OXFORD IB PREPARED

MATHEMATICS: ANALYSIS AND APPROACHES

IB DIPLOMA PROGRAMME

Paul Belcher
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Contents

Worked solutions to end-of-chapter practice questions and exam papers in this book can be found on your support website. Access the support website here:
www.oxfordsecondary.com/ib-prepared-support

Introduction

This book provides full coverage of the new (first examination in May 2021) IB diploma syllabus in Mathematics: Analysis and Approaches for both Higher Level and Standard Level. It complements the two Oxford University Press textbook course companions Mathematics: Analysis and Approaches, HL and SL, ISBN 978-0-19-842716-2 and 978-0-19-842710-0. There is a sister IB Prepared exam guide for the new Mathematics: Applications and Interpretations course, which complements the two Oxford University Press textbook course companions Mathematics: Applications and Interpretations, HL and SL, ISBN 978-0-19-842704-9 and 978-0-19-842698-1. This book offers support to students preparing for their examinations.
It will help you revise the study material, learn essential terms and concepts, strengthen your problem-solving skills and improve your approach to IB examinations. The book is packed with worked examples and exam tips that demonstrate best practices and warn against common errors. All topics are illustrated by annotated example student answers to questions informed by past examinations, which explain why marks may be scored or missed.

Practice questions for each chapter and a complete set of IB-style examination papers provide further opportunities to check your knowledge and skills, boost your confidence and monitor the progress of your studies. Full solutions to all problems and examination papers are given online at **www.oxfordsecondary.com/ib-prepared-support.**

As with any study guide, this book is not intended to replace course materials, such as textbooks, past papers, specimen papers and markschemes, the IB Mathematics: Analysis and Approaches syllabus and formula booklet, notation list, glossary of command terms and your own notes. Always have the IB material at hand whenever you are working. To succeed in the examination, you will need to use a broad range of resources, many of which are available online. The authors hope that this book will navigate you through this critical part of your studies, making your preparation for the exam less stressful and more efficient.

DP Mathematics: Analysis and Approaches assessment

All standard level (SL) and higher level (HL) students must complete the internal assessment and take two (SL) or three (HL) papers as part of their external assessment. Papers 1 and 2 are usually taken close to each other and Paper 3 a day or two later. The internal and external assessment marks are combined as shown in the table below to give your overall DP Mathematics grade, from 1 (lowest) to 7 (highest).

The final IB diploma score is calculated by combining grades for six subjects with up to three additional points from *Theory of knowledge* and *Extended essay* components.

Overview of the book structure

The book is divided into several sections that cover the core SL and additional higher level (HL) material in five chapters with a set of problems at the end of each chapter, the internal assessment and a complete set of practice examination papers.

The **Internal assessment** section outlines the nature of the Mathematical Exploration that you will have to carry out and explains how to select a suitable topic and present your Exploration in a suitable format to satisfy the marking criteria and achieve the highest grade.

The final section contains IB-style **practice examination Papers 1, 2 and 3**, written exclusively for this book. These papers will give you an opportunity to test yourself before the actual exam and at the same time provide additional practice problems for every topic.

The answers and solutions to all practice problems and examination papers are given online at **www.oxfordsecondary.com/ib-prepared-support**.

Assessment	Description	Topics	SL		HL	
			Marks	Weight	Marks	Weight
Internal	Mathematical Exploration	—	20	20%	20	20%
Paper 1	Short- and extended-response questions. No technology allowed	1,2,3,4,5 Including AHL topics for HL	80	40%	110	30%
Paper 2	Short- and extended-response questions. Technology allowed		80	40%	110	30%
Paper 3	Two extended-response, problem-solving questions. Technology allowed	All syllabus Especially AHL	–	–	55	20%

Command terms

Command terms are pre-defined words and phrases used in all IB Mathematics questions and problems. Each command term specifies the type and depth of the response expected from you in a particular question. See page vii for a full list of command terms.

Preparation and exam strategies

There are some simple rules you should follow during your preparation study and the exam itself.

1. Get ready for study. Have enough sleep, eat well, drink plenty of water and reduce your anxiety by positive thinking and physical exercise. A good night's sleep before the exam day is particularly important, as it can significantly improve your score.
2. Organize your study environment. Find a comfortable place with adequate lighting, temperature and ventilation. Eliminate all possible distractions. Keep your papers and computer files organized. Bookmark useful online and offline material.
3. Plan your studies. Make a list of your tasks and arrange them by importance. Break up large tasks into smaller, easily manageable parts. Create an agenda for your studying time and make sure that you can complete each task before the deadline.
4. Use this book as your first point of reference. Work your way through the topics systematically and identify the gaps in your understanding and skills. Spend extra time on the topics where improvement is required. Check your textbook and online resources for more information.
5. Read actively. Focus on understanding rather than memorizing. Recite key points and definitions using your own words. Try to solve every worked example and practice problem before looking at the answer. Make notes for future reference.
6. Get ready for the exams. Practice answering exam-style questions under a time constraint. Learn how to use the Mathematics formula booklet (allowed in all exams) quickly and efficiently. Solve as many problems from past papers as you can. Your school should give you a trial exam but you can create another using the papers at the end of this book. Know what each paper will involve. There will be 5 minutes reading time. SL Papers 1 and 2 are 1.5 hours long, HL Papers 1 and 2 are 2 hours long, Paper 3 is 1 hour long. Use of technology is allowed only for Papers 2 and 3. Make sure that your calculator is fully charged/has new batteries. Papers 1 and 2 come in two sections: questions in section A are short response questions to be answered on the exam paper underneath the question; questions in section B are extended response questions involving sustained reasoning to be answered in the answer booklets provided. Anything written on the exam paper in section B will not be seen by the examiner. Both sections carry the same number of marks. A common mistake is for students to work too long on section A and thus run out of time for section B. The marks allocated to each part of a question are a good guide to how many minutes should be spent on that part. Make sure that you have all the equipment required for the exam: pens, pencils, ruler and watch. If you are a non-native English speaker, consider bringing a paper dictionary to help with any words you are unsure of.
7. Optimize your exam approach. Read all questions carefully, paying extra attention to command terms. Keep your answers as short and clear as possible. Double-check all numerical values and units. Label axes in graphs and annotate diagrams. Use the exam tips from this book. Use the correct notation, e.g. put arrows on vectors. If you introduce a variable then explain what it stands for.
8. Take a positive attitude and concentrate on things you can improve. Set realistic goals and work systematically to achieve these goals. Be prepared to reflect on your performance and learn from your errors in order to improve your future results.

Key features of the book

Each chapter typically covers one topic, and starts with "**You should know**" and "**You should be able to**" checklists. Chapters contain the following features:

Note

Notes provide quick hints and explanations to help you better understand a concept.

Example

Examples offer solutions to typical problems and demonstrate common problem-solving techniques.

Assessment tip

Assessment tips assist in answering particular questions, warn against common errors and show how to maximize your score when answering particular questions.

Definitions to most rules and concepts are given in a grey box like this one, and explained in the text.

Sample student answers show typical student responses to IB-style questions (many of which are taken from past examination papers). Positive and negative feedback on the student's responses are given in the green and red pull-out boxes. The correct answer will always be given. An example is given below.

SAMPLE STUDENT ANSWER

Show that $x^2 + 6x + 13 \equiv (x+3)^2 + 4$

$x^2 + 6x + 13 = (x+3)^2 + 4$

$x^2 + 6x + 9 = (x+3)^2$

$x^2 + 6x + 9 = x^2 + 6x + 9$

$0 = 0$, so it is true

▲ The algebraic manipulation was useful and could have been part of a correct proof.

▼ The layout of the "proof" was incorrect. It started with the very statement that we were trying to prove and finished with $0 = 0$, which we already know. This working could be reconstructed into a proper proof as shown.

The answer above could have achieved 1/3 marks.

The correct solution should have been:

$\text{RHS} \equiv (x+3)^2 + 4 \equiv x^2 + 6x + 9 + 4 \equiv x^2 + 6x + 13 \equiv \text{LHS}$

Links provide a reference to relevant material, within another part of this book or the IB Mathematics: Analysis and Approaches syllabus, that relates to the text in question. The numbering refers to the IB syllabus and not the chapters in this book.

Questions similar to past IB examinations will have the exam paper icon.

Practice questions are given at the end of each chapter. These are similar to IB-style questions that provide you with an opportunity to test yourself and improve your problem-solving skills.

A formula book icon is used when a formula from the IB formula book is given.

IB command terms for Mathematics: Analysis and Approaches

Command term	Definition
Calculate	Obtain a numerical answer showing the relevant stages in the working.
Comment	Give a judgment based on a given statement or result of a calculation.
Compare	Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout.
Compare and contrast	Give an account of similarities and differences between two (or more) items or situations, referring to both (all) of them throughout.
Construct	Display information in a diagrammatic or logical form.
Contrast	Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout.
Deduce	Reach a conclusion from the information given.
Demonstrate	Make clear by reasoning or evidence, illustrating with examples or practical application.
Describe	Give a detailed account.
Determine	Obtain the only possible answer.
Differentiate	Obtain the derivative of a function.
Distinguish	Make clear the differences between two or more concepts or items.
Draw	Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Estimate	Obtain an approximate value.
Explain	Give a detailed account including reasons or causes.
Find	Obtain an answer showing relevant stages in the working.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.
Identify	Provide an answer from a number of possibilities.
Integrate	Obtain the integral of a function.
Interpret	Use knowledge and understanding to recognize trends and draw conclusions from given information.
Investigate	Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions.
Justify	Give valid reasons or evidence to support an answer or conclusion.
Label	Add labels to a diagram.
List	Give a sequence of brief answers with no explanation.
Plot	Mark the position of points on a diagram.
Predict	Give an expected result.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show	Give the steps in a calculation or derivation.
Show that	Obtain the required result (possibly using information given) without the formality of proof. "Show that" questions do not generally require the use of a calculator.
Sketch	Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.
Solve	Obtain the answer(s) using algebraic and/or numerical and/or graphical methods.
State	Give a specific name, value or other brief answer without explanation or calculation.
Suggest	Propose a solution, hypothesis or other possible answer.
Verify	Provide evidence that validates the result.
Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.

1 NUMBER AND ALGEBRA

1.1 NUMBER REPRESENTATION, PROOF AND THE BINOMIAL THEOREM

You must know:

- ✔ the difference between an equation and an identity
- ✔ the binomial theorem.

You should be able to:

- ✔ calculate with and express numbers in scientific notation
- ✔ construct simple deductive proofs
- ✔ apply the binomial theorem.

Assessment tip

The following statement will appear on the first page of all IB Mathematics papers.

Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

Remember this when answering any question. Many students lose marks by not following this instruction. Remember that non-zero digits are significant, zeros between non-zeros are significant, but leading zeros are not significant.

Assessment tip

After any final numerical answer, you should state the accuracy that the answer has been given to, as shown by "(3 sf)" in Example 1.1.1. This is a check to yourself that you have followed the instructions in the question or at the start of the paper.

Number representation

For very small and very large numbers it is convenient to represent them in the form

$$a\times10^k$$

where a is a real number, $1 \le a < 10$, and k is an integer. This is called scientific notation and is achieved by "moving" the decimal point, for example, $132\,000 = 1.32\times10^5$ and $0.000\,000\,456 = 4.56\times10^{-7}$

Note

Using scientific notation will alter the number of decimal places, but it will not affect the number of significant figures. This is why writing numbers in scientific notation provides a better indicator of the level of accuracy involved. The second example above confirms that leading zeros are not significant.

Note

If the question asks you to give your answer exactly, you may write, for example: 134, π, $\sqrt{2}$, $\frac{13}{7}$ and so on. Do not give a rounded decimal as your answer if the question asks for an exact answer.

Example 1.1.1

Earth's moon can be modelled as a sphere with radius $r = 1740\,\text{km}$. The formula for the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Use this formula to find the volume of the moon in km^3, giving your answer in the form $a \times 10^k$ where $a \in \mathbb{R}$, $1 \le a < 10$ and $k \in \mathbb{Z}$, i.e., in scientific notation.

Solution

$V = \frac{4}{3}\pi \times 1740^3 = 2.21 \times 10^{10}\,\text{km}^3$ (3 sf)

Note

Calculators can use, for example, E6 to represent 10^6, where the E indicates an exponent. Remember that your answers should be given with correct notation and not using calculator nomenclature.

Simple deductive proof

An equation is true for some values of the variables, and so uses an equals sign "=".

An identity is true for all values of the variables, and so uses the symbol "≡".

For example, $3x + 1 = 7$ is an equation, whereas $(x+1)^2 \equiv x^2 + 2x + 1$ is an identity.

An identity will have a right-hand side (RHS) and a left-hand side (LHS). To prove an identity, start with one side and use valid rules until that side has been transformed into the other side.

Assessment tip

It is often better to start with the side that looks the most complicated and try to simplify it to obtain the other side. Do not worry if the expression becomes longer before it eventually simplifies.

Example 1.1.2

Prove that $x^4 - 1 \equiv (x-1)(x^3 + x^2 + x + 1)$, $x \in \mathbb{R}$

Solution

$$\begin{aligned}\text{RHS} &\equiv (x-1)(x^3+x^2+x+1)\\ &\equiv x^4 + x^3 + x^2 + x - x^3 - x^2 - x - 1\\ &\equiv x^4 - 1 \equiv \text{LHS}\end{aligned}$$

Assessment tip

The command terms used when a deductive proof is required are "prove" or "show that".

Note

You *cannot* prove an identity just by verifying that it is true for some value(s) of the variables. In Example 1.1.2, stating that both sides equal 0 for $x = 1$ does not prove that this identity is true for all $x \in \mathbb{R}$.

Note

You also *cannot* prove an identity by starting with the very statement you are trying to prove, working with it until you reach something that is true and then declaring the original to be true.

For example, the following statement is faulty logic as $A \Rightarrow B$ does not mean that $B \Rightarrow A$:

"$3 = 4 \Rightarrow 4 = 3 \Rightarrow$ by adding the equations together, $7 = 7$ and that is true, so $3 = 4$ must be true as well."

Sometimes both sides will seem complicated. It is then permissible to work with the LHS and show that this equals expression P, for example, and then start again independently with the RHS and show that this also equals P. Since both sides equal P, both sides are equal. However, you *cannot* start with both sides at the same time with an equals sign between them.

▲ The algebraic manipulation was useful and could have been part of a correct proof.

▼ The layout of the "proof" was incorrect. It started with the very statement that we were trying to prove and finished with $0 = 0$, which we already know. This working could be reconstructed into a proper proof as shown.

SAMPLE STUDENT ANSWER

Show that $x^2 + 6x + 13 \equiv (x+3)^2 + 4$

$x^2 + 6x + 13 = (x+3)^2 + 4$

$x^2 + 6x + 9 = (x+3)^2$

$x^2 + 6x + 9 = x^2 + 6x + 9$

$0 = 0$, so it is true

The answer above could have achieved 1/3 marks.

The correct solution should have been:

$\text{RHS} \equiv (x+3)^2 + 4 \equiv x^2 + 6x + 9 + 4 \equiv x^2 + 6x + 13 \equiv \text{LHS}$

Binomial theorem

Note

The binomial theorem is given by

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + {}^nC_{n-1}a^1b^{n-1} + b^n$

or in summation notation $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}b^r$, where the ${}^nC_r$, the binomial coefficients (combinations or "choose" numbers), can be obtained from Pascal's triangle, the calculator or the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$

Assessment tip

In examinations, with "prove" or "show that" questions where the answer is given, your work will be checked carefully to ensure that each step follows logically from the previous step. Lay out your work methodically and do not miss out any steps. Remember, you are trying to communicate with the examiner. You are not just trying to convince yourself that the statement is true.

Assessment tip

The formula book will be available in all IB exams. Always work with it next to you. Get to know where each formula is within the book. Make sure that you have learned formulae that you will need that are not in the book.

Example 1.1.3

Expand $(1+2x)^4$ using the binomial theorem.

Solution

$(1+2x)^4 = 1^4 + 4 \times 1^3 \times (2x) + 6 \times 1^2 \times (2x)^2 + 4 \times 1^1 \times (2x)^3 + (2x)^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Assessment tip

When using the binomial theorem, identify what n, a and b are. For small values of n it is worth quickly writing down the start of Pascal's triangle:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

For ${}^nC_r$, when you are looking at row n, remember that it starts with $r = 0$. Put the binomial coefficients spaced out on a line, then fill in the powers of a and b. It is worth putting brackets around the expressions that are a and b so that you realize that it is the whole bracket that is raised to the appropriate power.

The construction of Pascal's triangle relies upon the fact that

$${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$$

This formula can be proved using the formula for ${}^nC_r$ that involves factorials, but a more informal proof follows.

A set of r people is to be chosen from $n + 1$ people. This can be done in ${}^{n+1}C_r$ different ways. The collection of $n + 1$ people has one very special person called Colin. The sets of r people can be divided into two disjoint subsets: those that include Colin and those that do not. For sets that include Colin, we will have to choose Colin and will then have to choose $r - 1$ people from the rest, so this can be done in ${}^{n}C_{r-1}$ different ways. For the sets that do not include Colin, all the r people will have to be chosen from the rest, so this can be done in ${}^{n}C_r$ different ways. Therefore, equating the two different ways of counting gives ${}^{n+1}C_r = {}^{n}C_{r-1} + {}^{n}C_r$

SAMPLE STUDENT ANSWER

Expand $(2 - 3x)^3$ using the binomial theorem.

$2^3 + 3 \times 2^2 \times -3x + 3 \times 2 \times -3x^2 + 1 \times -3x^3$

$= 8 - 36x - 18x^2 - 3x^3$

▲ The binomial theorem was used with the correct binomial coefficients and correct powers of a, ensuring that the first two terms were correct.

▼ Brackets were not placed around the expression for b, which meant that powers of the -3 were ignored.

The answer above could have achieved 2/4 marks.

The correct solution should have been:

$2^3 + 3 \times 2^2 \times (-3x) + 3 \times 2 \times (-3x)^2 + 1 \times (-3x)^3 = 8 - 36x - 54x^2 - 27x^3$

Example 1.1.4

Find the first three terms in the binomial expansion of $(1 - 4x)^{10}$ in ascending powers of x.

Solution

$(1 - 4x)^{10} = 1 + {}^{10}C_1\,(-4x) + {}^{10}C_2\,(-4x)^2 \ldots = 1 - 40x + 720x^2\ldots$

Assessment tip

For larger values of n, when evaluating ${}^{n}C_r$ use your calculator on the paper that allows the use of technology, and the formula involving factorials on the paper where technology is not allowed.

Example 1.1.5

Find the constant term in the binomial expansion of $\left(x + \frac{2}{x^3}\right)^8$

Solution

The general term is ${}^{8}C_r\, x^{8-r}\left(\frac{2}{x^3}\right)^r$

The power of x is $8 - r - 3r = 8 - 4r$

Require this to be 0, so $r = 2$ and the term is ${}^{8}C_2\, x^6\left(\frac{2}{x^3}\right)^2 = 112$

Assessment tip

With questions like Example 1.1.5, don't waste time writing down all the terms. Look at the general term and then fit it to what is required. The constant term is the one that does not involve x at all, because the exponent of x is zero.

1.2 ARITHMETIC AND GEOMETRIC SEQUENCES AND SERIES

You must know:

- ✔ the definition of an arithmetic sequence
- ✔ the definition of a geometric sequence.

You should be able to:

- ✔ use the formulae for the nth term and the sum of the first n terms of an arithmetic sequence
- ✔ apply arithmetic sequences and series to model situations
- ✔ use the formulae for the nth term and the sum of the first n terms of a geometric sequence
- ✔ apply geometric sequences and series to financial applications
- ✔ work with infinite, convergent geometric sequences and series.

Arithmetic sequences

An arithmetic sequence is defined as a sequence $\{u_n\}$ where the difference between consecutive terms $u_{n+1} - u_n$ always has a constant value of d, the common difference.

The nth term is given by $u_n = u_1 + (n-1)d$

The sum of the first n terms is given by
$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$

Note

In the formula for the n th term, the $(n-1)$ can be visualized by the fact that if you build a fence with n fence posts there will be $(n-1)$ gaps between the posts.

The formula $S_n = \frac{n}{2}(u_1 + u_n)$ for the sum of the first n terms can be thought of as the number of terms multiplied by the average of the first and last terms.
If plotted, the terms in an arithmetic sequence will lie on a straight line with a gradient of d.

Assessment tip

The command term "write down" is indicating that you do not need to do any further work, the calculator has already done it for you. In this case, just copy down the values.

Note

The formula $u_n = S_n - S_{n-1}$, which is true for any sequence, can be useful when the information given in a question is an expression for the sum of the first n terms. Summation notation can be used with $S_n = \sum_{i=1}^{n} u_i$

Example 1.2.1

An arithmetic sequence has second term 8 and sixth term 32.

(a) Find (i) the common difference and (ii) the first term.

(b) Write down and simplify an expression for (i) the nth term and (ii) the sum of the first n terms.

Solution

(a) $u_1 + d = 8,\ u_1 + 5d = 32 \Rightarrow 4d = 24$ (i) $d = 6$ (ii) $u_1 = 2$

(b) (i) $u_n = 2 + (n-1)6 = -4 + 6n$

(ii) $S_n = \frac{n}{2}(2 \times 2 + (n-1)6) = n(3n-1)$

SAMPLE STUDENT ANSWER

The sum of the first n terms of a sequence is $S_n = 4n^2 + 2n$

(a) Prove that this is an arithmetic sequence.

(b) Find the common difference and the first term.

(c) Write down a simplified expression for the nth term.

$u_1 = S_1 = 6,\ u_1 + u_2 = S_2 \Rightarrow 6 + u_2 = 20 \Rightarrow u_2 = 14 \Rightarrow d = 8$
$u_n = 6 + (n-1)8 = 8n - 2$

▲ The student has the correct answers to parts (b) and (c).

▼ The student has not labelled their answers (a), (b) and (c) and did not give the proof asked for in part (a). Their working assumed that it was arithmetic.

The answer above could have achieved 4/8 marks.

The correct solution for (a) is:

$u_n = S_n - S_{n-1} = n(4n + 2) - (n - 1)(4n - 2)$
$= 4n^2 + 2n - (4n^2 + 6n + 2) = 8n - 2$

$u_n - u_{n-1} = 8n - 2 - (8(n - 1) - 2) = 8$, which is constant,

so the sequence is arithmetic (with $d = 8$).

Example 1.2.2

An arithmetic sequence has first term 1 and common difference 5.

Find the value of n if:

(a) $u_n = 201$ (b) $S_n = 12\,145$

Solution

(a) $u_n = 201 = 1 + (n - 1)5 \Rightarrow n - 1 = 40 \Rightarrow n = 41$

(b) $S_n = 12\,145 = \frac{n}{2}(2 + (n-1)5) \Rightarrow 5n^2 - 3n - 24\,290 = 0$

Solving with technology, e.g., using a polynomial equation solver or a graph or using "table", and remembering that the answer must be a positive integer: $n = 70$

Example 1.2.3

The mass m_n in kilograms of a particular animal n months after its birth is given in the following table.

n	1	2	3	5
m_n	0.47	1.51	2.52	4.48

(a) Give the masses rounded to one decimal place and hence suggest an approximate mathematical model to fit this data.

(b) Use your model to estimate the mass of the animal (i) after 4 months (ii) after 7 months.

(c) Explain whether or not you would use your model to estimate the mass of the animal after 2 years.

(d) Suggest two limitations for the model that you have proposed.

Solution

(a)

n	1	2	3	5
m_n	0.5	1.5	2.5	4.5

This suggests that the data is approximately fitted by an arithmetic sequence with first term 0.5 and common difference $d = 1$.
Thus, the model is $m_n = 0.5 + (n - 1) \times 1 = n - 0.5$

(b) (i) $m_4 \simeq 3.5$ (ii) $m_7 \simeq 6.5$

(c) Would not use the model, since 24 months is too far away from the data given and we cannot assume that the increase in mass will continue to follow this pattern. Extrapolation is unreliable.

(d) Model would give the mass at birth as –0.5 kg, which clearly does not make sense.

As $n \to \infty$, $m_n \to \infty$, which is also unrealistic.

Geometric sequences

Note

Although both versions of the formula for S_n are equally valid, it is more convenient to use the first one if $r > 1$ and the second one if $r < 1$.

A geometric sequence is defined as a sequence $\{u_n\}$ where the ratio between consecutive terms $\frac{u_{n+1}}{u_n}$ always has constant value of r, the common ratio.

The nth term is given by $u_n = u_1 r^{n-1}$

The sum of the first n terms is given by $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, r \neq 1$

Example 1.2.4

A geometric sequence has second term 6 and fifth term 48.

(a) Find (i) the common ratio and (ii) the first term.

(b) Hence write down a formula for (i) the nth term and (ii) the sum of the first n terms.

Solution

(a) $u_1 r = 6,\ u_1 r^4 = 48 \Rightarrow r^3 = 8$

(i) $r = 2$ (ii) $u_1 = 3$

(b) (i) $u_n = 3 \times 2^{n-1}$ (ii) $S_n = 3(2^n - 1)$

SAMPLE STUDENT ANSWER

A geometric sequence has second term 5 and fourth term 45.
Find the possible values of the fifth term.

$ar = 5,\ ar^3 = 45 \Rightarrow r^2 = 9 \Rightarrow r = 3$, so $u_5 = 45 \times 3 = 135$

The answer above could have achieved 2/4 marks.

▲ The student had the correct method of dividing the two expressions for the terms, to obtain $r^2 = 9$ and had one of the correct answers.

▼ The student forgot the solution $r = -3$ when taking the square root and thus missed the other solution of $u_5 = -135$. The student also did not notice the plural word "values" in the question, which was a hint that the answer was not unique.

Example 1.2.5

The sum of the first n terms of a sequence is $S_n = \frac{4^n - 1}{3}$

(a) Prove that this sequence is geometric.

(b) Find the values of (i) the common ratio and (ii) the first term.

(c) Write down an expression for the nth term.

Solution

(a) $u_n = S_n - S_{n-1} = \left(\frac{4^n - 1}{3}\right) - \left(\frac{4^{n-1} - 1}{3}\right) = 4^{n-1}$

$\frac{u_{n+1}}{u_n} = \frac{4^n}{4^{n-1}} = 4$, which is a constant, proving that this is a geometric sequence.

(b) (i) $r = 4$ (ii) $u_1 = S_1 = 1$

(c) $u_n = 4^{n-1}$

Example 1.2.6

A geometric sequence has first term $\frac{1}{2}$ and common ratio -2.

Find the value of n if:

(a) $u_n = -256$ (b) $S_n = -682.5$

Link to Logarithms SL 1.7

Solution

(a) $u_n = \frac{1}{2}(-2)^{n-1} = -256$

Solving (by using "table" or with logs and absolute values) gives $n = 10$

(b) $S_n = \frac{1}{2}\frac{\left(1-(-2)^n\right)}{3} = -682.5$

Solving (by using "table" or with logs and absolute values) gives $n = 12$

Financial applications of geometric progressions

When an amount is increased by r% then the original amount has been multiplied by the constant $1 + \frac{r}{100}$ and so use of geometric sequences will be helpful. This is the case with compound interest.

The formula for compound interest is

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

where FV is the future value, PV is the present value, n is the number of years, k is the number of compounding periods per year and r% is the nominal annual rate of interest.

Example 1.2.7

Ruth deposits £500 into a bank that gives her 4% compound interest at the end of each year.

(a) Calculate how much her investment will be worth at the end of 10 years, giving your answer to two decimal places.

(b) Find the number of complete years it will take for her money to have doubled.

Solution

(a) $500 \times \left(1 + \frac{4}{100}\right)^{10} = £740.12$ (2 dp)

(b) $500 \times \left(1 + \frac{4}{100}\right)^{n} = 1000 \Rightarrow 1.04^{n} = 2$

Solving using logs, “table” or financial package on the calculator gives $n = 18$ (£1012.91 is the value when $n = 18$)

Assessment tip

In financial questions, you will often have to give your final answer to two decimal places. Showing that you have done this after writing out the final answer is good practice. If the question does not specifically ask for two decimal places, then the normal ‘three significant figures’ rule would apply.

It is well worth learning how to use the “Finance” capabilities of your calculator.

There is sometimes confusion because the nth term formula for a geometric progression has a power of $n-1$, whereas the compound interest formula has a power of n. This is because after n years it will be the $(n+1)$th term, since the first term was the present value. After n years you will have had n interest payments.

Example 1.2.8

Martin deposits \$1000 into a bank that gives him a nominal annual compound interest rate of 10%. The interest is compounded monthly.

(a) Calculate the value of his savings after one year, giving your answer to two decimal places.

(b) State his actual compound interest rate.

(c) Find the number of complete months that it will take for his savings to have doubled.

Solution

(a) $1000 \times \left(1 + \frac{10}{100 \times 12}\right)^{12 \times 1} = \$1104.71$ (2 dp)

(b) $1000 \times \left(1 + \frac{r}{100}\right) = 1104.71 \Rightarrow 1 + \frac{r}{100} = 1.10471 \Rightarrow r = 10.471$

Actual interest rate is 10.5% (3 sf)

(c) $1000 \times \left(1 + \frac{10}{100 \times 12}\right)^{m} = 2000$

Solving, for example, using “table”, it will take 84 months (\$2007.92 is the value after 84 months)

Note

Be careful: When the interest is not compounded yearly, make sure you use the correct values of k and n in the formula.

If an amount is depreciating in value rather than appreciating in value, then r will be negative.

SAMPLE STUDENT ANSWER

Geoff has just bought a new car for \$15 000. It depreciates in value at the rate of 4% per year. At the same time, he also bought a vintage motorbike for \$8000 which appreciates in value at 3% per year. In this question give monetary answers to two decimal places.

(a) Find the value of his car after 5 years.

(b) Find the value of his motorbike after 5 years.

(c) Find the number of complete years it will take before his motorbike is worth at least as much as his car.

(a) $15000\ (0.96)^5 = 12230.59$

(b) $8000\ (1.03)^5 = 9274.19$

(c) $15000\ (0.96)^n = 8000\ (1.03)^n \Rightarrow 15\ (0.96)^n = 8\ (1.03)^n$

$\Rightarrow \log 15 + n \log 0.96 = \log 8 + n \log 1.03$

$\Rightarrow \log 15 - \log 8 = n\ (\log 1.03 - \log 0.96)$

$n = \dfrac{\log \frac{15}{8}}{\log \frac{1.03}{0.96}} = 8.93$

The answer above could have achieved 9/10 marks.

▲ Correct answers in parts (a) and (b). Confident, correct use of logs in part (c).

▼ In parts (a) and (b) it would have been better practice to have used the \$ sign and to have written (2 dp) after each answer. No marks were lost for this, but a mark was lost in part (c) because the question asked how many complete years for the bike to be worth at least as much as the car. So the answer should have been 9 years. This answer could have been obtained using "table" if a candidate did not wish to use logs.

(After 9 complete years, the motorbike was worth \$10 438, which is greater than \$10 388, the value of the car.)

Assessment tip

See Example 1.2.7 (b), Example 1.2.8 (c) and the end of the red box above, where the values put in brackets at the end (even though they are not asked for) show the method and that "table" had been used correctly.

Infinite geometric series

The infinite geometric series $\sum_{i=1}^{\infty} u_1 r^{i-1}$ (for $r \neq 0$) will converge provided that $|r| < 1$ and the sum to infinity is given by $S_\infty = \dfrac{u_1}{1-r}$

Example 1.2.9

Find the sum of the following infinite geometric series.

(a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

(b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

Solution

(a) $r = \frac{1}{2}$ and $\frac{1}{2} < 1$ so $S_\infty = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

(b) $r = -\frac{1}{3}$ and $\frac{1}{3} < 1$ so $S_\infty = \dfrac{1}{1+\frac{1}{3}} = \dfrac{3}{4}$

Example 1.2.10

Find an expression for the sum of the following infinite geometric series

$1 - x + x^2 - x^3 + x^4 - \dots$

and state the condition for this series to converge to this sum.

Solution

$r = -x \Rightarrow S_\infty = \dfrac{1}{1+x}$

This will converge if $|x| < 1$

Example 1.2.11

Using a method involving an infinite geometric series, show that the repeating decimal $X = 0.232\,323\,23\dots$ can be written as a fraction.

Solution

$X = \dfrac{23}{100} + \dfrac{23}{10\,000} + \dfrac{23}{10\,00\,000} + \dots$

This is an infinite geometric series with $u_1 = \dfrac{23}{100}$, $r = \dfrac{1}{100}$

$\dfrac{1}{100} < 1$, so $S_\infty = \dfrac{\frac{23}{100}}{1 - \frac{1}{100}} = \dfrac{23}{99}$, which is a fraction.

SAMPLE STUDENT ANSWER

An infinite geometric series has a non-zero first term.

(a) Find the value of the common ratio if the sum to infinity is three times the first term.

(b) Find the value of the common ratio if the sum to infinity is $\frac{4}{5}$ times the first term.

(c) State if the sum to infinity can be $\frac{1}{3}$ times the first term.

(a) $\dfrac{u_1}{1-r} = 3u_1 \Rightarrow 1 - r = \dfrac{1}{3} \Rightarrow r = \dfrac{2}{3}$

(b) $\dfrac{u_1}{1-r} = \dfrac{4}{5}u_1 \Rightarrow 1 - r = \dfrac{5}{4} \Rightarrow r = \dfrac{-1}{4}$

(c) $\dfrac{u_1}{1-r} = \dfrac{1}{3}u_1 \Rightarrow 1 - r = 3 \Rightarrow r = -2$

The answer above could have achieved 5/6 marks.

▲ Correct method and answers in parts (a) and (b). Correct method in part (c).

▼ In part (c) the student forgot about the condition for convergence. The student should have gone on to say: But $|-2| > 1$ so the series would not converge in this case and hence the answer to the question is "No".

The student did not pick up on the difference used in the wording in part (c), which was designed to be helpful.

1.3 EXPONENTIALS AND LOGARITHMS

You must know:

- ✔ the laws of exponents
- ✔ the basic definition of a logarithm
- ✔ the laws of logarithms.

You should be able to:

- ✔ work with exponents
- ✔ work with logarithms
- ✔ change the base of a logarithm
- ✔ solve equations using logarithms.

A logarithm is a power, so it is useful to recall the rules of exponents.

Rules of exponents

Let $a \in \mathbb{R}, a > 0, x, y \in \mathbb{R}, p, q \in \mathbb{Z}^+$

$a^x \times a^y = a^{x+y}$ $\quad \frac{a^x}{a^y} = a^{x-y}$ $\quad (a^x)^y = a^{xy}$ $\quad a^0 = 1$ $\quad \frac{1}{a^x} = a^{-x}$

$\sqrt[p]{a} = a^{\frac{1}{p}}$ $\quad a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

These rules are not given in the formula book, so need to be learned.

Example 1.3.1

For $a \in \mathbb{R}, a > 0$, simplify as far as possible the expression $\frac{\sqrt{a} \times a^{\frac{3}{2}}}{(\sqrt[3]{a})^6}$

Solution

$$\frac{\sqrt{a} \times a^{\frac{3}{2}}}{(\sqrt[3]{a})^6} = \frac{a^{\frac{1}{2}+\frac{3}{2}}}{a^2} = a^0 = 1$$

Basic definition of a logarithm

If $a, b \in \mathbb{R}, a > 0, b > 0, a \neq 1$, then if $a^x = b$ we say that $x = \log_a b$, where a is called the base of the logarithm.

Note

If you find logarithms difficult, keep going back to this basic definition: a logarithm is a power.

Example 1.3.2

Find the value of the following logarithms:

(a) $\log_2 8$ (b) $\log_3 81$ (c) $\log_4 \frac{1}{16}$

(d) $\log_2 \frac{1}{\sqrt{2}}$ (e) $\log_9 3$ (f) $\log_{10} 0.001$

Solution

(a) $8 = 2^3 \Rightarrow \log_2 8 = 3$ (b) $81 = 3^4 \Rightarrow \log_3 81 = 4$

(c) $\frac{1}{16} = 4^{-2} \Rightarrow \log_4 \frac{1}{16} = -2$ (d) $\frac{1}{\sqrt{2}} = 2^{-\frac{1}{2}} \Rightarrow \log_2 \frac{1}{\sqrt{2}} = -\frac{1}{2}$

(e) $3 = 9^{\frac{1}{2}} \Rightarrow \log_9 3 = \frac{1}{2}$ (f) $0.001 = 10^{-3} \Rightarrow \log_{10} 0.001 = -3$

Note

The base of a logarithm must be a positive real number not equal to 1.
You cannot take the logarithm of a negative number.
However, the logarithm of a number can be negative.

Note

Logarithms to base 10 can be found using a calculator and are denoted by log.
Logarithms to base e can be found using a calculator and are denoted by ln.

As logarithms are powers, the rules of exponents can be transformed into rules about logarithms.

Rules of logarithms

$$\log_a xy = \log_a x + \log_a y, \quad \log_a \frac{x}{y} = \log_a x - \log_a y, \quad \log_a x^m = m\log_a x$$

Example 1.3.3

Simplify the following expressions as far as possible.

(a) $\log_6 2 + \log_6 3$ (b) $\log_5 100 - \log_5 4$

(c) $2\log_{12} 3 + 4\log_{12} 2$ (d) $\log\frac{1}{5} + \log\frac{1}{20}$

Solution

(a) $\log_6 2 + \log_6 3 = \log_6 6 = 1$

(b) $\log_5 100 - \log_5 4 = \log_5 25 = 2$

(c) $2\log_{12} 3 + 4\log_{12} 2 = \log_{12} 9 + \log_{12} 16 = \log_{12} 144 = 2$

(d) $\log\frac{1}{5} + \log\frac{1}{20} = \log\frac{1}{100} = -2$

SAMPLE STUDENT ANSWER

Let $\log 2 = p$ and $\log 7 = q$.
Find an expression, in terms of p and q, for:

(a) $\log 14$ (b) $\log 8$ (c) $\log 49$

(d) $\log\frac{1}{28}$ (e) $\log 5$

(a) $\log 14 = \log 2 + \log 7 = p + q$

(b) $\log 8 = \log 2^3 = 3\log 2 = 3p$

(c) $\log 49 = \log 7^2 = 2\log 7 = 2q$

(d) $\log\frac{1}{28} = \log\frac{1}{2^2} \times \frac{1}{7} = -2\log 2 - \log 7 = -2p - q$

(e) $\log 5 = \log(7-2) = \log 7 - \log 2 = q - p$

▲ Good use of the rules of logs to obtain the correct answers to all parts except part (e).

▼ The student could not see how to build up 5 multiplicatively and so invented a rule that is not true. The answer of $q - p = \log\frac{7}{2}$ and so must be incorrect. As the last part of the question it is likely to be the hardest and relies on the fact that it is easy to know the logs of powers of 10. The correct solution to part (e) is $\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - p$.

The answer above could have achieved 8/10 marks.

The logarithm rule $\log_a x^m = m \log_a x$ is very useful to apply when we have an unknown that is an exponent. By taking logs of both sides and employing this rule, the exponent becomes a multiplicative constant. You would normally take logs to base 10 or base e of both sides, as they are given on the calculator. However, most graphical calculators can work with any base.

Example 1.3.4

Solve $1.04^x = 2$ for $x \in \mathbb{R}$

Solution

$1.04^x = 2 \Rightarrow \log 1.04^x = \log 2 \Rightarrow x \log 1.04 = \log 2$

$$\Rightarrow x = \frac{\log 2}{\log 1.04} = 17.7 \text{ (3 sf)}$$

Link to Example 1.2.7 part (b)

Example 1.3.5

Solve the inequality $0.9^x < 0.01$ for $x \in \mathbb{R}$

Solution

Taking logs of both sides: $\log 0.9^x < -2 \Rightarrow x \log 0.9 < -2 \Rightarrow x > \dfrac{-2}{\log 0.9}$

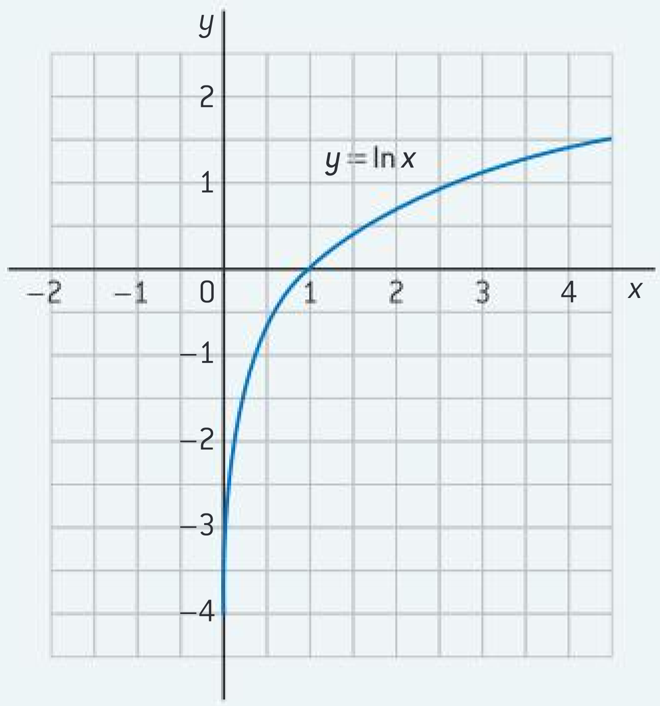
Hence, $x > 43.7$ (3 sf)

Assessment tip

It is worth knowing what the graph of $y = \log x$ looks like.

In Example 1.3.5, the fact that $y = \log x$ is an increasing function was used initially.

The graph also confirms that if $0 < x < 1$ then $\log x$ is negative. Hence $\log 0.9$ is negative which explains why the inequality "turns around" when we divide by $\log 0.9$. At $x = 0$ the graph has a vertical asymptote and there is an intercept on the x-axis at (1, 0).



Link to Logarithmic functions SL 2.9

Graphical calculators have the ability to find logs to bases other than 10 or e, but this can also be achieved using the change of base formula.

Log change of base formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We often take the new base b as 10 or e. However, this does not always have to be the case as it is a good idea to have all the logs in an equation in the same base, as shown in the sample answer below.

SAMPLE STUDENT ANSWER

Solve the equation $\log_3 x - 6\log_x 3 + 1 = 0$ for $x \in \mathbb{R}$.

Using the change of base rule $\log_x 3 = \frac{\log_3 3}{\log_3 x}$. Therefore, the equation becomes $\log_3 x - \frac{6}{\log_3 x} + 1 = 0$

Letting $\log_3 x = y$ $\quad y^2 + y - 6 = 0$

$(y-2)(y+3) = 0 \Rightarrow y = 2 \text{ or } -3$

$y = -3$ is rejected so $\log_3 x = 2 \Rightarrow x = 3^2 = 9$

The answer above could have achieved 6/8 marks.

▲ Good insight shown to change the logs to base 3 and to then recognize a disguised quadratic leading to one of the correct answers. Method was also well explained with words.

▼ The rejection of $\log_3 x = -3$ was wrong. The log can be negative, it is the x that cannot be negative. So the solution $\log_3 x = -3 \Rightarrow x = 3^{-3} = \frac{1}{27}$ was missed.

Assessment tip

If you use a variable that was not part of the original question, then define what it stands for. This was done well in the above sample answer by saying "Letting $\log_3 x = y$".

Also stick with the same variables and notation that are used in the question.

1.4 ALGEBRA (HL)

You must know:

- ✔ the difference between combinations and permutations
- ✔ the formula for the extended binomial theorem
- ✔ what a counterexample is.

You should be able to:

- ✔ count the number of ways of arranging objects
- ✔ apply the extended binomial theorem
- ✔ decompose a rational function into partial fractions
- ✔ construct a formal proof by induction
- ✔ construct a proof by contradiction
- ✔ solve systems of linear equations.

Counting principles

The combination number ${}^nC_r$ represents the number of ways of choosing r objects from n objects. The permutation number ${}^nP_r$ represents the number of different ways of arranging r objects from n objects in order. These can both be found on a calculator.

Formulae for combinations and permutations

$${}^nC_r = \frac{n!}{r!(n-r)!} \qquad {}^nP_r = \frac{n!}{(n-r)!}$$

If one operation can be done in n ways and a second operation can be done in m ways then the total number of ways of doing both operations is $n \times m$. Think of walking down a road that divides into n paths and then each of these paths divides into m paths. There would be $n \times m$ different routes that you could take.

Example 1.4.1

Find the number of ways of selecting a committee of three females and two males from a set of seven females and six males.	Solution We need to choose 3 females from 7 and 2 males from 6. ${}^7C_3 \times {}^6C_2 = 525$

Example 1.4.2

Find the number of ways of selecting three people to the positions of: chair-person, treasurer and deputy chair-person, from a committee of five people.	**Solution** ${}^5P_3 = 60$ The positions the persons would hold matters, so this is a permutation not a combination.

Assessment tip

Think of the C numbers as "choose" or "select" numbers, where the order does not matter.

Think of the P numbers as "put" or "arrange" numbers, where the order does matter.

Link to Probability SL 4.5

The difference between combinations and permutations can be illustrated as follows. If a team of four relay race runners is to be chosen from a group of six athletes, then this is a combination and there would be ${}^6C_4 = 15$ ways of selecting them.

If it is to be decided who carries the baton 1st, 2nd, 3rd and 4th then this is a permutation and the number of different ways in which this could be done would be ${}^6P_4 = 360$. The ${}^nP_r$ numbers are always bigger than the ${}^nC_r$ numbers by a factor of r!

▲ Good counting method shown for dealing with "at least one", looking at all possibilities and then taking away those that cannot happen.

SAMPLE STUDENT ANSWER

A team of four children is to be chosen from six girls and five boys. Find the number of ways in which this can be done if the team must contain at least one boy and at least one girl.

${}^{11}C_4 = 330$

But we cannot have all boys or all girls so

$330 - {}^5C_4 - {}^6C_4 = 330 - 5 - 15 = 310$

This answer could have achieved 5/5 marks.

Extension of the binomial theorem

The binomial theorem gives a finite expansion for $(a+b)^n$, where $n \in \mathbb{Z}^+$. This can be extended to negative integers and fractions and an infinite expansion is obtained. We utilize $(a+b)^p = a^p\left(1+\frac{a}{b}\right)^p$, where $p \in \mathbb{Q}$ and the following expansion.

Extended binomial theorem

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \ldots$$

where $p \in \mathbb{Q}$, provided that the RHS converges, so we need $|x| < 1$.

This expansion is not given in the formula book and so should be remembered.

Link to Binomial Theorem SL 1.9 and Maclaurin Expansions AHL 5.19

Example 1.4.3

(a) Find the first three terms in the extended binomial expansion for $(1+x)^{\frac{1}{2}}$, in ascending powers of x.

(b) Hence find a rational approximation for $\frac{\sqrt{3}}{2}$

Solution

(a) $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\times-\frac{1}{2}}{2}x^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{6}x^3 + \ldots = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$

(b) Taking $x = -\frac{1}{4}$ $\quad \left(\frac{3}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2} \simeq 1 - \frac{1}{8} - \frac{1}{128} - \frac{1}{1024} = \frac{887}{1024}$

Example 1.4.4

Show that $(2+3x)^{-1} = \frac{1}{2} - \frac{3x}{4} + \frac{9x^3}{8} - \frac{27x^3}{16} + \ldots$

Solution

$$(2+3x)^{-1} = 2^{-1}\left(1+\frac{3x}{2}\right)^{-1} = \frac{1}{2}\left(1 + -1\times\frac{3x}{2} + \frac{-1\times-2}{2!}\left(\frac{3x}{2}\right)^2 + \frac{-1\times-2\times-3}{3!}\left(\frac{3x}{2}\right)^3 + \ldots\right)$$

$$= \frac{1}{2} - \frac{3x}{4} + \frac{9x^2}{8} - \frac{27x^3}{16} + \ldots$$

Partial fractions

Just as fractions can be decomposed into simpler fractions, e.g., $\frac{7}{12} = \frac{1}{3} + \frac{1}{4}$, rational functions can be decomposed into simpler expressions.
A simple deductive proof could be used to show that, for instance:

$$\frac{3x+4}{(x+1)(x+2)} \equiv \frac{1}{x+1} + \frac{2}{x+2}$$

Partial fractions is a method that allows a rational function to be decomposed into simpler expressions. This is often done to assist in some other process, e.g., differentiation or integration.

Link to Simple Deductive Proof SL 1.6

Link to Partial Fractions to rearrange the integrand AHL 5.15

Example 1.4.5

Express $\frac{5x+16}{x^2+7x+10}$ in partial fractions.

Solution

$x^2+7x+10 = (x+2)(x+5)$

Setting $\frac{5x+16}{(x+2)(x+5)} \equiv \frac{A}{x+2} + \frac{B}{x+5}$ for constants A and B gives

$5x+16 \equiv A(x+5) + B(x+2)$

Method 1: Equating the coefficients of the linear functions on both sides gives $5 = A + B$, $16 = 5A + 2B \Rightarrow A = 2$, $B = 3$

Method 2: Since the identity above is true for all values of x it will be true for any particular values of x.

Putting $x = -5 \Rightarrow B = 3$

Putting $x = -2 \Rightarrow A = 2$

Concluding $\frac{5x+16}{x^2+7x+10} \equiv \frac{2}{x+2} + \frac{3}{x+5}$

Note

Method 1 is more rigorous and the equating of coefficients would show if the proposed representation were impossible. Method 2 is often quicker to use if you are sure that the rational function can be split in this way.

Assessment tip

The syllabus states that in IB examples there will be a maximum of two linear terms in the denominator and the degree of the numerator will be less that the degree of the denominator.

Since in general you need as many constants as the degree of the denominator, in these examples you will need an A and a B.

Example 1.4.6

(a) Express $\frac{1}{r^2+r}$ in partial fractions.

(b) Hence find an expression, in terms of n, for the sum $\sum_{r=1}^{n} \frac{1}{r^2+r}$

Solution

(a) $r^2 + r = r(r+1)$ $\quad \frac{1}{r(r+1)} \equiv \frac{A}{r} + \frac{B}{r+1} \Rightarrow 1 \equiv A(r+1) + Br$

So $A = 1, B = -1$ $\quad \frac{1}{r^2+r} \equiv \frac{1}{r} - \frac{1}{r+1}$

(b) $\sum_{r=1}^{n} \frac{1}{r^2+r} = \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Note

Example 1.4.6 illustrates how partial fractions could be used to create a "telescoping" series.

Proof by induction

Proof by induction is the most formal proof that you will be expected to do. There is a set way of laying out the proof to comply with the rigorous logic of the proof. Proof by induction is used to show that a statement $P(n)$ is true for all $n \in \mathbb{Z}^+$. Essentially, we have an infinite number of statements that we have to prove.

This is how you lay out an induction proof.

Identify the statement $P(n)$ that you intend to prove, for all $n \in \mathbb{Z}^+$.

Prove that $P(1)$ is true.

Assume $P(k)$ is true and show that this implies that $P(k+1)$ is true. This is called the induction step.

Conclude with the following comment "$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$"

This standard final comment needs to be learned by heart.

Assessment tip

You must first show that $P(1)$ is true no matter how trivial it appears to be.

The algebra of the induction step working out well should give you confidence that you are doing the process correctly. Remember that you cannot use the statement that you are trying to prove. However, you can keep an eye on it and work towards it. Do not worry if expressions become longer before eventually simplifying.

There will be a reasoning mark for the concluding comment (provided that enough marks have been gained elsewhere) so remember it and give it.

Example 1.4.7

Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^2+r} = \frac{n}{n+1}$, for all $n \in \mathbb{Z}^+$

Solution

Let $P(n)$ be the statement $\sum_{r=1}^{n} \frac{1}{r^2+r} = \frac{n}{n+1}$

LHS of $P(1)$ is $\frac{1}{1^2+1} = \frac{1}{2}$ RHS of $P(1)$ is $\frac{1}{1+1} = \frac{1}{2}$

So $P(1)$ is true.

Assume $P(k)$ is true so $\sum_{r=1}^{k} \frac{1}{r^2+r} = \frac{k}{k+1}$

LHS of $P(k+1)$ is $\sum_{r=1}^{k+1} \frac{1}{r^2+r} = \sum_{r=1}^{k} \frac{1}{r^2+r} + \frac{1}{(k+1)^2+(k+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+1)+1}$$

which is the RHS of $P(k+1)$, as required.

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$

Assessment tip

If the question states "prove by induction" then you must do it this way, even if there are other methods of proof. If it just says "prove" and you decide to use induction, state that you are doing so.

For induction to be used, variable n must be a natural number and the statement that you wish to prove must be fully known. You might be asked to do an investigation and then generalize your results into a conjecture which you are then asked to prove by induction.

Example 1.4.8

Prove by induction that $17^n - 1$ is always exactly divisible by 8 for all $n \in \mathbb{Z}^+$

Solution

Let $P(n)$ be the statement "8 exactly divides $17^n - 1$"

$17^1 - 1 = 16 = 8 \times 2$ so $P(1)$ is true.

Assume $P(k)$ is true, so $17^k - 1 = 8s$ for $s \in \mathbb{Z}^+$

$17^{k+1} - 1 = 17 \times 17^k - 1 = 17(8s+1) - 1 = 8 \times 17s + 16 = 8(17s+2)$

$17s + 2 \in \mathbb{Z}^+$ so 8 exactly divides $17^{k+1} - 1$, showing that $P(k+1)$ is true.

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

SAMPLE STUDENT ANSWER

Prove by induction that $\sum_{i=1}^{n} i^2 + i = \frac{1}{3}n(n+1)(n+2)$, for all $n \in \mathbb{Z}^+$

For $n = 1$ $\quad 1^2 + 1 = \frac{1}{3} \times 1 \times 2 \times 3 \Rightarrow 2 = 2$ ✓

Assume the result for k $\quad \sum_{i=1}^{k} i^2 + i = \frac{1}{3}k(k+1)(k+2)$

So $\sum_{i=1}^{k+1} i^2 + i = \frac{1}{3}(k+1)(k+1+1)(k+1+2)$

$= \frac{1}{3}(k+1)(k+2)(k+3)$

So it is true.

▲ The student knew they should first look at the case $n = 1$ and then assume the result for $n = k$ and attempt to prove it for $n = k + 1$.

▼ The logic of the case $n = 1$ is poor, starting with what is to be proved and putting "tick". The induction step has not been done. You certainly cannot just substitute $k + 1$ for k, they are consecutive integers. The standard concluding sentence has not been given, and even if it had been, there were not enough marks elsewhere for it to have gained the reasoning mark.

The answer above could have achieved 2/8 marks.

The proof should have been as follows:

Let $P(n)$ be the statement $\sum_{i=1}^{n} i^2 + i = \frac{1}{3}n(n+1)(n+2)$

LHS of $P(1)$ is $1^2 + 1 = 2$. RHS of $P(1)$ is $\frac{1}{3} \times 1 \times 2 \times 3 = 2$

So $P(1)$ is true.

Assume $P(k)$ is true, so $\sum_{i=1}^{k} i^2 + i = \frac{1}{3}k(k+1)(k+2)$

LHS of $P(k + 1)$ is

$$\sum_{i=1}^{k+1} i^2 + i = \sum_{i=1}^{k}(i^2 + i) + (k+1)^2 + (k+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= \frac{1}{3}(k+1)(k+2)(k+3) = \frac{1}{3}(k+1)(k+1+1)(k+1+2) = \text{RHS of } P(k+1)$$

$P(1)$ is true and $P(k)$ true implies $P(k + 1)$ is true, hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Assessment tip

When dealing with induction on series (like the question here), in the induction step always look to see if the expression contains common factors (especially if you can see that they should be in the expression that you are working towards), rather than multiplying everything out.

Note

It is possible to have a variation to the standard induction proof where the first step is prove the result for $d \in \mathbb{Z}$ rather than for $n = 1$. The induction step would be the same. This would prove the result for all $n \in \mathbb{Z}^+, n \geq d$.

Link to the Binomial Theorem, as proof by induction can be used to prove this theorem utilising ${}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r$

Counterexamples

To prove that a statement is not always true, it is sufficient to give just one example when the statement is not true. Such an example is called a counterexample.

Example 1.4.9

Show that the statement "11 exactly divides $n^{10} - 1$, for all $n \in \mathbb{Z}^+$" is false.

Solution

When $n = 11$, $\frac{11^{10} - 1}{11} = 11^9 - \frac{1}{11}$, which is not an integer.

So $n = 11$ is a counterexample.

Note

In fact, in Example 1.4.9, $n = 11$ is the smallest counterexample that could be found.

Note

In fact, $n = 4$ is the only possible counterexample in Example 1.4.10.

Example 1.4.10

Find a counterexample to the statement "n exactly divides $(n-1)! + 1$ or n exactly divides $(n-1)!$, for all $n \in \mathbb{Z}^+$."

Solution

For $n = 4$, $3! + 1 = 7$ and $3! = 6$, neither of which is exactly divisible by 4. So $n = 4$ is a counterexample.

Proof by contradiction

The layout of a proof by contradiction is as follows. You are given a statement to prove. You assume it to be false and proceed to make logical deductions based on that assumption. If you then obtain a result that you know to be impossible, you can conclude that the original statement must be true.

This is because the original statement must be either true or false. If assuming it to be false leads to a contradiction, you can conclude that it must be true.

Assessment tip

If the question asks for "Proof by contradiction", then you must do it that way. If a question just asks for a proof and you decide to use proof by contradiction, then state that this is the method that you are using.

Example 1.4.11

Prove by contradiction that $\sqrt{3} \notin \mathbb{Q}$

Solution

Suppose that $\sqrt{3} \in \mathbb{Q}$. Then $\sqrt{3}$ can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$. Furthermore, it can be assumed that a and b are coprime; there is not a prime number that exactly divides both a and b (as they could have been cancelled out). Then $\sqrt{3} = \frac{a}{b} \Rightarrow 3b^2 = a^2$, which implies that 3 exactly divides a^2, which also implies that 3 exactly divides a (since 3 is a prime). Let $a = 3A$, where $A \in \mathbb{Z}$, then $3b^2 = (3A)^2 \Rightarrow b^2 = 3A^2$. This implies that 3 exactly divides b^2, which also implies that 3 exactly divides b (since 3 is a prime). This gives the required contradiction as we have shown that 3 exactly divides both a and b, but we assumed that a and b were coprime. Therefore, we conclude that $\sqrt{3} \notin \mathbb{Q}$.

Example 1.4.12

Let $a + b\sqrt{3} = c + d\sqrt{3}$, where $a, b, c, d \in \mathbb{Q}$
Prove that $b = d$ and hence that $a = c$

Solution

Using proof by contradiction, assume that $b \neq d$.
Then $a - c = (d - b)\sqrt{3} \Rightarrow \sqrt{3} = \frac{a-c}{d-b}$

But $\frac{a-c}{d-b} \in \mathbb{Q}$. This is a contradiction. Therefore, we can conclude that $b = d$. Which also gives $a + b\sqrt{3} = c + b\sqrt{3}$ and hence $a = c$.

Solution of systems of linear equations

Link to AHL 3.18

The equation $ax + by = c$ represents a straight line in 2 dimensions and the equation $ax + by + cz = d$ represents a plane in 3 dimensions. Solving a system of linear equations of this form simultaneously can be thought of geometrically as finding the points where these objects intersect. There could be no solutions, one unique solution or an infinite number of solutions.

Assessment tip

The method of solution will depend on whether it is a paper that allows the use of technology.

With a paper that allows the use of technology, the easiest method is to use the simultaneous equation solver but you could find the intersection of lines by drawing a graph if you were only working in two dimensions.

With a paper where technology is not allowed you would reduce the number of equations by eliminating a variable and then repeat this process.

Example 1.4.13

Solve these simultaneous equations:

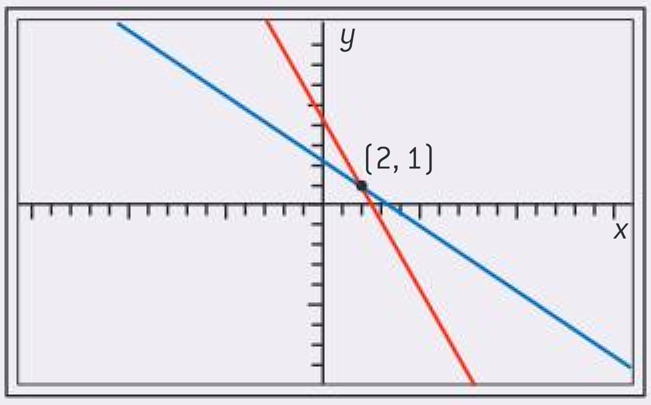
$$\begin{cases} 3x+4y=10 \\ 2x+y=5 \end{cases}$$

Solution

Technology allowed:

Using the simultaneous equations solver: $x=2$, $y=1$

Or, rearranging: $y=\frac{-3}{4}x+\frac{10}{4}$, $y=-2x+5$ and graphing to find the intersection.



No technology allowed:

$(2 \times \text{first equation}) - (3 \times \text{second equation}) \Rightarrow 5y=5 \Rightarrow y=1 \Rightarrow x=2$

The calculator will do the (reduced) row echelon form method for you, but it can also be done by hand on a paper where technology is not allowed. This is the same as solving the equations simultaneously, and it is useful to learn. A shorthand notation is used with the x, y, z, not being written and the equals signs being represented by a vertical line. A row (representing an equation) can be multiplied by a non-zero constant and multiples of one row can be subtracted or added to another row.

Example 1.4.14

Solve these simultaneous equations:

$$\begin{cases} x+3y+z=9 \\ 2x+y+4z=21 \\ x+5y-3z=-5 \end{cases}$$

Assessment tip

In IB questions you will have a maximum of three equations in three unknowns.

Assessment tip

Putting the description of what has been done with the rows as shown in Example 1.4.14 is good practice and makes it easier to follow your method.

To obtain row echelon form you are attempting to make the start of your notation look as much like

$$\begin{array}{ccc|}1&&\\0&1&\\0&0&1\end{array}$$

as possible.

To obtain reduced row echelon form you are attempting to make the start of your notation look as much like

$$\begin{array}{ccc|}1&0&0\\0&1&0\\0&0&1\end{array}$$

as possible.

Solution (no technology allowed)

Representing the equations by $\begin{array}{ccc|c}1&3&1&9\\2&1&4&21\\1&5&-3&-5\end{array}$

(Calculations go across the page, then down)

$$\begin{array}{l ccc|c} & 1&3&1&9\\ row\,2-2\,row\,1 & 0&-5&2&3\\ row\,3-row\,1 & 0&2&-4&-14\end{array} \qquad \begin{array}{l ccc|c} & 1&3&1&9\\ -\frac{1}{5}row\,2 & 0&1&-\frac{2}{5}&-\frac{13}{5}\\ & 0&2&-4&-14\end{array}$$

$$\begin{array}{l ccc|c} & 1&3&1&9\\ & 0&1&-\frac{2}{5}&-\frac{3}{5}\\ row\,3-2\,row\,2 & 0&0&-\frac{16}{5}&-\frac{64}{5}\end{array} \qquad \begin{array}{l ccc|c} & 1&3&1&9\\ & 0&1&-\frac{2}{5}&-\frac{3}{5}\\ -\frac{5}{16}row\,3 & 0&0&1&4\end{array}$$

Either: converting back into equations $z=4$

$y-\frac{2}{5}z=-\frac{3}{15}\Rightarrow y=-\frac{3}{5}+\frac{8}{5}=1 \quad x+3y+z=9\Rightarrow x=9-3-4=2$

Or: continue reducing rows in matrix form

$$\begin{array}{l ccc|c} row\,1-3\,row\,2 & 1&0&\frac{11}{5}&\frac{54}{5}\\ & 0&1&-\frac{2}{5}&-\frac{3}{5}\\ & 0&0&1&4\end{array}$$

$$\begin{array}{l ccc|c} row\ 1-\frac{11}{5}row\,3 & 1&0&0&2\\ row\ 2+\frac{2}{5}row\,3 & 0&1&0&1\\ & 0&0&1&4\end{array}$$

Reading off $x=2,\ y=1,\ z=4$

Note

A way to ensure that a calculator cannot be used is to include a parameter as demonstrated in Example 1.4.15

Example 1.4.15

(a) Find the value of λ for which the following system of equations is consistent (this means that there is at least one solution).

$$\begin{cases} x+3y+z=9\\ 2x+7y+4z=21\\ 4x+13y+6z=\lambda\end{cases}$$

(b) For the value of λ found in part (a), find the solutions to this system of equations.

