3 Geometry and trigonometry

Activity: Seven Bridges of Königsberg

There is a famous problem in mathematics based on the fact that the city of Königsberg (now Kaliningrad) lay on both sides of a river and also included two large islands. The different parts of the city were connected by a network of seven bridges.

The Swiss mathematician Leonhard Euler wondered it if was possible to plot a route that goes across all seven bridges without going over any bridge twice.

Below is a highlighted map of Königsberg, followed by a simplified form. The simplified form is called a **graph** or **network**.

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The dots in the graph are called **vertices**. The lines are known as **edges**.

In this network, the four vertices correspond to the four pieces of land, and the seven edges correspond to the seven bridges.

Tracing a path across all the bridges is equivalent to finding a path round the network. The conditions of the path are that

* you can start at any vertex
* you must pass along every edge exactly once (no doubling back).

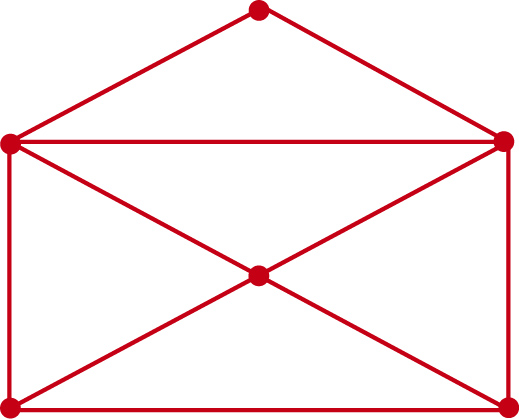
If such a path through a network exists, it is called a **Euler path***.*

If the path also begins and finishes at the same point, it is called a **Euler circuit**.

Can you find a Euler path for the Königsberg network? Try to do so before turning the paper over.

It is actually impossible to cross all seven bridges without crossing one bridge more than once. Maybe we can prove this. Let’s look at some other networks to get a general rule.

Consider the network below, which looks a bit like an envelope.



**How many vertices and edges does it have?**

There are six vertices and ten edges.

**Can you find an Euler path?**

This is equivalent to drawing the shape without taking your pen off the paper.

An Euler path does exist.

**Can you find a Euler circuit?**

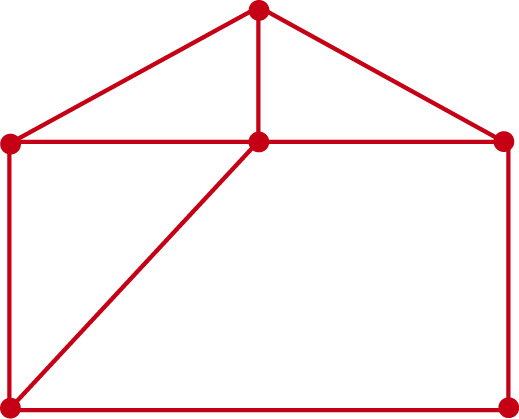
This is an Euler path that begins and ends at the same vertex.

This is impossible. The Euler path must start at one of the bottom corners and finish at the other bottom corner.

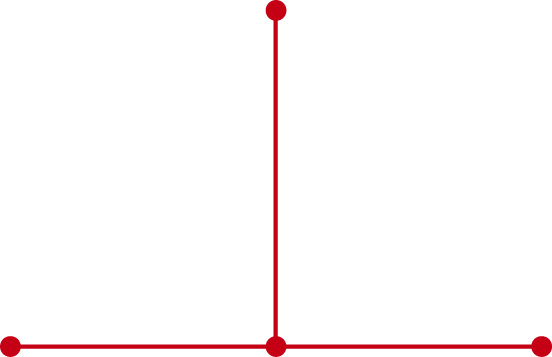
Now investigate some other networks.

Questions

1 Can you draw this network without taking your pen off the paper?



2 a How about this one?



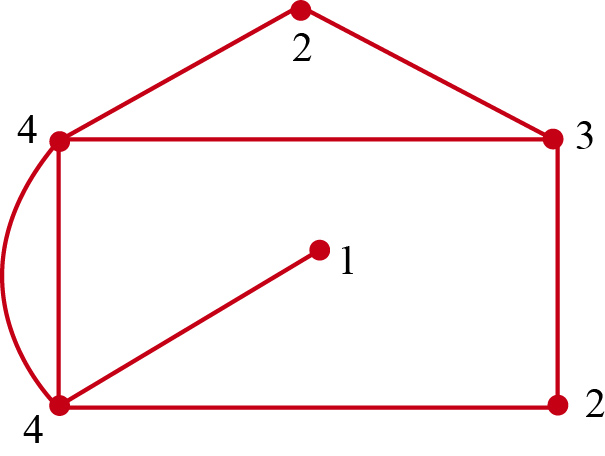
It is impossible!

b Can you explain why it is impossible?

3 For each network, attempt to draw it without taking your pen off the paper.   
If it is impossible, try and explain why.

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| --- | --- |
| a |  |
| b |  |
| c |  |

To understand why some shapes are possible and some are impossible we need to talk about the **degree** of each vertex. The degree is the number of lines coming out of a vertex. For example, in the graph below each vertex is labelled with its degree.



The degree of each vertex affects whether it could be part of a path.

* A vertex of degree 1 must be the start or an end point of any path, as it is a dead end.
* A vertex of degree 2 is easy to include in any path, as there is exactly one way in and one way out.
* A vertex of degree 3 causes a problem, as once we go in and out of it once we are left with only one route in or out. Hence any vertex of degree 3 must also be a start or end point, just like a vertex of degree 1.
* A vertex of degree 4 is easy to include in any path, as there are two routes in and two routes out.

**If the total number of vertices with odd degree is:**

* **more than 2 – no Euler path exists**
* **less than 2 – an Euler path exists**

This means that any edge with an odd degree must be either the start or end point of any path.

Since each path can have only one start and one end point, we can count the number of vertices with an odd degree to determine if an Euler path exists.

A special case of this is when there are no vertices with an odd degree, in which case not only does an Euler path exist, but so does an Euler circuit (starts and ends at the same place).

4 For each network, work out the degree of each vertex and determine if an Euler path or Euler circuit exists.

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| a |  |
| b |  |
| c |  |