (-k)^2 - 4(3)(6) &\geqslant 0 \\ k^2 &\geqslant 72 \\ -6\sqrt{2} \leqslant k &\leqslant 6\sqrt{2} \end{aligned}$$

15 a i Completing the square:

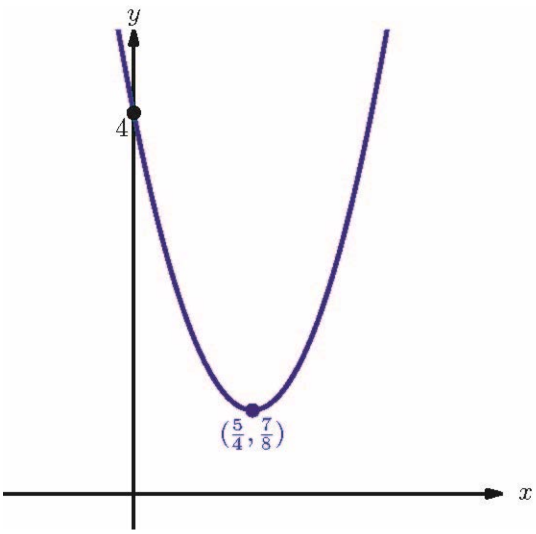
$$\begin{aligned} \mathrm{f}(x) &= 2\left(x^2 + \frac{k}{2}x\right) + 4 \\ &= 2\left(\left(x + \frac{k}{4}\right)^2 - \frac{k^2}{16}\right) + 4 \\ &= 2\left(x + \frac{k}{4}\right)^2 + \frac{32 - k^2}{8} \end{aligned}$$

Vertex is at $\left(-\frac{k}{4}, \frac{32-k^2}{8}\right)$ so $-\frac{k}{4} = 1.25$

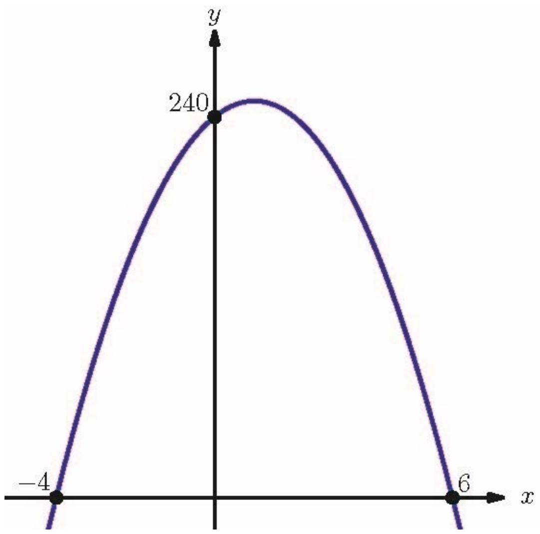
$k = -5$

ii The y-coordinate is $\frac{32-k^2}{8} = \frac{7}{8}$

b



16



a $f(x) = -10(x+4)(x-6)$

b Expanding and then completing the square:

$$\begin{aligned} f(x) &= -10(x^2 - 2x - 24) \\ &= -10(x^2 - 2x) + 240 \\ &= -10((x-1)^2 - 1) + 240 \\ &= -10(x-1)^2 + 250 \end{aligned}$$

c Expanding fully:

$$\begin{aligned} f(x) &= -10(x^2 - 2x - 24) \\ &= -10(x^2 - 2x) + 240 \\ &= -10x^2 + 20x + 240 \end{aligned}$$

17 a

$$\begin{aligned} \Delta &= (10-p)^2 - 4(p)\left(\frac{5p}{4} - 5\right) \\ &= 100 - 20p + p^2 - 5p^2 + 20p \\ &= 100 - 4p^2 \end{aligned}$$

b Equal roots when $\Delta = 0$

$$\begin{aligned} 4p^2 &= 100 \\ p &= \pm 5 \end{aligned}$$

18 a

$$\begin{aligned} x - 6x^2 + 15 &= (x-3)^2 - 9 + 15 \\ &= (x-3)^2 + 6 \end{aligned}$$

b This function has minimum value 6, so its reciprocal has maximum value $\frac{1}{6}$

Then $\frac{1}{x-6x^2+15}$ has maximum value $\frac{1}{2}$

19 When the line is tangent, the intersection of line and circle will have only one root.

Substituting for intersection: $x^2 + (2x+c)^2 - 3 = 0$

$$5x^2 + 4cx + c^2 - 3 = 0$$

Single root when $\Delta = 0$

$$\begin{aligned}(4c)^2 - 4(5)(c^2 - 3) &= 0\\ -4c^2 + 60 &= 0\\ c^2 &= 15\\ c &= \pm\sqrt{15}\end{aligned}$$

20 Multiplying through by $x \neq 0$ and rearranging:

$$ax^2 - 2x + 1 = 0$$

No real roots when $\Delta < 0$

$$\begin{aligned}(-2)^2 - 4(a)(1) &< 0\\ 4a &> 4\\ a &> 1\end{aligned}$$

21 $y = 2x^2 - 4kx + 3k^2$ is a positive quadratic.

$$\begin{aligned}\Delta &= (-4k)^2 - 4(2)(3k^2)\\ &= 16k^2 - 24k^2\\ &= -8k^2 < 0 \text{ for all } k\end{aligned}$$

Since $\Delta < 0$, it follows that there are no real roots to the quadratic, and therefore it lies entirely above the x-axis.

Worked solutions

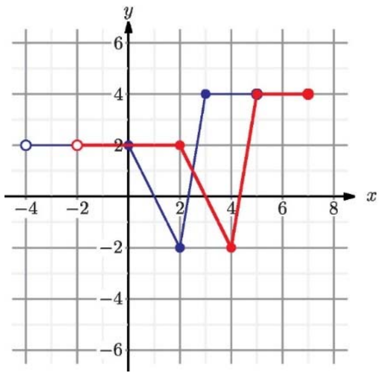
16 Analysis and approaches: Graphs

Worked solutions

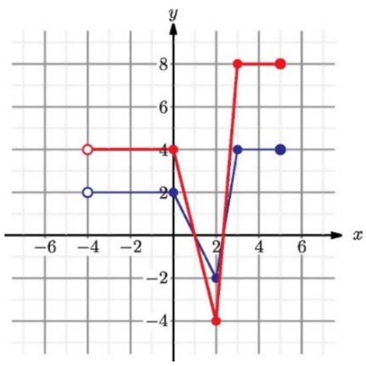
These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 16A

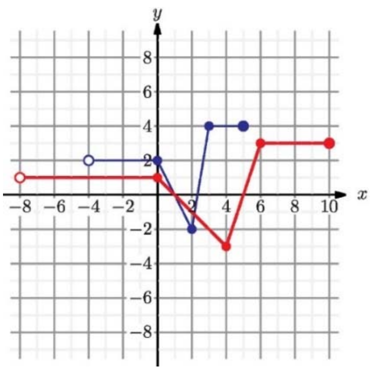
29 a $f(x-2)$ is $f(x)$ with a translation $\begin{pmatrix}2\\0\end{pmatrix}$:



b $2f(x)$ is $f(x)$ after a vertical stretch with scale factor 2:



c $y = f\left(\frac{1}{2}x\right) - 1$ is $f(x)$ after a horizontal stretch with scale factor 2 and a translation $\begin{pmatrix}0\\-1\end{pmatrix}$:



30 $y_1 = 3x^2 - 4x$

Translation $\begin{pmatrix}-3\\0\end{pmatrix}$: Replace x with $(x+3)$

$y_2 = 3(x+3)^2 - 4(x+3)$
$= 3x^2 + 14x + 15$

Vertical stretch scale factor 4: $y_3 = 4y_2$
$y_3 = 12x^2 + 56x + 60$

31 $y_1 = \mathrm{e}^x$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

$y_2 = \mathrm{e}^{x-2}$

Vertical stretch scale factor 3: $y_3 = 3y_2$

$y_3 = 3\mathrm{e}^{x-2}$

32 a $x^2 - 10x + 11 = (x-5)^2 - 14$

b $y_1 = x^2$

Translation $\begin{pmatrix}5\\0\end{pmatrix}$: Replace x with $(x-5)$:

$y_2 = (x-5)^2$

Translation $\begin{pmatrix}0\\-14\end{pmatrix}$: $y_3 = y_2 - 14$

$y_3 = (x-5)^2 - 14$

A translation $\begin{pmatrix}5\\-14\end{pmatrix}$ transforms $y = x^2$ to $y = x^2 - 10x + 11$

33 a $5x^2 + 30x + 54 = 5(x^2 + 6x + 9) = 5(x+3)^2$

b $y_1 = x^2$

Translation $\begin{pmatrix}-3\\0\end{pmatrix}$: Replace x with $(x+3)$:

$y_2 = (x+3)^2$

Vertical stretch with scale factor 5: $y_3 = 5y_2$

$y_3 = 5(x+3)^2$

A translation $\begin{pmatrix}-3\\0\end{pmatrix}$ and a vertical stretch with scale factor 5 transforms $y = x^2$ to $y = 5x^2 + 30x + 45$.

34 a $y_1 = x^2 = \mathrm{f}(x)$
Vertical stretch with scale factor 9: $y_2 = 9y_1$
$y_2 = 9x^2$

b $y_2 = (\pm 3x)^2 = \mathrm{f}(\pm 3x)$
From y_1 to y_2: Replace x with $3x$: Horizontal stretch with scale factor $\frac{1}{3}$
So to reverse the transformation would require a horizontal stretch with scale factor 3.

Tip: A stretch with scale factor $-\frac{1}{3}$ would cause x to be replaced by $-3x$ as an alternative solution, but this would normally be interpreted as a stretch and a reflection through the y-axis.

35 **a** $y_1 = 2x^3 = f(x)$

Vertical stretch with scale factor 8: $y_2 = 8y_1$

$y_2 = 16x^3$

b $y_2 = 2(2x)^3 = f(2x)$

Replace x with $2x$: Horizontal stretch with scale factor $\frac{1}{2}$

36 $y_1 = 2x^3 - 5x^2$

Reflection through x-axis: $y_2 = -y_1$

$y_2 = -2x^3 + 5x^2$

Reflection through y-axis: Replace x with $-x$

$y_3 = 2x^3 + 5x^2$

37 **a** $y_1 = f(x)$

Vertical stretch with scale factor 5: $y_2 = 5y_1$

$y_2 = 5f(x)$

Translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$: $y_3 = y_2 + 3$

$y_3 = 5f(x) + 3$

b $y_3 = 5\left(f(x) + \frac{3}{5}\right)$

Translation $\begin{pmatrix} 0 \\ 0.6 \end{pmatrix}$ followed by a vertical stretch with scale factor 5.

38 A stretch from the line $y = 1$ is equivalent to

Translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Stretch from the line $y = 0$

Translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y_1 = f(x)$

After (1):

$y_2 = f(x) - 1$

After (2):

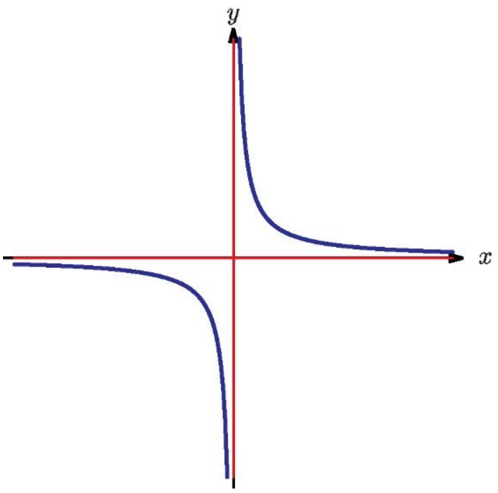
$y_3 = 2(f(x) - 1) = 2f(x) - 2$

After (3):

$y_4 = (2f(x) - 2) + 1 = 2f(x) - 1$

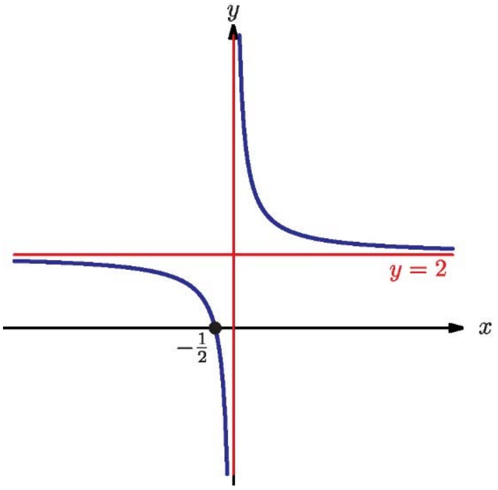
Exercise 16B

5 **a** Asymptotes $y = 0$ and $x = 0$



b Translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

c Asymptotes: $y = 2$ and $x = 0$



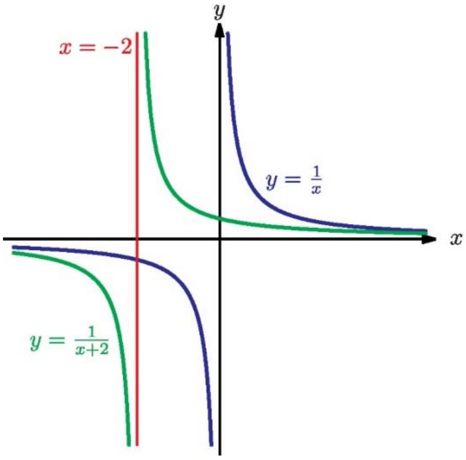
6 **a** Asymptotes $y = 0$ and $x = 0$

b Replace x with $(x - 3)$: Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

c Asymptotes $y = 0$ and $x = 3$

7 **a** Replace x with $(x + 2)$: Translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

b



8 **a** When $x = 0$, $y = -\frac{3}{2}$ so the y-intercept is $-\frac{3}{2}$

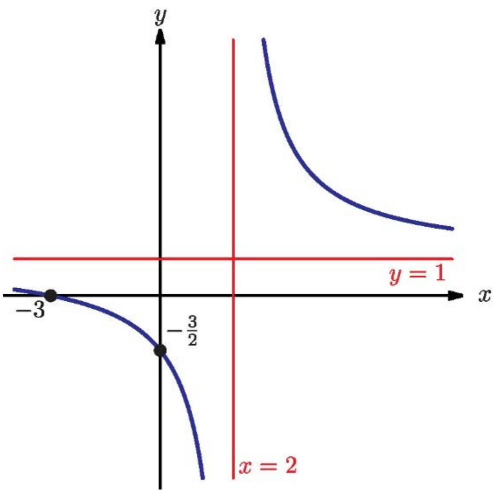
When $x = -3, y = 0$ so the x-intercept is -3

Axis intercepts are $(-3,0)$ and $\left(0, -\frac{3}{2}\right)$

b Vertical asymptote is root of denominator: $x = 2$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 1$

c



9 **a** Vertical asymptote is root of denominator: $x = -1$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 2$

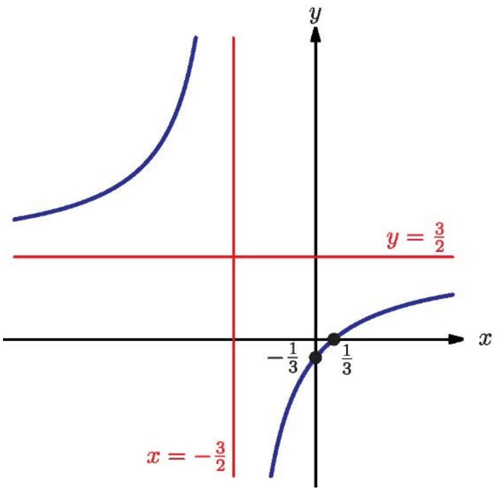
b When $x = -\frac{3}{2}, y = 0$ so the x-intercept is $-\frac{3}{2}$

10 Vertical asymptote is root of denominator: $x = -\frac{3}{2}$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = \frac{3}{2}$

When $x = \frac{1}{3}, y = 0$ so the x-intercept is $\frac{1}{3}$

When $x = 0, y = -\frac{1}{3}$ so the y-intercept is $-\frac{1}{3}$

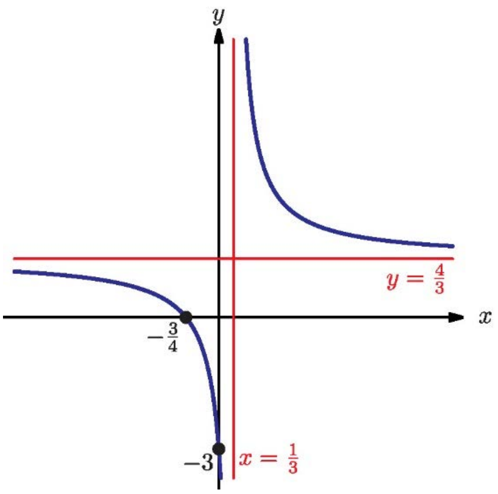


11 Vertical asymptote is root of denominator: $x = \frac{1}{3}$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = \frac{4}{3}$

When $x = -\frac{3}{4}, y = 0$ so the x-intercept is $-\frac{3}{4}$

When $x = 0, y = -3$ so the y-intercept is -3



12 a $\frac{2x+5}{x} = 2 + \frac{5}{x}$

b $y = \frac{1}{x}$

Horizontal stretch with scale factor 5: Replace x with $\frac{x}{5}$

$$y_1 = \frac{5}{x}$$

Translation $\begin{pmatrix}0\\2\end{pmatrix}$: $y_2 = y_1 + 2$

$$y_2 = 2 + \frac{5}{x}$$

This is one possible solution, but this sort of question often has many valid solutions.

Note that we could have answered that the first transformation was a vertical stretch:

Then $y_1 = \frac{5}{x}$

We should expect horizontal and vertical stretches to have the same effect on $y = \frac{1}{x}$ since the graph has the line of symmetry $y = x$

c $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

Applying the same transformations to these, the asymptotes to the curve $y = \frac{2x+5}{x}$ are

$x = 0$ and $y = 2$

13 a

$$\begin{aligned}\frac{2x-5}{x-3} &= \frac{(2x-6)+1}{x-3}\\ &= \frac{2x-6}{x-3} + \frac{1}{x-3}\\ &= 2 + \frac{1}{x-3}\end{aligned}$$

b $y = \frac{1}{x}$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x-3)$

$$y_1 = \frac{1}{x-3}$$

Translation $\begin{pmatrix}0\\2\end{pmatrix}$: $y_2 = y_1 + 2$

$$y_2 = 2 + \frac{1}{x-3}$$

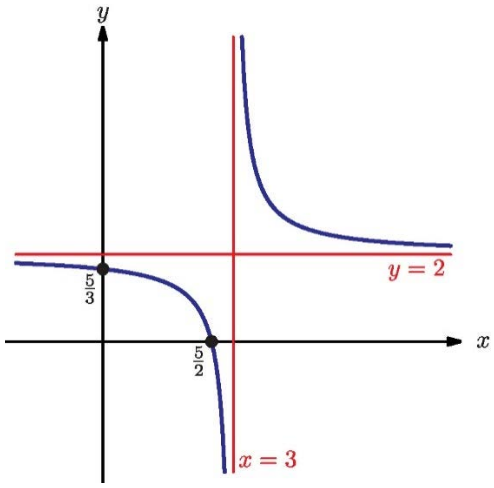
The transformation mapping $y = \frac{1}{x}$ to $y = \frac{2x-5}{x-3}$ is a translation $\begin{pmatrix}3\\2\end{pmatrix}$

c $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

Applying the same transformations to these, the asymptotes to the curve $y = \frac{2x-5}{x-3}$ are $x = 2$ and $y = 3$

When $x = \frac{5}{2}, y = 0$ so the x-intercept is $\frac{5}{2}$

When $x = 0, y = \frac{5}{3}$ so the y-intercept is $\frac{5}{3}$



14 a $\frac{5x-1}{x} = 5 - \frac{1}{x}$

b $y = \frac{1}{x}$

Reflection through the line $y = 0$: Replace x with $-x$

$y_1 = -\frac{1}{x}$

Translation $\begin{pmatrix}0\\5\end{pmatrix}$: $y_2 = y_1 + 5$

$y_2 = 5 - \frac{1}{x}$

Tip: Several alternatives exist, for example

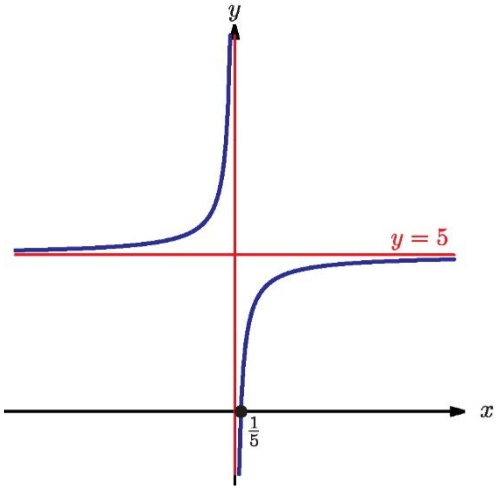
- a translation $\begin{pmatrix}0\\-5\end{pmatrix}$ followed by a reflection through $y = 0$
- a reflection through $x = 0$ followed by a translation $\begin{pmatrix}0\\5\end{pmatrix}$.

c $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

Applying the same transformations to these, the asymptotes to the curve $y = \frac{5x-1}{x}$ are

$x = 0$ and $y = 5$

When $x = \frac{1}{5}, y = 0$ so the x-intercept is $\frac{5}{2}$



15 a

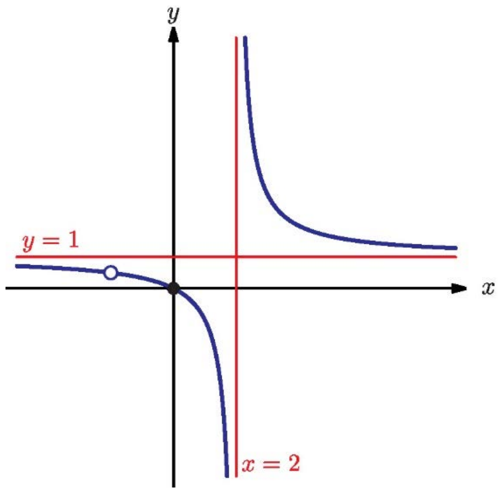
$$\frac{x^2+2x}{x^2-4} = \frac{x(x+2)}{(x-2)(x+2)}$$
$$= \frac{x}{x-2}, \qquad x \neq -2$$

b Vertical asymptote is root of denominator: $x = 2$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 1$

When $x = 0, y = 0$ so the curve passes through the origin.

There is a hole at $x = -2$, where the equation has no defined value.



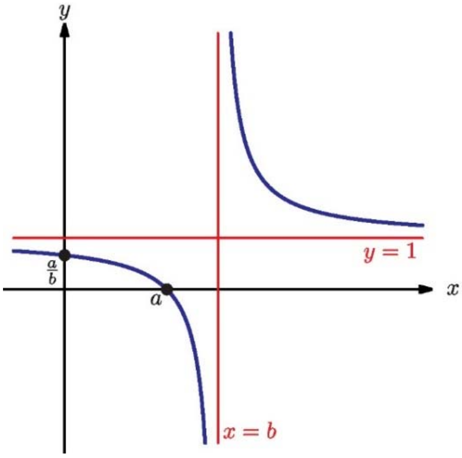
16 Vertical asymptote is root of denominator: $x = b$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 1$

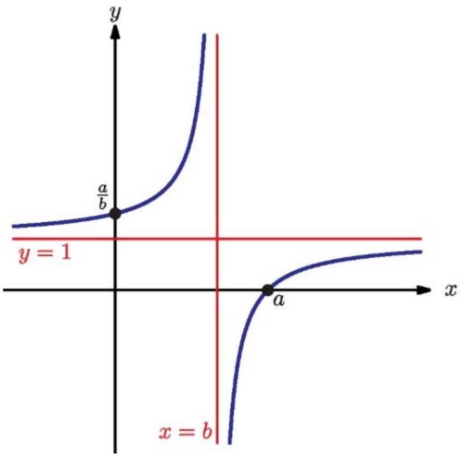
When $x = a, y = 0$ so the x-intercept is at $(a, 0)$.

When $x = 0, y = \frac{a}{b}$ so the y-intercept is at $\left(0, \frac{a}{b}\right)$.

a $0 < a < b$



b $0 < b < a$



Exercise 16C

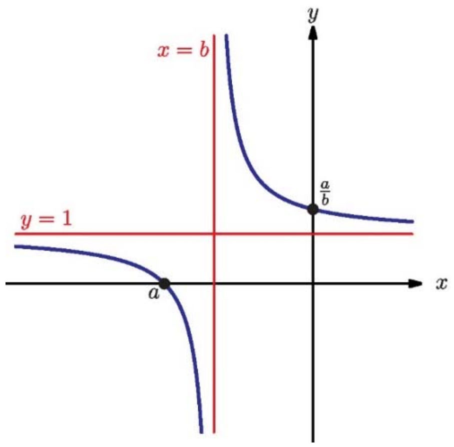
13 Since $2 < e$, the steeper positive exponential will be e^x

i $y = 2^x$ is Graph B.

ii $y = e^x$ is Graph A.

iii $y = 0.5^x$ (negative exponential for base < 1) is Graph C.

14



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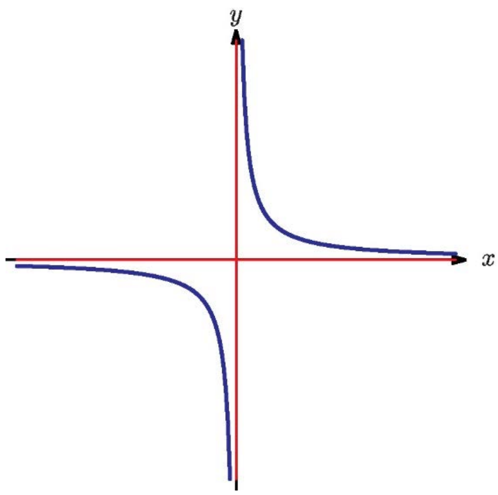
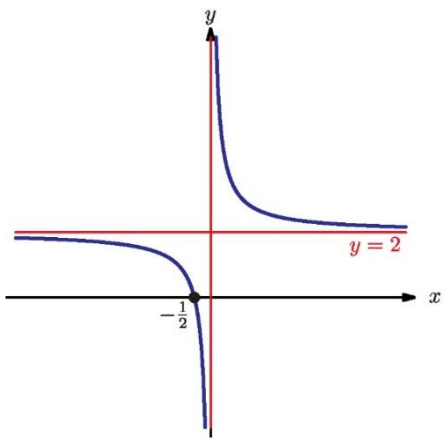
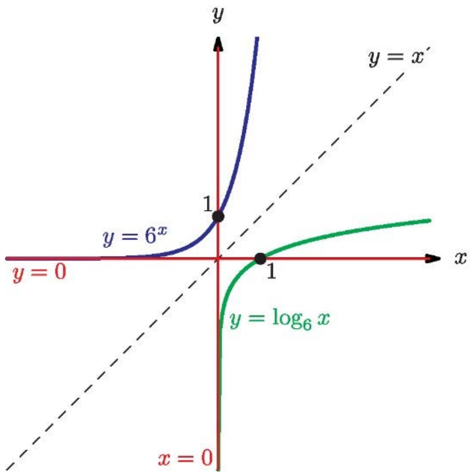


16



17



The two graphs are reflections through the line $y = x$.

18 The higher lines (for $x > 1$) will relate to lower bases; $2 < \mathrm{e} < 5$

i $y = \ln x$ is Graph B.

ii $y = \log_2 x$ is Graph C.

iii $y = \log_5 x$ is Graph A.

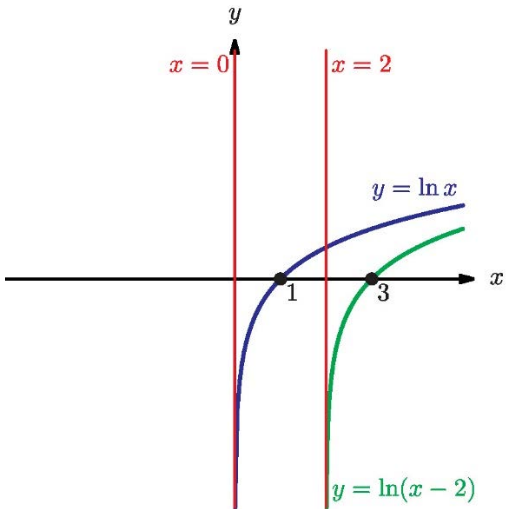
19 Using $\log_a x^{-1} = -\log_a x$:

i $y = \log_2 x$ is Graph B.

ii $y = \log_{0.5} x$ is Graph A.

iii $y = \log_{0.2} x$ is Graph C.

20



21 **a** $y = a + b\mathrm{e}^x$ has asymptote $y = a = 2$

b The y-intercept at $a + b = 5$ so $b = 3$

22 $y = e^x$ has y-intercept at $(0,1)$ and asymptote $y = 0$

After translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and vertical stretch scale factor 2

a new y-intercept is at $(0, 8)$

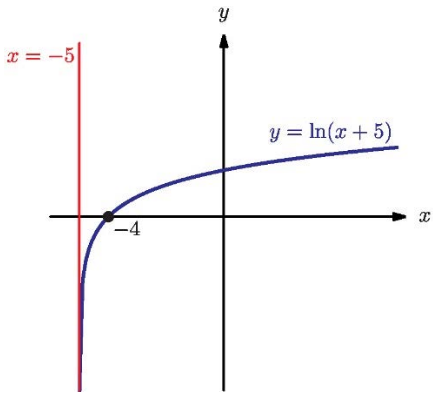
b new asymptote is at $y = 6$.

23 $y = \ln x$ has vertical asymptote $x = 0$ and x intercept at $(1,0)$

After a translation $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$, the graph will have

a vertical asymptote $x = -5$

b axis intercept $(-4,0)$ and new y-axis intercept $(0, \ln 5)$.



24 $y = Ca^x$ has y-intercept $C = 3$

And passes through $(2, Ca^2) = (2,48)$

So $a^2 = \frac{48}{3} = 16$

Since this is a positive exponential, $a = 4$

25 a $y = e^{kx} - c$ has y-intercept $1 - c = -1$ so $c = 2$

b $y(1) = e^k - 2 = 0.340$

So, $k = \ln 2.340 = 0.850$

26 a

$$5.2^x = \left(e^{\ln 5.2}\right)^x = e^{x \ln 5.2}$$
$$k = \ln 5.2 = 1.65$$

b Replace x with kx: Horizontal stretch with scale factor $\frac{1}{k} = 0.607$

27 a Horizontal asymptote is $y = -2$, and the graph lies above the line.

Range is $f(x) > -2$

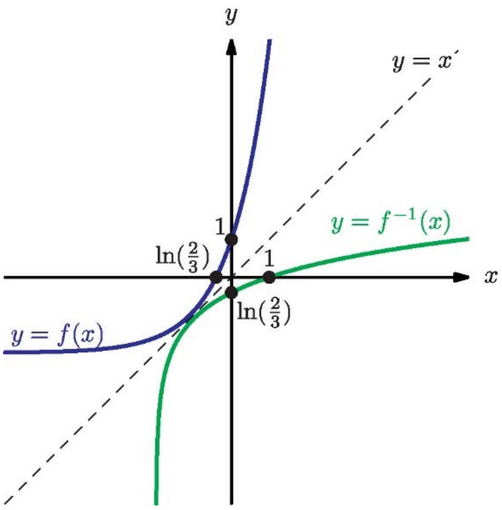
b The domain of $f^{-1}(x)$ is the range of $f(x)$.

If $y = f(x)$ then $y + 2 = 3e^x$

$$x = \ln\left(\frac{y+2}{3}\right) = f^{-1}(y)$$

So $f^{-1}(x) = \ln\left(\frac{x+2}{3}\right)$ with domain $x > -2$

c



28 a Horizontal stretch with scale factor $\frac{1}{3}$: Replace x with $3x$

New equation is $y = \ln 3x$

b Since $y = \ln 3x = \ln 3 + \ln x$, this can transformation also be achieved by a translation $\begin{pmatrix} 0 \\ \ln 3 \end{pmatrix}$.