Solution

(a) $\begin{array}{ccc|c}1&3&1&9\\2&7&4&21\\4&13&5&\lambda\end{array} \qquad \begin{array}{l ccc|c} & 1&3&1&9\\ row\,2-2\,row\,1 & 0&1&2&3\\ row\,3-4\,row\,1 & 0&1&2&\lambda-36\end{array}$

$$\begin{array}{l ccc|c} & 1&3&1&9\\ & 0&1&2&3\\ row\ 3-row\ 2 & 0&0&0&\lambda-39\end{array}$$

Last line is $0x+0y+0z=\lambda-39$

So to be consistent $\lambda=39$

(b) So $z=z,\ y+2z=3\Rightarrow y=3-2z,$

$x+3y+z=9\Rightarrow x=9-3\,(3-2z)-z=5z$

Solutions are of the form $\frac{x}{5}=\frac{y-3}{-2}=z$

Link to AHL 3.14 and AHL 3.15, the solution given in Example 1.4.15 (b) is a straight line through the point $(0, 3, 0)$ and parallel to the vector $\begin{pmatrix}5\\-2\\1\end{pmatrix}$

1.5 COMPLEX NUMBERS

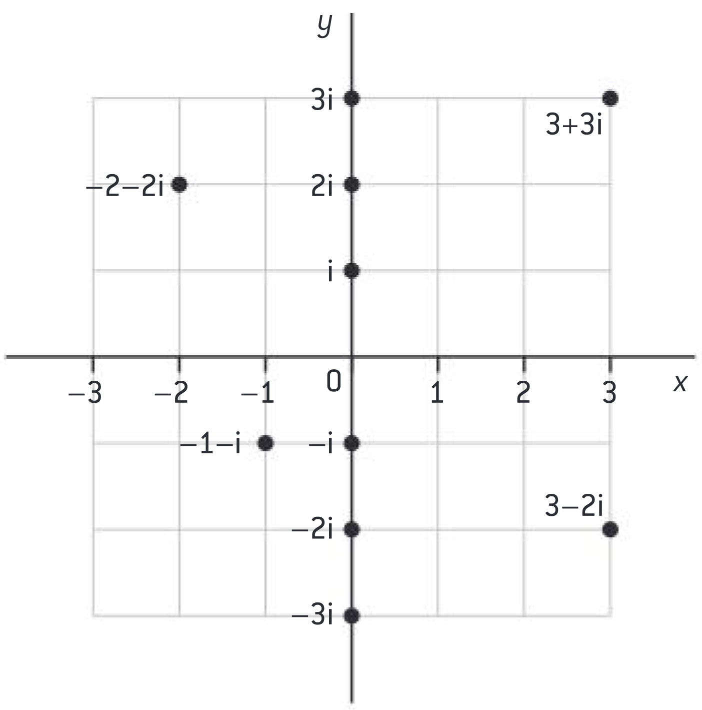
You must know:

- ✔ that $i^2 = -1$
- ✔ the terms 'real part', 'imaginary part', 'conjugate', 'modulus', and 'argument'
- ✔ De Moivre's theorem and its extension.

You should be able to:

- ✔ add, subtract, multiply and divide complex numbers in Cartesian form, that is $z = a + bi$
- ✔ multiply, divide and take powers of complex numbers in modulus-argument form, $z = r \text{ cis } \theta$ form and $z = re^{i\theta}$ form
- ✔ represent complex numbers in an Argand diagram
- ✔ solve polynomial equations with real coefficients
- ✔ find powers and roots of complex numbers.

The square root of negative one is not a real number. We will denote $\sqrt{-1}$ by i. This gives rise to a new set of numbers called complex numbers and they are denoted by $\mathbb{C}$, where $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. The real numbers can be represented geometrically by an infinite straight line but the complex numbers require a 2-dimensional plane to represent them. This is called an Argand diagram and is illustrated below.



Definitions and formulae for complex numbers

Let $z = a + bi$ then:

The real part of z, denoted by Re(z), is a.

The imaginary part of z, denoted by Im(z), is b.

The conjugate of z, denoted by z^*, is $a - bi$.

The modulus of z, denoted by $|z|$, is the distance from z to the origin and $|z| = \sqrt{a^2 + b^2}$ (using Pythagoras' theorem).

The argument of z, denoted by arg z, is the angle between the positive real axis and the line segment OZ, where Z represents the complex number z and O is the origin. It is conventional to give arg z with $-\pi < \arg z \leq \pi$. We have $\tan(\arg z) = \frac{b}{a}$.

Note

The IB has the convention that the $\sqrt{\ }$ sign always means the positive square root when we are dealing with real numbers. This is to ensure that it is a function.

Note

When finding the argument of a complex number z, first consider which quadrant z lies in. This is needed because the period of tangent is π. For example, if $z_1 = 1 + i$ and $z_2 = -1 - i$ then they both have tan (arg z) $= 1$ but z_1 is in the first quadrant, whereas z_2 is in the third quadrant. So $\arg z_1 = \frac{\pi}{4}$ and $\arg z_2 = -\frac{3\pi}{4}$.

Assessment tip

These should be learned as they are not in the formula book.

Link to Solving trig equations SL 3.8

Assessment tip

Example 1.5.1 shows that if you have an equation involving complex numbers then you can equate both the real parts and the imaginary parts. If you are ever stuck when working with complex numbers this might be a good thing to do, as you obtain two equations from one equation.

Example 1.5.1

Let $a + bi = c + di$, where $a, b, c, d \in \mathbb{R}$.
Prove that $b = d$ and hence that $a = c$.

Solution

Using proof by contradiction, assume that $b \neq d$.
Then $a - c = (d - b)\mathrm{i} \Rightarrow \mathrm{i} = \frac{a-c}{d-b}$

But $\frac{a-c}{d-b} \in \mathbb{R}$, giving a contradiction. So, we can conclude that $b = d$, which also gives $a + bi = c + bi$ and hence $a = c$.

For addition and subtraction of complex numbers you just use ordinary algebra, treating i as if it was a variable. The same is true for multiplication, but to tidy up you remember that $\mathrm{i}^2 = -1$. For division, you multiply the fraction, the numerator and the denominator by the conjugate of the denominator.

Assessment tip

A calculator can be used (when permitted) to perform operations like those shown in Example 1.5.2. Sometimes, even though you could use a calculator, it might be quicker to do the operation by hand.

Example 1.5.2

Let $z = 3 + 6\mathrm{i}$ and $w = 5 + 12\mathrm{i}$. Find the following, giving your answers in the form $a + bi$ where $a, b \in \mathbb{R}$.

(a) $z + w$ (b) $z - w$ (c) $3z + 2w$ (d) zw (e) $\frac{z}{w}$ (f) $|z|$ (g) z^*

Solution

(a) $8 + 18\mathrm{i}$ (b) $-2 - 6\mathrm{i}$ (c) $(9 + 18\mathrm{i}) + (10 + 24\mathrm{i}) = 19 + 42\mathrm{i}$

(d) $15 + 36\mathrm{i} + 30\mathrm{i} + 72\mathrm{i}^2 = -57 + 66\mathrm{i}$

(e) $\frac{3+6\mathrm{i}}{5+12\mathrm{i}} \times \frac{5-12\mathrm{i}}{5-12\mathrm{i}} = \frac{15-36\mathrm{i}+30\mathrm{i}-72\mathrm{i}^2}{25-144\mathrm{i}^2} = \frac{87-6\mathrm{i}}{169} = \frac{87}{169} - \frac{6\mathrm{i}}{169}$

(f) $\sqrt{9+36} = \sqrt{45}$ (g) $3 - 6\mathrm{i}$

Assessment tip

In complex number questions the symbol z will often be used instead of x and the question will make it clear that it is working in the set of complex numbers, $\mathbb{C}$.

Just because you have now met them, do not think that you now have to bring complex numbers into ordinary questions about real numbers.

Note

The method employed to divide complex numbers (multiplying both the numerator and the denominator by the conjugate of the denominator) will always work. This is because if $z = a + bi$ then:
$zz^* = (a + bi)(a - bi) = a^2 - b^2\mathrm{i}^2 = a^2 + b^2$ which is a positive real number, and we know how to divide by a real number. In fact, $zz^* = |z|^2$.

SAMPLE STUDENT ANSWER

Let $z = 3 + 2\mathrm{i}$ and $w = 6 + 5\mathrm{i}$. Find the following, giving your answers in the form $a + bi$ where $a, b \in \mathbb{R}$.

(a) $z + w$ (b) $z - w$ (c) zw (d) $\frac{z}{w}$

(a) $9 + 7\mathrm{i}$ (b) $-3 - 3\mathrm{i}$

(c) $18 + 15\mathrm{i} + 12\mathrm{i} + 10\mathrm{i}^2 = 8 + 27\mathrm{i}$

(d) $\frac{3+3\mathrm{i}}{6+5\mathrm{i}} = \frac{3}{6} + \frac{2\mathrm{i}}{5\mathrm{i}} = \frac{1}{2} + \frac{2}{5} = \frac{9}{10}$

▲ Correct method and answers for parts (a), (b) and (c).

▼ In part (d), the student did not use the conjugate of the denominator method. The student invented a rule for division that is not true.

The answer above could have achieved 4/6 marks.

Correct answer to part (d) is

$\frac{3+2\mathrm{i}}{6+5\mathrm{i}} \times \frac{6-5\mathrm{i}}{6-5\mathrm{i}} = \frac{18-15\mathrm{i}+12\mathrm{i}-10\mathrm{i}^2}{36+25} = \frac{28-3\mathrm{i}}{61} = \frac{28}{61} + \frac{-3}{61}\mathrm{i}$

As well the Cartesian form, $a + b\text{i}$, $a, b \in \mathbb{R}$, complex numbers can also be expressed in modulus-argument (polar) form, $r(\cos\theta + \text{i}\sin\theta)$, $r \in \mathbb{R}^+ \cup \{0\}$, $\theta \in \mathbb{R}$, where r is the modulus and θ is the argument.

How to convert between the two forms

$a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, $\theta = \arctan\frac{b}{a}$, being careful about which quadrant the complex number is in.

A shorthand notation for $r(\cos\theta + \text{i}\sin\theta)$ is $r\text{ cis }\theta$.

It can be shown that $r\text{ cis }\theta = r\text{e}^{\text{i}\theta}$ and this is called the exponential (Euler) form.

Note

When working in modulus-argument form, if you have an equation then you can equate both the moduli and the arguments. $r\text{ cis }\theta = s\text{ cis }\phi \Rightarrow r = s,\ \theta = \phi$

Note

As we are not doing calculus with complex numbers, it does not matter if the argument is given in radians or degrees, but just make sure that you have done whatever the question asked.

Often it is sensible just to consider the complex number's position on the Argand diagram in order to write it in modulus-argument form.

Rules for multiplication, division and taking powers when working in modulus-argument form

$r\text{ cis }\theta \times s\text{ cis }\phi = rs\text{ cis }(\theta + \phi)$ $\qquad r\text{ cis }\theta \div s\text{ cis }\phi = \frac{r}{s}\text{ cis }(\theta - \phi)$

When you multiply complex numbers together you multiply the moduli and add the arguments. When you divide complex numbers you divide the moduli and subtract the arguments.

$(r\text{ cis }\theta)^n = r^n\text{ cis}(n\theta)$, $n \in \mathbb{N}$ This is known as De Moivre's theorem.

Assessment tip

It is easier to work in Cartesian form when performing addition and subtraction, but easier to work in modulus-argument form when doing multiplication, division or taking powers.

Considering the complex numbers in the form $r\text{e}^{\text{i}\theta}$ explains why these rules are true.

Assessment tip

These rules are not given in the formula book and so should be learned.

Example 1.5.3

Let $z = -1 + \text{i}$ and $w = 1 + \text{i}\sqrt{3}$

(a) Write both z and w in modulus-argument form, working in radians.

(b) Hence express (i) zw (ii) $\frac{z}{w}$ (iii) z^3 (iv) w^6

in the form $r\text{ cis }\theta$, $r \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, $-\pi < \theta \le \pi$

Solution

(a) $z = \sqrt{2}\text{ cis }\frac{3\pi}{4}$, $w = 2\text{ cis }\frac{\pi}{3}$

(b) (i) $zw = 2\sqrt{2}\text{ cis}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) = 2\sqrt{2}\text{ cis }\frac{13\pi}{12} = 2\sqrt{2}\text{ cis } -\frac{11\pi}{12}$

(ii) $\frac{z}{w} = \frac{\sqrt{2}}{2}\text{ cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}\text{ cis }\frac{5\pi}{12}$

(iii) $z^3 = \left(\sqrt{2}\right)^3\text{ cis }\frac{9\pi}{4} = 2^{\frac{3}{2}}\text{ cis }\frac{\pi}{4}$

(iv) $w^6 = 2^6\text{cis }\frac{6\pi}{3} = 64\text{ cis }0\ (= 64)$

Link to Proof by Induction AHL 1.15 which can be used to prove De Moivre's theorem together with compound angle formulae AHL 3.10.

SAMPLE STUDENT ANSWER

Let $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. Find the smallest value of $n \in \mathbb{N}$ for which $z^n \in \mathbb{R}$.

$|z| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ $\tan\theta = \frac{1}{\sqrt{3}}$, 1^{st} quadrant, $\theta = \frac{\pi}{6}$ $z = 1 \operatorname{cis}\frac{\pi}{6}$

$z^n = 1^n \operatorname{cis}\frac{n\pi}{6} = \operatorname{cis}\frac{n\pi}{6}$.

For this to be real require $\frac{n\pi}{6}$ to be a multiple of 2π

So smallest value of $n = 12$.

▲ Well-explained, correct method of putting in modulus-argument form. Good application of De Moivre's theorem. The student realized that $z^n \in \mathbb{R}$ means that z^n lies on the real axis.

▼ In the final step, the student did not allow for z^n being negative and so on the negative real axis.

The answer above could have achieved 4/6 marks.

Last line of answer should have been: For this to be real require $\frac{n\pi}{6}$ to be a multiple of π. So smallest value of $n = 6$.

Rules about conjugates

$(z + w)^* = z^* + w^*$ $(z - w)^* = z^* - w^*$ $(zw)^* = z^*w^*$

$\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ $(z^n)^* = (z^*)^n, n \in \mathbb{Z}$

If $a \in \mathbb{R}$, $a^* = a$. (Real numbers are their own conjugate.)

$|z^*| = |z|$, $\arg z^* = -\arg z$

On an Argand diagram the complex number z^* is the reflection of z in the real axis.

Assessment tip

This information is not given in the formula book so should be memorized.

These rules about conjugates give rise to the following result.

Let $P(z) = 0$ be a polynomial equation with real coefficients. Then if z is a solution this implies that z^* must also be a solution.

Note

The condition that $P(z) = 0$ has real coefficients is important; the result is not true if this is not the case. An example of this would be taking the square root of a complex number.

You can think about this result as that complex numbers go around in pairs. A complex number "holds hands" with its conjugate. A real number "holds hands" with itself.

The fundamental theorem of algebra

If $P(z) = 0$ is a polynomial equation of degree $n \in \mathbb{Z}^+$, then it will have exactly n solutions (roots) in the set of complex numbers, $\mathbb{C}$.

These two results can be combined to give information such as:

If $P(z) = 0$ is a polynomial equation, with real coefficients, of degree 3, then there must be at least one real root.

In the above sentence "degree 3" can be replaced with "odd degree" and the same conclusion applies.

Example 1.5.4

One of the roots of the polynomial complex number equation $z^3 - 9z^2 + bz + c = 0$ where $a, b \in \mathbb{R}$ is $3 + \mathrm{i}$.

(a) Find the other roots.

(b) Find the values of b and c.

Solution

(a) Another root is $3 - \mathrm{i}$.

Let the third root be α which will be a real number.

The polynomial must factorize as

$$\big(z-(3+\mathrm{i})\big)\big(z-(3-\mathrm{i})\big)(z-\alpha) = (z^2 - 6z + 10)(z - \alpha)$$

$$= z^3 + (-6 - \alpha)z^2 + (10 + 6\alpha)z - 10\alpha$$

Equating coefficients of z^2 then $-6 - \alpha = -9 \Rightarrow \alpha = 3$

(b) $b = 10 + 6\alpha = 28 \qquad c = -10\alpha = -30$

Link to AHL 2.12, Sum and product of the roots of polynomial equations.

De Moivre's theorem can be extended to rational exponents $\frac{p}{q}$, which are expressed in their lowest form, as follows.

$$(r \text{ cis } \theta)^{\frac{p}{q}} = r^{\frac{p}{q}} \text{ cis}\left(\frac{p}{q}\theta + \frac{2\pi k}{q}\right),\ k = 0, 1, 2, \ldots, q-1$$

This has q values. The arguments can be converted to ones between $-\pi$ and π.

Assessment tip

This formula is not given in the formula book and so should be memorized.

Example 1.5.5

Solve the equation $z^2 = 2\mathrm{i}$ for $z \in \mathbb{C}$

(a) working in Cartesian form

(b) working in modulus-argument form.

Solution

(a) Let $z = a + b\mathrm{i}$, $a, b \in \mathbb{R}$. Require $(a + b\mathrm{i})^2 = a^2 - b^2 + 2ab\mathrm{i} = 0 + 2\mathrm{i}$

Equating real and imaginary parts: $\quad a^2 - b^2 = 0,\ 2ab = 2$

$ab = 1 \Rightarrow b = \frac{1}{a} \qquad a^2 - \frac{1}{a^2} = 0 \Rightarrow a^4 = 1 \qquad a^2 = 1$ (-1 rejected)

$a = 1$ or $-1 \qquad a = 1 \Rightarrow b = 1,\ a = -1 \Rightarrow b = -1 \qquad z = 1 + \mathrm{i}$ or $-1 - \mathrm{i}$

(b) Let $z = r \text{ cis } \theta$, $r \in \mathbb{R}^+$, $-\pi < \theta \le \pi$. Require $(r \text{ cis } \theta)^2 = r^2 \text{ cis } 2\theta = 2 \text{ cis } \frac{\pi}{2}$

Equating moduli and arguments: $r^2 = 2 \Rightarrow r = \sqrt{2}$

$2\theta = \frac{\pi}{2} + 2\pi k,\ k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{4} + \pi k \Rightarrow \theta = \frac{\pi}{4}$ or $-\frac{3\pi}{4}$ in the desired range.

$z = \sqrt{2} \text{ cis } \frac{\pi}{4}$ or $\sqrt{2} \text{ cis } -\frac{3\pi}{4}$

Assessment tip

With questions like Example 1.5.5 you can check that the answers to parts (a) and (b) agree by visualizing them geometrically. Another check is that the second answer is the first answer multiplied by -1.

▲ Good use of calculator to find one solution.

▼ The student did not correctly understand the condition for a solution to bring along its conjugate. The coefficients are not real in this case, so that rule does not apply.

SAMPLE STUDENT ANSWER

Solve the equation $z^2 = 2i$ for $z \in \mathbb{C}$

Using a calculator: $\sqrt{2i} = 1 + i$. Other solution is its conjugate $1 - i$.

The answer above could have achieved 2/4 marks.

The other solution will be $-1 - i$

Example 1.5.6

Solve the equation $z^4 = 1$ for $z \in \mathbb{C}$

(a) Use factorization and give the answer in Cartesian form.

(b) Work in modulus-argument form.

Solution

(a) $z^4 - 1 = 0 \Rightarrow (z^2 - 1)(z^2 + 1) = 0 \Rightarrow (z - 1)(z + 1)(z^2 + 1) = 0$

$z^2 + 1 = 0 \Rightarrow z^2 = -1$, so the four solutions are $z = 1, -1, i, -i$

(b) Let $z = r \operatorname{cis} \theta$, $r \in \mathbb{R}^+$, $-\pi < \theta \le \pi$.
Require $(r \operatorname{cis} \theta)^4 = r^4 \operatorname{cis} 4\theta = 1 \operatorname{cis} 0$

Equating moduli and arguments:
$r^4 = 1 \Rightarrow r = 1 \qquad 4\theta = 0 + 2\pi k, k \in \mathbb{Z}$

$\theta = \frac{\pi k}{2}$, so $\theta = 0, \frac{\pi}{2}, \pi$ or $-\frac{\pi}{2}$ in the desired range.

$z = 1 \operatorname{cis} 0, 1 \operatorname{cis} \frac{\pi}{2}, 1 \operatorname{cis} \pi$ or $1 \operatorname{cis} -\frac{\pi}{2}$

(These answers were also obtainable directly by applying the extended version of De Moivre's theorem.)

Assessment tip

Always keep track of which sets variables belong to. In Example 1.5.6, it might look that with the equation $r^4 = 1$ we are back to the same equation that we started with. However, the point is that $r \in \mathbb{R}^+$ and so the only solution is $r = 1$. Again, there is the check that the solutions to parts (a) and (b) agree. There is also the check that each solution comes with its conjugate.

Example 1.5.7

Use the exponential form to simplify $z = i^i$ as far as possible, giving the final exact answer in Cartesian form.

Solution

$|i| = 1, \arg i = \frac{\pi}{2} \qquad i = e^{i\frac{\pi}{2}} \qquad i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}}$

$z = e^{\frac{-\pi}{2}} + 0i$

Note

It might be surprising that such a seemingly complicated number as $z = i^i$ turns out to be a purely real number. This could be checked with the calculator, although not in exact form.

PRACTICE QUESTIONS

SL PAPER 1 SECTION A
NO TECHNOLOGY ALLOWED

Question 1 [5 marks]

An arithmetic sequence has third term 4 and sixth term 19.

Find: **a.** the common difference, d

b. the first term, u_1

c. the sum of the first five terms.

Question 2 [5 marks]

A geometric sequence has third term 3 and sixth term –24.

Find: **a.** the common ratio, r

b. the first term, u_1

c. the sum of the first five terms.

Question 3 [6 marks]

A geometric sequence with a non-zero common ratio has first term equal to 1.

An arithmetic sequence has first term equal to 0.

The two sequences have the same second term.

The fourth term of the geometric sequence equals the 10th term of the arithmetic sequence.

Find the possible values for the common ratio of the geometric sequence and the corresponding values for the common difference in the arithmetic sequence.

Question 4 [7 marks]

a. The sum to infinity of a geometric series with a non-zero first term is five times its first term. Find the value of the common ratio.

b. Determine whether or not it is possible for the sum to infinity of a geometric series with a non-zero first term and a real common difference to be equal to the second term. In other words, state, with a reason, if $S_\infty = u_2$ is possible.

Question 5 [6 marks]

Simplify each of these expressions as far as possible:

a. $\dfrac{2^4 \times 2^3}{2^6}$

b. $3^{-9} \times 3^7 \times \sqrt{3^2}$

c. $(2^2)^3 \times 1^{11}$

Question 6 [8 marks]

Simplify each of these expressions as far as possible:

a. $2\log 5 + \log 4$

b. $(\ln 7 + \ln 3) - \ln 21$

c. $e^{(\ln 7 + \ln 3) - \ln 21}$

Question 7 [8 marks]

a. By converting to logarithms with base 10, simplify $\log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 2$ as far as possible.

b. Solve the equation $\log_2 x + 2\log_x 2 = 3$

Question 8 [6 marks]

Using the binomial theorem, expand and simplify $(1 - 2x)^4$ in ascending powers of x.

Question 9 [6 marks]

a. Use a direct proof to show that:

$\dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-1} \equiv \dfrac{3x^2-1}{x^3-x}$, for $x \in \mathbb{R}$, $x \neq 0$, $x \neq -1$, $x \neq 1$

b. Hence, solve $\dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-1} = 0$

SL PAPER 2 SECTION A
TECHNOLOGY REQUIRED

Question 10 [6 marks]

The formula for the volume of a sphere is given by $V = \dfrac{4\pi r^3}{3}$

a. A proton can be considered to be a sphere with a radius of $r = 8.5 \times 10^{-16}$ m. Find the volume of a proton. Give your answer in the form $a \times 10^k$ m^3, where $a \in \mathbb{R}$, $1 \leq a < 10$, $k \in \mathbb{Z}$.

b. The Sun can be considered as a sphere with a volume of 1.412×10^{27} m^3. Find the radius of the sun. Give your answer in the form $a \times 10^k$ m, where $a \in \mathbb{R}$, $1 \leq a < 10$, $k \in \mathbb{Z}$.

Question 11 [6 marks]

In a cave that used to be inhabited by prehistoric humans, a research student finds some markings in the form of vertical lines. The markings are very indistinct and difficult to see. They are grouped together along a wall with a horizontal gap between each of the groups. The student counts the number of markings that she can identify in each group and records the following:

3 7 11 14 19 21 27

The student's professor suspects that the number of markings should form an arithmetic sequence.

a. Conjecture the common difference for the sequence. State which numbers the professor will ask the student to go back and check, and what the professor predicts these numbers will then be.

b. Using your conjecture, find the number of groups of markings that would have to be found for the total number of marking to be 253.

Question 12 [6 marks]

Xiaoming wants to buy a new guitar that costs £700. He has only £500, so he deposits this in a bank that pays 7% compound interest, compounded yearly.

a. Assuming Xiaoming makes no more deposits, calculate how many complete years he will have to wait until he has at least £700 available in the bank.

Xiaoming's grandmother gives him a gift voucher for use in his local music shop when he pays the £500 into the bank. A year later she gives him two vouchers. Another year later she gives him three vouchers, and so on.

b. Calculate how many gift vouchers she will have given him in total at the *beginning* of the year in which his account reaches £700.

With the money that Xiaoming has left over, having taken everything out of the bank and then paying for the guitar, he is going to buy his grandmother a set of noise-cancelling headphones so she wouldn't have to hear him practise on his new guitar in the house!

c. Calculate how much money he has available to spend on the headphones, giving your answer to two decimal places.

Question 13 [6 marks]

a. Find the first three terms in the expansion of $(1 + 3x)^{20}$ in ascending powers of x.

b. Hence, find the term in x in the expansion of $\left(2+\frac{1}{x}\right)(1+3x)^{20}$

Question 14 [7marks]

In the expansion of $\left(x+\frac{2}{x}\right)^6$, find:

a. the term in x^4

b. the constant term.

HL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 15 [9 marks]

Prove by induction that $2^{n+1}+3^{2n-1}$ is always exactly divisible by 7, for all $n \in \mathbb{Z}^+$

Question 16 [7 marks]

a. Solve $z^2 - 4z + 20 = 0$ for $z \in \mathbb{C}$

b. Hence, solve $w-3+\frac{20}{w+1}=0$ for $w \in \mathbb{Z}$

Question 17 [8 marks]

Let $w=\frac{z}{z^*}$, where z is a non-zero complex number of the form $z=a+b\mathrm{i}$, $a, b \in \mathbb{R}$, $a \neq b$.

Show that $\operatorname{Im}\left(\frac{2w}{1+w^2}\right)=0$.

HL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 18 [13 marks]

a. Decompose $f(x) = \frac{2x+1}{2x^2+5x+3}$ into partial fractions.

b. Hence, find:

i. the derivative of $f(x)$

ii. the integral of $f(x)$.

Question 19 [11 marks]

Find real numbers p and q such that $\frac{p}{1+\mathrm{i}}+\frac{q}{2-\mathrm{i}}=\frac{2+3\mathrm{i}}{\mathrm{i}}$

Question 20 [17 marks]

a. Let $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$

i. Show that $(z + w)^* = z^* + w^*$

ii. Show that $(z - w)^* = z^* - w^*$

iii. Show that $(zw)^* = z^*w^*$

b. Working in modulus-argument form, let $z = r \operatorname{cis} \theta$ and $w = s \operatorname{cis} \phi$.

i. Show that $\left(\frac{z}{w}\right)^*=\frac{z^*}{w^*}$

ii. Show that $(z^n)^*=(z^*)^n$, $n \in \mathbb{Z}^+$

Question 21 [13 marks]

Letting $z = r \operatorname{cis} \theta$, $\theta \in [-\pi, \pi]$ solve the equation $z^4 = -2 + 2\mathrm{i}$, and give your answers in modulus-argument form.

Question 22 [17 marks]

Note the style of this question would make it difficult for it to appear on an actual IB exam paper but it would still be instructive for you to do it.

The set of real numbers is "ordered". For any two real numbers x and y, we have either $x = y$ or $x < y$ or $x > y$. Some properties of an "ordering" are as follows:

If $x > y$ and $y > z$ then $x > z$

If $x > y$ then $x + z > y > z$

If $x > y$ and $z > 0$ then $xz > yz$

If $x > y$ and $z < 0$ then $xz < yz$

The set of complex numbers cannot be ordered.

a. Copy out the following proof of this result filling in all the missing blanks.

Proof by _______. Suppose $\mathbb{C}$ can be ordered.

Then either $i = 0$ or $i < 0$ or _______.

Case **i.** $i = 0 \Rightarrow i^2 = 0 \Rightarrow$ _______ (a contradiction)

Case **ii.** $i < 0$ then using a property of ordering $i \times i >$ _______ $\Rightarrow$ _______ (a contradiction)

Case (iii) _______ then using a property of ordering _______ $\Rightarrow$ _______ (a contradiction)

All three cases lead to a _______ and so we conclude that ______________.

b. Determine whether each of the following statements make sense or not. In other words, do they have a mathematical meaning? If they do make sense, state whether they are true or false.

i. $3 + 4i > 5 + 12i$

ii. $\mathrm{Re}(3 + 4i) > \mathrm{Re}(5 + 12i)$

iii. $|3 + 4i| > |5 + 12i|$

iv. $\arg(3 + 4i) > \arg(5 + 12i)$

v. $3 + 4i > 3$

Question 23 [11 marks]

a. Find the value of λ for which the following system of equations is consistent.

$x - 2y + 3z = 9$

$2x + y + 2z = 11$

$7x - 4y + 13z = \lambda$

b. For the value of λ that makes the equations consistent, find the solution to the system of equations.

HL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 24 [5 marks]

A committee of four people is to be chosen from three men and five women.

a. Find the number of possible committees that can be chosen.

b. Find the number of possible committees that can be chosen if the committee must contain two men and two women.

Question 25 [5 marks]

a. Find how many ways the letters in the word FAILURE can be arranged in a line, if the arrangement is to start with a vowel.

b. Find how many ways the letters in the word FAILURE can be arranged in a line, if all the vowels have to be together and all the constants have to be together.

Question 26 [8 marks]

a. Use the extension of the binomial theorem to find the first four terms in the expansion of $(1+2x)^{-\frac{1}{2}}$, in ascending powers of x.

b. Hence, find a rational approximation to $\frac{1}{\sqrt{2}}$

c. Find the absolute percentage error when using the approximation in part (b) to estimate $\frac{1}{\sqrt{2}}$

Question 27 [7 marks]

a. Consider ${}^pC_i$ where p is a prime number and $1 \le i \le p - 1$. Show that ${}^pC_i$ is always exactly divisible by p.

b. Let n be an odd natural number and let the natural number i satisfy the inequality $1 \le i \le n - 1$

Peter makes the conjecture that ${}^nC_i$ is always exactly divisible by n. Find a counterexample to disprove Peter's conjecture.

Question 28 [8 marks]

Solve the following systems of equations.

a. $x+y+2z=15$
$x+2y-z=3$
$2x+3y+2z=23$

b. $x+y+2z=15$
$x+2y-z=3$
$3x+4y+3z=32$

c. $x+y+2z=15$
$x+2y-z=3$
$3x+4y+3z=33$

HL PAPER 3 TECHNOLOGY REQUIRED

Question 29 [31 marks]

In this question you will investigate how rounding errors affect further calculations.

a. Let $x=1.235$ and $y=2.316$.

i. Add x and y together and then round the answer to 3 significant figures.

ii. First round x and y to 3 significant figures and then add them together.

iii. State what you notice. Write down a sentence that generalizes this result.

iv. Explain which of the methods of parts (i) and (ii) is the better one.

b. When a quantity t is measured and a result of T is obtained, we will use the notation $\varepsilon_t = T - t$ to represent the error made. We will suppose that ε_t is small in comparison to t.

i. Find ε_{a+b} in terms of ε_a and ε_b.

ii. Find ε_{a-b} in terms of ε_a and ε_b.

iii. Give an expression for the absolute maximum error in $a \pm b$.

From now on terms that are very small in comparison to another term can be neglected.

c. i. Find an approximate expression for $\varepsilon_{a\times b}$ in terms of a, b, ε_a and ε_b.

ii. State which rule in differentiation this expression is similar to.

iii. Give an expression for the absolute maximum error in ab.

d. i. Find an approximate expression for $\varepsilon_{\frac{a}{b}}$ in terms of a, b, ε_a and ε_b.

ii. State which rule in differentiation this expression is similar to.

iii. Give an expression for the absolute maximum error in $\frac{a}{b}$.

The absolute relative error for t is defined by $r_t = \frac{|\varepsilon_t|}{|t|}$

e. Using the answers to (c) (iii) and (d) (iii), find approximate expressions for **i.** $r_{a\times b}$ and **ii.** $r_{\frac{a}{b}}$ in terms of r_a and r_b. State in words what happens to absolute relative errors when you multiply or divide.

f. i. Use the binomial theorem to find an approximation for ε_{a^n} for $n \in \mathbb{Z}^+$.

ii. Hence give an approximation to the absolute relative error for r_{a^n} in terms of r_a.

2 FUNCTIONS

2.1 STRAIGHT LINES

You must know:

- ✔ the different forms of the equation of a straight line
- ✔ the conditions for parallel and perpendicular lines.

You should be able to:

- ✔ find gradients, intercepts and equations of straight lines.

The equation of a straight line can be expressed in gradient-intercept form $y = mx + c$, standard form $ax + by + c = 0$ or point-gradient form $y - y_1 = m(x - x_1)$. In the form $y = mx + c$, the m represents the gradient (slope) and the c represents the intercept on the y-axis. In the form $y - y_1 = m(x - x_1)$, the m represents the gradient (slope) and (x_1, y_1) is a particular point on the line.

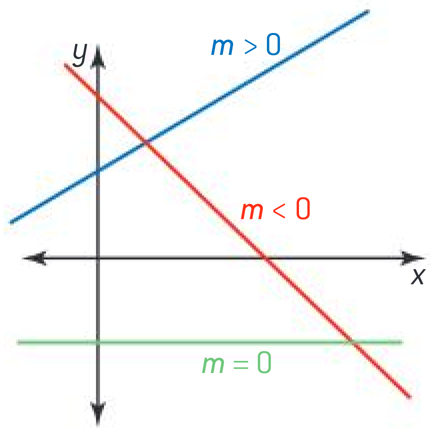
If $m > 0$, then the function is increasing; the line is going uphill as you travel from left to right.

If $m < 0$, then the function is decreasing; the line is going downhill as you travel from left to right.

If $m = 0$, then the line is horizontal.

If $|m|$ is large, then the line is very steep (either uphill or downhill).

If $|m|$ is small, then the line is very shallow (either uphill or downhill).



A straight line is uniquely defined if (i) two distinct points on the line are given, or (ii) the gradient of the line is given, together with one point on the line.

The gradient of the line through the points (x_1, y_1) and (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient can be thought of as $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{up}}{\text{along}}$ in a right-angled triangle.

Example 2.1.1

Find the equation of the straight line with gradient 3 that passes through the point (1, 4). Give your answer in the form $y = mx + c$.

Solution

Method 1: $y - 4 = 3(x - 1) \Rightarrow y = 3x + 1$

Method 2: $y = 3x + c$ through $(1, 4) \Rightarrow 4 = 3 + c \Rightarrow c = 1$ so $y = 3x + 1$

Assessment tip

In Example 2.1.2, a good check would be to make sure (4, 9) also satisfies your line equation.

Example 2.1.2

Find the equation of the straight line that passes through the points (2, 5) and (4, 9). Give your answer in the form $y = mx + c$.

Solution

Gradient $m = \frac{9-5}{4-2} = 2, \quad y = 2x + c$

through $(2, 5) \Rightarrow 5 = 4 + c \Rightarrow c = 1$

so $y = 2x + 1$

SAMPLE STUDENT ANSWER

Find the equation of the straight line that passes through the points (–2, 5) and (4, –9). Give your answer in the form $y = mx + c$.

$m = \frac{-9-5}{4+2} = \frac{-7}{3} \qquad 5 = \frac{-7}{3}x - 2 + c \qquad c = \frac{1}{3}$

▲ Correct method and values of m and c.

▼ Did not finish off by giving the final answer asked for.

The answer above could have achieved 4/5 marks.

The student should have concluded with $y = -\frac{7}{3}x + \frac{1}{3}$

Assessment tip

Always check that you have answered exactly what the question asked. Make sure that your answer is the correct type of object; do not give a number if it asks for an equation, or a point if it asks for a vector.

The intercept on the y-axis is found by putting $x = 0$ into the equation of the straight line. If the equation of the straight line is $y = mx + c$ then the intercept on the y-axis is $(0, c)$.

The intercept on the x-axis is found by putting $y = 0$ into the equation of the straight line. If the equation of the straight line is $y = mx + c$ then the intercept on the x-axis is $\left(-\frac{c}{m}, 0\right)$ provided that $m \neq 0$.

Note

An intercept is a point, so the answer will have an x-coordinate and a y-coordinate. The intercept on the x-axis is of the form $(x, 0)$. The intercept on the y-axis is of the form $(0, y)$.

Example 2.1.3

For each of the following straight lines, find (i) the intercept on the y-axis (ii) the intercept on the x-axis.

(a) $y = 5x + 2$ (b) $2x + 3y - 6 = 0$ (c) $y - 4 = -3(x - 6)$

Solution

(a) (i) (0, 2) (ii) $\left(-\frac{2}{5}, 0\right)$

(b) (i) (0, 2) (ii) (3, 0)

(c) (i) $x = 0 \Rightarrow y = 18 + 4 = 22$ so (0, 22)

(ii) $y = 0 \Rightarrow -4 = -3(x - 6) \Rightarrow \frac{4}{3} = x - 6 \Rightarrow x = \frac{22}{3}$ so $\left(\frac{22}{3}, 0\right)$

If line L_1 has gradient of m_1 and line L_2 has gradient of m_2 then the conditions for the lines to be parallel or perpendicular are as follows.

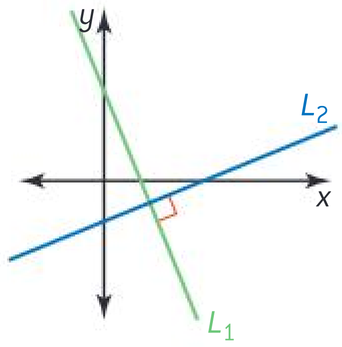
Assessment tip

Neither of these formulae is given in the formula book.

Two straight lines are parallel if and only if $m_1 = m_2$

Two straight lines are perpendicular if and only if $m_1 \times m_2 = -1$

The first statement is rather obvious; the second needs to be learned. The second can also be expressed as $m_2 = -\frac{1}{m_1}$ (the two gradients of perpendicular lines are negative reciprocals of one another). This formula seems reasonable for the following reasons. If one line is going uphill the other must be going downhill, explaining the negative sign. If one line is steep the other must be shallow, explaining the reciprocal.



Example 2.1.4

Consider the four straight lines given by:

(i) $y = 3x + 7$ (ii) $6x - 2y = 5$

(iii) $3(y-2) = 4 - x$ (iv) $y = \frac{1}{3}x + 1$

(a) State which pairs of lines are parallel.

(b) State which pairs of lines are perpendicular.

Solution

(i) The gradient is 3.

(ii) Rewrite the equation as $y = 3x - \frac{5}{2} \Rightarrow$ the gradient is 3.

(iii) Rewrite the equation as $y = -\frac{x}{3} + \frac{10}{3} \Rightarrow$ the gradient is $-\frac{1}{3}$.

(iv) The gradient is $\frac{1}{3}$.

(a) Lines (i) and (ii) are parallel.

(b) Lines (i) and (iii) are perpendicular, as are lines (ii) and (iii).

Note

To clearly see what the gradient of a straight line is, write it in the form $y = mx + c$.

Example 2.1.5

Line L_1 is given by $2x + 5y + 7 = 0$.

(a) Find the equation of the straight line parallel to L_1 that passes through the point (1, 2). Give your answer in the form $y = mx + c$.

(b) Find the equation of the straight line perpendicular to L_1 that passes through the point (1, 2). Give your answer in the form $ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.

Solution

$2x + 5y + 7 = 0 \Rightarrow y = -\frac{2}{5}x - \frac{7}{5}$ which has gradient of $-\frac{2}{5}$.

(a) Line equation is of the form $y = -\frac{2}{5}x + c$, through (1, 2).

$\Rightarrow 2 = -\frac{2}{5} + c \Rightarrow c = \frac{12}{5}$

Parallel line is $y = -\frac{2}{5}x + \frac{12}{5}$

(b) Line equation is of the form $y = \frac{5}{2}x + c$, through (1, 2).

$\Rightarrow 2 = \frac{5}{2} + c \Rightarrow c = -\frac{1}{2}$

Perpendicular line is $y = \frac{5}{2}x - \frac{1}{2}$

Multiplying by 2 to eliminate the fractions and writing the equation in standard form gives $-5x + 2y + 1 = 0$.

SAMPLE STUDENT ANSWER

Line L_1 is given by $3x + 4y - 12 = 0$.

(a) Find the equation of the straight line that does not intersect with L_1 and passes through the point $(1, -1)$. Give your answer in the form $y = mx + c$.

(b) Find the equation of the straight line perpendicular to L_1 that passes through the point $(1, -1)$. Give your answer in the form $y = mx + c$.

$3x + 4y - 12 = 0$ gradient $= -\frac{3}{4}$

(a) No intersect so parallel $y = -\frac{3}{4}x + c$ $\quad -1 = -\frac{3}{4} + c \Rightarrow c = -\frac{1}{4}$

$y = -\frac{3}{4}x - \frac{1}{4}$

(b) $y = -\frac{4}{3}x + c$ $\quad -1 = -\frac{4}{3} + c \Rightarrow c = \frac{1}{3}$ $\quad y = -\frac{4}{3}x + \frac{1}{3}$

▲ Correct method and solution for part (a) and good reasoning that they must be parallel. Had the correct general approach in part (b).

▼ Included a negative sign for the perpendicular gradient.

It should be $\frac{4}{3}$.

The answer above could have achieved 4/6 marks.

Correct solution to part (b) is $y = \frac{4}{3}x + c$

through $(1, -1) \Rightarrow -1 = \frac{4}{3} + c \Rightarrow c = -\frac{7}{3}$

Perpendicular line is $y = \frac{4}{3}x - \frac{7}{3}$

2.2 THE CONCEPT OF A FUNCTION

You must know:

- ✔ the definition of a function, its domain and range
- ✔ when an inverse function exists and its geometric interpretation
- ✔ what composite functions are
- ✔ what the identity function is
- ✔ the rules that composite and inverse functions obey.

You should be able to:

- ✔ find inverse functions
- ✔ sketch or draw graphs on paper or using technology
- ✔ find the intersection of two curves using a calculator
- ✔ find expressions for composite functions.

Functions

A relation from a set A to a set B is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

A function f is a relation (a mapping, a rule) from one set A called the domain (leaving set) to another set B called the codomain (arriving set).
The notation $f\colon A \to B$ and $f\colon x \mapsto f(x)$ could be used.
For example, $f\colon \mathbb{R} \to \mathbb{R}$, $f\colon x \mapsto 2x + 3$ is a linear function and would represent a straight line. This last piece of notation is usually abbreviated just to $f(x) = 2x + 3$.

To be a function, the relation **must** have the property that **all** elements in the domain are mapped to **one and only one** member in the codomain. If the domain and codomain are both subsets of $\mathbb{R}$ then the vertical line test can be employed. The vertical line test says that for a function, any vertical line will intersect the graph of the function at exactly one point.

Assessment tip

The IB has the convention that if the domain of a function is not stated then it can be assumed to be $\mathbb{R}$.

Note

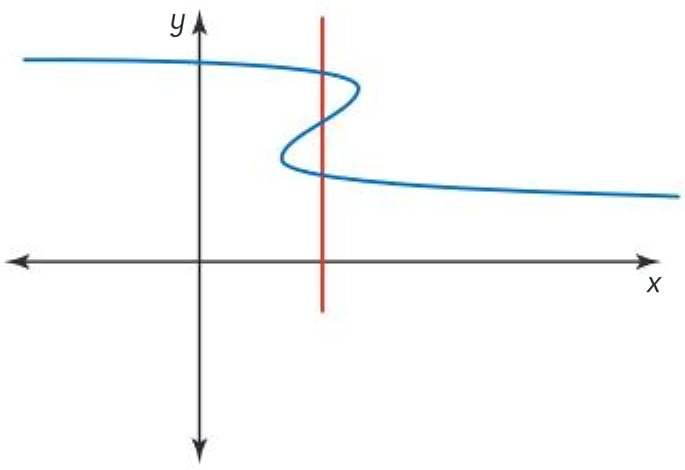
Always read what the domain is given as, as well as what the rule is, to decide if a relation is a function.

When considering the rule for the possible function realize that you cannot: divide by zero, take the square root of a negative number, or take the log of a negative or zero number.

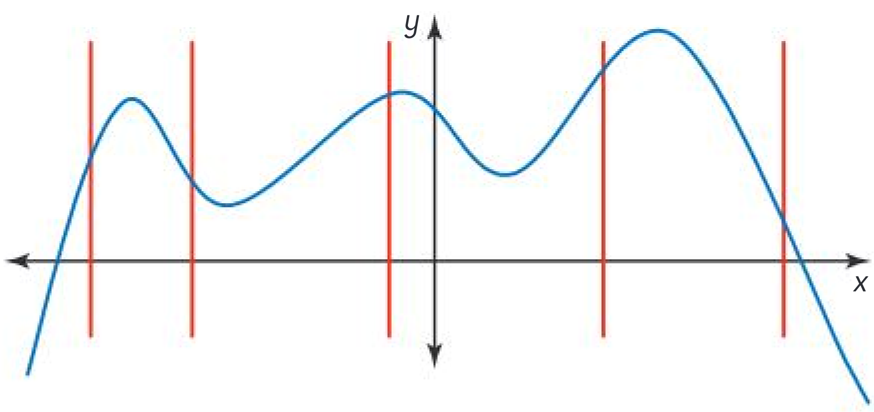
The relation $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \pm\sqrt{x}$ is not a function for two reasons. There are two values of $f(x)$ for every value of $x > 0$ and no values of $f(x)$ for $x < 0$. If it is changed to $f\colon \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$, $f(x) = +\sqrt{x}$ then it is a function.

The relation $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is not a function as it is undefined for $x = 0$.
If it is changed to $f\colon \{x \in \mathbb{R} \mid x \neq 0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ then it is a function.

Consider the graph on the right. It could not represent a function with domain of $\mathbb{R}$ as the vertical line test indicates.



This next graph (below) could represent a function with domain of $\mathbb{R}$ as the vertical line test indicates.



All finite polynomials
$f\colon \mathbb{R} \to \mathbb{R}, f\colon x \mapsto a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,\ n \in \mathbb{N},\ a_i \in \mathbb{R}$
are functions with domain of $\mathbb{R}$. They are also continuous and smooth.

Example 2.2.1

For each of the following mappings, if they are to be a function, state what the maximum domain would be.

(a) $f(x)=\sqrt{x-3}$

(b) $f(x)=\frac{1}{x+4}$

(c) $f(x)=\log(x-5)$

(d) $f(x)=\frac{2x}{\sqrt{x+5}}$

(e) $f(x)=\dfrac{1}{x^2-x-6}$

Solution

(a) $\{x\in\mathbb{R}\mid x\geq 3\}$

(b) $\{x\in\mathbb{R}\mid x\neq -4\}$

(c) $\{x\in\mathbb{R}\mid x>5\}$

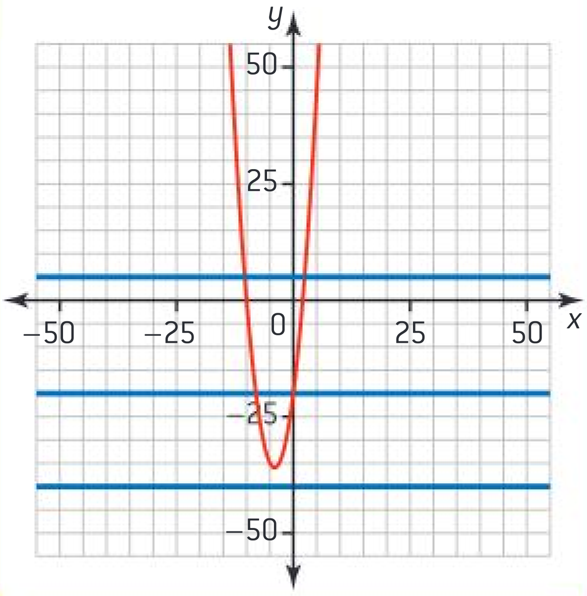
(d) $\{x\in\mathbb{R}\mid x>-5\}$

(e) $x^2-x-6=(x+2)(x-3)$, maximum domain is $\{x\in\mathbb{R}\mid x\neq -2, x\neq 3\}$

The range of a function is defined as the set of elements a in the codomain for which there is an x in the domain such that $f(x)=a$. If the function is from $\mathbb{R}\to\mathbb{R}$ then the range can be thought of as the height/depths that the graph reaches. In this case the horizontal line test is useful in determining the range.

Note

The horizontal line test



Example 2.2.2

State the range of the following functions:

(a) $f\colon \mathbb{R}\to\mathbb{R},\ f(x)=2x+1$

(b) $f\colon \mathbb{R}\to\mathbb{R},\ f(x)=7$

(c) $f\colon \{x\in\mathbb{R}\mid x\geq -4\}\to\mathbb{R},\ f(x)=\sqrt{x+4}$

(d) $f\colon \{x\in\mathbb{R}\mid x>-2\}\to\mathbb{R},\ f(x)=\ln(x+2)$

(e) $f\colon \mathbb{R}\to\mathbb{R},\ f(x)=x^2+8x-20$

(Hint: write $x^2+8x-20$ in the form $(x+p)^2+q$.)

Solution

(a) $\mathbb{R}$ (b) $\{y\in\mathbb{R}\mid y=7\}$ (c) $\{y\in\mathbb{R}\mid y\geq 0\}$ (d) $\mathbb{R}$

(e) $x^2+8x-20=(x+4)^2-36$

Therefore, the range is $\{y\in\mathbb{R}\mid y\geq -36\}$.

For a function f with domain and range both as subsets of $\mathbb{R}$, the expressions $(x, f(x))$ can be thought of as points of the form (x, y) and all these points represent the graph of the function.

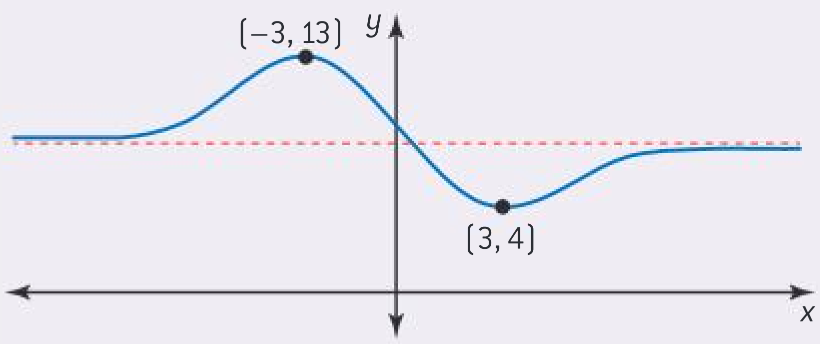
Note

You can think of arithmetic and geometric sequences as functions that act on the domain $\mathbb{Z}^+$.

Link to Arithmetic and geometric sequences, SL 1.2

Example 2.2.3

Given that the graph below represents a function with domain $\mathbb{R}$, state the range of this function.



Solution

By considering the intersection of the graph with horizontal lines the range is $\{y \in \mathbb{R} \mid 4 \le y \le 13\}$.

The function f may represent a mathematical model, for example, the height h of an object at time t. The notation used often alludes to this e.g., use of $h(t)$ in this example. So, $v(t)$ might be used to represent the velocity of an object at time t, or $C(n)$ to represent the capital in a savings account after n years.

Inverse functions

For a function f to have an inverse that is also a function, then for each element y in the range there can only be one element x in the domain such that $f(x) = y$. If this is the case then the original function is called "one to one" and the inverse function exists, with its domain being the range of the original function. If the domain and range are subsets of $\mathbb{R}$ then the horizontal line test can be used to determine if a function is "one to one" or not. The inverse function can be thought of as reversing or undoing the effect of a function. The notation for the inverse function is $f^{-1}(x)$.

If the domain and range are subsets of $\mathbb{R}$ then the graph of $f^{-1}(x)$ will be the reflection of the graph of $f(x)$ in the line $y = x$ as the points (x, y) are transformed to the points (y, x).

To find the inverse of a function in the form $y = y(x)$, interchange the x's and y's and use algebra to make the new y the subject of the equation. The stage at which the x's and y's are interchanged is not critical. The domain of the original function will become the range of the inverse function and the range of the original function will become the domain of the inverse function.

In order for the inverse relation to be a function it may be necessary to restrict the domain of the original function.

Assessment tip

The inverse function $f^{-1}(x)$ should not be confused with the reciprocal $\frac{1}{f(x)}$, which would be written as $(f(x))^{-1}$.

Note

To visualize the graph of the inverse of a function, consider the graph of the original function being drawn on paper in ink. Then, while the ink is still wet, the paper is folded along the line $y = x$. The transferred ink will give the graph of the inverse of the function.

The inverse relation of a function may or may not be a function.

Example 2.2.4

For each function $f(x)$, find the inverse function $f^{-1}(x)$, stating its domain and range. If the inverse function does not exist then give a reason why.

(a) $f: \mathbb{R} \to \mathbb{R}, f(x) = 5x - 3$

(b) $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 2$

(c) $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$

(d) $f: \{x \in \mathbb{R} \mid x \geq 0\} \to \{y \in \mathbb{R} \mid y \geq 0\}, f(x) = x^2$

(e) $f: \{x \in \mathbb{R} \mid x \neq 2\} \to \{y \in \mathbb{R} \mid y \neq -1\}, f(x) = \frac{4-x}{x-2}$

(f) $f: \{x \in \mathbb{R} \mid x > 3\} \to \mathbb{R}, f(x) = \ln(x-3)$

(g) $f: \{x \in \mathbb{R} \mid x \geq 5\} \to \{y \in \mathbb{R} \mid y \geq 3\}, f(x) = x^2 - 10x + 28$

Solution

(a) $y = 5x - 3$ interchanging $x = 5y - 3$ $\quad y = \frac{x+3}{5} = \frac{1}{5}x + \frac{3}{5}$

$f^{-1}(x) = \frac{1}{5}x + \frac{3}{5}, f^{-1}: \mathbb{R} \to \mathbb{R}$

(b) $y = x^3 + 2$ interchanging $x = y^3 + 2$ $\quad y = \sqrt[3]{x-2}$

$f^{-1}(x) = \sqrt[3]{x-2}, f^{-1}: \mathbb{R} \to \mathbb{R}$

(c) No inverse function, not "one to one", e.g., $f(2) = f(-2) = 4$

(d) $y = x^2$ interchanging $x = y^2$ $\quad y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}, f^{-1}: \{x \in \mathbb{R} \mid x \geq 0\} \to \{y \in \mathbb{R} \mid y \geq 0\}$

(e) $y = \frac{4-x}{x-2}$ interchanging $x = \frac{4-y}{y-2}$ $\quad xy - 2x = 4 - y$

$\Rightarrow xy + y = 4 + 2x \Rightarrow y = \frac{4+2x}{1+x}$ $\quad f^{-1}(x) = \frac{4+2x}{1+x}$

$f^{-1}: \{x \in \mathbb{R} \mid x \neq -1\} \to \{y \in \mathbb{R} \mid y \neq 2\}$

(f) $y = \ln(x-3)$ interchanging $x = \ln(y-3)$ $\quad y = 3 + e^x$

$f^{-1}(x) = 3 + e^x, f^{-1}: \mathbb{R} \to \{y \in \mathbb{R} \mid y > 3\}$

(g) $y = x^2 - 10x + 28$ interchanging $x = y^2 - 10y + 28$ $\quad x = (y-5)^2 + 3$

$\Rightarrow y = \sqrt{x-3} + 5$ $\quad f^{-1}(x) = \sqrt{x-3} + 5,$

$f^{-1}: \{x \in \mathbb{R} \mid x \geq 3\} \to \{y \in \mathbb{R} \mid y \geq 5\}$

If a function involves the variable x only once then its inverse function can be found using the "box" method. This method relies on the fact that to undo something you do the opposite things in the opposite order. An analogy would be putting on your socks and shoes. You could consider putting on your socks and then your shoes as the function, and then taking off your shoes and your socks as the inverse function. You start back where you began, with bare feet. The method considers the function as a conveyor belt going through operation boxes and the inverse is obtained by reversing the conveyor belt.

Note

On some calculators, a function and its inverse function are often given on the same button, e.g., $\log x$ and 10^x, $\ln x$ and e^x, x^2 and $\sqrt{x}$, $\sin x$ and $\arcsin x$.

Link to Inverse trig functions AHL 3.9

Note

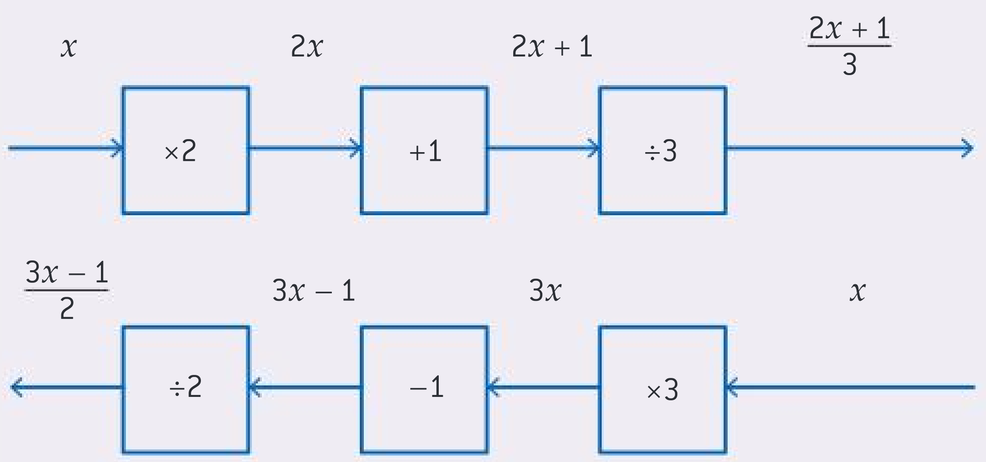
Again, note that it is important to check on the domain and range to see if a function will have an inverse. Part (g) here shows how the graph of a function can be restricted to insure that the inverse function exists. In this example just one "branch" of the quadratic is taken.

Example 2.2.5

Let a function be defined by $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \frac{2x+1}{3}$

Find the inverse function.

Solution



Inverse function is $f^{-1}(x) = \frac{3x-1}{2},\ f^{-1}\colon \mathbb{R} \to \mathbb{R}$

Link to Composite functions SL 2.5

It is clear that $(f^{-1})^{-1}(x) = f(x)$ is always true.

SAMPLE STUDENT ANSWER

Consider the function $f(x) = 5x - 3$.

(a) Find the intercept with (i) the x-axis (ii) the y-axis.

(b) State the range of this function.

(c) Find the inverse function $f^{-1}(x)$.

(a) (i) $5x - 3 = 0 \Rightarrow x = \frac{3}{5}$ (ii) $x = 0 \Rightarrow y = -3$

(b) $\mathbb{R}$

(c) $f^{-1}(x) = \frac{x}{5} + 3$

▲ Correct answers to parts (a) and (b). Had the concept of inverse in part (c).

▼ Would have been better to give intercepts in point notation in part (a). Incorrect inverse in part (c); did not give the reverse operations in the opposite order.

The answer above could have achieved 4/6 marks.

Correct answer to part (c) should have been $f^{-1}(x) = \frac{x+3}{5}$

Graphs of functions

When the IB asks for the representation of a graph to be given there are two command terms that could be used, draw or sketch, which are defined as follows:

Draw: Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.

This should be done on graph paper. It is not often asked for.

Sketch: Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship and should include relevant features.

This should be done on the paper that you are working on, next to your working.

You could be asked to sketch a graph on paper or using technology.

Assessment tip

It is often a good idea to make a quick sketch, even when not required, as this can assist your thinking and show the examiner that you have the correct method.

Assessment tip

You should study the IB Glossary of Command Terms as you revise so that you fully understand what each one means and what each one requires of you.

The same is true for the IB Notation Sheet. It would be a pity not to be able to do a question because you did not understand what the notation meant. Also remember that you must use correct mathematical notation and not calculator notation.

Neither the Glossary of Command Terms nor the Notation Sheet are allowed in the exams, and so the information on them needs to be understood in advance.

On a paper that allows the use of technology you can just obtain it on your calculator and then copy it down. Use the domain and the range of the function to assist in choosing an appropriate "window" for the graph.

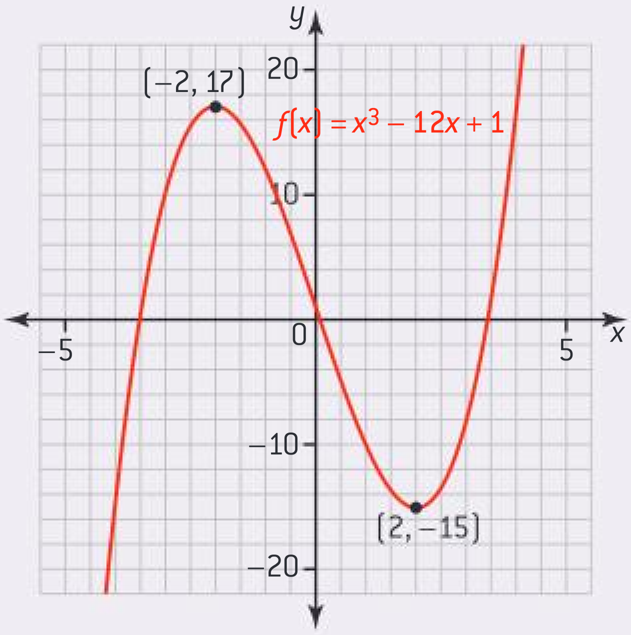
The calculator can find values for the intercepts on the axes, any maximums or minimums, and the y-value for any given x-value. It can also show the inverse function.

Example 2.2.6

Let a function be defined by $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 12x + 1$. Sketch the graph of $f(x)$, labelling any local maximum or minimum points.

Solution

Graphing the function on a calculator and using the calculator to find the maximums and minimums gives the following graph.



On a paper where technology is not allowed, it is worth considering any of the following ideas, if possible, to assist in sketching the graph.

- Are there any x-values where the function is undefined? What happens near these x-values? This will identify any vertical asymptotes.
- Find the intercepts with both axes. The y-axis intercept can usually be found.
- What is the range of the function?
- Find any local maximum or minimum points. When is the function (i) increasing (ii) decreasing? Find any points of inflexion. When is the function (i) concave up (ii) concave down? Calculus is a good tool for this investigation.
- Think big, i.e., what happens as (i) $x \to \infty$ (ii) $x \to -\infty$? This will find any horizontal asymptotes.
- Consider any other properties that the function might have, like is it odd or even, or periodic?
- Check that anything specifically mentioned in the question has been addressed.

Link to Differentiation, Chapter 5

Depending on the complexity of the function it might not be possible to do all of the above investigations. The order in which they are done does not matter. The different investigations should agree with each other and complement each other. If a mistake is made then there is often a contradiction between different pieces of information making it impossible to sketch the graph, rather than the wrong graph being sketched.

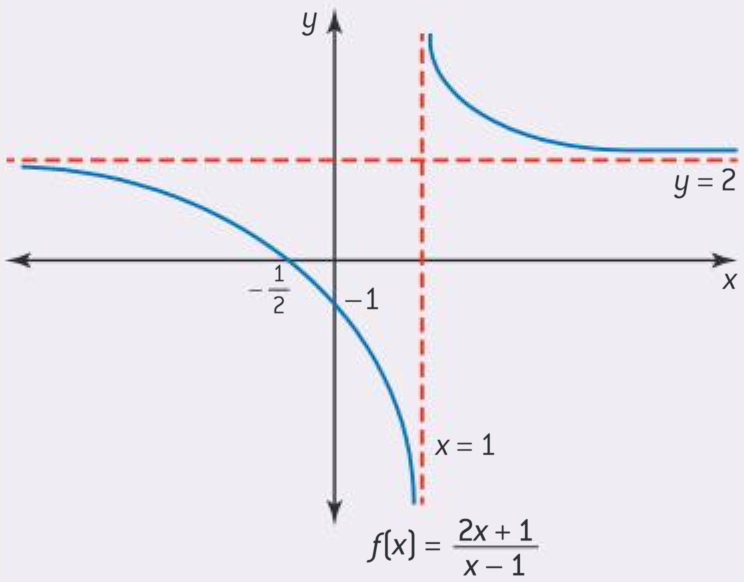
Example 2.2.7

Let a function be defined by
$f: \{x \in \mathbb{R} \mid x \neq 1\} \rightarrow \{y \in \mathbb{R} \mid y \neq 2\},\ f(x) = \frac{2x+1}{x-1}$
Sketch the graph of $f(x)$. State the equations of any vertical or horizontal asymptotes.

Solution

$f(x)$ is undefined for $x = 1$ As $x \rightarrow 1^+, y \rightarrow +\infty$ As $x \rightarrow 1^-, y \rightarrow -\infty$
So $x = 1$ is a vertical asymptote.
$x = 0 \Rightarrow y = -1$ y-axis intercept is $(0, -1)$
$y = 0 \Rightarrow x = -\frac{1}{2}$ x-axis intercept is $\left(-\frac{1}{2}, 0\right)$
As $x \rightarrow +\infty, y \simeq \frac{2H}{H}$, so $y \rightarrow 2$ As $x \rightarrow -\infty, y \simeq \frac{-2H}{-H}$, so $y \rightarrow 2$
(letting H stand for a huge number and neglecting +1 and − 1 as extremely small in comparison)
So $y = 2$ is a horizontal asymptote.

This would be sufficient to sketch the graph, but confirming with differentiation:
$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{2(x-1) - 1(2x+1)}{(x-1)^2} = \frac{-3}{(x-1)^2}$ This is always negative showing that the graph is always decreasing and therefore cannot have any max or mins. It also shows that $\frac{\mathrm{d}f}{\mathrm{d}x} \rightarrow 0$ as $x \rightarrow \pm\infty$.



Another way of sketching a graph without the use of a calculator is to relate it, via transformations, to a known graph.

Link to Transformations SL 2.11

This technique is often useful with trig functions.

Link to Composite trig functions SL 3.7

Example 2.2.8

Let a function be defined by $f: \mathbb{R} \rightarrow \mathbb{R},\ f(x) = -x^2 - 8x - 7$.
Sketch the graph of $f(x)$.
State the equation of the axis of symmetry.
Give the coordinates of all intercepts and any maximums or minimums.

Solution

$-x^2 - 8x - 7 = -(x^2 + 8x) - 7 = -(x+4)^2 + 9$

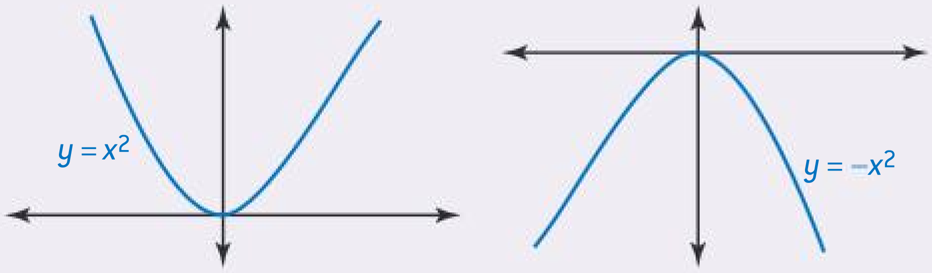
So the graph of $y = x^2$ can be taken, reflected to give $y = -x^2$ and then translated by the vector $\begin{pmatrix} -4 \\ 9 \end{pmatrix}$ to give $y = -(x+4)^2 + 9$.

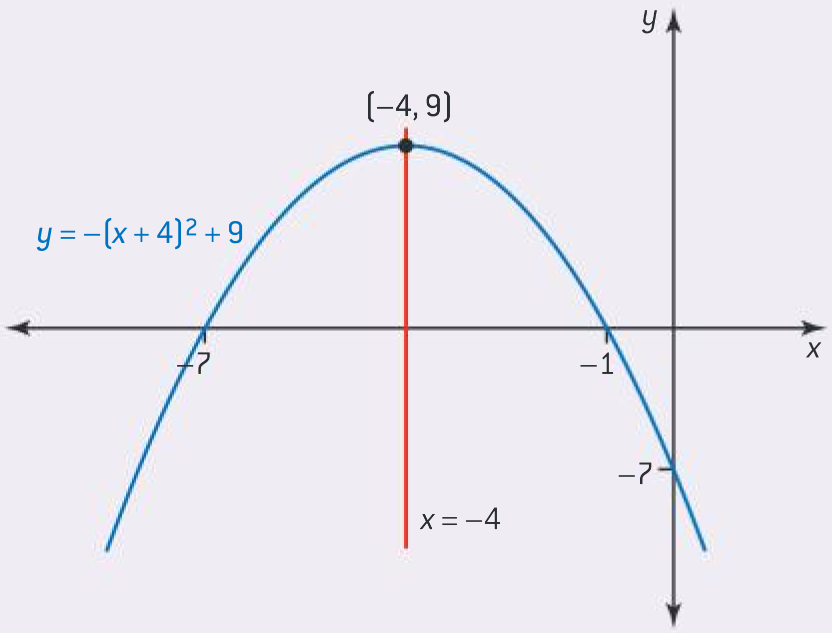
From this nested form it is clear that there is a maximum at $(-4, 9)$. Thus, the axis of symmetry is $x = -4$.

$x = 0 \Rightarrow y = -7$ $\quad$ y-axis intercept is $(0, -7)$

$y = 0 \Rightarrow (x+4)^2 = 9 \Rightarrow x+4 = \pm 3 \Rightarrow x = -7$ or $x = -1$

x-axis intercepts are $(-7, 0)$ and $(-1, 0)$



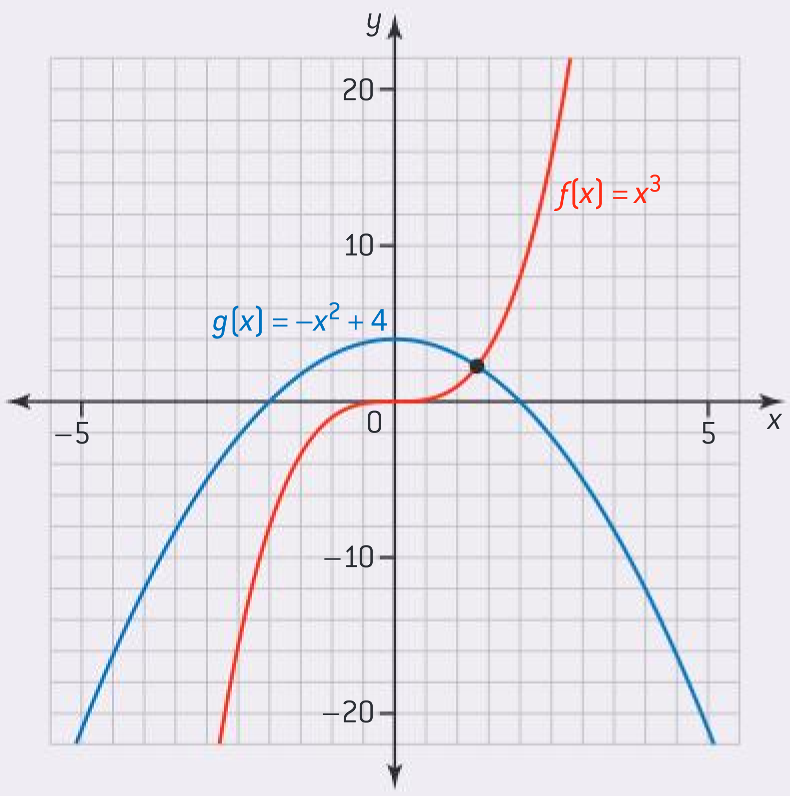


If a calculator is used to sketch the curves corresponding to functions, then it can also tell you the intersection points.

Example 2.2.9

Sketch the graphs of the functions $f(x) = x^3$ and $g(x) = -x^2 + 4$. Write down the point of intersection.

Solution



Using a calculator, the intersection point is (1.31, 2.27) (3 sf).

Given two functions $f(x)$ and $g(x)$, to sketch the graph of their sum with the calculator, then you would introduce a new function $h(x) = f(x) + g(x)$. You could type this function into the calculator and produce the graph. The same would apply if you wanted the difference but this time taking $h(x) = f(x) - g(x)$.

Considering the function that is the difference of two other functions and finding its zeros could be used as an alternative to finding the intersection of the two functions.

Example 2.2.10

A fast-growing pine tree has been planted in a garden.

Its height h in metres is modelled by $h(t) = \frac{10}{1+4e^{-t}}$, where t is time, measured in years after planting.

(a) Find the height of the tree when it was first planted.

(b) Find the height that the tree approaches as time goes on.

If the tree reaches 8 metres high then the neighbours will start to complain.

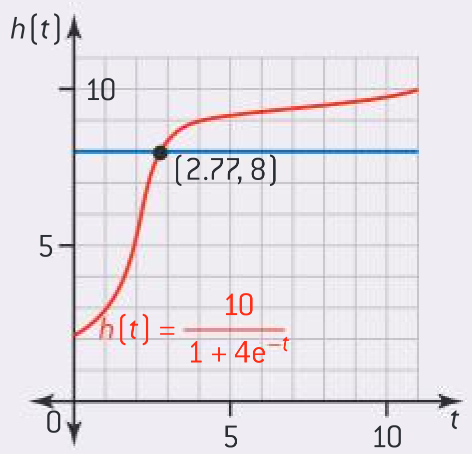
(c) Find how many years it will take before the neighbours start to complain.

Solution

(a) $h(0) = 2\,\text{m}$

(b) As $t \to \infty, e^{-t} \to 0, h \to 10\,\text{m}$

(c) Sketch the graph of $h(t) = \frac{10}{1+e^{-t}}$ and the line $h = 8$, and find the intersection. This is when $t = 2.77$ years (3 sf)



Assessment tip

In questions similar to Example 2.2.10, when considering a suitable "window" for the graph realize that $t \geq 0$ as it represents time. Parts (a) and (b) also assist with the vertical part of the "window".

Composite functions

If $g(x)$ is a function from $A \to B$ and $f(x)$ is a function from $C \to D$, then provided that the range of $g(x)$ is a subset of C, a new composite function can be formed. It will have the notation $(f \circ g)(x)$ and is defined by $(f \circ g)(x) = f(g(x))$.

In many cases the functions are from $\mathbb{R} \to \mathbb{R}$. You have already met composite functions many times; even the simple linear function $y(x) = 2x + 7$ can be thought of as $y(x) = (f \circ g)(x)$ where $g(x) = 2x$ and $f(x) = x + 7$.

Note

With $(f \circ g)(x)$ it is the function g that is the closest to x and this is the function that is applied first.

Example 2.2.11

Let $f(x)=4x$, $g(x)=x+2$ and $h(x)=x^3$. Find expressions for:

(a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$ (c) $(f\circ f)(x)$

(d) $(f\circ g)^{-1}(x)$ (e) $(f^{-1}\circ g^{-1})(x)$ (f) $(g^{-1}\circ f^{-1})(x)$

(g) $(f\circ g\circ h)(x)$

Solution

(a) $f(g(x))=f(x+2)=4x+8$ (b) $g(f(x))=g(4x)=4x+2$

(c) $f(f(x))=f(4x)=16x$

(d) $y=4x+8$ interchanging x and y, $x=4y+8\Rightarrow y=\frac{x}{4}-2$,

$(f\circ g)^{-1}(x)=\frac{x}{4}-2$

(e) $f^{-1}(g^{-1}(x))=f^{-1}(x-2)=\frac{x}{4}-\frac{1}{2}$

(f) $g^{-1}(f^{-1}(x))=g^{-1}\left(\frac{x}{4}\right)=\frac{x}{4}-2$

(g) $f(g(h(x)))=f(g(x^3))=f(x^3+2)=4x^3+8$

Note

Parts (a) and (b) show that, in general, $f\circ g\neq g\circ f$.

Note

Order is very important!
Consider the following:
$x+y=y+x, x-y\neq y-x, xy=yx,$
$(x+y)^2\neq x^2+y^2, x,y\in\mathbb{R}.$

Properties of composite and inverse functions

Let the Identity function that leaves every element in the domain completely unchanged be represented by $I(x)$, so $I(x)=x$ for all x.

$f\circ f^{-1}=I$ $f^{-1}\circ f=I$ $f\circ I=f$ $I\circ f=f$

$f\circ g\neq g\circ f$

$(f\circ g)^{-1}=g^{-1}\circ f^{-1}$ this can be thought of as the "socks and shoes" rule; you have to do the opposite operations in the opposite order.

If functions $f(x)$ and $g(x)$ satisfy $f\circ g=I$ and $g\circ f=I$ then $f^{-1}(x)=g(x)$.

For example, if $f(x)=4x+5$ and $g(x)=\frac{x-5}{4}$ then

$(f\circ g)(x)=f\left(\frac{x-5}{4}\right)=x-5+5=x$ and $(g\circ f)(x)=g(4x+5)=\frac{4x+5-5}{4}=x$

So $f^{-1}(x)=g(x)$ and $g^{-1}(x)=f(x)$.

Note

Recall that to prove that a statement is always true you have to produce a deductive proof, but to prove that a statement is not always true you only need to produce one counterexample.

Link to Deductive proof SL 1.6 and Counterexamples AHL 1.15

Link to the box method for finding inverse functions (page 42)

Note

Here is a slightly bizarre example to show that $f\circ g=g\circ f$ cannot be true in general. Let g be a function from $L\rightarrow E$, where E is a set of elephants and L is a set of lions. Let f be a function from $E\rightarrow M$, where M is a set of monkeys. Then $f\circ g$ will exist as a function and will send lions to monkeys. However, consider $(g\circ f)(x)=g(f(x))$.

Then x must be an elephant and so $f(x)$ is a monkey. But g has no idea what to do with a monkey as it is not in its domain; only lions are in the domain of g. So $g\circ f$ does not even exist.

2.3 QUADRATIC, RATIONAL, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

You must know:

- ✔ the quadratic formula
- ✔ the equation of the axis of symmetry
- ✔ the definition of the discriminant
- ✔ that the reciprocal function $f(x)=\frac{1}{x}$ is self-inverse
- ✔ what a rational function is
- ✔ what an exponential function is
- ✔ what a logarithmic function is.

You should be able to:

- ✔ solve quadratic equations and inequalities
- ✔ use the discriminant to determine the nature of the roots
- ✔ sketch quadratics, knowing which way up they are
- ✔ find the asymptotes of rational functions and sketch their graphs
- ✔ sketch the graph of an exponential function
- ✔ sketch the graph of a logarithmic function
- ✔ solve equations both graphically and analytically
- ✔ relate skills to real-life situations.

Quadratic functions

Quadratic functions are polynomials of degree two. They have the form $y=ax^2+bx+c$, where $a, b, c \in \mathbb{R}, a \neq 0$.

Properties of quadratic functions

If $a>0$, the graph is concave up and has a minimum.

If $a<0$, the graph is concave down and has a maximum.

The intercept on the y-axis is $(0, c)$.

The quadratic has an axis of symmetry given by the equation $x=-\frac{b}{2a}$

The quadratic equation $ax^2+bx+c=0$ has solutions given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

b^2-4ac is denoted by Δ and is called the discriminant. It discriminates between the 3 cases of how many solutions (roots) the quadratic equation has.

If $\Delta>0$ then the equation has two distinct real roots.

If $\Delta<0$ then the equation has no real roots (if working in $\mathbb{C}$ there will be a pair of conjugate roots).

If $\Delta=0$ then the equation has one real root. Sometimes called a repeated root, it will be a maximum or a minimum.

The sum of the roots is equal to $-\frac{b}{a}$

The product of the roots is equal to $\frac{c}{a}$

The formula for the max/min point is given by $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

If the quadratic can be factorized as $y=a(x-p)(x-q)$ then the x-axis intercepts are $(p, 0)$ and $(q, 0)$.

Assessment tip

On papers where technology is not allowed the quadratic equations given will often factorize. It is sometimes quicker to see if this is the case, before deciding to use the quadratic formula.

Note

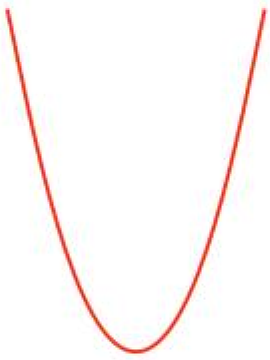
Use the quadratic formula in the case when $\Delta < 0$ to obtain the pair of complex conjugates if the question clearly states that you are working in the set of complex numbers, $\mathbb{C}$ (the variable is likely to be called z), otherwise just state that there are no real roots.

Link to Kinematics SL 5.9

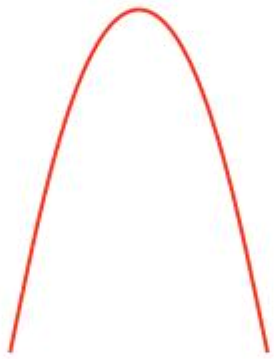
If the quadratic is written in the nested form $y = a(x-h)^2 + k$ then the vertex is the point (h, k).

Only the formulae for the axis of symmetry, the solutions of the quadratic equation and the discriminant are given in the formula book.

The basic shape of a quadratic is:



or



Quadratic functions frequently appear in all branches of science and mathematics.

Example 2.3.1

Delme, a giant, throws a stone vertically upwards with an initial speed of $30\,\mathrm{m\,s^{-1}}$. The height of the stone, measured in metres above his head, is given by $h(t) = 30t - 5t^2$ where t is time measured in seconds.

(a) Find the greatest height that the stone reaches.

(b) Find the length of time for which the stone will be over 25 m high.

(c) Find the time when the stone returns to hit Delme on the head.

Solution

(a) Axis of symmetry is $t = \frac{-30}{-10} = 3$ $\quad t = 3 \Rightarrow h = 45$

So greatest height is 45 m

(b) $h(t) = 30t - 5t^2 = 25 \Rightarrow t^2 - 6t + 5 = 0 \Rightarrow (t-5)(t-1) = 0$

So at $t = 1$ or 5 its height is 25 m and between these times the stone is above 25 m so the time duration is $5 - 1 = 4$ s

(c) By symmetry (or solving) h is next zero when $t = 6$ s

If a quadratic function has two distinct roots, then the axis of symmetry must be halfway between them. Letting the two roots be $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$ then $\alpha + \beta = -\frac{b}{a}$, confirming the formula for the sum of the roots. The axis of symmetry will be $x = \frac{\alpha+\beta}{2} = -\frac{b}{2a}$, confirming this formula as well.

The formula for the axis of symmetry is still this, even when there are not two distinct roots, but would have to be shown by a different method, for example, by differentiation or putting into nested form by completing the square.

Example 2.3.2

(a) Find the exact values of the roots of the following quadratic equations for $x \in \mathbb{R}$.

(i) $x^2+5x-14=0$ (ii) $x^2+9x+18=0$

(iii) $x^2+6x+4=0$ (iv) $x^2+8x+16=0$

(v) $x^2+5x+10=0$

(b) Find the exact values of the roots of the following quadratic equations for $z \in \mathbb{C}$.

(i) $z^2-2z+10=0$ (ii) $z^2+5z+10=0$

Solution

(a) (i) $x^2+5x-14=(x+7)(x-2)=0 \Rightarrow x=7$ or $x=2$

(ii) $x^2+9x+18=(x+6)(x+3)=0 \Rightarrow x=-6$ or $x=-3$

(iii) $x^2+6x+4=0 \Rightarrow x=\frac{-6\pm\sqrt{36-16}}{2}=\frac{-6\pm\sqrt{20}}{2}=-3\pm\sqrt{5}$

(iv) $x^2+8x+16=(x+4)^2=0 \Rightarrow x=-4$

(v) $x^2+5x+10=0$ $\Delta=25-40=-15$, so no real roots

(b) (i) $z^2-2z+10=0 \Rightarrow z=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm\sqrt{-36}}{2}=1\pm3\mathrm{i}$

(ii) $z^2+5z+10=0 \Rightarrow z=\frac{-5\pm\sqrt{25-40}}{2}=\frac{-5\pm\sqrt{-15}}{2}=-\frac{5}{2}\pm\frac{\sqrt{15}}{2}\mathrm{i}$

Assessment tip

If there are a variety of different types of answers to a problem and an exam question has many parts, then this might suggest that you will have to investigate different cases.

SAMPLE STUDENT ANSWER

Given that $x^2+kx+9\geq 0$ for all $x \in \mathbb{R}$, find the values that the constant k could take.

$\Delta=k^2-36\geq 0$ $k^2\geq 36$ $k\geq 6$

▲ The student had the correct idea of looking at the discriminant.

▼ The student had the discriminant inequality the wrong way around and also did not solve the inequality correctly. Needed to put in more words of explanation.

The answer above could have achieved 2/7 marks

Should have said the function is a concave up quadratic always greater or equal to zero, so it cannot cross the x-axis. Hence it must have either one or zero real roots, so

$\Delta=k^2-36\leq 0$ $k^2\leq 36 \Rightarrow -6\leq k\leq 6$

An alternative solution would be to use the fact that the y-coordinate of the minimum point must be ≥ 0.

Example 2.3.3

(a) Let the quadratic equation $w^2+2w+7=0$ have roots of α and β (which you are instructed not to find). Evaluate the following:

(i) $(\alpha+\beta)^2$ (ii) $\alpha^2+\beta^2$ (iii) $(\alpha-\beta)^2$

(b) Let the quadratic equation $aw^2+bw+c=0$, where $a, b, c \in \mathbb{R}$ $a\neq 0, c\neq 0$, have roots of α and β. Find a quadratic equation that has roots of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

(a) Using sum and product of roots $\alpha+\beta=-2$, $\alpha\beta=7$

Assessment tip

Some IB questions suggest or require that what you have done in one part of the question is used in the next part. See how this was done in Example 2.3.3 (a).

(i) $(\alpha+\beta)^2 = (-2)^2 = 4$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 4 - 14 = -10$

(iii) $(\alpha-\beta)^2 = \alpha^2+\beta^2 - 2\alpha\beta = -10 - 14 = -24$

(b) Method 1. Using sum and product of roots $\alpha+\beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$

$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-b}{c}$ $\qquad$ $\frac{1}{\alpha}\times\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c}$

So an equation with roots of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ would be $t^2 + \frac{b}{c}t + \frac{a}{c} = 0$

Which can be written as $ct^2 + bt + a = 0$

Method 2. Let $w = \frac{1}{t}$ then $a\left(\frac{1}{t}\right)^2 + b\left(\frac{1}{t}\right) + c = 0$ will have roots of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

This equation is equivalent to $a + bt + ct^2 = 0$ again giving $ct^2 + bt + a = 0$.

Assessment tip

If the command term "hence" is used, then you **must** use the last part to continue. If the command terms "hence or otherwise" are used, then you may use the last part to continue answering the question, but you do not have to. It is often safer to use "hence" rather than "otherwise". Even if these terms are not used, think to yourself, "Could what I have just done be useful with the next part?" With longer style questions you are often being led through the question. Try to see the path that you are being guided along and always ask yourself, "Why am I being asked to do that; is it going to be useful later?"

Note

When solving quadratic inequalities without a calculator, it is often good practice to solve the quadratic equation first and then consider the appropriate graph to find the inequality solution.

Example 2.3.4

Solve the inequality $x^2 - 20 > x$ for $x \in \mathbb{R}$.

Solution

(Note: If this had been on a paper that allows the use of technology, the graphs of $y = x^2 - 20$ and $y = x$ could have been sketched and the intersections found; this helps you decide when the first graph is "above" the second graph.)

$x^2 - 20 > x \Leftrightarrow x^2 - x - 20 > 0$

$x^2 - x - 20 = 0 \Rightarrow (x+4)(x-5) = 0 \Rightarrow x = -4$ or $x = 5$

We have a quadratic curve that is concave up and meets the x-axis at $x = -4$ and $x = 5$ so the solution to the inequality is $x < -4$ or $x > 5$. An alternative way of expressing this is using interval notation $]-\infty, -4[\cup]5, \infty[$

Note

If, when solving some problem (especially on a paper where technology is not allowed), the working leads to a quadratic equation that factorizes nicely, this is often an indication that you are doing the question as was intended. This is demonstrated in Example 2.3.5

Example 2.3.5

Working in Cartesian form, solve the equation $z^2 = 5 + 12\text{i}$, $z \in \mathbb{C}$.

Solution

Let $z = a + b\text{i}$, where $a, b \in \mathbb{R}$. Then $z^2 = a^2 - b^2 + 2ab\text{i}$

Equating real and imaginary parts: $a^2 - b^2 = 5$, $2ab = 12$

Eliminating a variable $b = \frac{6}{a}$, $a^2 - \frac{36}{a^2} = 5$

$a^4 - 5a^2 - 36 = 0$ This is a disguised quadratic in a^2 (note this will always be the case with this method)

$(a^2 + 4)(a^2 - 9) = 0$

We reject $a^2 = -4$ since $a \in \mathbb{R}$ so $a = \pm 3$

$a = 3 \Rightarrow b = 2$, $a = -3 \Rightarrow b = -2$, so solutions are $z = 3 + 2\text{i}$ or $-3 - 2\text{i}$ (as expected, one solution is the other one multiplied by -1)

Link to Roots of Complex numbers AHL 1.14

Rational functions

A rational function is of the form:

$$f(x) = \frac{ax + b}{cx + d}$$

where $a, b, c, d \in \mathbb{R}$, $c, d \neq 0$, and $x \neq -\frac{d}{c}$

The domain is $\left\{x \in \mathbb{R} \mid x \neq -\frac{d}{c}\right\}$ The range is $\left\{y \in \mathbb{R} \mid y \neq \frac{a}{c}\right\}$

The graph will have:

a vertical asymptote at $x = -\frac{d}{c}$

a horizontal asymptote at $y = \frac{a}{c}$

The intercept on the y-axis will be $\left(0, \frac{b}{d}\right)$

The intercept on the x-axis will be $\left(-\frac{b}{a}, 0\right)$

However, if $a = 0$ then the graph will not intercept the x-axis.

Assessment tip

If you have no idea what to do when asked for the equation of a vertical asymptote, look at the definition of the function given. Wherever you see a statement of the form $x \neq something$ replace it with $x = something$ for the equation of the vertical asymptote.

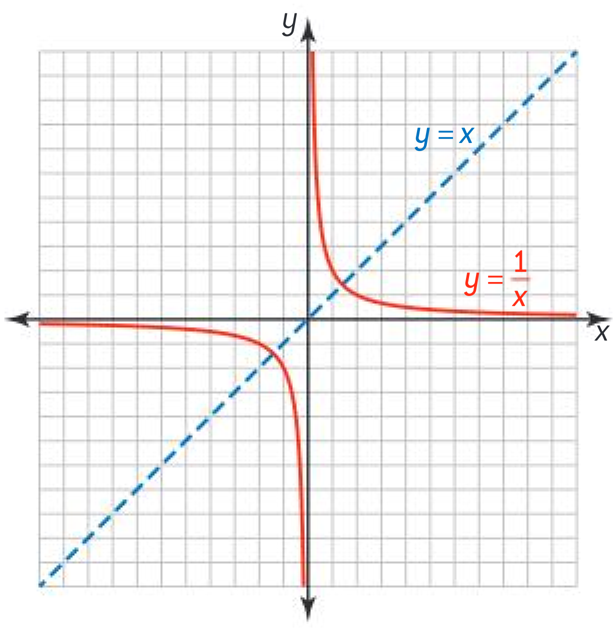
One of the simplest rational functions is the reciprocal function $f(x) = \frac{1}{x}$.

Its domain is $\{x \in \mathbb{R} \mid x \neq 0\}$ and its range is $\{y \in \mathbb{R} \mid y \neq 0\}$.

The graph will have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$, the two axes.

This function is its own inverse, since taking the reciprocal of the reciprocal gets you back to the original function. This explains the symmetry of the graph about $y = x$.

When sketching rational functions in general it is a good idea to start by finding the two asymptotes. Even if a calculator is allowed, this assists in choosing a "window" and ensuring that a branch of the graph is not missed.



Example 2.3.6

Let $f(x)=\frac{-2x+3}{x+4}$, $x\neq -4$.

(a) State (i) the maximum possible domain for this function and (ii) the range in this case.

(b) Find the equation of (i) the vertical asymptote and (ii) the horizontal asymptote.

(c) Find the intercept with (i) the x-axis (ii) the y-axis.

(d) Sketch the graph of the function.

(e) Hence solve the inequality $f(x) < 7$.

Solution

(a) (i) maximum possible domain $\{x \in \mathbb{R} \mid x \neq -4\}$
(ii) range $\{y \in \mathbb{R} \mid y \neq -2\}$

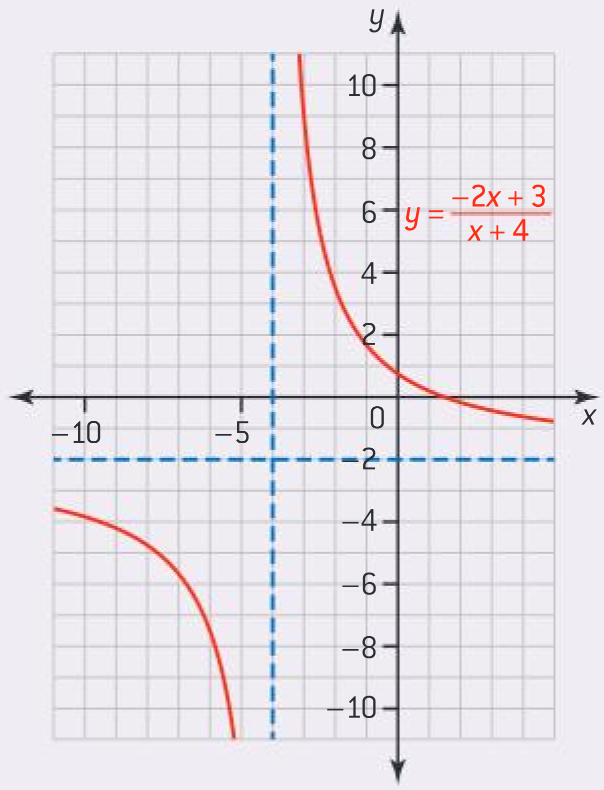
(b) (i) $x=-4$

(ii) $y=-2$

(c) (i) $\left(\frac{3}{2}, 0\right)$

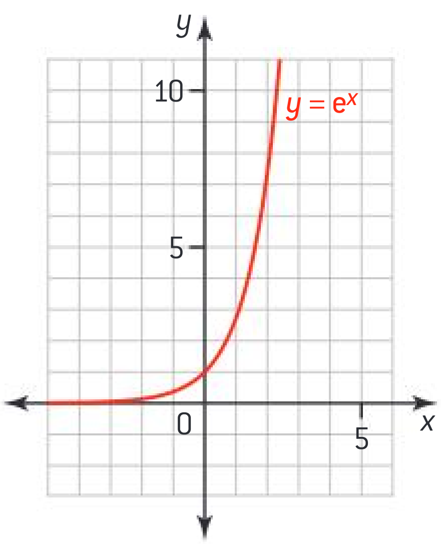
(ii) $\left(0, \frac{3}{4}\right)$

(d) $f(x)=\frac{-2x+3}{x+4}$, $x\neq -4$



(e) $y=7 \Rightarrow x=\frac{-25}{9}$ considering both branches, solution to $f(x)<7$ is $]-\infty,-4[\,\cup\,]\frac{-25}{9},\infty[$

Exponential functions



The exponential function $f(x)=e^x$ has a domain of $\mathbb{R}$ and a range of $\{y \in \mathbb{R} \mid y > 0\}$.

The intercept on the y-axis is (0, 1). The function is always increasing and $y=0$ is a horizontal asymptote. Its graph is as shown.

The function $y=e^x$ has the property that at any point on the graph the height y is the same as the gradient $\frac{dy}{dx}$ at that point. It is the only function that has this property.

Note

The fact that e^x is never zero or smaller than zero can be useful when solving equations after they have been factorized.

Example 2.3.7

Solve the equation $e^{3x}+3e^{2x}-10e^x=10$ for $x\in\mathbb{R}$, giving your answer in exact form.

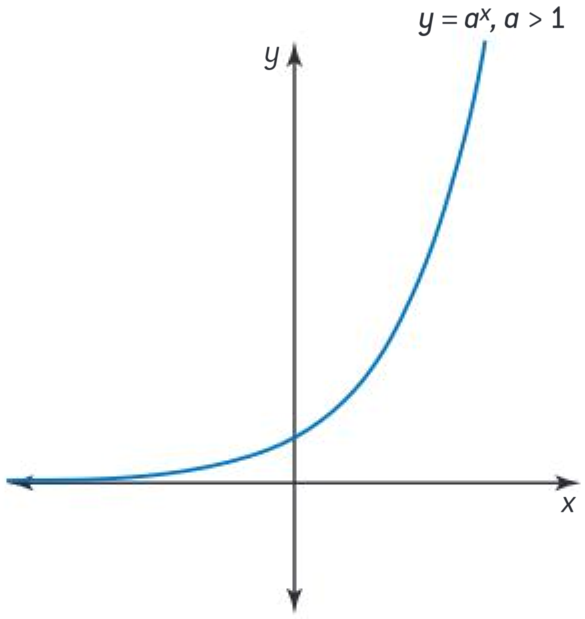
Solution

$e^{3x}+3e^{2x}-10e^x=0 \Rightarrow (e^x)^3+3(e^x)^2-10(e^x)=0 \Rightarrow (e^x)(e^x-2)(e^x+5)=0$, upon factorizing the disguised quadratic. So only possibility is $e^x=2$, giving exact solution as $x=\ln 2$.

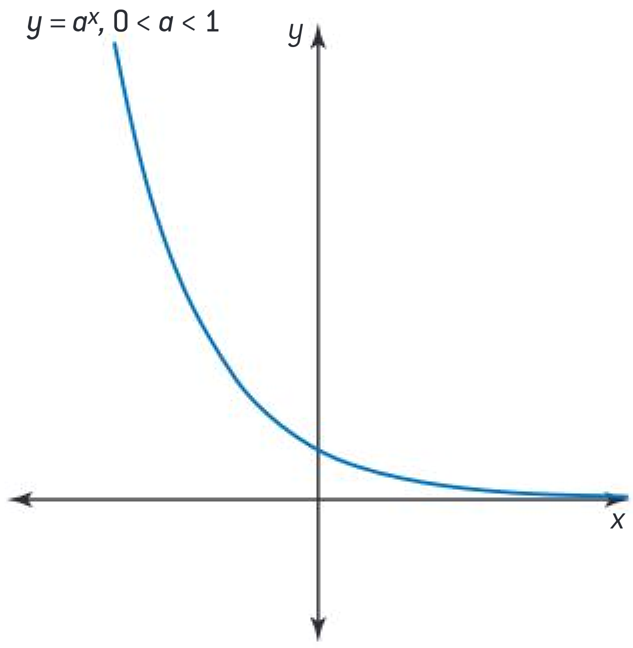
A more general exponential function is $f(x) = a^x$, where $a \in \mathbb{R}$, $a > 0$, $a \neq 1$

It has a domain of $\mathbb{R}$ and a range of $\{y \in \mathbb{R} \mid y > 0\}$. The intercept on the y-axis is $(0, 1)$ and $y = 0$ is the horizontal asymptote.

If $a > 1$ then the function is always increasing (it could model exponential population growth) and its graph is as shown:



If $0 < a < 1$ then the function is always decreasing (it could model exponential decay) and its graph is as shown:



The function $f(x) = a^x$, where $a \in \mathbb{R}$, $a > 0$, $a \neq 1$ can also be written in terms of e.

Let $\ln a = k$. Then $a = \mathrm{e}^k$. So $a^x = (\mathrm{e}^k)^x = \mathrm{e}^{kx} = \mathrm{e}^{x\ln a}$.

So the function $f(x) = a^x$ is equivalent to the function $f(x) = \mathrm{e}^{x\ln a}$.

This is given in the formula book and explains why the use of e is so common.

We often have exponential models of the form $y = A\mathrm{e}^{cx}$.

Link to Laws of exponents SL 1.7

Example 2.3.8

It is known that the population of krill in the Southern Ocean increases during the summer months. Anna, an oceanographer, takes readings every week to determine the number of krill in a particular bay. She believes that the number of krill can be modelled by $n(t) = A\mathrm{e}^{kt}$, where $n \times 10^9$ is the number of krill and t is the number of weeks after the first reading. Anna's supervisor Geraint believes that the model should be $n(t) = mt + c$.

(a) Using the information that for $t = 0$, $n = 14$ and for $t = 24$, $n = 92$

(i) find the constants A and k in Anna's model

(ii) find the constants m and c in Geraint's model.

(b) Using Anna's model, find the number of weeks it is predicted to take for the number of krill to be twice what it was when the first reading was taken. Give your answer accurate to 1 dp.

(c) Given that for $t = 12$, $n = 40$, state which model you consider to be the better one.

Solution

(a) (i) $t = 0,\ n = 14 \Rightarrow A = 14$

$$92 = 14e^{24k} \Rightarrow k = \frac{1}{24}\ln\frac{92}{14} = 0.078447\ldots$$

(ii) $t = 0,\ n = 14 \Rightarrow c = 14$

$$92 = 24m + 14 \Rightarrow m = 3.25$$

(b) Using $n(t) = 14e^{0.078447\ldots t}$, require $2 = e^{0.078447\ldots t}$

$$\Rightarrow t = \frac{\ln 2}{0.078447\ldots} = 8.8 \text{ (1 dp)}$$

(c) Using $n(t) = 14e^{0.078447\ldots t}$, $t = 12$ gives $n = 35.9$ (3 sf)

Using $n(t) = 14 + 3.25t$, $t = 12$ gives $n = 53$

So, conclude that Anna's is the better model.

Assessment tip

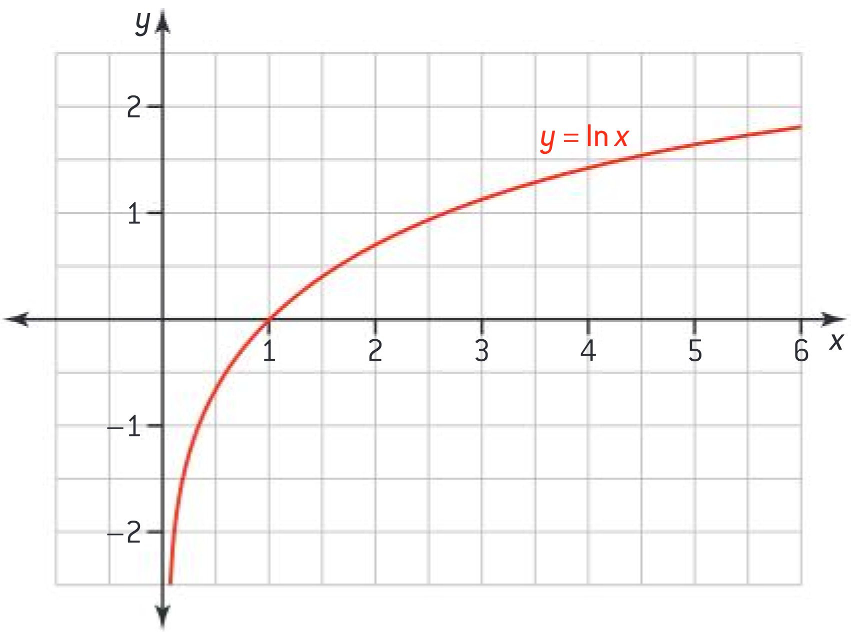
When using a numerical answer from one part of a question in subsequent parts of the question always use the more accurate, unrounded version rather than the rounded answer that you may have given in the first part.

This is to ensure that rounding errors do not creep into your work. This is what "..." indicates in Example 2.3.8. This notation is also used in IB mark schemes. When using the calculator, it has the capacity to use a previous answer (unrounded) in the next calculation and saves you having to type it in.

Logarithmic functions

A logarithmic function is of the form $f(x) = \log_a x$, where $a \in \mathbb{R}$, $a > 0$, $a \neq 1$.

Particular examples of this are with $a = e$ or $a = 10$, as these two logarithms are given by the calculator. The notation for $\log_e x$ is $\ln x$ and the notation for $\log_{10} x$ is $\log x$. The number a is called the base of the logarithm. It is more common to have $a > 1$ rather than $0 < a < 1$. The domain of this function is $\{x \in \mathbb{R} \mid x > 0\}$ and the range is $\mathbb{R}$. The y-axis is a vertical asymptote and the graph has an x-intercept at $(1, 0)$. The function $y(x) = \ln x$ is an increasing function and has a graph as shown.



Since by the change of base rule for logarithms (given in the formula book), $\log_a x = \frac{\ln x}{\ln a}$, the graph of $f(x) = \log_a x$ will be very similar to the graph of $y = \ln x$; it will just involve a vertical stretch. (When $0 < a < 1$ the transformation will also involve a reflection in the x-axis as $\ln a$ will be negative in this case.) This ability to change bases also explains why the most common logarithm used is $\ln x$.

Link to Laws of logarithms SL 1.7, Transformations SL2.11

Since we have the formula $\log_a(a^x) = x = a^{\log_a x}$ this shows that $\log_a x$ and a^x are inverse functions.

Example 2.3.9

A particular island is essentially a very tall mountain. The rainfall on the island increases the higher up the mountain you are. Jenny, a climate change scientist, has measured the rainfall at different heights. She believes that the amount of rain can be modelled by $r = ah^{1.5}$, $h > 0$, where r is the depth of rain each year in centimetres and h is the height in thousands of metres. Jenny wishes to plot the data she has gathered onto a graph and wants to obtain a straight line.

(a) State which of the following four choices of variables she should plot in order to obtain a straight line. Justify your answer.

(i) r against h (ii) r against $\ln h$

(iii) $\ln r$ against h (iv) $\ln r$ against $\ln h$

(b) For the choice made in part (a), state (i) the gradient of the straight line and (ii) how Jenny could obtain the value of the constant a from her straight line graph.

Solution

(a) $r = ah^{1.5} \Rightarrow \ln r = \ln a + 1.5 \ln h$, which represents a straight line. Jenny should plot $\ln r$ against $\ln h$.

(b) (i) The gradient of the straight line is 1.5

(ii) The intercept on the $\ln r$-axis will be $\ln a$. So Jenny should read this off and call it A, say, then $a = e^A$.

Solving equations, graphically and analytically

Any equation can be solved by using a calculator to draw its graph and then finding its zeros (roots). Calculators also have built in "solver" operations.

Assessment tip

It is often safer to use the graphical approach as that can ensure that solutions are not missed.

If the equation is a polynomial, then the calculator's polynomial equation solver is often the quickest way of obtaining the roots. The calculator also allows you to specify the form that you require for the answers.

There are equations that cannot be solved analytically and have to be solved with a calculator. Solving an equation analytically means not using the calculator but using your skill with mathematical techniques and operations. Solving analytically will lead to exact solutions whereas using a calculator may only give approximate decimal solutions.

Although an equation might have an analytic solution it might be quicker to solve it with the calculator.

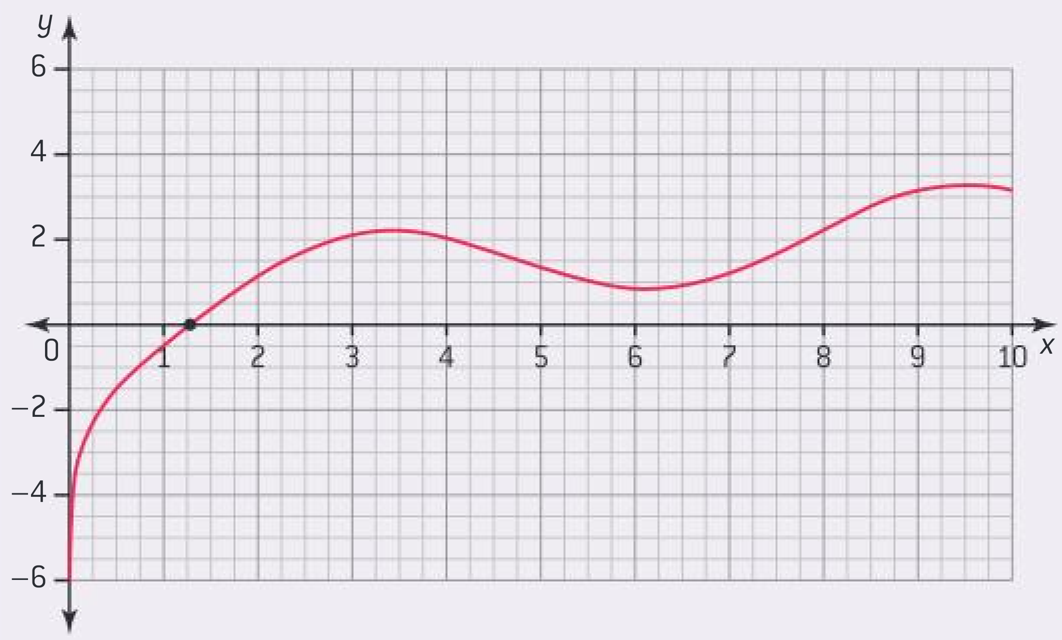
Assessment tip

On the front page of Paper 2 for SL and Paper 2 and Paper 3 for HL it will say "A graphical calculator is required for this paper". This is saying much more than "technology allowed". You are being examined on your calculator skills. You should be thinking, "When is the earliest I can pick up my calculator to assist me with this question?" There will be questions which can only be done with a calculator. A reluctance to use a calculator can waste a great deal of time in exams and often the question will expect you to use your calculator. When the calculator is used, there should still be words, diagrams and graphs used to explain the method and to thus gain method marks. Note that correct mathematical notation should be used and not calculator notation.

Example 2.3.10

Solve the equation $\ln x - \cos x = 0$ for $x > 0$.

Solution



Using a calculator to sketch the graph (as suggested by the Assessment tip in section 2.2, page 43) and finding its zero, $\ln x - \cos x = 0$ when $x = 1.30$ (3 sf)

Example 2.3.11

Find the exact solutions of the equation $e^x + 8e^{-x} - 6 = 0$ for $x \in \mathbb{R}$

Solution

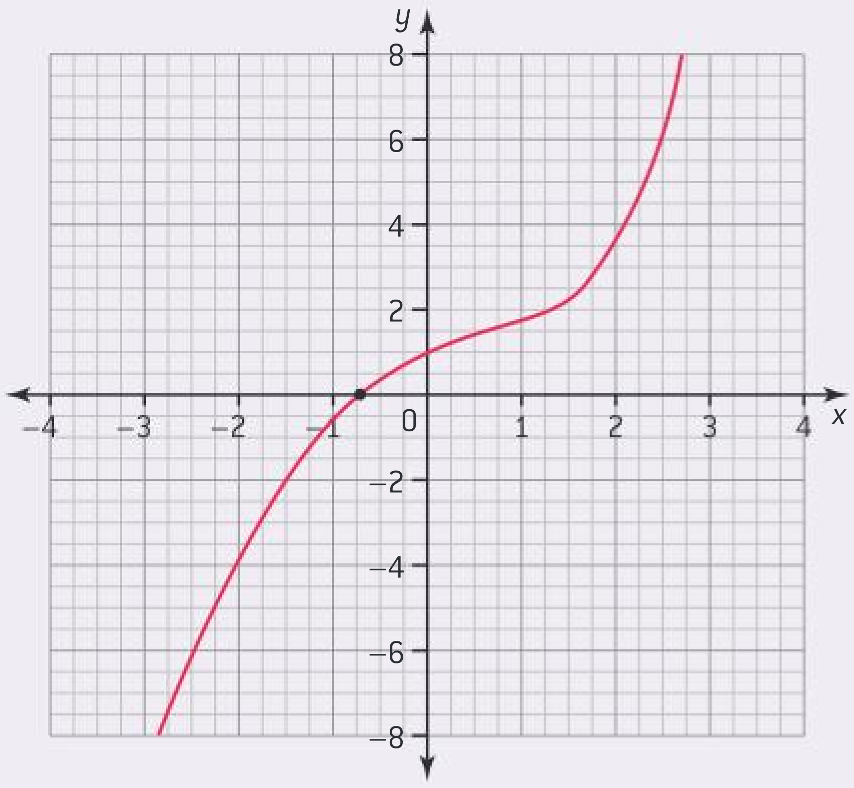
$e^x + 8e^{-x} - 6 = 0 \Rightarrow (e^x)^2 - 6e^x + 8 = 0 \Rightarrow (e^x - 4)(e^x - 2) = 0$

$e^x = 2$ or $4 \Rightarrow x = \ln 2$ or $x = \ln 4$

Example 2.3.12

Solve the equation $e^x - x^2 = 0$ for $x \in \mathbb{R}$

Solution



Using a calculator to sketch the graph (as suggested by the Assessment tip on page 43) and finding its zero, $e^x - x^2 = 0$ when $x = -0.703$ (3 sf).

SAMPLE STUDENT ANSWER

Solve the equation $e^{x^2-9} - 1 = 0$ for $x \in \mathbb{N}$

$e^{x^2-9} = 1 \qquad x^2 - 9 = 0 \qquad x^2 = 9 \qquad x = \pm 3$

▲ Good recognition of how to solve the problem and reducing it to a simple quadratic which was solved correctly.

▼ Did not reject $x = -3$ as should have since it said $x \in \mathbb{N}$.

This answer could have achieved 5/6 marks.

Assessment tip

Always read, very carefully, the description of the set that the question asks for the answers to be in. It is worth checking this again at the end of the question.

2.4 TRANSFORMATION OF GRAPHS

You must know:

- ✔ the meaning of the terms 'translation', 'reflection' and 'stretch/dilation'
- ✔ the importance of order.

You should be able to:

- ✔ use these transformations and composite transformations to transform graphs.

Link to Vectors AHL 3.12

Note

There are two types of linear transformations of graphs studied in this course: those that work on the "outside" of the function and affect the graph in a vertical direction, and those that work on the "inside" of the function and affect the graph in a horizontal direction.

Vertical transformations

Translations

$y = f(x) + b$ is a vertical translation of the graph of $y = f(x)$ by the amount b.

It could be considered as a translation by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$.

If $b > 0$ the graph is moved up ("lifted up") by b units, and if $b < 0$ the graph is moved down ("dropped down") by b units.

Reflections

$y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis. The graph is "turned over" vertically about the line $y = 0$.

Stretches/dilations

$y = pf(x),\ p > 0$ is a vertical "stretch"(dilation)of the graph of $y = f(x)$ with a scale factor of p. You could visualize the graph being drawn on elastic paper, stapled along the x-axis and then pulled from the top and the bottom with a "stretch" factor of p, if $p > 1$. If $0 < p < 1$ it could be described as a "compression" or "shrink" rather than a "stretch".

Note

The existence of $-a$ and $\frac{1}{q}$ in these descriptions shows that the transformation on the "inside" can be thought of as having the "opposite" effect to those on the "outside". Some examples:

$y = f(x) + 2$ represents a "lift up" by 2, whereas $y = f(x+2)$ represents a "move to the left" by 2.

$y = 2f(x)$ represents a vertical "stretch" by a factor of 2, whereas $y = f(2x)$ represents a horizontal "stretch" by a factor of $\frac{1}{2}$ (a "shrink").

Horizontal transformations

Translations

$y = f(x-a)$ is a horizontal translation of the graph of $y = f(x)$ by the amount a.

It could be considered as a translation by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

If $a > 0$ the graph is "shifted to the right" and if $a < 0$ the graph is "shifted to the left".

Reflections

$y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis. The graph is "turned from right to left" horizontally about the line $x = 0$.

Stretches/dilations

$y = f(qx), q > 0$ is a horizontal "stretch" (dilation) of the graph of $y = f(x)$ with a scale factor of $\frac{1}{q}$. You could visualize the graph being drawn on elastic paper, stapled along the y-axis and then pulled from the sides with a stretch factor of $\frac{1}{q}$ if $0 < q < 1$. If $q > 1$ it could be described as a "compression" or "shrink" rather than a "stretch".

These transformations can be combined to form composite transformations. For example:

- $y = -3f(x)$ represents a vertical "stretch" by a factor of 3 combined with a reflection in the x-axis.
- $y = f(3x) + 4$ represents a horizontal "stretch" by a factor of $\frac{1}{3}$ combined with a vertical "lift up" of 4.

In general, the order in which the transformations are carried out matters. For example, a vertical "stretch" by a factor of 2 followed by a vertical "lift" of 5 would be represented by $y = 2f(x) + 5$ and this is different to a vertical "lift" of 5 followed by vertical "stretch" by a factor of 2, which would be represented by $y = 2(f(x) + 5)$. The fact that $y = 2(f(x) + 5) = 2f(x) + 10$ shows how the transformations are different.

The order does not matter if one is a vertical transformation and the other is a horizontal transformation.

The order does not matter with a "stretch" and a reflection.

Link to Composite functions SL 2.5

Note

Again, those transformations on the "inside" are the "opposite" of those on the "outside" in terms of order. For example, $y = f\left(\frac{1}{3}(x+2)\right)$ represents a horizontal "stretch" by a factor of 3 followed by a "shift to the left" of 2. If we had wanted the transformations in the opposite order it would have been $y = f\left(\frac{1}{3}x + 2\right)$.

Example 2.4.1

The transformation $y = 4f(x) + 2$ is going to be applied to the graph of the function $f(x) = \ln x,\ x > 0$.

(a) Describe geometrically the two transformations that will achieve this, including the order in which they are performed.

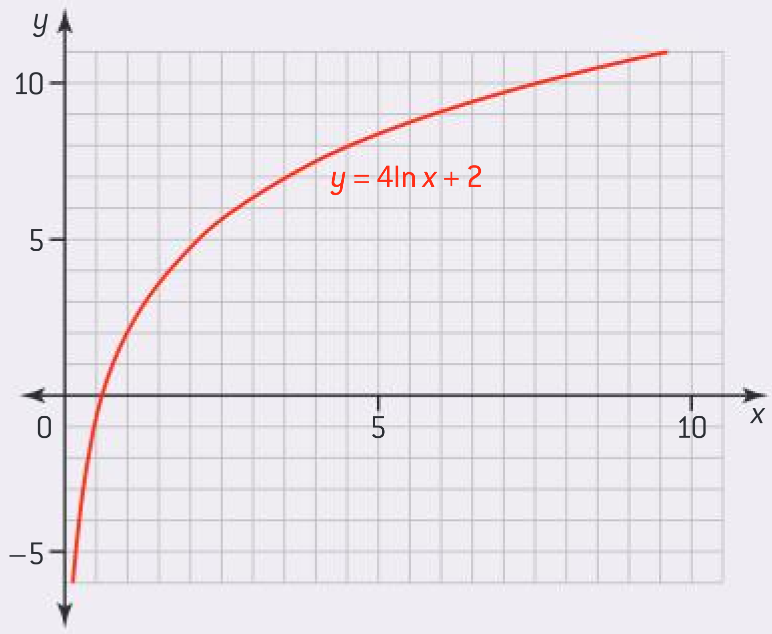
(b) Write down the explicit formula for y including its domain and sketch its graph.

Solution

(a) Vertical "stretch" by a factor of 4 followed by a vertical "lift up" by 2

(b) $y = 4\ln x + 2$

The domain is: $\{x \in \mathbb{R} \mid x > 0\}$



Assessment tip

If you are ever unsure about the order in which the transformations are done in, trace the movements of a convenient point.

SAMPLE STUDENT ANSWER

The transformation $y = f(x-3) + 2$ is going to be applied to the graph of the function $f(x) = e^x,\ x \in \mathbb{R}$.

(a) Describe geometrically the two transformations that will achieve this.

(b) Write down the explicit formula for y and state its range.

(a) Vertical "lift up" by 2 Horizontal "shift to left" by 3

(b) $y = e^{x-3} + 2$ Range $\{y \in \mathbb{R} \mid y > 2\}$

▲ Part (a) had the correct vertical transformation. Knew that the horizontal transformation was a "sideways shift". Had part (b) correct.

▼ Part (a) had the horizontal "shift" in the wrong direction.

The answer above could have achieved 5/6 marks.

Should have been horizontal "shift to right" by 3.

Example 2.4.2

(a) Express the function $y = 5x^2 + 10x + 9$ in the nested form $y = a(x-h)^2 + k$, where $a, h, k \in \mathbb{R}$.

(b) Hence describe geometrically, including the order, three transformations that would transform the graph of $f(x) = x^2$ into the graph of y.

(c) Hence write down the coordinates of the vertex of the graph of y.

Solution

(a) $y = 5x^2 + 10x + 9 = 5(x^2 + 2x) + 9 = 5(x+1)^2 + 4 = 5\left(x - (-1)\right)^2 + 4$

(b) A vertical "stretch" by a factor of 5, followed by a vertical "lift up" by 4 combined with a horizontal "shift" to the left by 1.

(c) Vertex is (–1, 4)

Example 2.4.3

Let $y = \frac{3x-2}{x+1}$, $x \neq -1$.

(a) Show that $y \equiv 3 + \frac{-5}{x+1}$

(b) Hence describe geometrically, including the order, four transformations that would transform the graph of $f(x) = \frac{1}{x}$ into the graph of y.

(c) Hence sketch the graph of y.

(d) Write down the equation of (i) the vertical asymptote and (ii) the horizontal asymptote.

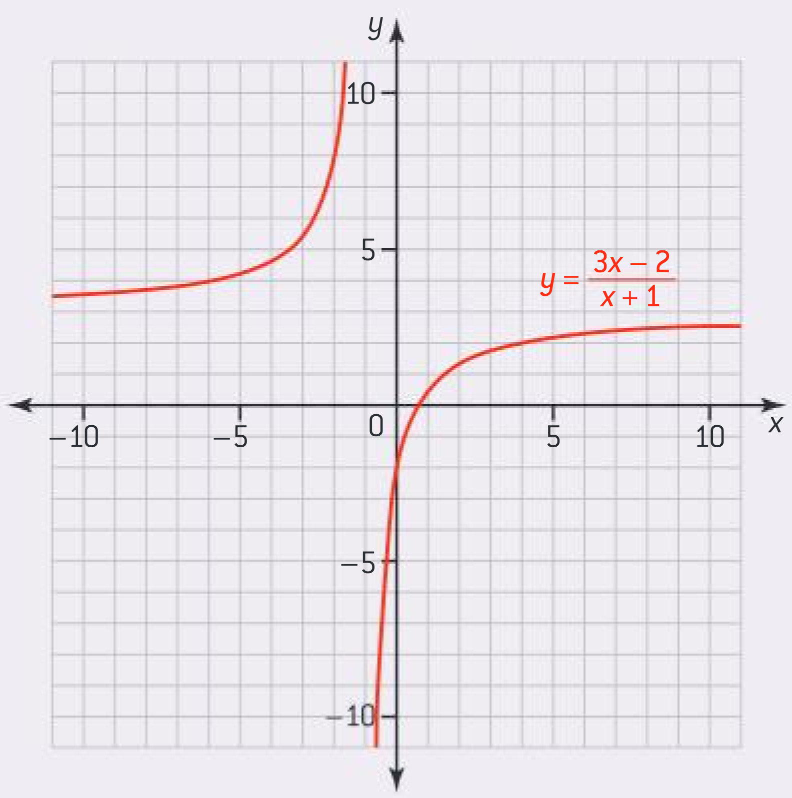
Solution

(a) $3 + \frac{-5}{x+1} \equiv \frac{3(x+1)-5}{x+1} \equiv \frac{3x-2}{x+1}$

(b) Horizontal "shift to left" by 1

Vertical reflection in the x-axis, vertical stretch by a factor of 5 followed by a "lift up" by 3

(c)



(d) (i) $x = -1$ (ii) $y = 3$

Note

These linear transformations are often used with trig functions such as $\sin x$ or $\cos x$.

A vertical "stretch" would affect the amplitude and a horizontal "stretch" would affect the period.

Link to Composite trig functions and transformations SL 3.7.

2.5 FUNCTIONS (HL)

You must know:

- ✔ the terms 'polynomial', 'zero', 'root' and 'factor'
- ✔ the sum and product of roots of a polynomial
- ✔ what odd and even functions are
- ✔ what a self-inverse function is.

You should be able to:

- ✔ apply the remainder and factor theorems
- ✔ investigate rational functions that are a linear function divided by a quadratic or vice versa
- ✔ find inverse functions (including considering a domain restriction)
- ✔ solve inequalities graphically and analytically
- ✔ sketch graphs of transformed functions using modulus, reciprocal, squaring and a linear function of the variable.

Polynomial functions

A polynomial function of degree n, $(n \in \mathbb{N})$ has the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \qquad a_i \in \mathbb{R}, a_n \neq 0.$$

In summation notation this is $\sum_{i=0}^{n} a_i x^i$

The domain is $\mathbb{R}$ and its graph is continuous and smooth.

Two polynomials are equal if and only if all the corresponding coefficients are equal.

When adding two polynomials the corresponding coefficients are added (subtraction is similar).

Two polynomials are multiplied by algebraically multiplying out the two expressions. If the polynomials are $p(x)$ and $q(x)$ then the degree of $p(x)q(x)$ equals the degree of $p(x)$ plus the degree of $q(x)$.

The division of two polynomials is similar to long division of integers in the way that it is laid out. With division, the degree of the remainder will be less than the degree of the denominator.

When solving a polynomial equation, the solutions are called zeros or roots of the equation; they are the x-intercepts on the graph of the polynomial.

A factor of a polynomial is another polynomial, of smaller degree, that divides the original exactly.

Example 2.5.1

Let $p(x) = x^3 + 3x^2 + 2x + 1$ and $q(x) = x^2 + 4x + 2$.

(a) Find (i) $p(x)+q(x)$ (ii) $p(x)-q(x)$ (iii) $p(x)q(x)$

(b) Divide $p(x)$ by $q(x)$ to obtain $\frac{p(x)}{q(x)} \equiv s(x) + \frac{r(x)}{q(x)}$ where $r(x)$ is the remainder.

Solution

(a) (i) $p(x)+q(x) = x^3 + 4x^2 + 6x + 3$ (ii) $p(x) - q(x) = x^3 + 2x^2 - 2x - 1$

(iii) $p(x)q(x) = x^5 + 7x^4 + 16x^3 + 15x^2 + 8x + 2$

(b)

$$\begin{array}{r} x-1 \\ x^2+4x+2 \overline{\smash{\big)}\, x^3+3x^2+2x+1} \\ \underline{x^3+4x^2+2x} \\ -x^2+0x+1 \\ \underline{-x^2-4x-2} \\ 4x+3 \end{array}$$

$$\frac{x^3+3x^2+2x+1}{x^2+4x+2} \equiv x-1+\frac{4x+3}{x^2+4x+2}$$

When solving a polynomial equation of degree n over the set of complex numbers it will have n solutions.

When solving it over the set of real numbers the number of solutions could be any natural number from 0 to n. Let $p(x)$ have k real roots. Since $p(x)$ has real coefficients, the remaining $n-k$ roots will come in complex pairs α and α^* where α is not real. Now $(x-\alpha)(x-\alpha^*) = x^2-(\alpha+\alpha^*)x+\alpha\alpha^*$ where both $\alpha+\alpha^*$ and $\alpha\alpha^*$ are real. This implies that over the set of real numbers $p(x)$ will factorize into k linear factors and $\frac{n-k}{2}$ irreducible quadratics. This also implies that if n is odd then there must be at least one linear factor and hence at least one real root.

The Remainder Theorem

If the polynomial $p(x)$ is divided by the linear polynomial $(x-c)$ then the remainder is $p(c)$. In particular, if $p(c) = 0$ then $(x-c)$ is a factor of $p(x)$; this is sometimes called the factor theorem. The remainder theorem allows you to find the remainder without having to do the division.

Example 2.5.2

(a) Find the remainder when $p(x) = x^4-2x^3+3x^2+5x-10$ is divided by $x-2$.

(b) When $q(x) = x^3+ax^2+bx+1$ is divided by $x-1$ the remainder is 10, and when it is divided by $x+2$ the remainder is –5. Find the remainder when it is divided by $x+1$.

Solution

(a) $p(2) = 16-16+12+10-10 = 12$

(b) $q(1) = 10 \Rightarrow 1+a+b+1 = 10 \Rightarrow a+b = 8$

$q(-2) = -5 \Rightarrow -8+4a-2b+1 = -5 \Rightarrow 4a-2b = 2$

Solving for a and b gives $a = 3$, $b = 5$, remainder $= q(-1) = -1+3-5+1 = -2$

Assessment tip

The formula for the remainder $p(c)$ when dividing $p(x)$ by $x-c$ is not given in the formula book and so should be learned. The value c that is put into $p(x)$ is the value of x that would make $x-c$ be zero. This is easier to remember if the proof of the remainder theorem has been seen and understood.

Note

If the polynomial $p(x)$ has integer coefficients with $a_n = 1$ and we are looking for linear factors of the form $x-c$, where c is an integer, then the integer c must exactly divide the integer a_0. This result means that there is usually only a small number of possible factors that we have to consider.

Example 2.5.3

Factorize $p(x)=x^3+7x^2+14x+8$ into linear factors.

Solution

Only possible factors are $x+1$, $x-1$, $x+2$, $x-2$, $x+4$, $x-4$, $x+8$, $x-8$

So we have to look at $p(-1)$, $p(1)$, $p(-2)$, $p(2)$, $p(-4)$, $p(4)$, $p(-8)$, $p(8)$

Now in this case p (positive number) will be positive as all the coefficients are positive, so four of the above can be eliminated. Looking at others

$p(-1)=-1+7-14+8=0$ so $x+1$ is a factor

$p(-2)=-8+28-28+8=0$ so $x+2$ is a factor

$p(-4)=-64+112-56+8=0$ so $x+4$ is a factor

There can only be three linear factors so we do not have to investigate any further $p(x)=x^3+7x^2+14x+8=(x+1)(x+2)(x+4)$

Assessment tip

Be clever, think and use knowledge in order to save time and avoid hard work. There are several examples of this in Example 2.5.3, deciding what possible factors there are, eliminating half of them without having to do any calculations, starting with $p(-1)$ rather than $p(-8)$ and realising when all the linear factors had been found. The formula for the product of the roots (or the sum of the roots) could also have been used to shorten the working or to check that the factors were correct.

Example 2.5.4

Factorize $p(x)=x^4+6x^3+22x^2+50x+33$ over the set of real numbers.

Solution

Only possible factors are:

$x+1$, $x-1$, $x+3$, $x-3$, $x+11$, $x-11$, $x+33$, $x-33$

$p(-1)=1-6+22-50+33=0$ so $x+1$ is a factor

$p(1)\neq 0$ so $x-1$ is not a factor

$p(-3)=81-162+198-150+33=0$ so $x+3$ is a factor

$(x+1)(x+3)=x^2+4x+3$

$$
\begin{array}{r|l}
 & x^2+2x+11 \\
x^2+4x+3 & x^4+6x^3+22x^2+50x+33 \\
 & x^4+4x^3+3x^2 \\
\hline
 & 2x^3+19x^2+50x+33 \\
 & 2x^3+8x^2+6x \\
\hline
 & 11x^2+44x+33 \\
 & 11x^2+44x+33 \\
\hline
 & 0
\end{array}
$$

Discriminant of $x^2+2x+11$ is $4-44=-40$ so this is an irreducible quadratic. $p(x)=x^4+6x^3+22x^2+50x+33=(x^2+2x+11)(x+1)(x+3)$

Link to Quadratics SL 2.7

Sums and products of the roots of polynomial equations

For the polynomial of degree n,

$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0,\quad a_n\neq 0$

The sum of the roots is $\frac{-a_{n-1}}{a_n}$ and the product of the roots is $\frac{(-1)^n a_0}{a_n}$

Note

Note how these formulae for the sum and product of the roots agree with those for the quadratic equation $ax^2+bx+c=0$, where $n=2$.

Example 2.5.5

The cubic equation $x^3+ax^2+bx+c=0$ has roots of $x=2$, 4 and 8.

(a) Find the values of a and c.

(b) Hence find the value of b.

Solution

(a) $-a=2+4+8 \Rightarrow a=-14$

$-c=2\times4\times8 \Rightarrow c=-64$

(b) 2 is a root so $8-56+2b-64=0 \Rightarrow b=56$

SAMPLE STUDENT ANSWER

The cubic equation $2x^3+px^2+qx+64=0$ has a repeated root of $x=\alpha$ and a third root of $x=\alpha^3$.