29 a $y_2 = 5y_1$: Vertical stretch with scale factor 5

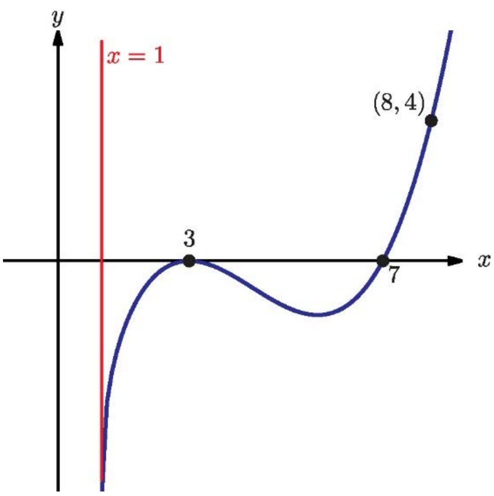
b $5\mathrm{e}^x = \mathrm{e}^{\ln 5} \times \mathrm{e}^x = \mathrm{e}^{x+\ln 5}$

Replace x with $(x + \ln 5)$: Translation $\begin{pmatrix} -\ln 5 \\ 0 \end{pmatrix}$

Mixed Practice

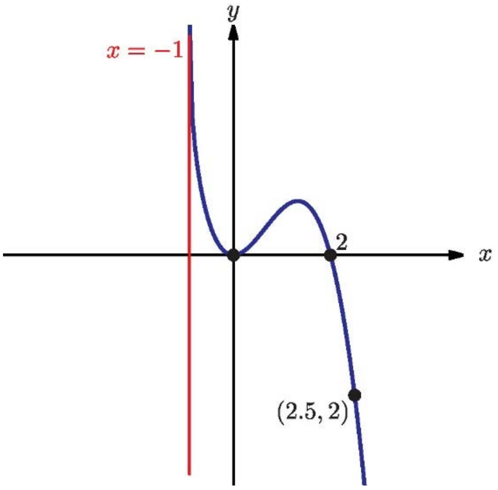
1 a Replace x with $(x - 3)$: Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$y_2 = 2y_1$: Vertical stretch with scale factor 2



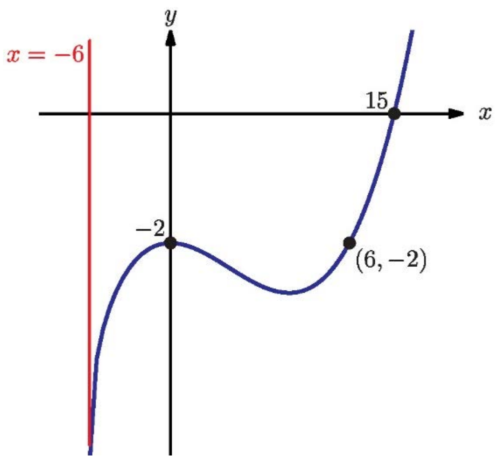
b Replace x with $(2x)$: Horizontal stretch with scale factor $\frac{1}{2}$

$y_2 = -y_1$: Reflection through $y = 0$



c Replace x with $\left(\frac{x}{3}\right)$: Horizontal stretch with scale factor 3

$y_2 = y_1 - 2$: Translation $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$



2 $y = x^3 - 2x$

Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: Replace x with $(x - 3)$

$$y_1 = (x-3)^3 - 2(x-3)$$
$$= x^3 - 9x^2 + 25x - 21$$

Vertical stretch scale factor 2: $y_2 = 2y_1$

$y_2 = 2x^3 - 18x^2 + 50x - 42$

3 **a** $x^2 + 4x + 9 = (x+2)^2 + 5$

b $y_1 = x^2$

Replace x with $(x + 2)$: Translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$y_2 = (x+2)^2$

$y_3 = y_2 + 5$: Translation $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$

The two transformations are both translations, equivalent to $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

4 a $y_1 = \frac{1}{x}$

Translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$: Replace x with $(x+2)$

$$y_2 = \frac{1}{x+2}$$

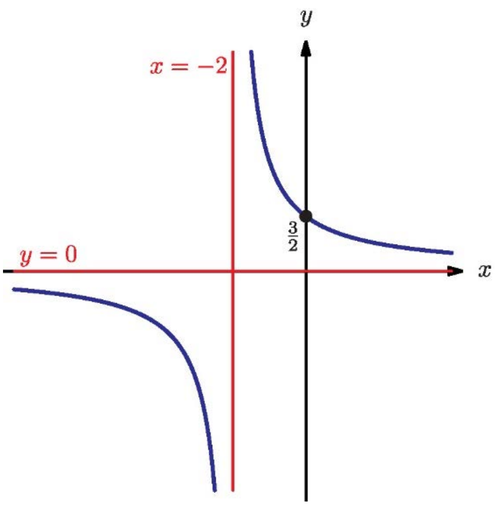
Vertical stretch with scale factor 3: $y_3 = 3y_2$

$$y_3 = \frac{3}{x+2}$$

b $y = \frac{1}{x}$ has asymptotes $y = 0$ and $x = 0$

After the transformations listed in part **a**, the new graph has asymptotes $y = 0$ and $x = -2$.

When $x = 0, y = \frac{3}{2}$ so the graph has y-intercept $\left(0, \frac{3}{2}\right)$



5 a

$$2 + \frac{1}{x-5} = \frac{2(x-5)+1}{x-5}$$
$$= \frac{2x-10+1}{x-5}$$
$$= \frac{2x-9}{x-5}$$

b $y_1 = \frac{1}{x}$

Replace x with $(x-5)$: Translation $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$$y_2 = \frac{1}{x-5}$$

$y_3 = y_2 + 2$: Translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

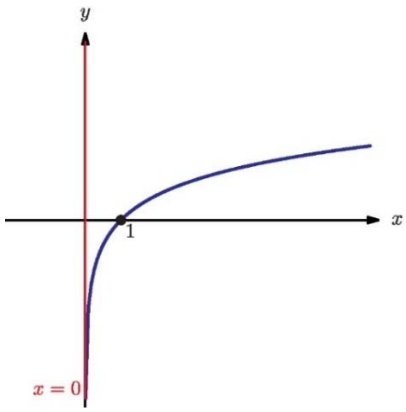
$$y_3 = 2 + \frac{1}{x-5}$$

The two transformations are both translations, equivalent to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

c $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$

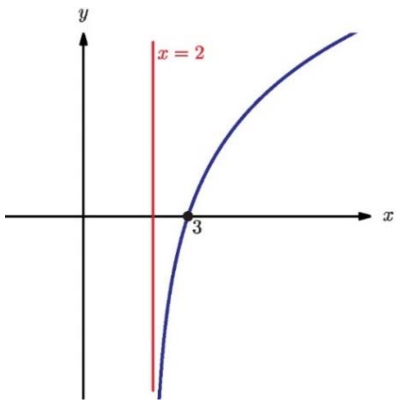
After the translations described in part **b**, the asymptotes to the new graph are $x = 5$ and $y = 2$.

6 **a** $y = \ln x$ has asymptote $x = 0$ and x-intercept at $(1, 0)$.



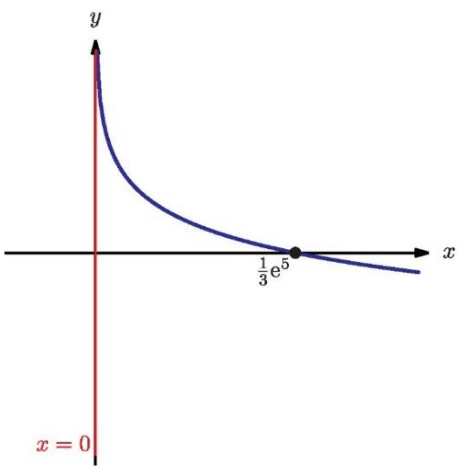
b The graph of $y = 3\ln(x - 2)$ is the graph of $y = \ln x$ after a translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and a vertical stretch with scale factor 3.

It has asymptote $x = 2$ and x-intercept $(3, 0)$.



c The graph of $y = 5 - \ln(3x)$ is the graph of $y = \ln x$ after a reflection through the y-axis, a translation $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$ and a horizontal stretch with scale factor $\frac{1}{3}$.

It has asymptote $x = 0$ and x-intercept $\left(\frac{e^5}{3}, 0\right)$.



7 $y_1 = ax + b$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x-3)$

$$y_2 = a(x-3)+b$$
$$= ax - 3a + b$$

Vertical stretch with scale factor 7: $y_3 = 7y_2$

$$y_3 = 7ax - 21a + 7b$$

Reflection in the x-axis: $y_4 = -y_3$

$$y_4 = -7ax + 21a - 7b$$
$$= 35 - 21x$$

Comparing coefficients:

$x^1: -7a = -21 \Rightarrow a = 3$

$x^0: 21a - 7b = 35 \Rightarrow b = 4$

8 $y_1 = 9(x-3)^2$

Replace x with $(x+5)$: Translation $\begin{pmatrix}-5\\0\end{pmatrix}$

$$y_2 = 9(x+2)^2$$

$y_3 = \frac{1}{3}y_2$: Vertical stretch with scale factor $\frac{1}{3}$

$$y_3 = 3(x+2)^2$$

The two transformations (in either order) are a horizontal translation of -5 units and a vertical stretch with scale factor $\frac{1}{3}$

9 $y_1 = \ln x$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

$$y_2 = \ln(x-2)$$

Translation $\begin{pmatrix}0\\3\end{pmatrix}$: $y_3 = y_2 + 3$

$$y_3 = 3 + \ln(x-2)$$

Vertical stretch with scale factor 2: $y_4 = 2y_3$

$$y_4 = 6 + 2\ln(x-2)$$
$$= \ln e^6 + \ln((x-2)^2)$$
$$= \ln(e^6(x-2)^2)$$

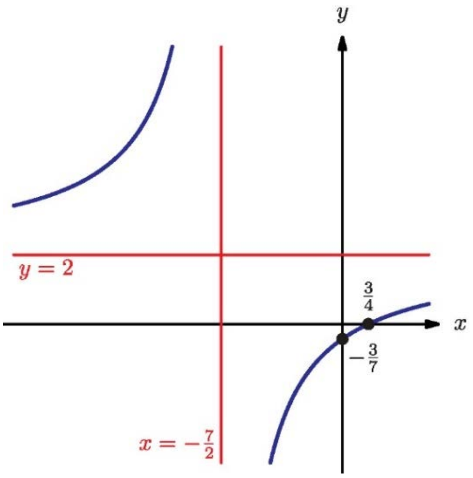
10 a Vertical asymptote is root of denominator: $x = -\frac{7}{2}$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 2$

b When $x = 0, y = -\frac{3}{7}$ so y-intercept is $-\frac{3}{7}$

When $x = \frac{3}{4}, y = 0$ so x-intercept is $\frac{3}{4}$

c



11 a Vertical asymptote is root of denominator: $x = -5$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 3$

b Domain of $\mathrm{f}(x)$ is $x \neq -5$

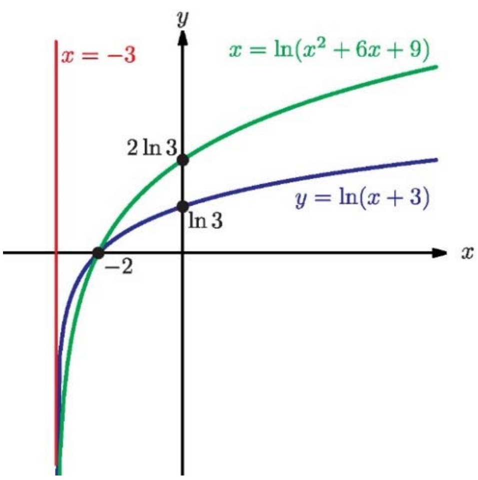
Range of $\mathrm{f}(x)$ is $f(x) \neq 3$

12 a Replace x with $(x + 3)$: Translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

b

$$\begin{aligned} y &= \ln(x^2 + 6x + 9) \\ &= \ln((x+3)^2) \\ &= 2\ln(x+3) \end{aligned}$$

over the domain $x > -3$



13 a Completing the square:

$$\begin{aligned} \mathrm{f}(x) &= p - (x^2 - qx) \\ &= p - \left(\left(x - \frac{q}{2}\right)^2 - \frac{q^2}{4}\right) \\ &= p + \frac{q^2}{4} - \left(x - \frac{q}{2}\right)^2 \end{aligned}$$

The maximum of this negative quadratic is at $\left(\frac{q}{2}, p + \frac{q^2}{4}\right) = (3,5)$

So $q = 6$ and $p = 5 - \frac{q^2}{4} = -4$

b $f(x) = -4 + 6x - x^2$ is transformed to $f_2(x)$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x-3)$

$$\begin{aligned} f_2(x) &= -4 + 6(x-3) - (x-3)^2 \\ &= -4 + 6x - 18 - x^2 + 6x - 9 \\ &= -31 + 12x - x^2 \end{aligned}$$

14 a $f(x) = p + \frac{9}{x-q}$ for $x \neq q$

Vertical asymptote is root of denominator: $x = q = 3$

b

$$f(0) = p + \frac{9}{-q} = 4$$

$$p = 4 + \frac{9}{q} = 7$$

c Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 7$

15 a

$$N(0) = 840 = 40 + a$$

$$a = 800$$

b

$$N(4) = 90 = 800 \times b^{-4} + 40$$

$$b^{-4} = \frac{50}{800} = \frac{1}{16}$$

$$b = 2$$

c The horizontal asymptote of the graph of $N(t)$ in this model is $N = 40$, below which the curve will not pass.

The minimum number of fish the model predicts is $N = 40$

16 $f(x) = \ln x$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x-3)$

$f_2(x) = \ln(x-3)$

Translation $\begin{pmatrix}0\\-2\end{pmatrix}$: $f_3(x) = f_2(x) - 2$

$$\begin{aligned} f_3(x) &= \ln(x-3) - 2 \\ &= \ln(x-3) - \ln(e^2) \\ &= \ln\left(\frac{x-3}{e^2}\right) \end{aligned}$$

Reflection through x-axis: $f_4(x) = -f_3(x)$

$$\begin{aligned} f_4(x) &= -\ln\left(\frac{x-3}{e^2}\right) \\ &= \ln\left(\frac{e^2}{x-3}\right) \end{aligned}$$

17 Vertical asymptote is root of denominator: $x = 5$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 2$

b

$$\text{f}(x) = \frac{2x - 10 + 1}{x - 5}$$
$$= 2 + \frac{1}{x - 5}$$

$\alpha = 2, \beta = 1$

c $y = \frac{1}{x}$

Replace x with $(x - 5)$: Translation $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$$y_2 = \frac{1}{x - 5}$$

$y_3 = y_2 + 2$: Translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$y_3 = 2 + \frac{1}{x - 5} = \text{f}(x)$$

d If $y = \text{f}(x) = 2 + \frac{1}{x-5}$

Then $\frac{1}{x-5} = y - 2$

$$x - 5 = \frac{1}{y - 2}$$
$$x = 5 + \frac{1}{y - 2} = \text{f}^{-1}(y)$$

$$\text{f}^{-1}(x) = 5 + \frac{1}{x - 2}$$
$$= \frac{5x - 9}{x - 2}$$

The inverse has domain equivalent to the range of the original function: $x \neq 2$

e The graph of $\text{f}(x)$ is mapped to the graph of $\text{f}^{-1}(x)$ by a reflection through the line $y = x$.

18

$$y = 4^x$$
$$= 2^{2x}$$

Transforming $y = 2^{2x}$ to $y = 2^x$

Replace x with $\frac{x}{2}$: Horizontal stretch with scale factor 2

19 Using change of base rule:

$$y = \log_{10} x = \frac{\ln x}{\ln 10}$$

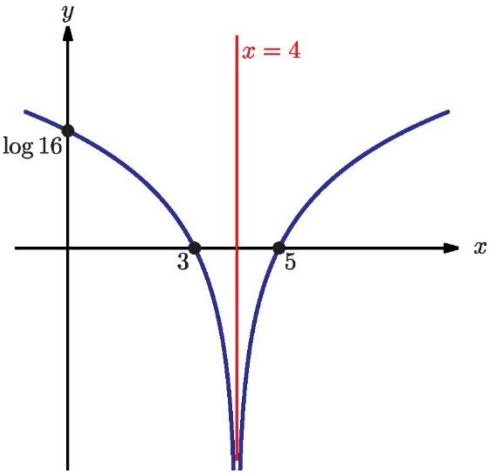
Transforming $y = \ln x$ to $y = \left(\frac{1}{\ln 10}\right) \ln x$: Vertical stretch, scale factor $\frac{1}{\ln 10}$

20

$$\log(x^2 - 8x + 16) = \log((x-4)^2)$$
$$= 2\log|x-4|$$

The graph of $y = \log(x-4)$ is the graph of $y = \log x$, translated $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and stretched vertically with scale factor 2.

The graph of $y = 2\log|x-4|$ is the same graph, together with its reflection through $x = 4$.



21 The graph of $y = \mathrm{f}(x)$ has roots at $-1, 0$ and 1

The graph of $y = x\mathrm{f}(x)$ will have a single root at -1 and 1, and a double root at 0.

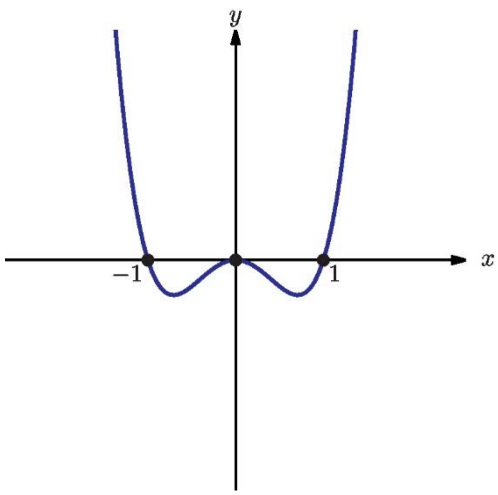
$\mathrm{f}(x) < 0$ for $x < -1$ and $0 < x < 1$

$x\mathrm{f}(x) < 0$ for $-1 < x < 0$ and $0 < x < 1$ (ie opposite sign for negative x, same sign for positive x)

End behaviour:

As $x \to -\infty$, $\mathrm{f}(x) \to -\infty$ so $x\mathrm{f}(x) \to \infty$

As $x \to \infty$, $\mathrm{f}(x) \to \infty$ so $x\mathrm{f}(x) \to \infty$



Tip: Strictly speaking, the graph of $\mathrm{f}(x)$ shows roots at $-1, 0$ and 1 with odd multiplicity, not necessarily 1, so it would be more accurate to say that $x\mathrm{f}(x)$ has roots at ± 1 with odd multiplicity and a root at 0 with even multiplicity, but the end interpretation is equivalent, for the purposes of a sketch.

22 To reflect in the line $y = 1$:

Translate $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$: $y_2 = \mathrm{f}(x) - 1$

Reflect in the x-axis: $y_3 = -y_2 = 1 - \mathrm{f}(x)$

Translate $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $y_4 = 1 + y_3 = 2 - \mathrm{f}(x)$

The new graph has equation $y = 2 - \mathrm{f}(x)$

23 If $\mathrm{f}(x^2) = x^2\mathrm{f}(x)$ for all x in the domain of f then

$\mathrm{f}(x) = \frac{\mathrm{f}(x^2)}{x^2}$ for $x \neq 0$

For any value $a \neq 0$ in the domain of f, $\mathrm{f}(-a) = \frac{\mathrm{f}((-a)^2)}{(-a)^2} = \frac{\mathrm{f}(a^2)}{a^2} = \mathrm{f}(a)$

If $\mathrm{f}(a) = \mathrm{f}(-a)$ for any $a \neq 0$ in the domain of f then the graph of $\mathrm{f}(x)$ is symmetrical across the y-axis.

17 Analysis and approaches: Equations

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 17A

19

$$x^3 - 5x = 0$$
$$x(x^2 - 5) = 0$$
$$x = 0 \text{ or } x^2 = 5$$
$$x = 0 \text{ or } \pm\sqrt{5}$$

20

$$5x \ln x = 8 \ln x$$
$$(5x - 8)\ln x = 0$$
$$5x - 8 = 0 \text{ or } \ln x = 0$$
$$x = \frac{8}{5} \text{ or } 1$$

21

$$(x^2)^2 - 10x^2 + 24 = 0$$
$$(x^2 - 4)(x^2 - 6) = 0$$
$$x^2 = 4 \text{ or } x^2 = 6$$
$$x = \pm 2 \text{ or } \pm\sqrt{6}$$

22

$$(\ln x)^2 - 2\ln x - 8 = 0$$
$$(\ln x - 4)(\ln x + 2) = 0$$
$$\ln x = 4 \text{ or } -2$$
$$x = e^4 \text{ or } e^{-2}$$

23

$$(e^x)^2 - 10e^x + 21 = 0$$
$$(e^x - 3)(e^x - 7) = 0$$
$$e^x = 3 \text{ or } 7$$
$$x = \ln 3 \text{ or } \ln 7$$

24

$$e^x(e^x + 3)(e^x - 3) = 0$$
$$e^x = 0 \text{ or } \pm 3$$

Since $e^x > 0$ for all real x, only $e^x = 3$ produces a real solution

$$x = \ln 3$$

25

$$(e^x+3)(e^x-2)=0$$
$$e^x=-3 \text{ or } 2$$

Since $e^x>0$ for all real x, only $e^x=2$ produces a real solution

$$x=\ln 2$$

26

$$(3^x)^2-3^x-72=0$$
$$(3^x-9)(3^x+8)=0$$
$$3^x=9 \text{ or } -8$$

Since $3^x>0$ for all real x, only $3^x=9$ produces a real solution

$$x=2$$

27

$$\left(x^{\frac{1}{3}}\right)^2-2x^{\frac{1}{3}}-3=0$$
$$\left(x^{\frac{1}{3}}-3\right)\left(x^{\frac{1}{3}}+1\right)=0$$
$$x^{\frac{1}{3}}=3 \text{ or } -1$$
$$x=27 \text{ or } -1$$

28

$$3^x(2^x-8)=0$$
$$3^x=0 \text{ or } 2^x=8$$

Since $3^x>0$ for all real x, only $2^x=8$ produces a real solution

$$x=2$$

29

$$x\left(\sqrt{x-1}-3\right)=0$$
$$x=0 \text{ or } \sqrt{x-1}=3$$
$$x=0 \text{ or } x=10$$

But if $x=0$ is not within the domain of real function $\sqrt{x-1}$ so is not a valid solution.

$x=10$ is the only valid solution.

30

$$e^{(x-3)\ln(x-1)}=1$$
$$(x-3)\ln(x-1)=0$$
$$x-3=0 \text{ or } \ln(x-1)=0$$
$$x=3 \text{ or } x=2$$

31 Using change of base, $\log_x 2=\frac{\log_2 2}{\log_2 x}=\frac{1}{\log_2 x}$

$$(\log_2 x)^2-2\log_2 x-3=0$$
$$(\log_2 x-3)(\log_2 x+1)=0$$
$$\log_2 x=3 \text{ or } -1$$
$$x=2^3 \text{ or } 2^{-1}$$
$$x=8 \text{ or } \frac{1}{2}$$

32

$$4^x(3 - 2 \times 3^x) = 0$$
$$4^x = 0 \text{ or } 2 \times 3^x = 3$$

$4^x > 0$ for all real x so only $2 \times 3^x = 3$ produces a real solution

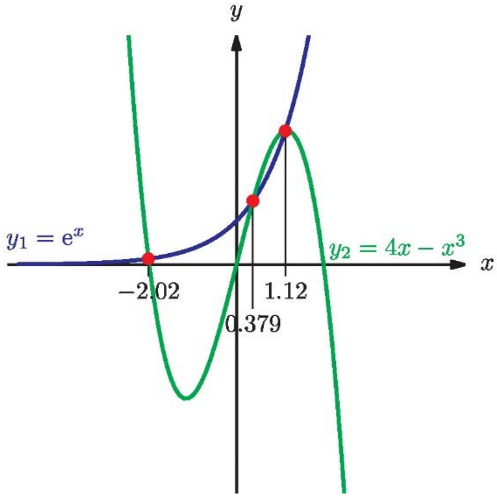
$$3^x = \frac{3}{2}$$
$$x = 1 - \log_3 2 = 0.369$$

33

$$(x - 7)^{x^2 - 16} = 0$$
$$x - 7 = 1 \text{ or } (x - 7 = -1 \text{ and } x^2 - 16 \text{ is even}) \text{ or } x^2 - 16 = 0$$
$$x = 8, 6 \text{ or } \pm 4$$

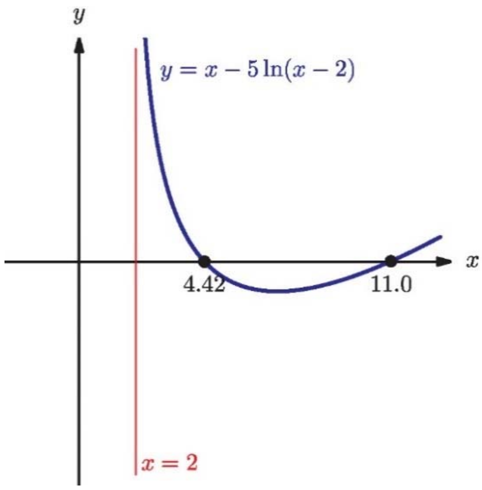
Exercise 17B

9 Graphing $y_1 = e^x$ and $y_2 = 4x - x^3$ and seeking intersections:



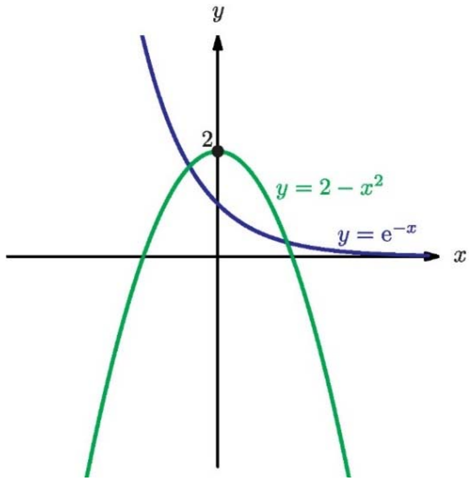
Solutions are $x = -2.02, 0.379, 1.12$

10 Graphing $y = x - 5\ln(x - 2)$ and seeking roots:



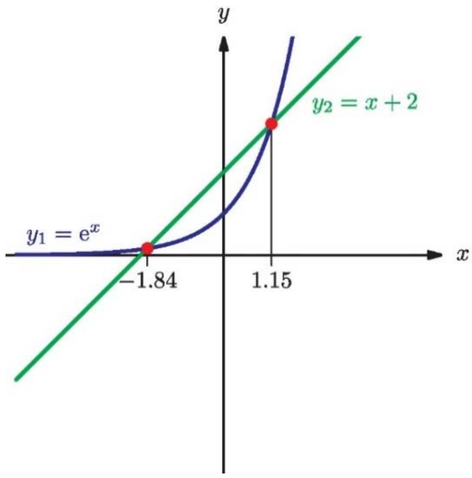
Solutions are $x = 4.42, 11.0$

11 a Graphing $y_1 = e^{-x}$ and $y_2 = 2 - x^2$



b There are two intersections to the two graphs, so two solutions to the equation $e^{-x} = 2 - x^2$

12 Graphing $y_1 = e^x$ and $y_2 = x + 2$ and seeking intersections:

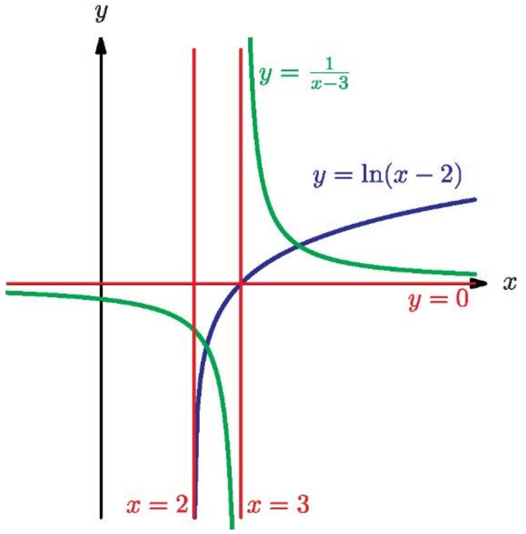


Solutions are $x = -1.84, 1.15$

13 a Graphing $y_1 = \ln(x-2)$ and $y_2 = \frac{1}{x-3}$

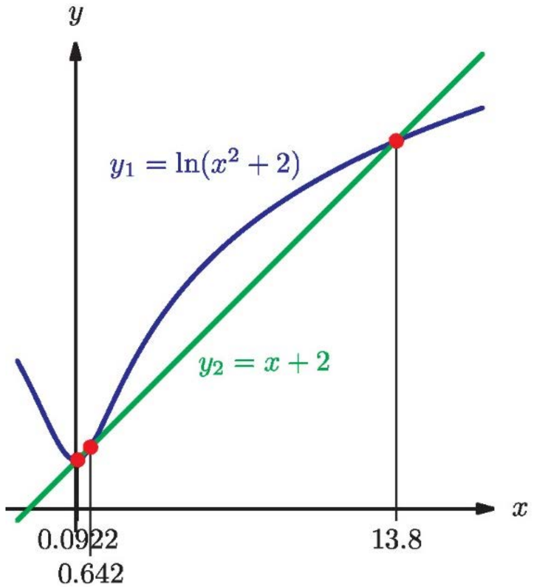
The graph of y_1 is the graph of $\ln x$ after a translation $\begin{pmatrix}2\\0\end{pmatrix}$

The graph of y_2 is the graph of $\frac{1}{x}$ after a translation $\begin{pmatrix}3\\0\end{pmatrix}$



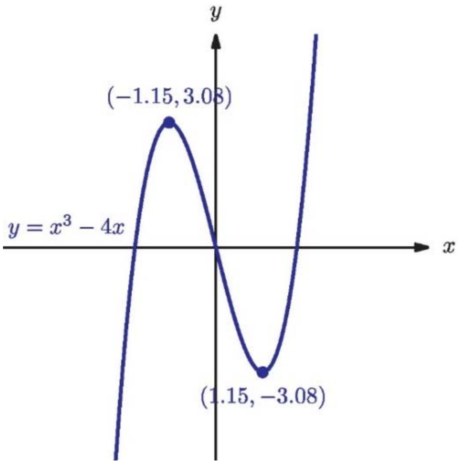
b There are two intersections to the two graphs, so two solutions to the equation $\ln(x-2) = \frac{1}{x-3}$.

14 Graphing $y_1 = 3\ln(x^2+2)$ and $y_2 = x + 2$ and seeking intersections:



Solutions are $x = 0.0922, 0.642, 13.8$

15 Graphing $y = x^3 - 4x$:



The local maximum is at $(-1.15, 3.08)$ and the local minimum is at $(1.15, -3.08)$.

The curve will have three intersections with $y = k$ for $-3.08 < k < 3.08$

16 $\frac{2e^x+3x}{e^x+1} = 2$

Since $e^x + 1 > 0$ for all real x, multiplying both sides by $e^x + 1$ neither loses nor introduces solutions.