(a) Find the value of α.

(b) Hence find the value of p.

(a) $\alpha^5=\frac{64}{2} \Rightarrow \alpha=2$

(b) $2+2+8=\frac{p}{2} \Rightarrow p=24$

▲ Good recognition of how to solve the problem using the sum and product of the roots.

▼ Forgot $(-1)^3$ in the product of roots formula.
So correct answers should have been $\alpha=-2$ and $p=-24$.

The answer above could have achieved 4/6 marks.

Assessment tip

The IB give follow-through marks. This is when a latter part of a question is done with the correct method, but, due to a mistake in an earlier part, it leads to the wrong answer. In the sample answer above, all marks would have been awarded for part (b) even though the answer was wrong.

More rational functions

The rational functions met earlier in this chapter were of the form of one linear function divided by another linear function.

This is now extended to a linear function divided by a quadratic function, and a quadratic function divided by a linear one. The general guidelines for what to investigate when sketching a graph, given in the bullet points on page 44 will apply in both cases.

Link to Rational functions SL 2.8

Link to Graphs of functions SL 2.3

Let $f(x)=\dfrac{ax+b}{cx^2+dx+e}$ where $c\neq0$.

This function could have two, one or zero vertical asymptotes depending on the discriminant of the quadratic in the denominator, indicating whether there are two, one or zero real roots.

Since for large $|x|$ the quadratic will always dominate the linear function, the x-axis will always be a horizontal asymptote.

The intercept on the y-axis will be $\left(0, \frac{b}{e}\right)$ so need $e\neq0$ for $f(x)$ to be a function.

The intercept on the x-axis will be $\left(\frac{-b}{a}, 0\right)$ so if $a=0$ the graph will not intersect with the x-axis.

Example 2.5.6

Let $f(x)=\frac{2x+6}{x^2+x-12}$

(a) Find all vertical and horizontal asymptotes.

(b) Find all intercepts with the axes.

(c) Hence sketch the graph of this function.

(d) Find $\frac{\mathrm{d}f}{\mathrm{d}x}$ and explain how this confirms your earlier findings.

(e) State (i) the maximum domain and (ii) the corresponding range of this function.

Solution

(a) $x^2+x-12=0 \Rightarrow (x+4)(x-3)=0 \Rightarrow x=-4$ or $x=3$
vertical asymptotes at $x=-4$ and $x=3$
horizontal asymptote at $y=0$

(b) $x=0 \Rightarrow y=-\frac{1}{2}$ $\quad$ y-axis intercept is $\left(0,-\frac{1}{2}\right)$

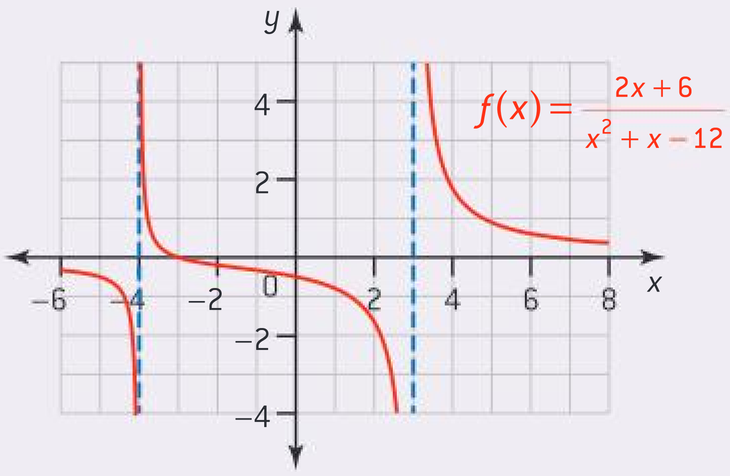
$y=0 \Rightarrow x=-3$ $\quad$ x-axis intercept is $(-3, 0)$

(c) To assist with drawing the graph

$x \to +\infty \Rightarrow y \to 0^+$ $\qquad$ $x \to -\infty \Rightarrow y \to 0^-$

$x \to -4^- \Rightarrow y \to -\infty$ $\qquad$ $x \to -4^+ \Rightarrow y \to +\infty$

$x \to 3^- \Rightarrow y \to -\infty$ $\qquad$ $x \to 3^+ \Rightarrow y \to +\infty$



(d) $\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{2(x^2+x-12)-(2x+6)(2x+1)}{(x^2+x-12)^2}=\frac{-2x^2-12x-30}{(x^2+x-12)^2}$

The discriminant of the quadratic in the numerator is $(-12)^2-4(-2)(-30)=-96$, showing that this quadratic is always negative and thus so is $\frac{\mathrm{d}f}{\mathrm{d}x}$. This confirms that the graph is always decreasing.

(e) (i) The maximum domain is $\{x \in \mathbb{R} \mid x \neq -4, x \neq 3\}$
(ii) The range is $\{y \in \mathbb{R}\}$

Let $f(x)=\frac{ax^2+bx+c}{dx+e}$ where $a, d \neq 0$.

This function will have a vertical asymptote at $x=-\frac{e}{d}$. Since for large $|x|$ the quadratic will always dominate the linear function, as $x \to \pm\infty$ then $f(x) \to \pm\infty$ depending on the sign of $\frac{a}{d}$. However we can be more precise than this by dividing ax^2+bx+c by $dx+e$ to show that there will be a straight line oblique asymptote with gradient of $\frac{a}{d}$.

The intercept on the y-axis will be $\left(0,\frac{c}{e}\right)$ so need $e \neq 0$ for $f(x)$ to be a function.

Note

There could be two, one or zero intercepts on the x-axis depending on the discriminant of the quadratic in the numerator, indicating whether it has two, one or zero real roots. If there is only one real root the graph will just touch the x-axis at this point rather than crossing it.

Example 2.5.7

Let $f(x)=\dfrac{x^2+x+4}{2x+8}$

(a) Find all vertical, horizontal and oblique asymptotes.

(b) Find all intercepts with the axes.

(c) Find $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ and hence find and classify all turning points.

(d) Hence sketch the graph of this function.

(e) State (i) the maximum domain and (ii) the corresponding range of this function.

Solution

(a) $2x+8=0 \Rightarrow x=-4$, so this is the vertical asymptote.
No horizontal asymptotes.

$$
\begin{array}{r|l}
 & \frac{1}{2}x-\frac{3}{2} \\
\hline
2x+8 & x^2+x+4 \\
 & \underline{x^2+4x} \\
 & \quad -3x+4 \\
 & \quad \underline{-3x-12} \\
 & \qquad 16
\end{array}
$$

This working shows that $\dfrac{x^2+x+4}{2x+8} \equiv \dfrac{1}{2}x-\dfrac{3}{2}+\dfrac{16}{2x+8}$ and so $y=\dfrac{1}{2}x-\dfrac{3}{2}$ is an oblique asymptote.

(b) $x=0 \Rightarrow y=\dfrac{1}{2}$ so the y-axis intercept is $\left(0,\dfrac{1}{2}\right)$

The quadratic equation $x^2+x+4=0$ has a negative discriminant of -15 so there are no intercepts on the x-axis.

(c) $\dfrac{\mathrm{d}f}{\mathrm{d}x}=\dfrac{(2x+1)(2x+8)-2(x^2+x+4)}{(2x+8)^2}=\dfrac{2x(x+8)}{(2x+8)^2}$

$2x(x+8)=0 \Rightarrow x=0$ or $x=8$

Using a sign table:

x	$x<-8$	-8	$-8<x<-4$	-4	$-4<x<0$	0	$x>0$
$\frac{\mathrm{d}f}{\mathrm{d}x}$	+ve	0	−ve	undefined	−ve	0	+ve

This shows that the function has a maximum at $\left(-8,-7\frac{1}{2}\right)$ and a minimum at $\left(0,\frac{1}{2}\right)$

Also note that as $x\to\pm\infty$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\to\dfrac{1}{2}$ in agreement with the oblique asymptote.

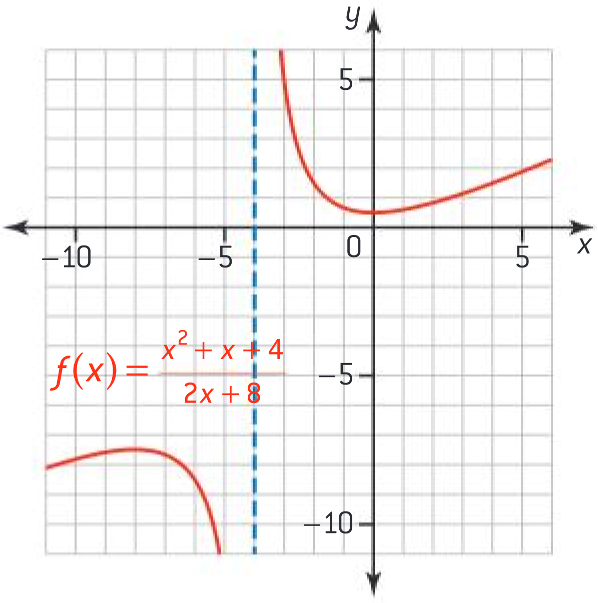
Link to Testing for max and mins SL 5.8

(d) To assist with drawing the graph:

$x\to+\infty \Rightarrow y\to+\infty \qquad x\to-\infty \Rightarrow y\to-\infty$

But more specifically as $x\to\pm\infty \Rightarrow y\to\dfrac{1}{2}x-\dfrac{3}{2}$

$x\to-4^- \Rightarrow y\to-\infty \qquad x\to-4^+ \Rightarrow y\to+\infty$



(e) (i) The maximum domain is $\{x\in\mathbb{R} \mid x\neq-4\}$

(ii) The range is $]-\infty,-7.5]\cup[0.5,\infty[$

Odd and even functions

An odd function is defined as a function for which $f(-x) = -f(x)$, for all x.

Examples are $f(x) = \sin x$, $f(x) = 3x^3 + x$.

The graphs will exhibit rotational symmetry of $180°$ about the origin.

An even function is defined as a function for which $f(-x) = f(x)$, for all x.

Examples are $f(x) = \cos x$, $f(x) = 4x^2 + 7$.

The graphs will exhibit reflectional symmetry about the y-axis.

Most functions are neither odd nor even.

A polynomial consisting of only odd powers of x will be an odd function. A polynomial consisting of only even powers of x will be an even function. This suggests where the names come from.

An even function multiplied or divided by an even function is an even function.

An odd function multiplied or divided by an odd function is an even function.

If an odd function and an even function are multiplied or divided, the resulting function is odd, e.g., $y = \tan x = \frac{\sin x}{\cos x}$ is an odd function.

Example 2.5.8

Given that $y = \cos x + ax + b$ is an even function, find the value of the constant a.

Solution

$\cos(-x) - ax + b \equiv \cos x + ax + b \Rightarrow -ax \equiv ax \Rightarrow a = 0$

Example 2.5.9

Given that $f(x)$ is an odd function with domain of $\mathbb{R}$, find the value of $f(0)$.

Solution

$f(-0) = -f(0) \Rightarrow f(0) = -f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$

This result also follows from the symmetry.

Note

The function $f(x) = 0, x \in \mathbb{R}$ is both an even function and an odd function. This is the only function with domain of $\mathbb{R}$ that has this property. This result can also be understood using symmetry.

Self-inverse functions

Earlier in this chapter it was explained that for a function to have an inverse function the original function must be "one to one". This can sometimes be achieved by placing restrictions on the domain and codomain.

A self-inverse function is a function that is its own inverse. This would mean that if $f(x)$ is a self-inverse function then its domain is the same as its range, its graph is symmetrical about the line $y = x$ and $(f \circ f)(x) = x$. Examples of self-inverse functions are $f(x) = x$, $f(x) = -x$ and f: $\{x \in \mathbb{R} \mid x \neq 0\} \to \{y \in \mathbb{R} \mid y \neq 0\}$, $f(x) = \frac{1}{x}$.

Link to Inverse functions SL 2.2, SL 2.5

Example 2.5.10

Show that f: $\{x \in \mathbb{R} \mid x \neq 0\} \rightarrow \{y \in \mathbb{R} \mid y \neq 0\}$, $f(x) = \frac{k}{x}$ is a self-inverse function, for any non-zero real constant k.

Solution

Method 1: $y = \frac{k}{x}$

Interchanging variables gives $x = \frac{k}{y}$ and solving for y gives $y = \frac{k}{x}$

Thus $f^{-1}(x) = \frac{k}{x} = f(x)$

Method 2: $(f \circ f)(x) = f(f(x)) = f\left(\frac{k}{x}\right) = \frac{k}{\frac{k}{x}} = x$ so $f(x)$ is self-inverse.

Example 2.5.11

Let $f(x) = \frac{ax+b}{x+c}$ for real constants a, b, c.

(a) (i) State the maximum domain if $f(x)$ is a function.
(ii) State the range in this case if $f(x)$ has an inverse function.

(b) Hence, for this function, write down the equation of (i) the vertical asymptote and (ii) the horizontal asymptote.

(c) Find conditions on the constants a, b, c if $f(x)$ is to be a self-inverse function.

Solution

(a) (i) maximum domain $\{x \in \mathbb{R} \mid x \neq -c\}$ (ii) range $\{y \in \mathbb{R} \mid y \neq a\}$

(b) (i) $x = -c$ (ii) $y = a$

(c) Since the vertical asymptote of $f(x)$ will become the horizontal asymptote of $f^{-1}(x)$, to be self-inverse require $c = -a$.
Also require $(f \circ f)(x) = x$

$$f\left(\frac{ax+b}{x-a}\right) = \frac{a\frac{ax+b}{x-a}+b}{\frac{ax+b}{x-a}-a} = \frac{(a^2+b)x}{(a^2+b)} = x$$ as required, provided that

$a^2 + b \neq 0$

Hence $f(x) = \frac{ax+b}{x-a}$, f: $\{x \in \mathbb{R} \mid x \neq a\} \rightarrow \{y \in \mathbb{R} \mid y \neq a\}$

Conditions are $c = -a$ and $a^2 + b \neq 0$.

If $b = -a^2$, then the function would be $f(x) = a$, which is not one to one.

Link to Rational functions SL 2.8

Solving inequalities

When working with inequalities, the following operations do not affect the direction of the inequality:

- adding or subtracting a number or expression from both sides
- multiplying or dividing both sides by a positive number
- simplifying a side or both sides.

However, multiplying or dividing both sides by a negative number will result in changing the direction of the inequality symbol.

SAMPLE STUDENT ANSWER

Solve the following inequalities for $x \in \mathbb{R}$.

(a) $3x+2>0$ (b) $-3x+2>0$ (c) $\ln\left(\frac{1}{2}\right)\times x+4>0$

(a) $3x+2>0 \Rightarrow 3x>-2 \Rightarrow x>-\frac{2}{3}$

(b) $-3x+2>0 \Rightarrow -3x>-2 \Rightarrow x<\frac{2}{3}$

(c) $\ln\left(\frac{1}{2}\right)\times x+4>0 \Rightarrow \ln\left(\frac{1}{2}\right)\times x>-4 \Rightarrow x>-\frac{4}{\ln\left(\frac{1}{2}\right)}$

▲ Parts (a) and (b) correct and generally the right method for part (c).

▼ Did not recognize that $\ln\left(\frac{1}{2}\right)$ is negative and so forgot to reverse the direction of the inequality.

The answer above could have achieved 5/6 marks.

The correct answer to part (c) should have been $x<-\frac{4}{\ln\left(\frac{1}{2}\right)}$ which is the same as $x<\frac{4}{\ln(2)}$.

For more complicated inequalities, first solve the equality and then consider the graph(s) to solve the inequality. If it is a paper that requires the use of technology then the calculator will help with the intersection of the curves and sketching the graphs. On a paper where technology is not allowed, although you work analytically to solve the equality, it is good practice to visualize or quickly sketch the graph(s) to assist with the inequality.

Note

When multiplying or dividing an inequality by a number or expression, make sure you know if it is positive or negative. For example, on first glance it might appear that multiplying both sides by $\ln\frac{1}{2}$ would be fine. However, $\ln\frac{1}{2}=-0.69\ldots3$ and since this is a negative number you would need to change the direction of the inequality symbol.

Example 2.5.12

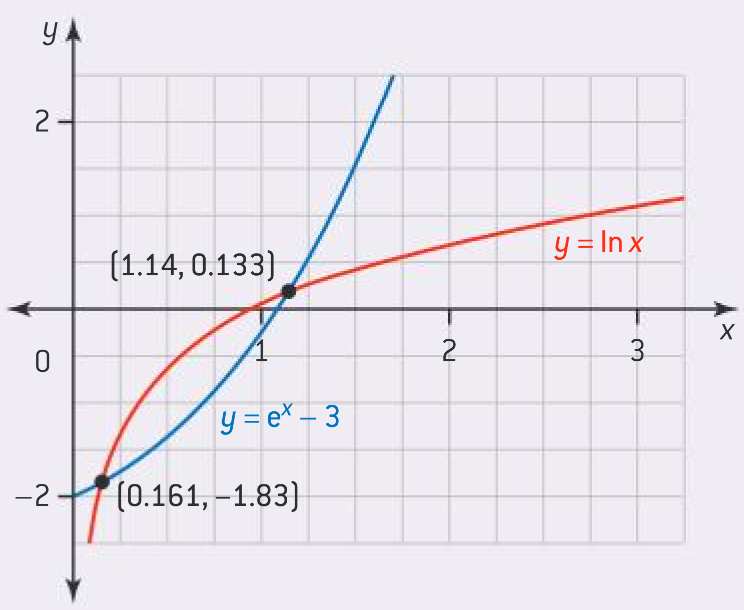
Solve the inequality $e^x-3<\ln x$ for $x>0$.

Solution

Sketch both curves and find where they intersect.

Using a calculator shows the curves intersect where $x=0.161$ and $x=1.14$ (3 sf)

Solution is:
$\{x\in\mathbb{R}\mid 0.161<x<1.14\}$



Link to Intersection of curves using technology SL 2.4, Solving equations graphically and analytically SL 2.10.

Example 2.5.13

Solve the inequality $x^2+4 \leq 2x^2+7x+16$.

Solution

$x^2+4 \leq 2x^2+7x+16 \Leftrightarrow x^2+7x+12 \geq 0$

Now solve the equation $x^2+7x+12=0$

$(x+3)(x+4)=0 \Rightarrow x=-3 \text{ or } -4$

Since $x^2+7x+12$ is concave up, the solution is $]-\infty,-4] \cup [-3,\infty[$

Further transformations of graphs

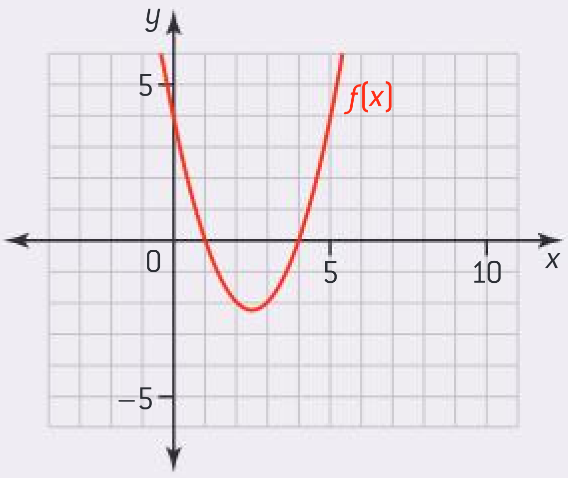
We can obtain the graph of a transformed function from the graph of the original function. If we know the equation of the original function and it is a paper that allows the use of technology then we can just put the new function into the calculator, sketch the curve and then copy it. If we know the equation of the original function and it is a paper where technology is not allowed, then having found the equation of the new function, we could go through the general guidelines for what to investigate when sketching a graph given in the bullet points on page 44. However, with some transformations, as given below, if we are given the graph of the original function (but not its equation) we can think geometrically to obtain the graph of the new function.

$\boldsymbol{y = |f(x)|}$

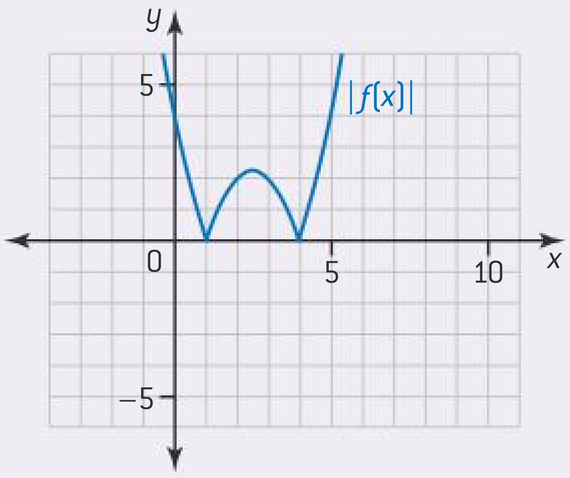
From the graph of $f(x)$, any portions of the curve that are below the x-axis are reflected in the x-axis. You can consider these portions as having been folded up, whereas the other portions remain the same. Any maximums or minimums on the reflected portions will generate minimums and maximums respectively. The resultant graph has the property that $y \geq 0$ for all x. As the modulus sign is on the "outside" of the original function the transformation occurs vertically.

Example 2.5.14

For the graph of $f(x)$ shown below, sketch the graph of $y=|f(x)|$



Solution

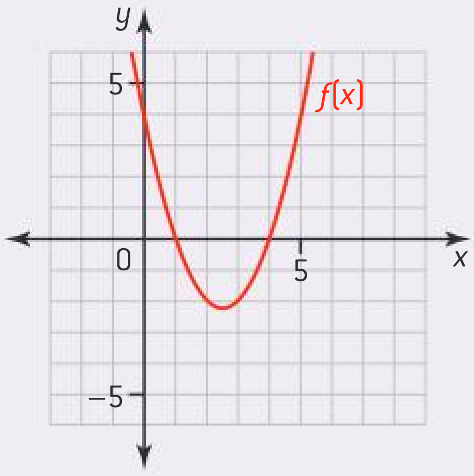


$y = f(|x|)$

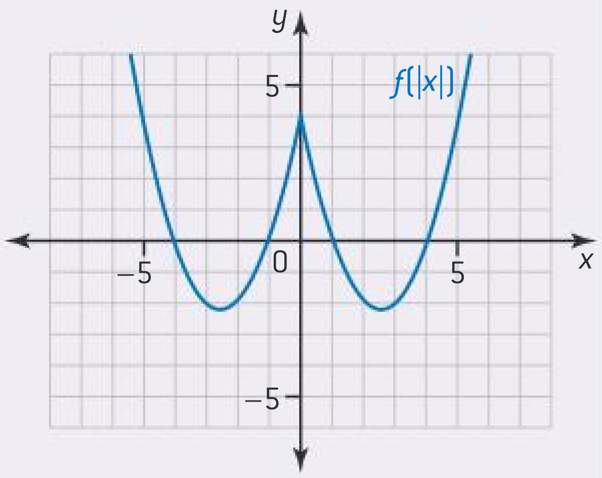
All of the graph of $f(x)$ to the right of the y-axis is reflected in the y-axis. The resultant graph will be an even function. What the original graph did for $x < 0$ is not relevant anymore and for $x \geq 0$ the new graph is the same as the original. As the modulus sign is on the "inside" of the original function the transformation occurs horizontally.

Example 2.5.15

For the graph of $f(x)$ shown below, sketch the graph of $y = f(|x|)$



Solution



$y = \frac{1}{f(x)}$

First, at any zeros of $f(x)$, the graph of y will have a vertical asymptote. Then the below guidelines can be followed:

If $f(x)$ is large and positive then y will be small and positive.

If $f(x)$ is small and positive then y will be large and positive.

If $f(x)$ is large and negative then y will be small and negative.

If $f(x)$ is small and negative then y will be large and negative.

All local maximums will become local minimums and vice versa.

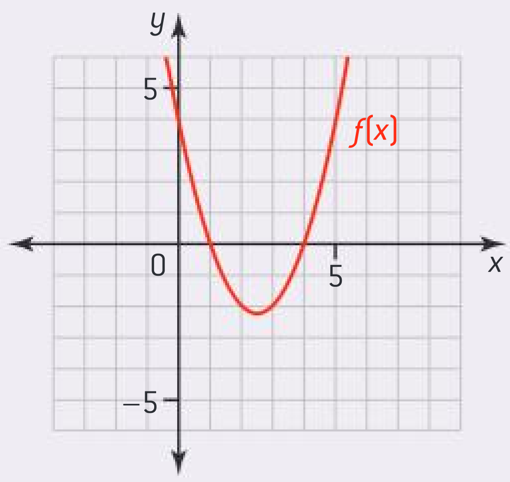
If $f(x)$ has a horizontal asymptote with equation $y = k,\ k \neq 0$ then y has a horizontal asymptote with equation $y = \frac{1}{k}$. The curve will approach the asymptote from the "other" side.

If $f(x)$ has a horizontal asymptote with equation $y = 0$, then $y \to \pm\infty$ as $x \to \pm\infty$ depending on which "side" the original curve was to $y = 0$.

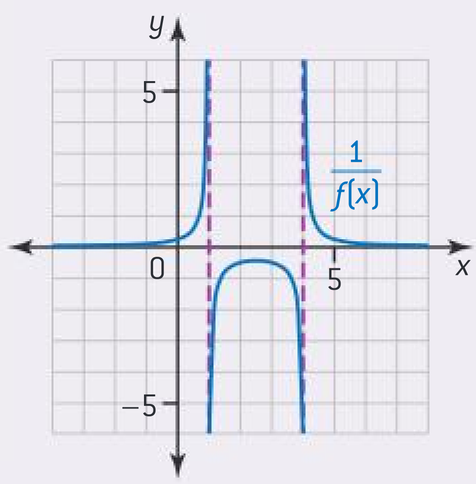
If $f(x) \to \pm\infty$ as $x \to \pm\infty$ then y will have a horizontal asymptote with equation $y = 0$, approaching from above if $f(x) \to +\infty$ and vice versa.

Example 2.5.16

For the graph of $f(x)$ shown below, sketch the graph of $y=\frac{1}{f(x)}$



Solution



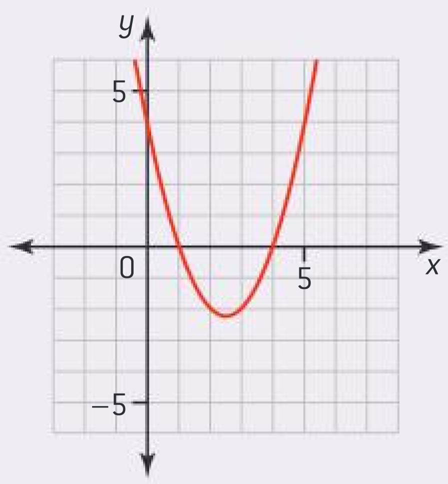
Link to Transformations of graphs SL 2.11

$y=f(ax+b),\ a\neq 0$

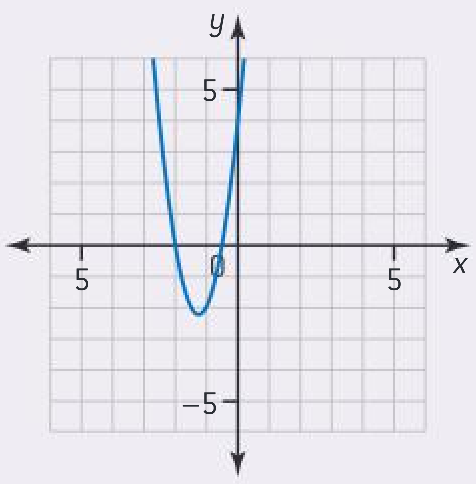
This "inside" linear transformation has already been covered in the SL work. It represents a horizontal translation by the vector $\begin{pmatrix}-b\\0\end{pmatrix}$ followed by a horizontal stretch by a factor of $\frac{1}{a}$.

Example 2.5.17

For the graph of $f(x)$ shown below, sketch the graph of $y=f(2x+5)$



Solution



Transformations are horizontal.

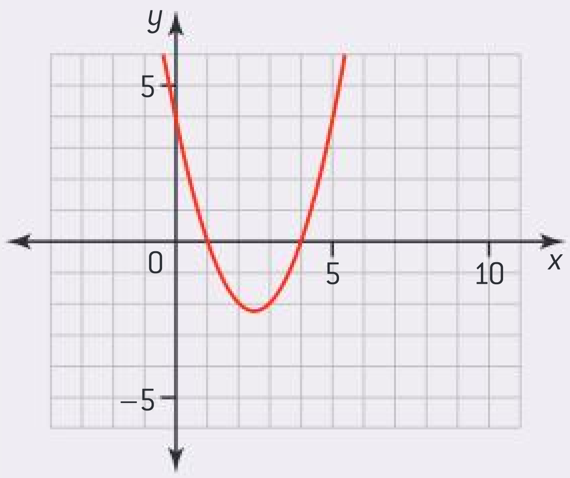
Shift to the left by 5 followed by horizontal stretch by a factor of $\frac{1}{2}$.

$y=[f(x)]^2$

This transformation is on the "outside" and so will work vertically. The resultant graph has the property that $y\geq 0$, for all x. If $f(x)=\pm 1$ then $y=1$. If $|f(x)|<1$ then the new point is "closer" to the x-axis and if $|f(x)|>1$ then the new point is "further" from the x-axis. Zeros of $f(x)$ will remain zeros of the function y. Maximums and minimums that are above the x-axis will generate maximums and minimums respectively. However, maximums and minimums that are below the x-axis will generate minimums and maximums, respectively.

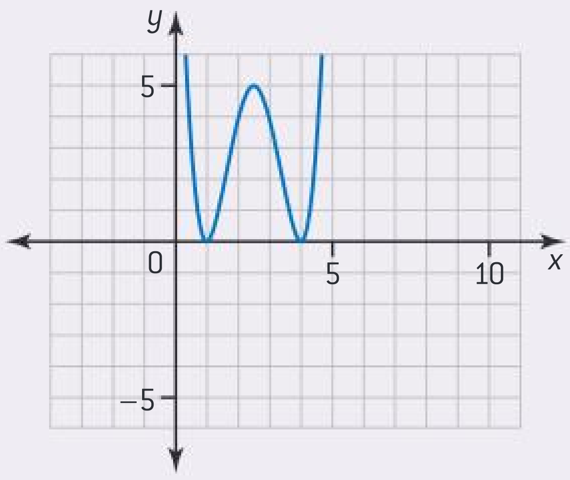
Example 2.5.18

For the graph of $f(x)$ shown below, sketch the graph of $y = [f(x)]^2$



Solution

Applying the guidance given above.



Solution of modulus equations and inequalities

The definition of the modulus function is $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

On a paper that requires the use of technology, an equation involving moduli could be solved by drawing the relevant graph or graphs. An inequality could be solved by considering the solutions to the equality and then taking the graph or graphs of the equality into account.

On a paper where technology is not allowed, the moduli signs can be removed using its definition above and splitting the working into different cases. It is still very useful to visualize or sketch the graph especially when solving an inequality.

Link to Solving equations graphically and analytically SL 2.10 , Finding the intersection of curves SL 2.4. and Solving inequalities both graphically and analytically AHL 2.15.

Note

Take care when "decoding" a modulus sign; *all* the x's in the definition become replaced with the new variable. For example,

$$|2x+6| = \begin{cases} 2x+6 \text{ if } 2x+6 \geq 0 \\ -(2x+6) \text{ if } 2x+6 < 0 \end{cases}$$

giving $$|2x+6| = \begin{cases} 2x+6 \text{ if } x \geq -3 \\ -2x-6 \text{ if } x < -3 \end{cases}$$

Also remember that when working with inequalities if you multiply or divide by a negative number then the inequalities reverse direction.

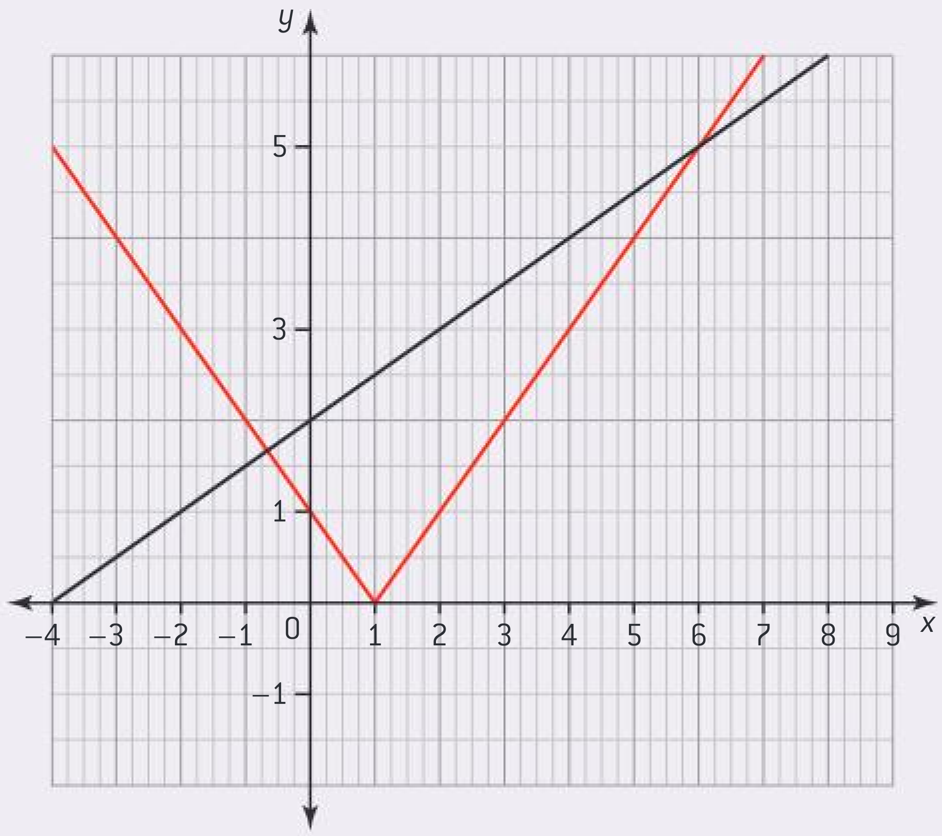
Example 2.5.19

(a) Solve the equation $|x-1|=\frac{1}{2}x+2$ for $x\in\mathbb{R}$.

(b) Hence solve the inequality $|x-1|\le\frac{1}{2}x+2$.

Solution

(a)



Using a calculator, the solutions are $x=-\frac{2}{3}$ or $x=6$

(b) By considering the graph the solution is $\left[-\frac{2}{3},6\right]$

Example 2.5.20

(a) Solve the equation $|x+3|=\left|\frac{1}{2}x-1\right|$ for $x\in\mathbb{R}$.

(b) Hence solve the inequality $|x+3|>\left|\frac{1}{2}x-1\right|$.

Solution

(a) $|x+3|=\begin{cases} x+3 \text{ if } x+3\ge 0 \\ -(x+3) \text{ if } x+3<0\end{cases}$ giving $|x+3|=\begin{cases} x+3 \text{ if } x\ge -3 \\ -x-3 \text{ if } x<-3\end{cases}$

$\left|\frac{1}{2}x-1\right|=\begin{cases} \frac{1}{2}x-1 \text{ if } \frac{1}{2}x-1\ge 0 \\ -\left(\frac{1}{2}x-1\right) \text{ if } \frac{1}{2}x-1<0\end{cases}$ giving $\left|\frac{1}{2}x-1\right|=\begin{cases} \frac{1}{2}x-1 \text{ if } x\ge 2 \\ -\frac{1}{2}x+1 \text{ if } x<2\end{cases}$

There will be three regions to consider

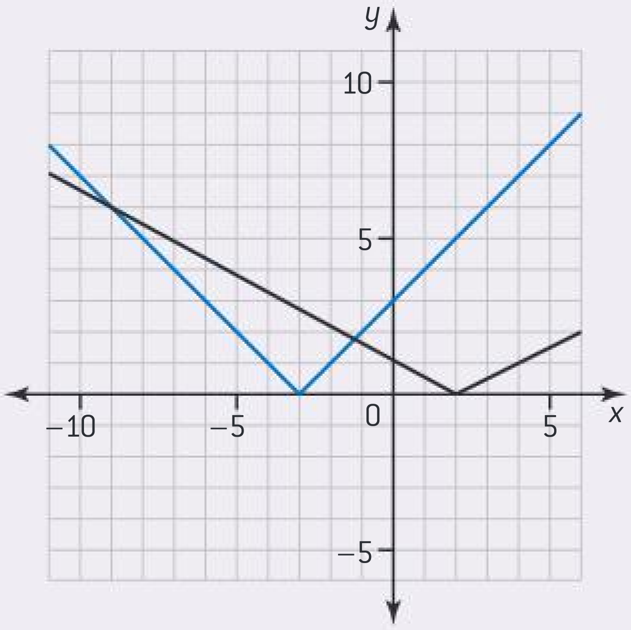
$x\ge 2$ This gives $x+3=\frac{1}{2}x-1$, but this implies $x=-8$, which is not in this region.

$-3\le x<2$ This gives $x+3=-\frac{1}{2}x+1\Rightarrow x=-\frac{4}{3}$

$x<-3$ This gives $-x-3=-\frac{1}{2}x+1\Rightarrow x=-8$

Thus, the solutions are $x=-8$ or $x=-\frac{4}{3}$

(b) Visualizing where the modulus curves "bounce" and their gradients as shown below.



Using part (a), the solution is $]-\infty,-8[\cup\left]-\frac{4}{3},\infty\right[$

SAMPLE STUDENT ANSWER

Find the exact solutions of the equation $|2x-6|=x^2-5$ for $x\in\mathbb{R}$.

$\pm(2x-6)=x^2-5 \qquad 2x-6=x^2-5\Rightarrow x^2-2x+1=0\Rightarrow$

$(x-1)^2=0\Rightarrow x=1$

$-2x+6=x^2-5\Rightarrow x^2+2x-11=0\Rightarrow x=\frac{-2\pm\sqrt{48}}{2}=-1\pm2\sqrt{3}$

Solutions are $x=1,-1\pm2\sqrt{3}$

▲ Had correct idea of wanting to remove the modulus. Two of the three answers given were correct.

▼ Did not take the two regions into account. The answer $x=1$ comes from the region $x\geq3$ and hence should be rejected. Visualization of the two curves should have indicated that there could only have been two solutions.

The answer above could have achieved 4/6 marks.

The correct solutions come from the region $x<3$ and are $-1\pm2\sqrt{3}$

PRACTICE QUESTIONS

SL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 1 [8 marks]

Consider the function $f(x) = \frac{ax+b}{x+c}$, $x \in \mathbb{R}$, $x \neq -c$.

a. Show that $f'(x) = \frac{ac-b}{(x+c)^2}$

b. Hence, state conditions involving the constants a, b and c for the graph of $f(x)$ to be:

i. always increasing

ii. always decreasing

iii. a horizontal straight line.

c. In case (b) (iii), simplify the expression for $f(x)$.

Question 2 [9 marks]

Consider the function $f(x) = \frac{x+2}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$.

a. Find the inverse function $f^{-1}(x)$.

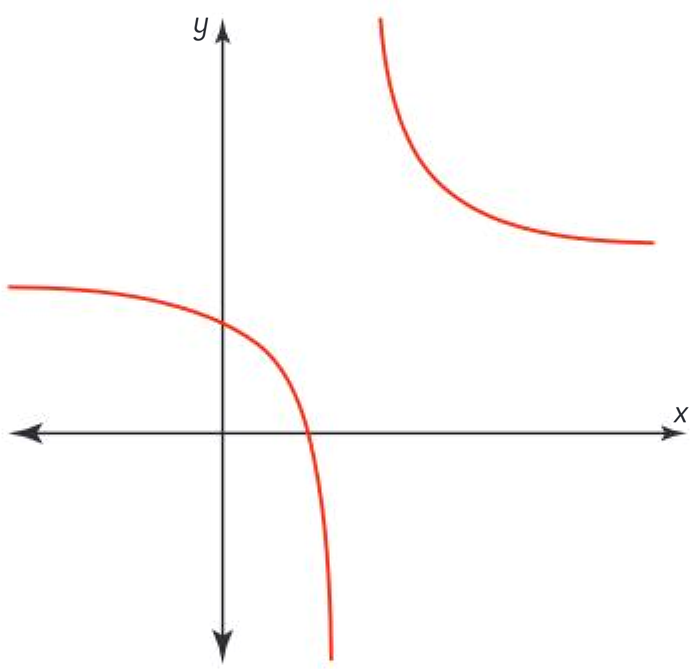
b. State the domain and range of:

i. $f(x)$ **ii.** $f^{-1}(x)$

c. Write down an expression for $(f \circ f)(x)$.

Question 3 [8 marks]

The graph of $y = \frac{ax+b}{x+c}$, $x \in \mathbb{R}$, $x \neq -c$ is shown below.



There is a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 4$. The intercept on the x-axis is $(2, 0)$.

a. Find the values of the constants a, b and c.

b. Find the intercept on the y-axis.

Question 4 [6 marks]

Let $f(x) = 2x^2 + 7x + c$.

a. Find the value of the constant c if the x-axis is a tangent to the graph of this function.

b. For this value of c, solve the equation $f(x) = 0$.

Question 5 [6 marks]

Solve the equation $x - 7\sqrt{x} + 10 = 0$, for $x \in \mathbb{R}^+$.

Question 6 [7 marks]

Determine the number of real roots of the equation $(x^2 + 6x + 4)(x^2 + 6x + 10) = 0$.

Question 7 [7 marks]

a. The equation $ax^2 + 8x + 4 = 0$ has only one real root. Find the value of this root.

b. Given that $4x^2 + 8x + b > 10$ for all x, find the inequality that b must satisfy.

Question 8 [9 marks]

Let $y = a(x + h)^2 + k$.

The graph of y has a vertex at $(2, -16)$ and an intercept on the y-axis of $(0, 4)$.

a. Find the values of the constants a, h and k.

b. Hence find the exact solutions to $y = 0$.

SL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 9 [15 marks]

A rectangle $ABCD$ has $A = (1, 3)$, $B = (5, 5)$ and $C = (2, p)$.

a. Find the equation of the straight line through A and B.

b. Find the equation of the straight line through B and C.

c. Hence, find the value of p.

d. Find the equation of the straight line through C and D.

e. Find the equation of the straight line through A and D.

f. Hence, find the point D.

g. Find the mid-point of $[BD]$.

Question 10 [12 marks]

For each of the following functions, state the maximum possible domain and the corresponding range.

a. $f(x)=\frac{1}{x}$

b. $f(x)=\sqrt{x+5}$

c. $f(x)=\frac{4}{x+3}$

d. $f(x)=\frac{1}{\sqrt{x-3}}$

e. $f(x)=\log_2(x-2)$

f. $f(x)=2^x$

Question 11 [11 marks]

Let $f(x)=\frac{x-3}{4},\ x\in\mathbb{R}$.

Find:

a. $f^{-1}(x)$

b. $(f\circ f)(x)$

c. $(f\circ f)^{-1}(x)$

d. $(f^{-1}\circ f^{-1})(x)$

Question 12 [13 marks]

Let $f(x)=x+4,\ g(x)=\frac{1}{x}$ for $x\neq 0$, and $h(x)=x+3$.

a. Find $(h\circ g\circ f)(x)$

b. State the domain and range of the function $h\circ g\circ f$

c. Find $(h\circ g\circ f)^{-1}(x)$, and state its domain and range.

d. Given that $(h\circ g\circ f)(x)=\frac{ax+b}{x+c}$, find the values of the constants a, b and c.

Question 13 [12 marks]

The function f is defined by $f(x)=2^x+1$.

a. Sketch the graph of $f(x)$, giving the equations of any asymptotes.

Four related functions are defined by:

i. $f_1(x)=2^x+3$

ii. $f_2(x)=2^{x-1}+1$

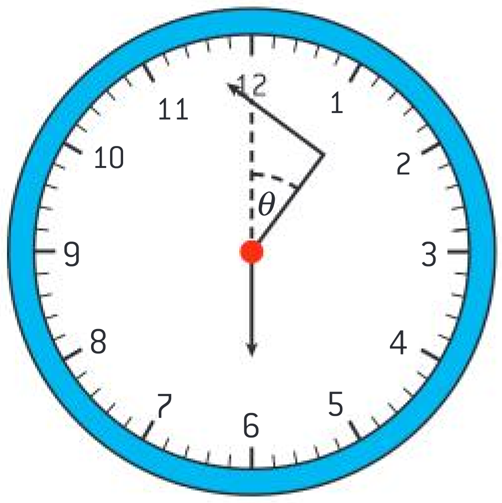
iii. $f_3(x)=2^{4x}+1$

iv. $f_4(x)=2^{x+1}+4$

b. For each of the four functions above, give a **single** geometric transformation that would transform the graph of $f(x)$ into the graph of this function.

Question 14 [12 marks]

An ornamental clock hangs vertically at a large train station. The minute hand is a metal rod of length two metres. A vandal bends the minute hand to the left, at its mid-point, so that the angle here is a right angle. The hour hand points to 6. Let the angle between the bottom section of the minute hand and the vertical line from the centre of the clock to the number 12 be θ.

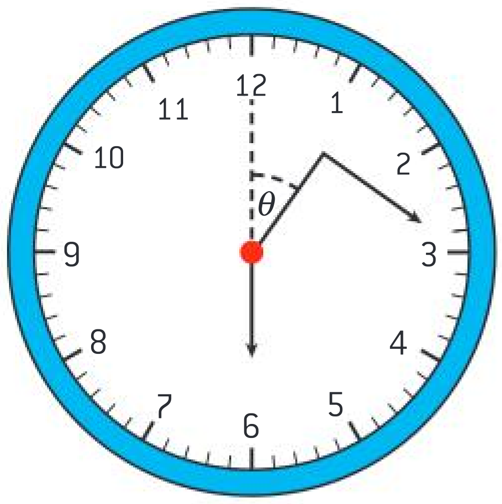


Let $h(\theta)$ be the height of the end of the bent minute hand above a horizontal line that goes through the centre of the clock.

a. Show that $h(\theta)=\cos\theta+\sin\theta$

b. By considering the graphs of $\cos\theta$ and $\sin\theta$, sketch the graph of $h(\theta)$ for $0\leq\theta\leq 2\pi$

Another vandal appears and she bends the minute hand back to its original position and then bends it to the right, at its mid-point, so that the angle here is a right angle.



c. Show that in this case $h(\theta)=\cos\theta-\sin\theta$

d. Sketch the graph of this version of $h(\theta)$ for $0\leq\theta\leq 2\pi$.

SL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 15 [5 marks]

Sketch the graphs of two functions on the same axes and hence solve the equation $e^x = 4\sin x$, for $x \geq 0$.

SL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 16 [15 marks]

On a Scottish island, the population of grey squirrels is modelled by $g(t) = 10(3^t - 2t)$ and the population of red squirrels is modelled by $r(t) = \frac{1000}{2^t - t}$, where t, measured in decades, is the time after the introduction of the grey squirrels onto the island.

a. Write down the initial population of each species of squirrel.

b. Find the population of grey squirrels after 10 years.

c. Find how many years it takes for the population of grey squirrels to double.

d. Find how many years it takes for the populations to be equal.

e. Find how many years it takes for the population of grey squirrels to be twice that of the red squirrels.

f. Find how many years it takes for this model to predict that the population of red squirrels will be zero, when rounded to the nearest integer.

HL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 17 [5 marks]

a. Find the remainder when $p(x) = x^3 + 2x^2 - 5x - 6$ is divided by $x - 1$.

b. When $q(x) = x^3 + x^2 + cx + 1$ is divided by $x - 1$, the remainder is the same as when it is divided by $x + 1$. Find the value of the constant c.

Question 18 [9 marks]

a. Find the vertex of the quadratic $f(x) = x^2 + 2x + 5$

The domain of $f(x)$ is now going to be restricted so that the inverse function $f^{-1}(x)$ exists.

b. Find the maximum possible domain of $f(x)$ in the form $\{x \in \mathbb{R} | x \geq k\}$ (for some $k \in \mathbb{R}$) so that $f^{-1}(x)$ exists.

c. Given the domain you found in part (b), find the equation of the inverse $f^{-1}(x)$, and state its domain and range.

Question 19 [8 marks]

a. On the same set of axes, sketch the graphs of $f(x) = 2x + 4$ and $g(x) = \frac{1}{2x+4}$, $x \neq -2$.

b. Hence, find the exact values of x for which $f(x) > g(x)$.

Question 20 [8 marks]

Consider the function $y = \left| |x| - 4 \right|$, $x \in \mathbb{R}$.

a. Find all intercepts with both the x- and y-axes.

b. Sketch the graph of this function.

HL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 21 [22 marks]

The function $f(x) = \frac{1}{x}$, $x \neq 0$ is a self-inverse function and an odd function.

a. Sketch the graph of $f(x)$ giving the equations of any asymptotes.

Four related functions are defined by:

i. $f_1(x) = \frac{1}{x+1}$, $x \neq -1$

ii. $f_2(x) = \frac{2}{x}$, $x \neq 0$

iii. $f_3(x) = \frac{1}{3x}$, $x \neq 0$

iv. $f_4(x) = \frac{1}{x} + 2$, $x \neq 0$

b. For each of the four functions above, give a geometric transformation that would transform the graph of $f(x)$ into the graph of this function.

c. For each of the four functions above, state if it is a self-inverse function.

d. For each of the four functions above, state if it is an odd function, an even function or neither.

Let $f_5(x) = \frac{1}{x^2}$, $x \neq 0$.

e. Sketch the graph of $f_5(x)$, giving the equations of any asymptotes.

f. State if $f_5(x)$:

i. is a self-inverse function

ii. is an odd function, an even function or neither.

Question 22 [17 marks]

Consider the function $f(x)=\frac{ax+b}{x+c}$, $x\in\mathbb{R}$, $x\neq -c$.

a. If $f(x)$ is an even function:

i. determine what can be deduced about the constants a, b and c.

ii. simplify the expression for $f(x)$ in this case.

b. If $f(x)$ is an odd function:

i. determine what can be deduced about the constants a, b and c.

ii. simplify the expression(s) for $f(x)$ in this case.

Question 23 [15 marks]

a. Factorize $p(x) = x^3 + 2x^2 - 5x - 6$ into linear factors.

b. Hence, find the values of x for which $2x^3 + 3x^2 - x - 2 \geq x^3 + x^2 + 4x + 4$

Question 24 [19 marks]

Let a cubic polynomial be given by
$y = ax^3 + bx^2 + cx + d$, $\quad a, b, c, d \in \mathbb{R}, a \neq 0$.

a. Find **i.** $\frac{dy}{dx}$ **ii.** $\frac{d^2y}{dx^2}$

b. Find the sum of the roots of the equation

i. $\frac{dy}{dx}=0$ **ii.** $\frac{d^2y}{dx^2}=0$

c. Hence, if y has two turning points, show that the x value of the point of inflexion is halfway between the x values of the maximum and minimum points.

Let a polynomial be given by

$p(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$,
$a_i\in\mathbb{R}$, $a_n\neq 0$, $n\geq 2$.

d. Find **i.** $p'(x)$ **ii.** $p''(x)$

e. Find the sum of the roots of the equation:

i. $p'(x) = 0$, and let the answer be S.

ii. $p''(x) = 0$, and let the answer be T.

f. Show that $T=\left(\frac{n-2}{n-1}\right)S$

Question 25 [16 marks]

Sketch a separate graph of each of the following functions and find any intercepts with the axes.

a. $y = \log_3 x$, $x > 0$

b. $y = \log_3 (x - 2)$, $x > 2$

c. $y = \log_3 (2x)$, $x > 0$

d. $y = \log_3 (x^2)$, $x \in \mathbb{R}$, $x \neq 0$

e. $y = (\log_3 x)^2$, $x > 0$

Question 26 [10 marks]

Let $f(x) = \frac{1}{x-2}$, $x\neq 2$.

a. Find the inverse function $f^{-1}(x)$ and state its domain and range.

b. Find the set of values of x for which $f(x) > f^{-1}(x)$

Question 27 [20 marks]

Consider the function $y = \frac{x+3}{x^2-4x+4}$, $x\in\mathbb{R}$, $x\neq 2$.

a. Find all vertical and horizontal asymptotes and investigate on which side(s) of each asymptote the graph approaches the asymptote.

b. Find all intercepts with both the x and y-axes.

c. Show that $\frac{dy}{dx}=-\frac{x+8}{(x-2)^3}$ and hence find and any maximum or minimum points.

d. Give the regions where y is **i.** increasing **ii.** decreasing.

e. Hence, sketch the graph of this function.

Question 28 [20 marks]

Consider the function $y=\frac{x+3}{x^2+x+4}$, $x\in\mathbb{R}$.

a. Find any vertical or horizontal asymptotes and investigate on which side(s) of each asymptote the graph approaches the asymptote.

b. Find all intercepts with both the x- and y-axes.

c. Show that $\frac{dy}{dx}=\frac{-x^2-6x+1}{(x^2+x+4)^2}$ and hence find the exact x-values of any maximum or minimum points.

d. Give the regions where y is **i.** increasing **ii.** decreasing.

e. Hence, sketch the graph of this function.

Question 29 [23 marks]

Let $f(x)=\frac{x^2-4x+4}{x+3}$

a. Find all vertical, horizontal and oblique asymptotes.

b. Find all intercepts with the axes.

c. Show that $f'(x)=\frac{(x-2)(x+8)}{(x+3)^2}$ and hence find and classify all turning points.

d. Hence, sketch the graph of this function.

e. State **i.** the maximum domain **ii.** the corresponding range of this function.

Question 30 [23 marks]

Let $f(x)=\frac{x^2+x-12}{2x+6}$

a. Find all vertical, horizontal and oblique asymptotes.

b. Find all intercepts with the axes.

c. Show that $f'(x)=\frac{2x^2+12x+30}{(2x+6)^2}$ and hence find and classify all turning points.

d. Hence, sketch the graph of this function.

e. State **i.** the maximum domain **ii.** the corresponding range of this function.

HL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 31 [9 marks]

Anna and Steve are going for a long walk in the mountains. The journey that they will make from one hut to another hut is modelled by the equation $h(x) = -0.25x^4 + 4x^3 - 22x^2 + 48x$, $0 \le x \le 8$

where x is the horizontal distance from the first hut measured in units of 5 kilometres and h is the height above the first hut, measured in 100-metre units. The snow line is at 3400 m, so if they are at or above this height they will be walking in snow.

a. Find the values of x for which they will be walking in snow.

b. Hence, find the horizontal distance travelled while they are walking in snow.

c. Find the difference in height between the first hut and the second hut.

HL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 32 [9 marks]

A cubic polynomial is defined by $p(x) = ax^3 + bx^2 + 2x + 1$.

When $p(x)$ is divided by $x - 2$, the remainder is 49.

When $p(x)$ is divided by $x + 1$, the remainder is -2.

a. Find the value of the constants a and b.

b. Hence, find the remainder when $p(x)$ is divided by $x - 1$.

HL PAPER 3 TECHNOLOGY REQUIRED

Question 33 [32 marks]

In this question you will investigate composite functions.

The notation $f^2(x)$ will used to represent $(f \circ f)(x)$. The notation $f^3(x)$ will used to represent $(f \circ f \circ f)(x)$ and so on.

Let $g(x) = 2x + 1$.

a. Find the functions **i.** $g^2(x)$ **ii.** $g^3(x)$ **iii.** $g^4(x)$

b. Using your answers to part (a), suggest a formula for $g^n(x)$.

c. Find the inverse function $g^{-1}(x)$.

d. Using your answer to part (b), suggest a formula for the inverse function $(g^n)^{-1}(x)$.

e. Show that $g^2(x) \neq [g(x)]^2$

Let $f(x) = ax + b$, $a \neq 0$, a $\neq 1$.

f. Prove by induction that $f^n(x)=a^n x+\frac{b(a^n-1)}{a-1}$, for all $n\in\mathbb{Z}^+$.

g. Find the formula for the inverse function $(f^n)^{-1}(x)$.

In the following parts, give any monetary answers to two decimal places.

h. On 1 January 2000, David invested £1000 in a bank that pays 5% compound interest, compounded yearly.

Every year, his interest for the year is paid into to his account on 31 December. Then, on the first day of every year (beginning 1 January 2001), David pays a further £100 into his account.

David wants to find the total amount of money in his account on 1 January 2010, just after he has paid in his tenth instalment of £100.

i. Explain why the formula obtained in part (f) can be used to find this amount.

ii. Calculate this amount.

i. John had a similar investment plan at the same bank during the same time period.

The difference between John and David was that John invested £Y on 1 January 2000 and added £300 to his account on 1 January each year.

The total amount in John's account on 1 January 2010 (after he had paid in his tenth instalment of £300) was £6000.

Using a suitable formula from a previous part, calculate the value of Y.

3 GEOMETRY AND TRIGONOMETRY

3.1 VOLUME AND SURFACE AREA OF 3D SOLIDS AND RIGHT-ANGLED TRIANGLE TRIGONOMETRY

You must know:

- ✔ formulae for the volume and surface area of common 3D solids
- ✔ the terms 'angle of elevation' and 'angle of depression'
- ✔ the definition of a bearing.

You should be able to:

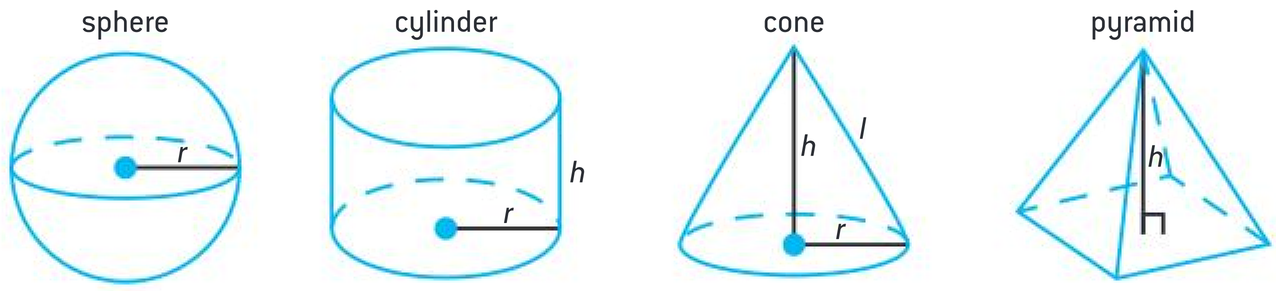
- ✔ find the distance between two points in 3D space and find their midpoint
- ✔ use the properties of right-angled triangles and the Pythagorean theorem to find side lengths and angles.

Note

You can say "right-angled triangle" or just "right triangle".

Volume and surface area of common solids

This section deals with formulas that are used with a 3D solids and right triangles and looks at the relationships between the angles and sides, their application to volume and surface area, real life situations and the area of a triangle.



sphere	cylinder	cone	pyramid
$SA = 4\pi r^2$	$SA = 2\pi r^2 + 2\pi rh$	$SA = \pi r^2 + \pi rl$	
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{1}{3}\textit{base area} \times h$

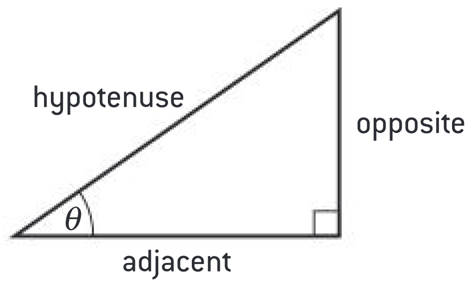
Distance between two points and their midpoint

The point that is the same distance from two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and is on the line segment $[AB]$ is called the midpoint of AB. You calculate the midpoint using the midpoint formula and the distance between the points, d, using the distance formula.

$$\text{Midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \quad d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Right-angled triangles

Naming the sides of a right triangle is an important skill. The longest side is called the hypotenuse and the other two sides are named by their relative position, opposite or adjacent (next to), a given angle. A common symbol used for an angle is the Greek letter theta, θ.



Three formulas connect the angle and the sides:

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\quad\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\quad\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

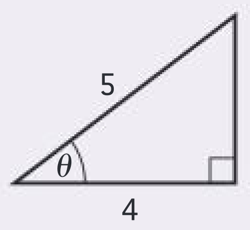
These formulas are often written in shorthand as:

$\sin\theta = \dfrac{\text{O}}{\text{H}}$ $\quad\cos\theta = \dfrac{\text{A}}{\text{H}}$ $\quad\tan\theta = \dfrac{\text{O}}{\text{A}}$

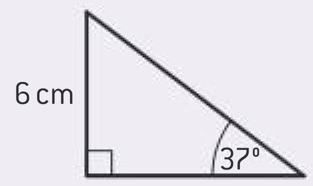
It is easy to remember these as SOHCAHTOA.

Example 3.1.1

(a) Find the value of θ.



(b) Find the length of the hypotenuse.



Note

Some others ways of saying "Find the value of …" are "Find the measure of …" or "Find the size of …".

Solution

(a) $\cos\theta = \dfrac{4}{5}$

$$\theta = \arccos\left(\frac{4}{5}\right) = 36.9° \text{ (3 sf)}$$

(b) $\sin 37° = \dfrac{6}{h}$

$$h = \frac{6}{\sin 37°} = 9.97\,\text{cm (3 sf)}$$

Assessment tip

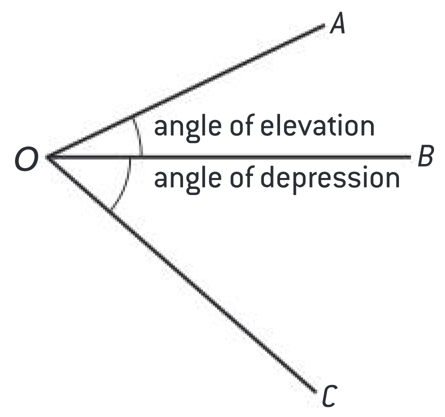
When using inverse trig functions to find an angle, most calculators use the notation $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, that is, the function with an exponent of -1. The IB does not use this notation because it is easily confused with the reciprocal trig functions. Hence, when using an inverse trig function to find an angle, use the correct mathematical notation: arcsin, arccos or arctan.

There are some special angles whose sizes you need to remember.

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Note

You will often use these on the paper where technology is not allowed.



Angles of elevation/depression and bearings

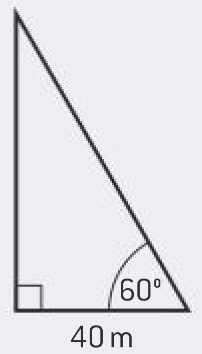
Some angles have special names.

Angle $B\hat{O}A$ is an angle of elevation (up from the horizontal) and angle $B\hat{O}C$ is an angle of depression (down from the horizontal).

Example 3.1.2

Lee stands 40 m from a cliff and measures the angle of elevation to the top of the cliff as 60°. Find the height of the cliff, as an **exact** value.

Solution

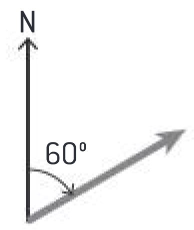


$$\tan 60° = \frac{\text{height}}{40}$$

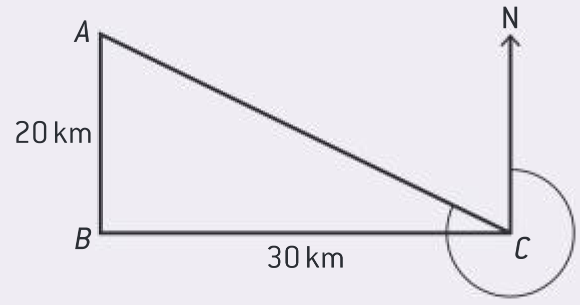
$$\Rightarrow \text{height} = 40 \tan 60° = 40\sqrt{3}\text{ m}$$

Three-figure bearings are measured clockwise from north.

This is a bearing of 060°. Note that it is written as 060°, not 60°.



Example 3.1.3



A ship sails from port A and travels for 20 km south to port B and then 30 km east to port C. Find the bearing of port A from port C.

Solution

Required bearing is 270° + angle $A\hat{C}B$

$\tan A\hat{C}B = \frac{20}{30}$

$A\hat{C}B = \arctan\left(\frac{20}{30}\right) = 33.7°$

The bearing of A from $C = 270° + 33.7° = 304°$

Example 3.1.4

The diagram shows a cone with height 15 cm and base radius 8 cm.

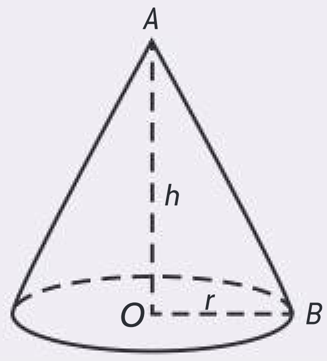
Find, correct to 2 decimal places where necessary:

(a) the slant height

(b) the volume of the cone

(c) the total surface area

(d) the measure of angle $A\hat{B}O$.



Solution

(a) $AB^2 = OA^2 + OB^2 = 15^2 + 8^2 = 289$
$\Rightarrow AB = 17\text{ cm}$

(b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(8)^2(15) = 1005.31\text{ cm}^3$ (2 dp)

(c) $SA = \pi r^2 + \pi rl = \pi(8)^2 + \pi(8)(17) = 628.32\text{ cm}^2$ (2 dp)

(d) $\tan A\hat{B}O = \frac{15}{8}$, $A\hat{B}O = 61.93°$ (2 dp)

3.2 TRIGONOMETRY RATIOS

You must know:

- ✔ that angles can be measured in radians
- ✔ exact values of sines, cosines and tangents of common angles
- ✔ the ambiguous case of the sine rule.

You should be able to:

- ✔ find side lengths and angles in triangles that are not right-angled
- ✔ find areas of triangles
- ✔ find arc length and area of a sector.