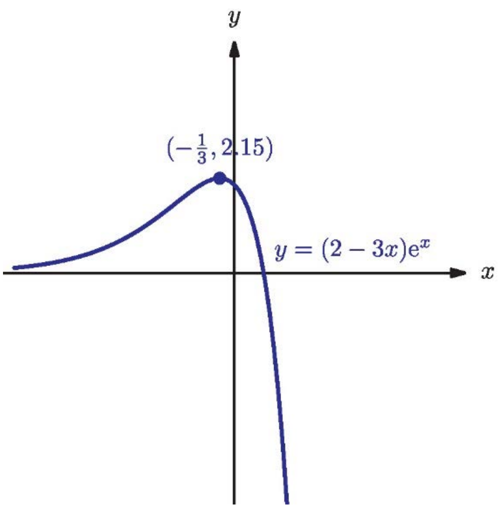
$$2e^x + 3x = 2e^x + 2$$
$$3x = 2$$
$$x = \frac{2}{3} = 0.667$$

17 $ke^{-x} + 3x = 2$

Rearranging:

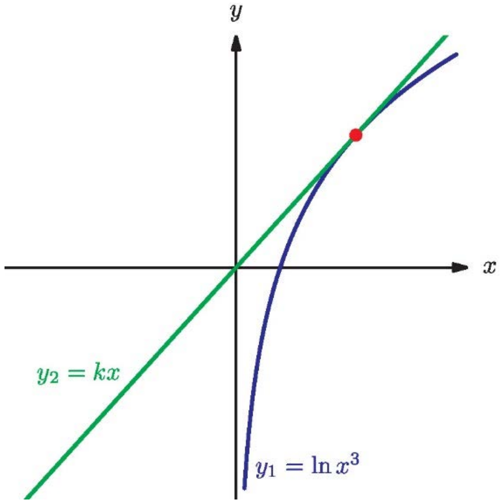
$(2 - 3x)e^x = k$

Graphing $y = (2 - 3x)e^x$



The maximum is at $\left(-\frac{1}{3}, 2.15\right)$ so the equation $(2 - 3x)e^x = k$ has no real roots for $k > 2.15$

18 Graphing $y_1 = \ln x^3 = 3\ln x$ and line $y_2 = kx$. For there to be only one point of intersection, the line y_2 must be tangent to the curve y_1.



a $\ln x^4 = 4\ln|x|$

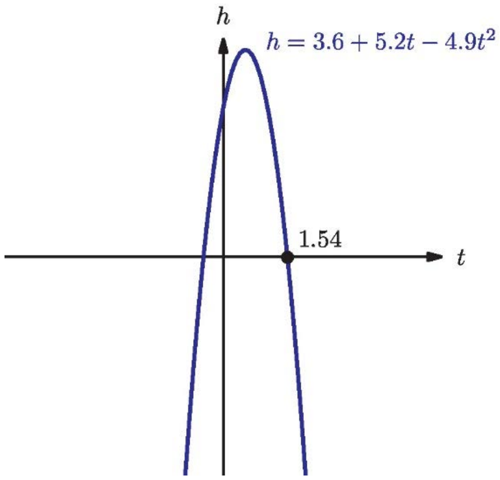
Since the graph of $y = \ln x^4$ is the graph of $y = \ln x^3$ after a vertical stretch with scale factor $\frac{4}{3}$, together with its reflection through the y-axis, it follows that the line $y = kx$ will intersect the graph in three places (twice for $x > 0$ and once for $-1 < x < 0$), so there will be three roots to the equation $\ln x^4 = kx$.

b $\ln\sqrt{x} = \frac{1}{2}\ln x = \frac{1}{6}\ln x^3$

Since the graph of $y = \ln\sqrt{x}$ is the graph of $y = \ln x^3$ after a vertical stretch with scale factor $\frac{1}{6}$, it follows that the line $y = kx$ will pass entirely above the curve of $y = \ln\sqrt{x}$, so there will be no roots to the equation $\ln\sqrt{x} = kx$

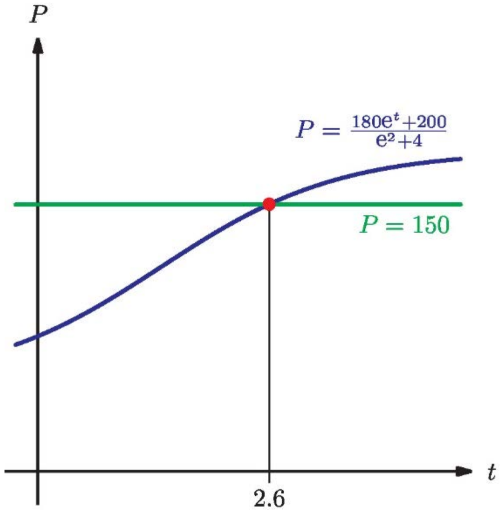
Exercise 17C

1 Graphing $h = 3.6 + 5.2t - 4.9t^2$ to find the positive root:



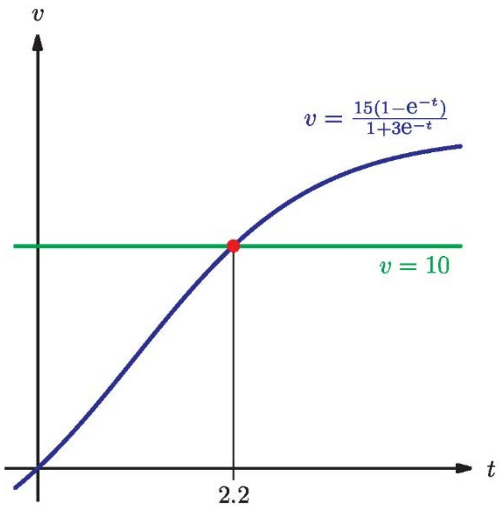
The ball reaches the ground at $t = 1.54$ seconds

2 Graphing $P = \frac{180e^t + 200}{e^t + 4}$ to find when $P = 150$:



$t = 2.6$ months

3 Graphing $v = \frac{15(1-e^{-t})}{1+3e^{-t}}$ to find when $v = 10$:



$t = 2.20$ seconds

4 $N = 1500 - 1000e^{-0.07t} = 1200$

$$t = -\frac{1}{0.07}\ln\left(\frac{300}{1000}\right) = 17.2 \text{ hours}$$

5 Volume of a cube with side length x is x^3

Surface area of the cube is $A = 6x^2$

Total edge length is $l = 12x$

If $12x = x^3$ then $x(x^2 - 12) = 0$

Rejecting $x \leqslant 0$ from context, the only valid solution is $x = \sqrt{12}$

So $A = 72 \text{ cm}^2$

6 Area $A = \frac{5}{2}x\ \text{cm}^2$

Perimeter $P = 5 + x + \sqrt{x^2 + 25}$

$$5 + x + \sqrt{x^2 + 25} = \frac{5x}{2}$$
$$\frac{3x}{2} - 5 = \sqrt{x^2 + 25}$$
$$3x - 10 = \sqrt{4x^2 + 100}$$
$$(3x - 10)^2 = 4x^2 + 100$$
$$9x^2 - 60x + 100 = 4x^2 + 100$$
$$5x^2 = 60x$$
$$x = 12$$

7 Let the series have first term u_1 and common ratio r.

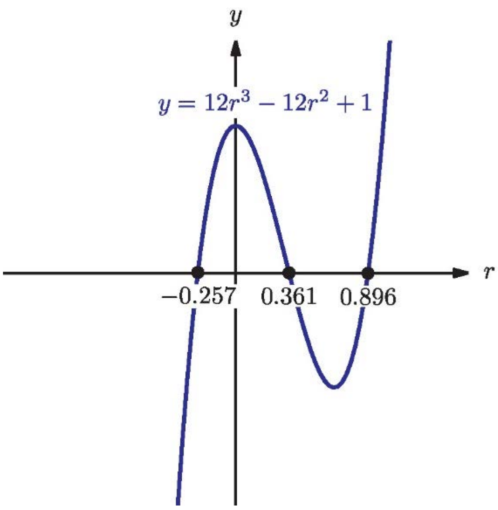
Then $u_3 = u_1 r^2 = 1$

$$S_\infty = \frac{u_1}{1 - r} = 12$$

Then $u_1 = 12 - 12r$

Substituting: $12r^2(1 - r) = 1$

Graphing $y = 12r^3 - 12r^2 + 1$ and looking for roots:



$r = -0.257, 0.361, 0.896$

Require that $|r| < 1$ for the sum to infinity to converge, so all these are valid solutions.

Tip: Even though the check does not eliminate any of the possible values, you should ALWAYS make a check on solution validity and write the results explicitly in your answer.

8 **a** $C(n) = 600 \times 1.03^n$

b Require that $600 \times 1.03^n = 750$

$1.03^n = 1.25$

$n = \log_{1.03} 1.25 = \frac{\ln 1.25}{\ln 1.03}$ using the logarithm change of base formula.

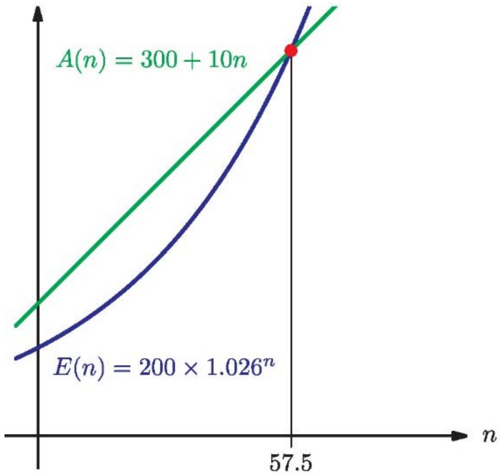
9 Let $A(n)$ be the amount in Asher's account at the end of year n

Let $E(n)$ be the amount in Elsa's account at the end of year n.

$A(n) = 200 \times 1.026^n$

$E(n) = 300 + 10n$

Graphing $A(n)$ and $E(n)$:



The two graphs intersect at $n = 57.5$

After 58 years, Asher will have more money than Elsa.

10 a Let $d(v)$ be the stopping distance of a car travelling at v km per hour as described.

$d(30) = 12$ m

b Let $u = \frac{v}{5}$

Then $d(u) = u + \frac{u^2}{6} = 36$

$$u^2 + 6u - 216 = 0$$
$$(u - 12)(u + 18) = 0$$
$$u = 12 \text{ or } -18$$

Rejecting the negative solution as being invalid for the context, $u = 12$ so $v = 60$ km per hour.

c

$$\frac{v^2}{25} + \frac{v^4}{6{,}250{,}000} = 101$$
$$u^2 + \frac{u^4}{10000} = 101$$
$$u^4 + 10000u^2 - 1010000 = 0$$
$$(u^2 - 100)(u^2 + 10100) = 0$$
$$u^2 = 100 \text{ or } -10100$$

Rejecting the negative root, $u^2 = 100$ so $u = \pm 10$.

Again rejecting the negative root, $u = 10$ so $v = 50$ km per hour

11 a

$$e^x + e^{-x} = 4$$
$$(e^x)^2 - 4e^x + 1 = 0$$
$$e^x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = 2 \pm \sqrt{3}$$
$$x = \ln(2 + \sqrt{3}) \text{ or } \ln(2 - \sqrt{3})$$

b

$$\ln(2+\sqrt{3}) + \ln(2-\sqrt{3}) = \ln\left((2+\sqrt{3})(2-\sqrt{3})\right)$$
$$= \ln(4-3)$$
$$= 0$$

Since the sum of the two solutions is zero, it follows that their mean is zero.

12 Let u_1 be the first term of the geometric series and r be the common ratio

$$u_4 = -3 = u_1 r^3 \qquad (1)$$

$$S_\infty = \frac{u_1}{1-r} = 20$$

a

Substituting $u_1 = 20(1-r)$ into (1):

$$20r^3(1-r) = -3$$
$$20r^4 - 20r^3 - 3 = 0$$

b From calculator, the solutions to this quartic are $r = -0.468$ or 1.110

For the series sum to converge, $|r| < 1$ so reject the positive root.

$$(1): u_1 = -\frac{3}{r^3} = 29.4$$

13 a $\sum_{k=0}^{\infty} e^{-(2k+1)x} = e^{-x}(1 + e^{-2x} + e^{-4x} + \cdots) = \frac{2}{3}$

Geometric series sum, first term $u_1 = e^{-x}$ and common ratio $r = e^{-2x}$

$$S_\infty = \frac{u_1}{1-r} = \frac{2}{3}$$
$$\frac{e^{-x}}{1-e^{-2x}} = \frac{2}{3}$$
$$3e^{-x} = 2 - 2e^{-2x}$$
$$2e^{-2x} + 3e^{-x} - 2 = 0$$

Multiplying through by e^{2x} and changing signs:

$$2e^{2x} - 3e^x - 2 = 0$$

b

$$(2e^{-x} - 1)(e^{-x} + 2) = 0$$

$$e^{-x} = \frac{1}{2} \text{ or } -2$$

Since $e^{-x} > 0$ for all real x, reject the negative solution.

$$e^{-x} = \frac{1}{2}$$

$$x = \ln 2$$

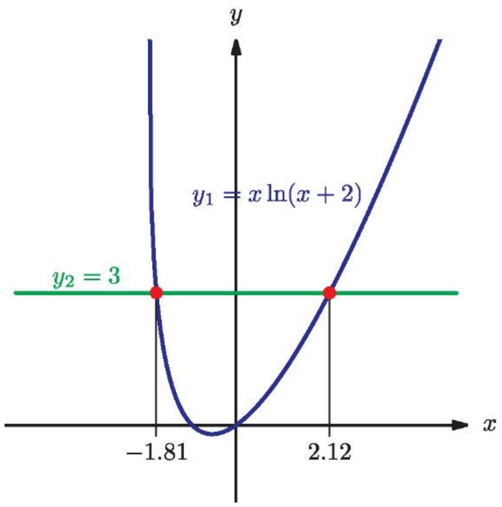
Mixed Practice

1 $x(\ln(x+1) - 5) = 0$

$x = 0$ or $\ln(x+1) = 5$

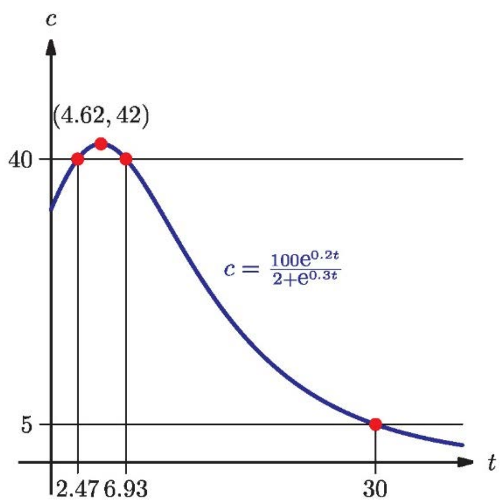
$x = 0$ or $x = e^5 - 1$

2 Graphing $y_1 = x\ln(x+2)$ and $y_2 = 3$ and seeking intersections:



Solutions are $x = -1.81, 2.12$

3 **a** Graphing $c = \frac{100e^{0.2t}}{2+e^{0.3t}}$ and seeking the maximum:



The maximum concentration of the drug is 42 units.

b From GDC, $c = 40$ when $t = 2.47$ or 6.93 hours

c From GDC, $c < 5$ for $t > 30.0$

It takes 30 hours for the concentration to fall below 5 units.

4

$$x^4 - 2x^2 - 63 = 0$$
$$(x^2 - 9)(x^2 + 7) = 0$$
$$(x + 3)(x - 3)(x^2 + 7) = 0$$

Real solutions: $x = \pm 3$

5 **a** Height is zero when the projectile is launched and lands.

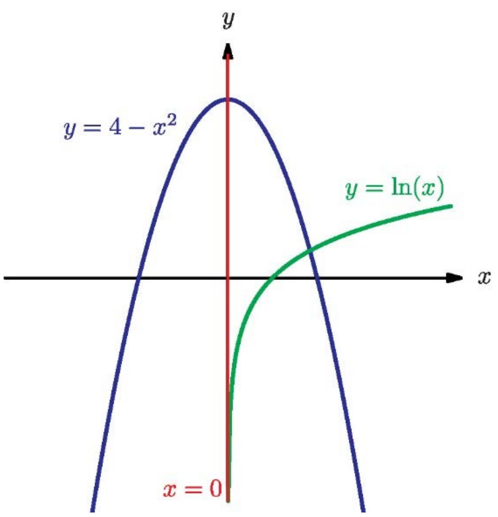
$t(1 - 5t) = 0$ has roots $t = 0$ (launch) and $t = 0.2$ (landing)

b $h = t(1 - 5t) + \ln(t + 1)$

From GDC, this has roots at $t = 0$ and $t = 0.370$

The projectile is in the air 0.17 seconds longer.

6 **a**



b From the graph, there is only one intersection point and so there is only one solution to the equation $x^2 + \ln x = 4$

7 **a** Let P be the population of big cats in Africa, and t be the number of years since 2004.

$$P = 10\,000 \times 1.05^t$$
$$P(1) = 10\,500$$

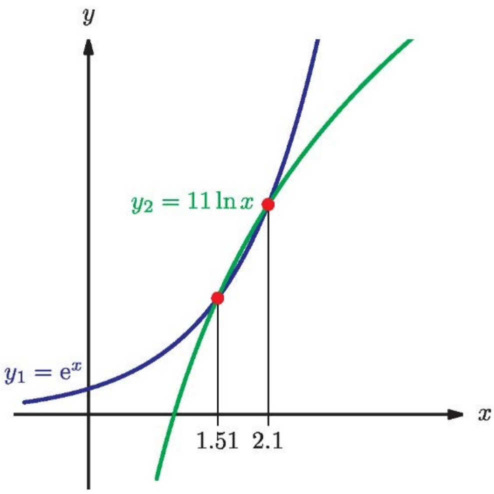
b $P(6) = 13\,400$

c

$$10\,000 \times 1.05^t > 50\,000$$
$$1.05^t > 5$$
$$t > \log_{1.05}(5)$$
$$t > 32.99$$

It takes 33 years for the population to exceed 50 000

8 Graphing $y_1 = e^x$ and $y_2 = 11\ln x$ and seeking intersections:



Solutions are $x = 1.51, 2.10$

9

$$(\sqrt{x}-1)(\sqrt{x}-5) = 0$$
$$\sqrt{x} = 1 \text{ or } 5$$
$$x = 1 \text{ or } 25$$

10

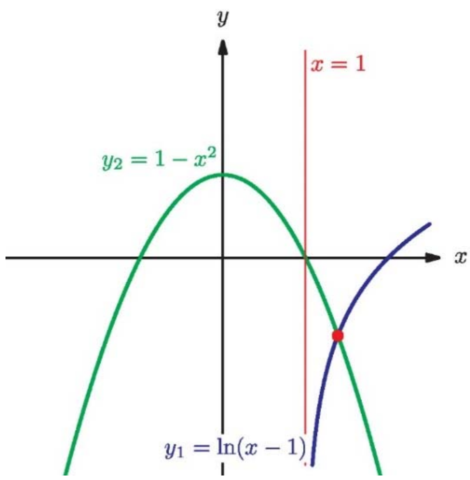
$$(e^x-2)(e^x-5) = 0$$
$$e^x = 2 \text{ or } 5$$
$$x = \ln 2 \text{ or } \ln 5$$

11

$$\log_3 x + \log_3(x+6) = \log_3 27$$
$$\log_3(x^2+6x) = \log_3 27$$
$$x^2+6x-27 = 0$$
$$(x-3)(x+9) = 0$$
$$x = 3 \text{ or } -9$$

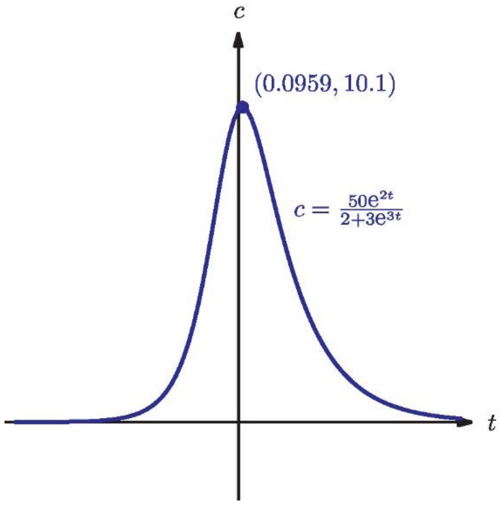
But the domain of $\log_3 x$ is restricted to $x > 0$ so the only valid solution is $x = 3$

12 Graphing $y_1 = \ln(x-1)$ and $y_2 = 1-x^2$ and seeking intersections:



There is a single solution.

13 Graphing $y = \frac{50e^{2x}}{2+3e^{3x}}$:



From GDC, the maximum point is at $(0.0959, 10.1)$ and $y = 0$ is an asymptote for $x \to \pm\infty$.

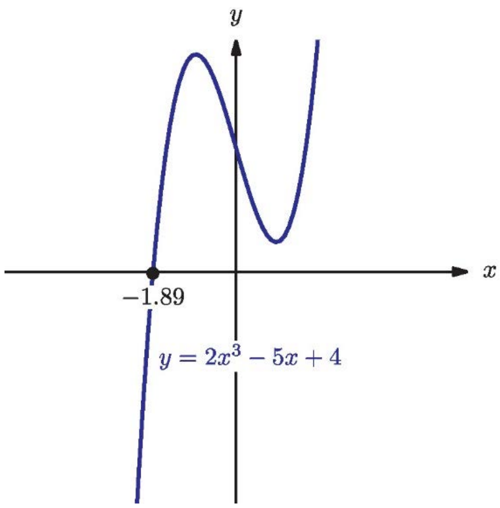
Therefore there will be two real solutions to $\frac{50e^{2x}}{2+3e^{3x}} = k$ for $0 < k < 10.1$

14 **a** Two solutions

b Two solutions

c Zero solutions

15 Graphing $y = 2x^3 - 5x + 4$



The cubic has a single root at $x = -1.89$, and is positive for $x > -1.89$

Therefore $\ln(2x^3 - 5x + 4)$ has vertical asymptote $x = -1.89$ and has domain $x > -1.89$

16 Vertical asymptotes are at roots of the denominator.

$$\begin{aligned} e^{3x} - 9e^x &= e^x(e^{2x} - 9) \\ &= e^x(e^x + 3)(e^x - 3) \end{aligned}$$

Since $e^x > 0$ for all real x, the only root to $e^{3x} - 9e^x = 0$ is $e^x = 3$

The vertical asymptote is $x = \ln 3$

17 a Let r be the common ratio of the geometric sequence.

$u_7 = u_1 \times r^6 = 108 \qquad (1)$

$u_8 = u_1 \times r^7 = 36 \qquad (2)$

$$(2) \div (1): r = \frac{1}{3}$$

b $(1): u_1 = \frac{108}{r^6} = 78\,732$

c $S_k = \frac{u_1(1-r^k)}{1-r} = 118\,096$

Rearranging:

$$r^k = 1 - 118\,096\frac{(1-r)}{u_1}$$

$$k = \log_r\left(1 - 118\,096\frac{(1-r)}{u_1}\right) = 10$$

18

$$\begin{aligned} 4^x &= 4 \times 2^x + 32 \\ (2^x)^2 - 4 \times 2^x - 32 &= 0 \\ (2^x - 8)(2^x + 4) &= 0 \\ 2^x &= 8 \text{ or } -4 \end{aligned}$$

Since $2^x > 0$ for all real x, the only valid solution is $2^x = 8$, so $x = 3$

19

$$\begin{aligned} 4^x + 32 &= 12 \times 2^x \\ (2^x)^2 - 12(2^x) + 32 &= 0 \\ (2^x - 4)(2^x - 8) &= 0 \\ 2^x &= 4 \text{ or } 8 \\ x &= 2 \text{ or } 3 \end{aligned}$$

20 $e^{2x} - 2ke^x + 1 = 0$

Quadratic in e^x, solve using the completed square:

$$\begin{aligned} (e^x - k)^2 &= k^2 - 1 \\ e^x &= k \pm \sqrt{k^2 - 1} \end{aligned}$$

Since $0 < \sqrt{k^2 - 1} < k$ for $k > 1$, both these solutions are positive and so are both valid.

$$x = \ln\left(k \pm \sqrt{k^2 - 1}\right)$$

21 Let the first geometric sequence be $\{u\}$ and the second be $\{v\}$

$$\sum_{k=1}^{\infty} u_k = \frac{a}{1-r} = 76 \qquad (1)$$

$$\sum_{k=1}^{\infty} v_k = \frac{a}{1-r^3} = 36 \qquad (2)$$

$(1): a = 76(1-r)$

$(2): a = 36(1-r^3)$

Then $$76(1-r) = 36(1-r^3)$$
$$76(1-r) = 36(1+r+r^2)(1-r)$$
$$(9r^2+9r-10)(1-r) = 0$$
$$(3r+5)(3r-2)(1-r) = 0$$
$$r = -\frac{5}{3} \text{ or } \frac{2}{3} \text{ or } 1$$

But since the series converge, $|r| < 1$ so the only valid solution is $r = \frac{2}{3}$

22 Let $f(x) = x^3 - x + 4$, so that $f(x) = 0$ has solution -1.796

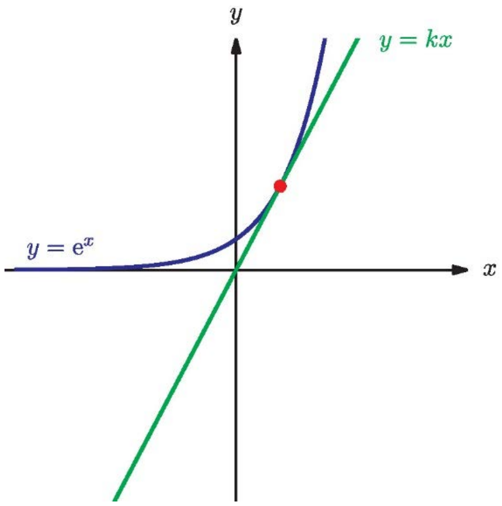
a $f(x-1)$ has the graph of $f(x)$ after a translation one unit to the right.

$f(x-1) = 0$ for $x = -0.796$

b $f(2x)$ has the graph of $f(x)$ after a horizontal stretch with scale factor $\frac{1}{2}$

$f(2x) = 0$ for $x = -0.898$

23 If $e^x = kx$ has only one solution then $y = kx$ is tangent to $y = e^x$, with point of tangency for some value $x > 0$.



a $e^{2x} > e^x$ for $x > 0$ so will lie entirely above the line $y = kx$.

There are no solutions to $e^{2x} = kx$

b $y = -kx$ is the reflection of $y = kx$ through either axis, so will intersect $y = e^x$ for some $x < 0$.

There is one solution to $e^x = -kx$

24 Let the three sides have lengths $x-1, x$ and $x+1$ where x is an integer.

By Pythagoras Theorem, $(x+1)^2 = (x-1)^2 + x^2$

$$x^2 + 2x + 1 = x^2 - 2x + 1 + x^2$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0 \text{ or } x = 4$$

The only solution in context is $x = 4$.

The 3, 4, 5 triangle is the only right-angled triangle whose sides are consecutive integers.

25 Let the a be the length of the hypotenuse.

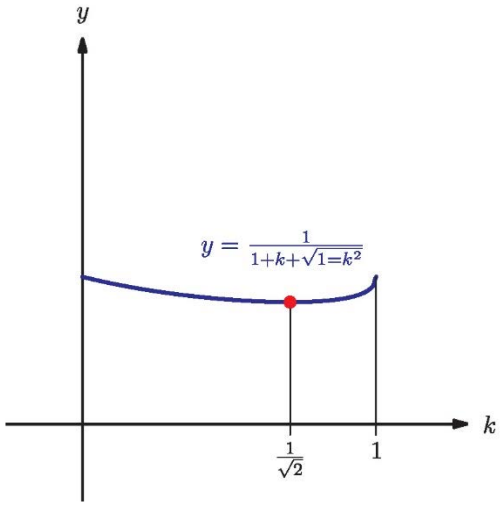
Let the shortest side have length ka for some k such that $0 < k < 1$.

Then the other side has length $\sqrt{a^2 - (ka)^2} = a\sqrt{1-k^2}$

The ratio of the hypotenuse to the perimeter is $\frac{a}{a+ak+a\sqrt{1-k^2}} = \frac{1}{1+k+\sqrt{1-k^2}}$

Let $y(k) = \frac{1}{1+k+\sqrt{1-k^2}}$ for $0 < 1 < k$

Graphing this function:



$$y(k) = \frac{1}{1 + k + \sqrt{1-k^2}}$$

$y(k)$ has a maximum at $k = \frac{1}{\sqrt{2}}$ (ie for the isosceles right-angled triangle)

Maximum ratio is $\frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$

Tip: You could alternatively express the ratio in terms of one of the angles, and calculate the maximum using trigonometry, without a calculator.

If the right-angled triangle has one angle θ then the ratio $\frac{\text{Hyp}}{\text{Opp+Adj+Hyp}} = \frac{1}{\sin\theta + \cos\theta + 1}$

Looking at the denominator:

$1 + \sin\theta + \cos\theta = 1 + \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$, which within $0 < \theta < \frac{\pi}{2}$ has maximum value $1 + \sqrt{2}$ at $\theta = \frac{\pi}{4}$.

Then the maximum value of the ratio is $\frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$

Worked solutions

18 Analysis and approaches: Trigonometry

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 18A

25 Perimeter $= r\theta + 2r = 20.8$ cm

Area $= \frac{1}{2}r^2\theta = 19.2$ cm^2

26 Perimeter $= r\theta + 2r = 27.9$ cm

Area $= \frac{1}{2}r^2\theta = 48.05$ cm^2 $= 48.1$ cm^2 to 3.s.f.

27 Arc length $= r\theta = 1.2r = 12.3$

$r = \frac{12.3}{1.2} = 10.25$ cm $= 10.3$ cm to 3.s.f.

28 a Arc length $= r\theta = 5\theta = 7$

$\theta = \frac{7}{5} = 1.4$ radians

b Area $= \frac{1}{2}r^2\theta = 17.5$ cm^2

29 Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(23)^2\theta = 185$

$$\theta = \frac{2 \times 185}{23^2} = 0.699 \text{ radians}$$

30 Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(2.7)r^2 = 326$

$$r = \sqrt{\frac{2 \times 326}{2.7}} = 15.5 \text{ cm}$$

31 a Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(1.3)r^2 = 87.3$

$$r = \sqrt{\frac{2 \times 87.3}{1.3}} = 11.6 \text{ cm}$$

b Perimeter $= r\theta + 2r = 38.2$ cm

32 Sector Area $= \frac{1}{2}(0.9)(7^2) = 22.05$ cm^2

Rectangle Area $= 7 \times 5 = 35$ cm^2

Total Area $= 57.05 \text{ cm}^2$

Arc length $= 0.9 \times 7 = 6.3$ cm

Straight perimeter $= 2 \times 5 + 2 \times 7 = 28$ cm

Total perimeter $= 34.3$ cm

33 Perimeter $= r\theta + 2r = 3.8r = 26$ cm

$$r = \frac{26}{3.8} = 6.84 \text{ cm}$$

34 Perimeter $= r\theta + 2r = (2+\theta)r = 30$ cm (1)

Area $= \frac{1}{2}r^2\theta = 18 \text{ cm}^2$ (2)

$$(1): \theta = \frac{30}{r} - 2$$

Substituting into (2):

$$r^2\left(\frac{30}{r} - 2\right) = 36$$
$$2r^2 - 30r + 36 = 0$$
$$r^2 - 15r + 18 = 0$$
$$r = \frac{15 \pm \sqrt{(-15)^2 - 4(1)(18)}}{2(1)}$$
$$= 1.32 \text{ cm or } 13.7 \text{ cm}$$

Corresponding values of θ would be $\theta = \frac{30}{r} - 2 = 20.8$ or 0.192

In context, $\theta \leqslant 2\pi$ so reject the first solution.

$r = 13.7$ cm is the only valid solution.

35 Each of the sides is a $60°$ ($\frac{\pi}{3}$ radian) arc centred at the opposite vertex.

$$\text{Perimeter} = 3 \times \frac{\pi}{3} \times 12 = 12\pi \text{ cm}$$

36 Arc length $= r\theta = 0.9 \times 8 = 7.2$ cm

Cosine Rule:

$$PQ = \sqrt{8^2 + 8^2 - 2(8)(8)\cos 0.9} = 6.96 \text{ cm}$$

Difference in lengths is $7.2 - 6.96 = 0.241$ cm

37 Area:

$$\text{Sector area} = \frac{1}{2}r^2\theta = \frac{1}{2}(12)^2(0.6) = 43.2 \text{ cm}^2$$

Sine Rule for area:

$$\text{Triangle } OAB \text{ area} = \frac{1}{2}(12)(12)\sin 0.6 = 40.7 \text{ cm}^2$$

$$\text{Shaded region} = 43.2 - 40.7 = 2.55 \text{ cm}^2$$

Perimeter:

Arc length: $r\theta = 12 \times 0.6 = \frac{36}{5}$

Let O denote the origin

Let X be the midpoint of AB, let the distance $AX \coloneqq x$

$\angle BOA = 0.6 \;\therefore\; \angle XOB = 0.3$

Using trig. $\sin(0.3) = \frac{x}{12}$

$x = 12\sin(0.3)$

Now, the shaded perimeter: $2x + \frac{36}{5} = 24\sin(0.3) + \frac{36}{5} = 14.292\ldots$

Perimeter: 14.3cm

38 a Sector area $= \frac{1}{2}r^2\theta = 8\theta$

Sine Rule for area:

Triangle OPQ area $= \frac{1}{2}(4)(4)\sin\theta = 8\sin\theta$

Shaded area $= 8(\theta - \sin\theta) = 6$

$$\theta - \sin\theta = \frac{6}{8} = 0.75$$

b From GDC, $\theta = 1.74$ radians

c Perimeter of shaded region $= r\theta + \sqrt{2r^2 - 2r^2\cos\theta} = 13.1$ cm

39 The shaded region is the difference between two sector areas:

$$\text{Shaded area} = \frac{1.2}{2}((15+x)^2 - 15^2)$$
$$0.6(x^2 + 30x) = 59.4$$
$$x^2 + 30x - 99 = 0$$
$$(x-3)(x+33) = 0$$

$x = 3$ is the only positive solution

40 Radius r of the sector becomes the slant length of the cone

$r = \sqrt{22^2 + 8^2} = 23.4$ cm

Arclength of the sector becomes the circumference of the cone base

$$r\theta = 2\pi \times 8$$
$$\theta = \frac{16\pi}{r} = 2.15 \text{ radians}$$

41 If the two circle intersections are P and Q then PC_1C_2 is equilateral, since each side is a radius of one circle.

Then $P\widehat{C_1}Q = \frac{2\pi}{3}$ radians

The shaded region is the sum of two identical segments.

$$\text{Segment area} = \frac{r^2}{2}(\theta - \sin\theta)$$

$$\text{Shaded area} = r^2(\theta - \sin\theta) = 8^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) = 8^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 78.6\text{ cm}^2$$

$$\text{Perimeter} = 2r\theta = \frac{32\pi}{3} = 33.5\text{ cm}$$

Exercise 18B

25 $\sin x + \sin(2\pi - x) = \sin x - \sin x = 0$

26 $\sin 60° \cos 30° + \cos 60° \sin 30° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = 1 = \sin 90°$

27 $\left(\tan\left(\frac{\pi}{3}\right) - 1\right)^2 = (\sqrt{3} - 1)^2 = 3 - 2\sqrt{3} + 1 = 4 - 2\sqrt{3}$

28

$$\begin{aligned}\tan\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) &= \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\\ &= \frac{5}{6}\sqrt{3}\end{aligned}$$

29 $\cos x + \cos\left(x + \frac{\pi}{2}\right) + \cos(x + \pi) + \cos\left(x + \frac{3\pi}{2}\right) + \cos(x + 2\pi)$

$$\begin{aligned}&= \cos x - \sin x - \cos x + \sin x + \cos x\\ &= \cos x\end{aligned}$$

30 Cosine Rule:

$$\begin{aligned}BC &= \sqrt{AB^2 + AC^2 - 2(AB)(AC)\cos A\hat{B}C}\\ &= \sqrt{81 + 16 - 72 \times \frac{1}{2}}\\ &= \sqrt{61}\text{ cm}\end{aligned}$$

31 Sine Rule for area

$$35 = \frac{1}{2}(8)(11)\sin\theta°$$

$$\sin\theta° = \frac{35}{44}$$

$$\theta_1 = 52.7, \theta_2 = 180 - \theta_1 = 127$$

32 Cosine Rule:

$$5.3^2 = 7.5^2 + AC^2 - 2(7.5)(AC)\cos 44°$$

$$AC^2 - 2(7.5)(AC)\cos 44° + 7.5^2 - 5.3^2 = 0$$

$$AC = 7.5\cos 44° \pm \sqrt{(7.5\cos 44°)^2 + (5.3^2 - 7.5^2)}$$
$$= 4.42 \text{ cm or } 6.37 \text{ cm}$$

33 Sine Rule:

$$K\hat{M}L = \sin^{-1}\left(\frac{12\sin 55°}{15}\right) = 40.9°$$

There is no second possibility; $KL < LM$ so $K\hat{M}L < M\hat{K}L$, since side lengths are ranked in the same order as opposite angles.

$$K\hat{L}M = 180 - 55 - 40.9 = 84.1°$$

Sine Rule:

$$KM = \frac{15\sin 84.1°}{\sin 55°} = 18.2 \text{ cm}$$

34 Cosine Rule:

$$c = \sqrt{2^2 + 5^2 - 2(2)(5)\cos 120°}$$
$$= \sqrt{4 + 25 - 20\left(-\frac{1}{2}\right)}$$
$$= \sqrt{39}$$

35 If angle is 30° then the gradient is $\tan 30° = \frac{1}{\sqrt{3}}$

The equation is $x - y\sqrt{3} = 0$ or $y = \frac{x}{\sqrt{3}}$

36 **a** Gradient $= \frac{\text{rise}}{\text{tread}}$

If the angle is θ then for a tread of δx, rise $= \tan\theta \ \delta x$

So the gradient is $\frac{\delta x \tan\theta}{\delta x} = \tan\theta$

Hence the line equation is $y = x\tan\theta$

b The line making an angle of $180° + \theta$ with the x-axis is the same line, so has the same gradient.

Since the gradient is given as $\tan\theta$, it follows that $\tan\theta = \tan(180° + \theta)$

c $\tan\theta \times \tan(90° + \theta)$ is the product of gradients of two perpendicular lines, one at an angle θ to the x-axis and one at right angles to it.

The product of two perpendicular gradients is always -1.

$$\tan\theta \times \tan(90° + \theta) = -1$$

Exercise 18C

15 a For $\pi < \theta < 2\pi$, $\sin\theta < 0$

$$\begin{aligned}\sin\theta &= -\sqrt{1-\cos^2\theta}\\ &= -\sqrt{1-\frac{4}{25}}\\ &= -\frac{\sqrt{21}}{5}\end{aligned}$$

b

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{2}{5}\right)\left(-\frac{\sqrt{21}}{5}\right)\\ &= -\frac{4\sqrt{21}}{5}\end{aligned}$$

16 a For $\frac{\pi}{2} < x < \pi$, $\cos x < 0$

$$\begin{aligned}\cos x &= -\sqrt{1-\sin^2 x}\\ &= -\sqrt{1-\frac{9}{49}}\\ &= -\frac{2\sqrt{10}}{7}\end{aligned}$$

b

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ &= \frac{3}{-2\sqrt{10}}\\ &= -\frac{3\sqrt{10}}{20}\end{aligned}$$

17 a

$$\begin{aligned}\cos 2\theta &= 1-2\sin^2\theta\\ &= 1-2\left(\frac{16}{81}\right)\\ &= \frac{49}{81}\end{aligned}$$

b

$$\begin{aligned}\cos\theta &= \pm\sqrt{1-\sin^2\theta}\\ &= \pm\sqrt{1-\frac{16}{81}}\\ &= \pm\frac{\sqrt{65}}{9}\end{aligned}$$

18 $3\sin^2 x + 7(1 - \sin^2 x) = 7 - 4\sin^2 x$

19 $4\cos^2 x - 5(1 - \cos^2 x) = 9\cos^2 x - 5$

20

$$\begin{aligned} 1 + \tan^2\theta &\equiv 1 + \frac{\sin^2 x}{\cos^2 x} \\ &\equiv \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &\equiv \frac{1}{\cos^2 x} \end{aligned}$$

21 $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

$$\begin{aligned} &\equiv (\sin^2 x + 2\sin x\cos x + \cos^2 x) + (\sin^2 x - 2\sin x\cos x + \cos^2 x) \\ &\equiv 2(\sin^2 x + \cos^2 x) \\ &\equiv 2 \end{aligned}$$

22 Let $x = 15°$

$$\cos 2x = \cos 30° = \frac{\sqrt{3}}{2}$$

But $\cos 2x = 2\cos^2 x - 1$

$$\begin{aligned} \text{So } \cos x &= \sqrt{\frac{1 + \cos 2x}{2}} \\ &= \sqrt{\frac{2 + \sqrt{3}}{4}} \end{aligned}$$

23 **a** $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

b Let $x = \frac{\pi}{8}$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\begin{aligned} \text{So } \sin x &= \sqrt{\frac{1 - \cos 2x}{2}} \\ &= \sqrt{\frac{2 - \sqrt{2}}{4}} \end{aligned}$$

24 If $0 < x < \frac{\pi}{2}$ then $\cos x > 0$

$$\cos 2x = 2\cos^2 x - 1$$

$$\text{So } \cos x = \sqrt{\frac{1+\cos 2x}{2}}$$

$$= \sqrt{\frac{11}{12}}$$

$$= \sqrt{\frac{33}{36}}$$

$$= \frac{\sqrt{33}}{6}$$

25 If $0 < x < \frac{\pi}{2}$ then $\sin x > 0$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\text{So } \sin x = \sqrt{\frac{1-\cos 2x}{2}}$$

$$= \sqrt{\frac{1}{3}}$$

$$= \frac{\sqrt{3}}{3}$$

26

$$\begin{aligned}\cos 4x &= 2\cos^2 2x - 1\\ &= 2(2\cos^2 x - 1)^2 - 1\\ &= 2\left(-\frac{1}{9}\right)^2 - 1\\ &= -\frac{79}{81}\end{aligned}$$

27 a $\cos 2x \equiv 1 - 2\sin^2 x$

b $\cos 2x \equiv 2\cos^2 x - 1$

$$\begin{aligned}\cos 4x &\equiv 2\cos^2 2x - 1\\ &\equiv 2(1-2\sin^2 x)^2 - 1\\ &\equiv 2 - 8\sin^2 x + 4\sin^4 x - 1\\ &\equiv 1 - 8\sin^2 x + 4\sin^4 x\end{aligned}$$

28

$$\begin{aligned}\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} &\equiv \frac{1}{\sin^2 A} - \frac{\cos^2 A}{\sin^2 A}\\ &\equiv \frac{1-\cos^2 A}{\sin^2 A}\\ &\equiv \frac{\sin^2 A}{\sin^2 A}\\ &\equiv 1\end{aligned}$$

29

$$\frac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \frac{2\sin^2\theta}{2\cos^2\theta}$$
$$\equiv \tan^2\theta$$

30

$$\frac{\sin 2x}{\tan x} \equiv 2\sin x\cos x \times \frac{\cos x}{\sin x}$$
$$\equiv 2\cos^2 x$$

31

$$\frac{1}{\cos\theta} - \cos\theta \equiv \frac{1-\cos^2\theta}{\cos\theta}$$
$$\equiv \frac{\sin^2\theta}{\cos\theta}$$
$$\equiv \sin\theta\tan\theta$$

32

$$2\sin 4x \equiv 4\sin 2x\cos 2x = 3\sin 2x$$

Then $\sin 2x = 0$ or $\cos 2x = \frac{3}{4}$

There are no solutions for $\sin 2x = 0$ for $0 < x < \frac{\pi}{2}$

$$\cos 2x \equiv 2\cos^2 x - 1 = \frac{3}{4}$$

$$\cos^2 x = \frac{7}{8}$$

$\cos x > 0$ for $0 < x < \frac{\pi}{2}$

$$\cos x = \frac{\sqrt{7}}{\sqrt{8}} = \frac{\sqrt{14}}{4}$$

33 a $\sin\frac{3\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}}$

$\cos x > 0$ for $0 < x < \frac{\pi}{2}$ so $\cos\frac{3\pi}{8} > 0$

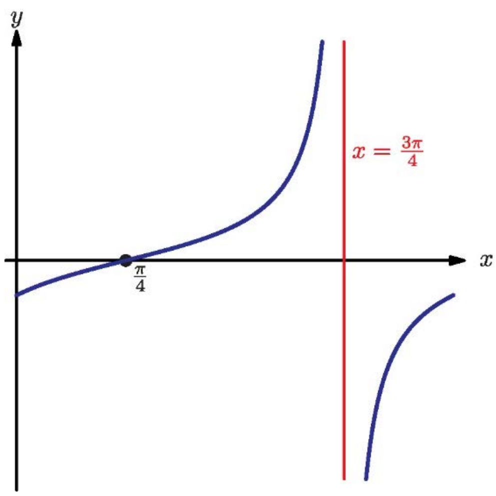
$$\cos\frac{3\pi}{8} = \sqrt{1-\sin^2\frac{3\pi}{8}}$$
$$= \sqrt{1-\frac{2+\sqrt{2}}{4}}$$
$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

b

$$\tan\frac{3\pi}{8} = \frac{\sin\frac{3\pi}{8}}{\cos\frac{3\pi}{8}}$$
$$= \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$
$$= \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$
$$= \sqrt{\frac{\left(2+\sqrt{2}\right)^2}{4-2}}$$
$$= \frac{2+\sqrt{2}}{\sqrt{2}}$$
$$= \sqrt{2}+1$$

Exercise 18D

13



14 $\sin x$ has period 2π so $3\sin 4x$ has period $\frac{\pi}{2}$

15 Range of $\cos 2x$ is $-1 < \cos 2x < 1$

So range of $g(x) = 10\cos 2x - 7$ is $-10 - 7 < g(x) < 10 - 7$

Minimum value is -17 and maximum value is 3.

16 Range of $\cos \pi t$ is $-1 < \cos \pi t < 1$

So range of $h = 0.8 - 0.6\cos \pi t$ is $0.8 - 0.6 < h < 0.8 + 0.6$

Maximum height is 1.4 m

17 a Range of $\sin\left(\frac{\pi t}{12}\right)$ is $-1 < \sin\left(\frac{\pi t}{12}\right) < 1$

So range of $d = 5 + 1.6\sin\left(\frac{\pi t}{12}\right)$ is $5 - 1.6 < d < 5 + 1.6$

High tide is 6.6 m.

b $\sin\left(\frac{\pi t}{12}\right) = 1$ when $\frac{\pi t}{12} = \frac{\pi}{2}$

$t = 6$

High tide occurs at 6 a. m.

18 a Period of $\cos\left(\frac{\pi t}{4}\right)$ is $\frac{2\pi}{\left(\frac{\pi}{4}\right)} = 8$ minutes

b Maximum height is $6.2 + 4.8 = 11$ m

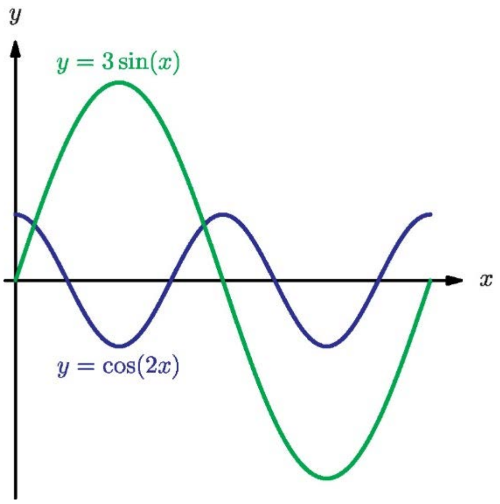
c When $t = 2\frac{2}{3}$, $h = 6.2 - 4.8\cos\left(\frac{2\pi}{3}\right) = 8.6$ m above ground.

19 Maximum value is 5 and minimum value is -5 so $a = 5$

Period is 180° so $b = \frac{360}{180} = 2$

20 Period is $\frac{\pi}{3}$ so $b = \frac{\pi}{\left(\frac{\pi}{3}\right)} = 3$

21 a



b The graphs intersect in two places in that interval so there are two solutions.

c The smallest common period is 2π so in three periods there will be three times as many: 6

22 a Maximum h is $1.4 + 0.2 = 1.6$ m

b Period is $\frac{2\pi}{15}$

$$3 \div \frac{2\pi}{15} = \frac{45}{2\pi} \approx 7.2$$

There are 7 complete oscillations.

c

$$1.4 - 0.2\cos(15t) = 1.5$$
$$\cos 15t = -\frac{1}{2}$$
$$15t = \frac{4\pi}{3}$$
$$t = \frac{4\pi}{45} = 0.279 \text{ seconds}$$

23 $\sin 2x$ has period π.

$\sin 6x$ has period $\frac{\pi}{3}$.

The lowest common multiple of the two (where periods values are equal after integer multiplication) is π, so the period of any composite function of the two will be π.

24 **a** Vertical asymptotes appear when the denominator is zero.

Since $3\sin 2x$ has range $-3 < 3\sin 2x < 3$, this cannot happen, so there are no vertical asymptotes.

b Since the denominator is always positive, the maximum value of $f(x)$ will occur when the denominator is minimum.

Require $\sin 2x = -1$

$$2x = \frac{3\pi}{2}$$
$$x = \frac{3\pi}{4}$$
$$f\left(\frac{3\pi}{4}\right) = \frac{3}{4}$$

Exercise 18E

33 From GDC, solutions $x = 0.702, 4.41$

34 $\sin x = \frac{1}{2}$

$x = 30°, 150°$

35 $\tan x = \pm 1$

$x = \pm 45°, \pm 135°$

36 $\cos(x - 10°) = -\frac{1}{2}$

$$x - 10° = 120°, 240°$$
$$x = 130°, 250°$$

37 **a** $\cos 2x \equiv 2\cos^2 x - 1$

So $2\cos^2 x - 1 = \cos x - 1$

$2\cos^2 x - \cos x = 0$

b $\cos x\,(2\cos x - 1) = 0$

$\cos x = 0$ or $\cos x = \frac{1}{2}$

$x = 90°$ or $270°$ or $60°$ or $360°$

38 $\sin x = \cos x$

$\tan x = 1$

$x = 45°, 225°$

39

$$\sin 2x \equiv 2\sin x\cos x$$
$$2\sin x\cos x = \sin x$$
$$\sin x\,(2\cos x - 1) = 0$$
$$\sin x = 0 \text{ or } \cos x = \frac{1}{2}$$
$$x = 0 \text{ or } \pi \text{ or } \frac{\pi}{3}$$

40

$$\cos^2\theta \equiv 1 - \sin^2\theta$$
$$2(1-\sin^2\theta) - \sin\theta - 1 = 0$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or} - 1$$
$$\theta = 30° \text{ or } 150° \text{ or } 270°$$

41

$$\cos^2 x + 3\cos x = 3(1-\cos^2 x)$$
$$4\cos^2 x + 3\cos x - 3 = 0$$
$$\cos x = \frac{-3 \pm \sqrt{3^2 + 4(4)(3)}}{8} = -\frac{3}{8} \pm \frac{\sqrt{57}}{8}$$

Since $\sqrt{57} > 7$, the negative root will be less than -1 and so outside the range of $\cos x$.

The only valid solution is $\cos x = \frac{-3+\sqrt{57}}{8}$

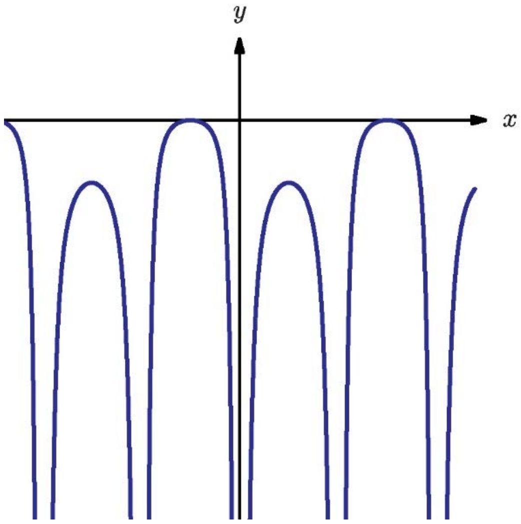
42

$$\cos 2x \equiv 2\cos^2 x - 1$$
$$2\cos^2 x - 1 = \cos x$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$\cos x = -\frac{1}{2} \text{ or } 1$$
$$x = \pm\frac{2\pi}{3} \text{ or } 0$$

43 Rearranging:

$$k = -\frac{1+\sin x}{\sin^2 x} = g(x)$$

Plotting the graph of $g(x)$:



The graph has range $g(x) \leqslant 0$

Therefore there are solutions for $k \leqslant 0$

Tip: If you had to solve this algebraically without a calculator:

$$k = \frac{-1-\sin x}{\sin^2 x}$$

Numerator has range $-2 \leqslant -1 - \sin x \leqslant 0$ with the zero value when $x = 1.5\pi$

Denominator has range $0 \leqslant \sin^2 x \leqslant 1$ with the zero value when $x = n\pi$ for $n \in \mathbb{Z}$

Since the numerator and denominator do not have common roots, the function, where defined, is never positive but can equal zero.

Allowing the denominator to get close to zero will make the overall value arbitrarily small (and negative). The range is therefore $k \leqslant 0$

Mixed Practice

1 **a** Amplitude is 1.3 m

b Period $= \frac{2\pi}{2.5} = 2.51$ m

2

$$\begin{aligned}\cos\left(\frac{\pi}{6}\right) - \tan\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\\ &= \frac{3}{2\sqrt{3}} - \frac{2}{2\sqrt{3}}\\ &= \frac{1}{2\sqrt{3}}\\ &= \frac{\sqrt{3}}{6}\\ &= \frac{1}{6}\tan\left(\frac{\pi}{3}\right)\end{aligned}$$

3 $\sin x = \frac{\sqrt{3}}{2}$

$x = 60°, 120°$

4 Sine Rule:

$$\sin\theta = \frac{6\sin 38°}{4.5}$$

$\theta = 55.2°$ or $125°$

5 From calculator, solutions are $0.695, 2.45, 3.54, 5.88$

6

$$\cos\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$$
$$2x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \ldots$$
$$2x = 0, \frac{4\pi}{3}, 2\pi, \frac{10\pi}{3}, 4\pi, \ldots$$
$$x = 0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$

7 Obtuse angle: $90° < A < 180°$

a Obtuse angle: $90° < A < 180°$

$\cos A < 0$ for angle in this interval.

$$\cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}$$

b $\tan A \equiv \frac{\sin A}{\cos A} = -\frac{5}{12}$

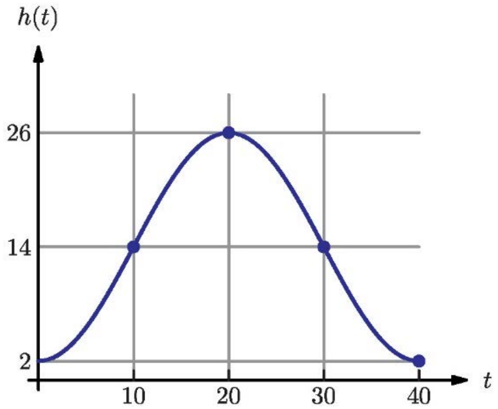
c $\sin 2A \equiv 2\sin A\cos A = -\frac{120}{169}$

8 **a** **i** 14 m

ii 26 m

b Complete rotation takes 40 seconds, so the seat is at A after 10 seconds and at B after 30 seconds.

c

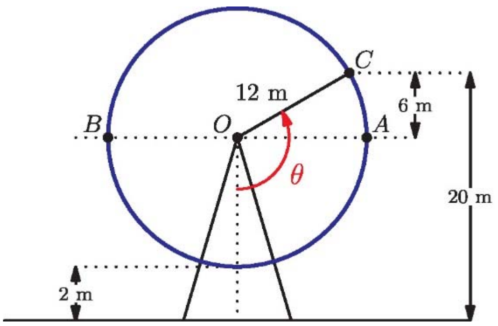


d **i** Amplitude $a = 12$

ii Frequency $b = \frac{2\pi}{\text{period}} = \frac{2\pi}{40} = \frac{\pi}{20}$

iii Central value $c = 14$

e **i**



ii Angle $\theta = 90° + \sin^{-1}\left(\frac{6}{12}\right) = 120°$

iii

$$bT = \frac{2\pi}{3}$$
$$T = \frac{2\pi}{3b} = \frac{40}{3}$$
$$T = 13.3 \text{ seconds}$$

9 **a** $\frac{\pi}{2} < x < \pi$ so $\cos x < 0$

$$\cos x = -\sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

b

$$\cos 2x = 1 - 2\sin^2 x$$
$$= 1 - \frac{18}{16}$$
$$= -\frac{1}{8}$$

10

$$\tan x = \pm\frac{1}{\sqrt{3}}$$
$$x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$$

11

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$
$$\equiv \frac{1}{\sin\theta\cos\theta}$$
$$\equiv \frac{2}{\sin 2\theta}$$

12 Cosine Rule:

$$15^2 = 10^2 + x^2 - 2(10)x\cos 120°$$
$$225 = 100 + x^2 + 10x$$
$$x^2 - 10x - 115 = 0$$
$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-125)}}{2(1)}$$
$$= -5 \pm \sqrt{150}$$

Since $x > 0$ there is only one solution

$$x = 5\left(\sqrt{6} - 1\right) \text{ cm}$$

13 Sine Rule:

$$\sin A\hat{C}B = \frac{10\sin 40°}{7}$$
$$A\hat{C}B = 66.7 \text{ or } 113°$$
$$B\hat{A}C = 180 - 40 - A\hat{C}B = 73.3° \text{ or } 26.7°$$

Sine Rule for area:

$$\text{Area } ABC = \frac{1}{2}(AB)(AC)\sin B\hat{A}C$$
$$= 33.5 \text{ cm}^2 \text{ or } 15.7 \text{ cm}^2$$

The difference in area of the two possible triangles is $33.5 - 15.7 = 17.8 \text{ cm}^2$

14 **a**

$$\tan 2x \equiv \frac{\sin 2x}{\cos 2x}$$
$$\equiv \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$$

Dividing numerator and denominator by $\cos^2 x$:

$$\tan 2x \equiv \frac{2\tan x}{1 - \tan^2 x}$$

b

$$\tan 2x + \tan x \equiv \frac{2\tan x}{1 - \tan^2 x} + \tan x$$
$$\equiv \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x}$$
$$\equiv \frac{\tan x\,(3 - \tan^2 x)}{1 - \tan^2 x}$$

If $\tan 2x + \tan x = 0$ then $\tan x\,(3 - \tan^2 x) = 0$

$$\tan x = 0 \text{ or } \tan x = \pm\sqrt{3}$$
$$x = 0 \text{ or } \pi \text{ or } \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

15

$$\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\equiv\frac{(1-\cos x)+(1+\cos x)}{(1+\cos x)(1-\cos x)}$$
$$\equiv\frac{2}{1-\cos^2 x}$$
$$\equiv\frac{2}{\sin^2 x}$$

16

$$y(0)=3=a\tan c \qquad (1)$$
$$y\left(\frac{3\pi}{4}\right)=0=a\tan\left(\frac{3\pi}{4}+c\right) \qquad (2)$$
$$(2): c=\frac{\pi}{4}$$
$$(1): 3=a$$

17 a $f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{2}\right)+k=1+k=6$

$k=5$

b Minimum value is $k-1=4$

c $p=-\frac{\pi}{4}, q=5$

18

$$2\cos^2 x-5x+2=0$$
$$(2\cos x-1)(\cos x-2)=0$$
$$\cos x=\frac{1}{2}\text{ or }2$$

Reject the value 2 as it lies outside the range of $\cos x$ for real x.

For $0<x<2\pi$, $\cos x=0.5$ has solutions $\frac{\pi}{3}$ or $\frac{5\pi}{3}$

19 If the two endpoints of the circle arc are PQ and it is assumed that the centre O of the circle from which the sector is taken lies at the midpoint of the right side of the rectangle then the height of the rectangle is the length of chord PQ and the width of the rectangle equals the radius of the circle.

$$\text{Sector area}=\frac{1}{2}r^2\theta=\frac{1}{2}r^2=7$$
$$r=\sqrt{14}=3.74\text{ cm}\approx 37\text{ mm}$$

Cosine Rule on triangle OPQ

$$PQ=\sqrt{OP^2+OQ^2-2(OP)(OQ)\cos P\hat{O}Q}$$
$$=\sqrt{r^2+r^2-2r^2\cos 1}$$
$$=3.59\text{ cm}\approx 36\text{ mm}$$

Challenge to student: Find the height and width of the rectangle with least area which can contain this sector.

20 a $OC = 4 = OB, BC = 3$

Cosine Rule:

$$B\hat{O}C = \cos^{-1}\left(\frac{4^2 + 4^2 - 3^2}{2(4)(4)}\right) = 44.0° \text{ (0.769 radians)}$$

b Segment BC is the area of sector OBC less the area of triangle OBC. The shaded region is double that.

$$\begin{aligned}\text{Shaded area} &= r^2(\theta - \sin\theta)\\ &= 4^2(0.769 - \sin 0.769)\\ &= 1.18 \text{ cm}^2\end{aligned}$$

21

$$\begin{aligned}\sin 4x &\equiv 2\sin 2x\cos 2x\\ &\equiv 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)\\ &\equiv 4\sin x\cos^3 x - 4\sin^3 x\cos x\end{aligned}$$

22

Tip: Remember that when establishing an identity, it is usually best to start at the more convoluted expression and simplify towards the other. In this case, showing the identity is more intuitive when starting from the right side.

$$\begin{aligned}\frac{\tan x}{1+\tan^2 x} &\equiv \frac{\sin x}{\cos x\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)}\\ &\equiv \frac{\sin x\cos x}{\cos^2 x + \sin^2 x}\\ &\equiv \frac{\sin x\cos x}{1}\\ &\equiv \frac{2\sin x\cos x}{2}\\ &\equiv \frac{\sin 2x}{2}\end{aligned}$$

23 $\sin\left(\frac{x}{2}\right)$ has period 4π.

$\cos\left(\frac{x}{5}\right)$ has period 10π.

The lowest common multiple of the two (where periods values are equal after integer multiplication) is 20π, so the period of any composite function of the two will be π.

24 $f(x) = (4\cos(x-\pi) - 1)^2$

Let $g(x) = 4\cos(x-\pi) - 1$ so that $f(x) = \left(g(x)\right)^2$

$g(x)$ has range $-5 \leqslant g(x) \leqslant 3$

So $f(x)$ has range $0 \leqslant f(x) \leqslant 25$

25

$I(t) = a \sin bt$

$I(1) = a \sin b = 0.2 = \frac{1}{5}$

$I(2) = a \sin 2b = -0.1 = -\frac{1}{10}$

Now,

$a \sin 2b \equiv 2a \sin b \cos b$

Therefore, we have $2a \sin b \cos b = -\frac{1}{10}$

$$2 \cos b \, (a \sin b) = -\frac{1}{10}$$
$$2 \cos b \left(\frac{1}{5}\right) = -\frac{1}{10}$$
$$\cos b = -\frac{1}{4}$$

Using $\sin^2 x + \cos^2 x \equiv 1$

$$\sin b = \sqrt{1 - \left(-\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

$a \sin b = \frac{1}{5}$ ∴ we now know that $a\frac{\sqrt{15}}{4} = \frac{1}{5} \Rightarrow a = \frac{4}{5\sqrt{15}}$

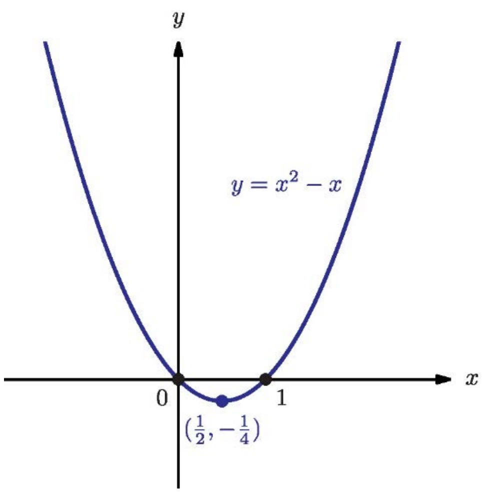
$I(t) = a \sin bt$ will have a maximum of $|a| = \frac{4}{5\sqrt{15}} = 0.2065 \ldots = 0.207$ amps

26 In the double period, there will be four solutions.

If $k = \arcsin x$ then the other three solutions are $\pi - k, 2\pi + k, 3\pi - k$

The sum of these four solutions is $k + (\pi - k) + (2\pi + k) + (3\pi - k) = 6\pi$

27 **a** Positive quadratic, roots at 0 and 1, vertex at $x = \frac{1}{2}$.



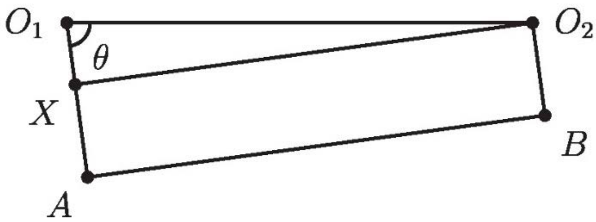
b If $\sin^2 x - \sin x = k$ then there are only solutions for $-1 \leqslant \sin x \leqslant 1$

The range of the function in part **a** for domain restricted to $[-1,1]$ is $[-0.25,2]$ as shown in the diagram.

So there for solutions to the equation $\sin^2 x - \sin x = k$, k must lie in the interval $-0.25 \leqslant k \leqslant 2$

28 a AB must be perpendicular to both O_1A and O_2B, since the line segment AB is tangent to the circles.

Then O_1O_2BA is a trapezium, with parallel side lengths 6 and 10, perpendicular base and oblique side length 30.



Considering the trapezium as a right-angled triangle O_1XO_2 atop a rectangle O_2XAB, the side lengths of the right triangle are $O_1X = 4, O_1O_2 = 30$

So angle $X\hat{O}_1O_2 = \cos^{-1}\left(\frac{4}{30}\right) = 82.3° = 1.44$ radians

b Considering the same triangle, $O_2X = \sqrt{30^2 - 4^2} = \sqrt{884} = AB = CD$

Then the bicycle chain consists of the large arc AD, the small arc BC and twice the length of the line segment AB

Since ABO_2O_1 is a trapezium with $O_1A \parallel O_2B$, $B\hat{O}_2O_1 = \pi - A\hat{O}_1O_2 = 1.70$

$$\text{arc } AD = (2\pi - 2 \times 1.44) \times 10 = 34.1 \text{ cm}$$
$$\text{arc } BC = (2\pi - 2 \times 1.70) \times 6 = 17.2 \text{ cm}$$

Total bike chain length $= 34.1 + 17.2 + 2\sqrt{884} = 111$ cm

29 $\sin x + \cos x = \frac{2}{3}$

Squaring:

$$\sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{4}{9}$$
$$1 + 2\sin x \cos x = \frac{4}{9}$$
$$2\sin x \cos x = -\frac{5}{9}$$

But $2\sin x \cos x \equiv \sin 2x$

$$\sin 2x = -\frac{5}{9}$$

$$\begin{aligned}\cos 4x &\equiv 1 - 2\sin^2 2x \\ &= 1 - 2\left(-\frac{5}{9}\right)^2 \\ &= 1 - \frac{50}{81} \\ &= \frac{31}{81}\end{aligned}$$

19 Analysis and approaches: Statistics and probability

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 19A

5 **a** $x = 15.5 - 0.0620y$

b $15.5 - 0.0620(205) = 2.8$ km

c $r^2 = 0.789$ so the correlation is reasonable.

The estimate value lies within the interval of the data values.

The estimate is reliable.

6 Predicting number from temperature: Use $x = 12.5y + 68.2$

$12.5(23.6) + 68.2 = 363$ visitors estimated

7 **a** Regression line: arm length $= 0.376(\text{height}) + 5.07$

If height $= 161$ then estimate arm length $= 65.6$ cm

b Regression line: height $= 0.984(\text{arm length}) + 93.7$

If arm length $= 56$ then estimate height $= 149$ cm

8 Paper 1 score is x, paper 2 score is y.

Regression line: $x = -0.756 + 1.30y$

Predicted value: $-0.756 + 1.30(64) = 76$ marks estimate in paper 1.

9 **a** When predicting the x value from a known y value, you should use the x-on-y regression line, with y as the predictor variable and x as the response variable.

b Regression line: $x = 26.6 + 0.835y$

Predicted value: $26.6 + 0.835(55) = 75$

c No. 72 lies outside the data value interval for chemistry scores; extrapolating beyond the known data is not reliable.

10 **a** When $y = 2.5$, the x-on-y regression line predicts profit $1.01(2.5) + 1.32 = 3.845$

Estimate a profit of \$3845 the following month.

b The two regression lines cross at the mean point $(\bar{x}, \bar{y})$

From GDC, the intersection of the two lines is at $(4.15, 2.80)$.

So the mean monthly profit is \$4150 and the mean advertising spend is \$280.

11 a Since $r^2 = 1$, it follows that $r = \pm 1$

Since the slope of the regression line is negative, $r = -1$

b The regression lines are the same. Rearranging:

$$\begin{aligned} y &= -0.5x + 4 \\ -2y &= x - 8 \\ x &= -2y + 8 \end{aligned}$$

12 a The gradient of the y-on-t line is 6, which indicates that on the best fit linear model, the snail travels at approximately 6 cm min^{-1}.

This is only an estimate; the regression equation does not cross the origin (intercept is 5) and the r^2 values is only 0.6, indicating that the linear model may not be ideal, and that the snail is not showing a constant speed.

b

$$\begin{aligned} Y &= \frac{y}{100}, T = \frac{t}{60} \\ \frac{y}{100} &= \frac{6t + 5}{100} \\ Y &= \frac{36t + 30}{600} \\ Y &= 3.6T + 0.05 \end{aligned}$$

c Correlation coefficient is unaffected by a change of scale so the correlation coefficient between Y and T is also 0.6.

Exercise 19B

16 a $\text{P}(A|B) = \frac{\text{P}(A \cap B)}{\text{P}(B)} = \frac{1}{3}$

b $\text{P}(B|A) = \frac{\text{P}(A \cap B)}{\text{P}(A)} = \frac{1}{4}$

17 a

$$\begin{aligned} \text{P}(A \cap B) &= \text{P}(A) \times \text{P}(B|A) \\ &= 0.3 \times 0.5 = 0.15 \end{aligned}$$

b

$$\begin{aligned} \text{P}(B|A) &= \frac{\text{P}(A \cap B)}{\text{P}(A)} \\ &= \frac{0.15}{0.5} = 0.3 \end{aligned}$$

18 a

$$\begin{aligned} \text{P}(A \cap B) &= \text{P}(A) + \text{P}(B) - \text{P}(A \cup B) \\ &= 0.5 + 0.4 - 0.8 = 0.1 \end{aligned}$$

b $\text{P}(A) \times \text{P}(B) = 0.5 \times 0.4 = 0.2 \neq \text{P}(A \cap B)$

Therefore A and B are not independent events

Worked solutions

c

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$
$$= \frac{0.1}{0.4} = 0.25$$

19 a $\mathrm{P}(A|B) = \mathrm{P}(A)$, so the result of B does not affect the probability of A. That is, A is independent of B.

b

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) \text{ for independent events}$$
$$= 0.3 \times 0.5 = 0.15$$

c

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$
$$= 0.3 + 0.5 - 0.15 = 0.65$$

20

$$\mathrm{P}(\text{pet}) = 0.28 + 0.12 = 0.4$$
$$\mathrm{P}(\text{sibling}) = 0.28 + 0.42 = 0.7$$
$$\mathrm{P}(\text{pet} \cap \text{sibling}) = 0.28 = 0.4 \times 0.7 = \mathrm{P}(\text{pet}) \times \mathrm{P}(\text{sibling})$$

The two events are independent.