SOHCAHTOA and the Pythagorean theorem may be used in right triangles, but for non-right triangles you must use the sine rule and the cosine rule.

The sine rule is used when you are given either two angles and one side or two sides and a non-included angle. The cosine rule is used when you are given either all three sides or two sides and the included angle.

f

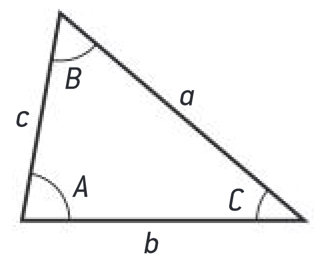
The sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule:

$a^2 = b^2 + c^2 - 2bc\cos A$, to find a side

or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, to find an angle.

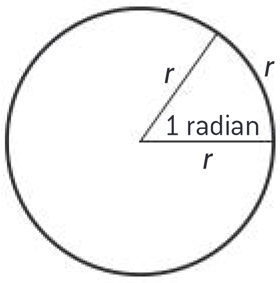


On your calculator, you might notice that both $\sin 30°$ and $\sin 150°$ are equal to 0.5, but if you find $\arcsin(0.5)$, the calculator only shows 30°. You need to be aware that if two angles add up to 180°, they have the same sine and both might fit into your triangle. This is called the 'ambiguous case' of the sine rule. A couple of useful tips are that the three angles in a triangle sum to 180° and that the largest angle is always opposite to the longest side.

There can be an ambiguous case when you use the sine rule if:

- you are given two sides and a non-included acute angle
- the side opposite the given acute angle is the shorter of the two given sides.

The area of a triangle may be found using area $= \frac{1}{2}ab \sin C$.



Angles can be measured in degrees or radians. One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle.

One complete rotation around a circle, or 360°, is equal to 2π radians.

To convert radians to degrees, multiply the measurement in radians by $\frac{180}{\pi}$

To convert degrees to radians, multiply the measurement in degrees by $\frac{\pi}{180}$

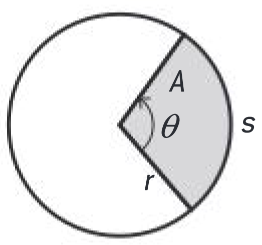
Some helpful angles to remember:

Degrees	30	45	60	90	180
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π

The central angle in a circle is a fraction of 2π, so you can calculate the length of the arc the angle subtends as a fraction of the circumference. Using radian measure for the central angle, you can write simple formulae for finding the length of an arc and the area of a sector.

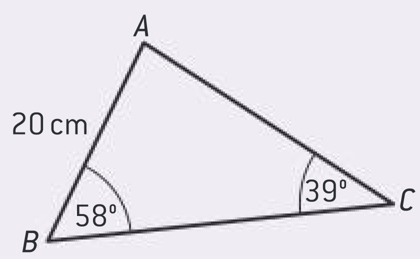
Arc length $s = r\theta$

Area of a sector $A = \frac{1}{2}r^2\theta$



Example 3.2.1

Find (a) the length of side AC and (b) the area of triangle ABC.



Solution

(a) $\frac{b}{\sin B} = \frac{c}{\sin C}$

$\frac{b}{\sin 58°} = \frac{20}{\sin 39°}$

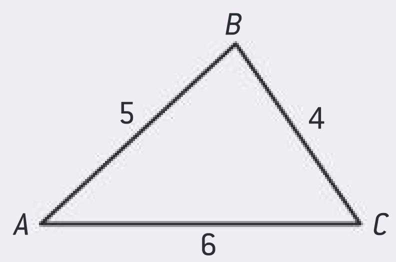
$b = \frac{20 \sin 58°}{\sin 39°} = 27.0\,\text{cm (3 sf)}$

(b) Angle $\hat{A} = 180 - 58 - 39 = 83°$

Area $= \frac{1}{2} bc \sin A = \frac{1}{2}(20)(26.95...) \sin 83° = 268\,\text{cm}^2$ (3 sf)

Example 3.2.2

Find in radians the smallest angle in triangle *ABC*.



Solution

The smallest angle is opposite the smallest side, so angle $\hat{A}$ is the smallest angle.

$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$

$= \frac{6^2 + 5^2 - 4^2}{2(6)(5)} = \frac{45}{60} = 0.75$

$\hat{A} = \arccos(0.75) = 0.723$ (3 sf)

Example 3.2.3

In triangle *ABC*, angle $\hat{A} = 40°$, $a = 14\,\text{cm}$ and $c = 20\,\text{cm}$.
Give two possible values for angle *C*.

Solution

$\frac{\sin \hat{C}}{20} = \frac{\sin 40°}{14}$

$\sin \hat{C} = \frac{20 \sin 40°}{14}$

$\hat{C} = \arcsin\left(\frac{20 \sin 40°}{14}\right) = 66.7°, 113°$ (3 sf)

Assessment tip

Questions that use the sine rule and ask for two possible values or all possible angles often refer to the use of the ambiguous case.

Note

The calculator gives 66.7°, correct to 3 sf. The second answer is $180° - 66.7° = 113°$

Assessment tip

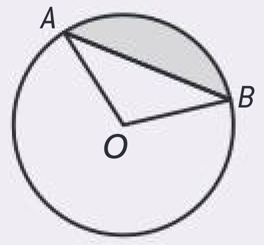
This is a popular style of question for examinations. First you have to find the area of the whole sector. Next you will have to subtract the area of triangle OAB.

Example 3.2.4

This diagram shows a circle with centre O and radius $OA = 5\,\text{cm}$.

Angle $A\hat{O}B = \frac{3\pi}{4}$

Find the area of the shaded segment.



Solution

Area of sector $OAB = \frac{1}{2}(5)^2\left(\frac{3\pi}{4}\right) = 29.5$

Area of triangle $OAB = \frac{1}{2}(5)(5)\sin\left(\frac{3\pi}{4}\right) = 8.84$

Segment area = area of sector OAB – area of triangle OAB

$= 20.6\,\text{cm}^2$ (3 sf)

Note that $29.5 - 8.84 = 20.7$ to three significant figures. The correct answer is 20.6 because the full answers of 29.45243… and 8.83883… should be used.

Example 3.2.5

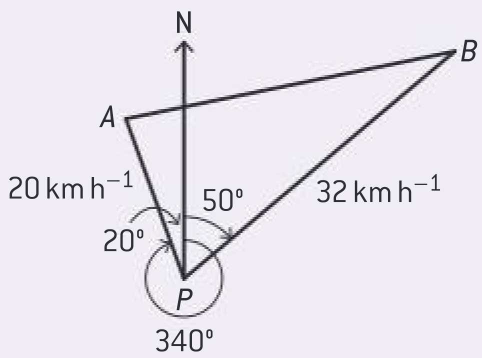
Two fishing boats, A and B, leave port P at 8:00 am. Boat A travels on a bearing of 340° at $20\,\text{km}\,\text{h}^{-1}$ and boat B travels on a bearing of 050° at $32\,\text{km}\,\text{h}^{-1}$.

(a) Sketch a diagram to show this information.

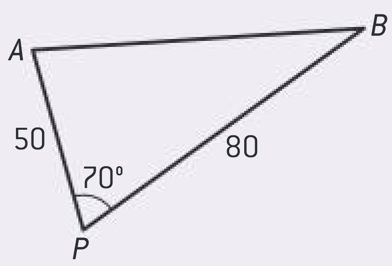
(b) Find the distance between the two boats at 10:30 am.

Solution

(a)



(b)



$PA = 2.5 \times 20$ and $PB = 2.5 \times 32$

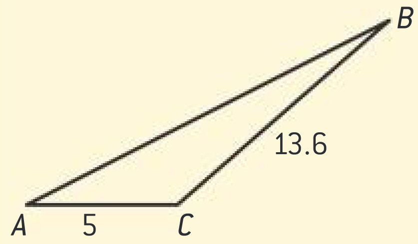
Using the cosine rule:

$AB^2 = 50^2 + 80^2 - (2 \times 50 \times 80 \times \cos 70°)$

$AB = 78.5\,\text{km}$ (3 sf)

SAMPLE STUDENT ANSWER

The diagram shows triangle *ABC*.



The angle at C is obtuse, $AC = 5\,\text{cm}$, $BC = 13.6\,\text{cm}$ and the area is $20\,\text{cm}^2$

(a) Find angle $A\hat{C}B$.

(b) Find AB.

(a) Area $= \frac{1}{2}ab\sin C$

$20 = \frac{1}{2}(5)(13.6)\sin C$

$\sin C = 0.588 = 36.0°$

(b) $(AB)^2 = 5^2 + 13.6^2 - 2(5)(13.6)\cos 36°$

$AB = 13.7$

▲ The correct method for finding the angle was used.

▼ The sine of an angle lies between 0 and 1. $\sin\hat{C} = 0.588$ is correct but the next line should read $\hat{C} = \arcsin(0.588) = 180° - 36° = 144°$ (3 sf)

▲ The correct choice of the cosine rule was used.

▼ The question asked for the obtuse angle which was clearly stated and shown in the diagram. The student should have used $(AB)^2 = 5^2 + 13.6^2 - 2(5)(13.6)\cos 143.968\ldots$ giving the correct answer of $AB = 17.9\,\text{cm}$ (3 sf).

The answer above could have achieved 3/6 marks.

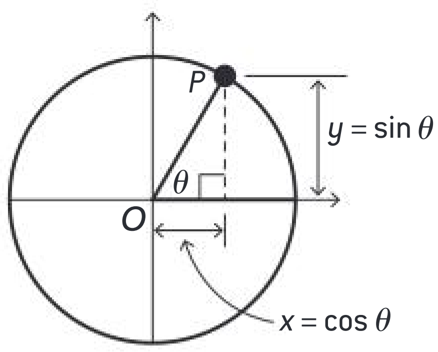
3.3 TRIGONOMETRIC IDENTITIES AND EQUATIONS

You must know:

- ✔ the definitions of sin and cos in terms of the unit circle
- ✔ the Pythagorean identity and the double angle identities for sin and cos.

You should be able to:

- ✔ solve trig equations in a finite interval
- ✔ prove trig identities.



The unit circle and trigonometric identities

The unit circle is a circle of radius one unit with its centre at the origin.

In the unit circle, the radius is one unit, therefore $OP = 1$. Using the Pythagorean theorem, $\sin^2\theta + \cos^2\theta \equiv 1$. This is called a trigonometric identity.

We have the definitions

$\sin\theta =$ projection onto the y-axis

$\cos\theta =$ projection onto the x-axis

The diagram on the next page illustrates the sign of the trig functions in each quadrant of the unit circle.

There are some useful patterns for supplementary angles (angles that add up to 180°).

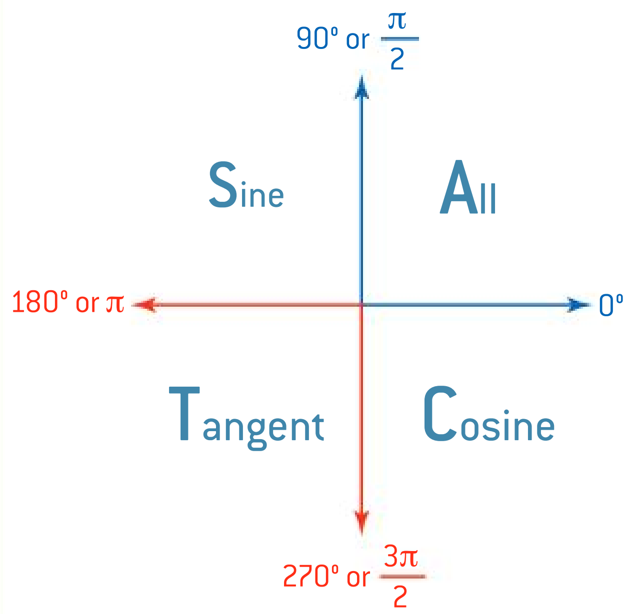
When $A + B = 180°$ or π radians,

$\sin A = \sin B$ and $\cos A = -\cos B$

Link to Odd and even functions AHL 2.14

Note

Try to memorize this diagram. It will help you remember which trig ratios are positive in each of the four quadrants.



You can think of a mnemonic such as 'All Students Take Calculus' to help you remember this.

Also:

sin is an odd function because $\sin(-\theta) = -\sin\theta$

cos is an even function because $\cos(-\theta) = \cos\theta$

tan is an odd function because $\tan(-\theta) = -\tan(\theta)$

Assessment tip

Always keep a copy of the formula booklet with you. Use it in class, when doing homework and during assessments as your teacher permits. You will find out what is in there and what is not; you will be able to recognize formulae, facts and identities much easier. You will also see what you have to remember that is not in the formula booklet.

Below are the formulae required. Remember that all formulae are required for HL but only the SL ones for the SL assessment.

SL formulae

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad\quad$ $\sin^2\theta + \cos^2\theta = 1$

The double angle identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

HL formulae

Reciprocal trigonometric identities

$\sec\theta = \dfrac{1}{\cos\theta}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\quad$ This identity is not in the formula book but is easily remembered.

Pythagorean identities

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Compound angle identities

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Double angle identity for tan

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Example 3.3.1

$\cos\theta = \frac{\sqrt{3}}{2}$, and $\tan\theta$ is negative. Without the use of a calculator, find:

(a) $\sin\theta$

(b) θ, where $0 \le \theta \le 360°$

(c) $\sin 2\theta$

Solution

(a) $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\theta = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$$

$\sin\theta = \pm 0.5$

As the cosine is positive and the tangent is negative, θ must lie between 270° and 360° where sine is negative.

$\therefore \sin\theta = -0.5$

(b) $\arcsin(0.5) = 30°$

$\theta = 360° - 30° = 330°$

(c) $\sin 2\theta = 2(-0.5)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$

You will be asked to solve trigonometric equations for a finite interval, both graphically and analytically, with some equations leading to quadratic equations. You should know that all trigonometric functions are periodic, so a trigonometric equation will generally have an infinite number of solutions, which is why questions will restrict the domain, saying things such as $0 \le \theta \le 2\pi$. If the domain is in radians, you should answer in radians. If the domain is in degrees, then answer in degrees.

Example 3.3.2

Solve $2\sin^2\theta + \sin\theta = 0$ for $0 \le \theta \le 4\pi$

Solution

$2\sin^2\theta + \sin\theta = 0$

$\sin\theta(2\sin\theta + 1) = 0$

$\sin\theta = 0$ or $2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \arcsin(0)$ or $\theta = \arcsin\left(-\frac{1}{2}\right)$

$$\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi, 3\pi, \frac{19\pi}{6}, \frac{23\pi}{6}, 4\pi$$

Note

0 and $\frac{7\pi}{6}$ are called the primary values.

π and $\frac{11\pi}{6}$ are called the secondary values. You now take each value and add or subtract 2π until you reach the ends of the given domain.

Example 3.3.3

Solve $\cos 2\theta = \sin\theta - 2,\ 0 \le \theta \le \pi$

Solution

$1 - 2\sin^2\theta = \sin\theta - 2$

$2\sin^2\theta + \sin\theta - 3 = 0$

$(2\sin\theta + 3)(\sin\theta - 1) = 0$

$\sin\theta = -\frac{3}{2}, 1$

$-\frac{3}{2}$ is not possible as $-1 \le \sin\theta \le 1$ for all θ

$\therefore\ \sin\theta = 1$ and $\theta = \frac{\pi}{2}$

SAMPLE STUDENT ANSWER

Let $f(x) = 6x\sqrt{1 - x^2}$, for $-1 \le x \le 1$, and $g(x) = \cos(x)$, for $x \in \mathbb{R}$.

Let $h(x) = (f \circ g)(x)$.

(a) Write $h(x)$ in the form $a\sin(bx)$, where $a, b \in \mathbb{Z}$.

(b) Hence find the range of h.

(a) $h(x) = 6(\cos x)\sqrt{1 - (\cos x)^2}$

$= 6(\cos x)\sqrt{\sin^2 x}$

$= 6\cos x^2 \sin x^2$

$= 6\sin x$

(b) $\{y \in \mathbb{R}, y \neq 0\}$

The answer above could have achieved 2/7 marks.

▲ The student successfully applied g first, followed by f, and used the identity $\sin^2 x \equiv 1 - \cos^2 x$.

▼ The student should have written $\sqrt{\sin^2 x} = \sin x$, and this line would have been $6\sin x\cos x$ to give $3\sin 2x$ as the correct answer.

▼ The range of $\sin x$ is from -1 to 1. Even with the incorrect answer of $6\sin x$, the range would be -6 to 6 to gain follow through (FT) marks. Can you find the correct answer?

Example 3.3.4 (HL)

Prove that $\tan x\sin x + \cos x \equiv \sec x$

Solution

LHS: $\tan x\sin x + \cos x$

$\equiv \frac{\sin x}{\cos x} \times \sin x + \cos x \equiv \frac{\sin^2 x}{\cos x} + \cos x \equiv \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \equiv \frac{\sin^2 x + \cos^2 x}{\cos x}$

$\equiv \frac{1}{\cos x} \equiv \sec x$

$\equiv$ RHS

Assessment tip

When proving an identity, it is usually best to work on one side **only** until it is the same as the other side. Sometimes, however, you may have to work on both sides, until they are both the same, but make sure you clearly identify which side you are working on. Do not place an equals sign between the two sides until you have actually proved they are the same.

Example 3.3.5 (HL)

Show that the exact value of $\tan 165° = -2 + \sqrt{3}$

Solution

$165° = (120° + 45°)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(120° + 45°) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$

$$= \frac{-\sqrt{3}+1}{1-(-\sqrt{3})(1)}$$

$$= \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{1-2\sqrt{3}+3}{1-3}$$

$$= \frac{4-2\sqrt{3}}{-2}$$

$$\Rightarrow \tan(165°) = -2 + \sqrt{3}$$

Assessment tip

When a question says "the exact value", do not round your answers to decimal values. Keep your answer as a surd or, when finding angles, in terms of π.

3.4 TRIGONOMETRIC FUNCTIONS

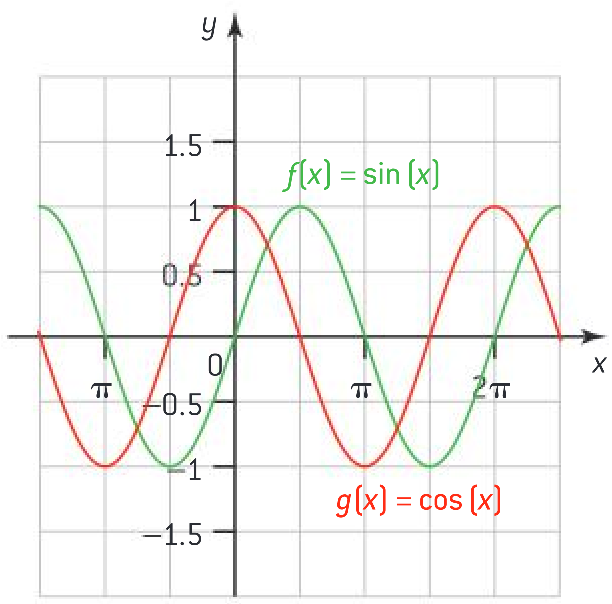
You must know:

- ✔ the maximum, minimum and zero values for sine and cosine
- ✔ how to show a graph on your calculator
- ✔ the basic shapes of the sin, cos and tan functions
- ✔ the terms 'amplitude' and 'period'.

You should be able to:

- ✔ model practical situations with trig equations
- ✔ sketch any given function using a calculator in degree or radian mode.

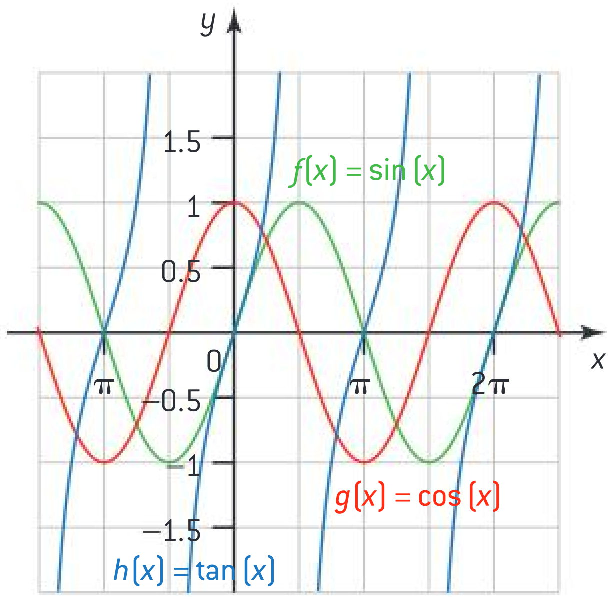
This graph shows the functions of $\sin x$ and $\cos x$. Note their similarities in maximum and minimum values and general shape. Can you see that translating the sine curve 90° or $\frac{\pi}{2}$ to the left would put it directly over the cosine curve?



Curves that repeat their pattern as we follow them along the horizontal axis are called **periodic functions**.
The functions $\sin x$ and $\cos x$ both have a period of 2π.

The general equation of the sine curve is $f(x) = a\sin(b(x-c)) + d$

- a is the amplitude and is found by $a = \frac{\text{max } y \text{ value} - \text{min } y \text{ value}}{2}$
- b is the frequency. This means the number of complete periods between 0 and 2π. The period is $\frac{2\pi}{|b|}$. The period is also called the wavelength.



- c is the horizontal shift
- d is the vertical shift

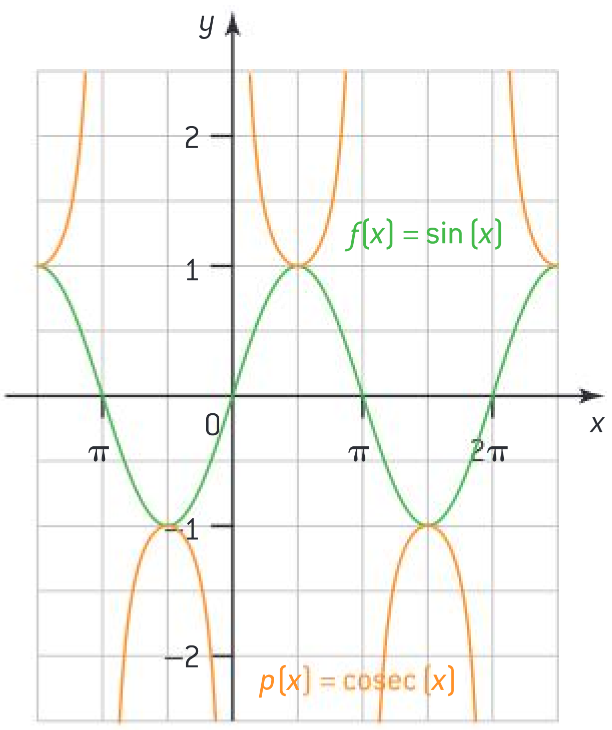
Can you see what happens when you add a tangent curve with the vertical asymptotes?

The asymptotes are at ... $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$...

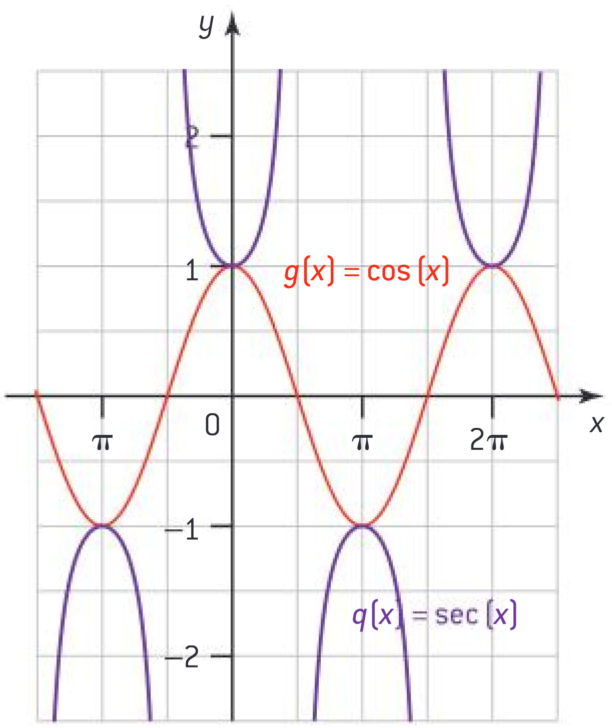
The period of tan is π.

Below are the graphs of the trigonometric functions and their reciprocal functions.

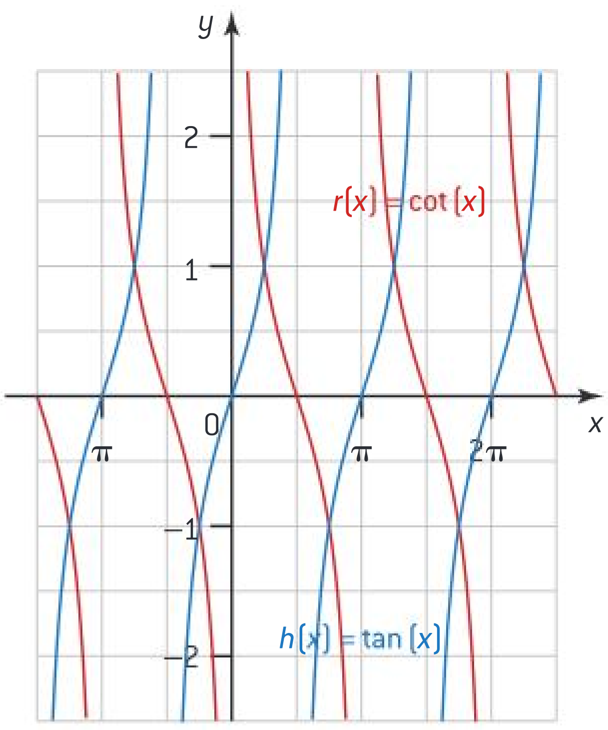
sine and cosecant



cosine and secant



tangent and cotangent



Assessment tip

When you have multiples of an angle (in the form ax), write out the normal solutions for x and then keep on going until you have met a times the given domain. In this case you have $2x$, so instead of going from 0 to 2π, you go from 0 to 4π.

Example 3.4.1

For the function $f(x) = 3\sin 2x$, $0 \le x \le 2\pi$:

(a) write down the amplitude

(b) find the period

(c) solve $f(x) = 0$

(d) sketch $f(x)$.

Solution

(a) The amplitude is 3

(b) The period $= \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$

(c) $3\sin 2x = 0$

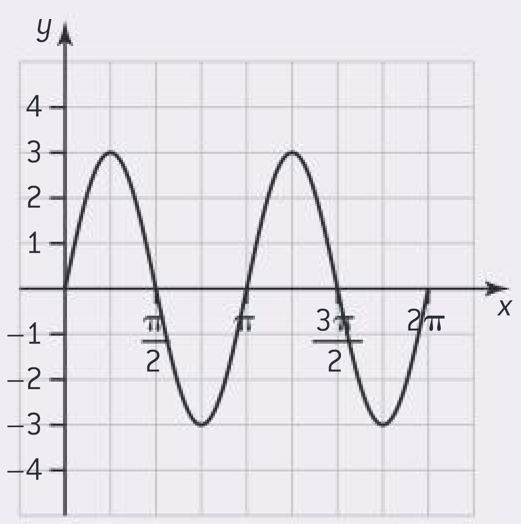
$\sin 2x = 0$

$2x = \arcsin(0)$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(d)



Example 3.4.2

Describe the transformations necessary to produce the graph of $g(x) = 1 - \sin(\theta + \frac{\pi}{2})$ from $f(x) = \sin\theta$

Solution

The negative coefficient of sin reflects the sine curve in the x-axis.

$+\frac{\pi}{2}$ translates the curve $\frac{\pi}{2}$ units to the left.

The constant term, 1, translates the curve 1 unit up (this translation happens last).

Link to Transformation of graphs SL 2.11

Example 3.4.3

The depth of the water at the base of a cliff is measured as 7 m at low tide and 15 m at high tide. If low tide was at 6am on Sunday and high tide occurred at noon:

(a) find a cosine function to model the water levels if the cycle repeats itself at approximately 12-hour intervals

(b) sketch the function from midnight to noon.

Cliff divers need at least 10 m of water and will not arrive until 7 am.

(c) Draw a suitable line on your sketch to find the earliest time, to the nearest minute, that they can dive and state the earliest time that they can dive.

Solution

(a) The amplitude of the function.

$$a = \frac{\text{maximum depth} - \text{minimum depth}}{2} = \frac{15 - 7}{2} = 4$$

The period if the cycle repeats every 12 hours.

$$12 = \frac{2\pi}{b}$$

$$b = \frac{2\pi}{12} = \frac{\pi}{6}$$

The vertical shift $= \frac{15 + 7}{2} = 11$ (the midpoint of the depth is at 11 m).

There is no horizontal shift as, if the greatest depth is at noon (12 pm) and the cyclic function has a period of 12 hours, it was also at a greatest depth of 15 m at the previous midnight.

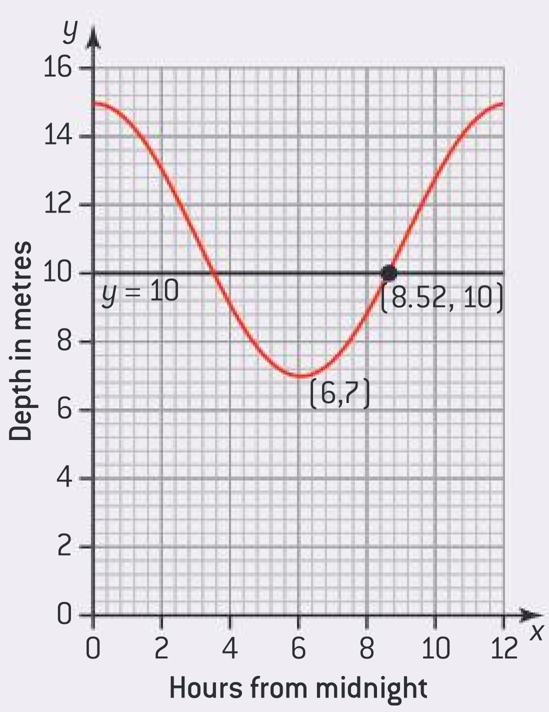
The function for the depth of the tide is modelled by

$f(x) = a\cos(b(x - c)) + d$

$f(x) = 4\cos(\frac{\pi}{6}x) + 11$,

where x is the number of hours after midnight.

(b) and (c)



The cliff divers have sufficient depth after 8.52 hours. They can start diving at 8:31am.

Assessment tip

For part (c), you should use your calculator, draw the line and find the points of intersection.

Note

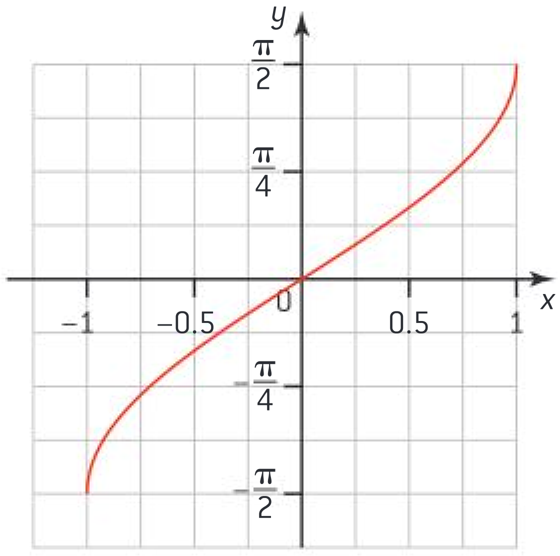
A common student error is to think that 8.52 hours is the same as 8:52am. There are 60 minutes in an hour, so you have to know that this is 8 hours and 52 hundredths of an hour. Hence 8 hours and 31 minutes, to the nearest minute.

Link to Inverse functions AHL 2.14

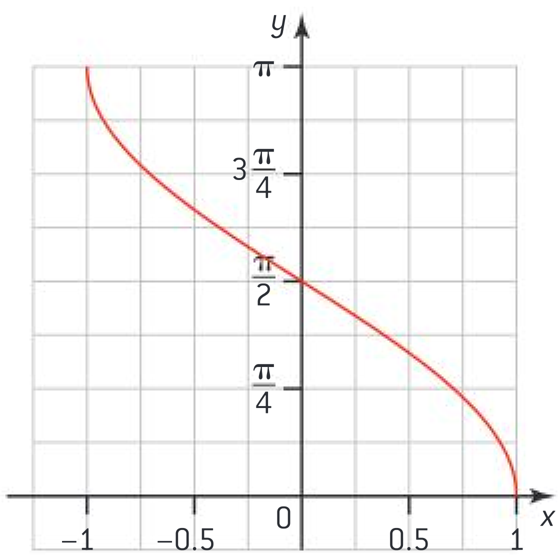
Inverse trig functions

The inverse functions of the three main trig functions sin, cos and tan are denoted by arcsin, arccos and arctan, respectively.

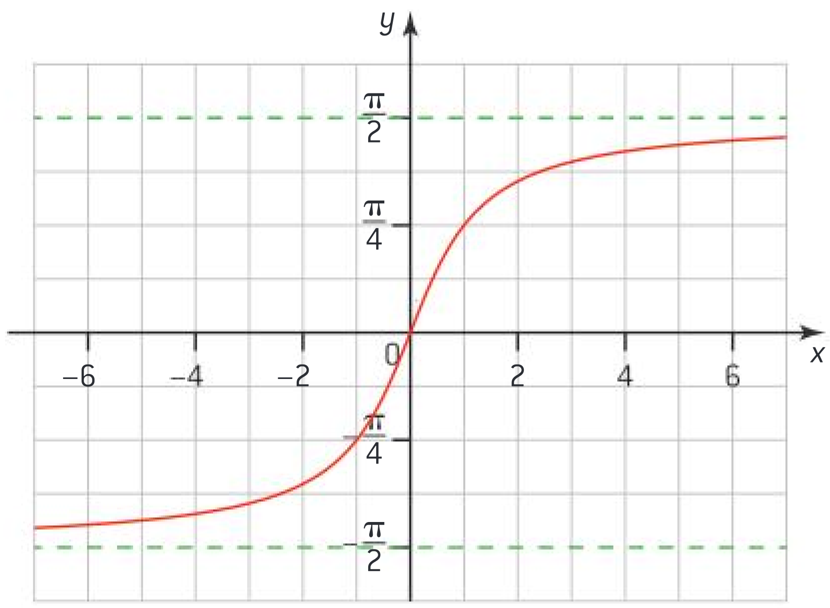
The function $y = \arcsin x$ has a domain of $[-1, 1]$ and a range of $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, with a graph as shown:



The function $y = \arccos x$ has a domain of $[-1, 1]$ and a range of $[0, \pi]$, with a graph as shown:



The function $y = \arctan x$ has a domain of $\mathbb{R}$ and a range of $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, with a graph as shown. The graph has horizontal asymptotes at $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$



Example 3.4.4

Evaluate $\arctan(\sqrt{3}) - \operatorname{arcsec}(-2)$

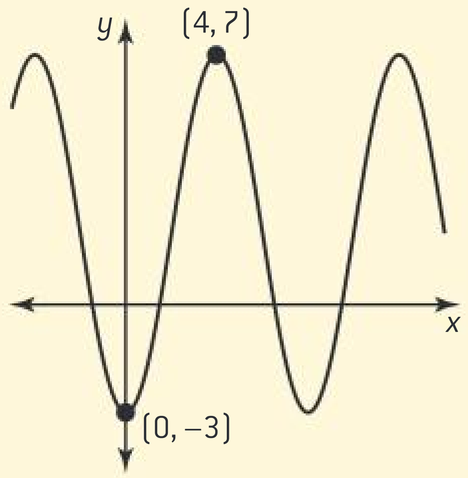
Solution

$\arctan(\sqrt{3}) - \operatorname{arcsec}(-2) = \arctan(\sqrt{3}) - (\pi - \operatorname{arcsec}(2))$

$\frac{\pi}{3} - \pi + \arccos\left(\frac{1}{2}\right) = -\frac{2\pi}{3} + \frac{\pi}{3} = -\frac{\pi}{3}$

SAMPLE STUDENT ANSWER

The graph of $y = p\cos qx + r$, $-5 \le x \le 14$ is shown.



Find the values of
(a) p (b) q, given that it is positive (c) r

(a) Amplitude $(p) = \frac{7+3}{2} = 5$

▲ The correct method for finding the amplitude was used.

▼ As this is a reflection of the cosine curve in the x-axis, p should be -5.

(b) Period $(q) = 8$

▲ This is the correct period.

▼ q is not the period, but $q = \frac{2\pi}{\text{period}} = \frac{2\pi}{8} = \frac{\pi}{4}$

(c) $r = \frac{7-3}{2} = 2$

▲ This is the correct value for r.

The answer above could have achieved 4/6 marks.

3.5 VECTORS IN TWO AND THREE DIMENSIONS (HL)

You must know:

- ✔ the difference between a vector and a scalar
- ✔ the definition of the scalar product (dot product) and vector product (cross product)
- ✔ conditions for perpendicular and parallel vectors.

You should be able to:

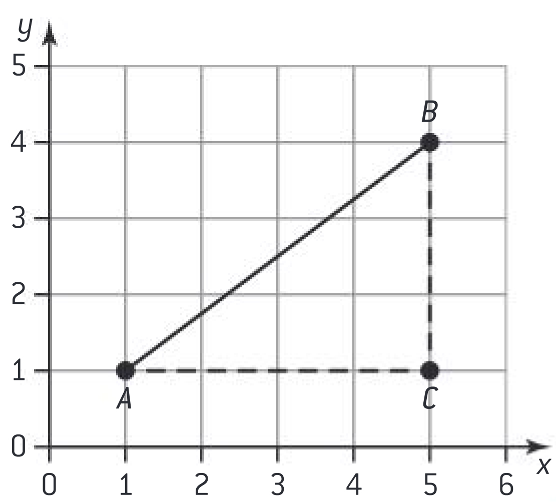
- ✔ perform algebraic manipulations with vectors
- ✔ calculate the angle between two vectors
- ✔ find the vector equations of lines and planes.

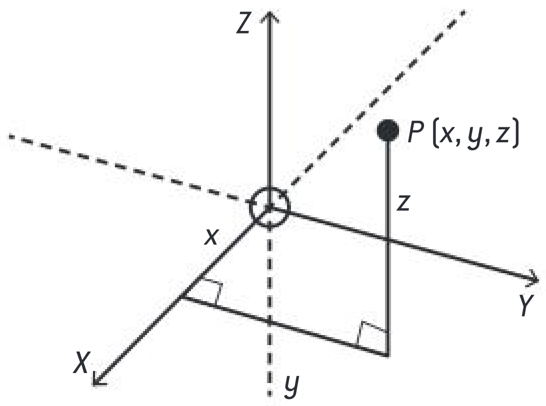
Vectors represent the movement or displacement between points. The points on the graph show $A(1, 1)$ and $B(5, 4)$.

If you start at A and travel 4 units horizontally to the right and then 3 units vertically up, you arrive at point B.

In vector form this is $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ or in unit vector form $4\mathbf{i} + 3\mathbf{j}$.

The top number on a vector indicates horizontal movement (+ for right, – for left) and the bottom number indicates vertical movement (+ for up, – for down).





$\overrightarrow{AB}$ may also be written as a single, bold letter like **u**. In your working, you should put an arrow over a letter to show it is a vector.

The journey from A to C and then from C to B also starts at A and finishes at B and so $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$. This shows how vectors are added.

As well as objects that move along a flat surface (a plane) in two dimensions, you can also think about objects that move around in three-dimensional space. In two dimensions we have the x- and y-axes, and a third dimension adds a z-axis. You can represent a vector in three dimensions in a similar way and introduce the letter **k** for the vector of length one unit in the z-direction.

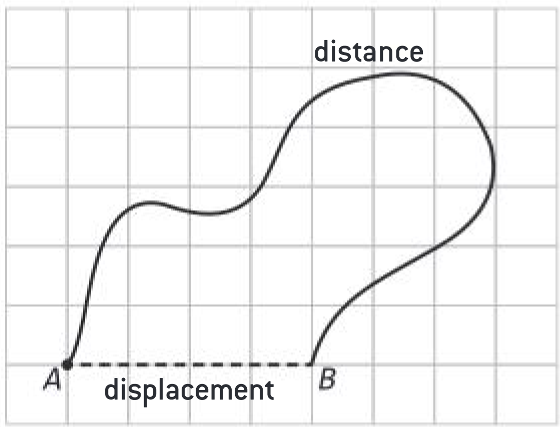
The vector from O to point P can be represented as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Assessment tip

A vector is written as a column whereas a point is written as a row. Always check that you have used the correct notation for the object you are describing.

A vector has size (magnitude) and direction. Examples of vectors are displacement and velocity.

- Distance is a scalar quantity which refers to how far an object has moved during its motion.
- Displacement is a vector quantity that depends only on the difference between the initial and final position of the body.



A scalar quantity, often represented by k, has size but no direction. Examples of scalars are distance and speed.

The **magnitude** of $\overrightarrow{AB}$ is the length of the vector and is denoted by $|\overrightarrow{AB}|$

The magnitude is found by using the Pythagorean theorem and is given by the formula:

Assessment tip

Always put arrows over all the vector symbols use use. You will not be able to use bold print as is used in books. It is important to distinguish between vectors and scalars.

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}, \text{ where } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Vectors are added by adding the three corresponding coefficients, for example:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \text{ or } (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) = 5\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$$

A vector can be multiplied by a scalar and in this case all the three coefficients are multiplied by the scalar, for example:

$$9\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \\ 27 \end{pmatrix} \text{ or } 9(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 9\mathbf{i} + 18\mathbf{j} + 27\mathbf{k}$$

The vectors **a** and k**a** will be parallel. The magnitude will have increased by $|k|$. If k is negative the direction will be reversed.

Scalar multiplication of a vector is distributive over vector addition, which means that $k(\mathbf{a}+\mathbf{b}) = k\mathbf{a}+k\mathbf{b}$

The zero vector has magnitude of 0 and is denoted by **0**.

Hence $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$

A **unit vector** is a vector of length one unit. For example, the unit vector in the direction of $\overrightarrow{AB}$ is $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.

Example 3.5.1

(a) Find the magnitude of $(3\mathbf{i} - 2\mathbf{j} + \sqrt{3}\mathbf{k})$.

(b) Find a unit vector parallel to $\begin{pmatrix}6\\8\end{pmatrix}$.

Solution

(a) $\sqrt{3^2+(-2)^2+(\sqrt{3})^2} = 4$

(b) $\sqrt{6^2+8^2} = 10$

The vector has length 10. To reduce it to length one (a unit vector), multiply by a scalar of $\frac{1}{10}$

$$\frac{1}{10}\begin{pmatrix}6\\8\end{pmatrix} = \begin{pmatrix}0.6\\0.8\end{pmatrix}$$

Example 3.5.2

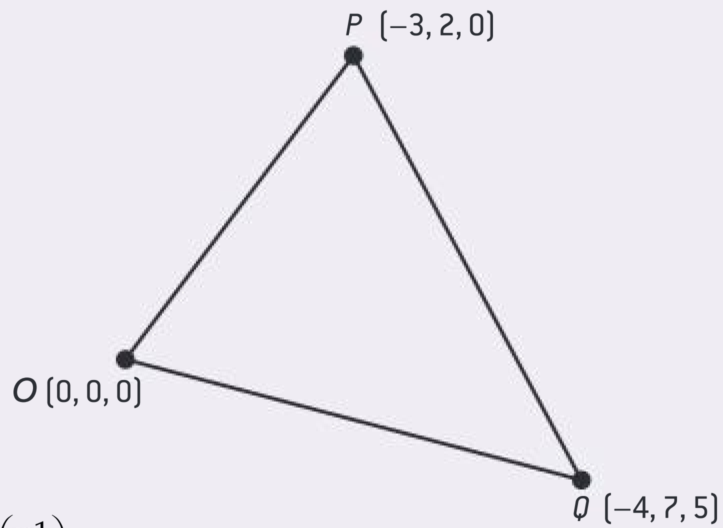
Points P and Q have coordinates $(-3, 2, 0)$ and $(-4, 7, 5)$. Find the vector $\overrightarrow{PQ}$.

Solution

The origin is (0, 0, 0).
Writing position vectors for $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

$$\overrightarrow{OP} = \begin{pmatrix}-3\\3\\0\end{pmatrix} \Rightarrow \overrightarrow{PO} = \begin{pmatrix}3\\-2\\0\end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix}-4\\7\\5\end{pmatrix}$$



$$\overrightarrow{PQ} = \overrightarrow{PO}+\overrightarrow{OQ} = \begin{pmatrix}3\\-2\\0\end{pmatrix}+\begin{pmatrix}-4\\7\\5\end{pmatrix} = \begin{pmatrix}-1\\5\\5\end{pmatrix}$$

Note

$\overrightarrow{PQ} = -\overrightarrow{OP}+\overrightarrow{OQ} = \overrightarrow{PO}+\overrightarrow{OQ}$

A common mistake is to write $\overrightarrow{PQ} = \overrightarrow{OP}+\overrightarrow{OQ}$

Note

The scalar product is unusual as when you take the scalar product of two vectors the answer is a scalar. Usually when you combine two objects the answer is the same type of object.

Note

Often you will just be working in two dimensions when finding the angle between two vectors.

Assessment tip

It is common to see a question asking to show that two vectors are perpendicular.

f

The scalar product (sometimes called the dot product) of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined by:

$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

The scalar product can be evaluated using the following formula:

$$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3, \text{ where } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_2 \end{pmatrix}$$

You can combine these two formulae to find the angle between two vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_2 \end{pmatrix}$:

$$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|}$$

For two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \bullet \mathbf{b} = 0$ if and only if the vectors are perpendicular.

When $\mathbf{a} \bullet \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}|$, $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

Properties of the scalar product

$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

$(k\mathbf{a}) \bullet \mathbf{b} = k(\mathbf{a} \bullet \mathbf{b})$, where k is a scalar

$\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$

Example 3.5.3

$\overrightarrow{AB} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{AC} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$

Assessment tip

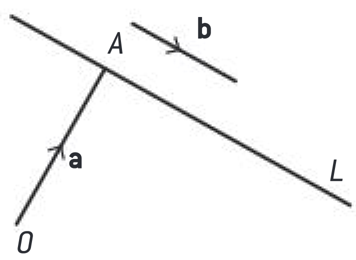
Another term for perpendicular vectors is orthogonal vectors.

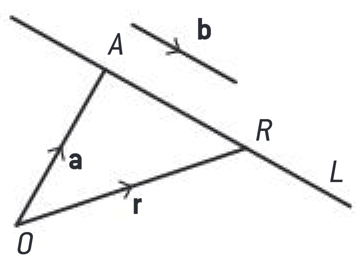
Solution

$\overrightarrow{AB} \bullet \overrightarrow{AC} = (6 \times (-2)) + ((-2) \times (-3)) + (3 \times 2) = -12 + 6 + 6 = 0$

$\cos\theta = 0$ and $\theta = 90°$

Therefore $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.





The vector equation of a line

A straight line L parallel to vector $\mathbf{b}$ passes through point A, where A has a position vector $\mathbf{a}$.

If we let R be any point on the line, then $\overrightarrow{AR}$ is parallel to $\mathbf{b}$.

Any point R on the line L can be found by starting at the origin, moving along vector $\mathbf{a}$ to reach the line, then moving through some multiple of vector $\mathbf{b}$ to reach the point R. Now there must be some number λ such that $\overrightarrow{AR} = \lambda\mathbf{b}$.

Therefore, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

where:

- $\mathbf{r}$ is the general position vector of a point on the line
- $\mathbf{a}$ is a given position vector of a point on the line
- $\mathbf{b}$ is a direction vector parallel to the line
- λ is called the parameter.

Note

It is a common mistake to think that $\mathbf{r}$ is the line. In fact, $\mathbf{r}$ is actually a variable position vector. It is the end of $\mathbf{r}$ that traces out the line L.

In kinematics, the equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be interpreted as the straight line trajectory of an object, with $\mathbf{b}$ representing velocity ($|\mathbf{b}|$ representing speed) and λ representing time.

Link to Kinematics SL 5.9

Example 3.5.4

Find the vector equation of the line passing through the points A (–3, 4, 1) and B (2, –2, 2)

Solution

The direction vector $\mathbf{b}$ of the line can be found by finding $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{OA} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$$

You can now choose one of the points, A or B, to find position vector $\mathbf{a}$.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

$$\mathbf{r} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$$

An equation of a line can be represented in terms of a set of equations known as **parametric equations,** where the x, y and z distances are defined in terms of a fourth variable, such as λ.

A vector expressed in the form of a set of parametric equations is said to be expressed in parametric form.

The vector form and the parametric form of the equation of the line are related in the following way.

Using the equation from the previous example, if

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$$

then the parametric equations are

$x = -3 + 5\lambda$

$y = 4 - 6\lambda$

$z = 1 + \lambda$

Note

With vector equations you can equate the coefficients of the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components, thus creating three scalar equations from one vector equation. This is what was done here.

In general, the parametric form of the equation of a line is

$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

which, when eliminating λ, gives the Cartesian equations of a line:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

If the line is given in Cartesian form then it can be read off that the line is parallel to the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ and goes through the point $(x_0,\ y_0,\ z_0)$.

This information is very useful. The parametric equations from the previous example can be converted into Cartesian form as

$$\frac{x+3}{5} = \frac{y-4}{-6} = \frac{z-1}{1}$$

Any two lines can be classified as one of parallel, intersecting or skew. Skew lines are non-parallel lines that do not intersect in 3D space.

The vector product (sometimes called the cross product) of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined by $\mathbf{v} \times \mathbf{w} = |\mathbf{v}||\mathbf{w}|\sin\theta\,\hat{\mathbf{n}}$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ with direction given by the right-hand screw rule. This gives rise to the formula $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$.

The vector product can be evaluated using the formula:

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \text{where } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_2 \end{pmatrix}$$

Properties of the vector product

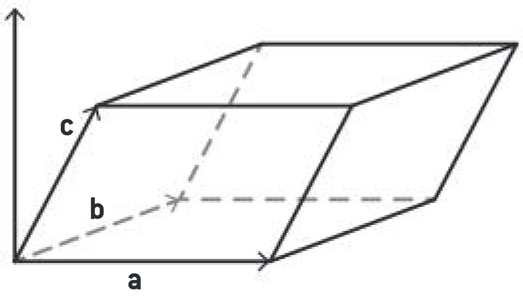
- The vector products $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same magnitude but are in opposite directions, so $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- The vector product of non-zero parallel vectors is zero.
- The vector product is distributive over vector addition. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$.
- $t(\mathbf{a} \times \mathbf{b}) = (t\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (t\mathbf{b})$, where t is a a scalar.
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

For two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if the vectors are parallel.

The magnitude of the vector product measures the area of the parallelogram that the two vectors create, so if $\mathbf{v}$ and $\mathbf{w}$ form two adjacent sides of a parallelogram the area of the parallelogram is given by:

$A = |\mathbf{v} \times \mathbf{w}|$ where $\mathbf{v}$ and $\mathbf{w}$ form two adjacent sides of a parallelogram.

Dividing this formula by two would give the area of a triangle.



A **parallelepiped** is a 3D shape formed by six parallelograms.

The volume of a parallelepiped is given by the formula $V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$

The volume of a pyramid bounded by three non-coplanar vectors is $V = \frac{1}{6}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$

Example 3.5.5

Find a unit vector that is perpendicular to both of the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{k}$.

Solution

The vector $\mathbf{a} \times \mathbf{b}$ will be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (3 \times -3) - (-2 \times 0) \\ (-2 \times 5) - (1 \times -3) \\ (1 \times 0) - (3 \times 5) \end{pmatrix} = \begin{pmatrix} -9 \\ -7 \\ -15 \end{pmatrix} = -9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}$$

$-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

Finding its magnitude:

$|-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}| = \sqrt{(-9)^2 + (-7)^2 + (-15)^2} = \sqrt{355}$

The unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{\sqrt{355}}(-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k})$

Assessment tip

Many students use a calculator to evaluate coefficients here such as $\frac{1}{\sqrt{355}} \times -9 = -0.4776704032\ldots$

It is better to leave your result as the exact answer. This applies throughout for irrational numbers such as roots, logs, π, etc.

Example 3.5.6

Find the volume of the parallelepiped with edges $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Solution

First finding $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (1 \times 4) - (1 \times 2) \\ (1 \times 1) - (2 \times 4) \\ (2 \times 2) - (1 \times 1) \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$$

Finding the scalar product of $(\mathbf{a} \times \mathbf{b})$ with $\mathbf{c}$.

$V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |(2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})| = |6 - 14 + 3| = |-5| = 5$

The parallelepiped has a volume of 5 cubic units.

Vector equations of planes

A plane is created by two intersecting lines, producing an infinite flat surface. Any straight line joining any two points on the plane will lie entirely in the plane. This surface can be vertical, horizontal or sloping. An equation defining the position of a plane can be found using:

- three non-collinear points
- a point and two non-parallel vectors
- a vector perpendicular (normal) to the plane, and a point that lies in the plane.

The parametric form of the vector equation of the plane that passes through a point A with position vector **a** and contains two non-parallel vectors **b** and **c** is:

$$r = a + \lambda b + \mu c$$

From this vector equation, three parametric equations can be found by equating the coefficients of **i**, **j** and **k**.

Eliminating the parameters λ and μ would give you the Cartesian equation of the plane.

The **Cartesian equation** of a plane is

$ax + by + cz = d$ where $a, b, c, d \in \mathbb{R}$

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be read off as a normal vector to the plane.

When **n** is a **normal vector** to a plane and **a** is a fixed position vector of a point on the plane, then the equation of the plane can be written as:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Note

Since $ax + by = c$ is the equation of a line in 2D, it is a common mistake to think that $ax + by + cz = d$ is also the equation of a line in 3D. It is not, it is a plane. The Cartesian equation of a line in 3D has two equal signs.

Assessment tip

It is very useful to know how to read off information from the Cartesian equations of both lines and planes.

Example 3.5.7

Find the acute angle between the planes $x + 2y - 2z = 3$ and $4x + 3y = -9$

Solution

Reading off, the first plane is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and the second plane is perpendicular to $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

The angle between the two planes will be the angle between the two perpendiculars.

$$\cos\theta = \frac{|(1\times 4)+(2\times 3)+(-2\times 0)|}{\sqrt{1^2+2^2+(-2)^2}\sqrt{4^2+3^2}} = \frac{2}{3}$$

$$\theta = \arccos\left(\frac{2}{3}\right) = 48.2° \text{ (3 sf)}$$

Example 3.5.8

The vector equation of line l is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

Find the Cartesian equation of the plane containing l and the point (4, –2, 5).

Solution

First, find the vector on the plane through (1, 3, 6) and (4, –2, 5):

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

Finding the vector product ($\mathbf{a} \times \mathbf{b}$) of the two direction vectors:

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}$$

Using the dot product to find d in in the equation $ax + by + cz = d$.

$$\begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 25$$

Or by substituting either of the points into the equation $7x + 4y + z = d$

The Cartesian equation is $7x + 4y + z = 25$

SAMPLE STUDENT ANSWER

Let A and B be points such that $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}$

(a) Show that $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

(b) Let C and D be points such that $ABCD$ is a rectangle.

Given that $\overrightarrow{AD} = \begin{pmatrix} 4 \\ p \\ 1 \end{pmatrix}$, show that $p = 3$

(a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

(b) $p = 3$

$\overrightarrow{AB} \bullet \overrightarrow{AD} = (1 \times 4) + ((-2) \times 3) + (2 \times 1) = 0$

$4 - 6 + 2 = 0$

▲ Showing a formula and substituting values is a good method for a "show that" question.

▼ This student worked backwards, using the given value of 3 for p to find the coordinates of point D. Candidates who work backwards on a "show that" question will earn no marks.

The answer above could have achieved 2/5 marks.

The correct response to part (b) would have been:

For perpendicular vectors, $\overrightarrow{AB} \bullet \overrightarrow{AD} = 0$

$\overrightarrow{AB} \bullet \overrightarrow{AD} = (1 \times 4) + ((-2) \times p) + (2 \times 1) = 0$

$4 - 2p + 2 = 0$

$6 - 2p = 0$

$p = 3$

Link to Solutions of systems of linear equations AHL 1.16

The following rules apply when working in three dimensions:

- A line and a plane will intersect in a point, provided that the line does not lie completely in the plane.
- Two non-parallel planes will intersect in a line.
- Three different planes will, in general, intersect in a point, but there will be cases where the intersection is a line or there is no intersection at all.

Example 3.5.9

Find the intersection of the line $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ and the plane

$\mathbf{s}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$

Solution

Require $\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$

So $\lambda=\alpha$, $1=\alpha+\beta$, $2+\lambda=\beta$ implying $1=\lambda+2+\lambda$, $\lambda=-\frac{1}{2}$

Hence the intersection is at the point $\left(\frac{1}{2}, 2, \frac{5}{2}\right)$

Example 3.5.10

Find the intersection of the two planes given by the equations
$\mathbf{s}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\mathbf{r}\bullet\begin{pmatrix}1\\0\\1\end{pmatrix}=2$

Solution

Require $\left(\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}\right)\bullet\begin{pmatrix}1\\0\\1\end{pmatrix}=2$

This implies $2+\alpha+\beta=2,\ \beta=-\alpha$

So, the intersection is the line given by $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}-\alpha\begin{pmatrix}0\\1\\1\end{pmatrix}$,

which is $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$ or in Cartesian form $\frac{x-1}{1}=\frac{z-1}{-1}, y=1$.

Example 3.5.11

Find the intersection of the three planes $\mathbf{s}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=2$ and plane Π.

(a) when Π is given by $x+y+z=4$

(b) when Π is given by $x+y+z=3$

(c) when Π is given by $x+y+2z=6$

Solution

From Example 3.5.10 the first two planes intersect in the line

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

(a) Substituting into Π gives $(1+\alpha)+1+(1-\alpha)=4\Rightarrow 3=4$ so there are no intersection points

(b) Substituting into Π gives $(1+\alpha)+1+(1-\alpha)=3$, which is satisfied for all values of α so the intersection is the line

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

(c) Substituting into Π gives $(1+\alpha)+1+2(1-\alpha)=6\Rightarrow -\alpha=2$, $\alpha=-2$ the intersection is the point $(-1, 1, 3)$

SAMPLE STUDENT ANSWER

Find the acute angle between the plane given by $x+y+2z=6$ and the line given by $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$

$$\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=3=\sqrt{1^2+1^2+2^2}\sqrt{1^2+1^2}\cos\theta,\ \cos\theta=\frac{3}{\sqrt{6}\sqrt{2}},\ \theta=30$$

▲ The student had the correct method of using the dot product.

▼ The student did not explain their working at all. They have found the angle between the perpendicular to the plane and the direction of the line.

This solution could have achieved 5/7 marks.

The correct explained solution should be:

The plane is perpendicular to the vector $\begin{pmatrix}1\\1\\2\end{pmatrix}$

and the line is parallel to the vector $\begin{pmatrix}1\\0\\1\end{pmatrix}$

Using the dot product and letting the angle between these two vectors be θ, then

$$\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=3=\sqrt{1^2+1^2+2^2}\sqrt{1^2+1^2}\cos\theta\Rightarrow\cos\theta=\frac{3}{\sqrt{6}\sqrt{2}}\Rightarrow\theta=30$$

Thus, the angle required between the line and the plane is $90-30=60°$

PRACTICE QUESTIONS

SL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 1 [5 marks]

A straight line passes through the origin (0, 0) and through the point with coordinates (6, 2). This line forms an acute angle of θ with the x-axis.

a. Write down the value of $\tan\theta$.

b. The point $(10, b)$ also lies on this line. Find the value of b.

c. The point $(c, 3)$ also lies on this line. Find the value of c.

Question 2 [5 marks]

A ski slope is 600 m long horizontally, with an angle of elevation of 30° measured from the bottom.

a. Find the vertical height of the ski slope.

b. The inclined length of the ski slope is $k\sqrt{3}$, where $k \in \mathbb{N}$. Find the value of k.

c. Write down the angle of depression measured down the ski slope from the top.

Question 3 [6 marks]

Let A and B be two points on a circle of radius r and centre O. The angle $A\hat{O}B = \theta$.

The length of arc AB is 12 cm. The area of sector OAB is 360 cm^2.

Find the value of r and the value of θ.

Question 4 [6 marks]

Solve the equation $(\sin x + \cos x)^2 = 1 + 2\cos x$, for $0 \le x \le 3\pi$.

Question 5 [5 marks]

Solve the equation $6 - \cos 2\theta + 7\sin\theta = 0$ for $0 \le \theta \le 2\pi$.

Question 6 [9 marks]

Let triangle ABC have angles of A, B and C, where $A > B > C$, and sides of length a, b and c, respectively.

a. If $A \le \frac{\pi}{2}$, use the sine rule to show that the largest side is opposite the largest angle and the smallest side is opposite the smallest angle.

b. If $A > \frac{\pi}{2}$, state if the statement is part (a) is still true and justify your answer.

Question 7 [9 marks]

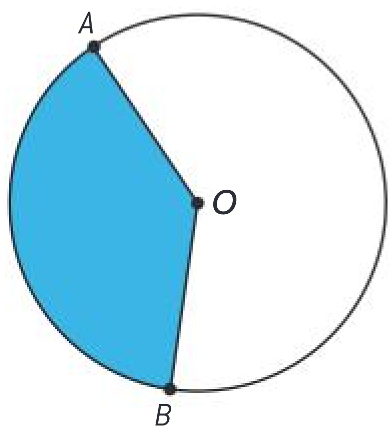
The **median** of a triangle is defined as the line segment joining a vertex to the midpoint of the opposite side. Let triangle ABC have angles of A, B and C and sides of length a, b and c, respectively.

a. Let $[AD]$ be a median of triangle ABC. Using the cosine rule in triangle ABD and triangle ACD, show that $4AD^2 = 2b^2 + 2c^2 - a^2$.

b. A triangle has sides of length 3 m, 4 m and 5 m. Find the length of its smallest median.

Question 8 [8 marks]

Consider a shaded sector of a circle of radius l and centre at the point O that has been drawn on paper and is as shown in the diagram below.



Let angle $A\hat{O}B = \theta$ radians.

a. Write down in terms of l and θ:

i. the area of the shaded region

ii. the arc length from A to B of the shaded sector.

The shaded region is cut out and made to form a 3D right cone, with point O becoming the vertex and point A moving next to point B. The circular base of this cone has radius of r.

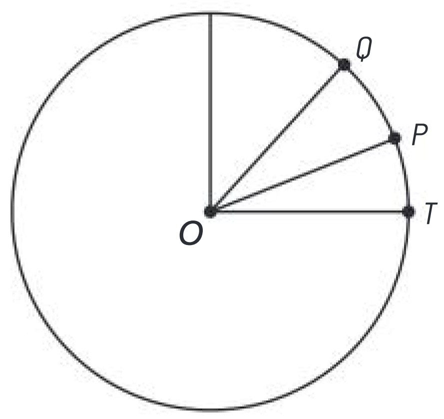
b. Using part (a), show that the curved surface area of the cone will be given by area $= \pi rl$

SL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 9 [11 marks]

Consider two points P and Q on a circle of radius 1, with centre at O.

T is the point (1, 0). Let $T\hat{O}P = B$ and $T\hat{O}Q = A$.



a. Write down the coordinates of the points P and Q in terms of A and B.

b. Hence, find and simplify an expression for the distance PQ.

c. Use the cosine rule in triangle OPQ to find an expression for PQ^2.

d. Hence, show that $\cos(A - B) = \cos A \cos B + \sin A \sin B$

SL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 10 [5 marks]

Monique is standing on level ground 15 metres away from the base of a tall, vertical tree. The angle of elevation from Monique's feet to the top of the tree is 55°.

a. Find the height of the tree.

Pam is standing 20 metres away from the base of the tree, on the other side of the tree to Monique. Monique's feet, the base of the tree and Pam's feet lie in a straight line.

b. Find the angle of depression from the top of the tree to Pam's feet.

Question 11 [8 marks]

A quadrilateral $ABCD$ in 3D coordinate space has vertices A (0, 0, 0), B (1, 2, 3), C (3, 4, 7) and D (1, 1, 1).

a. Find M, the midpoint of [BC].

b. Find the length of CD.

c. Use the cosine rule to find the angle $C\hat{A}D$.

Question 12 [6 marks]

A stand-up paddle boarder travels 3 kilometres from a harbour on a bearing of 180°. He then travels another 1 kilometre on a bearing of 090°. He is now completely exhausted and needs to be rescued by a boat that will start from the harbour and travel in a straight line to him.

a. Sketch a diagram to represent this situation.

b. Find the distance that the rescue boat will have to travel to reach him.

c. Find the bearing (to the nearest degree) that the rescue boat will have to travel along to reach him.

Question 13 [7 marks]

A metal cuboid of dimensions $2\,\text{m} \times 3\,\text{m} \times 4\,\text{m}$ is to be melted down and made into identical spheres each of radius 1 m, in preparation for a strongest man competition.

a. Find the maximum number of spheres that can be created.

b. This number of spheres is created. Find the volume of metal that is left over.

c. This left-over metal is reformed into a cube. Find the side length of the cube.

Question 14 [7 marks]

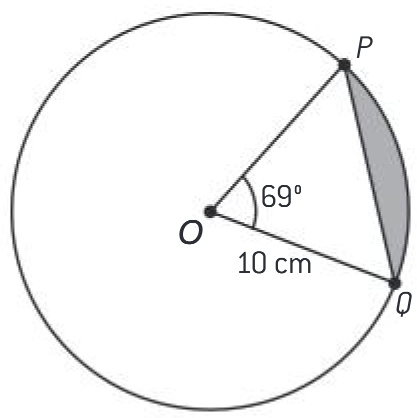
The height of a right circular cone is equal to twice its radius. Its volume is $100\,\text{cm}^3$.

a. Find the radius of the cone.

b. Find the surface area of the cone, including the circular base.

Question 15 [8 marks]

Sue is doing some craft work. On paper, she draws a circle with centre O and radius 10 cm, and then draws in a cord [PQ], as shown in the diagram. Angle $P\hat{O}Q = 69°$.



a. Find the area of the shaded segment of the circle contained between the arc PQ and the cord [PQ].

b. Sue cuts out the circle and then cuts off the shaded area. Calculate the area of the cut-off shaded area as a percentage of the total area of the circle.

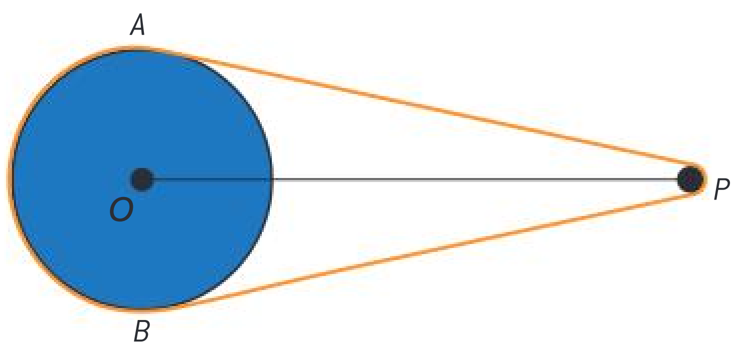
SL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 16 [10 marks]

An orange rope is wound around a circular disc of radius 1 m and a peg of negligible diameter. This information is represented in the diagram below.

The rope leaves the disc at points A and B. The centre of the disc is O and the peg is at point P. The distance $OP = 5$ m. $O\hat{A}P = O\hat{B}P = \frac{\pi}{2}$

Find the length of the rope.



HL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 17 [9 marks]

a. Show that $\tan 4\theta = \frac{4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta}{1 - 8\sin^2\theta\cos^2\theta}$

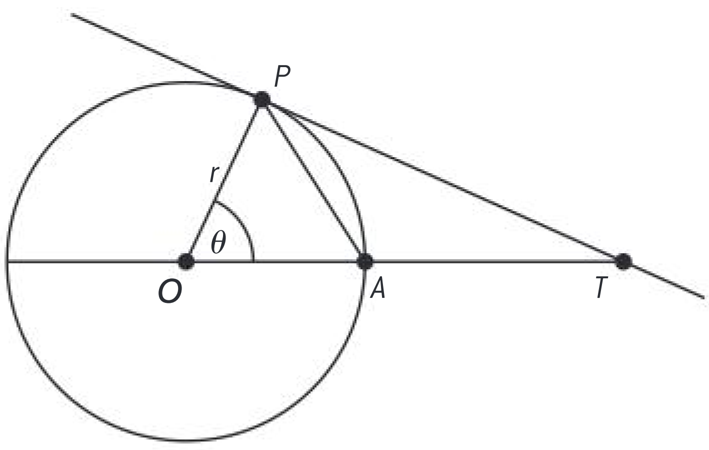
b. Hence, solve $4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta = 1 - 8\sin^2\theta\cos^2\theta$, for $0 \le \theta \le \frac{\pi}{2}$

Question 18 [6 marks]

If x satisfies the equation $\cos\left(x + \frac{\pi}{6}\right) = 5\cos x$, find the exact value of $\tan x$.

Question 19 [8 marks]

The diagram given shows a tangent TP to the circle with center O and radius r. The size of angle $P\hat{O}A$ is θ radians, where $0 < \theta < \frac{\pi}{2}$



a. Find the area of triangle AOP in terms of r and θ.

b. Find the area of triangle POT in terms of r and θ.

c. Find the area of the sector of the circle with angle θ at the center, in terms of r and θ.

d. Using your results from parts (a), (b) and (c), show that $\sin\theta < \theta < \tan\theta$, where $0 < \theta < \frac{\pi}{2}$

Question 20 [8 marks]

a. Use the compound angle formulae for $\sin(A + B)$ and $\cos(A + B)$ to prove that $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$

b. Hence, find the exact value of $\tan 75°$.

Question 21 [8 marks]

Consider a circle with radius r and centre at O. Let A and B be two points on the circle such that the line segment $[AB]$ is a diameter. Let P be a general point on the circle other than A or B. Let $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OA} = \mathbf{a}$.

a. Find vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ in terms of $\mathbf{p}$ and $\mathbf{a}$.

b. By considering $\overrightarrow{AP} \bullet \overrightarrow{PB}$, prove that the angle subtended on the arc of a semicircle is 90°.

HL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 22 [11 marks]

Let ABC be a triangle with $\overrightarrow{AC} = \mathbf{b}$, $\overrightarrow{CB} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{c}$.

a. Find vector c in terms of **b** and **a**.

b. If angle $A\hat{C}B = 90°$, by considering $\mathbf{c}\bullet\mathbf{c}$, prove Pythagoras's theorem for triangle ABC.

c. If angle $A\hat{C}B = C°$, by considering $\mathbf{c}\bullet\mathbf{c}$, obtain the cosine rule for triangle ABC.

Question 23 [12 marks]

Let a triangle have sides of a, b and c and opposite angles of A, B and C, respectively.

Let $s = \frac{a+b+c}{2}$ be half the perimeter of the triangle. Heron's formula states that the area of the triangle is given by $\sqrt{s(s-a)(s-b)(s-c)}$.

a. Use Heron's formula to find the area of a triangle with sides of lengths 4, 13 and 15.

b. Starting with area $= \frac{1}{2}ab\sin C$ and using the cosine rule involving angle C, show that area $= \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}$

c. Using the formula from part (b) and repeated factorization of the difference of two squares, complete the proof of Heron's formula that area $= \sqrt{s(s-a)(s-b)(s-c)}$

Question 24 [12 marks]

Consider the two vectors

$\mathbf{v} = \cos\alpha\,\mathbf{i} + \mathbf{j} + \sin\alpha\,\mathbf{k}$, $\mathbf{w} = \cos\alpha\,\mathbf{i} - \mathbf{j} + \mathbf{k}$, $0 \le \alpha \le 2\pi$

a. Find $\mathbf{v} \bullet \mathbf{w}$ in terms of α.

b. Hence, find the exact values of α for which $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

c. Find $\mathbf{v} \times \mathbf{w}$ in terms of α.

d. Hence, find the exact values of α for which $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Question 25 [12 marks]

Three of the four equations below have no solutions. Identify, with a reason, which three equations these are. For the equation that does have solutions, find all the solutions in the interval [0°, 360°]

a. $\tan^2 x + \sec^2 x + 1 = 0$

b. $\sin x \cos x - 1 = 0$

c. $4\cos x \sin^2 x - \cos x = 0$

d. $3\sin x + 4\cos x - 8 = 0$

Question 26 [13 marks]

Let triangle ABC have angles of A, B and C, and sides of length a, b and c respectively. The point X divides the side $[BC]$ in the ratio $m : n$ where $m + n = 1$.

Let $B\hat{A}X = \alpha$, $C\hat{A}X = \beta$, $A\hat{X}C = \theta$

a. Using the sine rule in triangle ABX and triangle ACX, show that $\dfrac{m\sin(\theta-\alpha)}{\sin\alpha} = \dfrac{n\sin(\theta+\beta)}{\sin\beta}$

b. Hence, use the compound angle formula to show that $\cot\theta = m\cot\alpha - n\cot\beta$

c. Hence, if X is the midpoint of side $[BC]$, show that $2\cot\theta = \cot\alpha - \cot\beta$

Question 27 [11 marks]

a. For $x, y \in \mathbb{R}$, show that

$\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$

Hint: Let $\arctan x = \alpha$ and $\arctan y = \beta$, then consider $\tan(\alpha+\beta)$.

b. State the relationship between x and y if the right-hand side of the above expression is undefined.

c. For $x \in \mathbb{R}$, $x > 0$, show that $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$

Hint: Let $\arctan x = \alpha$ and sketch a suitable right-angled triangle.

d. Explain why the expression obtained in part (a) could not be used directly to obtain the result of part (c).

Question 28 [13 marks]

Determine, with reasons, for each of the pairs of lines below, whether they are:

(i) parallel (ii) intersecting or (iii) skew.

If they intersect, find the point of intersection.

a. $\mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix}0\\-5\\4\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$

b. $\mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\-1\end{pmatrix}$ and $\dfrac{x-4}{6} = \dfrac{y-0}{-2} = \dfrac{z+6}{2}$

c. $\mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$

Question 29 [15 marks]

a. Show that $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$, $A \in \left[0, \dfrac{\pi}{2}\right]$.

b. Let A, B and C be the angles in the triangle ABC.

i. Show that $\sin A\cos B + \cos A\sin B = \sin C$.

ii. If $C = 90°$, use your answer to part (i) to obtain $\sin^2 A + \cos^2 A = 1$.

HL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 30 [9 marks]

Find the acute angle, in degrees, between the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $2x = 1 - y = z + 5$

HL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 31 [19 marks]

Ali and Ben are both playing with their own radio-controlled model car. With respect to a set of coordinate axes, Ali's car starts at point (0, 10), and Ben's car starts at the same time at point (0, –5). Distances are measured in metres, and time t (from when the cars both start) is measured in seconds. Ali's car has a constant velocity vector of $\begin{pmatrix}1\\-1\end{pmatrix}$ m s^{-1} and Ben's car has a constant velocity vector of $\begin{pmatrix}2\\1\end{pmatrix}$ m s^{-1}.

a. **i.** Write down a vector equation in the form $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$ for the line that Ali's car travels along.

ii. Write down a vector equation in the form $\mathbf{s} = \mathbf{c} + \mu\,\mathbf{d}$ for the line that Ben's car travels along.

b. Hence, find the intersection point of these two lines.

Let $\mathbf{r}_A$ represent the displacement of Ali's car at time t, and let $\mathbf{r}_B$ represent the displacement of Ben's car at time t.

c. Transform the vector equations in part (a) to find vector equations for $\mathbf{r}_A$ and $\mathbf{r}_B$ in terms of the parameter t. Hence show that the two cars do not collide.

d. **i.** Find the time when the two cars are nearest to each other.

ii. Find the distance between the two cars at this time.

iii. Find the position of each car at this time.

Question 32 [12 marks]

Consider the two planes given by

$$\Pi_1 : \mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\Pi_2 : \mathbf{s} = \begin{pmatrix}0\\-5\\3\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\0\end{pmatrix} + \beta\begin{pmatrix}3\\1\\4\end{pmatrix}$$

a. Show that these two planes are parallel by showing that they are both perpendicular to the vector $\mathbf{n} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

b. Write down the vector parametric equation of the line L that goes through the point (2, –1, 0) and is parallel to **n**.

c. Find the intersection point of L and Π_2.

d. Hence, find the exact value of the distance between the two planes.

Question 33 [15 marks]

The height of water, h metres, in a harbour, at time, t hours, after 0:00am on a particular day can be modelled by $h = a\sin(b(t-3)) + c$, where $a, b > 0$.

The period of this function is 12 hours. At 0:00am the height is 1 m and at 6:00am the height is 11 m.

a. Find the values of each of a, b and c.

An old sailor called Sinbad believes in "the rule of twelves". This states that the height change in each of first 6 hours will be in the ratio 1 : 2 : 3: 3 : 2 : 1

b. **i.** Find what the height change would be in each of the first 6 hours according to the model.

ii. Convert these six answers into a ratio of the form $p : q : r : s : u : v$ where $p + q + r + s + u + v = 12$ and each value is given to two significant figures.

iii. Comment very briefly on how the model compares with Sinbad's rule.

c. **i.** Find $\frac{dh}{dt}$, the rate of change of the height with respect to time.

ii. Hence, find the maximum rate of change in the height during the first 6 hours. Also state the time at which this will occur.

HL PAPER 3 TECHNOLOGY REQUIRED

Question 34 [28 marks]

Let a triangle have side lengths of a, b and c, and opposite angles of A, B and C, respectively. As well as the sine rule and the cosine rule, there is a less well-known rule, known as the tangent rule. This states that

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

This question will investigate this rule. A particular triangle has $a = 5$, $b = 3$, $C = 80$.

a. Explain why the sine rule cannot be used initially to find any other angles or sides.

b. Use the cosine rule to find the length of side c.

c. Use the tangent rule to find angles A and C.

d. Use the answers from part (c) and the sine rule to find the length of side c. Check that your answer agrees with the answer obtained in part (b).

The tangent rule is now going to be proved for the general triangle.

e. Use the sine rule to show that $\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$

f. Show that

i. $\sin(C + D) + \sin(C - D) = 2\sin C\cos D$

ii. $\sin(C + D) - \sin(C - D) = 2\cos C\sin D$

g. Use your answer to part (f) with suitable substitutions to obtain similar expressions for $\sin A + \sin B$ and $\sin A - \sin B$.

h. Hence, complete the proof of the tangent rule.

4 STATISTICS AND PROBABILITY

4.1 DESCRIPTIVE STATISTICS

You must know:

✔ the terms 'population', 'sample', 'random sample', 'discrete and continuous data'

✔ the interpretation of outliers

✔ sampling techniques and their effectiveness.

You should be able to:

✔ calculate measures of central tendency

✔ calculate measures of dispersion

✔ draw box-and-whisker plots

✔ use cumulative frequency graphs.

Note

Graphs and diagrams are an essential part of statistics and must be drawn clearly and accurately.

Note

Technically, "data" is plural, and "datum" is singular, although most people use the word "data" for both the singular and plural.

Statistics is the branch of mathematics dealing with the collection, analysis, interpretation and presentation of numerical data so that you can make inferences about the population the data comes from.

A population includes all of the elements from a set of data.

A sample consists one or more observations drawn from the population.

Qualitative data, or categorical data, are data organized into categories based on non-numerical types such as music genre, feelings, colours or anything that does not have a number associated with it.

Quantitative data are data that can be counted or measured. Quantitative data describes information that can be counted, such as "How many people live in your house?" or "How long does it take you to get home after school?"

Quantitative data can be **discrete** or **continuous**.

Discrete data can usually be counted, such as how many cars pass the school in an hour.

Continuous data is not restricted to certain fixed values, such as integers and can be fractions or decimals.

Continuous data usually refers to something that can be measured, such as length, time and mass.

With continuous data, if two values are possible then usually there are values between them that are also possible.

Primary data are collected first-hand by you, using tools like experiments and surveys. Secondary data are collected by someone else and might be found online or in publications.

When analyzing a data set containing missing values or errors, the causes of the missing data must be considered in order to handle the data properly.

Sampling techniques

- Random sampling occurs when every sample of the same size has an equal chance of being selected. Example: choosing ten students from your school by putting all their names into a hat and drawing out ten.
- Systematic sampling is easier to do than random sampling, as you do not need to generate random numbers to choose your sample. Instead, the list of elements is "counted off". For example, you could take an A to Z list of students in your school, choose a starting point and then select every 10th student.
- Convenience sampling is perhaps the least useful technique to use. An example of this would be to survey the first 20 students that you see after school. An advantage of this kind of sampling is that it is very easy to do. A disadvantage is that the data is unlikely to be representative of the population.
- Stratified sampling involves dividing the population into smaller groups known as strata. The strata are formed based on members' shared characteristics. You then choose a random sample from each stratum and put them together to form your sample.

  Example: In a high-school with 1000 students, you could choose 25 students from each of the four year groups to form a sample of 100.

  Now, if you know that 60% of the 1000 students are female and 40% are male, you could use a random sampling technique to choose a sample that is also 60% female and 40% male.
- Quota sampling is like stratified sampling but the sampling is not random. This can make quota sampling biased and unreliable.

  You can organize data into groups in a grouped frequency table. For continuous data you can draw a histogram, which is similar to a bar chart but doesn't have gaps between the bars.

Assessment tip

You should include a discussion of the type of data and your sampling method in your internal assessment if you use statistics.

Example 4.1.1

A group of 17 students were asked about their journey time to school. The times, in minutes, were:

7, 21, 22.5, 26, 31.5, 13.6, 41, 23, 24, 33, 16, 18, 16.5, 17, 37.5, 12.5, 26

(a) State if the data is continuous or discrete.

(b) Copy and complete this frequency table.

Time, t min	Frequency
$0 \le t < 5$	0
$5 \le t < 10$	1
$10 \le t < 15$	

(c) Draw a histogram to represent this data.

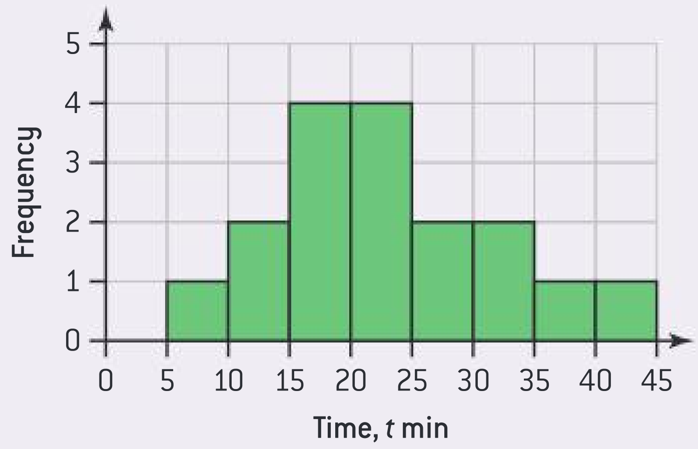
Solution

(a) The data is continuous.

(b)

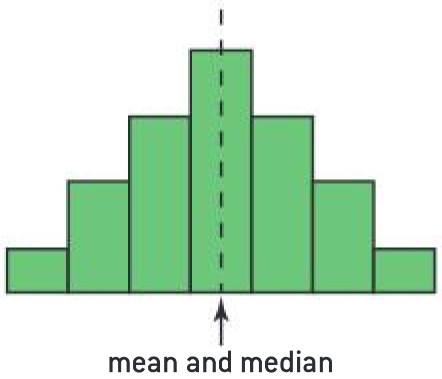
Time, t min	Frequency
$0 \le t < 5$	0
$5 \le t < 10$	1
$10 \le t < 15$	2
$15 \le t < 20$	4
$20 \le t < 25$	4
$25 \le t < 30$	2
$30 \le t < 35$	2
$35 \le t < 40$	1
$40 \le t < 45$	1

(c)



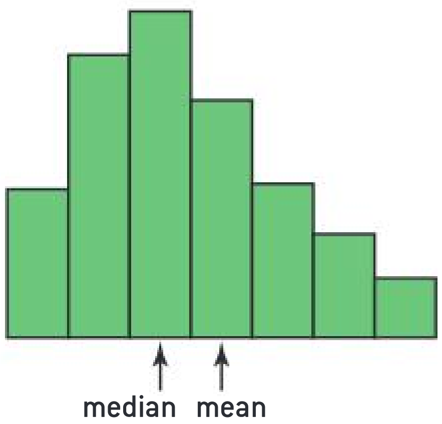
Skewness

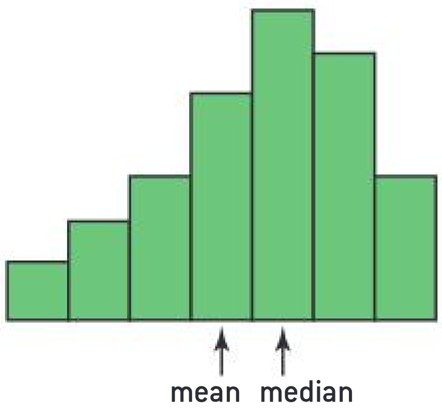
In a non-skewed distribution, the mean and median have the same value, and the histogram for the data shows a vertical line of symmetry through the centre of the data.



A right-skewed (positively skewed) distribution will have the mean to the right of the median.

A left-skewed (negatively skewed) distribution will have the mean to the left of the median.





Measures of central tendency

A measure of central tendency is a single value that represents the middle or centre of its distribution. The three most common measures of central tendency are the mode, the mean and the median.

The mode is the value that occurs most often in a data set.

The median is the middle value when the data is arranged in order.

The mean is the "common average" of the data.

Example 4.1.2

Find the mode, median and mean of the data set:
5, 12, 11, 7, 3, 5, 4, 10, 6

Solution

The mode is 5 as there are more 5's than any other number.

To find the median, arrange the data in order and find the one in the middle.

The median is the $\left(\frac{n+1}{2}\right)^{th}$ value when arranged in order, where n is the number of values in a set of data.

In ascending order: 3, 4, 5, 5, 6, 7, 10, 11, 12

6 is the median.

The mean $= \frac{\text{sum of the data items}}{\text{number of data items}} = \frac{\sum x}{n} = \frac{3+4+5+5+6+7+10+11+12}{9} = 7$

The mean is often written using the symbol $\bar{x}$ or μ

Note

You can have more than one mode. If the data set consists of the numbers 2, 2, 3, 4, 5, 6, 6 then there are two modes: 2 and 6. This means the data is bi-modal.

Note

If the number of data items is even, then the median is in between two numbers, and you take their average. If the number of data items is odd, then there is an exact middle value.

If data values are repeated with value x_i having frequency f_i then the mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}, \text{ where } n = \sum_{i=1}^{k} f_i$$

Example 4.1.3

Jie asked 50 students about the number of times that they visited the library in a month and displayed the data in a grouped frequency table.

Visits, v	Frequency, f
$0 \le v < 10$	12
$10 \le v < 20$	20
$20 \le v < 30$	10
$30 \le v < 40$	8

(a) Find the modal class for the number of visits.

(b) Find the mean number of visits.

Solution

(a) The modal class is $10 \le v < 20$

(b) To find the mean of grouped data, you first find the midpoint of each class in a third column. For each class, multiply the midpoint by the frequency and enter this product in a fourth column. The mean is found by dividing the total of the fourth column by the total frequency.

Visits, v	Frequency, f	Midpoint, m	$m \times f$
$0 \le v < 10$	12	5	60
$10 \le v < 20$	20	15	300
$20 \le v < 30$	10	25	250
$30 \le v < 40$	8	35	280
Total	**50**		**890**

$$\text{Mean} = \frac{\Sigma fm}{\Sigma f} = \frac{890}{50} = 17.8 \text{ visits}$$

Note

Questions often say "estimate" the mean because when we see a grouped frequency we assume the data to be equally spread around the midpoint of the group, which might not always be true and can lead to some inaccuracy.

Assessment tip

Reflecting on the possible inaccuracies in your statistics internal assessment might be beneficial.

Cumulative frequency

The median of a grouped data set displayed in a data table may be found with the use of a cumulative frequency diagram. This also allows you to explore the measures of spread such as the range, quartiles and percentiles.

Measures of dispersion

The **range** is the difference between the largest and smallest values.

The median of a data set divides the data into a top half and a bottom half. Quartiles separate the original set of data into four equal parts. Each of these parts contains one-quarter (25%) of the data. Percentiles divide the data into hundredths. For example, the 60th percentile is a number such that 60% of the data is below this number when the data has been placed in ascending order.

The median is the 50th percentile, the lower quartile is the 25th percentile and the upper quartile is the 75th percentile.

The lower quartile, Q_1, is the median of the lower half of the data. The upper quartile, Q_3, is the median of the upper half of the data. The interquartile range, IQR, is the difference between the upper quartile and the lower quartile. $\text{IQR} = Q_3 - Q_1$. Half of the data lie between the two quartiles.

Example 4.1.4

40 students took a class test with maximum mark of 100. Their scores are displayed in this table.

21	9	40	34	35	50	17	33
60	40	81	53	52	66	77	49
23	78	27	59	66	72	58	88
77	53	77	57	48	36	79	51
56	18	55	30	13	50	39	79

(a) Construct a cumulative frequency table.

(b) Draw a cumulative frequency diagram.

(c) Use your graph to estimate the minimum score needed for an A grade if this is awarded to the top 20% of students.

(d) Use your graph to estimate the interquartile range.

(e) Use your graph to estimate the median score.

Solution

(a)

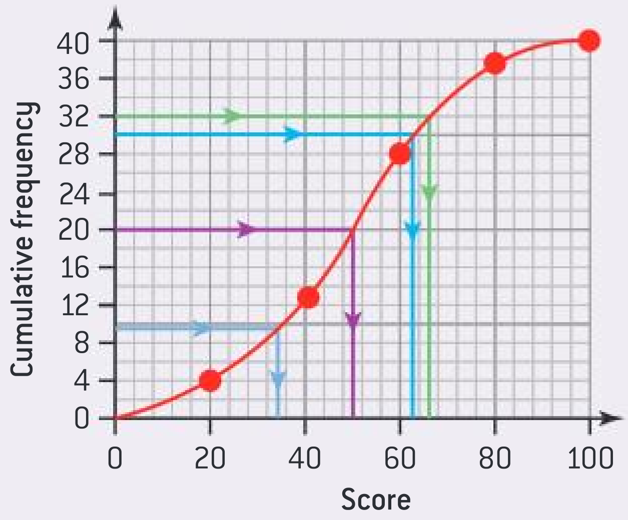
Score	$0 \le s < 20$	$20 \le s < 40$	$40 \le s < 60$	$60 \le s < 80$	$80 \le s < 100$
f	4	9	15	10	2
CF	4	13	28	38	40

(b) Cumulative frequency diagram is shown to the right.

(c) There are 32 students not in the top 20%, so reading off from the graph, the answer is 66

(d) $\text{IQR} = Q_3 - Q_1 = 64 - 36 = 28$

(e) Reading off from the graph, median = 50

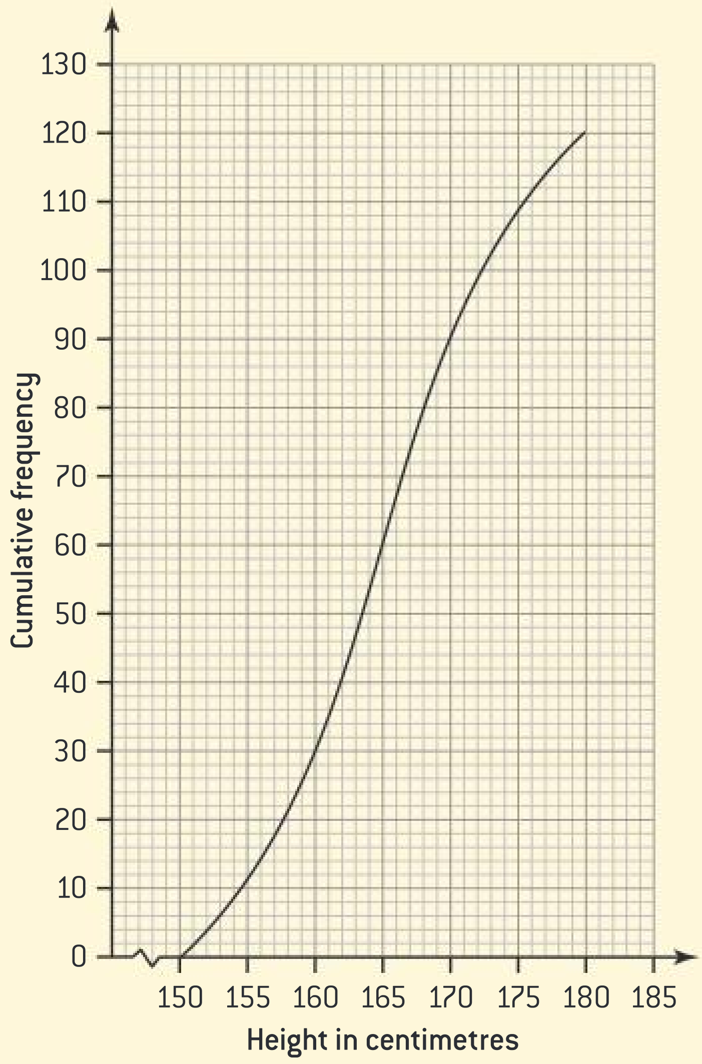


SAMPLE STUDENT ANSWER

The cumulative frequency graph shows the heights of 120 girls at a school.

(a) Using the graph:
  (i) write down the median
  (ii) find the interquartile range.

(b) Given that 60% of the girls are taller than a cm, find the value of a.



(a) (i) Median = 165 cm

(ii) IQR = max − min = 185 − 150 = 35

(b) 60% of 120 = 72nd student. Height = 167 cm

The answer above could have achieved 2/6 marks.

▲ The student would have drawn a horizontal line from 60 to the curve and then vertically down to 165 cm.

▼ This is wrong. The student has misinterpreted the question and found the range when asked for the interquartile range. The correct answer is:

Lower quartile = 160
Upper quartile = 170
IQR = 170 − 160 = 10 cm

▼ This is an easy error to make as we need to use the 40th percentile, or 48th student, by drawing a horizontal line through (0, 48). Thus, the correct answer is $a = 163$ cm.

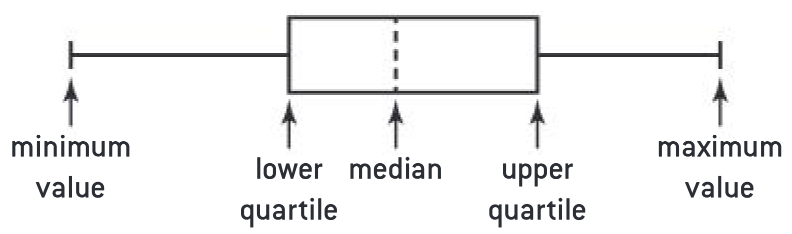
> **Note**
>
> Box-and-whisker plots do not show frequency; each half of the box, and each whisker, contains 25% of the population. These diagrams show how spread out the data is.

> **Note**
>
> Although formulae for the variance and standard deviation are given in the formula booklet, you will always calculate them using a calculator.

> **Note**
>
> Different calculators use different symbols for the standard deviation and will give two values for the variance. For the purpose of this course, it is always the lower one that is required.

Box-and-whisker plots (box plots)

The interquartile range is a measure of spread about the median and can be seen in a cumulative frequency diagram. It can also be represented in a box-and-whisker plot using a "five number summary" of minimum value, lower quartile, median, upper quartile and maximum value.



An **outlier** is any value at least $1.5 \times$ IQR above Q_3 or below Q_1. Outliers are marked with a cross on a box-and-whisker plot.

The variance and standard deviation are measures of spread associated with the mean.

The variance is a measure of spread based on the square of the distance between each data value and the mean. It is the square of the standard deviation.

Example 4.1.5

A data set for the number of pages in 11 different novels is given below:

290, 320, 500, 200, 240, 370, 336, 326, 328, 300, 325

(a) Find the interquartile range for this data and hence identify any outliers.

(b) Construct a labelled box-and-whisker plot for this data.

Solution

(a) Putting the data in order gives:

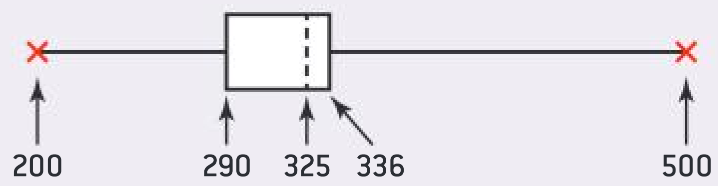
200, 240, 290, 300, 320, 325, 326, 328, 336, 370, 500

$Q_1 = 290$, $Q_3 = 336 \Rightarrow \text{IQR} = 46$

$290 - 1.5 \times 46 = 221$, $336 - 1.5 \times 46 = 405$

So the outliers are 200 and 500.

(b) Median is 325. Minimum is 200. Maximum is 500.



Example 4.1.6

The exam results for a group of students are given here:

Mark	1	2	3	4	5	6	7
Frequency	2	4	6	9	x	9	4

The mean grade was 4.5

(a) Find x.

(b) Write down the standard deviation.

(c) Find the variance.

(d) If all of the marks have 3 added to them, write down what will happen to the standard deviation. Explain your reasoning.

Solution

(a) Mean $= \frac{\Sigma fx}{\Sigma f}$

$$4.5 = \frac{146 + 5x}{34 + x}$$

$$4.5(34 + x) = 146 + 5x$$

$$x = 14$$

(b) Standard deviation $= 1.5411... = 1.54$ (3 sf)

(c) Variance $= 1.5411...^2 = 2.375$

(d) The standard deviation will remain the same as all of the data points have been increased by the same amount.

Note

"Write down" means marks will be given for your answers only, as you will work this out using your calculator.

"Find" indicates that you should show your working.

Effect of constant changes to the original data

If you add/subtract a constant value c to/from all the numbers in a list, the arithmetic mean and the median increase/decrease by c but the standard deviation, variance, range and interquartile range all remain the same.

If you multiply/divide all the numbers in the list by a constant value c, the arithmetic mean and the median are multiplied/divided by c. The standard deviation, range and interquartile range are multiplied/divided by the modulus of c. The variance is multiplied/divided by c^2.

Assessment tip

In the paper that allows the use of technology, all measures of central tendency and spread, except the mode, range and interquartile range, can be found with your calculator. As a calculator gives the max, min and quartiles, it only requires a subtraction to find the range or interquartile range.

4.2 CORRELATION AND REGRESSION

You must know:

- ✔ the terms 'linear correlation', 'interpolation' and 'extrapolation'
- ✔ descriptions of linear correlation such as: positive, negative, strong, weak or no correlation.

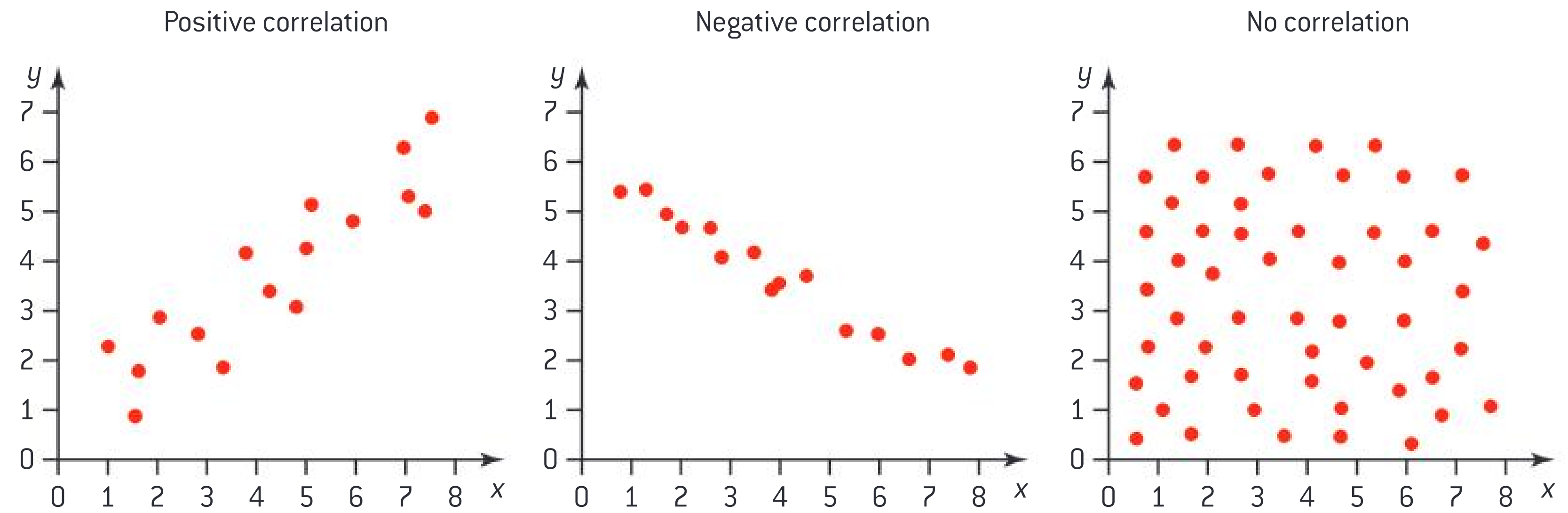
You should be able to:

- ✔ draw scatter diagrams
- ✔ calculate Pearson's product moment correlation coefficient
- ✔ calculate regression lines
- ✔ make predictions from a regression line.

So far, we have used univariate data, which is data consisting of only one variable. In a bivariate data set, each item of data consists of two variables, which may or may not be related in some way.

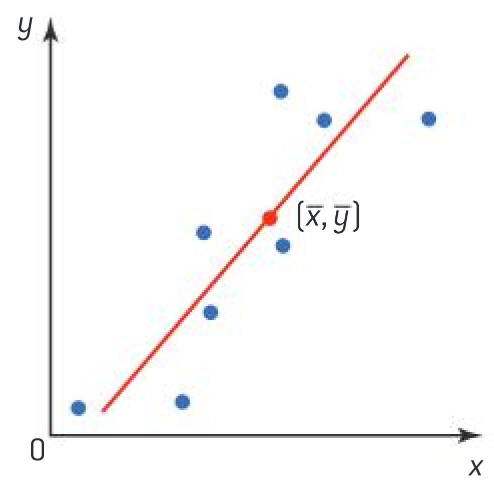
Bivariate data can be plotted on a scatter diagram, with one variable represented by the x-coordinates and the other by the y-coordinates. You can use a scatter diagram to investigate the possible relationship between two variables and to see if there might be a trend.

The relationship between the two data sets is called a correlation.



Assessment tip

When conducting an internal assessment involving bivariate data you have to remember that correlation does not imply causation. A typical example is the number of ice creams sold at a European beach each month and the number of cases of sunburn. This would be likely to yield a positive correlation without causation and is a good opportunity to use critical reflection. There could be a third factor—number of hours of hot sunshine—as the cause.



In mathematics you will often see an idea built on with the next step and the next step and so on. Our next step is to draw a line between the dots to represent the relationship between the data points. This line is made easier to draw by finding the mean of the x values and the mean of the y values and plotting this as a "mean point" that your line of best fit will go through. To draw a line of best fit by eye, draw a line that balances the number of points above the line with the number of points below the line. You should note that lines may vary a little between students. This line of best fit can then be used to make predictions.

Be careful: the line of best fit may or may not go through the origin—even with lines showing positive correlation. You need to look at the context of the problem to decide this.

For example, the best fit line for a scatter diagram showing a class's score in a test against the number of hours they spent studying *might* not necessarily go through the origin. Students might need to study for a minimum of 3 hours just to get any marks at all, for example.

Can you think of two variables where the line of best fit *would* pass through the origin?

You can use two points to find the equation of this line in the form $y = mx + c$ and your calculator can also be used to find the equation of the line called a regression line.

There are two lines of regression, that of **y on x** and **x on y.** The line of regression of y on x is given by $y = a + bx$ and is used to predict the value of y for a given value of x. If you want to predict the value of x from a given value of y, you should use the line of regression of x on y which is given by $x = c + dy$. Often, only one of these lines make sense.

Note

In the regression line y on x given by $y = a + bx$, the b represents the change in y when the x is increased by one unit and the a is where the line crosses the y-axis.

Consider the two variables "rice crop yield, y" and "rainfall, x". For this situation, the construction of regression line of y on x would make sense and you would be able to demonstrate the dependence of the rice crop yield on rainfall. You would then be able to estimate crop yield given rainfall.

A regression line of x on y would show that rainfall is dependent on the rice crop yield and would lead to the suggestion that if you grow really big crops you will be assured of a heavy rainfall, which is absurd.

The process of trying to predict outside of the given data is called extrapolation and can be inaccurate. Using the equation of the regression line to predict a data value within the range of the given data is called interpolation. It is generally more reliable than extrapolation.

Pearson's product-moment correlation coefficient, *r*

Pearson's product-moment correlation coefficient is a number between –1 and 1, which tells you the strength of the correlation between the two data sets you have plotted. It is widely used in the sciences as a measure of the strength of **linear** dependence between two variables. The stronger the r value, the more accurate you would expect your prediction to be. The value of r can be found using your calculator. This table gives a quick way to interpret the r value.

Range, r	Correlation
$0 < \|r\| \le 0.25$	very weak
$0.25 < \|r\| \le 0.5$	weak
$0.5 < \|r\| \le 0.75$	moderate
$0.75 < \|r\| \le 1$	strong

Note

The r value and the equation of the regression line will be found using a calculator.

A positive r value suggests that y increases as x increases and a negative r value suggests that y decreases as x increases.

Linear regression does not directly test whether data is linear. It finds the slope and the intercept, assuming that the relationship between the independent and dependent variable can be best explained by a straight line.

You can construct a scatter diagram to confirm this conjecture. If the scatter diagram reveals a non-linear relationship, often a suitable transformation can be used to attain linearity.

Example 4.2.1

In an experiment, a vertical spring was fixed at its upper end. It was stretched by hanging different weights on its lower end. The length of the spring was then measured. The following readings were obtained.

Load, x kg	0	1	2	3	4	5	6	7	8
Length, y cm	23.5	25.0	26.5	27.0	28.5	31.5	34.5	36.0	37.5

(a) Plot these pairs of values on a scatter diagram, taking 1 cm to represent 1 kg on the horizontal axis and 1 cm to represent 2 cm on the vertical axis.

(b) (i) Write down the mean value of the load, $\bar{x}$.
(ii) Write down the mean value of the length, $\bar{y}$.

(c) Plot the mean point $(\bar{x}, \bar{y})$ on the scatter diagram. Label the mean point as M.

(d) (i) Write down the correlation coefficient r for these readings.
(ii) Comment on this result.

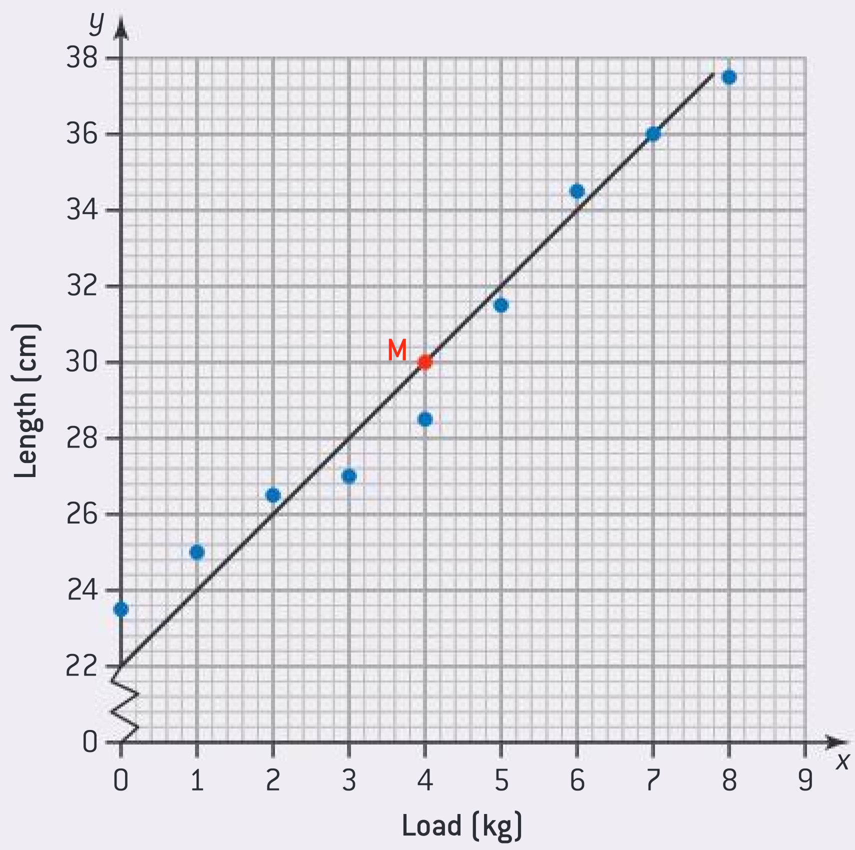
(e) Find the equation of the regression line of y on x.

(f) Draw the line of regression on the scatter diagram.

(g) (i) Using your diagram or otherwise, estimate the length of the spring when a load of 5.4 kg is applied.
(ii) Sandy uses the equation to claim that a weight of 30 kg would result in a length of 62.8 cm. Comment on this claim.

Solution

(a), (c) and (f)



(b) (i) 4 (ii) 30

(d) (i) $r = 0.986$ (3 sf)
(ii) This is a very strong positive linear correlation.

(e) $y = 1.835x + 22.7$

(g) (i) $y = 1.835(5.4) + 22.7 = 32.6$ cm (3 sf)
(ii) It would not be reliable to predict as 30 kg lies far outside the given set of data.

SAMPLE STUDENT ANSWER

The table shows the results of a class that conducted a physics and a biology experiment and were then graded on each.

Physics	45	56	73	30	67	65	60	85	40	79
Biology	60	65	85	50	75	71	70	95	48	81

(a) Write down the mean score for physics.

(b) Write down the mean score for biology.

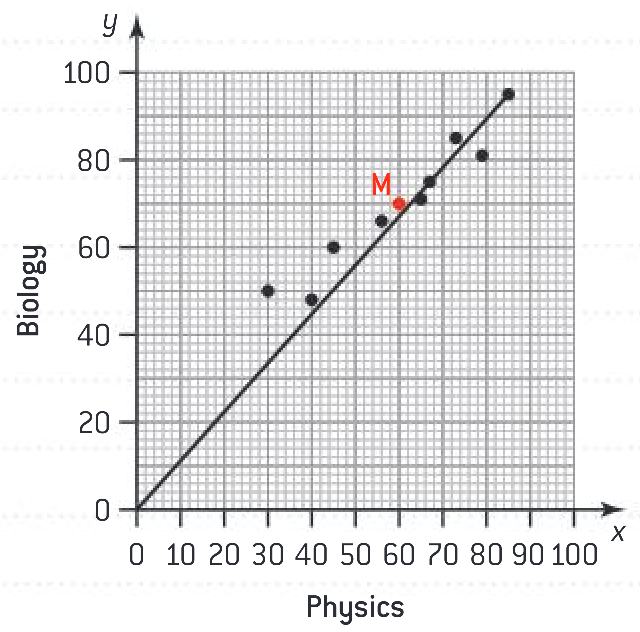
(c) Draw a scatter diagram and label the mean point as M.

(d) Draw a line of best fit on the scatter diagram.

(a) 60

(b) 70

(c) and (d)



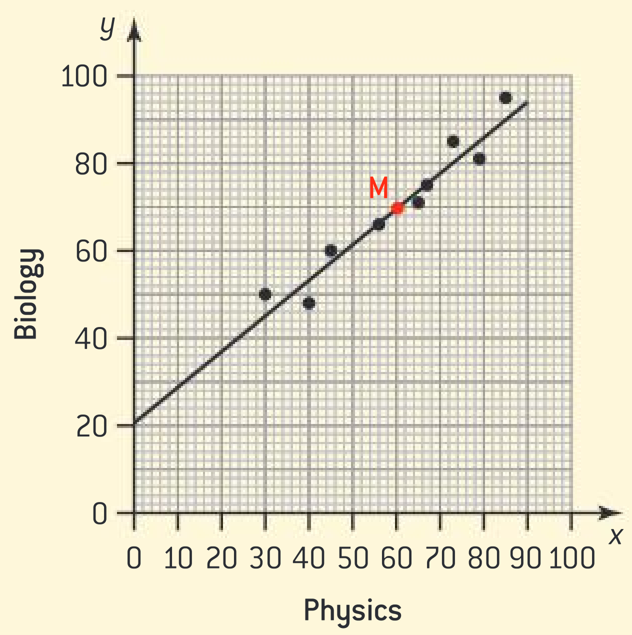
▲ These are the correct answers from the calculator.

▲ The scatter diagram is correct and the mean point is in the right place.

▼ The line of best fit does not pass through the mean point and it incorrectly goes through the origin.

The answer above could have achieved 7/8 marks.

Correct line on scatter diagram is shown below.



Assessment tip

Students often want to start the line of best fit from the origin even when there is no good reason to do this.

4.3 PROBABILITY

You must know:

- ✔ the concept of the probability of an event
- ✔ the meaning of terms 'independent' and 'mutually exclusive'.

You should be able to:

- ✔ find probabilities of combined events
- ✔ work with discrete probability distributions, including the binomial distribution
- ✔ work with continuous probability distributions, including the normal distribution
- ✔ find conditional probabilities.

Probability measures the chance that some event will occur as a fraction or decimal between 0 and 1, where 0 means the event is impossible, and 1 means the event is 100% certain. If we use $n(A)$ to represent the number of ways that an event A can occur, and $n(U)$ to represent the number of all possible events in the sample space, then the probability of event A, $P(A)$, is defined as:

$$P(A) = \frac{n(A)}{n(U)}$$

provided that all the outcomes in the sample space are equally likely. The probability of an event not happening is written $P(A')$, therefore:

$$P(A) + P(A') = 1$$

Venn diagrams are used to illustrate the probabilities of single or combined events.

We will consider two special types of events: independent and mutually exclusive events.

Two events A and B are **independent** if the occurrence of one does not affect the chance that the other will occur. For example, if we toss a coin two times, the first time it may show a head, but this does not affect the probability of a head on the second toss. The outcome of the first toss does not change the probability for the outcome of the second toss.

The definition of two events A and B being independent is:

$$P(A \cap B) = P(A) \times P(B)$$

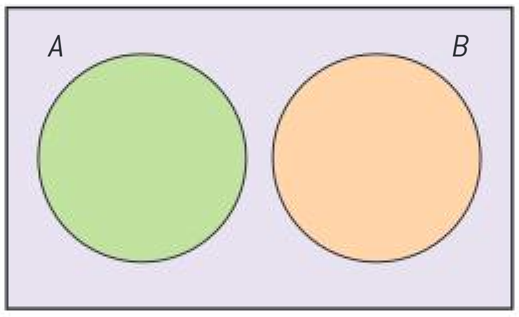
Assessment tip

Exam questions often ask you to show that two events are independent. You should substitute the values into this equation to show that the left-hand side is equal to the right-hand side.

A and B are **mutually exclusive** events if they cannot occur at the same time.

For example, if we toss a coin it can only show a head *or* a tail, not both, hence:

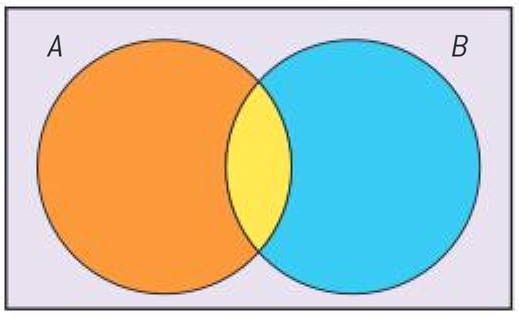
$$P(A \cap B) = 0$$



General probability rule for two events

The probability that either event A **or** event B will occur is shown as the union of sets A and B on a Venn diagram, and calculated with the general formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



The intersection is subtracted here because it has been counted twice, once in $P(A)$ and again in $P(B)$.

For mutually exclusive events we have:

$$P(A \cup B) = P(A) + P(B)$$

Example 4.3.1

Two events A and B are independent. $P(A) = \frac{1}{4}$ and $P(A \cup B) = \frac{2}{5}$.

Find $P(B)$.

Solution

Independence implies $P(A \cap B) = P(A) \times P(B)$

So general rule of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ becomes

$\frac{2}{5} = \frac{1}{4} + P(B) - \frac{1}{4}P(B)$

So $\frac{3}{20} = \frac{3}{4}P(B)$ giving $P(B) = \frac{1}{5}$

Conditional probability is used to calculate the probability of event A, given that event B has happened, and the symbol $P(A|B)$ is used.

In general, for two events A and B, the probability of A occurring given that B has occurred can be found using:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Assessment tip

The two phrases "independent" and "mutually exclusive" are often confused, so it is important to learn their definitions.

Assessment tip

This formula is often the starting point for examination questions where set notation is seen.

Assessment tip

A common mistake is to use the formula for independent events or mutually exclusive events when two events are neither independent nor mutually exclusive. It is always safer to use the general formula.

Note

For independent events, $P(A \cap B) = P(A) \times P(B)$

So $P(A|B) = \frac{P(A) \times P(B)}{P(B)} = P(A)$

Assessment tip

Students (especially non-native English speakers) often do not realize that a conditional probability is being asked for. The words "given that" can often be taken as trigger words to show that the question is one of conditional probability.

SAMPLE STUDENT ANSWER

A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{7}{8}$.

(a) Calculate $P(A \cap B)$.

(b) Calculate $P(A|B)$.

(c) State whether or not events A and B are independent.
Give a reason for your answer.

▲ Parts (a) and (b) were answered using the correct formulae and the given information.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{7}{8} = \frac{1}{2} + \frac{3}{4} - P(A \cap B)$

$P(A \cap B) = \frac{1}{2} + \frac{3}{4} - \frac{7}{8} = \frac{3}{8}$

(b) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{3}{8}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{2}$

(c) Yes, because of part (a).

▼ Part c is correct in that they are independent, but the reason is insufficient and would be better as:

"Yes because $P(A \cap B) = P(A) \times P(B)$ or $P(A|B) = P(A)$."

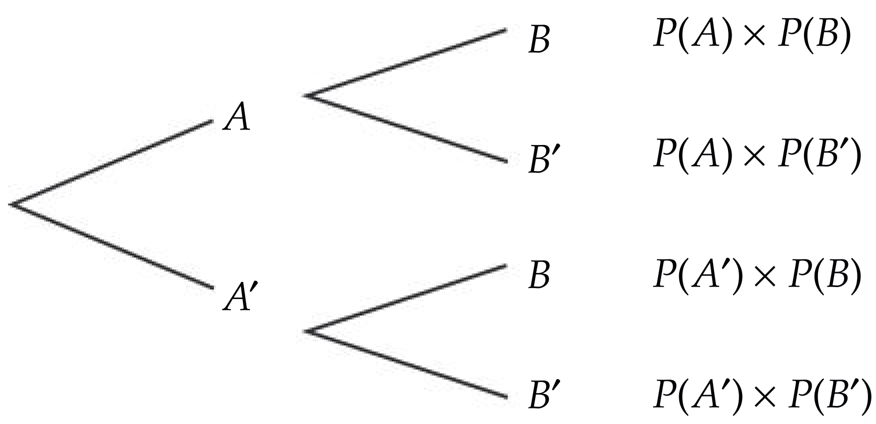
The answer above could have achieved 5/6 marks.

Assessment tip

You should have the formula booklet with you at all times when doing these and other problems to become familiar with which formulae are there, where they are and what format they are presented in.

Tree diagrams can be used to illustrate the probability of a sequence of events and help you ensure that you have accounted for all the different possibilities of two or three events occurring.

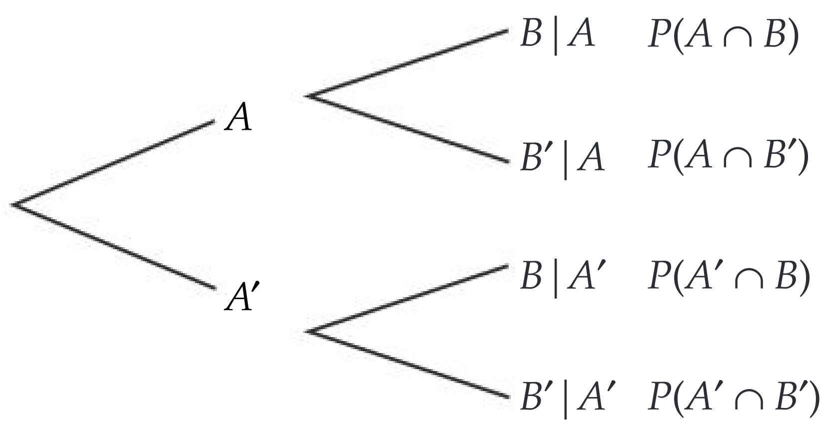
For independent events A and B:



Note

As all of the outcomes are covered, these four probabilities should add up to 1.

For the general case where events A and B are not independent:



Example 4.3.2

Kelly has a bag containing four red balls and five blue balls. Two balls are randomly drawn one after the other without replacement.

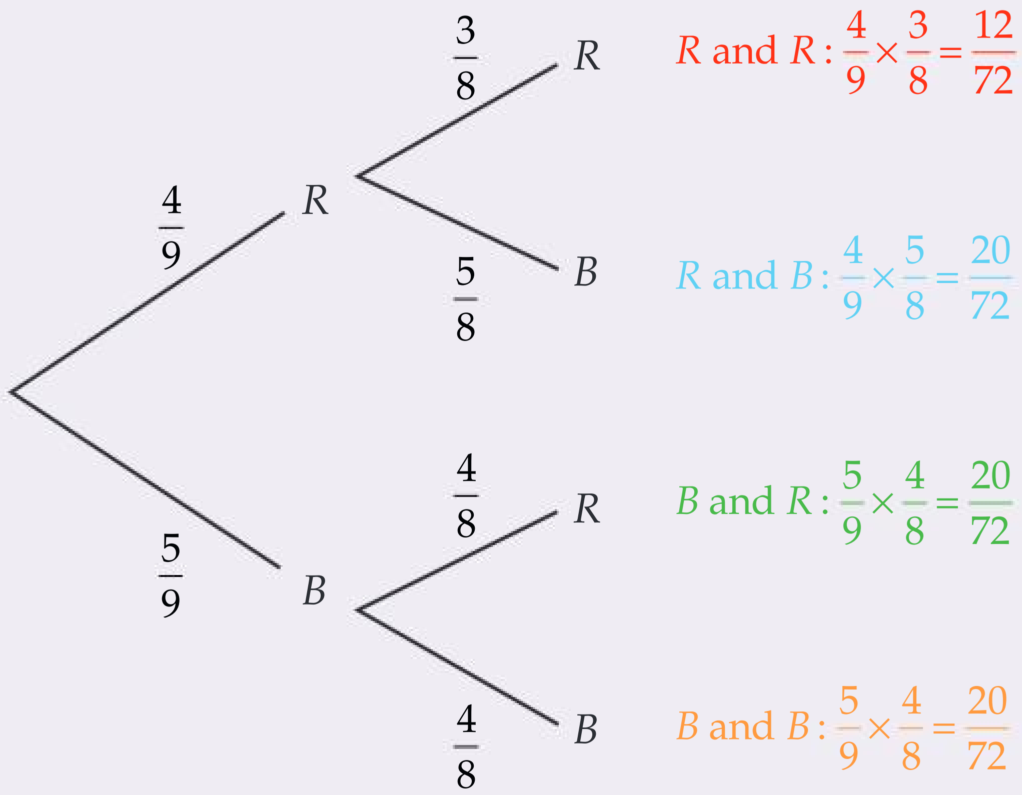
(a) Draw a tree diagram for Kelly's choices.

(b) Find the probability that Kelly draws:

(i) two red balls
(ii) different coloured balls
(iii) at least one red ball
(iv) one blue ball, given that a red is chosen.

Solution

(a)



(b) (i) $\frac{4}{9}\times\frac{3}{8}=\frac{12}{72}=\frac{1}{6}$

(ii) $P(RB)+P(BR)=\frac{20}{72}+\frac{20}{72}=\frac{40}{72}=\frac{5}{9}$

(iii) $1-P(BB)=1-\frac{20}{72}=\frac{52}{72}=\frac{13}{18}$

(iv) $P(B|R)=\frac{P(B\cap R)}{P(R)}=\frac{\left(\frac{40}{72}\right)}{\left(\frac{52}{72}\right)}=\frac{40}{52}=\frac{10}{13}$

Note

Note the words "without replacement" are important in this question. It would have been a different question if they had been replaced.

Note

The words "given that" in this type of question usually indicates conditional probability.

Assessment tip

When constructing tree diagrams, it is a good idea to think about the sequence in which events happen to help you construct the branches.

Note

It is easier to keep unsimplified fractions in the tree diagram to check that all of the probabilities add up to 1, but final answers in further parts should be simplified.

Assessment tip

At least one red allows you to sum all cases with one red or evaluate 1 — P (no reds). The latter is often a necessary shortcut to save time during an examination.

A sample space diagram (sometimes called a lattice diagram) is another way of dealing with two events where the events are represented on a horizontal and vertical axis or in a table.

Example 4.3.3

Two fair tetrahedral dice, one red and one blue, with sides numbered 1 to 4 are thrown and their scores are added together. Show the possible outcomes in a sample space diagram and find the probability that the score is more than 5.

Solution

Blue die					
	4	5	6	7	8
	3	4	5	6	7
	2	3	4	5	6
	1	2	3	4	5
		1	2	3	4
		Red die			

$P(\text{score} > 5) = \frac{6}{16} = \frac{3}{8}$

Probability distributions

A probability distribution for a discrete random variable is a table containing each value of the random variable and the probability that each value occurs.

The lowercase x is used to denote the values and $P(X = x)$ represents the probability that they occur, where X is the random variable.

Here is a probability distribution for throwing an ordinary, fair die.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Notice that $\sum P(X = x) = 1$

The mean of the distribution is called the expected value, $E(X)$, and is found by:

$$E(X) = \sum x \times P(X = x)$$

$$= \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = \frac{21}{6} = 3.5$$

Distributions may be defined by a function such as $P(X = x) = kx^3$ for $x = 1, 2, 3$.

This would lead to the table:

x	1	2	3
$P(X = x)$	k	$8k$	$27k$

So, as $\sum P(X = x) = 1$, then $k + 8k + 27k = 1$ giving $k = \frac{1}{36}$

SAMPLE STUDENT ANSWER

The probability distribution of the discrete random variable X is given by the following table.

x	1	2	3	4	5
$P(X=x)$	0.4	p	0.2	0.07	0.02

(a) Find the value of p.

(b) Calculate the expected value of X.

(a) $0.4+p+0.2+0.07+0.02=1$

$p=0.31$

(b) Mean $=\dfrac{(1\times 0.4)+(2\times 0.31)+(3\times 0.2)+(4\times 0.07)+(5\times 0.02)}{5}=\dfrac{2}{5}=0.4$

▲ This is correct. The probabilities add up to 1.

▼ This is incorrect. Although the expected value is the mean of the distribution, it should not be divided by 5 but by the sum of the probabilities which is 1, giving $E(X)=2$.

The answer above could have achieved 3/5 marks.

The binomial distribution

A binomial distribution is used when there is a fixed number of trials and where each trial has two outcomes: success (with probability p) and failure (with probability $1-p$). The trials must be independent of each other. The random variable X is the number of successes.

In a binomial distribution where the total number of trials is n, the probability of r successes out of n trials is given by the formula:

$$P(X=r)={}^{n}C_{r}\,p^{r}q^{n-r}\text{, where } q=1-p$$

A binomial distribution may also be represented as $X\sim \mathrm{B}(n,p)$. The expected value (mean) of the distribution is $E(X)=np$ and the variance $\mathrm{Var}(X)=npq$

Note

Students can often miss that it is a binomial distribution question. Look for the trigger words "the number of …".

Example 4.3.4

A factory makes lightbulbs and the probability that one is defective is 0.04. The factory tests a random sample of 100 lightbulbs. Let the random variable X be the number of defective lightbulbs.

(a) Find the expected value of X.

(b) Find the probability that there are exactly six defective lightbulbs in the sample.

(c) Find the probability that there is at least one defective lightbulb in the sample.

(d) Find the standard deviation of X.

Assessment tip

You can also use your calculator to find binomial probabilities and cumulative probabilities.

Solution

(a) $E(X)=np=100\times 0.04=4$

(b) $P(X=r)={}^{100}C_{6}(0.04)^{6}(0.96)^{94}=0.105$ (3 sf)

(c) $1-P(X=0)=1-(0.96)^{100}=0.983$ (3 sf)

(d) $\mathrm{Var}(X)=npq=100\times 0.04\times 0.96=3.84$

Standard deviation $=\sqrt{3.84}=1.96$ (3 sf)

Assessment tip

If you see the words "at least" it is often worth considering doing 1 – the opposite probability.

SAMPLE STUDENT ANSWER

A box holds 240 eggs. The probability that an egg is brown is 0.05.

(a) Find the expected number of brown eggs in the box.

(b) Find the probability that there are 15 brown eggs in the box.

(c) Find the probability that there are at least 10 brown eggs in the box.

(a) $E(X) = np = 0.05 \times 240 = 12$

(b) $P(X = 15) = {}^{240}C_{15}\,(0.05)^{15}(0.95)^{225} = 2.97 \times 10^{-25}$

(c) $P(X \le 9) = 0.236$

The answer above could have achieved 3/7 marks.

▲ This is correct.

▼ The student has the correct substitution into the formula but forgot to multiply by ${}^{240}C_{15}$. The correct answer is 0.733 (3 sf)

▼ Incorrect as the student did not understand the phrase "at least 10" and should now be using

$P(X \ge 10) = 1 - P(X \le 9)$

$= 1 - 0.235\,67...$

$= 0.764$ (3 sf)

Note

The binomial distribution is discrete and the normal distribution is continuous.

The normal distribution

The normal distribution is used to model many frequently occurring continuous distributions, such as heights or weights, and has a "bell" shape that is symmetrical about the mean.

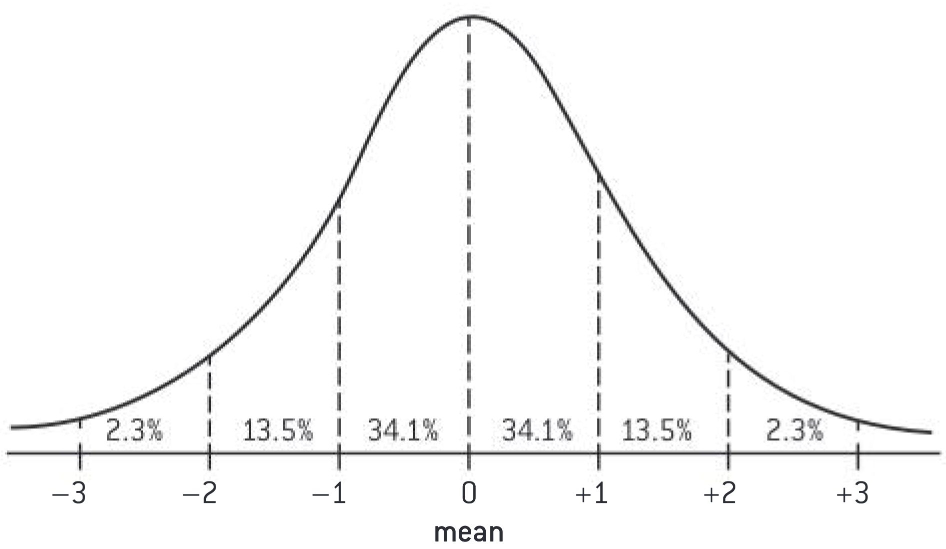
You can use your calculator to find the area under the curve between two values and this shows the probability that the variable lies in that range.