21 Sum of all probabilities in the event space is 1, so $x + y = 0.53$

A and B are independent, so $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

$$x = (x + 0.21)(x + 0.26)$$
$$x^2 + 0.47x + 0.0546 = x$$
$$x^2 - 0.53x + 0.0546 = 0$$

From GDC, $x = 0.14$ or 0.39

So the solutions are $(x, y) = (0.14, 0.39)$ or $(0.39, 0.14)$

22

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$$
$$= \frac{\mathrm{P}(A|B) \times \mathrm{P}(B)}{\mathrm{P}(A)}$$
$$= \frac{\mathrm{P}(A|B) \times \left(\mathrm{P}(B|A') \times \mathrm{P}(A') + \mathrm{P}(B|A)\mathrm{P}(A)\right)}{\mathrm{P}(A)}$$
$$= \frac{\mathrm{P}(A|B) \times \left(\mathrm{P}(B|A') \times \left(1 - \mathrm{P}(A)\right) + \mathrm{P}(B|A)\mathrm{P}(A)\right)}{\mathrm{P}(A)}$$

Tip: You may find this easier to keep track of by drawing a probability tree. Label the branch for $B|A$ as having probability x, and therefore $B'|A$ as $1 - x$.

Label all other branches, then equate the expression for $\mathrm{P}(A|B)$ with the information in the question.

If $\mathrm{P}(B|A) = x$ then

$$x = \frac{\left(\frac{2}{11}\right)\left(\left(\frac{2}{5}\right)\left(\frac{2}{3}\right) + x\left(\frac{1}{3}\right)\right)}{\left(\frac{1}{3}\right)}$$

$$x = \frac{2}{11}\left(\frac{4}{5} + x\right)$$

$$\frac{9}{11}x = \frac{8}{55}$$

$$x = \mathrm{P}(B|A) = \frac{8}{45}$$

23

$$\begin{aligned}\mathrm{P}(1|\mathrm{Red}) &= \frac{\mathrm{P}(1 \cap \mathrm{Red})}{\mathrm{P}(\mathrm{Red})} \\ &= \frac{\mathrm{P}(1)\mathrm{P}(\mathrm{Red}|1)}{\mathrm{P}(1)\mathrm{P}(\mathrm{Red}|1) + \mathrm{P}(2)\mathrm{P}(\mathrm{Red}|2)} \\ &= \frac{\left(\frac{1}{6}\right)\left(\frac{4}{9}\right)}{\left(\frac{1}{6}\right)\left(\frac{4}{9}\right) + \left(\frac{5}{6}\right)\left(\frac{7}{10}\right)} \\ &= \frac{40}{40 + 315} \\ &= \frac{8}{71}\end{aligned}$$

24

$$\begin{aligned}\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(B)} \\ &= \frac{0.6 \times 0.5}{0.4} \\ &= 0.75\end{aligned}$$

25

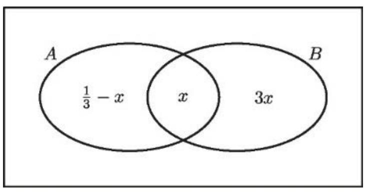
Let $P(A \cap B) = x$

$$\mathrm{P}(A|B) = \frac{1}{4} = \frac{x}{\mathrm{P}(B)} \text{ so } \mathrm{P}(B) = 4x = \mathrm{P}(A' \cap B) + \mathrm{P}(A \cap B) = \mathrm{P}(A' \cap B) + x$$

$$\mathrm{P}(A' \cap B) = 3x$$

$$\mathrm{P}(A) = \frac{1}{3} = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B') = x + \mathrm{P}(A \cap B')$$

$$\mathrm{P}(A \cap B') = \frac{1}{3} - x$$



$$P(A \cup B) = P(B) + P(A \cap B')$$
$$= 4x + \frac{1}{3} - x$$
$$= \frac{1}{3} + 3x = \frac{3}{4}$$
$$3x = \frac{5}{12}$$
$$x = \frac{5}{36}$$
$$P(B) = 4x = \frac{5}{9}$$

26 If A and B are mutually exclusive then $P(A \cap B) = 0$

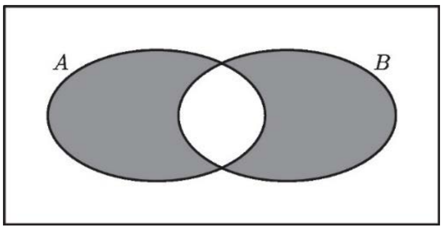
But if $P(A) \neq 0$ and $P(B) \neq 0$ then $P(A)P(B) \neq 0$

So $P(A)P(B) \neq P(A \cap B)$, which means A and B are not independent.

27

$$\begin{aligned} A \text{ and } B \text{ are independent } &\Leftrightarrow P(A \cap B) = P(A)P(B) \\ &\Leftrightarrow P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) \\ &\Leftrightarrow P(A \cup B) = (1 - P(A))(P(B) - 1) + 1 \\ &\Leftrightarrow 1 - P(A' \cap B') = 1 - (1 - P(A))(1 - P(B)) \\ &\Leftrightarrow P(A' \cap B') = (1 - P(A))(1 - P(B)) \\ &\Leftrightarrow P(A' \cap B') = P(A')P(B') \\ &\Leftrightarrow A' \text{ and } B' \text{ are independent} \end{aligned}$$

28



$$\begin{aligned} P(A \cup B) - P(A \cap B) &= P(A \cap B') + P(B \cap A') \\ &= P(A)P(B'|A) + P(B)P(A'|B) \\ &= P(A)(1 - P(B|A)) + P(B)(1 - P(A|B)) \end{aligned}$$

Exercise 19C

21 $X \sim N(\mu, \sigma^2)$

$P(\mu + \sigma < X < \mu + 1.5\sigma) = P(1 < Z < 1.5) = 0.0918$

22 X is the diameter of a bolt. $X \sim N(\mu, 0.065^2)$

$P(X > 2.5) = 0.12$

$P(Z > z_1) = 0.12$

$$z_1 = 1.175 = \frac{2.5 - \mu}{0.065}$$
$$\mu = 2.5 - 0.065 \times 1.175 = 2.42 \text{ cm}$$

23 $X \sim \mathrm{N}(21.4, \sigma^2)$

$$\mathrm{P}(X < 20) = \mathrm{P}\left(Z < \frac{20 - 21.4}{\sigma}\right) = 0.065 = \mathrm{P}(Z < -1.514)$$

$$\sigma = \frac{1.4}{1.514} = 0.925$$

24 $X \sim \mathrm{N}(36, \sigma^2)$

$$\mathrm{P}(X < 30) = P\left(Z < \frac{30 - 36}{\sigma}\right) = 0.15 = \mathrm{P}(Z < -1.036)$$

$$\sigma = \frac{6}{1.036} = 5.79$$

25 $X \sim \mathrm{N}(25, \sigma^2)$

$$P(10 < X < 25) = P(X < 25) - P(X < 10)$$
$$= 0.5 - P(X < 10) = 0.48$$

$$\mathrm{P}(X < 10) = P\left(\frac{10 - 25}{\sigma}\right) = 0.02 = \mathrm{P}(Z < -2.05)$$

$$\sigma = \frac{5}{2.05} = 2.43$$

26 Let X be the mass of an apple. $X \sim \mathrm{N}(130, \sigma^2)$

$$\mathrm{P}(110 < X < 140) = P(X < 150) - P(X < 110)$$
$$= P\left(Z < \frac{150 - 130}{\sigma}\right) - P\left(Z < \frac{110 - 130}{\sigma}\right)$$
$$= \mathrm{P}\left(Z < \frac{20}{\sigma}\right) - P\left(Z < \frac{-20}{\sigma}\right)$$
$$= 1 - 2\mathrm{P}\left(Z < -\frac{20}{\sigma}\right) \text{ by symmetry of } Z$$
$$= 0.99$$

$$P\left(Z < -\frac{20}{\sigma}\right) = 0.005 = \mathrm{P}(Z < -2.576)$$

$$\sigma = \frac{20}{2.576} = 7.76 \text{ g}$$

27 Let X be the score on a test. $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(Z < 35) = \mathrm{P}\left(Z < \frac{35 - \mu}{\sigma}\right) = 0.07 = \mathrm{P}(Z \leq -1.476)$$

$$\mathrm{P}(Z < 70) = \mathrm{P}\left(Z < \frac{70 - \mu}{\sigma}\right) = 0.95 = \mathrm{P}(Z \leq 1.645)$$

$$\begin{cases} 35 - \mu = -1.476\sigma & (1) \\ 70 - \mu = 1.645\sigma & (2) \end{cases}$$

$(2) - (1): 35 = 3.121\sigma$

$\sigma = 11.2$

$(1): \mu = 35 + 1.476\sigma = 51.6$

variance $\sigma^2 = 126$

28 Let X be the wingspan of a pigeon from the species. $X \sim \mathrm{N}(60, \sigma^2)$

$$\begin{aligned}\mathrm{P}(55 < X < 65) &= P(X < 65) - P(X < 55) \\ &= P\left(Z < \frac{65-60}{\sigma}\right) - P\left(Z < \frac{55-60}{\sigma}\right) \\ &= \mathrm{P}\left(Z < \frac{5}{\sigma}\right) - P\left(Z < \frac{-5}{\sigma}\right) \\ &= 1 - 2\mathrm{P}\left(Z < -\frac{5}{\sigma}\right) \text{ by symmetry of } Z \\ &= 0.6\end{aligned}$$

$$P\left(Z < -\frac{5}{\sigma}\right) = 0.2 = \mathrm{P}(Z < -0.8416)$$

$$\sigma = \frac{5}{0.8416} = 5.94 \text{ cm}$$

$$\mathrm{P}(X < 50) = \mathrm{P}\left(Z < \frac{50-60}{5.94}\right) = 4.62\%$$

29 Centring the distribution at the mean, for $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(\mu - 3\sigma < X < \mu + 3\sigma) = \mathrm{P}(-3 < Z < 3) = 99.7\%$$

30 Let $X \sim \mathrm{N}(\mu, \sigma^2)$

Since the distribution is symmetric about μ, the lower quartile and upper quartile are equal distance from the mean. Let Q_3 be the upper quartile. Then $\mathrm{IQR} = 2(Q_3 - \mu) = 12.8$

$$Q_3 - \mu = 6.4$$

$$\mathrm{P}(X < Q_3) = \mathrm{P}\left(Z < \frac{(Q_3 - \mu)}{\sigma}\right) = 0.75 = \mathrm{P}(Z < 0.6745)$$

$$\frac{6.4}{\sigma} = 0.6745$$

$$\sigma = \frac{6.4}{0.6745}$$

$$\text{variance } \sigma^2 = \left(\frac{6.4}{0.6745}\right)^2 = 90.0$$

Mixed Practice

1 **a** $r = 0.949$

b Let x be mark in the first test and y the mark in the second test.

x-on-y regression line is $x = 1.13\,y - 5.04$

Then when $y = 53$, predict $x = 1.13(53) - 5.04 = 55$

c The correlation coefficient is very high so there is strong correlation.

The value $y = 53$ lies within the interval of the data so the estimate is an interpolation not extrapolation.

The estimate is reliable.

2 **a** Low correlation, so the line fit is not an accurate model of the data.

b The correlation line given is y-on-x, so is not suitable for predicting a value of x from a given value of y.

c The value $y = 23$ lies outside the data interval; extrapolation is not reliable.

3 **a**

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B|A)$$
$$= \frac{2}{3} \times \frac{3}{20} = \frac{1}{10}$$

b

$$P(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$
$$= \frac{0.1}{0.2}$$
$$= \frac{1}{2}$$

4 **a** G is the event of green eyes, B is the event of brown hair.

$$P(G) + P(B) = P(G \cup B) + P(G \cap B)$$
$$= 1 - \mathrm{P}(G' \cap B') + \mathrm{P}(G \cap B)$$

$$0.6 + 0.4 = 1 - 0.2 + \mathrm{P}(G \cap B)$$

$$\mathrm{P}(G \cap B) = 0.2$$

b $P(G) \times P(B) = 0.6 \times 0.4 = 0.24 \neq P(G \cap B)$

So the two events are not independent.

5 Let X be a race time. $X \sim \mathrm{N}(\mu, \sigma^2)$

a

$$\mathrm{P}(X < 26.2) = P\left(Z < \frac{26.2 - \mu}{\sigma}\right) = 0.88 = \mathrm{P}(Z < 1.175)$$

$$\frac{26.2 - \mu}{\sigma} = 1.175$$

$$\mu + 1.175\sigma = 26.2 \qquad (1)$$

b

$$\mathrm{P}(X < 22.5) = P\left(Z < \frac{22.5 - \mu}{\sigma}\right) = 0.05 = \mathrm{P}(Z < -1.645)$$

$$\frac{22.5 - \mu}{\sigma} = -1.645$$

$$\mu - 1.645\sigma = 22.5 \qquad (2)$$

$$(1) - (2): 2.820\sigma = 3.7$$

$$\sigma = \frac{3.7}{2.820} = 1.31 \text{ seconds}$$

$$(1): \mu = 26.2 - 1.175\sigma = 24.7 \text{ seconds}$$

6 **a**

$$\begin{aligned} P(A \cap B) &= P(B) \times P(A|B) \\ &= \frac{11}{20} \times \frac{2}{11} = \frac{1}{10} \end{aligned}$$

b

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{2}{5} + \frac{11}{20} - \frac{1}{10} \\ &= \frac{17}{20} \end{aligned}$$

c

$$P(A) \times P(B) = \frac{11}{20} \times \frac{2}{5} = \frac{11}{50} \neq P(A \cap B)$$

A and B are not independent.

7 **a** $y = 10.7x + 121$

b i The gradient indicates the estimated additional cost per box.

ii The y-intercept indicates fixed costs.

c $y(60) = 121 + 10.7(60) = \$760$

d Revenue $= 19.99x$

Profit $=$ Revenue $-$ Costs $= 9.29x - 121$

For profit to be positive, find least integer x such that profit > 0

$$9.29x > 121$$

$$x > \frac{121}{9.29} = 12.94$$

Least number of boxes needed for profit is 13.

e **i** $x = 5000$ is well outside the interval of the data values, so this estimate would be an extrapolation, and unreliable.

ii Estimating x from $y = 540$ would require the x-on-y regression line, not the y-on-x regression line given in part **a**.

8

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.6 + 0.4 - 0.76 \\ &= 0.24 \\ &= P(A) \times P(B) \end{aligned}$$

Since $P(A \cap B) = P(A) \times P(B)$, the events are independent.

9 Converting all units to cm:

$X \sim N(\mu, 1.7^2)$

$$P(X < 4.5) = P\left(Z < \frac{(4.5 - \mu)}{1.7}\right) = \frac{1}{6} = P(Z < -0.967)$$

$$\mu = 4.5 + 0.967 \times 1.7 = 6.14 \text{ cm}$$

10 $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(X < 12) = \mathrm{P}\left(Z < \frac{12 - \mu}{\sigma}\right) = 0.1 = \mathrm{P}(Z < -1.28)$$

$$\frac{12 - \mu}{\sigma} = -1.28 \qquad (1)$$

$$\mathrm{P}(X < 25) = \mathrm{P}\left(Z < \frac{25 - \mu}{\sigma}\right) = 0.85 = \mathrm{P}(Z < 1.04)$$

$$\frac{25 - \mu}{\sigma} = 1.04 \qquad (2)$$

$$(2) - (1): \frac{13}{\sigma} = 2.32$$

$$\sigma = \frac{13}{2.32} = 5.61$$

$$(1): \mu = 12 + 1.28\sigma = 19.2$$

11 Let X be the mass of a dog. $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(X < 2.4) = P\left(Z < \frac{2.4 - \mu}{\sigma}\right) = 0.3 = \mathrm{P}(Z < -0.5244)$$

$$\frac{2.4 - \mu}{\sigma} = -0.5244 \qquad (1)$$

$$\mathrm{P}(X < 10.7) = P\left(Z < \frac{10.7 - \mu}{\sigma}\right) = 0.83 = \mathrm{P}(Z < 1.48)$$

$$\frac{10.7 - \mu}{\sigma} = 1.48 \qquad (2)$$

$$(2) - (1): \frac{8.3}{\sigma} = 2.00$$

$$\sigma = \frac{8.3}{2.00} = 4.15 \text{ kg}$$

$$(1): \mu = 2.4 + 0.5244\sigma = 4.58 \text{ kg}$$

$$\mathrm{P}(\mathrm{X} > 8) = \mathrm{P}\left(Z > \frac{8 - 4.58}{4.15}\right) = 0.205 = 20.5\%$$

12 Let $X \sim \mathrm{N}(\mu, \sigma^2)$

Since the distribution is symmetric about μ, the lower quartile and upper quartile are equal distance from the mean. Let Q_3 be the upper quartile. Then $\mathrm{IQR} = 2(Q_3 - \mu) = 17$

$$Q_3 - \mu = 8.5$$

$$\mathrm{P}(X < Q_3) = \mathrm{P}\left(Z < \frac{(Q_3 - \mu)}{\sigma}\right) = 0.75 = \mathrm{P}(Z < 0.6745)$$

$$\frac{8.5}{\sigma} = 0.6745$$

$$\sigma = \frac{8.5}{0.6745}$$

$$\text{variance } \sigma^2 = \left(\frac{8.5}{0.6745}\right)^2 = 159$$

13 Let X be the time (in minutes) to complete a test. $X \sim \mathrm{N}(45, 10^2)$

a $\mathrm{P}(X > 50) = \mathrm{P}\left(Z > \frac{50-45}{10}\right) = 0.309$

b

$$\mathrm{P}(X > 60 | X > 50) = \mathrm{P}\left(\frac{X > 60 \cap X > 50}{X > 50}\right)$$
$$= \mathrm{P}\left(\frac{X > 60}{X > 50}\right)$$
$$= \frac{0.0668}{0.309}$$
$$= 0.217$$

14 a Let X be the mass (in grams) of an apple. $X \sim \mathrm{N}(126, 12^2)$

$$\mathrm{P}(X > 150) = \mathrm{P}\left(Z > \frac{150 - 126}{12}\right) = 2.28\%$$

b

$$\mathrm{P}(X > 160 | X > 150) = \mathrm{P}\left(\frac{X > 160 \cap X > 150}{X > 150}\right)$$
$$= \mathrm{P}\left(\frac{X > 160}{X > 150}\right)$$
$$= \frac{0.0023}{0.0228}$$
$$= 10.1\%$$

15 Let X be a mark in the test. $X \sim \mathrm{N}(100, \sigma^2)$

$$\mathrm{P}(X < 124) = \mathrm{P}\left(Z < \frac{124 - 100}{\sigma}\right) = 0.68 = \mathrm{P}(Z < 0.4677)$$
$$\frac{24}{\sigma} = 0.4677$$
$$\sigma^2 = \left(\frac{24}{0.4677}\right)^2 = 2630$$

16 Let X be the duration of a flight in minutes. $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(X < 13 \times 60) = P\left(Z < \frac{780 - \mu}{\sigma}\right) = 0.92 = \mathrm{P}(Z < 1.405)$$
$$\frac{780 - \mu}{\sigma} = 1.405 \qquad (1)$$
$$\mathrm{P}(X < 755) = P\left(Z < \frac{755 - \mu}{\sigma}\right) = 0.12 = \mathrm{P}(Z < -1.175)$$
$$\frac{755 - \mu}{\sigma} = -1.175 \qquad (2)$$
$$(1) - (2): \frac{25}{\sigma} = 2.580$$
$$\sigma = \frac{25}{2.580} = 9.69 \approx 10 \text{ minutes}$$
$$(1): \mu = 780 - 1.405\sigma = 766 \text{ minutes}$$

17 a Let X be the number of defective lamps in a sample of 30.

$$\begin{aligned}P(X>0)&=1-P(X=0)\\&=1-0.95^{30}\\&=0.785\end{aligned}$$

b

$$\begin{aligned}P(X\le 2|X>0)&=\frac{P(X\le 2\cap X>0)}{P(X>0)}\\&=\frac{P(X=1\cup X=2)}{P(X>0)}\\&=\frac{30(0.05)(0.95)^{29}+{}^{30}C_2(0.05)^2(0.95)^{28}}{0.785}\\&=\frac{0.598}{0.785}\\&=0.761\end{aligned}$$

18

$$P(B\cap A')=P(A\cup B)-P(A)=\frac{1}{8}$$

$$P(A'|B)=\frac{P(A'\cap B)}{P(B)}=\frac{\frac{1}{8}}{P(B)}=\frac{1}{4}$$

$$P(B)=\frac{1}{2}$$

19 a

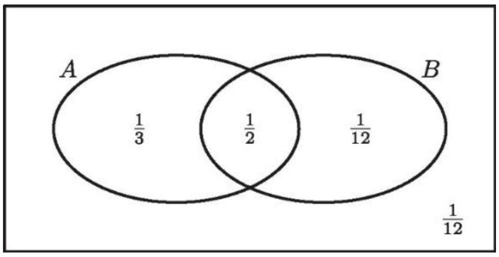
$$P(B|A)=\frac{P(B\cap A)}{P(A)}=\frac{P(B\cap A)}{P(B\cap A)+P(B'\cap A)}=\frac{3}{5}$$

$$P(B\cap A)=\frac{3}{5}\left(P(B\cap A)+P(B'\cap A)\right)$$

$$\frac{2}{5}P(B\cap A)=\frac{3}{5}P(B'\cap A)=\frac{1}{5}$$

$$P(B\cap A)=\frac{1}{2}$$

b



$$P(A\cap B')=\frac{1}{3}, P(B\cap A)=\frac{1}{2}, P(A\cup B)=\frac{11}{12}\text{ so }P(B\cap A')=\frac{11}{12}-\frac{1}{3}-\frac{1}{2}=\frac{1}{12}$$

$$P(A)=\frac{1}{3}+\frac{1}{2}=\frac{5}{6}, P(B)=\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$$

$$P(B\cap A)=\frac{1}{2}\ne P(A)\times P(B)$$

So events A and B are not independent.

20 Let X be the length of a wire cut by the machine (in cm). $X \sim \mathrm{N}(30, 2.5)$.

$$\begin{aligned}\mathrm{P}(X > 34 | X < 35) &= \frac{\mathrm{P}(34 < X < 35)}{\mathrm{P}(X < 35)} \\ &= \frac{\mathrm{P}(X < 35) - \mathrm{P}(X < 34)}{\mathrm{P}(X < 35)} \\ &= \frac{\mathrm{P}\left(Z < \frac{35-30}{2.5}\right) - \mathrm{P}\left(Z < \frac{34-30}{2.5}\right)}{\mathrm{P}\left(Z < \frac{35-30}{2.5}\right)} \\ &= \frac{0.97725 - 0.9452}{0.97725} \\ &= 0.0328\end{aligned}$$

Let Y be the number of wires in a box of ten with length greater than 34 cm.

$$\begin{aligned}\mathrm{P}(Y > 0) &= 1 - \mathrm{P}(Y = 0) \\ &= 1 - (1 - 0.0328)^{10} \\ &= 0.284\end{aligned}$$

21 **a** $r_{x,y} = -0.0209$

b $r_{x^2,y} = 0.983$

c $y = 2.07x^2 - 1.52$

22 $L \sim \mathrm{N}(50, \sigma^2)$

a $\mathrm{P}(50 - \sigma < L < 50 + 2\sigma) = \mathrm{P}(-1 < Z < 2) = \mathrm{P}(Z < 2) - \mathrm{P}(Z < -1) = 0.819$

b

$$\mathrm{P}(L < 53.92) = \mathrm{P}\left(Z < \frac{53.92 - 50}{\sigma}\right) = 0.975 = \mathrm{P}(Z < 1.96)$$

$$\frac{3.92}{\sigma} = 1.96$$

$$\sigma = \frac{3.92}{1.96} = 2.00 \text{ mm}$$

c

$$\mathrm{P}(L < t) = P\left(Z < \frac{t - 50}{2.00}\right) = 0.25 = \mathrm{P}(Z < -0.6745)$$

$$\frac{t - 50}{2} = -0.6745$$

$$t = 50 - 2(0.6745) = 48.7 \text{ mm}$$

Worked solutions

d i

$$\begin{aligned}
P(L < 50.1 | L > t) &= \frac{P(t < L < 50.1)}{P(L > t)} \\
&= \frac{P(L < 50.1) - 0.25}{0.75} \\
&= \frac{P\left(Z < \frac{50.1 - 50}{2.00}\right) - 0.25}{0.75} \\
&= 0.360
\end{aligned}$$

ii Let Y be the number of large nails shorter than 50.1 mm in a sample of ten large nails.

$$\begin{aligned}
P(Y \geq 2) &= 1 - P(Y = 0) - P(Y = 1) \\
&= 1 - (1 - 0.360)^{10} - 10(1 - 0.360)^{9(0.360)} \\
&= 1 - 0.01154 - 0.06491 \\
&= 0.924
\end{aligned}$$

20 Analysis and approaches: Differentiation

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 20A

16 $y' = \cos x$ so $y'\left(\frac{\pi}{3}\right) = \frac{1}{2}$

17

$$h = 10t^{0.5} + 20t^{-0.5}$$
$$\frac{dh}{dt} = 5t^{-0.5} - 10t^{-1.5}$$
$$\frac{dh}{dt}(1) = 5 - 10 = -5 \text{ m s}^{-1}$$

Negative value means that h is decreasing when $t = 1$; the falcon is diving.

18

$$y' = 5 - e^x = 4$$
$$e^x = 1$$
$$x = 0$$

The point on the curve is $(0,1)$

19

$$y' = e^x + 3$$
$$y'(\ln 2) = 5$$

20 Line $2y + 3x + 3 = 0$ can be written as $y = -1.5x - 1.5$, so has gradient -1.5

$$y = 3\cos x$$
$$y' = -3\sin x = -1.5$$
$$\sin x = 0.5$$
$$x = \frac{\pi}{6}$$

Then the point of tangency is $\left(\frac{\pi}{6}, 3\cos\left(\frac{\pi}{6}\right)\right) = \left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$

Tangent equation is $y - \frac{3\sqrt{3}}{2} = -\frac{3}{2}\left(x - \frac{\pi}{6}\right)$

$$y = -\frac{3x}{2} + \frac{\pi}{4} + \frac{3\sqrt{3}}{2}$$

21 $y = -x + 4$ has gradient -1, so the perpendicular has gradient 1.

$y' = \mathrm{e}^x - 2 = 1$ so $x = \ln 3$

Point of tangency is $(\ln 3, 3 - 2\ln 3)$

Tangent equation is $y - 3 + 2\ln 3 = (x - \ln 3)$

$$y = x + 3 - 3\ln 3$$

22 Tangent gradient is $\frac{1}{6}$

$$y = x^{0.5} \text{ so } y' = 0.5x^{-0.5} = \frac{1}{6}$$

$$x^{-0.5} = \frac{1}{3}$$
$$x = 9$$

Point of tangency is $(9,3)$

23 **a** $y = 2x - \mathrm{e}^x$ so $y' = 2 - \mathrm{e}^x$

$y'(\ln 2.5) = 2 - 2.5 = -0.5$

The gradient of the normal at this point is 2.

Point of intersection of the normal is $(\ln 2.5, 2\ln 2.5 - 2.5)$

Equation of the normal is $y + 2.5 - 2\ln 2.5 = 2(x - \ln 2.5)$

$$y = 2x - \frac{5}{2}$$

b Intersections of the normal and the curve:

$$2x - \mathrm{e}^x = 2x - 2.5$$
$$\mathrm{e}^x = 2.5$$

This has only a single solution, at $x = \ln 2.5$, so the normal only intersects the curve at one point.

24 **a** $N(0) = 1$

b

$$\left(1 + \sqrt{t}\right)^2 = 9$$
$$1 + \sqrt{t} = 3$$
$$t = 4 \text{ hours}$$

c

$$N = 1 + 2t^{0.5} + t$$
$$\frac{\mathrm{d}N}{\mathrm{d}t} = t^{-0.5} + 1$$
$$\frac{\mathrm{d}N}{\mathrm{d}t}(4) = 1.5 \text{ million bacteria per hour}$$

d The model assumes growth without restriction, imposing no upper limit on the population, which would after a short time exceed the space available on the agar dish.

25 $y = \ln x - 2x$ so $y' = x^{-1} - 2$

This is a decreasing function when $y' < 0$

$$x^{-1} - 2 < 0$$
$$x > 0.5$$

26 $y_1 = x^{0.5}$ so $y_1' = 0.5x^{-0.5}$

Tangent at $x = k$ has gradient $0.5k^{-0.5}$

$y_2 = x^{-0.5}$ so $y_2' = -0.5x^{-1.5}$

Tangent at $x = k$ has gradient $-0.5k^{-1.5}$

If the two tangents are perpendicular then $0.5k^{-0.5} \times (-0.5\,k^{-1.5}) = -1$

$$-0.25k^{-2} = -1$$
$$k^{-2} = 4$$
$$k = 0.5$$

27 **a** $\cos x = 0$ for $x = \pm\frac{\pi}{2}$

$y' = -\sin x$ so the gradients at these points are ∓ 1

By symmetry, the tangents meet at the y-axis

$$P{:}\left(-\frac{\pi}{2}, 0\right), Q{:}\left(\frac{\pi}{2}, 0\right), R{:}\left(0, \frac{\pi}{2}\right)$$

Triangle PQR has base length π and perpendicular height $\frac{\pi}{2}$ so has area $\frac{\pi^2}{4}$

b Since the graph of $y = \sin x$ is a translation of the graph of $y = \cos x$, the area of ABC will also be $\frac{\pi^2}{4}$

28 $y = x^{0.5}$ so $y' = 0.5x^{-0.5}$

The tangent at point (a^2, a) has gradient $\frac{1}{2a}$

It has equation $y - a = \frac{1}{2a}(x - a^2)$

$$y = \frac{1}{2a}x + \frac{a}{2}$$

If this passes through $(0,1)$ then $a = 2$

The point on the curve is therefore $(4, 2)$

29

$$y = \sin x - \frac{1}{2}$$
$$y' = \cos x$$

Let P have coordinates $\left(p, \sin p - \frac{1}{2}\right)$

If the tangent is $y = \frac{\sqrt{3}}{2}x - \frac{13\pi}{4\sqrt{3}}$ then the gradient at P is $y'(p) = \frac{\sqrt{3}}{2}$

$\cos p = \frac{\sqrt{3}}{2}$ has solutions $p = \pm\frac{\pi}{6} + 2n\pi$

Substituting into the tangent equation:

$$y\left(2n\pi \pm \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\left(2n\pi \pm \frac{\pi}{6}\right) - \frac{13\pi}{4\sqrt{3}} = \frac{\pi}{\sqrt{3}}\left(3n + \frac{-13 \pm 1}{4}\right)$$

Substituting $p = \pm\frac{\pi}{6} + 2n\pi$ into the curve equation:

$$y\left(2n\pi \pm \frac{\pi}{6}\right) = \pm\frac{1}{2} - \frac{1}{2}$$

If these two values are to be equal, both must equal zero, since the second cannot be any non-zero rational multiple of $\frac{\pi}{\sqrt{3}}$

So $p = 2\pi + \frac{\pi}{6}$

P has coordinates $\left(\frac{13\pi}{6}, 0\right)$

30 $y_1 = -3x + 9$ has gradient -3.

$y_2 = x^2 - 2\ln x$

$y_2' = 2x - 2x^{-1}$

Require that $2x - 2x^{-1} = -3$

$$2x^2 + 3x - 2 = 0$$
$$(2x - 1)(x + 2) = 0$$
$$x = \frac{1}{2} \text{ or } -2$$

Reject the negative solution as outside the domain of $\ln x$.

$x = \frac{1}{2}$ is the only solution.

Exercise 20B

15

$$f(x) = e^{-x}$$
$$f'(x) = -e^{-x}$$

16

$$y = e^{2x} + 5x$$
$$y' = 2e^{2x} + 5$$
$$y'(\ln 3) = 2(3)^2 + 5 = 23$$

17

$$y = (x^2 + 9)^{0.5}$$
$$y' = 0.5(2x)(x^2 + 9)^{-0.5} = \frac{x}{\sqrt{x^2 + 9}}$$
$$y'(4) = \frac{4}{\sqrt{25}} = \frac{4}{5}$$
$$y(4) = \sqrt{25} = 5$$

Tangent is $y - 5 = \frac{4}{5}(x - 4)$

$$4x - 5y = -9$$
$$y = \frac{4x}{5} + \frac{9}{5}$$

18

$$y = \ln(2x - 5)$$
$$y' = \frac{2}{2x-5} = 2$$

So $x = 3$

$$y(3) = \ln(1) = 0$$

Tangent at $(3, 0)$.