We use $X \sim N(\mu, \sigma^2)$ to indicate that X satisfies a normal distribution, where μ represents the mean and σ the standard deviation.

The standardized value for X is calculated with the z-score, $Z = \dfrac{X - \mu}{\sigma}$

If $X \sim N(\mu, \sigma^2)$ then $Z \sim N(0, 1^2)$

The diagram below shows the curve for the normal distribution with mean of 0 and standard deviation of 1.



Assessment tip

Always draw a picture of the normal curve to help your thinking and ensure that your answers look reasonable.

Example 4.3.5

The volume V of liquid in a can of a particular brand of root beer is normally distributed with a mean of 330 mL and a standard deviation of 2 mL.

(a) If a can is chosen at random, find:

(i) $P(328 < V < 333)$

(ii) $P(V > 327)$

(b) It is known that 70% of all cans have a volume of liquid greater than w mL.

Find the value of w.

(c) Find the probability that $V < 334$, given that $V > w$.

(d) The cans of root beer are sold in packs of six. Abi buys a six-pack. Find the probability that this pack contains:

(i) exactly two cans with $V > w$

(ii) at least one can with $V > w$.

Solution

(a) $V \sim \mathrm{N}(330, 2^2)$

(i) By calculator, $P(328 < V < 333) = 0.775$ (3 sf)

(ii) By calculator, $P(V > 327) = 0.933$ (3 sf)

(b) By calculator $P(V > w) = 0.7 \Rightarrow w = 329$ mL (3 sf)

(c) $P(V > 334 | V > w) = \dfrac{P(V > 334 \text{ and } V > w)}{P(V > w)} = \dfrac{P(V > 334)}{0.7} = 0.0325$ (3 sf)

(d) Let X be the number of cans with $V > w$. $X \sim \mathrm{B}(6, 0.7)$

(i) By calculator $P(x = 3) = 0.185$ (3 sf)

(ii) $P(X \geq 1) = 1 - P(X = 0) = 0.999$ (3 sf)

Note

In this example, notice how a question can involve both the continuous normal distribution and the discrete binomial distribution, as well as a conditional probability.

Also note that part (d) did not depend on the answers to earlier parts.

Assessment tip

When using a calculator for the normal distribution, note that:

- if you want the probability of being between certain limits, you use “normal cdf”
- if you know the probability and want to find a limit, use “inverse norm”
- you will never use “normal pdf”.

When using a calculator for the binomial distribution, note that:

- if you want the probability of a particular value, use “binomial pdf”
- if you want the probability of a range of values, use “binomial cdf”.

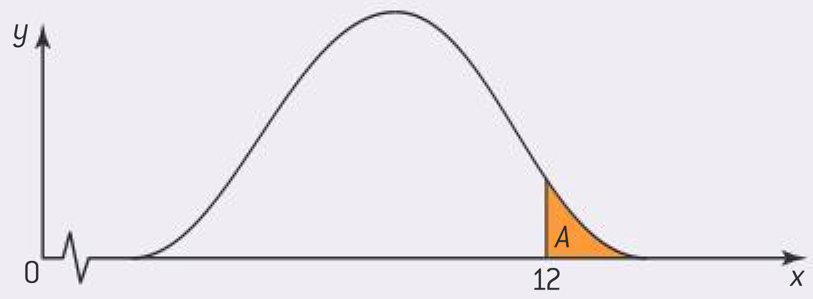
Assessment tip

When drawing the normal curve you will automatically draw two points of inflection symmetrically either side of the mean which is at the maximum. These points of inflection will be at $\mu + \sigma$ and $\mu - \sigma$.

This will assist you with the scale of your diagram.

Example 4.3.6

The graph shows a normal curve for the random variable X, with mean μ and standard deviation σ.



It is known that $P(X \geq 12) = 0.1$

(a) The shaded region A is the region under the curve where $x \geq 12$. Write down the area of the shaded region A.

It is also known that $P(X \leq 7) = 0.0272$.

(b) Find the value of μ, showing your method in full.

(c) Show that $\sigma = 1.56$, accurate to three significant figures.

(d) Find $P(X \leq 11)$.

Solution

(a) 0.1

(b) $P(X \geq 12) = 0.1 \Rightarrow Z = 1.281\,55\ldots$

$P(X \leq 7) = 0.0272 \Rightarrow Z = -1.923\,67\ldots$

$\therefore -1.923\,67\ldots = \frac{7-\mu}{\sigma}; 1.281\,55\ldots = \frac{12-\mu}{\sigma}$

$-1.923\,67\ldots\sigma = 7 - \mu$

$1.281\,55\ldots\sigma = 12 - \mu$

$\Rightarrow \mu = 10$ (solving with a calculator)

(c) Substituting $\mu = 10$ into one of the above equations,

$\sigma = 1.56$ (3 sf)

(d) $P(X \leq 11) = 0.739$ (3 sf)

Additional higher level probability

Using Bayes' theorem for a maximum of three events

Bayes' theorem for two events is:

$$P(B|A) = \frac{P(B)\,P(A|B)}{P(B)\,P(A|B) + P(B')\,P(A|B')}$$

Bayes' theorem for three events is:

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3)\,P(A|B_3)}$$

Example 4.3.7

A company is making batteries using three machines: A, B and C. Machine A produces 25% of the total output, machine B produces 35% and machine C produces 40%. In daily trials, 5% of the batteries from machine A are defective, 4% from machine B are defective and 2% from machine C are defective.

If a battery is chosen at random and found to be defective, find the probability that it came from machine A.

Solution

Let:
D = {a battery is defective}
A = {the battery is from machine A}
B = {the battery is from machine B}
C = {the battery is from machine C}

$P(A) = 0.25$, $P(B) = 0.35$ and $P(C) = 0.4$

$P(D|A) = 0.05$, $P(D|B) = 0.04$, $P(D|C) = 0.02$

$$P(A|D) = \frac{P(A) \times P(D|A)}{P(D|A)\,P(A) + P(D|B)\,P(B) + P(D|C)\,P(C)}$$

$$= \frac{0.25 \times 0.05}{(0.05 \times 0.25) + (0.4 \times 0.35) + (0.02 \times 0.04)} = 0.362 \text{ (3 sf)}$$

Assessment tip

It is often easier to draw a probability tree and just use the standard formula for conditional probability rather than use Bayes' theorem.

Variance of a discrete random variable

The variance of a discrete random variable X is often written as $\text{Var}(X)$ or σ^2 and is defined as:

$\text{Var}(X) = \sum (x - \mu)^2 P(X = x)$

which leads to the formula $\text{Var}(X) = E(X^2) - [E(X)]^2$

The square root of the variance is equal to the standard deviation.

Assessment tip

You can use your calculator to find the expected value and standard deviation of a discrete random variable by putting the probabilities in as though they were frequencies. A check is that it should give $n = 1$ as the sum of the probabilities must be 1.

SAMPLE STUDENT ANSWER

Random variable X has the following probability function.

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.4	0.3

Find:

(a) $E(X)$

(b) $E(X^2)$

(c) $\mathrm{Var}(X)$

(d) the standard deviation of X.

using $\mathrm{var}(X) = E(X^2) - [E(X)]^2$

(a) $E(X) = (0 \times 0.1) + (1 \times 0.2) + (2 \times .4) + (3 \times 0.3) = 1.9$

▲ This has been answered correctly.

(b) $E(X^2) = (0 \times 0.1)^2 + (1 \times 0.2)^2 + (2 \times 0.4)^2 + (3 \times 0.3)^2 = 1.49$

▼ This is incorrect. The whole brackets have been squared where it should only be the X value squared to give:
$E(X^2) = (0^2 \times 0.1) + (1^2 \times 0.2) + (2^2 \times 0.4) + (3^2 \times 0.3) = 4.5$

(c) $\mathrm{var}(X) = E(X^2) - [E(X)]^2 = 1.49 - 1.9 = -0.41$

▼ This is the correct method and would earn follow-through credit; however, the $E(X)$ value has not been squared. The correct solution is:
$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
$= 4.5 - (1.9)^2 = 0.89$

(d) $\sigma = \sqrt{-0.41} = 0.640.$

▼ Correct method but with the wrong value brought forward. The candidate should have realized that something was wrong as the variance cannot be negative.

The correct answer is
$\sigma = \sqrt{0.89} = 0.943$ (3 sf)

The answer above could have achieved 5/8 marks.

Assessment tip

You should always show your working by using a formula, explanation or writing a sentence to gain method marks in case you have made an error in calculations.

Continuous random variables

To move from discrete to continuous distributions, you replace the sums in the formulae by integrals.

Let X be a continuous random variable with range $[a, b]$ and probability density function (pdf) of $f(x)$. The density function $f(x)$ is always greater than or equal to zero and the total area under the curve equals 1. The probability that X is between two values is equal to the area under the curve between these two values.

Link to Integration SL 5.5

The expected value (mean or μ) of X is defined by:

$$\mu = E(X) = \int_a^b x f(x)\,dx$$

The mode occurs at the local maximum of $f(x)$.

The median m occurs at the point that divides the area under the graph into two equal parts.

$$\int_a^m f(x)\,dx = \frac{1}{2}$$

The variance is defined by:

$$\mathrm{Var}(X) = \int_a^b x^2 f(x)\,dx - \mu^2$$

Example 4.3.8

For the given pdf in the range [1, 4], $f(x)=\begin{cases} c(x-1)^2, & 1 \le x \le 2 \\ c(8-x), & 2 < x \le 4 \end{cases}$

Find:

(a) the exact value of constant c

(b) the mean

(c) $P(1.5 \le x \le 3)$

Solution

(a) $\int_1^2 c(x-1)^2\,dx + \int_2^4 c(8-x)\,dx = 1$

$$c\left[\frac{1}{3}(x-1)^3\right]_1^2 + c\left[8x-\frac{x^2}{2}\right]_2^4 = 1$$

$$c\left[\left(\frac{1}{3}-0\right)+(24-14)\right]=1$$

$$c=\frac{3}{31}$$

(b) $\mu = \int_1^2 cx(x-1)^2\,dx + \int_2^4 cx(8-x)\,dx$

$$\mu = \frac{3}{31}\left(\int_1^2 (x^3-2x^2+x)\,dx + \int_2^4 (8x-x^2)\,dx\right) = 2.90 \text{ (3 sf)}$$

(c) $P(1.5 \le x \le 3) = \dfrac{3}{31}\left(\int_{1.5}^2 (x^3-2x^2+x)\,dx + \int_2^3 (8x-x^2)\,dx\right) = 0.802 \text{ (3 sf)}$

Note

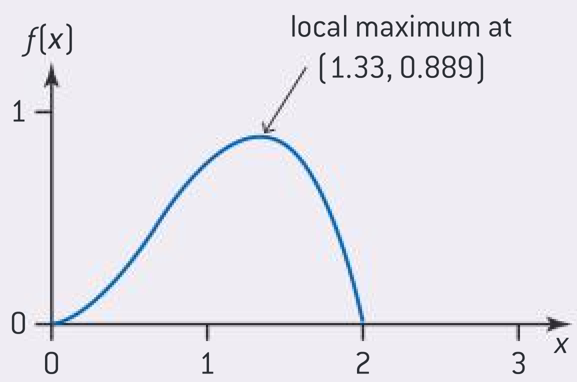
On Paper 2 or 3 you will be able to use a calculator to evaluate such expressions.

Example 4.3.9

The continuous random variable X has a pdf given by $f(x)=\frac{3}{4}x^2(2-x), 0 \le x \le 2$. Find the mode of the distribution.

Solution

On Paper 2 or 3 you can locate the mode at the maximum point on your calculator and show your working in a sketch.



On Paper 1 you would use calculus to find the stationary point.

$$f(x)=\frac{3}{4}(2x^2-x^3)$$

$$f'(x)=\frac{3}{4}(4x-3x^2)=\frac{3}{4}x(4-3x)=0$$

$x=\dfrac{4}{3}$ as this would give the max.

The mode of the distribution is $\dfrac{4}{3}$

Example 4.3.10

The continuous random variable X has a pdf given by $f(x) = 0.5x$, within the range [0, 2].

Find (a) the median and (b) the standard deviation.

Solution

(a) $\int_0^m 0.5x\,\mathrm{d}x = \frac{1}{2}$

$$\left[0.25x^2\right]_0^m = \frac{1}{2}$$

$$0.25m^2 = \frac{1}{2}$$

$$m = \sqrt{2}$$

(b) $\mathrm{Var}(X) = \int_a^b x^2 f(x)\,\mathrm{d}x - \mu^2$

$$= \int_0^2 0.5x^3\,\mathrm{d}x - \left(\int_0^2 0.5x^2\,\mathrm{d}x\right)^2 = 0.222...$$

$$\sigma = \sqrt{0.222...} = 0.471 \text{ (3 sf)}$$

The effect of a linear transformation on a single random variable

If a linear transformation is applied to a discrete or a continuous random variable X, then we have the formulae:

$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$

$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Example 4.3.11

A discrete random variable X has $E(X) = 2$ and $E(X^2) = 13$.
A new random variable $Y = aX + b$ has been designed to represent the winnings in a fair game, with $\mathrm{Var}(Y) = 36$

Find the values of the constants a and b given that a is positive.

Solution

For a fair game require $E(Y) = 0$.

So $E(Y) = aE(X) + b \Rightarrow 2a + b = 0$.

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 13 - 4 = 9$

$\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X) \Rightarrow 36 = 9a^2 \Rightarrow a = 2$ (since a is positive)

So solution is $a = 2$, $b = -4$

PRACTICE QUESTIONS

SL PAPER 1 SECTION A
NO TECHNOLOGY ALLOWED

Question 1 (7 marks)

The data set p, q, r, s, t, where $p \le q \le r \le s \le t$, has a median of 5, a mode of 7, a mean of 5 and a range of 5.

Find the values of each of p, q, r, s and t.

Question 2 [7 marks]

a. Simplify as far as possible:

i. $P(A|A \cap B)$

ii. $P(A|A')$

iii. $P(A \cup B|A)$

iv. $P(A|B) + P(A'|B)$

b. Show that $P(A \cap B|B) = P(A|B)$

Question 3 [8 marks]

Consider the data set
$A = \{25, 2, 32, 37, 65, 60, 35, 34, 1, 36, 33\}$

a. Identify any outliers in set A.

These outliers are then removed from set A to create set B.

b. Identify any outliers in set B.

Question 4 [6 marks]

Mr Forgetful is going for a walk with his umbrella. He will visit three different shops in turn, and then return home. When he is in any particular shop, the probability that he will leave his umbrella there is always $\frac{1}{3}$.

Leave all your answers as fractions in this question.

a. Construct a labelled probability tree to represent this information.

b. Find the probability that he does not have his umbrella with him when he returns home.

c. Given that he does not have his umbrella with him when he returns home, find the probability that he lost it in the second shop that he visited.

Question 5 [9 marks]

Four six-sided dice, A, B, C and D, are marked as follows:

A: 0, 0, 4, 4, 4, 4 $\quad$ C: 2, 2, 2, 2, 6, 6

B: 3, 3, 3, 3, 3, 3 $\quad$ D: 1, 1, 1, 5, 5, 5

For each dice, the probability that any particular face lands uppermost is $\frac{1}{6}$. The uppermost face when rolled is called the score of the dice.

Leave all your answers as fractions in this question.

a. For each dice, find the expected value of its score when it is rolled once.

If two dice are thrown, we say that one dice *beats* the other dice if its score is higher.

b. With the two dice being thrown once each, find the probability that:

i. dice A beats dice B

ii. dice B beats dice C

iii. dice C beats dice D

iv. dice D beats dice A.

c. You and your worst enemy are going to settle a dispute where each of you rolls one of these dice to determine the winner. State what your strategy would be: opting to be first to choose one of the dice, or letting your enemy have first choice.

Question 6 [8 marks]

a. Let A and B be two events, both with non-zero probabilities. Show that if A and B are independent events then they cannot be mutually exclusive events.

b. If $P(A) = 0$, show that event A and any other event B must be:

i. mutually exclusive

ii. independent.

Question 7 [6 marks]

a. Three coins are placed in a bag. One coin is double-headed, one coin is double-tailed and the third coin has a head on one side and a tail on the other. A coin is randomly taken out of the bag and placed on a table, without the underside of the coin being seen. The top side can be seen as a head.

Kate claims that, due to symmetry, the probability that the other side of the coin is also a head must be $\frac{1}{2}$.

State, with reasons, whether or not Kate's claim is true.

b. For the purposes of this question it can be assumed that the probability that a child is a boy equals the probability that the child is a girl equals $\frac{1}{2}$.

A family has two children. Find the probability that both children are boys if:

i. the elder child is a boy

ii. at least one of the children is a boy.

Question 8 [5 marks]

A toy train is travelling at constant speed around a circular track of radius r. There is a station next to point S on the track. Find the probability that at a random moment in time the distance of the train from the station is more than $\sqrt{3}r$.

Question 9 [6 marks]

It is known that two children have their birthdays in the same week.

Find the probability that the two birthdays are:

a. both on Monday

b. on the same day

c. on consecutive days.

Question 10 [8 marks]

A sample of three players is to be taken from the players A, B, C, D, E and F of a six-a-side football team.

a. Find the number of different samples that could be chosen.

b. The first proposed method of sampling is to throw a coin; if the coin lands on heads, choose the sample $\{A, B, C\}$, and if the coin lands tails choose the sample $\{D, E, F\}$.

i. State the probability that any particular player will be chosen in the sample.

ii. Explain whether or not this first proposed method of sampling creates a random sample.

c. Two of the players are female and four are male. The second proposed method of sampling is quota sampling based on gender.

i. State the number of females and the number of males that will be chosen in the sample.

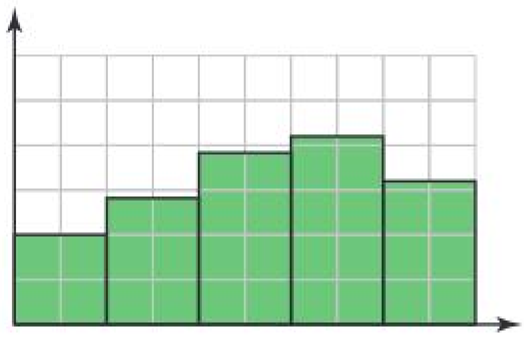
ii. Find the number of different samples that could be chosen using this method.

d. The third proposed method of sampling is to take the first three players who come out of the changing room. State the name of this type of sampling.

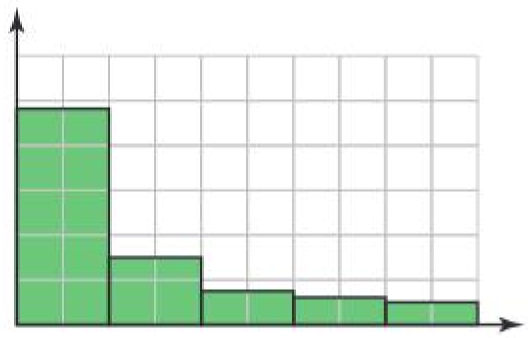
Question 11 [6 marks]

Three data sets are represented by the histograms A, B and C below. Match each histogram with the correct cumulative frequency diagram and box and whisker plot.

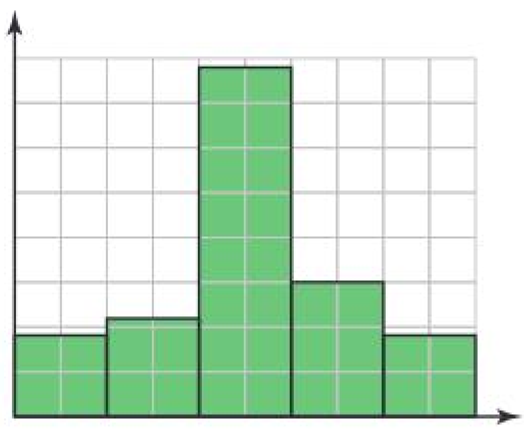
A



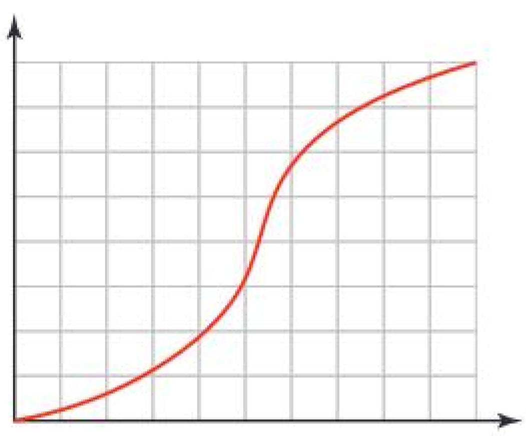
B

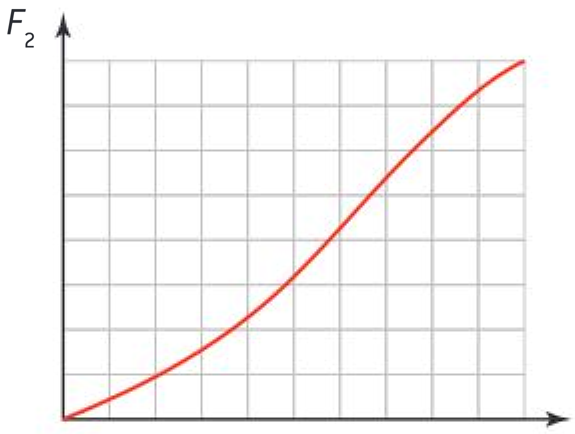


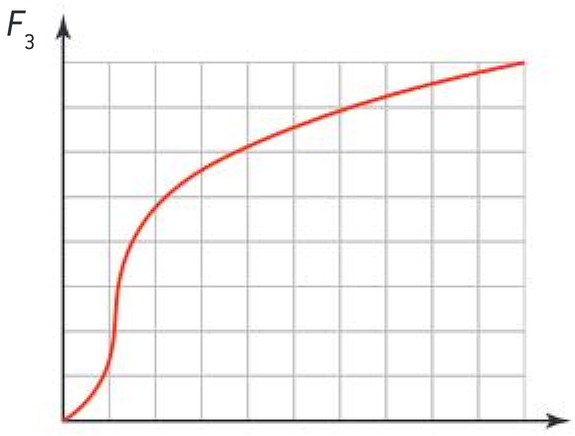
C

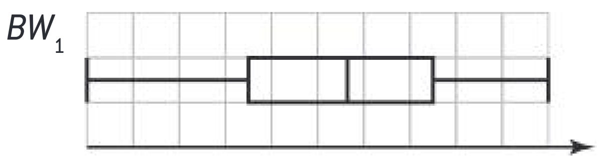


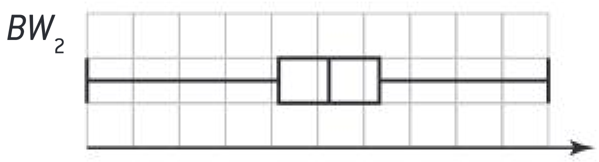
F_1

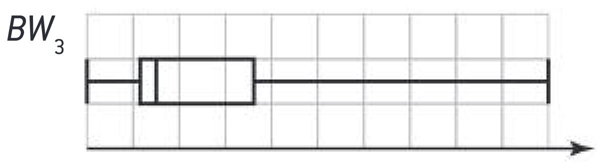












SL PAPER 1 SECTION B
NO TECHNOLOGY ALLOWED

Question 12 [18 marks]

For two events A and B, the general probability rule is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

a. Use this rule to show that the comparable rule for three events A, B and C is:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

> *Hint: Start with* $A \cup B \cup C = (A \cup B) \cup C$
>
> *The distributive law*
>
> $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ *could assist.*

b. If A and B are mutually exclusive, B and C are mutually exclusive and A and C are mutually exclusive, state what the general rule shown in part (a) reduces to.

c. If A and B are mutually exclusive and B and C are mutually exclusive, state what the general rule shown in part (a) reduces to.

d. If A and C are mutually exclusive, state what the general rule shown in part (a) reduces to.

e. Construct four separate labelled Venn diagrams designated (i), (ii), (iii) and (iv) to represent each of the cases in parts (a), (b), (c) and (d), respectively.

f. A box contains 300 cards with an integer value written on each card. A card is taken out at random.

A is the event that the integer is a prime number greater than 10, where $P(A) = \frac{30}{300}$.

B is the event that the integer is a natural number smaller than 200, where $P(B) = \frac{100}{300}$

C is the event that the integer is an even integer where $P(C) = \frac{150}{300}$.

$P(A \cap B) = \frac{2}{300}$ and $P(B \cap C) = \frac{50}{300}$

Find **i.** $P(A \cup B \cup C)$ **ii.** $P((B \cap A') \cap C')$

Leave your answers as fractions with 300 as the denominator.

Question 13 [10 marks]

Let 100 cards be labelled with the natural numbers 1 to 100 inclusive.

A card is chosen at random.

A is the event that the number is exactly divisible by 4.

B is the event that the number is exactly divisible by 5.

C is the event that the number is exactly divisible by 10.

a. Determine whether or not events A and B are independent.

b. Determine whether or not events A and C are independent.

c. Explain any difference in your answers to parts (a) and (b) in terms of the divisors.

Question 14 [11 marks]

The lengths of mango leaves from a particular tree are recorded in the frequency table.

Length, l cm	Frequency
$8 < l \leq 10$	12
$10 < l \leq 12$	16
$12 < l \leq 14$	24
$14 < l \leq 16$	32
$16 < l \leq 18$	16

a. State how many mango leaves were measured.

b. Construct a cumulative frequency table for this data.

c. Construct a cumulative frequency graph for this data.

d. Use your graph to estimate:

i. the median **ii.** the interquartile range.

e. Construct a box and whisker diagram for this data.

SL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 15 [7 marks]

Owen is practicing taking penalty kicks at goal in rugby from a particular spot. The probability that he succeeds in kicking the ball between the posts is always 0.9.

He takes 10 practice kicks.

a. Find the expected number of successful kicks.

b. Find the probability of exactly eight successful kicks.

c. Find the probability of less than six successful kicks.

d. If he actually had nine successful kicks, write down the probability that his one miss was on his last attempt.

Question 16 [8 marks]

A factory makes a large number of cylinders. The diameters of the cylinders are normally distributed with mean of 2.60 cm and standard deviation of 0.08 cm.

a. Find the percentage of cylinders with a diameter of less than 2.56 cm.

b. Find the limits of the acceptable values for the diameters, if 5% are rejected for being oversized and 4% are rejected for being undersized.

c. Given that the diameter of a cylinder is greater than 2.60 cm, find the probability that the diameter is less than 2.70 cm.

Question 17 [9 marks]

A data set with 15 values is given below:

1 20 7 3 8 5 7 10

11 19 3 7 25 19 17

a. Find the **i.** mode **ii.** median **iii.** mean.

b. Find the **i.** range **ii.** interquartile range **iii.** standard deviation.

c. Construct a box-and-whisker diagram for this data.

Question 18 [8 marks]

Anna, in Antarctica, measured the heights of 100 adult Emperor penguins, in metres, and recorded the information in the tables below. Unfortunately melting ice obliterated several of her entries.

Height, h m	Frequency
$1.05 < h \leq 1.10$	1
$1.10 < h \leq 1.15$	20
$1.15 < h \leq 1.20$	
$1.20 < h \leq 1.25$	
$1.25 < h \leq 1.30$	
$1.30 < h \leq 1.35$	2

Height, h m	Cumulative frequency
$h \leq 1.10$	
$h \leq 1.15$	
$h \leq 1.20$	51
$h \leq 1.25$	76
$h \leq 1.30$	
$h \leq 1.35$	100

a. Copy the two tables and fill in the missing values.

b. Write down the modal class.

c. Find estimates for:

i. the mean **ii.** the standard deviation.

SL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 19 [12 marks]

a. For a large data set, state the percentage of data points that will lie between the lower quartile, Q_1, and the upper quartile, Q_3.

b. If $X : N(0, 1^2)$ find l such that $P(-l < X < l) = 0.5$

c. For the large data set of part (a), show that a data point y would be classified as an outlier if:

$y > 2.5Q_3 - 1.5Q_1$ or $y < 2.5Q_1 - 1.5Q_3$

d. Using part (b), find $P(X > 4l \text{ or } X < -4l)$

e. Considering the connections between the earlier parts, explain why the number $4l$ has been chosen in part (d).

f. If $W : N(\mu, \sigma^2)$, find $P(W > \mu + 4l\sigma \text{ or } X < \mu - 4l\sigma)$

Question 20 [11 marks]

Abi is playing a board game with her grandfather. In order to start the game, she first has to throw a six on an ordinary, fair, six-sided dice. If she has not thrown a six on her fourth attempt, her grandfather allows her to start anyway (this avoids tantrums and the game being thrown on the floor!). Let the discrete random variable X be the number of times that Abi rolls the dice before she is allowed to start.

a. Construct, with explanations, a table for the probability distribution function of X.

b. Suggest and perform a check on the probabilities in your table.

c. Write down **i.** the mode of X **ii.** $E(X)$ **iii.** $\text{Var}(X)$

Question 21 [14 marks]

Two data sets A and B are given by the frequency tables below:

A

Score	1	10	11	20	21	30	31	40
Frequency	1	10	1	10	10	1	10	1

B

Score	1	10	11	20	21	30	31	40
Frequency	10	1	10	1	1	10	1	10

a. Construct a frequency histogram for each data set.

b. For each of the two data sets, find the:

i. mean **ii.** median
iii. range **iv.** interquartile range
v. standard deviation

c. By considering the two histograms, explain briefly which of the three measures of dispersion above tells you more about the two data sets.

Question 22 [12 marks]

In a factory, a machine produces a large number of tent pegs. The length of a tent peg, L, is normally distributed with mean of 20 cm and standard deviation of 0.5 cm.

a. Find $P(19.6 < L < 20.8)$

b. 95% of the tent pegs have a length greater than l. Find the value of l.

A clumsy employee in the factory drives a fork lift truck into the machine. It is believed that this has altered the mean but not the standard deviation. Assume that this is true. Now 80% of the tent pegs are longer than 19 cm.

c. Find the new value of the mean.

d. With this new value of the mean, find what percentage of the tent pegs could be expected to have a length less than 20 cm. The factory could use this expected percentage to see if the standard deviation had remained unaltered.

Question 23 [10 marks]

Paula believes that there is correlation between an adult's intelligence quotient (IQ), x, and the number of tattoos on their body, y. She takes a random sample of ten adults and obtains the following data:

x	120	110	100	90	80	70	103	95	85	125
y	0	0	1	2	3	10	2	3	4	0

a. Calculate the Pearson product moment correlation coefficient.

b. State, in words, a description for this correlation.

c. Another adult has an IQ of 105. Estimate the number of tattoos they will have.

d. Yet another adult has five tattoos. Estimate their IQ.

HL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 24 [8 marks]

a. Show that if A and B are two independent events then A' and B are also two independent events.

b. Show that if A and B are two independent events then A' and B' are also two independent events.

Question 25 [7 marks]

The seven letters in the word SUCCESS are each written on a piece of card. The seven pieces of card are then randomly placed in a line. Leave all your answers as fractions in this question.

a. Find the probability that the three S's are next to each other.

b. Find the probability that the line of letters starts and finishes with a C.

Question 26 [7 marks]

In a television game show, there are three closed doors. Behind two of them there is a goat and behind the third is a million pounds. The contestant picked a door but did not open it. The game show host then opened one of the other two doors to reveal a goat. The contestant was then given the opportunity to change their decision to the other available door if they wished. Using a probability tree and reasoning, state what your advice would be to the contestant, as to whether or not to change their choice.

Question 27 [7 marks]

Dawn believes that there is correlation between the intelligence quotient (IQ), x, an adult has and their height in centimetres, y. She takes a random sample of ten adults and obtains the following data:

x	120	110	100	90	80	70	103	95	85	125
y	175	180	182	183	157	173	184	183	185	155

a. Calculate the Pearson product moment correlation coefficient.

b. State, in words, a description for this correlation.

c. Dawn uses the regression line y on x to estimate the IQ of an adult with a height of 135 cm. State three reasons why she is mistaken in doing this.

HL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 28 [15 marks]

Let $f(x)$ be the probability density function of a continuous random variable, X, that is defined on the interval $[a, b]$. Let $E(X) = \mu$.

a. Show that

$$\int_a^b (x-\mu)^2 f(x)\,\mathrm{d}x = \int_a^b x^2 f(x)\,\mathrm{d}x - \mu^2$$

This indicates that these two formulae for $\mathrm{Var}(X)$ are equivalent.

b. Show that $E(X^2) \geq (E(X)^2)$

c. Investigate whether $E(X^2) = (E(X))^2$ is possible for a continuous random variable.

d. Let the discrete random variable Y have a probability distribution function given by:

y	7
$P(Y = y)$	1

Find **i.** $E(Y)$ **ii.** $E(Y^2)$ **iii.** $\mathrm{Var}(Y)$ and comment on these results.

HL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 29 [9 marks]

When Eric goes to his local restaurant, he always choses one of the following meals: beef burger, chicken pie or vegetarian pasta, with probabilities of $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$, respectively. Eric always wants everything done perfectly. If the beef burger is undercooked, which happens with a probability of $\frac{1}{4}$, he sends it back to the kitchen. If the chicken pie has soft pastry, which happens with a probability of $\frac{1}{3}$, he also sends it back. If the vegetarian pasta is burnt, which happens with a probability of $\frac{1}{6}$, he sends that back. In all other cases he is happy to eat the meal. If he does send his meal back, he goes home to cook for himself.

Leave all your answers as fractions in this question.

a. Construct a labelled probability tree to represent his information.

b. Find the probability that Eric sends his meal back.

c. Given that he does send his meal back, find the probability that he chose the chicken pie.

d. Eric is seen happily eating his meal in the restaurant. Find the probability that he chose the beef burger.

Question 30 [9 marks]

A bag contains four red counters, five blue counters and six green counters. Three counters are taken out of the bag at the same time. Find the probability that:

a. they are all green

b. two are blue and one is red

c. all three are different colours

d. all three are the same colour.

Leave all your answers as fractions in this question.

HL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 31 [12 marks]

A discrete random variable X has a probability distribution function given by:

x	1	2	3	4
$P(X=x)$	$\frac{4}{13}$	$\frac{3}{13}$	$\frac{2}{13}$	k

a. Show that $k=\frac{4}{13}$

b. Find **i.** $E(X)$ **ii.** $V(X)$.

Let the random variable Y be given by $Y=\frac{1}{X}$

c. Find $E(Y)$.

d. Hence show that $E\left(\frac{1}{X}\right)\neq\frac{1}{E(X)}$

The discrete random variable W represents the winning in a fair game where the possibilities for W are –5, –3, 2, 4.

e. Write down $E(W)$.

f. Hence explain why $E\left(\frac{1}{W}\right)\neq\frac{1}{E(W)}$ in this example.

Question 32 [13 marks]

The probability density function for a continuous random variable X is given by

$$f(x)=\begin{cases}k|\sin x|, & 0\leq x\leq 2\pi\\ 0, \text{elsewhere}\end{cases}$$

a. Show that $k=\frac{1}{4}$

b. Sketch the graph of this probability density function.

c. **i.** Write down the value of $E(X)$, giving a reason.

ii. Write down the value of the median of X, giving a reason.

iii. Explain why $f(x)$ is bi-modal and state what the two bi-modal values are.

d. Calculate $\text{Var}(X)$.

Question 33 [12 marks]

Mavis, who has sight problems, has a tin with three chocolate and three plain biscuits in it. She is going to take biscuits one at a time, at random, and eat them until she eats her first chocolate biscuit. Let the discrete random variable, X, be the number of plain biscuits she eats before the chocolate one. Give all your answers as fractions in this question.

a. Construct a labelled probability tree to represent this situation.

b. Hence, construct with explanations, a table for the probability distribution function of X.

c. Suggest and perform a check on the probabilities in your table.

d. Find $E(X)$.

e. Find $\text{Var}(X)$.

Question 34 [12 marks]

20% of the inhabitants of a particular country are infected with a certain virus. There is a test available. If a person has the virus, the test will say that they have it 98% of the time. This means that the probability of a false negative (the test saying that the person does not have it when actually they do) is 2%. If a person does not have the virus, the test will say that they do not have it 99% of the time. This means that the probability of a false positive (the test saying that the person has the virus when actually they do not) is 1%.

a. Construct a probability tree to represent this information.

b. Find the probability that a person chosen at random is told that they have the virus when they take the test.

c. If a randomly selected person takes the test and is told that they have the virus, find the probability that they actually do have the virus.

d. If a randomly selected person takes the test and is told that they do not have the virus, find the probability that they actually do have the virus.

A school with 300 pupils, where none of them actually have the virus, decides to test them all for the virus. If the test says that one or more pupils has the virus then the school will be closed.

e. Write down the expected number of students that the test will say that they have the virus.

f. Find the probability that the school will be closed due to the test results.

Question 35 [12 marks]

Give all your answers as fractions in this question.

I have a bag containing six blue pens and two red pens.

I take out a pen, note its colour, and then place it on my table, not back in the bag. I repeat this action two more times.

a. Find the probability that I obtain:

i. three blue pens **ii.** three red pens.

b. Find the expected value for the number, X, of blue pens that I obtain.

Now, starting with all the pens back in the bag, I then take out a pen, note its colour, and then place it back in the bag. I repeat this action two more times.

c. Find the probability that I obtain:

i. three blue pens **ii.** three red pens.

d. Find the expected value for the number, Y, of blue pens that I obtain this time.

e. Comment briefly on your answers to parts (b) and (d).

HL PAPER 3 TECHNOLOGY REQUIRED

Question 36 [28 marks]

Let the two discrete random variables X and Y represent the numbers obtained when rolling two fair ordinary dice, one red and one blue, respectively.

a. Find **i.** $E(X)$ and **ii.** $\text{Var}(X)$, giving your answers as fractions.

The two numbers obtained are added together to give a new random variable, the total $T = X + Y$.

b. Construct a table (lattice diagram) for T and hence find $P(T = 6)$.

c. Use your table to create the probability distribution function for T, giving your answer in the form of a distribution table, starting:

t	
$P(T = t)$	

Hence write down the mode of T.

d. Use the distribution table to find **i.** $E(T)$ and **ii.** $\text{Var}(T)$, giving your answers as fractions.

e. For two random variables U and V, it is always true that $E(U + V) = E(U) + E(V)$.
Verify this rule for X and Y.

f. For two independent, random variables U and V, it is always true that $\text{Var}(U + V) = \text{Var}(U) + \text{Var}(V)$.
Verify this rule for X and Y.

g. The central limit theorem states that: For any random variable, whose distribution has mean of μ and variance of σ^2, if a random sample of size n is taken, with n sufficiently large, then the sample mean is approximately normally distributed with mean of μ and variance of $\frac{\sigma^2}{n}$.

100 observations of T are to be taken. Use the central limit theorem to find an approximation for $P(6.8 \le \overline{T} \le 7.2)$.

Let a new random variable M be defined as the minimum of X and Y. For example, if $X = 4$ and $Y = 2$ then $M = 2$; and if $X = 3$ and $Y = 3$ then $M = 3$.

h. Find $P(M = 2)$.

i. Find the probability that $2M = T$.

5 CALCULUS

5.1 DIFFERENTIATION

You must know:

- ✔ that differentiation involves taking a limit
- ✔ the derivatives of basic functions
- ✔ rules for differentiating functions.

You should be able to:

- ✔ differentiate composite expressions of basic functions
- ✔ find and classify stationary points
- ✔ decide where a graph is increasing/decreasing
- ✔ find and classify points of inflexion
- ✔ find tangents and normals
- ✔ solve optimization problems.

A derivative is the instantaneous rate of change of one variable with respect to another. The definition involves a limiting process. The derivative function of a function $f(x)$ with respect to x represents the gradient (slope) function of the graph of $f(x)$. The derivative function is denoted by $\frac{dy}{dx}$ or $f'(x)$. If $s(t)$ represents the displacement of an object moving in a straight line, then $s'(t)$ or $\frac{ds}{dt}$ will represent its velocity. A necessary (but not sufficient) condition for a function $f(x)$ to be differentiable at a point given by $x = c$ is that $f(x)$ must be continuous at $x = c$, i.e., its graph can be drawn without taking the pen off the paper. If a function is differentiable everywhere, then its derivative is a continuous function and the graph of $f(x)$ will be smooth.

The derivative of x^n is given by $\frac{d(x^n)}{dx} = nx^{n-1}, n \in \mathbb{R}$

This is called "the power rule". The power becomes the coefficient in the differentiated expression, and the new power is one less than the original.

Note

If you have expressions involving root signs, change them to rational powers first, and then apply the power rule.

Example 5.1.1

Write down the derivative of $f(x) = \sqrt{x}$.

Solution

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

Link to Differentiation from first principles AHL 5.12

Link to Kinetics SL 5.9

Note

Do not confuse average speed with instantaneous velocity. The average speed is the total distance divided by the total time, whereas the instantaneous velocity refers to the velocity at a specific time.

Assessment tip

Differentiation can be thought of as a language where (thankfully) there are few words together with some rules for combining the words which can be thought of as a syntax. The words are the derivatives of the basic functions, which are given in the formula booklet.

Assessment tip

The command term "write down" means that no working needs to be shown. It should be relatively easy and will probably only carry one mark.

Assessment tip

Always work with your formula booklet next to you, as it is allowed in all exams. You need to get used to what is in it and where, as well as what is not in it.

When differentiating, if there is a multiplicative constant in front of a function then it just remains in front. You can differentiate term by term, since the derivative of a sum or a difference of functions is the sum or the difference of the derivatives of the functions. These rules can be expressed as follows:

$$\frac{\mathrm{d}kf(x)}{\mathrm{d}x} = k\frac{\mathrm{d}f(x)}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}(f(x) \pm g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} \pm \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

Example 5.1.2

$f(x) = 3x^4 + 4x^2 + \frac{1}{x^2}$

Find $\frac{\mathrm{d}f}{\mathrm{d}x}$

Solution

$f(x) = 3x^4 + 4x^2 + x^{-2}$

$\frac{\mathrm{d}f}{\mathrm{d}x} = 3(4x^3) + 4(2x) + (-2)x^{-3}$

$= 12x^3 + 8x - 2x^{-3}$

Assessment tip

Although these basic derivatives are given in the formula booklet you should learn them anyway. This will help you (especially when doing integration) to identify expressions that are derivatives of basic functions.

Other basic derivatives

$\frac{\mathrm{d}(\sin x)}{\mathrm{d}x} = \cos x \qquad \frac{\mathrm{d}(\cos x)}{\mathrm{d}x} = -\sin x \qquad \frac{\mathrm{d}(\tan x)}{\mathrm{d}x} = \frac{1}{\cos^2 x} = \sec x^2$

$\frac{\mathrm{d}(\mathrm{e}^x)}{\mathrm{d}x} = \mathrm{e}^x \qquad \frac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \frac{1}{x}$

Note

The formulae for the derivatives of trig functions are true **only** when working in radians.

Assessment tip

The IB uses the convention that all angles are given in radians unless otherwise stated.

Put your calculator in radian mode at the start of the exam. Change to degree mode only when a question specifically calls for degrees, and then immediately change back to radians.

If the domain of a trig function is given in degrees then your answer(s) should be in degrees. Likewise, if the domain is given in radians, your answer should be in radians.

Example 5.1.3

$f(x) = 3\sin x + \mathrm{e}^x - 4\ln x$

Find $\frac{\mathrm{d}f}{\mathrm{d}x}$

Solution

$\frac{\mathrm{d}f}{\mathrm{d}x} = 3\cos x + \mathrm{e}^x - \frac{4}{x}$

Rules to differentiate composite functions are the product rule, the quotient rule and the function of a function (chain) rule.

The product rule

The product rule for differentiation

If $y = uv$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

The rule states that the derivative of the product of two functions is the 1st function times the derivative of the 2nd function plus the 2nd function times the derivative of the 1st function.

Example 5.1.4

Differentiate $y = (x^2 + 1)\sin x$

Solution

$u = (x^2 + 1)$, $v = \sin x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos x$

$\Rightarrow \frac{dy}{dx} = (x^2 + 1)\cos x + \sin x \times 2x$

SAMPLE STUDENT ANSWER

Differentiate $y = (x^3 + 4)e^x$

$\frac{dy}{dx} = (x^3 + 4)e^x + 3x^2 e^x$

The answer above could have achieved 3/3 marks.

▲ The product rule was used correctly, with both functions differentiated correctly.

SAMPLE STUDENT ANSWER

Differentiate $y = (x^4 + 2)\sin x$

$\frac{dy}{dx} = 4x^3 \cos x$

The answer above could have achieved 0/3 marks.

▼ This is wrong, as the product rule was not used correctly. If the student had written out explicitly that $\frac{d(x^4+2)}{dx} = 4x^3$; $\frac{d(\sin x)}{dx} = \cos x$, then A1 would have been awarded for the first derivative. The 2nd would not have received any marks since it is in the formula book. Answer should have been $4x^3 \sin x + (x^4 + 2)\cos x$.

Note

To decide which rule to use, consider what was the last operation that was performed. For example, with $(x^2 + 1)\sin x$, the last operation is multiplication so use the product rule. Many students use a mnemonic like BODMAS (Brackets, Order, Division, Multiplication, Addition, Subtraction) to decide the order in which operations are performed. This will assist in deciding what was the last operation.

Example 5.1.5

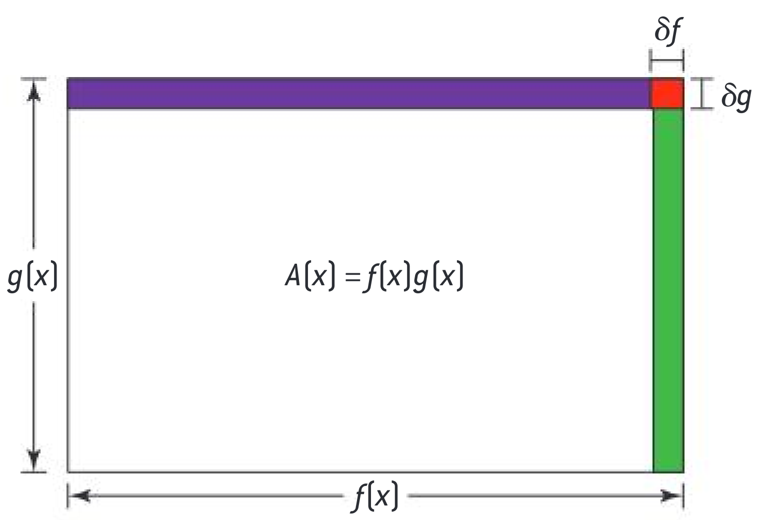
Differentiate $y = e^x \ln x$

Solution

$\frac{dy}{dx} = e^x \times \frac{1}{x} + e^x \ln x = e^x\left(\frac{1}{x} + \ln x\right)$

Visualization of the product rule

Suppose that a child has a magic carpet that grows in size as they grow older. Let its length be $f(x)$ and its width be $g(x)$, where x represents time. Then its area $A(x) = f(x)g(x)$. As time changes by a small amount δx, the length will change by a small amount δf, the width will change by a small amount δg and the area will change by a small amount δA. This is shown in the diagram below. The δf and δg are much smaller than shown.



The extra area is given by the three rectangles: $\delta f \times g(x) + f(x) \times \delta g + \delta f \times \delta g$

Now $\delta f \times \delta g$ is extremely small and can be ignored.
Therefore, the change in area $\delta A \approx \delta f \times g(x) + f(x) \times \delta g$.

Note

This visualization helps you recall the correct rule and shows that $\frac{d(fg)}{dx} = \frac{df}{dx}\frac{dg}{dx}$ is clearly wrong.

Dividing by δx we have $\frac{\delta A}{\delta x} \simeq \frac{\delta f}{\delta x} \times g(x) + f(x) \times \frac{\delta g}{\delta x}$

Now letting $\delta x \to 0$ and using $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$ (which explains the notation used by Leibnitz) gives $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \times g(x) + f(x) \times \frac{\mathrm{d}g}{\mathrm{d}x}$ which is the product rule since $A(x) = f(x)g(x)$.

The quotient rule

The quotient rule for differentiation

If $y = \frac{u}{v}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Assessment tip

When using the quotient rule, start by putting the $(v(x))^2$ in the denominator and remember that in the numerator there is a negative sign when the $v(x)$ is differentiated. Unlike the product rule, the quotient rule is not symmetrical.

Example 5.1.6

Differentiate $y = \frac{x^2+1}{\sin x}$

Solution

$u = x^2 + 1, \ v = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x \times 2x - (x^2+1) \times \cos x}{(\sin x)^2}$

SAMPLE STUDENT ANSWER

Differentiate $y = \frac{2x+3}{4x-2}, x \neq \frac{1}{2}$. Hence, state with a reason, whether this function is increasing or decreasing.

$\frac{dy}{dx} = \frac{2(4x-2) - 4(2x+3)}{(4x-2)^2} = \frac{-16}{(4x-2)^2}$ Since a square number is greater or equal to 0 the derivative is always negative and so the function is decreasing for all x.

▲ There is correct application of the quotient rule. The expression is then simplified, which allows the correct conclusion to be obtained and the reasoning is good.

The answer above could have achieved 5/5 marks.

SAMPLE STUDENT ANSWER

Differentiate $y(x) = \frac{\sin x}{2x+3}$

$\frac{dy}{dx} = \frac{2\sin x - \cos x(2x+3)}{(2x+3)^2}$

▲ The correct method using the quotient rule was used and both derivatives were correct.

▼ The rule was applied incorrectly with the minus sign in the wrong place.

The answer above could have achieved 2/3 marks.

The correct answer should have been $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x(2x+3) - 2\sin x}{(2x+3)^2}$

The chain rule

The chain rule for differentiation

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, where $u(x)$ is an "inside" function.

The chain rule is used when you have a function of another function.

Example 5.1.7

Differentiate $y = \cos(x^3 + 2x)$

Solution

Let $u = x^3 + 2x$ then $\frac{du}{dx} = 3x^2 + 2$ and as $y = \cos u$, $\frac{dy}{du} = -\sin u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times (3x^2 + 2)$

Now subsituting $x^3 + 2x$ for u gives $\frac{dy}{dx} = -\sin(x^2 + 2x) \times (3x^2 + 2)$

Note, it is better to rearrange this to be $\frac{dy}{dx} = -(3x^2 + 2)\sin(x^3 + 2x)$ so that is it clear what the argument of the sine function is.

In practice we do not set it out so formally. We first differentiate the outside function and then multiply by the derivative of the inside function.

For example, if $f(x) = \underbrace{\sin\,(\overbrace{e^x + 8x}^{\text{inside function}})}_{\text{outside function}}$, then $f'(x) = \cos(e^x + 8x) \times (e^x + 8)$

Example 5.1.8

Differentiate $f(x) = (x^2 + 3x + 1)^4$

Solution

To find $f'(x)$, we differentiate the outside function, then multiply by the derivative of the inside function, where $x^2 + 3x + 1$ is the inside function.

$f'(x) = 4(x^2 + 3x + 1)^3\,(2x + 3)$

Assessment tip

You should practice differentiating using the chain rule until you have mastered it. Make sure you do problems that include trig functions, powers, the natural log and the exponential function in various combinations. Mastering the chain rule will assist you in understanding the inspection (lookout) method for integration, which will appear later.

Example 5.1.9

Differentiate $f(x) = (\cos x + 2)^5$

Solution

$f'(x) = 5(\cos x + 2)^4\,(-\sin x) = -5\sin x(\cos x + 2)^4$

Example 5.1.10

Differentiate (a) $y = \cos(3x)$ (b) $y = \tan(4x)$ (c) $y = \ln(5x)$

Solution

(a) $\frac{dy}{dx} = -3\sin(3x)$ (b) $\frac{dy}{dx} = 4\sec^2(4x)$

(c) $\frac{dy}{dx} = \frac{5}{5x} = \frac{1}{x}$ as confirmed by the fact that $\ln(5x) = \ln 5 + \ln x$

Assessment tip

Although it is in your formula booklet, the chain rule can be remembered since it looks as though the du's cancel (due to the notation that Leibnitz used) despite the fact that technically we could not do such a thing, as it is not really the product of two fractions.

Note

Be careful when there is an implied set of brackets, as it is easy to forget to multiply by the derivative of the inside function when there are no brackets. For example, if $y = e^{2x}$ then $\frac{dy}{dx} = 2e^{2x}$ rather than the common mistake of just e^{2x}. It is better to put the brackets in and think of the function as $y = e^{(2x)}$. You will then be less likely to forget about the derivative of the inside function.

The differentiation rules can be combined together as shown in the following examples.

Example 5.1.11

Differentiate $y = \ln x \cos x^2$

Solution

$\frac{dy}{dx} = \frac{1}{x}\cos x^2 - 2x \ln x(\sin x^2)$, applying the chain rule inside the product rule.

Assessment tip

If the question just says differentiate then do not spend too much time attempting to simplify the answer unless the question specifically asks for simplification, or you need to in order to continue to work further with it.

Example 5.1.12

Differentiate $y = \frac{\sin x}{(x^2+1)^3}$ and simplify your answer.

Solution

$$\frac{dy}{dx} = \frac{\cos x(x^2+1)^3 - \sin x \times 3(x^2+1)^2 2x}{(x^2+1)^6} = \frac{\cos x(x^2+1) - 6x\sin x}{(x^2+1)^4}$$

SAMPLE STUDENT ANSWER

Differentiate $y = (\sin 2x + x)^4$

$\frac{dy}{dx} = 4(\sin 2x + x)^3 (\cos 2x + 1)$

The answer above could have achieved 2/3 marks.

▲ The student applied the chain rule.

▼ The student has missed the second disguised function of a function, namely $\cos(2x)$. The correct answer should be $\frac{dy}{dx} = 4(\sin 2x + x)^3 (2\cos 2x + 1)$

One of the main purposes of differentiation is to optimize an expression by finding its maximums and minimums.

Assessment tip

A maximum or a minimum at $x = c$ implies that $f'(c) = 0$ but it is a one-way implication. Just because $f'(c) = 0$, it does not mean that we must have a maximum or a minimum at $x = c$. A common mistake is to think that the two statements are equivalent.

At a maximum or a minimum, the derivative of $y = f(x)$ must be zero.

If $f'(x) > 0$ for all $x \in \,]a, b[$ then the function is increasing on $]a, b[$

If $f'(x) < 0$ for all $x \in \,]a, b[$ then the function is decreasing on $]a, b[$

If $f'(x) = 0$, the function has either a maximum or a minimum or a horizontal point of inflexion.

A point with $x = c$ where $f'(c) = 0$ is called a stationary point; if it is a maximum or a minimum it is called a turning point.

To classify a stationary point, the use of a sign diagram is strongly recommended, as shown in Example 5.1.13.

Example 5.1.13

Let $y = \frac{x^3}{3} - 3x^2 + 8x$

(a) Find and classify all stationary points.

(b) State, with a reason, the x intervals where the function is (i) increasing (ii) decreasing.

Solution

(a) $\frac{dy}{dx} = x^2 - 6x + 8 = (x-2)(x-4) = 0 \Rightarrow x = 2$ or $x = 4$

x	$x<2$	$x=2$	$2<x<4$	$x=4$	$x>4$
$\frac{dy}{dx}$	+ve	0	–ve	0	+ve

In the table, +ve means *positive* and –ve means *negative*.

The values in the top row are placed in ascending order. When deciding upon the sign of $\frac{dy}{dx}$, any suitable x value can be used to represent that interval, e.g., $x = 3$ for the interval $2 < x < 4$.

There is a maximum at $x = 2$ since $\frac{dy}{dx}$ goes from +ve to –ve and and a minimum at $x = 4$, since $\frac{dy}{dx}$ goes from –ve to +ve.

The maximum point is $\left(2, \frac{20}{3}\right)$ and the minimum point is $\left(4, \frac{16}{3}\right)$

(b) (i) The function is increasing in the intervals $x < 2$ and $x > 4$ since $\frac{dy}{dx}$ is positive

(ii) The function is decreasing in the interval $2 < x < 4$ since $\frac{dy}{dx}$ is negative.

Example 5.1.14

Let $y = x - 1 + \frac{1}{x}$, $x \neq 0$

(a) Find and classify all stationary points.

(b) Find the x intervals where the function is (i) increasing (ii) decreasing.

Solution

(a) $\frac{dy}{dx} = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

x	$x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
$\frac{dy}{dx}$	+ve	0	–ve	undefined	–ve	0	+ve

So (–1, –3) is a maximum and (1, 1) is a minimum.

Note: In addition to evaluating the signs of the gradients to the left and right of the zeros of the function, you also need to do the same at any x where the function is undefined.

(b) (i) Increasing in the intervals $x < -1$ and $x > 1$

(ii) Decreasing in the intervals $-1 < x < 0$ and $0 < x < 1$

Assessment tip

A point has an x value and a y value, so if you only give the x value you will lose marks.

Assessment tip

On a paper that allows the use of technology, the graph could be drawn and the maximum and minimum points read off. Observing the graph indicates the intervals where the function is increasing or decreasing. The question could hint at this by using the words "write down" indicating that no working needs to be shown.

Note

The IB has the convention that the domain of the function will be $\mathbb{R}$ unless stated otherwise.

If the function is undefined for any x value, then put that value in your sign table since the sign of $\frac{dy}{dx}$ could alter as you move though this x value.

SAMPLE STUDENT ANSWER

Let $y = x^2 + \frac{1}{x^2}$, $x \neq 0$. Find the x intervals where the function is (a) increasing (b) decreasing.

$\frac{dy}{dx} = 2x - \frac{2}{x^3} = 0 \Rightarrow x = \pm 1$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	$-$ve	0	$-$ve	0	+ve

(a) Increasing for $x > 1$

(b) Decreasing for $x < -1$ and $-1 < x < 1$

▲ The student has differentiated correctly and found the correct x values for the stationary points.

▼ The student has not put $x = 0$ in the sign table and has thus missed that the function is increasing for $-1 < x < 0$ and decreasing for $0 < x < 1$.

The answer above could have achieved 4/6 marks.

Assessment tip

A graphical calculator can find $\frac{dy}{dx}$ for a particular value of x but cannot find an analytic expression for the function itself.

Example 5.1.15

Let $y = e^x \sin x$. Find $\frac{dy}{dx}$ at $x = 0$.

Solution

Paper 1: $\frac{dy}{dx} = e^x \sin x + e^x \cos x \Rightarrow \frac{dy}{dx}$ at $x = 0$ is 1

Paper 2: $\frac{dy}{dx}$ at $x = 0$ is 1

Link to Chapter 3 Trigonometry

For papers where technology is not allowed, if trig functions are involved then they will be evaluated at standard angles. So learn again the sin, cos and tan of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$ (and how to evaluate related angles) given in the table in section 3.1, but you will have to convert the angles from degrees to radians. It cannot be emphasized enough that you *must* do this (see conversion table in section 3.2). Also remember the diagram for the sign of trig functions in section 3.3, and be able to sketch the graphs of sine and cosine functions.

When drawing graphs of functions that are continuous and differentiable (all polynomials are) there must be a maximum between two minimums and there must be a minimum between two maximums. A polynomial of degree n will have a derivative of degree $n - 1$ and so can have at most $n - 1$ turning points.

A cubic function will either have no turning points or two (a max and a min).

Being able to find the derivative of a function at a particular point allows us to find the equation of the tangent and the normal at that point. This is because the gradient of the tangent will be the same as the gradient of the curve at the point of tangency (contact).

Example 5.1.16

Find the equation of (a) the tangent and (b) the normal to the graph of $f(x) = \sqrt{x^3 + 1}$ at the point where $x = 2$.

Solution

$f'(x) = \frac{1}{2} 3x^2 (x^3 + 1)^{-\frac{1}{2}} \Rightarrow f'(2) = 2$

(a) Equation of tangent must be $y = 2x + c$

$f(2) = \sqrt{2^3 + 1} = \sqrt{9} = 3$

Substituting (2, 3) gives $c = -1$

Tangent is $y = 2x - 1$

(b) Gradient of the normal is $-\frac{1}{2}$, so the equation of the normal must be $y = -\frac{1}{2}x + d$

Substituting (2, 3) gives $d = 4$ Normal is $y = -\frac{1}{2}x + 4$

The second derivative of $y = f(x)$, denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$, is the derivative of the first derivative and thus indicates if the gradient is increasing or decreasing. This defines the concavity of the function.

If $f''(x) > 0$ for all $x \in\ ]a, b[$ then the graph is concave up on this interval.

If $f''(x) < 0$ for all $x \in\ ]a, b[$ then the graph is concave down on this interval.

If $f''(x) = 0$ the graph is momentarily straight.

This gives rise to what is called the second derivative test.
If at a particular point:

$f'(c) = 0$ and $f''(c) > 0$ there is a minimum at $x = c$

$f'(c) = 0$ and $f'(c) < 0$ there is a maximum at $x = c$

$f'(c) = 0$ and $f''(c) = 0$ the second derivative test fails (at $x = c$ there could be a horizontal point of inflexion or a minimum or a maximum). In this case go back to the first derivative sign diagram to classify the stationary point.

A point of inflexion is a point where the concavity changes from concave up to concave down or vice versa. Thus at this point $f''(c) = 0$ and it actually changes in sign.

There are two types of inflexion point. In the general case $f'(x) \neq 0$ at the point, so the tangent line is not horizontal and this is called an oblique point of inflexion. In the special case $f'(x) = 0$ at the point, so the tangent line is horizontal and this is called a horizontal point of inflexion. When making the sign table for $f'(x)$ this would show up as the sign of $f'(x)$ going from positive to positive through 0, or negative to negative through 0. For a continuous and differentiable function (for example, a polynomial function) there must be a point of inflexion between a maximum point and a minimum point because the concavity of the function must change.

Example 5.1.17

Let $f(x) = \frac{x^3}{3} - x^2 - 8x + 1$

(a) Find and classify all stationary points.

(b) Find all points of inflexion.

Solution

(a) $f'(x) = x^2 - 2x - 8 = (x + 2)(x - 4) = 0 \Rightarrow x = -2$ or 4

$f''(x) = 2x - 2$

$f''(-2) = 2(-2) - 2 = -6$ $f''(x) < 0 \Rightarrow$ at $x = -2$, $f(x)$ has a maximum.

$f''(4) = 2(4) - 2 = 6$ $f''(x) > 0 \Rightarrow$ at $x = 4$, $f(x)$ has a minimum.

$\left(-2, 10\frac{1}{3}\right)$ is a maximum, $\left(4, -25\frac{2}{3}\right)$ is a minimum.

Note

Make sure that you have answered the question. For a tangent or a normal the final answer must be the equation of a straight line.

Assessment tip

When using a calculator, having drawn the graph, it will also draw in the tangent line at a particular point and state its equation. Hence if Example 5.1.16 was on Paper 2 you could just write down the answer for the tangent. Some calculators will also directly give the equation of the normal.

Note

A common mistake is to think that $f'(c) = 0$ and $f''(c) = 0$ means that there must be a point of inflexion at $x = c$.

Note

A point of inflexion implies that $f''(c) = 0$ but again it is a one way implication. Just because $f''(c) = 0$ does not mean that there must be a point of inflexion at $x = c$.
We must check that $f''(x)$ actually changes in sign (the concavity changes when passing through the point where $x = c$).

Using a sign table for the second derivative is a good way of checking this.

(b) $f''(x) = 2x - 2 = 0 \Rightarrow x = 1$

x	$x<1$	$x=1$	$x>1$
$f''(x)$	–ve	0	+ve

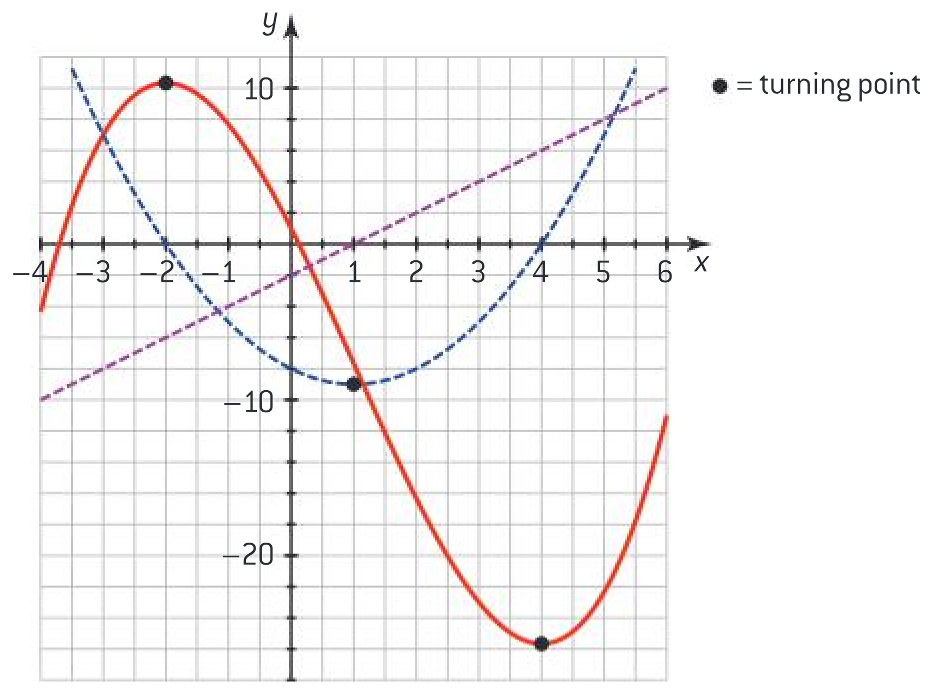
$\left(1,\ -7\frac{2}{3}\right)$ is an oblique point of inflexion.