19

$$y = \cos(x^{-1})$$
$$\begin{aligned} y' &= (-x^{-2})(-\sin(x^{-1})) \\ &= x^{-2}\sin(x^{-1}) \end{aligned}$$

$$y'\left(\frac{2}{\pi}\right) = \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$y\left(\frac{2}{\pi}\right) = 0$$

Tangent at $\left(\frac{2}{\pi}, 0\right)$ has equation $y = \frac{\pi^2}{4}\left(x - \frac{2}{\pi}\right)$

$$y = \frac{\pi^2 x}{4} - \frac{\pi}{2}$$

20 Gradient of $y = -3x + 2$ is -3 so the perpendicular has gradient $\frac{1}{3}$

$$y = \ln(x - 2)$$
$$y' = \frac{1}{x-2} = \frac{1}{3}$$
$$x = 5$$

Tangent at $(5, \ln 3)$ has equation $y - \ln 3 = \frac{1}{3}(x - 5)$

$$y = \frac{x}{3} - \frac{5}{3} + \ln 3$$

21 a $\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x) = 2\sin x\cos x$

b $2\sin x\cos x = 0$ has solutions $x = \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ for $0 < x < 2\pi$

So the points on the curve with zero gradient are $\left(\frac{\pi}{2}, 1\right), (\pi, 0)$ and $\left(\frac{3\pi}{2}, 1\right)$

22 a

$$\begin{aligned} (\cos x + \sin x)^2 &= \cos^2 x + 2\sin x\cos x + \cos^2 x \\ &= (\cos^2 x + \sin^2 x) + 2\sin x\cos x \\ &= 1 + \sin 2x \end{aligned}$$

b Using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\cos x + \sin x)^2 &= 2(-\sin x + \cos x)(\cos x + \sin x) \\ &= 2(\cos^2 x - \sin^2 x) \end{aligned}$$

But using the identity in part **a**:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x + \sin x)^2 = \frac{d}{dx}(1 + \sin 2x)$$
$$= 2\cos 2x$$

Equating these two: $\cos^2 x - \sin^2 x = \cos 2x$

23 $\sin 3x = 3\sin x - 4\sin^3 x$

Differentiating both sides of the equation using the chain rule:

$$3\cos 3x = 3\cos x - 4(3\cos x)(\sin^2 x)$$
$$= 3\cos x\,(1 - 4\sin^2 x)$$
$$= 3\cos x\left(1 - 4(1 - \cos^2 x)\right)$$
$$= 3\cos x\,(4\cos^2 x - 3)$$

So $\cos 3x = 4\cos^3 x - 3\cos x$

24

$$y = \mathrm{e}^{kx}$$
$$y' = k\mathrm{e}^{kx}$$

$y'\left(\frac{1}{k}\right) = k\mathrm{e}$ and $y\left(\frac{1}{k}\right) = \mathrm{e}$

Tangent at $\left(\frac{1}{k}, \mathrm{e}\right)$ has equation $y - \mathrm{e} = k\mathrm{e}\left(x - \frac{1}{k}\right)$

$$y - \mathrm{e} = k\mathrm{e}x - \mathrm{e}$$
$$y = k\mathrm{e}x$$

The tangent always passes through the origin, for any non-zero k.

25

$$y = \ln(x^2 - 8)$$
$$y' = \frac{2x}{x^2 - 8} = 6$$
$$6x^2 - 2x - 40$$
$$3x^2 - x - 24 = 0$$
$$x^2 - \frac{1}{3}x - 8 = 0$$
$$(x - \frac{1}{6})^2 - \left(\frac{1}{6}\right)^2 = 8$$
$$x = \frac{1}{6} \pm \sqrt{8 + \frac{1}{36}}$$
$$x = 3, x = -\frac{8}{3}$$

Reject the second solution, since $\left(-\frac{8}{3}\right)^2 - 8 < 0 \therefore$ is not in the domain of $\ln(x^2 - 8)$.

The only point where the gradient is 6 is $(3, 0)$.

26 a $y = \mathrm{e}^{2x}$

Let $u = 2x$, so $y = \mathrm{e}^{u}$

By chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \mathrm{e}^{u} \times 2$$
$$= y \times 2$$

b Not true; if $y = a\mathrm{e}^{2x}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2a\mathrm{e}^{2x} = 2y$ for any $a \neq 0$

Exercise 20C

13 Using the quotient rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}$$

$$y'\left(\frac{\pi}{2}\right) = \frac{0-1}{\left(\frac{\pi}{2}\right)^2} = -\frac{4}{\pi^2}$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

Tangent at $\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$ has equation $y - \frac{2}{\pi} = -\frac{4}{\pi^2}\left(x - \frac{\pi}{2}\right)$

$$y = \frac{4}{\pi} - \frac{4x}{\pi^2}$$

14

$$y' = \cos 2x - 2x\sin 2x$$

$$y'\left(\frac{\pi}{4}\right) = 0 - \frac{\pi}{2}$$

Gradient of the normal is $\frac{2}{\pi}$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \times 0$$

Equation of the normal at $\left(\frac{\pi}{4}, 0\right)$ is $y = \frac{2}{\pi}\left(x - \frac{\pi}{4}\right)$

$$y = \frac{2x}{\pi} - \frac{1}{2}$$

15

$$\mathrm{f}(x) = x\mathrm{e}^{2x}$$

$$\mathrm{f}'(x) = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$$
$$= \mathrm{e}^{2x}(1 + 2x)$$

$$\mathrm{f}'(3) = 7\mathrm{e}^{6}$$

16 Let $u = xe^x$ and $v = \ln x$

Then $u' = e^x + xe^x = (1+x)e^x$ and $v' = \frac{1}{x}$

$$\begin{aligned}\frac{d}{dx}(uv) &= uv' + u'v \\ &= \frac{xe^x}{x} + (1+x)e^x \ln x \\ &= e^x(1 + (1+x)\ln x)\end{aligned}$$

17 Let $u = x \sin x$

Then $u' = \sin x + x\cos x$

$$\begin{aligned}\frac{d}{dx}(e^u) &= \frac{d}{du}(e^u) \times \frac{du}{dx} \\ &= e^u(\sin x + x\cos x) \\ &= (\sin x + x\cos x)e^{x\sin x}\end{aligned}$$

18 $y' = 3e^{3x} - 11 = 13$

So $e^{3x} = 8 = 2^3$

$x = \ln 2$

The point of tangency is $(\ln 2, 8 - 11\ln 2)$

19

$$y = xe^{-x}$$

$$y' = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

If $y' = 0$ then $x = 1$ or $e^{-x} = 0$ (which has no solutions).

The only point on the curve with zero gradient is $\left(1, \frac{1}{e}\right)$.

20

$$y = x\ln x$$

$$y' = \ln x + \frac{x}{x} = 1 + \ln x$$

If $y' = 2$ then $\ln x = 1$

The point on the curve with gradient 2 is (e, e)

21

$$f(x) = x(k + x^2)^{-\frac{1}{2}}$$

$$\begin{aligned}f'(x) &= (k+x^2)^{-\frac{1}{2}} - \frac{1}{2}(2x)x(k+x^2)^{-\frac{3}{2}} \\ &= (k + x^2 - x^2)(k+x^2)^{-\frac{3}{2}} \\ &= \frac{k}{\left(\sqrt{k+x^2}\right)^{1.5}}\end{aligned}$$

22

$$f(x) = x(2+x)^{\frac{1}{2}}$$

$$f'(x) = (2+x)^{\frac{1}{2}} + \frac{1}{2}x(2+x)^{-\frac{1}{2}}$$

$$= \frac{2+x+\frac{1}{2}x}{\sqrt{2+x}}$$

$$= \frac{3x+4}{x\sqrt{2+x}}$$

$a = 3, b = 4$

23 $f(x) = \frac{x^2}{1+x}$

Using the quotient rule:

$$f'(x) = \frac{2x(1+x) - x^2}{(1+x)^2}$$

$$= \frac{x^2 + 2x}{(1+x)^2}$$

Since the denominator is zero at $x = -1$ and positive for all $x \neq -1$, the sign of the derivative is the same as the sign of the numerator.

$$x^2 + 2x = x(x+2)$$

A positive quadratic takes negative values between the roots.

The gradient is positive for $x < -2$ and for $x > 0$

$f(x)$ is an increasing function for $x < -2$ or $x > 0$.

$$a = 0, b = -2$$

24 a

$$P = 4(1 + 3e^{-n})^{-1}$$

$$\frac{dP}{dn} = -4(-3e^{-n})(1+3e^{-n})^{-2}$$

$$= \frac{12e^{-n}}{(1+3e^{-n})^2}$$

Both numerator and denominator only take positive values for all n so the function P is always increasing, since $P' > 0$ for all n.

b i $P(0) = \frac{4}{4} = 1$

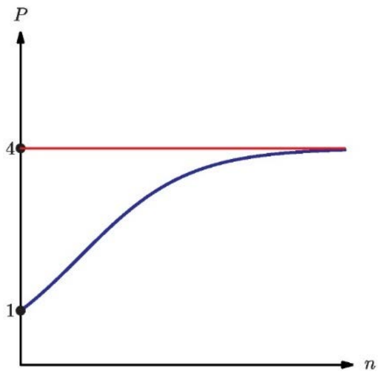
The initial population is 1000 rabbits.

ii $P'(0) = \frac{12}{4^2} = 0.75$

The initial growth rate is 750 rabbits per year.

c As $n \to \infty, P \to 4$ so the population tends towards 4000 rabbits.

d



25 a For $x > 0$, $\ln x$ is a real value.

$$x^x = (x)^x$$
$$= \left(e^{\ln x}\right)^x$$
$$= e^{x \ln x}$$

b Let $u = x \ln x$

Then $u' = \ln x + \frac{x}{x} = 1 + \ln x$

$$y = x^x = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u(1 + \ln x)$$
$$= x^x(1 + \ln x)$$

$$\frac{dy}{dx}(1) = 1(1 + 0) = 1$$

$$y(1) = 1$$

Tangent at $(1,1)$ is $y = x$

Exercise 20D

10

$$f'(x) = 3x^2 + 2kx + 3$$
$$f'(1) = 3 + 2k + 3 = 10$$
$$k = 2$$

11

$$f'(x) = 3x^2 + 2ax + b$$

$$f'(1) = 3 + 2a + b = 4 \qquad (1)$$

$$f''(x) = 6x + 2a$$

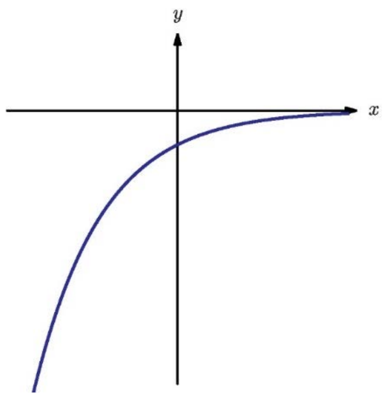
$f''(-1) = -6 + 2a = -4$ so $a = 1$

(1): $5 + b = 4$ so $b = -1$

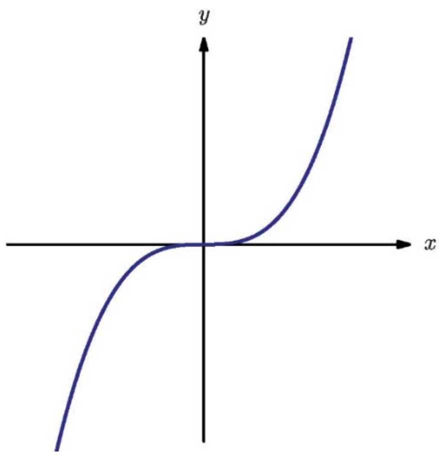
12

$$y = xe^x$$
$$y' = e^x + xe^x$$
$$y'' = e^x + e^x + xe^x = (2 + x)e^x$$

13 For example, $y = -e^{-x}$



14 For example, $y = x^3$



15

$$y = x^3 - kx^2 + 4x + 7$$
$$y' = 3x^2 - 2kx + 4$$
$$y'' = 6x - 2k$$

The graph is concave-up ($y'' > 0$) for $x > 1$

$$k = 3$$

16

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$
$$f''(1) = 12 = n(n-1)$$
$$n^2 - n - 12 = 0$$
$$(n+3)(n-4) = 12$$
$$n = -3 \text{ or } 4$$

Worked solutions

17

$$y = x^2 + bx + c$$
$$y' = 2x + b$$

$y'' = 2$ so the graph is always concave-up, since $y''(x) > 0$ for all x.

18 $y = \sin 3x + 2\cos 3x$

$y' = 3\cos 3x - 6\sin 3x$

$$\begin{aligned} y'' &= -9\sin 3x - 18\cos 3x \\ &= -9(\sin 3x + 2\cos 3x) \\ &= -9y \end{aligned}$$

19 $y = \ln(a + x)$

$$y' = \frac{1}{a+x} = (a+x)^{-1}$$

$y'' = -(a+x)^{-2} = -\frac{1}{(a+x)^2} < 0$ for all x, so the graph is always concave-down.

20

$$y = x^2 \ln x$$

$$y' = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$$

$$y'' = 2\ln x + 2x\left(\frac{1}{x}\right) + 1 = 2\ln x + 3$$

$$\begin{aligned} 2\ln x + 3 &= 1 \\ \ln x &= -1 \\ x &= e^{-1} \end{aligned}$$

The point on the graph with gradient 1 is $\left(\frac{1}{e}, \frac{1}{e^2}\right)$

21

$$y = x^3 - 5x^2 + 4x - 2$$
$$y' = 3x^2 - 10x + 4$$
$$y'' = 6x - 10$$

Gradient is decreasing where $6x - 10 < 0$

$$x < \frac{5}{3}$$

22 a

$$y = x^3 + ax^2 + bx + 7$$
$$y' = 3x^2 + 2ax + b$$
$$y'' = 6x + 2a$$

Graph is concave-up where $y'' > 0$

$6x + 2a > 12 + 2a$ for $x > 2$ so $12 + 2a = 0$

$a = -6$

b Curve is strictly increasing if $y' > 0$ for all x

$3x^2 - 12x + b > 0$ for all x.

Require that there are no roots to the quadratic $3x^2 - 12x + b$

No roots: Discriminant $\Delta < 0$

$$\Delta = (-12)^2 - 4(3)(b) = 144 - 12b < 0$$

$$b > 12$$

23 a

$$y = ax^3 - bx^2$$
$$y' = 3ax^2 - 2bx = x(3ax - 2b)$$

$y' > 0$ for $x < 0$ and $x > 4$

So the roots of y' are 0 and 4

$$\frac{2b}{3a} = 4$$
$$\frac{b}{a} = 6$$

b $y'' = 6ax - 2b = 6a\left(x - \frac{2b}{6a}\right) = 6a(x - 2)$

$a > 0$ so $y'' > 0$, and therefore the curve is concave-up, for $x > 2$.

24 $y'' = 10$

$y' = 10x + b$ for some constant b

$y'(0) = 10 = b$

$y = 5x^2 + bx + c$ for some constant c

$y(0) = 2 = c$

$$y = 5x^2 + 10x + 2$$

Exercise 20E

6 $y' = 2x + b$

Minimum at $x = 1$ so $y'(1) = 2 + b = 0$

$b = -2$

$y(1) = 2 = 1 + b + c$ so $c = 3$

7 $y' = 4x^3 + b$

Minimum at $x = 1$ so $y'(1) = 4 + b = 0$

$b = -4$

$y(1) = -2 = 1 + b + c$ so $c = 1$

8 $P' = 20x - 40x^3 = 20x(1 - 2x^2)$

Stationary points when $P' = 0$ occur at $x = 0, \pm\frac{1}{\sqrt{2}}$

Only valid solution in context is positive, so the solution is $x = \frac{1}{\sqrt{2}}$

Check that this is a maximum:

$P''(x) = 20 - 120x^2$ so $P''\left(\frac{1}{\sqrt{2}}\right) = 20 - 60 < 0$ and the point is a local maximum.

$$P\left(\frac{1}{\sqrt{2}}\right) = 10\left(\frac{1}{2} - \frac{1}{4}\right) = 2.5$$

Maximal profit is \$2.5 million

Tip: Alternatively, use a substitution: $u = x^2$ so that $P = 10u - 10u^2 = 10u(1-u)$.

This is a negative quadratic in u with roots at $u = 0$ and $u = 1$, so by symmetry will have its maximum at $u = 0.5$.

Hence maximum P is $P(u = 0.5) = 5(0.5) = 2.5$

9 **a** $P = 2x + 2(20 - x) = 40$ cm

b $A = x(20 - x)$

Negative quadratic has maximal value midway between the roots, by symmetry.

Roots are at $x = 0$ and $x = 20$

Maximal A is at $x = 10$: $A(10) = 100$ cm^2

10

$$y = \frac{x^2}{1+x}$$
$$y' = \frac{2x(1+x) - x^2(1)}{(1+x)^2} = \frac{x^2 + 2x}{(1+x)^2}$$

Stationary points where $y' = 0$

$$x^2 + 2x = 0 = x(x+2)$$

Roots of the derivative are at $x = 0$ and $x = -2$

$$y(0) = \frac{0}{1} = 0, y(-2) = \frac{4}{-1} = -4$$

Stationary points are $(0,0)$ and $(-2,-4)$.

11

$$y = x - 2x^{0.5}$$
$$y' = 1 - x^{-0.5}$$

Stationary points where $y' = 0$

$$x^{-0.5} = 1$$
$$x = 1$$
$$y(1) = 1 - 2 = -1$$

Stationary point is $(1,-1)$

$$y'' = 0.5x^{-1.5}$$

$y''(1) = 0.5 > 0$ so $(1,-1)$ is a local minimum.

12 $y = \sin x - 0.5x$

$y' = \cos x - 0.5$

Stationary point where $y' = 0$

$\cos x = 0.5$ has solution $x = \frac{\pi}{3}$ for $0 < x < \pi$

$y'' = -\sin x$

$y''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0$ so $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ is a local maximum.

13

$$V = x^2(9 - x)$$
$$V' = 18x - 3x^2 = 3x(6 - x)$$
$$V'' = 18 - 6x = 6(3 - x)$$

Maximum volume occurs where $V' = 0$ and $V'' < 0$

$V' = 0$ has solutions $x = 0$ or 6

$V''(6) = -18 < 0$

$V(6) = 108\text{ cm}^3$ is the maximal volume of the cuboid.

14 a

$$\mathrm{f}'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

b Stationary point where $f'(x) = 0$

$1 - \ln x = 0$

$x = \mathrm{e}$

Coordinates are $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$

c Using quotient rule, with $u = 1 - \ln x$, $v = x^2$ so $u' = x^{-1}, v' = 2x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

$$\mathrm{f}''(x) = \frac{x^2(x^{-1}) - 2x(1 - \ln x)}{x^4} = \frac{2x\ln x - 3x}{x^4} = \frac{2\ln x - 3}{x^3}$$

d $\mathrm{f}''(\mathrm{e}) = \frac{2-3}{\mathrm{e}^3} = -\mathrm{e}^{-3} < 0$

So the stationary point $(\mathrm{e}, \mathrm{e}^{-1})$ is a local maximum.

15 $y = e^{\sin x}$

Let $u = \sin x$ so $y = e^u$ and $u' = \cos x$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\begin{aligned} y' &= e^u \cos x \\ &= \cos x\, e^{\sin x} \end{aligned}$$

Stationary point where $y' = 0$: $\cos x = 0$ or $e^{\sin x} = 0$

$\cos x = 0$ at $x = (n + 0.5)\pi$ for $n \in \mathbb{Z}$. $e^{\sin x} = 0$ has no solutions.

$$\begin{aligned} y'' &= -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x} \\ &= (\cos^2 x - \sin x)e^{\sin x} \end{aligned}$$

For $0 < x < 2\pi$, the stationary points are $\left(\frac{\pi}{2}, e\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$

$y''\left(\frac{\pi}{2}\right) = -e < 0$ so $\left(\frac{\pi}{2}, e\right)$ is a local maximum

$y''\left(\frac{3\pi}{2}\right) = \frac{1}{e} > 0$ so $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$ is a local minimum

16 **a** $N(0) = 1 - 0 + 5 = 6$

Initial population is 6 million

b $N'(t) = e^t - 2$

Stationary point when $N' = 0$: $t = \ln 2$

$N(\ln 2) = 2 - 2\ln 2 + 5 = 7 - 2\ln 2 = 5.61$

This is lower than the initial population, so is the global minimum.

Minimum population is 5.61 million.

17 **a**

$$\begin{aligned} \text{Profit } P &= (x - 4)(1000 - 100x) \\ &= -100x^2 + 1400x - 4000 \\ &= -100(x^2 - 14x + 40) \\ &= -100((x - 7)^2 - 9) \\ &= 900 - 100(x - 7)^2 \end{aligned}$$

Maximal P is 900 when $x = 7$.

The optimal price is \$7 per widget.

b In this model, the greater the value of x, the lower the number sold, but the model never predicts a negative number of units sold (the model in part **a** suggests number sold is negative for $x > 10$).

c

$$\begin{aligned} P &= (x - 4)\left(\frac{1000}{(x + 1)^2}\right) \\ &= 1000\frac{x - 4}{(x + 1)^2} \end{aligned}$$

Using quotient rule:

$$P' = 1000\frac{\left((x+1)^2 - (x-4)2(x+1)\right)}{(x+1)^4}$$
$$= 1000\frac{\left((x+1) - 2(x-4)\right)}{(x+1)^3}$$
$$= 1000\frac{9-x}{(x+1)^3}$$

$P' = 0$ when $x = 9$

Maximum profit is seen when the price is \$9 per widget.

18 a distance = speed × time

$$d = \left(4 - \frac{1}{m}\right) \times \frac{200}{m}$$
$$= \frac{800}{m} - \frac{200}{m^2}$$

b

$$\frac{\mathrm{d}d}{\mathrm{d}m} = -800m^{-2} + 400m^{-3}$$
$$= 400m^{-3}(1-2m)$$

Stationary point at $m = \frac{1}{2}$

Since d is a negative quadratic in m^{-1}, this must be the maximum point.

The drone should be 500 g mass to maximise the distance it can fly.

19

$$\mathrm{f}(x) = \mathrm{e}^x - 3x + 7$$
$$\mathrm{f}'(x) = \mathrm{e}^x - 3$$

$\mathrm{f}'(x) = 0$ when $x = \ln 3$

$\mathrm{f}''(x) = \mathrm{e}^x > 0$ for all x, so this is a local minimum.

$\mathrm{f}(\ln 3) = 3 - 3\ln 3 + 7 = 10 - 3\ln 3$

So the range of $\mathrm{f}(x)$ is $\mathrm{f}(x) \geqslant 10 - 3\ln 3$

20

$$\mathrm{f}'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x+1)(x-2)$$

Turning points are $x = -1, 0, 2$

Positive quartic has turning points minimum, maximum, minimum

$\mathrm{f}(-1) = 4 + 5 - 12 + 2 = -1$

$\mathrm{f}(2) = 48 - 32 - 48 + 2 = -30$

The least minimum is -30 so the range of the function is $\mathrm{f}(x) \geqslant -30$

21

$$y = e^{2x} - 6e^{x} + 4x + 8$$

$$\begin{aligned} y' &= 2e^{2x} - 6e^{x} + 4 \\ &= 2(e^{x} - 2)(e^{x} - 1) \end{aligned}$$

Stationary points are at $x = \ln 2$ and $x = \ln 1 = 0$

When $x = \ln 2, y = 4 - 12 + 4\ln 2 + 8 = 4\ln 2$

When $x = 0, y = 1 - 6 + 0 + 8 = 3$

Stationary points are $(\ln 2, 4\ln 2)$ and $(0,3)$

$$y'' = 4e^{2x} - 6e^{x}$$

$y''(\ln 2) = 16 - 12 = 4 > 0$ so $(\ln 2, 4\ln 2)$ is a local mínimum.

$y''(0) = 4 - 6 = -2 < 0$ so $(0, 3)$ is a local máximum.

22

$$\begin{aligned} h &= e^{x} + e^{-2x} \\ h' &= e^{x} - 2e^{-2x} \end{aligned}$$

Stationary points when $h' = 0$

$$e^{-2x}(e^{3x} - 2) = 0$$

Only solution is $x = \frac{1}{3}\ln 2 = \ln \sqrt[3]{2}$

$$\begin{aligned} h(\ln \sqrt[3]{2}) &= \sqrt[3]{2} + \frac{1}{(\sqrt[3]{2})^2} \\ &= \sqrt[3]{2}\left(1 + \frac{1}{2}\right) \end{aligned}$$

The minimum height is $1.5\sqrt[3]{2}$ m ≈ 1.89 m

Exercise 20F

5

$$\begin{aligned} y &= x^3 + 9x^2 + x - 1 \\ y' &= 3x^2 + 18x + 1 \\ y'' &= 6x + 18 \end{aligned}$$

Inflexion occurs when $y'' = 0$

$$\begin{aligned} x &= -3 \\ y(-3) &= -27 + 81 - 3 - 1 = 50 \end{aligned}$$

Point of inflexion is $(-3, 50)$.

6

$$\begin{aligned} y &= x^5 - 80x^2 \\ y' &= 5x^4 - 160x \\ y'' &= 20x^3 - 160 \end{aligned}$$

Inflexion occurs when $y'' = 0$

$x^3 = 8$ so $x = 2$

$y(2) = 32 - 320$

Point of inflexion is $(2, -288)$.

7

$$y = x^3 + bx^2 + c$$
$$y' = 3x^2 + 2bx + c$$
$$y'' = 6x + 2b$$

Inflexion occurs when $y'' = 0$

$y''(1) = 0 = 6 + 2b$ so $b = -3$

$y(1) = 1 + b + c = 3$ so $c = 5$

8

$$y = 2x^3 - ax^2 + b$$
$$y' = 6x^2 - 2ax$$
$$y'' = 12x - 2a$$

Inflexion occurs when $y'' = 0$

$y''(1) = 0 = 12 - 2a$ so $a = 6$

$y(1) = 2 - a + b = -7$ so $b = -3$

9

$$y = x^3 + ax^2 + bx + c$$
$$y' = 3x^2 + 2ax + b$$
$$y'' = 6x + 2a$$

Inflexion occurs when $y'' = 0$

$y''(1) = 0 = 6 + 2a$ so $a = -3$

Stationary point occurs when $y' = 0$

$y'(1) = 0 = 3 + 2a + b$ so $b = 3$

$y(1) = 3 = 1 + a + b + c$ so $c = 2$

10 a

$$\mathrm{f}(x) = 8\ln x + x^2$$
$$\mathrm{f}'(x) = 8x^{-1} + 2x$$
$$\mathrm{f}''(x) = 2 - 8x^{-2}$$

Inflexion occurs when $y'' = 0$

$2 - 8x^{-2} = 0$ so $x = 2$

Verify that this is a point of inflexion:

$\mathrm{f}''(1) = 2 - 8 < 0$

$\mathrm{f}''(4) = 2 - \frac{1}{2} > 0$

$(2, 4 + 8\ln 2)$ is a point of inflexion, as the curve is concave-down for $x < 2$ and concave-up for $x > 2$.

b $f'(2) = 4 + 4 \neq 0$

This is a non-stationary point of inflexion.

11 a

$$y = 2\ln x + x^2$$
$$y' = 2x^{-1} + 2x = 2x^{-1}(1 + x^2)$$

Since $x^{-1} \neq 0$ for any x and $1 + x^2 > 0$ for all real x, it follows that $y' \neq 0$ for any real x and there is no stationary point.

b $y'' = -2x^{-2} + 2 = 2x^{-2}(x^2 - 1)$

The domain of the original function is $x > 0$ so the only point where $y'' = 0$ is at $x = 1$.

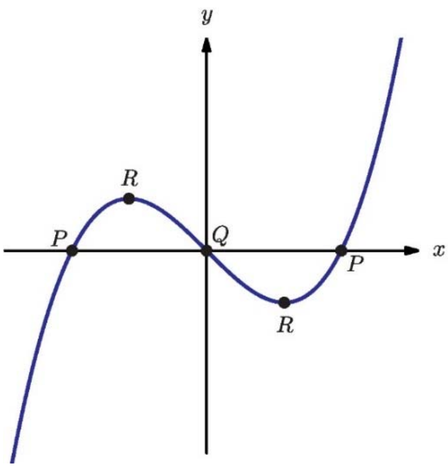
$(1, 1)$ is the point of inflexion.

12 a, b, c

Local minimum of $f(x)$ occurs when $f'(x) = 0$ and $f'(x)$ moves from negative to positive.

Local maximum of $f(x)$ occurs when $f'(x) = 0$ and $f'(x)$ moves from positive to negative.

Point of inflexion occurs when $f'(x)$ reaches a local maximum or minimum



d Neither inflexion point occurs when $f'(x) = 0$ so neither is a stationary inflexion point.

13 a

$$f(x) = x^4 - 6x^2 + 8x - 3$$
$$f'(x) = 4x^3 - 12x + 8$$

Stationary point occurs when $f'(x) = 0$

Tip: For solving the cubic, if you cannot quickly see a root or a means of factoring, just use your calculator to find the roots.

$$4(x^3 - 3x + 2) = 0$$
$$4(x + 2)(x^2 - 2x + 1) = 0$$
$$4(x + 2)(x - 1)^2 = 0$$

Stationary points occur at $x = -2$ or 1.

b $\quad f''(x) = 12x^2 - 12$

$f''(-2) = 48 - 12 > 0$ so there is a local minimum at $x = -2$

$$f''(1) = 12 - 12 = 0$$
$$f''(0) = -12 < 0$$

$f''(3) = 96 > 0$ so the curve changes from concave-down to concave-up at $x = 1$.

There is a stationary point of inflexion at $x = 1$

14 a

$$f(x) = 5x^4 - x^5$$
$$f'(x) = 20x^3 - 5x^4 = 5x^3(4 - x)$$

Stationary point where $f'(x) = 0$: $x = 0$ or $x = 4$

b

$$f''(x) = 60x^2 - 20x^3$$
$$f''(0) = 0$$
$$f''(1) = 60 - 20 > 0$$
$$f''(-1) = 60 + 20 > 0$$

The curve is concave-up for $x < 0$ and for $x > 0$

There is a local minimum at $x = 0$, not a point of inflexión.

$f''(4) = -320 < 0$ so there is local maximum at $x = 4$

15

$$y = e^{-x^2}$$
$$y' = -2xe^{-x^2}$$
$$y'' = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}$$

Point of inflexion occurs where $y'' = 0$

Since $e^{-x^2} \neq 0$ for any real x, the possible points of inflexion are at $x = \pm\frac{1}{\sqrt{2}}$

$$y''(0) = -2 < 0$$
$$y''(1) = 2e^{-1} > 0$$

The curve changes from convex-down to convex-up at $\left(\frac{\sqrt{2}}{2}, e^{-\frac{1}{2}}\right)$ so this is a point of inflexion.

Since the graph is symmetrical about the y-axis, $\left(-\frac{\sqrt{2}}{2}, e^{-\frac{1}{2}}\right)$is also a point of inflexion.

16

$$y = x^2 \ln x$$
$$y' = 2x \ln x + x^2(x^{-1}) = x + 2x \ln x$$
$$y'' = 1 + 2\ln x + 2xx^{-1} = 3 + 2\ln x$$

y'' is a strictly increasing function, so will pass through $y'' = 0$ from negative to positive exactly once. The point of inflexion occurs where $y'' = 0$

$3 + 2\ln x = 0$ so $x = e^{-\frac{2}{3}}$

Point of inflexion is at $\left(e^{-\frac{2}{3}}, -\frac{3}{2}e^{-3}\right)$.

17

$$y = 3x^5 - 10x^4 + 10x^3 + 2$$
$$y' = 15x^4 - 40x^3 + 30x^2$$
$$y'' = 60x^3 - 120x^2 + 60x$$
$$= 60x(x^2 - 2x + 1)$$
$$= 60x(x - 1)^2$$

Point of inflexion occurs where $y'' = 0$

Checking second derivative either side of possible points:

$$y''(-1) = -240 < 0$$
$$y''(0.5) = 7.5 > 0$$
$$y''(2) = 120 > 0$$

The graph changes from concave-down to concave-up at $x = 0$ so $(0, 2)$ is a (stationary) point of inflexion.

The graph continues concave-up either side of $x = 1$ so $(1,5)$ is not a point of inflexion.

18

$$f(x) = x^3 + ax^2 + bx + c$$
$$f'(x) = 3x^2 + 2ax + b$$
$$f''(x) = 6x + 2a$$

Point of inflexion occurs where $f''(x) = 0$ so $x = -\frac{a}{3}$

Stationary point occurs where $f'(x) = 0$

$$f'\left(-\frac{a}{3}\right) = \frac{a^2}{3} - \frac{2a^2}{3} + b = 0$$
$$b = \frac{a^2}{3}$$
$$a^2 = 3b$$

Mixed Practice

1 $f(x) = ax^3 - 3x + 5$

a $f'(x) = 3ax^2 - 3$

b $f'(0) = -3$

c Maximum at $x = -2$: $f'(-2) = 0 = 12a - 3$

$$a = 0.25 = \frac{1}{4}$$

2

$$y = x^3 + 6x^2 - 4x + 1$$
$$y' = 3x^2 + 12x - 4$$
$$y'' = 6x + 12$$

Point of inflexion occurs where $y'' = 0$

$$x = -2$$
$$y(-2) = -8 + 24 + 8 + 1 = 25$$

The point of inflexion is at $(-2, 25)$.

3

$$y = x^2 + bx + c$$
$$y' = 2x + b$$

Minimum point at $(2,3)$ so $y'(2) = 0$ and $y(2) = 3$

$$y'(2) = 4 + b = 0 \text{ so } b = -4$$
$$y(2) = 4 + 2b + c = 3 \text{ so } c = 7$$

4

$$y = x^3 + bx^2 + cx + 5$$
$$y' = 3x^2 + 2bx + c$$
$$y'' = 6x + 2b$$

Point of inflexion at $(2, -3)$ so $y''(2) = 0$ and $y(2) = -3$

$$y''(2) = 12 + 2b = 0 \text{ so } b = -6$$

$$y(2) = 8 + 4b + 2c + 5 = -3 \text{ so } c = 4$$

5

$$y = (x - 1)^{0.5}$$
$$y' = 0.5(x - 1)^{-0.5}$$
$$y'(5) = 0.25$$

6

$$y = \ln x$$
$$y' = x^{-1}$$
$$y'(1) = 1, y(1) = 0$$

Tangent at $(1,0)$ has equation $y = x - 1$

7

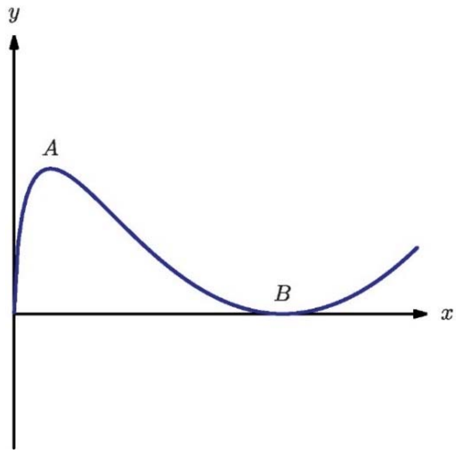
$$y = \mathrm{e}^{2x}$$
$$y' = 2\mathrm{e}^{2x}$$
$$y'(0) = 2, y(0) = 1$$

Normal at $(0,1)$ has gradient -0.5

Normal at $(0,1)$ has equation $y - 1 = -0.5(x - 0)$

$$x + 2y = 2$$
$$y = 1 - \frac{x}{2}$$

8



a Let $u = x, v = (\ln x)^2$ so $u' = 1$ and $v' = 2x^{-1}\ln x$

$$y = x(\ln x)^2 = uv$$

By product rule,

$$\begin{aligned} y' &= uv' + u'v \\ &= 2\ln x + (\ln x)^2 \\ &= (2 + \ln x)\ln x \end{aligned}$$

Stationary points occur where $y' = 0$

$2 + \ln x = 0$ gives point A at $x = \mathrm{e}^{-2}$, so $A = (\mathrm{e}^{-2}, 4\mathrm{e}^{-2})$.

$\ln x = 0$ gives point B at $x = 1$, so $B = (1, 0)$.

b

$$\begin{aligned} y'' &= (2 + \ln x)x^{-1} + (x^{-1})\ln x \\ &= \frac{2(1 + \ln x)}{x} \end{aligned}$$

Point of inflexion occurs where $y'' = 0$

$$\begin{aligned} 1 + \ln x &= 1 \\ x &= \mathrm{e}^{-1} \end{aligned}$$

The point of inflexion is at $(\mathrm{e}^{-1}, \mathrm{e}^{-1})$.

9 $C = \frac{2t}{3+t^2}$

Using the quotient rule:

$$\begin{aligned} C' &= \frac{2(3 + t^2) - 2t(2t)}{(3 + t^2)^2} \\ &= \frac{6 - 2t^2}{(3 + t^2)^2} \end{aligned}$$

Maximum concentration when $C' = 0$

$$\begin{aligned} t^2 &= 3 \\ t &= \sqrt{3} \end{aligned}$$

Maximum concentration is $C(\sqrt{3}) = \frac{\sqrt{3}}{3}$ mg l^{-1}

10 Let s be the distance between the cyclists at time t.

s is the length of the hypotenuse of a right-angled triangle with orthogonal side lengths $20t$ and $15t$.

$$s = \sqrt{(20t)^2 + (15t)^2}$$
$$= 25t$$

$$\frac{ds}{dt} = 25$$

The distance between the cyclists increases at a constant rate of 25 km h^{-1}

11 $f(x) = \ln(x^4 + 1)$

a $f(0) = \ln 1 = 0$

b $f'(x) = \frac{4x^3}{x^4+1}$

$f(x)$ is increasing for $f'(x) > 0$.

$f(x)$ is increasing for $x > 0$

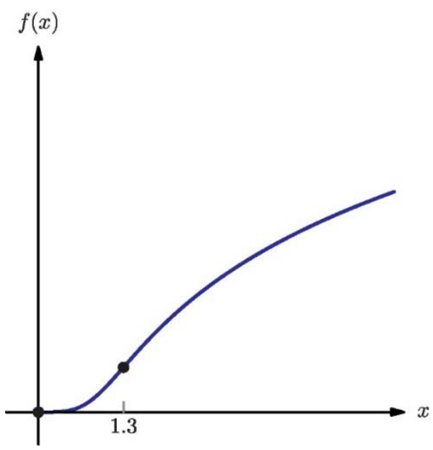
c $f''(x) = \frac{4x^2(3-x^4)}{(x^4+1)^2}$

i $f''(1) = \frac{4(1)(2)}{(2)^2} = 2$

ii $f''(-1) = \frac{4(1)(2)}{(2)^2} = 2$

Since the curve is convex-up either side of the point $(0,0)$, there is no point of inflexion at $(0,0)$; this is instead a minimum of the curve.

d



12 $f(x) = \frac{x}{-2x^2+5x-2}$

a Let $u = x, v = -2x^2 + 5x - 2$ so $u' = 1, v' = 5 - 4x$

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$

$$\begin{aligned} f'(x) &= \frac{(1)(-2x^2 + 5x - 2) - x(5 - 4x)}{(-2x^2 + 5x - 2)^2} \\ &= \frac{-2x^2 + 5x - 2 - 5x + 4x^2}{(-2x^2 + 5x - 2)^2} \\ &= \frac{2x^2 - 2}{(-2x^2 + 5x - 2)^2} \end{aligned}$$

b Stationary points occur when $f'(x) = 0$

$$\begin{aligned} 2x^2 - 2 &= 0 \\ x &= \pm 1 \end{aligned}$$

So B has coordinates $\left(1, f(1)\right) = \left(1, \frac{1}{9}\right)$

c From the graph, the range of $f(x)$ excludes the interval $\frac{1}{9} < y < 1$ so $y = k$ will not meet the graph for $\frac{1}{9} < k < 1$

13

$$\begin{aligned} y &= x\cos x \\ y' &= \cos x - x\sin x \\ y'(0) &= 1 \\ y(0) &= 0 \end{aligned}$$

So the tangent at $(0,0)$ has equation $y = x$

14 $y = x^2e^{2x}$

$$\begin{aligned} y' &= 2xe^{2x} + 2x^2e^{2x} \\ &= (2x + 2x^2)e^{2x} \end{aligned}$$

$$\begin{aligned} y'' &= (2 + 4x)e^{2x} + (4x + 4x^2)e^{2x} \\ &= (2 + 8x + 4x^2)e^{2x} \\ &= 2e^{2x}(2x^2 + 4x + 1) \end{aligned}$$

15 a

$$A = \frac{1}{2}x(6 - x) = 3x - \frac{x^2}{2}$$

$$A' = 3 - x$$

Stationary point on curve of A occurs when $A' = 0$: $x = 3$

Maximum area is $A(3) = 4.5\ \text{cm}^2$

(A is a negative quadratic, so the only stationary point is the global maximum)

b

$$P = x + (6 - x) + \sqrt{x^2 + (6 - x)^2}$$
$$= 6 + \sqrt{2x^2 - 12x + 36}$$
$$P' = \frac{4x - 12}{\sqrt{2x^2 - 12x + 36}}$$

Stationary point on curve of P occurs when $P' = 0$: $x = 3$

Minimum perimeter is $P(3) = 3 + 3 + 3\sqrt{2} = 6 + 3\sqrt{2}$ cm

16

$$f(x) = xe^{kx}$$
$$f'(x) = e^{kx} + kxe^{kx}$$
$$= (1 + kx)e^{kx}$$
$$f''(x) = ke^{kx} + k(1 + kx)e^{kx}$$
$$= (2k + k^2x)e^{kx}$$
$$f''(0) = 10 = 2k \text{ so } k = 5$$
$$f'(1) = (1 + k)e^k = 6e^5$$

17

$$f(x) = x^3 - kx^2 + 8x + 2$$
$$f'(x) = 3x^2 - 2kx + 8$$
$$f''(x) = 6x - 2k$$

$f(x)$ is concave-up where $f''(x) > 0$

$$6x - 2k > 0$$

$x > \frac{k}{3}$ is equivalent to $x > 4$

$$k = 12$$

18 Let $u = xe^x, v = \sin x$ so $u' = e^x + xe^x = (1 + x)e^x, v' = \cos x$

Product rule: $\frac{d}{dx}(uv) = uv' + u'v$

$$\frac{d}{dx}(xe^x \sin x) = xe^x \cos x + (1 + x)e^x \sin x$$
$$= e^x\big((1 + x)\sin x + x\cos x\big)$$

19 $f(x) = \frac{x}{\sqrt{4+x}} = x(4 + x)^{-0.5}$

Using the Product Rule and Chain Rule:

$$f'(x) = (4 + x)^{-0.5} - 0.5x(4 + x)^{-1.5}$$
$$= (4 + x)^{-1.5}(4 + x - 0.5x)$$
$$= \frac{8 + x}{2(4 + x)^{1.5}}$$

$a = 1, b = 8, c = 1.5$

20 a

$$\mathrm{f}(x) = x\mathrm{e}^{-x}$$
$$\mathrm{f}'(x) = \mathrm{e}^{-x} - x\mathrm{e}^{-x} = (1-x)\mathrm{e}^{-x}$$
$$\mathrm{f}''(x) = -\mathrm{e}^{-x} - (1-x)\mathrm{e}^{-x} = (x-2)\mathrm{e}^{-x}$$

$\mathrm{f}(x)$ is concave-down where $\mathrm{f}''(x) < 0$

Since $\mathrm{e}^{-x} > 0$ for all real x, the curve is concave-down for $x < 2$

b The point of inflexion is at $(2, 2\mathrm{e}^{-2})$, since at that point the curve changes from concave-down to concave-up.

21 $y = \mathrm{e}^{-x}\sin x$

$$y' = \mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x$$
$$= \mathrm{e}^{-x}(\cos x - \sin x)$$

Stationary point occurs where $y' = 0$.

Since $\mathrm{e}^{-x} > 0$ for all real x, there is a stationary point where $\cos x - \sin x = 0$

$$\sin x = \cos x$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}$$

Stationary point is $\left(\frac{\pi}{4}, \frac{\mathrm{e}^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$.

22 a $V(0) = 5$

Initial volume of the lake is 5 million m^3

b $\frac{\mathrm{d}V}{\mathrm{d}t} = 2\mathrm{e}^{-t} - 2t\mathrm{e}^{-t} = 2\mathrm{e}^{-t}(1-t)$

Stationary point occurs when $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$: $t = 1$

$$V(1) = 5 + \frac{2}{\mathrm{e}} \approx 5.74 \text{ million m}^3$$

c Greatest rate of emptying is a maximum of $\frac{\mathrm{d}V}{\mathrm{d}t}$, so $\frac{\mathrm{d}^2V}{\mathrm{d}t^2} = 0$

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2} = -2\mathrm{e}^{-t} - 2(1-t)\mathrm{e}^{-t} = 2\mathrm{e}^{-t}(t-2)$$

Greatest rate of emptying is at $t = 2$ hours.

23

$$y = \ln x + kx$$
$$y' = x^{-1} + k$$
$$y'(1) = 1 + k$$
$$y(1) = k$$

Tangent at $(1, k)$ has equation $y - k = (1+k)(x-1)$

$$y = (1+k)x - 1$$

The y-intercept of this tangent is -1, independent of the value of k.

24 $f(x) = \frac{100}{1+50e^{-0.2x}}$

a $f(0) = \frac{100}{51}$

b

$$\begin{aligned} 95 &= \frac{100}{1+50e^{-0.2x}} \\ 1+50e^{-0.2x} &= \frac{100}{95} \\ e^{-0.2x} &= \frac{5}{95 \times 50} \\ x &= -5\ln\left(\frac{1}{950}\right) = 34.3 \end{aligned}$$

c As $x \to \infty, f(x) \to 100$

Range of $f(x)$ is $0 < f(x) < 100$

d Using Chain Rule:

$f(x) = 100(1+50e^{-0.2x})^{-1}$

$$\begin{aligned} f'(x) &= -100(-0.2 \times 50e^{-0.2x})(1+50e^{-0.2x})^{-2} \\ &= \frac{1000e^{-0.2x}}{(1+50e^{-0.2x})^2} \end{aligned}$$

e From GDC, maximum value of $f'(x)$ is $f'(x) = 5$

Tip: You may be required to complete this type of answer without a GDC; in that case, the shortcut below is the type of working you need to look for.

Let $u = e^{-0.2x}$

$f'(x) = 1000u(1+50u)^{-2}$

By Chain Rule,

$$\begin{aligned} f''(x) &= \frac{d}{du}\big(f'(x)\big) \times \frac{du}{dx} \\ &= 1000\left((1+50u)^{-2} - 100u(1+50u)^{-3}\right) \times \frac{du}{dx} \\ &= 1000\frac{(1-50u)}{(1+50u)^3} \times \frac{du}{dx} \end{aligned}$$

Maximum rate of change of $f(x)$ occurs when $f''(x) = 0$

$\frac{du}{dx} = -0.2e^{-0.2x} \neq 0$ for any real x

So the only solution for a stationary point of $f'(x)$ occurs when $u = \frac{1}{50}$

The graph shows that $f(x)$ is concave-up and then concave-down, it is clear that the stationary point of $f'(x)$ must be a maximum.

When $u = \frac{1}{50}, f'(x) = \frac{20}{2^2} = 5$

The maximum rate of change of $f(x)$ is 5.

25 a

$$f(x) = \frac{x-a}{x-b}$$
$$= 1 + \frac{b-a}{x-b}$$
$$f'(x) = -(b-a)(x-b)^{-2}$$
$$= \frac{a-b}{(x-b)^2}$$

If $a > b$ then this is always positive, since $(x-b)^2 > 0$ for all x in the domain.

b If $p > q$ and both lie the same side of b then $f(p) > f(q)$.

However, the discontinuity at $x = b$ means that if $q < b < p$ then $f(q) > f(p)$.

26 a For a real function, require that the argument of the square root is non-negative.

$$x^2 \leqslant 4$$

The largest possible domain is $-2 \leqslant x \leqslant 2$

b

$$y = (4-x^2)^{0.5}$$
$$y' = -x(4-x^2)^{-0.5}$$

At point $\left(a, \sqrt{4-a^2}\right)$, the gradient is $-\frac{a}{\sqrt{4-a^2}}$

The gradient of the normal is therefore $\frac{\sqrt{4-a^2}}{a}$

The equation of the normal is $y - \sqrt{4-a^2} = \frac{\sqrt{4-a^2}}{a}(x-a)$

$$y = \frac{\sqrt{4-a^2}}{a}x$$

The y-intercept of the normal is at 0, independent of the value of a.

27 a

$$y = kx^n$$
$$y' = nkx^{n-1}$$

At point (a, ka^n), the gradient is nka^{n-1}

The product of the gradient and y-value is $nk^2a^{2n-1} = 1$

For this to be true for all values a, it follows that $2n - 1 = 0$.

$n = \frac{1}{2}$ (in which case the domain of the function is $x \geqslant 0$ and the gradient is not defined at $x = 0$).

b Then $\frac{1}{2}k^2 = 1$ so $k = \pm\sqrt{2}$

21 Analysis and approaches: Integration

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 21A

23

$$y = \int 3x^{0.5}\ \mathrm{d}x$$
$$= 2x^{1.5} + c$$
$$y(9) = 12 = 2 \times 27 + c$$
$$c = -42$$
$$y = 2x^{\frac{3}{2}} - 42$$

24

$$\mathrm{f}(x) = \int 2\cos x - 3\sin x\ \mathrm{d}x$$
$$= 2\sin x + 3\cos x + c$$
$$\mathrm{f}(0) = 5 = 3 + c$$
$$c = 2$$
$$\mathrm{f}(x) = 2\sin x + 3\cos x + 2$$

25

$$y = \int 2\mathrm{e}^{x} - 5x^{-1}\ \mathrm{d}x$$
$$= 2\mathrm{e}^{x} - 5\ln x + c$$
$$y(1) = 0 = 2\mathrm{e} + c$$
$$c = -2\mathrm{e}$$
$$y = 2\mathrm{e}^{x} - 5\ln|x| - 2\mathrm{e}$$

26 $\int 2x + 1.5x^{-1}\,\mathrm{d}x = x^2 + \frac{3}{2}\ln|x| + c$

27

$$V = \int t - 0.5\sin t \; \mathrm{d}t$$
$$= 0.5t^2 + 0.5\cos t + c$$
$$V(0) = 2 = 0.5 + c$$
$$c = 1.5$$
$$V = \frac{1}{2}t^2 + \frac{1}{2}\cos t + \frac{3}{2}$$

28

$$x = \int 5 - 2\mathrm{e}^t \; \mathrm{d}t$$
$$= 5t - 2\mathrm{e}^t + c$$
$$x(0) = 5 = c - 2$$
$$c = 7$$
$$x = 5t - 2\mathrm{e}^t + 7$$

29 If $\mathrm{g}(x) = 5 - 2x$ then $\mathrm{g}'(x) = -2$

$$\int (5-2x)^6 \; \mathrm{d}x = -\frac{1}{2}\int \left(\mathrm{g}(x)\right)^6 \mathrm{g}'(x) \; \mathrm{d}x$$
$$= -\frac{1}{2}\left(\frac{1}{7}\left(\mathrm{g}(x)\right)^7\right) + c$$
$$= -\frac{1}{14}(5-2x)^7 + c$$

30 $\int 3\sin 2x - 2\cos 3x \; \mathrm{d}x = -\frac{3}{2}\cos 2x - \frac{2}{3}\sin 3x + c$

31 If $\mathrm{g}(x) = x^2 + 1$ then $\mathrm{g}'(x) = 2x$

$$\int \left(\mathrm{g}(x)\right)^5 g'(x) \; \mathrm{d}x = \frac{1}{6}\left(\mathrm{g}(x)\right)^6 + \mathrm{c}$$
$$= \frac{1}{6}(x^2+1)^6 + \mathrm{c}$$

32 If $\mathrm{g}(x) = 3x^2$ then $\mathrm{g}'(x) = 6x$

$$y' = \frac{1}{6}g'(x)\sin\left(g(x)\right)$$
$$y = -\frac{1}{6}\cos\left(g(x)\right) + c$$
$$= -\frac{1}{6}\cos(3x^2) + c$$

33 If $g(x) = x^2 + 2$ then $g'(x) = 2x$

$$\int \frac{3x}{\sqrt{x^2+2}} \; \mathrm{d}x = \int 1.5\, g'(x)\left(g(x)\right)^{-0.5} \; \mathrm{d}x$$
$$= 3\left(g(x)\right)^{0.5} + c$$
$$= 3\sqrt{x^2+1} + c$$

34

$$\begin{aligned} f(x) &= \int (2x+3)^{-2}\ dx \\ &= -\left(\frac{1}{2}\right)(2x+3)^{-1} + c \\ &= c - \frac{1}{4x+6} \end{aligned}$$

$$f(-1) = 1 = c - \frac{1}{2}$$

$$c = \frac{3}{2}$$

$$\begin{aligned} f(x) &= \frac{3}{2} - \frac{1}{4x+6} \\ &= \frac{6x+8}{4x+6} \\ &= \frac{3x+4}{2x+3} \end{aligned}$$

35 Let $g(x) = \ln x$ then $g'(x) = x^{-1}$

$$f' = \frac{2}{3} g'(x) g(x)$$

$$\begin{aligned} f &= \frac{1}{3}\left(g(x)\right)^2 + c \\ &= \frac{1}{3}(\ln x)^2 + c \end{aligned}$$

$$f(1) = 1 = c$$

$$f(x) = 1 + \frac{(\ln x)^2}{3}$$

36

$$\begin{aligned} f(x) &= \int 2x^{-1}\ dx \\ &= 2\ln|x| + c \end{aligned}$$

$$f(-1) = 5 = c$$

$$f(-3) = 5 + 2\ln 3$$

37 $\int 4(\cos x)^3 \sin x + kx^2 + 1 = -\cos^4 x + \frac{k}{3}x^3 + x + c$

38 $\int e^x - e^{-3x}\, dx = e^x + \frac{1}{3} e^{-3x} + c$

39 Let $g(x) = x^2 + 1$ then $g'(x) = 2x$

$$\begin{aligned} \int x\sqrt{x^2+1}\, dx &= \frac{1}{2}\int \left(g(x)\right)^{0.5} g'(x)\, dx \\ &= \frac{1}{3}\left(g(x)\right)^{1.5} + c \\ &= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c \end{aligned}$$

40 a $\cos 2x = 1 - 2\sin^2 x$

b

$$\int \sin^2 x \ \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \ \mathrm{d}x$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

41 a $\cos 2x = 1 - 2\sin^2 x$

b From part **a**: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

42 Let $\mathrm{g}(x) = \cos x$ then $\mathrm{g}'(x) = -\sin x$

$$\int \tan x \ \mathrm{d}x = -\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)} \ \mathrm{d}x$$
$$= -\ln|\mathrm{g}(x)| + c$$
$$= c - \ln|\cos x|$$

43 $\int \sin x\,(1 - \cos^2 x) \ \mathrm{d}x = -\cos x + \frac{1}{3}\cos^3 x + c$

44 Let $\mathrm{g}(x) = \ln x$ then $\mathrm{g}'(x) = x^{-1}$

$$\int \frac{1}{x \ln x} \ \mathrm{d}x = \int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)} \ \mathrm{d}x$$
$$= \ln|\mathrm{g}(x)| + c$$
$$= \ln|ln\, x| + c$$

Exercise 21B

16

$$\int_1^5 (3x + 1)^{0.5} \ \mathrm{d}x = \left[\frac{1}{3}\left(\frac{1}{1.5}\right)(3x + 1)^{1.5}\right]_1^5$$
$$= \frac{16^{1.5} - 4^{1.5}}{4.5}$$
$$= \frac{112}{9}$$

17

$$\int_0^{\frac{\pi}{2}} 3\sin 2x \ \mathrm{d}x = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{3}{2} + \frac{3}{2} = 3$$

18 $\int_2^5 \sqrt{\ln x} \ \mathrm{d}x = 3.29$ (GDC)

19

$$\int_1^a 3x^{-0.5}\ dx = [6\sqrt{x}]_1^a$$
$$= 6(\sqrt{a} - 1) = 24$$
$$\sqrt{a} - 1 = 4$$
$$a = 25$$

20 a $y = (3 - x)(3 + x)$ has roots $(\pm 3, 0)$.

b

$$\int_{-3}^{3} 9 - x^2\ dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$$
$$= (27 - 9) - (-27 + 9)$$
$$= 36$$

21 a

$$\int_{-1}^{1} x^3 - 2x^2 - x + 2\ dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{1}$$
$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right)$$
$$= \frac{8}{3}$$

$$\int_{1}^{2} x^3 - 2x^2 - x + 2\ dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{1}^{2}$$
$$= \left(4 - \frac{16}{3} - 2 + 4\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)$$
$$= -\frac{5}{12}$$

b Total area enclosed $= \frac{8}{3} + \frac{5}{12} = \frac{37}{12}$

22 $\int_{-3}^{3} \ln(10 - x^2)\ dx = 11.0$ (GDC).

Tip: Solving exactly without a GDC:

$$\int_{-3}^{3} \ln(10 - x^2)\ dx = \int_{-3}^{3} \ln\left((\sqrt{10} + x)(\sqrt{10} - x)\right)\ dx$$
$$= \int_{-3}^{3} \ln(\sqrt{10} + x) + \ln(\sqrt{10} - x)\ dx$$

Using $\int \ln x\ dx = x\ln x - x + c$:

$$\int_{-3}^{3} \ln(\sqrt{10} + x) + \ln(\sqrt{10} - x)\ dx$$

$$= \left[(\sqrt{10} + x)\ln(\sqrt{10} + x) - (\sqrt{10} + x) - (\sqrt{10} - x)\ln(\sqrt{10} - x) + (\sqrt{10} - x)\right]_{-3}^{3}$$
$$= \left[\sqrt{10}\left(\ln(\sqrt{10} + x) - \ln(\sqrt{10} - x)\right) + x\left(\ln(\sqrt{10} + x) + \ln(\sqrt{10} - x)\right) - 2x\right]_{-3}^{3}$$
$$= \left[\sqrt{10}\ln\left(\frac{\sqrt{10} + x}{\sqrt{10} - x}\right) + x\ln(10 - x^2) - 2x\right]_{-3}^{3}$$

$$= \sqrt{10}\ln\left(\frac{\sqrt{10}+3}{\sqrt{10}-3}\right) - \sqrt{10}\ln\left(\frac{\sqrt{10}-3}{\sqrt{10}+3}\right) - 12$$
$$= 2\sqrt{10}\ln\left(\frac{\sqrt{10}+3}{\sqrt{10}-3}\right) - 12$$

23

$$\int_0^{\ln 3} 2e^{-3x}\,dx = \left[-\frac{2}{3}e^{-3x}\right]_0^{\ln 3}$$
$$= \frac{2}{3}\left(1 - e^{-3\ln 3}\right)$$
$$= \frac{2}{3}\left(1 - \frac{1}{27}\right)$$
$$= \frac{52}{81}$$

24

$$\int_{-9}^{-3} 5x^{-1}\,dx = [5\ln|x|]_{-9}^{-3}$$
$$= 5(\ln 3 - \ln 9)$$
$$= -5\ln 3$$

25

$$\int_k^{2k} x^{-1}\,dx = [\ln x]_k^{2k}$$
$$= \ln 2k - \ln k$$
$$= \ln\left(\frac{2k}{k}\right)$$
$$= \ln 2$$

26 a By symmetry, the graphs intersect at $x = \frac{\pi}{4}$

$$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$$

b Again using symmetry:

$$\text{Shaded area} = 2\int_0^{\frac{\pi}{4}} \sin x\ dx$$
$$= 2[-\cos x]_0^{\frac{\pi}{4}}$$
$$= 2\left(1 - \frac{\sqrt{2}}{2}\right)$$
$$= 2 - \sqrt{2}$$

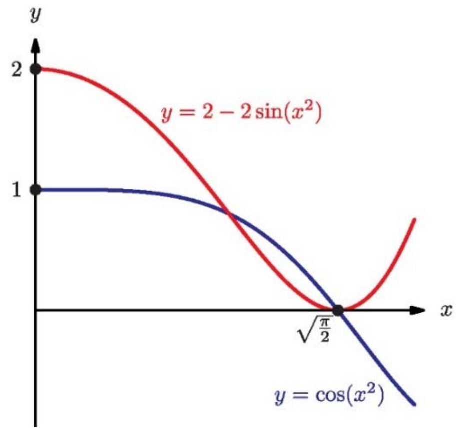
27 a At the intersections, $e^{-x^2} = e^{-x}$

Taking logarithms: $x^2 = x$ so $x = 0$ or 1

$A: (0,1), B(1, e^{-1})$

b $\int_0^1 \left(e^{-x^2} - e^{-x}\right)\ dx = 0.115$ (GDC)

28 a



b Intersections where $\cos x^2 = 2 - 2\sin x^2$

From GDC: $x = \cos^{-1}(0.8) = 0.802$ or $\sqrt{\frac{\pi}{2}} = 1.25$

$$\int_{0.802}^{1.25} (2 - 2\sin x^2 - \cos x^2)\ \mathrm{d}x = 0.0701$$

29 Intersections when $x^3 - 4x = 2x - x^2$

$x^3 + x^2 - 6x = 0$

$x(x-2)(x+3) = 0$

The enclosed area for $x \geqslant 0$ is between $x = 0$ and $x = 2$.

$$\begin{aligned}\int_0^2 6x - x^2 - x^3\ \mathrm{d}x &= \left[3x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\ &= 12 - \frac{8}{3} - 4 \\ &= \frac{16}{3} = 5.33\end{aligned}$$

30 By symmetry, the enclosed area is

$$\begin{aligned}2\int_0^{\frac{\pi}{2}} \sin 2x\ \mathrm{d}x &= 2\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} \\ &= 2\end{aligned}$$

31 Let $\mathrm{g}(x) = x^2 + 1$ so $\mathrm{g}'(x) = 2x$

$$\begin{aligned}\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{3}} \sqrt{\mathrm{g}(x)}\ \mathrm{g}'(x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{2}{3}(\mathrm{g}(x))^{1.5}\right]_0^{\sqrt{3}} \\ &= \left[\frac{1}{3}(x^2+1)^{1.5}\right]_0^{\sqrt{3}} \\ &= \frac{8}{3} - \frac{1}{3} \\ &= \frac{7}{3}\end{aligned}$$

32

$$\text{Shaded area} = \int_{-2}^{0} e^x \, dx + \int_{0}^{1} e^{-2x} \, dx$$
$$= [e^x]_{-2}^{0} + \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= 1 - e^{-2} - \frac{1}{2}e^{-2} + \frac{1}{2}$$
$$= \frac{3(1 - e^{-2x})}{2}$$

33

$$\int_1^4 (2x+3)^{-1} \, dx = \left[\frac{1}{2}\ln|2x+3|\right]_1^4$$
$$= \frac{1}{2}(\ln 11 - \ln 5)$$
$$= \frac{1}{2}\ln\left(\frac{11}{5}\right)$$

34

$$\int_2^3 (x-5)^{-1} \, dx = [\ln|x-5|]_2^3$$
$$= \ln 2 - \ln 3$$
$$= \ln\left(\frac{2}{3}\right)$$

35

$$\int_2^5 (3f(x) - 1) \, dx = 3\int_2^5 f(x) \, dx - \int_2^5 1 \, dx$$
$$= 30 - [x]_2^5$$
$$= 30 - (5 - 2)$$
$$= 27$$

36 Let $u = 2x$ so $du = 2dx$

When $x = 3, u = 6$ and when $x = 0, u = 0$

$$\int_{x=0}^{x=3} 5f(2x) \, dx = \frac{1}{2} \times 5 \int_{u=0}^{u=6} f(u) \, du$$
$$= \frac{5}{2}(7)$$
$$= 17.5$$

37 a $\frac{d}{dx}(x \ln x) = \ln x + x\left(\frac{1}{x}\right) = 1 + \ln x$

b

$$\int_1^e \ln x \, dx = \int_1^e 1 + \ln x - 1 \, dx$$
$$= [x \ln x - x]_1^e$$
$$= (e - e) - (0 - 1)$$
$$= 1$$

38

$$\int_a^d |f(x)|\,dx = 17 = \int_a^b f(x)\,dx + \int_c^d f(x)\,dx - \int_b^c f(x)\,dx$$
$$= \int_a^d f(x)\,dx - 2\int_b^c f(x)\,dx$$
$$= 5 - 2\int_b^c f(x)\,dx$$
$$\Rightarrow \int_b^c f(x)\,dx = -6$$

$$\int_a^d f(x)\,dx = 5 = \int_a^b f(x)\,dx + \int_b^d f(x)\,dx$$
$$= \int_a^b f(x)\,dx + 1$$
$$\Rightarrow \int_a^b f(x)\,dx = 4$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = -2$$

Exercise 21C

14 a $s(2) = 20 - 4 = 16$ m

b

$$v = \frac{ds}{dt} = 10 - 2t$$
$$v(3) = 4 \text{ m s}^{-1}$$

c

$$a = \frac{dv}{dt} = -2$$
$$a(4) = -2 \text{ m s}^{-2}$$

15

$$s(3) = \int_0^3 v\ dt$$
$$= \int_0^3 e^{-0.5t}\ dt$$
$$= [-2e^{-0.5t}]_0^3$$
$$= 2 - 2e^{-1.5} \approx 1.55 \text{ m}$$

16 a

$$s(6) = \int_0^6 v\ dt$$
$$= -48 \text{ m (GDC)}$$

b

$$d(6) = \int_0^6 |v| \; \mathrm{d}t$$
$$= 58.7 \text{ m}$$

17

$$a = 2t + 1$$
$$v = \int a \; \mathrm{d}t = t^2 + t + c$$
$$v(0) = 3 = c$$
$$v = t^2 + t + 3$$
$$v(4) = 16 + 4 + 3 = 23 \text{ m s}^{-1}$$

18 a $v(0) = 256 \text{ m s}^{-1}$

b $a = \frac{\mathrm{d}v}{\mathrm{d}t} = -4t^3$

$a(2) = -32 \text{ m s}^{-2}$

c Bullet stops when $v = 0$

$$t^4 = 256$$
$$t = 4 \text{ s}$$

d Since all the motion is in the same direction, distance travelled is the same as displacement.

$$s(4) = \int_0^4 v \; \mathrm{d}t$$
$$= \int_0^4 256 - t^4 \; \mathrm{d}t$$
$$= \left[256t - \frac{1}{5}t^5\right]_0^4$$
$$= \frac{4096}{5} = 819.2 \text{ m}$$

19

$$d(2) = \int_0^2 |v| \; \mathrm{d}t$$
$$= 2.39 \text{ (GDC)}$$

20 a $t^2(6 - t) = 0$

Displacement is zero at $t = 0$ and $t = 6$ seconds

b

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 12t - 3t^2$$
$$3t(4 - t) = 0$$

Velocity is zero at $t = 0$ and $t = 4$ seconds.

c Displacement is a negative cubic, so has a local maximum at the second stationary point.

The maximum displacement for $x \geqslant 0$ therefore occurs at $t = 4$ seconds.

21 a $v = \frac{ds}{dt} = \sin t + t\cos t$

b $v(0) = 0 \text{ m s}^{-1}$

c

$$v\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\pi}{4} \times \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}(4+\pi)}{8} = \frac{4+\pi}{4\sqrt{2}} \text{ m s}^{-1}$$

d

$$a = \frac{dv}{dt} = 2\cos t - t\sin t$$
$$a(0) = 2 \text{ m s}^{-2}$$

e $v = 1$ at $t = 0.556, 1.57$ or 5.10 seconds (GDC)

f

$$d = \int_0^{2\pi} |v| \, dt$$
$$= 13.3 \text{ m}$$

22 a

$$v = \int a \, dt$$
$$= At + c$$

$v(0) = c = u$

So $v = At + u$

b

$$s = \int v \, dt$$
$$= \frac{1}{2}At^2 + ut + k$$

$s(0) = 0 = k$

$$s = \frac{1}{2}At^2 + ut$$

23 $s = \sin \omega t$

$$v = \frac{ds}{dt} = \omega \cos \omega t$$

$$a = \frac{dv}{dt} = -\omega^2 \sin \omega t = -\omega^2 s$$

24 a $a = \frac{dv}{dt} = -10 \text{ m s}^2$

b Maximum height occurs when $v = 0$, so at $t = 0.5$ s

$$s = \int v \, dt$$
$$= 5t - 5t^2 + s(0)$$
$$s = 5t(1-t) + 60$$
$$s(0.5) = 61.25 \text{ m}$$

c $s = 0$ when $5t^2 - 5t - 60 = 0$

$$t^2 - t - 12 = 0$$
$$(t+3)(t-4) = 0$$

The ball hits the sea at $t = 4$ seconds

d $d = \int_0^4 |v| \, dt$

Using the context, the ball rises 1.25 m and then falls 61.25 m, so the total vertical distance travelled is 62.5 m

e The model assumes no obstructions and no significant air resistance or air movement interfering with the projected movement of the ball.

The model also uses acceleration due to gravity of 10 m s^{-2}, which is an approximation.

25 a $v(0) = 18 \text{ m s}^{-1}$

b $18 - 2t^2 = 0$ for $t = 3$ seconds

c

$$a = \frac{dv}{dt} = -4t$$
$$a(2) = -8 \text{ m s}^{-2}$$

d $d(6) = \int_0^6 |v| \, dt = 108 \text{ m}$

e $s(6) = \int_0^6 v \, dt = -36 \text{ m}$

f Since the bicycle turns at $t = 3$, and travels 108 m in total, ending 36 m behind its start position, it must travel 36 m forwards for the first 3 seconds, then 72 m in the second 3 seconds.

It travels 36 m in the first 3 seconds.

26 $v = \frac{ds}{dt} = -3t^2 + 12t - 2$

$$a = \frac{dv}{dt} = -6t + 12$$

Maximum velocity occurs when $a = 0$: $t = 2$

$v(2) = -12 + 24 - 2 = 10 \text{ m s}^{-1}$

27

$$\begin{aligned} v(2) &= \int_0^2 a \ \mathrm{d}t \\ &= \int_0^2 2t \ \mathrm{d}t \\ &= [t^2]_0^2 \\ &= 4 \text{ m s}^{-1} \end{aligned}$$

$$\begin{aligned} v(4) &= v(2) + \int_2^4 a \ \mathrm{d}t \\ &= 4 + \int_2^4 16t^{-2} \ \mathrm{d}t \\ &= 4 + [-16t^{-1}]_2^4 \\ &= 4 + (8 - 4) \\ &= 8 \text{ m s}^{-1} \end{aligned}$$

28 Root of the equation $v = \frac{1}{t} - 1$ occurs at $t = 1$.

$$\begin{aligned} d &= \int_{0.5}^2 |v| \ \mathrm{d}t \\ &= \int_{0.5}^1 v \ dt - \int_1^2 v \ dt \\ &= [\ln t - t]_{0.5}^1 - [\ln t - t]_1^2 \\ &= -1 - \ln 0.5 + 0.5 - \ln 2 + 2 - 1 \\ &= 0.5 - \ln(0.5 \times 2) \\ &= 0.5 \text{ m} \end{aligned}$$

29 a $s(10) = 0$ m

b $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 10 - 2t$

The movement is symmetrical, turning back at $t = 5$

$d(10) = 2s(5) = 2 \times 25 = 50$ m

30 From the GDC:

a Max speed $= 6.06$ m s^{-1}

b Min speed $= 0$ m s^{-1}

c Ave speed $= \frac{1}{4}\int_0^4 |v| \ \mathrm{d}t = 2.96$ m s^{-1}

31 a i

$$\begin{aligned} v &= \int a \ \mathrm{d}t \\ &= \frac{1}{2}At^2 + v(0) \\ &= \frac{1}{2}At^2 + u \end{aligned}$$

ii

$$s = \int v \; \mathrm{d}t$$
$$= \frac{1}{6}At^3 + ut + s(0)$$
$$= \frac{1}{6}At^3 + ut$$

b Since acceleration is positive from $t = 0$, $v > u > 0$ for all $t > 0$.

Then average speed is total distance divided by time taken

$$\text{Ave speed} = \frac{s}{t}$$
$$= \frac{1}{6}At^2 + u$$
$$= \frac{1}{3}\left(\frac{1}{2}At^2 + u\right) + \frac{2}{3}u$$
$$= \frac{1}{3}v + \frac{2}{3}u$$
$$= \frac{v + 2u}{3}$$

32 Let Jane's distance from Aisla's start line be s_J and Aisla's distance from her start line be s_A

$$s_J = \int v_J \; \mathrm{d}t = \frac{1}{2}t^2 + 2t + s_J(0) = \frac{1}{2}t^2 + 2t + 42$$

$$s_A = \int v_A \; \mathrm{d}t = \frac{1}{3}t^3 + s_A(0) = \frac{1}{3}t^3$$

Aisla passes Jane when $S_A = s_J$

From GDC, the solution to $\frac{1}{2}t^2 + 2t + 42 = \frac{1}{3}t^3$ is $t = 6$ seconds

33 Let the position of the first ball be given by

$$x_1(t) = \int v \; \mathrm{d}t = 5t - 5t^2$$

Let the position of the second ball from $t = 1$ be given by x_2

$$x_2(t) = x_1(t - 0.5) = 5(t - 0.5) - 5(t - 0.5)^2$$

The two collide when $x_1 = x_2$

$$5t(1 - t) = 5(t - 0.5)(1.5 - t)$$

From GDC, or from considering the symmetry, since the first ball is back at zero displacement at $t = 1$, the intersection occurs when $t = 0.75$.

$$x_1(0.75) = x_2(0.75) = 0.9375 \text{ m}$$

34 Consider an object which moves a distance a in time b, and then a further distance c in time d.

Then the average speed in the first period of movement is $\frac{\text{distance}}{\text{time}} = \frac{a}{b}$

The (greater) average speed in the second period of movement is $\frac{\text{distance}}{\text{time}} = \frac{c}{d}$

Then the average speed over the whole period must be between these two values.

$$\begin{aligned}\text{Average speed} &= \frac{\text{total distance}}{\text{total time}} \\ &= \frac{a+c}{b+d}\end{aligned}$$

Therefore,

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Mixed Practice

1

$$\begin{aligned}\int_1^4 3x^2 - 4\,\mathrm{d}x &= [x^3 - 4x]_1^4 \\ &= (64 - 16) - (1 - 4) \\ &= 51\end{aligned}$$

2

$$\begin{aligned}y &= \int \frac{1}{4}x^{-0.5}\ \mathrm{d}x \\ &= 2\left(\frac{1}{4}\right)x^{0.5} + c\end{aligned}$$

$$y(4) = 3 = \frac{2}{4}\sqrt{4} + c$$

$$c = 2$$

$$y = \frac{1}{2}\sqrt{x} + 2$$

3

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos x - \sin x\ \mathrm{d}x &= [\sin x + \cos x]_0^{\frac{\pi}{6}} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} - (0 + 1) \\ &= \frac{\sqrt{3} - 1}{2}\end{aligned}$$

4 $\int \frac{3}{4}x^{-1} - \frac{1}{2}x^{-2}\,\mathrm{d}x = \frac{3}{4}\ln|x| + \frac{1}{2}x^{-1} + c$

5

$$v = \frac{ds}{dt} = 3 - 0.18t^2$$
$$v(8.6) = -10.3 \text{ m s}^{-1}$$
$$a = \frac{dv}{dt} = -0.36t$$
$$a(8.6) = -3.10 \text{ m s}^{-1}$$

6 Let $g(x) = 2x - 5$ so $g'(x) = 2$

$$f(x) = 6\int \frac{2}{2x-5}\,dx$$
$$= 6\int \frac{g'(x)}{g(x)}\,dx$$
$$= 6\ln|g(x)| + c$$
$$= 6\ln|2x-5| + c$$

$$f(4) = 0 = 6\ln 3 + c$$

$$c = -6\ln 3$$

$$f(x) = 6\ln\left(\frac{|2x-5|}{3}\right)$$

Given the domain of $f(x)$ is restricted to $x > \frac{5}{2}$, the modulus signs are superfluous.

$$f(x) = 6\ln\left(\frac{2x-5}{3}\right)$$

7 **a**

$$a = \frac{dv}{dt} = 12 - 6t^2$$
$$a(2.7) = -31.7 \text{ cm s}^{-2}$$

b

$$s = \int v\,dt = 6t^2 - \frac{1}{2}t^4 - t + s(0)$$
$$s(0) = 0 \text{ so } s = 6t^2 - \frac{1}{2}t^4 - t$$
$$s(1.3) = 7.41 \text{ cm}$$

8 $\int 3x^{0.5} - 2x^{-1.5}\,dx = 2x^{\frac{3}{2}} + 4x^{-\frac{1}{2}} + c$

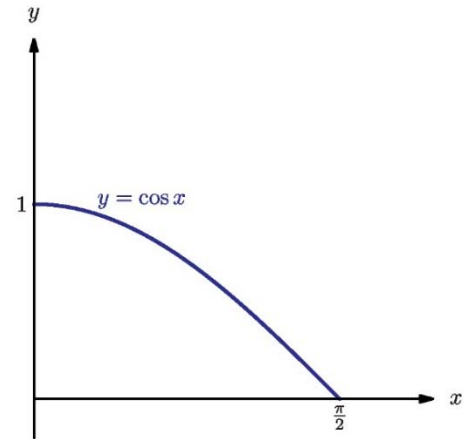
9 Let $g(x) = \sin x$ so $g'(x) = \cos x$

$$y = \int 3\cos x \sin^2 x\,dx$$
$$= \int 3g'(x)\big(g(x)\big)^2\,dx$$
$$= \big(g(x)\big)^3 + c$$
$$= \sin^3 x + c$$

$$y(\pi) = 2 = c$$

$$y(x) = \sin^3 x + 2$$

10 a



b $A(0,1), B\left(\frac{\pi}{2}, 0\right)$

c Gradient $= \frac{1-0}{0-\frac{\pi}{2}} = -\frac{2}{\pi}$

Equation is $y - 1 = -\frac{2}{\pi}(x - 0)$

$$y = 1 - \frac{2}{\pi}x$$

d Enclosed area is the area under the curve, less the area of triangle AOB.

$$\text{Enclosed area} = \int_0^{\frac{\pi}{2}} \cos x \;\mathrm{d}x - \frac{1}{2}\left(1 \times \frac{\pi}{2}\right)$$
$$= [\sin x]_0^{\frac{\pi}{2}} - \frac{\pi}{4}$$
$$= 1 - \frac{\pi}{4}$$

11 Require $\int_0^2 f(x) \;\mathrm{d}x = 0$

$$\int_0^2 x^2 - kx \;\mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{1}{2}kx^2\right]_0^2$$
$$= \frac{8}{3} - 2k$$
$$k = \frac{4}{3}$$

12 a Intersection point B: $x^2 + 1 = 7 - x$

$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = 2$$

$B\ (2, 5)$

$A\ (0, 1)$

$C\ (7, 0)$

b Shaded region is the area of the triangle DBC where D has coordinates $(2,0)$ plus the area between the curve and the x-axis for $0 \leqslant x \leqslant 2$.

$$\begin{aligned}\text{Shaded area} &= \frac{1}{2}(5 \times 5) + \int_0^2 x^2 + 1 \ \mathrm{d}x \\ &= \frac{25}{2} + \left[\frac{1}{3}x^3 + x\right]_0^2 \\ &= \frac{25}{2} + \frac{8}{3} + 2 \\ &= \frac{103}{6} = 17.2\end{aligned}$$

13 a

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 9 - 6t$$

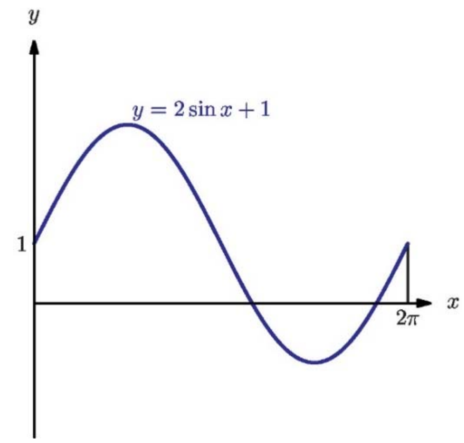
$$a(3) = -9 \text{ m s}^{-2}$$

b

$$\begin{aligned}s &= \int v \ \mathrm{d}t \\ &= \frac{9}{2}t^2 - t^3 + s(0)\end{aligned}$$

$$s(3) = \frac{81}{2} - 27 + 5 = 18.5 \text{ m}$$

14 a



b $v(2.5) = 2.20 \text{ m s}^{-1}$

c

$$\begin{aligned}s &= \int v \ \mathrm{d}t \\ &= t - 2\cos t + s(0)\end{aligned}$$

Defining displacement as distance from the initial position,

$$\begin{aligned}s &= t - 2\cos t + c \\ s(0) &= c - 2 \text{ so } c = s(0) + 2 \\ s &= t - 2\cos t + 2 + s(0) \\ s(2\pi) - s(0) &= 2\pi - 2 + 2 = 6.28 \text{ m}\end{aligned}$$

d $d(2\pi) = \int_0^{2\pi} |v| \ \mathrm{d}t = 9.02 \text{ m}$

15 $R = \int_0^4 \frac{x}{x^2+1} \ \mathrm{d}x$

Let $\mathrm{g}(x) = x^2 + 1$ so $\mathrm{g}'(x) = 2x$

$$R = \frac{1}{2}\int_0^4 \frac{\mathrm{g}'(x)}{\mathrm{g}(x)} \ \mathrm{d}x$$
$$= \left[\frac{1}{2}\ln(\mathrm{g}(x))\right]_0^4$$
$$= \frac{1}{2}(\ln 17 - \ln 1)$$
$$= \frac{1}{2}\ln 17$$

16

$$\int_{\frac{3\pi}{2}}^{b} \cos x \ \mathrm{d}x = [\sin x]_{\frac{3\pi}{2}}^{b}$$
$$= \sin b + 1$$
$$= 1 - \frac{\sqrt{3}}{2}$$

$$\sin b = -\frac{\sqrt{3}}{2} \text{ for some } \frac{3\pi}{2} < b < 2\pi$$

$$b = \frac{5\pi}{3}$$

17 a Graph B is linear with a root at $t = 3$, C is quadratic with maximum at $t = 3$ and A appears cubic with inflexion at $t = 3$.

It is reasonable to suppose that B represents acceleration, C represents velocity and A represents displacement.

b $t = 3$

18 $\int_5^8 2\mathrm{f}(x-3) \ \mathrm{d}x = 2\int_2^5 \mathrm{f}(x) \ \mathrm{d}x = 6.$

19 a $\mathrm{f}(x)$ is decreasing where $\mathrm{f}'(x) < 0$: $0 < x < d$

b $\mathrm{f}(x)$ is concave-up where $\mathrm{f}'(x)$ is increasing: $a < x < b$ and $x > c$

c

$$\int_a^0 \mathrm{f}'(x) \ \mathrm{d}x - \int_0^d \mathrm{f}'(x) \ \mathrm{d}x = \mathrm{f}(0) - \mathrm{f}(a) - (\mathrm{f}(d) - \mathrm{f}(0))$$
$$= 2\mathrm{f}(0) - \mathrm{f}(a) - \mathrm{f}(d)$$
$$= 2\mathrm{f}(0) - 8 - 2 = 20$$

$$2\mathrm{f}(0) = 30$$
$$\mathrm{f}(0) = 15$$

20 Intersection points where $2 - x^2 = x^3 - x^2 - bx + 2$

$$x^3 - bx = 0$$
$$x = 0, \pm\sqrt{b}$$

Let L be the left enclosed area and R the right enclosed area.

$$L = \int_{-\sqrt{b}}^{0} g(x) - f(x)\ \mathrm{d}x$$
$$= \int_{-\sqrt{b}}^{0} x^3 - bx\ \mathrm{d}x$$
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}bx^2\right]_{-\sqrt{b}}^{0}$$
$$= \frac{1}{4}b^2$$

$$R = \int_{0}^{\sqrt{b}} f(x) - g(x)\ \mathrm{d}x$$
$$= \int_{0}^{\sqrt{b}} bx - x^3\ \mathrm{d}x$$
$$= \left[\frac{1}{2}bx^2 - \frac{1}{4}x^4\right]_{0}^{\sqrt{b}}$$
$$= \frac{1}{4}b^2$$

As shown, $L = R$ irrespective of the value of b.

21 **a**

$$\cos 2x = 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\int \sin^2 3x\, dx = \frac{1}{2}\int (1 - \cos 6x) dx$$
$$= \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + \mathrm{c}$$
$$= \frac{1}{2}x - \frac{1}{12}\sin 6x + c$$

b

$$\int \cos^2 3x\, dx = \int (1 - \sin^2 3x) dx$$
$$= x - \left(\frac{1}{2}x - \frac{1}{12}\sin 6x + c\right)$$
$$= \frac{1}{2}x + \frac{1}{12}\sin 6x + c'$$

Analysis and approaches SL: Practice Paper 1

1 a 13 cm

b 38 (any answer between 30 and 40 is reasonable)

c $\frac{125}{200} = \frac{5}{8} = 0.625$

2 a $c = 15$

b $A(5,0)$

c $(2.5, 7.5)$

3 a 80

b $\mathrm{P}(1 \cap \mathrm{HL}) = \frac{10}{80} = \frac{1}{8} = 0.125$

c $\mathrm{P}(\mathrm{SL}|2) = \frac{30}{50} = \frac{3}{5} = 0.6$

4 a $\log_2 \frac{1}{4} = -2$

b Factorising:

$$(\log_2 x - 3)(\log_2 x + 2) = 0$$
$$\log_2 x = 3 \text{ or } -2$$
$$x = 8 \text{ or } \frac{1}{4}$$

5 a $\mathrm{f} \circ \mathrm{f}(3) = \mathrm{f}(0) = 1$

b $\mathrm{f}'(2) < 0$ so f is decreasing

c Stationary points where $\mathrm{f}'(x) = 0$:

$(1, 4)$ and $(4, -3)$

Classify using $\mathrm{f}''(x)$

$\mathrm{f}''(1) < 0$ so $(1, 4)$ is a local maximum

$\mathrm{f}''(4) > 0$ so $(4, -3)$ is a local minimum

d Inflection occurs where $\mathrm{f}''(x) = 0$ so $(2, 1)$ is a point of inflection

6 a $\tan 60° = \sqrt{3}$

$$\sin 60° = \frac{\sqrt{3}}{2}$$

$$\frac{\tan 60°}{\sin 60°} - \sin 60° \tan 60° = 2 - \frac{3}{2} = \frac{1}{2}$$

b

$$\frac{\tan x}{\sin x} - \sin x \tan x \equiv \frac{\left(\frac{\sin x}{\cos x}\right)}{\sin x} - \frac{\sin^2 x}{\cos x}$$
$$\equiv \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$$
$$\equiv \frac{\cos^2 x}{\cos x}$$
$$\equiv \cos x$$

7 Shaded area $= \int_0^a \left|(2x-4)\mathrm{e}^{-(x^2-4x)}\right| \mathrm{d}x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-(x^2-4x)}\right) = -(2x-4)\mathrm{e}^{-(x^2-4x)}$$

The curve passes through the x-axis at $x = 2$ so the integral can be split into two parts

$$\text{Shaded area} = -\int_0^2 (2x-4)\mathrm{e}^{-(x^2-4x)}\,\mathrm{d}x + \int_2^a (2x-4)\mathrm{e}^{-(x^2-4x)}\,\mathrm{d}x$$
$$= -\left[-\mathrm{e}^{-(x^2-4x)}\right]_0^2 + \left[-\mathrm{e}^{-(x^2-4x)}\right]_2^a$$
$$= \mathrm{e}^4 - 1 + \left(\mathrm{e}^4 - \mathrm{e}^{a^2-4a}\right)$$
$$= 2\mathrm{e}^4 - 1 - \mathrm{e}^{a^2-4a}$$

$2\mathrm{e}^4 - 1 - \mathrm{e}^{a^2-4a} = 2\mathrm{e}^4 - 2$

So $\mathrm{e}^{a^2-4a} = 1$

$a^2 - 4a = 0$

$a = 0$ or 4; from context, the correct solution is $a = 4$.

8 **a** Vertex is $(3, 0)$

b Original curve is $y = (x-3)^2$

Translate $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

New curve is $y = (x-5)^2$

Translate $\begin{pmatrix}0\\4\end{pmatrix}$: $\mathrm{f}(x)$ becomes $\mathrm{f}(x) + 4$

New curve is $y = (x-5)^2 + 4$

i Vertex moves from $(3, 0)$ to $(5, 4)$

ii New equation is $y = x^2 - 10x + 29$

c Intersection of $y = x^2 - 6x + 9$ and $y = x^2 - 10x + 29$

$$4x = 20$$
$$x = 5$$

Intersection point is $(5, 4)$

d **i** Substituting $y = x - k$ into the equation for P:

$$x - k = x^2 - 6x + 9$$
$$x^2 - 7x + (9 + k) = 0$$

ii If L is a tangent then there is a single repeated root to this quadratic

Repeated root: Discriminant $\Delta = 0$

$$(-7)^2 - 4(9+k) = 0$$

$$49 - 36 - 4k = 0$$

$$k = \frac{13}{4}$$

9 a $\text{P(Alessia wins on first attempt)} = \text{P}(D'A) = \frac{1}{2}(1-p)$

b

$$\begin{aligned}\text{P(Alessia wins on second attempt)} &= \text{P}(D'A'D'A)\\ &= (1-p)\times\frac{1}{2}\times(1-p)\times\frac{1}{2}\\ &= (1-p)^2\left(\frac{1}{2}\right)^2\end{aligned}$$

$a = 2, b = 2$

c i $\text{P(Alessia wins on third attempt)} = \text{P}(D'A'D'A'D'A) = (1-p)^3\left(\frac{1}{2}\right)^3$

ii $\text{P}\left(\text{Alessia wins on } k^{\text{th}} \text{ attempt}\right) = (1-p)^k\left(\frac{1}{2}\right)^k$

d $\text{P(Alessia wins)} = \sum(1-p)^k\left(\frac{1}{2}\right)^k$

This is the sum of a geometric series, with common ratio $r = \frac{1-p}{2}$ and first term $u_1 = r$

$$\text{P(Alessia wins)} = u_1\left(\frac{1}{1-r}\right) = \frac{1}{5}$$

$$\frac{r}{1-r} = \frac{1}{5}$$

$$\frac{1-r}{r} = 5$$

$$\frac{1}{r} - 1 = 5$$

$$r = \frac{1}{6} = \frac{1-p}{2}$$

$$1-p = \frac{1}{3}$$

$$p = \frac{2}{3}$$

10 a $f'(x) = \frac{3x^2}{x^3+1}$

Over the domain $x > -1$, numerator is always non-negative and the denominator is always positive.

Then $f'(x) \geqslant 0$ for the whole domain

This means that the function never decreases.

b $f''(x) = \frac{d}{dx}\left(\frac{3x^2}{x^3+1}\right)$

Using the quotient rule:

$$\begin{aligned} f''(x) &= \frac{6x(x^3+1) - 3x^2(3x^2)}{(x^3+1)^2} \\ &= \frac{6x - 3x^4}{(x^3+1)^2} \\ &= \frac{3x(2-x^3)}{(x^3+1)^2} \end{aligned}$$

c At the point $(0,0)$, $f''(x) = 0$ and (from part **a**), $f'(x) > 0$ either side of the point.

Therefore $(0,0)$ is a (stationary) point of inflection.

d $f''(x) = 0$ at $x = 0$ and $x = \sqrt[3]{2}$, so the other point of inflection is $\left(\sqrt[3]{2}, \ln 3\right)$

Again, this is known to be a point of inflection because $f''(x) = 0$ and $f'(x) > 0$ in the vicinity.

e The function is increasing over the whole domain, so is 1-1 and has an inverse.

$$\begin{aligned} y &= f(x) = \ln(x^3+1) \\ e^y &= x^3 + 1 \\ x &= \sqrt[3]{e^y - 1} = f^{-1}(y) \end{aligned}$$

Changing variable:

$$f^{-1}(x) = \sqrt[3]{e^x - 1}$$

The domain of the inverse is the range of the original function: $x \in \mathbb{R}$

The range of the inverse is the domain of the original function: $f^{-1}(x) > -1$

Analysis and Approaches SL: Practice Paper 2

1 a $CD = r\theta = 13 \times 0.75 = 9.75$ cm

b Triangle area $= \frac{1}{2}(18)(13)\sin 0.75 = 79.8\ \text{cm}^2$

Sector area $= \frac{1}{2}r^2\theta = \frac{1}{2}(13)^2 \times 0.75 = 63.4\ \text{cm}^2$

Shaded area $= 79.8 - 63.4 = 16.4\ \text{cm}^2$

2 a $\text{f}'(x) = \frac{2}{x}\ln x$

b If $\text{g}(x) = x^3 - x^2 - 6x$ then $\text{g}'(x) = 3x^2 - 2x - 6$

The tangents of the two curves are parallel for the same value of x when the gradients have the same value.

That is, $\text{f}'(x) = \text{g}'(x)$ so

$$\frac{2}{x}\ln x = 3x^2 - 2x - 6$$

Multiplying by x (noting that there is no solution $x = 0$ since this is outside the domain of $\text{f}(x)$)

$$2\ln x = 3x^3 - 2x^2 - 6x$$

c From GDC, solutions are $x = 0.340, 1.86$

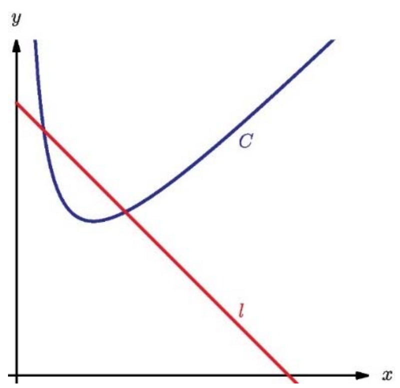
3 Using sine rule:

$$\sin B\hat{A}C = BC \times \frac{\sin A\hat{C}B}{AB} = 0.704$$

Primary solution: $B\hat{A}C = \sin^{-1} 0.704 = 44.8°$

Since $BC > AB$, the secondary solution is also valid: $B\hat{A}C = 180 - 44.8 = 135°$

4 a



b Intersections occur where $x + 2x^{-1} = 5 - x$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = 0.5$ or 2

$$\begin{aligned}\text{Enclosed area} &= \int_{0.5}^{2} (5 - x) - (x + 2x^{-1})\ \mathrm{d}x \\ &= \int_{0.5}^{2} 5 - 2x - 2x^{-1}\ \mathrm{d}x \\ &= [5x - x^2 - 2\ln x]_{0.5}^{2} \\ &= (10 - 4 - 2\ln 2) - (2.5 - 0.25 + 2\ln 2) \\ &= 3.75 - 4\ln 2 \approx 0.977\end{aligned}$$

5 **a** Vertical asymptote at root of denominator is $x = -1$

Maximal domain is $x \neq -1$

b Horizontal asymptote shown by end behaviour as $x \to \pm\infty$: $y = 2$

Range is $\mathrm{f}(x) \neq 2$

c

$$y = \mathrm{f}(x) = \frac{2x + 3}{x + 1}$$

$$(x + 1)y = 2x + 3$$

$$xy - 2x = 3 - y$$

$$x(y - 2) = 3 - y$$

$$x = \mathrm{f}^{-1}(y) = \frac{3 - y}{y - 2}$$

Changing variable:

$$\mathrm{f}^{-1}(x) = \frac{3 - x}{x - 2}$$

6 **a**

$v = \mathrm{e}^t \cos 2t$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^t \cos 2t - 2\mathrm{e}^t \sin 2t$$

b $d(2) = \int_0^2 |v|\,\mathrm{d}t = 4.76$ m

7 **a** Let the base of the pyramid be $ABCD$ and the vertex E, with the midpoint of the base X lying vertically below E so that $EX = 50$ cm

Then triangle AXE has angle $A\hat{X}E = 90°$ and $X\hat{A}E = 75°$

By trigonometry $AE = \frac{50}{\sin 75°} = 51.8$ cm

b And $AX = \frac{50}{\tan 75°} = 13.4$ cm

Then the base diagonal $AC = 2AX = 26.8$ cm

c The base area is given by $\frac{AC^2}{2} = 359\text{ cm}^2$

$$\begin{aligned}\text{Volume} &= \frac{1}{3}(\text{base area}) \times \text{height}\\ &= \frac{1}{3} \times 359 \times 50\\ &= 5\,983\text{ cm}^3\end{aligned}$$

8 a i $y = \frac{12-3x}{2} = 6 - \frac{3}{2}x$

Gradient of l_1 is $-\frac{3}{2}$

ii $N(0,6)$

b l_2 has gradient $\frac{2}{3}$ and passes through $P(3,-5)$

l_2 has equation $y + 5 = \frac{2}{3}(x-3)$

$3y + 15 = 2x - 6$

$2x - 3y - 21 = 0$

c $3x + 2y = 12 \qquad (1)$

$2x - 3y = 21 \qquad (2)$

Intersection:

$3(1) + 2(2): 13x = 78$

$x = 6$

Q has coordinates $(6,-3)$

d i $NQ = \sqrt{(6-0)^2 + (-3-6)^2} = \sqrt{117} \approx 10.8$

ii $PQ = \sqrt{(6-3)^2 + (-3+5)^2} = \sqrt{13} \approx 3.61$

e Since $NP \perp PQ$, area $NPQ = \frac{1}{2}(NP)(PQ) = 19.5$

9 a i Let X be the waiting time, in minutes.

$X \sim \text{N}(\mu, \sigma^2)$

$$\text{P}(X < 19) = \text{P}\left(Z < \frac{19-\mu}{\sigma}\right) = 0.9 = \text{P}(Z < 1.2816)$$

$$\text{So } \frac{19-\mu}{\sigma} = 1.2816$$

$\mu + 1.2816\sigma = 19 \qquad (1)$

ii $\text{P}(X < 3) = \text{P}\left(Z < \frac{3-\mu}{\sigma}\right) = 0.05 = \text{P}(Z < -1.6449)$

$$\text{So } \frac{3-\mu}{\sigma} = -1.6449$$

$\mu - 1.6449\sigma = 3 \qquad (2)$

iii $(1) - (2): 2.9264\sigma = 16$

$\sigma = 5.47$

$\mu = 3 + 1.6449\sigma = 12.0$

b $\mathrm{P}(X < t) = \mathrm{P}\left(Z < \frac{t-\mu}{\sigma}\right) = 0.2 = \mathrm{P}(Z < -0.8416)$

$t = \mu - 0.8416\sigma = 7.40$ minutes

c i $\mathrm{P}(X > 15) = 1 - \mathrm{P}\left(Z < \frac{15-\mu}{\sigma}\right) = 0.291$

ii Let Y be the number of patients, in a sample of 10, who wait more than 15 minutes.

Assuming independence between patient times, $Y \sim \mathrm{B}(10, 0.291)$

$$\begin{aligned}\mathrm{P}(Y \geq 5) &= 1 - \mathrm{P}(Y \leqslant 4)\\ &= 1 - 0.8647\ldots\\ &= 0.135\end{aligned}$$

10 a i

$$\begin{aligned}y &= x^{-1}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -x^{-2}\\ \frac{\mathrm{d}y}{\mathrm{d}x}(p) &= -\frac{1}{p^2}\end{aligned}$$

ii Then the tangent at (p, p^{-1}) will have gradient $-p^{-2}$ and so will have equation

$y - p^{-1} = -p^{-2}(x - p)$

$y - p^{-1} = -p^{-2}x + p^{-1}$

$p^2y - 2p = -x$

$x + p^2y - 2p = 0$

b i On the tangent line:

When $x = 0, y = \frac{2}{p}$ so R has coordinates $\left(0, \frac{2}{p}\right)$

When $y = 0, x = 2p$ so Q has coordinates $(2p, 0)$

ii Then area $OQR = 0.5(2p)(2p^{-1}) = 2$, independent of the value of p.

c $QR = \sqrt{(2p^{-1})^2 + (2p)^2} = 2\sqrt{p^2 + p^{-2}}$

d

Tip: Find minimum using GDC or calculus method such as shown below. Remember that when finding a minimum distance, where the distance has been given by a formula including a square root, it is usually easier to differentiate the square of the distance. Since distance is non-negative, the minimum of the distance and the minimum of its square must occur at the same value of the variable.

QR will have a minimum when QR^2 has a minimum

$$\begin{aligned}QR^2 &= 4p^2 + 4p^{-2}\\ \frac{\mathrm{d}}{\mathrm{d}p}(QR^2) &= 8p - 8p^{-3} = 8p^{-3}(p^4 - 1)\\ \frac{\mathrm{d}}{\mathrm{d}p}(QR^2) &= 0 \text{ when } p^4 = 1, \text{so } p = 1 \text{ (since } p > 0)\end{aligned}$$

This represents a minimum since:

$\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) < 0$ for $0 < p < 1$, and $\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) > 0$ for $p > 1$.

Minimum $QR = 2\sqrt{1+1} = 2\sqrt{2} \approx 2.83$