Note how the different pieces of information each agree with each other and the sign of the second derivative confirms that it is a maximum followed by a minimum.

Note

In the graph below, the curve of $f(x)$ is solid red, the first derivative of $f(x)$ is dashed blue and the second derivative of $f(x)$ is dashed purple.

Drawing the graphs of $f(x)$, $f'(x)$ and $f''(x)$ on the same set of axes demonstrates the connections between these functions. Note how the turning points of $f(x)$ are vertically in line with $f'(x) = 0$ and a point of inflexion is vertically in line with $f''(x) = 0$ and a turning point of $f'(x)$.



Example 5.1.18

Link to Axis of symmetry of quadratics in Chapter 2

Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Use the second derivative test to investigate conditions for the quadratic to be concave up or concave down. Find the coordinates of the turning point in terms of a, b and c.

Solution

$f'(x) = 2ax + b$, $f''(x) = 2a$

The quadratic will be concave up if $a > 0$ and concave down if $a < 0$

The turning point occurs where $x = \frac{-b}{2a}$, so the coordinates of the turning point are: $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

Optimization problems

Being able to find maximums and minimums allows us to solve optimization problems.

Example 5.1.19

A right circular cone has radius r cm and height h cm. The variables r and h are connected by the equation $r + h = 6$ cm. Let the volume of the cone be V. Find the maximum value of V and the value of r that gives this maximum.

Solution

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(6 - r) = 2\pi r^2 - \frac{1}{3}\pi r^3$$

$$\frac{dV}{dr} = \pi(4r - r^2) = 0 \Rightarrow r = 4$$

$r = 0$ is rejected as it would be a minimum and does not fit the context.

The derivative goes from positive to negative through 0, so at $r = 4$ there is a maximum.

(An alternative would be to use the second derivative test.)

So $h = 2$ and the maximum volume is $\frac{32\pi}{3} = 33.5\,\text{cm}^3$ (3 sf) achieved when $r = 4$ cm.

Assessment tip

With optimization problems involving two variables, eliminate one of the variables using the information given before starting to differentiate. Make sure that your answer makes sense with the context (reject answers that do not).

Link to AHL 5.14 Example 5.4.6

SAMPLE STUDENT ANSWER

This is the same question from Example 5.1.19.

A right circular cone has radius r cm and height h cm. The variables r and h are connected by the equation $r + h = 6$ cm. Let the volume of the cone be V. Find the maximum value of V and the value of r that gives this maximum.

$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dr} = \frac{2}{3}\pi hr = 0 \Rightarrow r = 0$ so max of V is 0.

▲ The student had the correct formula and knew that they should differentiate.

This answer to the previous question (from Example 5.1.19) could have achieved 1/6 marks.

▼ The student treated h as though it was a constant, leading to an answer that did not make sense in the context of the question.

5.2 INTEGRATION

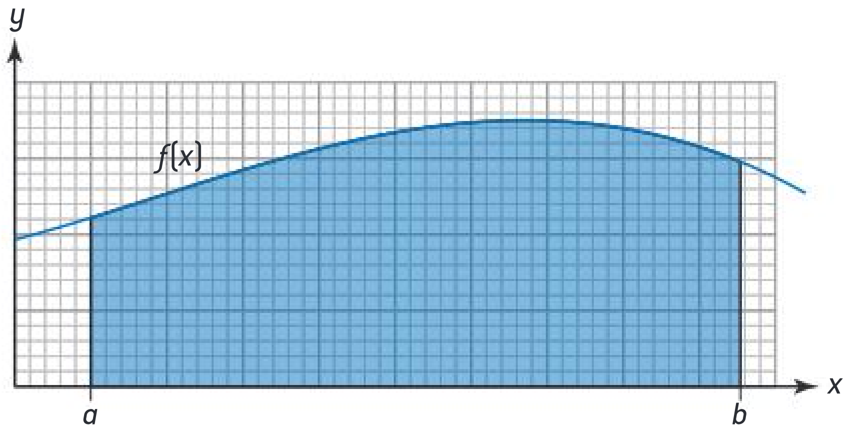
You must know:

- ✔ that integration is the reverse process of differentiation
- ✔ that areas can be found using integration
- ✔ the integrals of basic functions.

You should be able to:

- ✔ find indefinite and definite integrals
- ✔ find areas under curves
- ✔ find areas between curves
- ✔ recognize "lookout" integrals.

The Fundamental Theorem of Calculus tells us that integration, which can be used to find areas, is the reverse process of differentiation. There are two types of integral: indefinite integrals denoted by $\int f(x)\,dx$, where the answer is a function of x, $F(x)$, plus an arbitrary constant c; and definite integrals denoted by $\int_a^b f(x)\,dx$, where the answer is a number representing the area between the graph of $f(x)$, the x-axis and two vertical lines at $x = a$ and $x = b$, as shown in the diagram below (allowing for the fact that area below the x-axis is treated as negative).



Just as with differentiation, we can integrate term by term when terms are added together and also multiplicative constants will stay in front. With integration, we can put down what we think the answer is, visualize it being differentiated and then multiply by the necessary constant to make the answer correct.

The equivalent "power rule" for integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \in \mathbb{R},\ n \neq -1$$

Assessment tip

Although a calculator will give you the answer to a definite integral, you need to be able to do it yourself if it is on a paper where technology is not allowed or if the question asks for the *exact* answer.

Assessment tip

With indefinite integrals don't forget the $+c$, as you could be penalized. There might be conditions given, e.g., in kinematics, that allow you to calculate it.

Link to Kinematics SL 5.9

Example 5.2.1

Find $\int x^5 + 3x^4 - 2x^2\,dx$

Solution

$$\frac{x^6}{6} + \frac{3}{5}x^5 - \frac{2}{3}x^3 + c$$

Example 5.2.2

The function $f(x)$ passes through the point $(1, 9)$.
The derivative of $f(x)$ is $f'(x) = 2x + 3$. Find $f(x)$.

Solution

$f(x) = \int 2x + 3\,dx = x^2 + 3x + c$

$f(1) = 9 \Rightarrow 1 + 3 + c = 9 \Rightarrow c = 5$

$\Rightarrow f(x) = x^2 + 3x + 5$

Example 5.2.3

Find $\int \sqrt{x} + \frac{3}{x^3} \mathrm{d}x$

Solution

$$\int \sqrt{x} + \frac{3}{x^3} \mathrm{d}x = \int x^{\frac{1}{2}} + 3x^{-3} \mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} - \frac{3}{2}x^{-2} + c$$

Example 5.2.4

Find the value of $\int_0^1 x^3 + x^2 + 1 \, \mathrm{d}x$

Solution

Paper 1: $\left[\frac{x^4}{4} + \frac{x^3}{3} + x\right]_0^1 = \left(\frac{1}{4} + \frac{1}{3} + 1\right) - (0) = \frac{19}{12}$

Paper 2: 1.58 (3 sf)

Note

Write surds as powers of x and use negative powers if necessary to allow you to use the power rule.

Note

The notation $[F(x)]_a^b$ is used as a shorthand for $F(b) - F(a)$, the difference between the function evaluated at its upper limit and lower limit.

Other basic indefinite integrals

$\int \sin x \, \mathrm{d}x = -\cos x + c$ $\quad$ $\int \cos x \, \mathrm{d}x = \sin x + c$

$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$ $\quad$ $\int \frac{1}{x} \mathrm{d}x = \ln|x| + c$

Assessment tip

Although these integrals are given in the formula booklet, they should be remembered.

Note that $\int \frac{1}{x} \mathrm{d}x = \int x^{-1} \mathrm{d}x$ deals with the case when $n = -1$, and the power rule cannot be used for integration in this case.

Example 5.2.5

Find $\int 3\sin x - 4\cos x + 2\mathrm{e}^x + x^2 + \frac{5}{x} \mathrm{d}x$ for $x > 0$

Solution

$$-3\cos x - 4\sin x + 2\mathrm{e}^x + \frac{x^3}{3} + 5\ln x + c$$

Assessment tip

The standard integrals of trig functions are valid only when working in radians, so keep your calculator in radian mode when doing calculus.

Example 5.2.6

Find $\int \frac{x + x^2}{x^3} \mathrm{d}x$ for $x > 0$

Solution

$$\int \frac{x + x^2}{x^3} \mathrm{d}x = \int \frac{1}{x^2} + \frac{1}{x} \mathrm{d}x = \int x^{-2} + \frac{1}{x} \mathrm{d}x = -x^{-1} + \ln x + c$$

Assessment tip

Sometimes applying algebra to the function to be integrated first allows it to be in a form that is easier to integrate.

Note

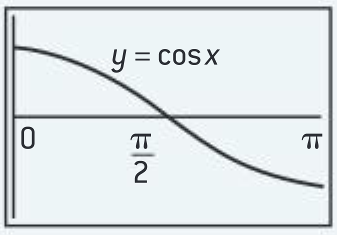
Integration treats area below the x-axis as negative. If this is not required then $\int_a^b |f(x)| \, \mathrm{d}x$ should be taken or the area split into portions above the axis and portions below, with the portions below then converted into positive quantities.

Link to Kinematics SL 5.9

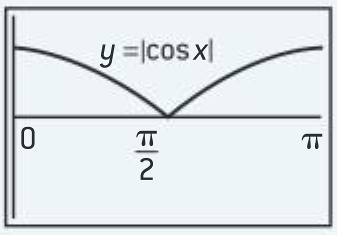
Assessment tip

Sketches as shown below assist with knowing what is happening.

(a)



(b)



Example 5.2.7

Find (a) $\int_0^{\pi} \cos x \, dx$ (b) $\int_0^{\pi} |\cos x| \, dx$

Solution

(a) $[\sin x]_0^{\pi} = (0) - (0) = 0$

(b) $\int_0^{\pi} |\cos x| \, dx = 2$ using a calculator, or by writing as

$$\int_0^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$$

$$= (1) - (0) - ((0) - (1)) = 2$$

Note

When integrating, the inside function is $ax + b$. We can think of the $\frac{1}{a}$ as putting an a in the denominator, ready to cancel the a that we know will appear when differentiating.

We know that differentiating $f(ax + b)$ gives $af'(ax + b)$ by using the chain rule and remembering to multiply by the derivative of the inside function. When integrating, we need to take this derivative of the inside function into account. Hence if $\int f(x)dx = F(x) + c$ then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+c$, where a and b are real constants, $a \neq 0$.

Example 5.2.8

Find $\int (2x + 1)^7 + 5 \cos (3x - 1) + e^{(4x-2)} \, dx$

Solution

$$\frac{1}{2} \times \frac{1}{8}(2x+1)^8 + \frac{1}{3} 5 \sin (3x-1) + \frac{1}{4}e^{(4x-2)} + c$$

$$= \frac{1}{16}(2x+1)^8 + \frac{5}{3} \sin (3x-1) + \frac{1}{4}e^{(4x-2)} + c$$

Note

This method works only for constants, not functions of x.

When taking the derivative of the inside function into account, be careful when there are implied brackets.

Forgetting to allow for the constant derivative of the inside function is a very common mistake. You should do a large number of examples until this becomes second nature.

SAMPLE STUDENT ANSWER

Find $\int \frac{1}{(2x+3)^2}\,\mathrm{d}x$

$-\frac{1}{2}(2x+3)^{-1}+c$

▲ The student had the correct application of the integration power rule. They allowed for derivative of the inside function. They remembered the constant of integration.

The student answer above could have achieved 3/3 marks.

$\ln(2x+3)^2$

▼ The student incorrectly thought that because the expression was in the denominator the integral would be a logarithm. They did not allow for derivative of the inside function. They did not remember the constant of integration.

The different student answer above could have achieved 0/3 marks.

Integration by inspection (the "lookout" method)

Since the derivative of $(x^3+5)^7$ is $21x^2(x^3+5)^6$, the integral of $21x^2(x^3+5)^6$ must be $(x^3+5)^7$.

When integrating, we notice that we have the derivative of the inside function conveniently sitting in front for us. In general, we are "looking out" for derivatives of the form $\int kg'(x)f(g(x))\mathrm{d}x$. We can easily deal with the constant k.

Example 5.2.9

Find the following integrals:

(a) $\int 2x(x^2+7)^4\,\mathrm{d}x$

(b) $\int \mathrm{e}^x\cos(\mathrm{e}^x+4)\,\mathrm{d}x$

(c) $\int \cos x(4\sin x+3)^4\,\mathrm{d}x$

Solution

(a) $\frac{1}{5}(x^2+7)^5+c$

(b) $\sin(\mathrm{e}^x+4)+c$

(c) $\frac{1}{20}(4\sin x+3)^5+c$

Note

Note that in each case the derivative of the function in brackets is contained within the integral, except for a constant, which is easy to provide for.

Example 5.2.10

Find the integral $\int 2x(x^2+7)^4\,\mathrm{d}x$ from Example 5.2.9, but this time using substitution.

Solution

Let $x^2+7=u$ $\qquad \frac{\mathrm{d}u}{\mathrm{d}x}=2x \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{2x}$

The integral becomes $\int 2xu^4\frac{1}{2x}du = \int u^4\mathrm{d}u = \frac{u^5}{5}+c = \frac{(x^2+7)^5}{5}+c$

Note

Integrals done by the lookout method can also be done using integration by substitution, using the formula

$$\int f(x)\mathrm{d}x = \int f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$$

upon applying a substitution $x=x(u)$.

Link to AHL 5.16 Integration by substitution

Assessment tip

The lookout method is often quicker and easier than integration by substitution. The answer can almost be written straight down and valuable time saved. Lookout method integrals are very common on IB papers and good students should have become adept at spotting them.

Finding the area enclosed between two curves

If the graphs of two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, with the functions being continuous in the interval $[a, b]$ and $f(x) \geq g(x)$ throughout this interval, then the area enclosed by the graphs of the functions is given by $\int_a^b (f(x) - g(x))\,\mathrm{d}x$.

Note

The x values of the points of intersection can be found either by equating the equations or using a calculator, if allowed.

Example 5.2.11

Find the area enclosed between the curves $y = x^2 - 8x + 6$ and $y = -x^2 + 4x - 10$.

Solution

Since the first quadratic is concave up and the second quadratic is concave down, the second quadratic is the upper curve.

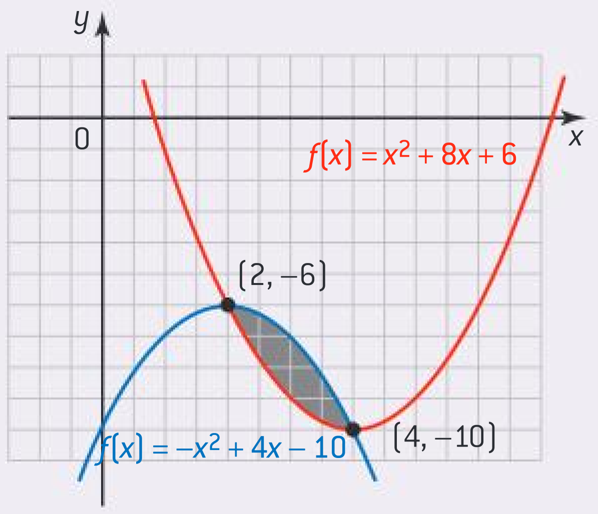
$x^2 - 8x + 6 = -x^2 + 4x - 10 \Rightarrow 2x^2 - 12x + 16 = 0$
$\Rightarrow 2(x - 2)(x - 4) = 0 \Rightarrow x = 2$ or $x = 4$

$$\int_2^4 (-x^2 + 4x - 10) - (x^2 - 8x + 6)\,\mathrm{d}x = \int_2^4 (-2x^2 + 12x - 16)\,\mathrm{d}x$$

$$= \left[\frac{-2x^3}{3} + 6x^2 - 16x\right]_2^4$$

$$= \left(\frac{-128}{3} + 96 - 64\right) - \left(\frac{-16}{3} + 24 - 32\right) = \frac{8}{3}$$

If using a calculator, first find the points of intersection:



Then the integral can be evaluated directly as 2.67 (3 sf)

Note

Even if not asked to, showing a quick sketch of the two functions can be useful whether integrating by hand or using a calculator. The enclosed area is the area that is trapped between the two curves with a finite boundary.

5.3 KINEMATICS

You must know:

✔ the meaning of the terms ‘displacement’, ‘velocity’ and ‘acceleration’.

You should be able to:

✔ find displacement, velocity and acceleration using differentiation and integration

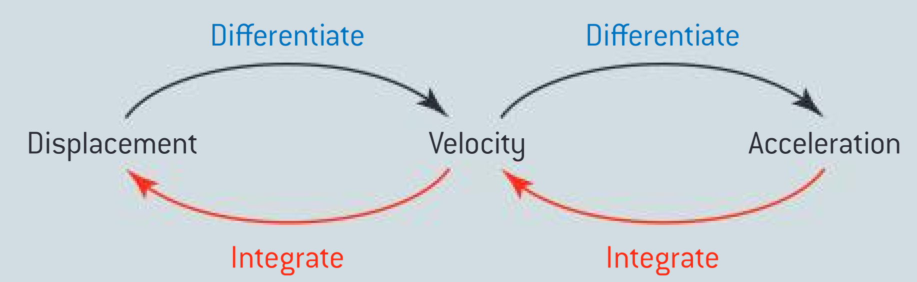
✔ answer such questions in context.

For an object moving in a straight line, its displacement (s) is measured from a fixed origin. Displacement can be either positive or negative. The magnitude of displacement is distance and hence distance is always positive.

The derivative of displacement with respect to time (t) is velocity (v). This again can be positive or negative. The magnitude of velocity is speed and hence speed is always positive.

The derivative of velocity with respect to time is acceleration (a) and again this can be positive or negative.

Moving between displacement, velocity and acceleration



Link to Differentiation (section 5.1) and Integration (section 5.2)

Example 5.3.1

The displacement of an object s, in metres, moving in a straight line relative to a fixed origin is given by $s(t) = 200 + 8t - 5t^2$, where t is in seconds.

(a) Write down the initial displacement.

(b) (i) Find an expression for the velocity at time t, and hence
(ii) write down the initial velocity.

(c) Find the acceleration of the object at time t.

Solution

(a) $s(0) = 200\,\text{m}$

(b) (i) $s'(t) = v(t) = 8 - 10t$ (ii) $v(0) = 8\,\text{m}\,\text{s}^{-1}$

(c) $s''(t) = v'(t) = a(t) = -10\,\text{m}\,\text{s}^{-2}$

Assessment tip

Notice the difference in the command terms. “Write down” means little or no calculation is required and working does not need to be shown. “Find” means to obtain an answer and show the relevant stages in your working.

Note

Example 5.3.1 could have been placed in the context of a stone being thrown vertically upwards from the top of a cliff.

Example 5.3.2

The acceleration of an object moving in a straight line relative to a fixed origin is given by $a(t) = 3\sin t \text{ m s}^{-2}$. Its initial velocity is 2 m s^{-1} and its initial displacement is 1m.

(a) Find an expression for its velocity in terms of t.

(b) Find an expression for its displacement in terms of t.

Solution

(a) $v(t) = \int a(t)\, dt = -3\cos t + c$ $\quad t = 0, v = 2 \Rightarrow 2 = -3 + c \Rightarrow c = 5$

$v = -3\cos t + 5$

(b) $s(t) = \int v(t)\, dt = -3\sin t + 5t + d$ $\quad t = 0, s = 1 \Rightarrow d = 1$

$s = -3\sin t + 5t + 1$

Note

Note how vital it is to remember the constants of integration when solving this problem. It is also good practice to give them different letters.

There is a distinction to be made between the displacement of an object from time t_1 to time t_2 and the total distance that the object has travelled in this time. This is because the object could change from moving towards the origin to moving away (or vice versa) as velocity could be positive or negative.

Displacement is given by $\int_{t_1}^{t_2} v(t)\, dt$ whereas distance is given by $\int_{t_1}^{t_2} |v(t)|\, dt$

Assessment tip

Formulae for both displacement and distance are given in the formula booklet. A common mistake is to forget the modulus when asked for the total distance travelled. The answer cannot be negative (unlike the displacement) and it is best to use a calculator for the total distance travelled.

Example 5.3.3

The velocity of an object moving in a straight line relative to a fixed origin is given by $v = \sin 2t \text{ m s}^{-1}$

(a) Find the displacement of the object after the first π seconds.

(b) Find the total distance travelled by the object in the first π seconds.

Solution

(a) $\int_0^{\pi} \sin 2t\, dt = \left[-\frac{1}{2}\cos 2t\right]_0^{\pi} = \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 0$

(b) $\int_0^{\pi} |\sin 2t|\, dt = 2 \text{ m}$

SAMPLE STUDENT ANSWER

The acceleration of an object moving in a straight line is given by $a = -\frac{4}{t+1} \text{ m s}^{-2}$. The initial velocity is 2 m s^{-1}

(a) Find an expression for the velocity at time t.

(b) Hence find the total distance travelled in the first 5 seconds.

(a) $v = -4\ln(t+1) + c$

$t = 0, v = 2 \Rightarrow c = 2$

$v = -4\ln(t+1) + 2$

(b) -13.0 m

▲ Part (a) was integrated correctly and the constant of integration found.

▼ In part (b) the modulus was missed out when using the formula. Answer should have been

$\int_0^5 |2 - 4\ln(t+1)|\, dt = 14.2$ m (3 sf)

The student should have recognized that a mistake had been made due to the negative answer.

The answer above could have achieved 4/6 marks.

5.4 ADDITIONAL DIFFERENTIATION AND INTEGRATION (HL)

You must know:

- ✔ derivatives of trig, inverse trig, exponential and log functions
- ✔ basic Maclaurin expansions.

You should be able to:

- ✔ differentiate from first principles
- ✔ differentiate implicitly
- ✔ obtain related rates of change
- ✔ solve optimization problems
- ✔ recognize integrals of derivatives
- ✔ integrate with partial fractions
- ✔ integrate by substitution
- ✔ integrate by parts
- ✔ find volumes of solids of revolution
- ✔ solve variable separable, homogeneous and first order linear differential equations
- ✔ employ Euler's method for differential equations
- ✔ obtain Maclaurin series
- ✔ evaluate limits using L'Hôpital's rule or Maclaurin series.

Differentiation from first principles

The derivatives of basic functions $y=f(x)$ are obtained from first principles using the definition

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 5.4.1

From first principles, find the derivative of $f(x)=3x^2+x+3$

Solution

Require $\lim_{h\to 0}\dfrac{\left(3(x+h)^2+(x+h)+3\right)-(3x^2+x+3)}{h}$

$$=\lim_{h\to 0}\frac{6xh+3h^2+h}{h}=\lim_{h\to 0}\frac{h(6x+3h+1)}{h}=\lim_{h\to 0}(6x+1+3h)$$

$$=6x+1 \text{ (as expected)}$$

Example 5.4.2

From first principles, find the derivative of $f(x)=\dfrac{1}{x}$

Solution

Require $\lim_{h\to 0}\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\dfrac{\frac{x-(x+h)}{(x+h)x}}{h}=\lim_{h\to 0}\dfrac{-1}{(x+h)x}$

$$=-\frac{1}{x^2} \text{ (as expected)}$$

Assessment tip

Do not use differentiation by first principles unless the exam question specifically asks you to.

Note

Do not let $h\to 0$ too soon as you may find you're trying to divide by zero. Instead, do the algebra (being careful about brackets), do not worry if the expressing becomes longer, then simplify, finally letting $h\to 0$.

Implicit differentiation

Implicit expressions can be differentiated exactly as they appear (using the usual rules) rather than requiring y to be an explicit function of x.

Example 5.4.3

Given that $x^2y + y^2x = 4$, find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

Differentiating both sides implicitly w.r.t. x gives:

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx}x + y^2 = 0 \qquad \frac{dy}{dx} = \frac{-(2xy + y^2)}{x^2 + 2xy}$$

This technique can be used to find the derivatives of inverse functions.

Example 5.4.4

Find the derivative of $y = \arctan x$

Solution

$y = \arctan x \Rightarrow \tan y = x$. Differentiating implicitly:

$$\sec^2 y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

Related rates of change

If two variables that are connected by an equation are both functions of time, then that equation can be differentiated implicitly with respect to time to find a connection between the two rates of change of the variables.

Example 5.4.5

The area A engulfed by a forest fire remains in a circular shape of radius r. The area is constantly increasing at a rate of $1000\,\mathrm{m^2\,h^{-1}}$. Find the rate at which the radius is increasing at the moment in time when $r = 50\,\mathrm{m}$.

Solution

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$1000 = 2 \times 50\pi\frac{dr}{dt}$$

$$\frac{dr}{dt} = 3.18\,\mathrm{m\,h^{-1}}\ (3\text{ sf})$$

Optimization problems

Differentiation can be used to find maximum and minimum points which gives the optimum solution to a problem.

Link to SL 5.8 Example 5.1.19

Example 5.4.6

A straight road runs from west to east for 40 kilometres from point A to point B. Helen, a triathlete, is in a field at point C, 5 kilometres directly south of point A. Helen runs at a constant speed of $15\,\text{km}\,\text{h}^{-1}$ and cycles at a constant speed of $30\,\text{km}\,\text{h}^{-1}$. Helen will run in a straight line across the field to point P between A and B on the road where she will leap onto a bike and then cycle on the road to point B. Let the distance from A to P be x km. Let the travel time from C to B be T hours.

(a) Show that $T = \dfrac{\sqrt{25+x^2}}{15} + \dfrac{40-x}{30}$

(b) Find $\dfrac{\mathrm{d}T}{\mathrm{d}x}$ and hence find the exact value of x to minimize T.

Solution

(a) Running distance is $\sqrt{25+x^2}$, so running time is $\dfrac{\sqrt{25+x^2}}{15}$

Cycling distance is $40 - x$, so cycling time is $\dfrac{40-x}{30}$

Total time = running time + cycling time

Hence $T = \dfrac{\sqrt{25+x^2}}{15} + \dfrac{40-x}{30}$

(b) $\dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{x}{15\sqrt{25+x^2}} - \dfrac{1}{30} = 0 \Rightarrow 2x = \sqrt{25+x^2} \Rightarrow 4x^2 = 25 + x^2$

$\Rightarrow 3x^2 = 25 \Rightarrow x = \dfrac{5}{\sqrt{3}}$ km

The negative solution is discarded as inappropriate to the context and a check on the signs of the derivative show that the solution is indeed a minimum.

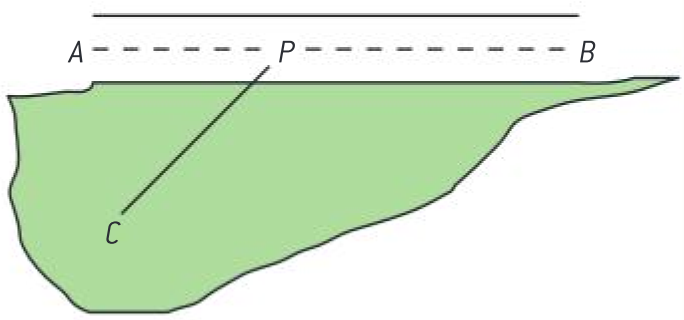
Assessment tip

Note the use of "Show that" in part (a) with the answer given. This allows you to do part (b) even if you could not do part (a). Never give up on a question even if you cannot do the first part or parts.

The use of "exact" in part (b) is a good indication that the question is designed to be done without the need for a calculator.

Note

Draw yourself a sketch to get an idea of the relative positions of the various points on Helen's triathalon.



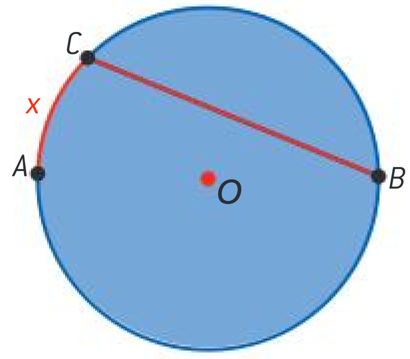
SAMPLE STUDENT ANSWER

Rich is standing at point A at the edge of a circular lake with radius 1 km and centre O. He wants to get to point B, directly on the opposite side of the lake. He can swim at $4\,\text{km}\,\text{h}^{-1}$ and run at $8\,\text{km}\,\text{h}^{-1}$. He will run a distance x, where $0 \le x \le \pi$, around the edge of the lake to point C and then dive in and swim in a straight line across to point B. See the diagram to the left.

(a) Show that the total time T to run from A to C and then swim from C to B is given by:

$T = \dfrac{x}{8} + \dfrac{1}{2}\sin\left(\dfrac{\pi - x}{2}\right)$

(b) Find the value of x that minimizes the total time T.



▲ This is a correct, well-explained answer to part (a). There was good differentiation in part (b) and the stationary point was found.

▼ A sign table was not used, which would have identified the stationary point as a maximum. Hence, the optimum solution is an end point. Letting $x = 0$ gives $T = \frac{1}{2}$ or 30 minutes. Letting $x = \pi$ gives $T = \frac{\pi}{8}$ or 23.6 minutes, so this is the best solution: running all the way around.

(a) Running distance is x, so running time is $\frac{x}{8}$. Since radius is 1, $A\hat{O}C = x$, $C\hat{O}B = \pi - x$. Triangle OCB is isosceles, cutting it in half gives swimming distance $CB = 2\sin\left(\frac{\pi - x}{2}\right)$.

Swimming time is $\frac{1}{2}\sin\left(\frac{\pi - x}{2}\right)$. So total time is

$T = \frac{x}{8} + \frac{1}{2}\sin\left(\frac{\pi - x}{2}\right)$.

(b) $\frac{dT}{dx} = \frac{1}{8} - \frac{1}{4}\cos\left(\frac{\pi - x}{2}\right) = 0 \Rightarrow \cos\left(\frac{\pi - x}{2}\right) = \frac{1}{2}$

$\frac{\pi - x}{2} = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}$

The answer above could have achieved 8/11 marks.

Other derivatives

Derivatives of other trig, inverse trig, exponential and logarithmic functions

$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$

$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$

$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$

$f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$

$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x\ln a}$

$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1 - x^2}}$

$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1 + x^2}$

$f(x) = \arccos x \Rightarrow f'(x) = \frac{-1}{\sqrt{1 - x^2}}$

Assessment tip

Although these formula are given in the formula book, it is worth remembering them. This ability to recognize derivatives is important when using the Inspection (Lookout) method in integration.

Note that trig and inverse trig functions with a "co" in them all differentiate to an expression starting with a negative sign. These are similarities that assist the learning process.

Composite functions of these functions with a linear function can be differentiated using the chain rule. Similarly, the above formula incorporating linear functions can be used for integration, remembering always to take the derivative of the inside function into account as well.

Link to Composite functions SL 2.5

Example 5.4.7

Find the derivatives of the following functions:

(a) $y = \operatorname{cosec}(3x + 5)$ (b) $y = 3^{4x}$ (c) $y = \arctan(2x + 1)$

Solution

(a) $\frac{dy}{dx} = -3\operatorname{cosec}(3x + 5)\cot(3x + 5)$

(b) $\frac{dy}{dx} = 4(\ln 3)3^{4x}$

(c) $\frac{dy}{dx} = \frac{2}{1 + (2x + 1)^2}$

Example 5.4.8

Find the following indefinite integrals:

(a) $\int \sec^2(5x)\,dx$ (b) $\int \frac{1}{\sqrt{-4x^2+4x}}\,dx$ (c) $\int a^x\,dx$

Solution

(a) $\frac{1}{5}\tan(5x)+c$

(b) $\int \frac{1}{\sqrt{-4x^2+4x}}\,dx = \int \frac{1}{\sqrt{1-(2x-1)^2}}\,dx = \frac{1}{2}\arcsin(2x-1)+c$

(c) $\frac{1}{\ln a}a^x + c$

Assessment tip

Integral (c) is given in the formula booklet. Integral (b) is an example of how it can be beneficial to do some algebra first so that the expression is then in a format where its integral can be found.

Partial fractions

Taking a rational expression and decomposing it into simpler rational expressions is called partial fraction decomposition. This gives us yet another method for integrating expressions.

Link to Partial fractions AHL 1.11

Example 5.4.9

Express $\frac{1}{x^2+4x+3}$ in partial fractions and hence find $\int \frac{1}{x^2+4x+3}\,dx$ giving your answer in the form of a single logarithm.

Solution

$$\frac{1}{x^2+4x+3} \equiv \frac{1}{(x+1)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+3)} \Rightarrow 1 \equiv A(x+3)+B(x+1)$$

$$\Rightarrow A=\frac{1}{2},\ B=-\frac{1}{2} \qquad \frac{1}{x^2+4x+3} \equiv \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x+3}$$

$$\int \frac{1}{x^2+4x+3}\,dx \int \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x+3}\,dx = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + c$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + \ln k = \ln k\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}}$$

Assessment tip

At the end of each question always check that you have answered precisely what the question asked and that your answer has been put in the required format.

Here, use of the laws of logs were required to change the answer into the required format.

Link to Laws of logs AHL 1.9

Integration by substitution

This method changes an integral that is proving difficult into an easier one by making a substitution of variables.

The formula for integration by substitution is $\int f(x)\mathrm{d}x = \int f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ upon applying a substitution $x = x(u)$.

Link to SL 5.11

Example 5.4.10

Find $\int 2x\sqrt{3x+1}\,dx$ using the substitution $u = 3x+1$

Solution

$$x = \frac{u-1}{3},\ \frac{dx}{du} = \frac{1}{3}$$

The integral becomes:

$$\int \frac{2(u-1)}{9}u^{\frac{1}{2}}\,du = \int \frac{2}{9}u^{\frac{3}{2}} - \frac{2}{9}u^{\frac{1}{2}}\,du = \frac{4}{45}u^{\frac{5}{2}} - \frac{4}{27}u^{\frac{3}{2}} + c$$

$$= \frac{4}{45}(3x+1)^{\frac{5}{2}} - \frac{4}{27}(3x+1)^{\frac{3}{2}} + c$$

Assessment tip

The formula for integration by substitution is not given in the formula booklet so it is important that you learn it. It is important not to forget the $\frac{dx}{du}$. It can be remembered as it looks the same if you imagine the du's being canceled, but you cannot actually do this mathematically.

Assessment tip

On exam papers, the substitution required will be provided, unless the integral is obvious as an inspection (lookout) type. Use the substitution provided and expect the new integral to be easier.

Note

The standard integrals $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$ and $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c, |x| < a$, which are given in the formula booklet, can be proved using the substitutions $x = a\tan u$ and $x = a\sin u$, respectively.

SAMPLE STUDENT ANSWER

Use the substitution $u = 4x - 1$ to find $\int \frac{4x}{\sqrt{4x-1}}\,dx$

$$\int \frac{4x}{\sqrt{4x-1}}\,dx = \int \frac{u+1}{u^{\frac{1}{2}}}\,du = \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$$

$$= \frac{2}{3}(4x-1)^{\frac{3}{2}} + 2(4x-1)^{\frac{1}{2}} + c$$

▲ The student used the given substitution. There was good use of the power rule for integration.

▼ The student forgot to put in $\frac{dx}{du}$ which is $\frac{1}{4}$, so answer given is 4 times the correct answer.

The answer above could have achieved 3/5 marks.

If a definite integral is done by substitution (without a calculator) then the limits can be changed from the x limits to u limits, which saves going back to the x's at the end.

Assessment tip

When changing the limits, use x limits when there is a dx at the end of the integral and u limits when there is a du at the end of the integral. The use of a table to show the conversion of the limits is a good idea.

Example 5.4.11

Use the substitution $x = \cos 2u$ to find the exact value of $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Solution

x:	0	→	1
u:	$\frac{\pi}{4}$	→	0

$x = \cos 2u \Rightarrow \frac{du}{dx} = -2\sin 2u$

Integral is $\int_{\frac{\pi}{4}}^{0} -\sqrt{\frac{1-\cos 2u}{1+\cos 2u}}\,2\sin 2u\,du = \int_{\frac{\pi}{4}}^{0} -\sqrt{\frac{2\sin^2 u}{2\cos^2 u}}\,4\sin u\cos u\,du$

$$= \int_{\frac{\pi}{4}}^{0} -4\sin^2 u\,du = \int_{\frac{\pi}{4}}^{0} -2 + 2\cos 2u\,du$$

$$= [-2u + \sin 2u]_{\frac{\pi}{4}}^{0}$$

$$= (0) - \left(-\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1$$

Note

Example 5.4.11 shows the importance of knowing the double angle formulae and being able to move in both directions with them.

Link to SL 3.6 Double angle identities

Integration by parts

This method exchanges an integral that is proving difficult to do for the difference of an expression minus a new integral, which is hopefully easier.

The integration by parts formula

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Example 5.4.12

Find $\int (2x+1)\, e^{3x}\, dx$

Solution

$u = 2x + 1 \qquad v = \frac{e^{3x}}{3}$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = e^{3x}$

$$\int (2x+1)\, e^{3x}\, dx = (2x-1)\frac{e^{3x}}{3} - \int \frac{2}{3} e^{3x}\, dx = (2x+1)\frac{e^{3x}}{3} - \frac{2}{9} e^{3x} + c$$

Assessment tip

When doing integration by parts, there are two possibilities for the function that gets differentiated. Generally, one choice will lead to an easier integral and the other to a harder one. To decide which one to differentiate, use the mnemonic ILATE. This stands for Inverse, Log, Algebraic, Trig, Exponential. The expression that comes first in the list is the one that is differentiated. In Example 5.4.12, the algebraic expression $2x+1$ is differentiated rather than the exponential e^{3x}.

Always use a table as shown in the Note box, and start by filling in u and $\frac{dv}{dx}$. Then differentiate and integrate to find $\frac{du}{dx}$ and v, respectively, before substituting into the formula.

Note

When integrating by parts, it is helpful to construct a table for each component, as shown.

$u =$	$v =$
$\frac{du}{dx} =$	$\frac{dv}{dx} =$

SAMPLE STUDENT ANSWER

Find $\int 5x \sin x\, dx$

$$\int 5x \sin x\, dx = \frac{5x^2}{2}\sin x - \int \frac{5x^2}{2}\cos x\, dx$$

$$= \frac{5x^2}{2}\sin x - \frac{5x^3}{6}\sin x + c$$

▲ There was an attempt at integration by parts.

▼ Although the formula for integration by parts has been used and the first line is true, it is not at all helpful. The new integral is harder than the original. On the last line, out of desperation, the student has invented an integration rule that is not true. The integral of a product is certainly not the product of the integrals.

The answer above could have achieved 1/5 marks.

Correct answer should be:

Using ILATE, $5x$ is algebraic and $\sin x$ is trig.

$u = 5x \qquad v = -\cos x$

$\frac{du}{dx} = 5 \qquad \frac{dv}{dx} = \sin x$

$$\int 5x \sin x\, dx = -5x\cos x + \int 5\cos x\, dx = -5x\cos x + 5\sin x + c$$

It is possible that integration by parts might have to be used more than once (the IB are not likely to ask for more than two applications) with each application making the new integral easier.

Example 5.4.13

Find $\int (x^2+1)\cos x \,\mathrm{d}x$

Solution

Integrate by parts:

$u = x^2+1$	$v = \sin x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$$\int (x^2+1)\cos x \,\mathrm{d}x = (x^2+1)\sin x - \int 2x\sin x\,\mathrm{d}x$$

Integrating by parts again:

$u = 2x$	$v = -\cos x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$$\int 2x\sin x\,\mathrm{d}x = -2x\cos x + \int 2\cos x\,\mathrm{d}x = -2x\cos x + 2\sin x + c$$

$$\begin{aligned}\int (x^2+1)\cos x\,\mathrm{d}x &= (x^2+1)\sin x - (-2x\cos x + 2\sin x) + c \\ &= (x^2+1)\sin x + 2x\cos x - 2\sin x + c\end{aligned}$$

Note

Note how ILATE was applied twice and two tables written down.
Be careful when substituting one formula into the previous one, especially with minus signs.

Integration by parts can also be used for integrals that do not immediately look as though they are two expressions multiplied together, e.g., $\int \ln x\,\mathrm{d}x$ or $\int \arctan x\,\mathrm{d}x$. This is done, for example, by writing $\int \ln x\,\mathrm{d}x$ as $\int 1\times \ln x\,\mathrm{d}x$ and then considering ILATE.

Volumes of solids of revolution

If the graph of $y(x)$ from $x = a$ to $x = b$ is rotated through 2π radians about the x-axis then a solid of revolution is formed. The volume of this solid is given by $V = \int_a^b \pi y^2\,\mathrm{d}x$.

If the graph of $y(x)$ from $y = a$ to $y = b$ is rotated through 2π radians about the y-axis then the volume of this solid is given by $V = \int_a^b \pi x^2\,\mathrm{d}y$.

Example 5.4.14

The curve $y = \mathrm{e}^{x^2}$ from $x = 0$ to $x = 1$ is rotated through 2π radians about the x-axis. Find the volume of the solid of revolution that is generated.

Solution

$$V = \int_0^1 \pi \mathrm{e}^{2x^2}\,\mathrm{d}x = 7.43 \text{ (3 sf)}$$

Assessment tip

On a paper that allows the use of technology, go directly to the calculator for definite integrals. The integral in Example 5.4.14 could not be evaluated analytically anyway. Do not waste time when the calculator can do the work.

SAMPLE STUDENT ANSWER

By the rotation of a suitable function, find the volume of a right circular cone with height h and radius r.

Consider the straight line $y = \frac{h}{r}x$ from $y = 0$ to $y = h$ being rotated through 2π about the y-axis. This will generate the desired cone.

$$V = \int_0^h \pi \frac{r^2}{h^2} x^2\,dy = \left[\pi \frac{r^2}{h^2}\frac{x^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h$$

▲ The student had a good choice of function and limits to obtain the cone. There was excellent explanation of what they were doing (students are sometimes poor at explaining what they are doing). The integration and final answer were both correct.

The answer above could have achieved 6/6 marks.

First order differential equations

Solving a differential equation will involve integration to produce a connection between the variables that does not involve derivatives. There are three types of differential equations that this course investigates: variable separable, homogeneous and first order linear.

Variable separable

In this type of differential equation it is possible to get all the x's on one side of the equation and all the y's on the other side. Then two integrations solve the problem.

Example 5.4.15

Solve the differential equation $\frac{dy}{dx} = y^2e^{-x}$ given that $y(1) = 1$.

Give your answer in the form $y = y(x)$

Solution

$$\frac{dy}{dx} = y^2e^{-x} \Rightarrow \int y^{-2}\,dy = \int e^{-x}\,dx \Rightarrow -y^{-1} = -e^{-x} + c$$

$$y(1) = 1 \Rightarrow -1 = \frac{-1}{e} + c \Rightarrow c = -1 + \frac{1}{e} \Rightarrow \frac{1}{y} = e^{-x} + 1 - \frac{1}{e}$$

$$y = \frac{1}{e^{-x} + 1 - \frac{1}{e}}$$

Assessment tip

Remember the constant of integration. There will sometimes be conditions given that allow this to be found. Do not spend time after doing the integrals to make y the subject unless the question specifically asks you to do so.

Note

If the differential equation only involves one variable then you must be able to separate the variables. For example, $\frac{dP}{dt} = 7P$

SAMPLE STUDENT ANSWER

Solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$. Give the answer in the form $y = y(x)$.

$$\int y\,dy = \int x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c \Rightarrow y = x + k$$

▲ The student separated the variables and correctly integrated, including the constant of integration.

▼ The student was let down by poor algebraic manipulation on the final step. Answer should be $y = \sqrt{x^2 + k}$

The answer above could have achieved 4/5 marks.

Homogeneous

These can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. The substitution $v = \frac{y}{x}$ is always made, then $\frac{dy}{dx}$ is replaced with $v + x\frac{dv}{dx}$ and the equation will always be of the variable separable form, and will involve $\int \frac{1}{x}\,dx$.

Example 5.4.16

Solve $\frac{dy}{dx} = \frac{y^3 + x^3}{xy^2}$, for $x, y > 0$

Solution

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2$$

Let $\frac{y}{x} = v$, $v + x\frac{dv}{dx} = v + \frac{1}{v^2}$

$$\int v^2\,dv = \int \frac{1}{x}\,dx \Rightarrow \frac{v^3}{3} = \ln x + c,\ \frac{y^3}{3x^3} = \ln x + c \Rightarrow y^3 = 3x^3(\ln x + c)$$

Assessment tip

Remember to replace v when giving the final solution.

A test to see if $\frac{dy}{dx} = g(x, y)$ is homogeneous is to consider $g(tx, ty)$. If $g(tx, ty) \equiv g(x, y)$ due to the t's canceling, then the differential equation is homogeneous.

The substitution $v = \frac{y}{x}$ is not given in the formula booklet so should be remembered.

First order linear

First order linear differential equations are of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

These are solved by multiplying by the integrating factor $e^{\int P(x)dx}$

This turns the left-hand side into the **exact** derivative of a product.

Note

If the differential equation has only one y in it, suspect that it could be first order linear. Good understanding of the laws of logarithms is required to simplify the integrating factor. Check in your working that the left-hand side does indeed give the exact derivative of a product. Be careful with the final algebra involving c.

Example 5.4.17

Solve $\frac{dy}{dx} - \frac{5}{x}y = x^7$, giving your answer in the form $y = y(x)$

Solution

$$\text{I.F.} = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = e^{\ln x^{-5}} = x^{-5}$$

Multiplying by $\frac{1}{x^5}$ gives

$$\frac{1}{x^5}\frac{dy}{dx} - \frac{5}{x^6}y = x^2 \Rightarrow \frac{d\left(\frac{y}{x^5}\right)}{dx} = x^2$$

$$\Rightarrow \frac{y}{x^5} = \int x^2\,dx = \frac{x^3}{3} + c$$

$$y = \frac{x^8}{3} + cx^5$$

Euler's method

This is a numerical method for finding an approximation to the solution of a differential equation at a particular x value. The iterative formula, given in the formula booklet, which will be used on the differential equation $\frac{dy}{dx} = f(x,y)$ is $x_{n+1} = x_n + h$, $y_{n+1} = y_n + h \times f(x_n, y_n)$, where h is the step length. A calculator will be used for this method employing sequences and a table.

Assessment tip

Give the particular iteration formula that you are using. It is a good idea to work out the first new y value independently of the sequences in the table. This ensures that you gain marks for method and is a check that you have put the formulae into the calculator correctly. It is very easy to make a mistake when entering the iterative formula into the calculator, especially with the brackets, so do it with great care and attention to detail.

Example 5.4.18

Use Euler's method with a step length of 0.25 to find an approximation for $y(2)$, given that $y(x)$ satisfies the differential equation $\frac{dy}{dx} = x + y^2$ and $y(1) = 0$

Solution

Using $x_{n+1} = x_n + 0.25$, $y_{n+1} = y_n + 0.25\,(x_n + y_n^{\,2})$

$x_0 = 1$, $y_0 = 0$ so $x_1 = 1.25$, $y_1 = 0 + 0.25\,(1 + 0^2) = 0.25$

Further iteration by calculator gives $y\,(2) = 1.74$ (3 sf)

Calculator is using u for x and v for y.

n	$u(n)$	$v(n)$
0	1	0
1	1.25	0.25
2	1.5	0.57813
3	1.75	1.0367
4	2	1.7429
5	2.25	3.0022
6	2.5	5.8181

$v(n) = 1.742\,859\,588$

Maclaurin series

A Maclaurin series expresses a function $f(x)$ as an infinite polynomial. It does this using the value of the function and its derivatives at $x = 0$.

The Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

Maclaurin series have already been encountered with infinite geometric series and the extension to the binomial theorem.

The Maclaurin expansion is valid only when the series on the right-hand side converges. The notation for the nth derivative of $f(x)$ is $f^{(n)}(x)$.

Link to Infinite geometric series SL 1.8, Extended binomial theorem AHL 1.10

Example 5.4.19

Find the first four terms in the Maclaurin expansion for $f(x) = \sec x$

Solution

$f(x) = \sec x$	$f(0) = 1$
$f'(x) = \sec x \tan x$	$f'(0) = 0$
$f''(x) = \sec x \tan^2 x + \sec^3 x$	$f''(0) = 1$
$f'''(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$	$f'''(0) = 0$

$$\sec x = 1 + 0x + \frac{1}{2}x^2 + 0x^3 + \dots$$

Assessment tip

Make sure that you lay out your working of the derivatives in a fashion that is easily followed, as in the above example. If the Maclaurin series is asked for, that implies there is a pattern and the general term can be identified. If this is not the case, then you could be asked for the first so many terms or the first so many non-zero terms. Make sure that you have answered what is actually asked. Known properties of the function, e.g., odd or even, can be used as checks.

Series for famous functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots,\ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

The right-hand side of the first three series converges for all values of x.

The Maclaurin expansion that is the extended binomial theorem is given by

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 \dots \qquad p \in \mathbb{Q}$$

This expansion is not in the formula booklet.

Maclaurin series for other functions can be obtained from known ones by simple substitution, multiplication, integration and differentiation.

SAMPLE STUDENT ANSWER

(a) Explain why $1 - x + x^2 - x^3 + x^4 \ldots = \frac{1}{1+x}$ and give the values of x for which this statement is valid.

(b) Write down the Maclaurin expansion for $\frac{1}{1+x^2}$

(c) Hence find the Maclaurin expansion for $\arctan x$.

(a) This is an infinite geometric series with $a = 1$ and $r = -x$.

$S_\infty = \frac{a}{1-r} = \frac{1}{1+x}$

Condition for convergence is $|r| < 1$ so require $|x| < 1$

(b) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \ldots$

(c) Integrating both sides

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \ldots$

▲ Parts (a) and (b) were answered and explained well. In part (c) it was recognized that integration was required and they had the integrals correct.

▼ The student did not mention the constant of integration $+c$ or explain that putting in $x = 0$ shows that $c = 0$. Although the final answer was correct this omission of method caused a loss of 2 marks.

The answer above could have achieved 9/11 marks.

Maclaurin series can be developed from differential equations by using implicit differentiation and the standard formula.

Example 5.4.20

Find the first four terms in the Maclaurin expansion for $y(x)$ given that $y(x)$ satisfies the differential equation $\frac{dy}{dx} = x + y^2$ and $y(0) = 1$

Solution

$y(0) = 1$ $\qquad$ $y'(0) = 0 + 1^2 = 1$

$y''(x) = 1 + 2y(x)\,y'(x)$ $\qquad$ $y''(0) = 3$

$y'''(x) = 2y'(x)\,y'(x) + 2y(x)\,y''(x)$ $\qquad$ $y'''(0) = 8$

$$y(x) = 1 + x + \frac{3x^2}{2} + \frac{8x^3}{6} + \ldots = 1 + x + \frac{3x^2}{2} + \frac{4x^3}{3} + \ldots$$

L'Hôpital's rule

This allows you to evaluate limits of the form $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ when they are in the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If this is the case then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$. The rule might have to be used more than once. The rule can also deal with the case where the limit is of the form $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to-\infty}$.

Example 5.4.21

Use L'Hôpital's rule to determine

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{\sin x-x}$$

Solution

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{\sin x-2}=\frac{0}{0} \qquad \lim_{x\to 0}\frac{e^x-1-x}{\cos x-1}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{e^x-1}{-\sin x}=\frac{0}{0} \qquad \lim_{x\to 0}\frac{e^x}{-\cos x}=-1$$

So the answer is –1.

SAMPLE STUDENT ANSWER

Use L'Hôpital's rule to determine $\lim_{x\to\infty}\frac{\ln x}{x}$

The limit is of form $\frac{\infty}{\infty}$ so equals $\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0$.

▲ The student recognized the indeterminate form and had the correct derivatives and final answer.

The answer above could have achieved 4/4 marks.

L'Hôpital's rule can also be adapted to deal with the indeterminate form $\infty\times 0$.

Example 5.4.22

Use L'Hôpital's rule to determine

$$\lim_{x\to\frac{\pi}{2}}\sec x\times\left(x-\frac{\pi}{2}\right)$$

Solution

The limit is of form $\infty\times 0$. Rearranging

equals $\lim_{x\to\frac{\pi}{2}}\frac{\left(x-\frac{\pi}{2}\right)}{\cos x}$

This is now of the form $\frac{0}{0}$

Differentiating gives

$$\lim_{x\to\frac{\pi}{2}}\frac{1}{-\sin x}=-1$$

So the answer is –1.

If the limit is of the form $\lim_{x\to 0}\frac{f(x)}{g(x)}$ then an alternative to using L'Hôpital's rule is to insert the Maclaurin expansions for $f(x)$ and $g(x)$.

Example 5.4.23

Determine $\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}}{\sin x-2}$

Solution

Inserting Maclaurin expansions gives

$$\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)-1-x-\frac{x^2}{2}}{\left(x-\frac{x^3}{6}+\ldots\right)-x}$$

$$=\lim_{x\to 0}\frac{\frac{x^3}{6}\ldots}{-\frac{x^3}{6}+\ldots}=-1$$

PRACTICE QUESTIONS

SL PAPER 1 SECTION A
NO TECHNOLOGY ALLOWED

Question 1 [6 marks]

Find the derivatives of the following functions:

a. $y = 3x^2 + 2x + 11$

b. $y = x^3 + \frac{1}{x^3}, x \neq 0$

c. $y = \sqrt[3]{x}$

Question 2 [8 marks]

Let $y = x^4 + 1$

a. Find the equation of the tangent to the graph of this function at $x = 1$.

b. Find the equation of the normal to the graph of this function at $x = -1$.

Question 3 [8 marks]

Let $y = \frac{x^3}{3} - \frac{3}{2}x^2 - 10x + 1$

a. Find the x values of all maximum and minimum points and classify them.

b. State the regions where the function y is

i. increasing **ii.** decreasing.

Question 4 [9 marks]

Let $y = (x-1)^3 + 5$

a. Find the coordinates of the point where $\frac{d^2y}{dx^2} = 0$

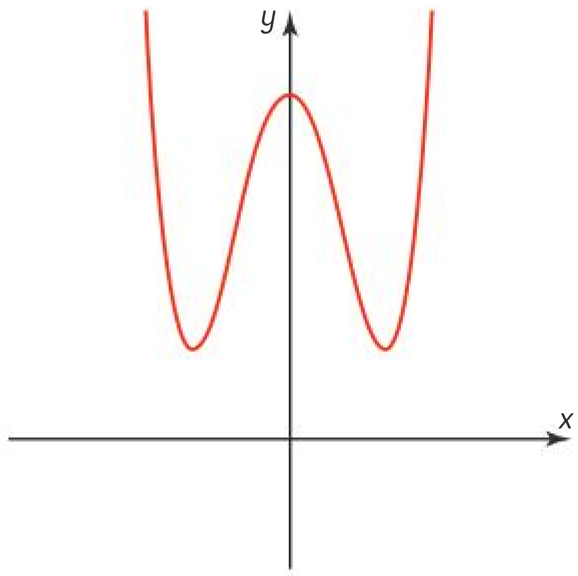
b. Show that this is a point of inflexion and state with a reason whether it is a horizontal or an oblique point of inflexion.

c. State the intervals where the graph of this function is

i. concave up **ii.** concave down.

Question 5 [6 marks]

A function $f(x)$, the equation of which you are not told, has a graph as shown.



Copy this graph and add to the diagram a sketch of the graph of $f'(x)$.

Still on the same diagram, also sketch the graph of $f''(x)$.

Make it clear which curve represents which function.

Question 6 [7 marks]

A function $f(x)$ passes through the point (1, 2) and has derivative $f'(x) = 3x^2 + e^{x-1} + 1$

a. Find $f(x)$.

b. Find $f''(x)$.

Question 7 [7 marks]

Find the following indefinite integrals:

a. $\int x^2 + \frac{1}{x^2} dx$

b. $\int \sqrt{x}\, dx$

c. $\int \sin x + \cos 3x\, dx$

Question 8 [7 marks]

a. On the same diagram, sketch the graphs of $y = 3x^2$, $y = 20$, $x = -2$, and $x = 2$.

b. Find the area enclosed by these four graphs.

Question 9 [9 marks]

Find the following indefinite integrals:

a. $\int \sin x (\cos x)^5 dx$

b. $\int x e^{x^2+3} dx$

c. $\int \frac{x^2}{(x^3+2)^2} dx$

SL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 10 [12 marks]

Find the derivatives of the following functions:

a. $y = x^3 \cos x$

b. $y = \dfrac{2x+1}{\sin x}$

c. $y = e^{(x^2+1)}$

d. $y = x\ln(3x)$

Question 11 [13 marks]

A parcel, which is to be sent through the post, is in the form of a cuboid with a square base. Its dimensions, measured in metres, are $x \times x \times h$, as it has width of x, length of x and height of h. Regulations state that: *the sum of the width plus the length plus the height cannot exceed 1 metre.*

a. Find the values of x and h that will maximize the volume of the parcel.

b. Find **i.** the maximum volume of the parcel and **ii.** the corresponding surface area of the parcel.

Question 12 [12 marks]

A farmer has her farmhouse at the origin on a straight track running from east to west. She owns land to the north and the south of the track. The boundaries that enclose her fields are given by the equations $y = x^2 - 4$, $x = -3$, $x = 4$ and $y = 0$.

a. Sketch a diagram to represent her three fields.

b. Find the total area of the land that she owns. The answer will be in hectares.

SL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 13 [5 marks]

The equation of a curve is given by $y = e^{\cos x \ln x}$, $x > 0$

a. Find the value of the gradient at the point where $x = 2$.

b. Hence find the gradient of the normal to this curve at $x = 2$.

Question 14 [5 marks]

a. Sketch the graph of the region enclosed between the curve $y = e^{2x^2}$ and the lines $y = 0, x = 0$ and $x = 1$.

b. Find the area of this region.

Question 15 [9 marks]

a. On the same diagram, sketch the graphs of $y = e^x$ and $y = 4 - x^2$

b. Find the points of intersection between these two curves.

c. Hence, find the area enclosed between the two curves.

Question 16 [9 marks]

An object that starts at the origin and moves in a straight line has velocity given by $v(t) = 3\sin 4t$, measured in metres per second.

a. Find an expression for the acceleration $a(t)$.

b. Find an expression for the displacement $s(t)$.

c. Find the displacement when $t = 5$.

d. Find the total distance travelled in the first 5 seconds.

SL PAPER 2 SECTION B TECHNOLOGY REQUIRED

Question 17 [12 marks]

A cubic equation is given by $y = x^3 + ax^2 + bx + c$

It has a y-axis intercept at (0, 5), a maximum at $x = 1$ and a minimum at $x = 4$.

a. Find the values of the constants a, b and c.

b. Find the coordinates of the maximum and the minimum points on the curve of y.

c. Find the coordinates of the point of inflexion on the curve of y.

HL PAPER 1 SECTION A NO TECHNOLOGY ALLOWED

Question 18 [5 marks]

Use the definition of differentiation by first principles to find the derivative of $f(x) = x^2 - 4$

Question 19 [9 marks]

Find the exact value of the definite integral $\displaystyle\int_{-1}^{0} \frac{x}{\sqrt{3x+4}}\,dx$

Question 20 [9 marks]

Use L'Hôpital's rule to find the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2}$

b. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^2}$

Question 21 [6 marks]

Solve the differential equation $\frac{dy}{dx} = e^{-y}\tan^3 x \sec^2 x$, given that $y = 0$ when $x = 0$.

Give the answer in the form $y = y(x)$.

HL PAPER 1 SECTION B NO TECHNOLOGY ALLOWED

Question 22 [12 marks]

a. If $y\tan x + y^2 \arctan x = 4$, use implicit differentiation to find an expression for $\frac{dy}{dx}$ in terms of x and y.

b. If $y = \operatorname{arccosec} x$, $x > 1$, find an expression for $\frac{dy}{dx}$ in terms of x.

Question 23 [11 marks]

Find the following indefinite integrals:

a. $\int \frac{1}{1+x^2}\,dx$

b. $\int \frac{x}{1+x^2}\,dx$

c. $\int \frac{1}{1-x^2}\,dx$

Question 24 [12 marks]

Find the following indefinite integral.

$\int 2x^2 e^{2x}\,dx$

Question 25 [12 marks]

Solve the differential equation $\frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x}$

Give the answer in the form $y = y(x)$.

Question 26 [12 marks]

a. Solve the differential equation $\frac{dy}{dx} + 2xy = 4x$, given that $y = 5$ when $x = 0$.

Give the answer in the form $y = y(x)$

b. Hence state $\lim_{x\to\pm\infty} y(x)$.

Question 27 [17 marks]

a. Write down the Maclaurin expansion for $y = \arctan x$

b. Find $\int \arctan x\,dx$

c. Hence, find the Maclaurin expansion for $y = x\arctan x - \frac{1}{2}\ln(1+x^2)$

d. Using your answer to part (a), write down the Maclaurin expansion for $y = x\arctan x$

e. Hence, find the Maclaurin expansion for $y = \arctan x + \frac{x}{1+x^2}$

f. Find the Maclaurin expansion for $y = \frac{1}{2}\ln(1+x^2)$

g. Use your answers to parts (d) and (f) to confirm your answer to part (c).

Question 28 [10 marks]

a. Find the first three derivatives of $f(x) = a^x$, and hence write down the first four terms in the Maclaurin expansion of $f(x) = a^x$

b. Using the fact that $a = e^{\ln a}$, use the Maclaurin expansion for $y = e^x$ to confirm your answer from part (a).

HL PAPER 2 SECTION A TECHNOLOGY REQUIRED

Question 29 [7 marks]

Consider the function $y = 2^{x^2}$

a. Find the area of the region enclosed by the graph of this function, the x-axis and the vertical lines $x = 0$ and $x = 1$.

b. This region is then rotated through 2π about the x-axis. Find the volume of the solid of revolution that is generated.

Question 30 [8 marks]

Consider the differential equation $\frac{dy}{dx} = \sqrt{x}\sqrt{y} + 1$

Given that $y(0) = 1$, use Euler's method with a step length of $h = 0.25$ to find an approximation for $y(1)$.

HL PAPER 2 SECTION B TECHNOLOGY REQUIRED

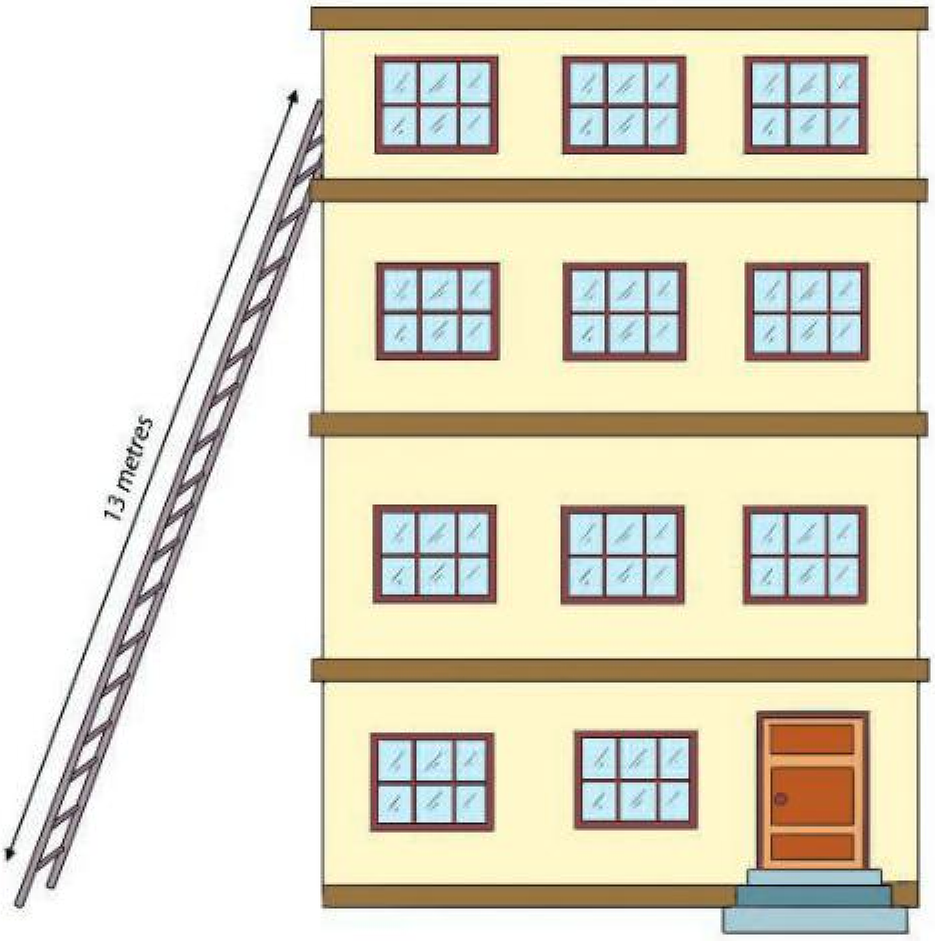
Question 31 [12 marks]

Almu is standing on the top of a ladder that is 13 metres long. Her end of the ladder is resting on a vertical wall; the other end of the ladder is resting on the ground. Tom is pulling the bottom end of the ladder away from the wall in a horizontal direction at a constant speed of $0.5\,\text{m}\,\text{s}^{-1}$.

a. Find the rate at which Almu is descending when the distance from the bottom of the ladder to the bottom of the wall is 5 m.

b. Find the rate at which Almu is descending when the vertical distance from the top of the ladder to the ground is 1 m.

c. Explain why this model suggests that Tom could not keep pulling the bottom of the ladder away at this constant speed.



Question 32 [13 marks]

a. The function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = x^2y^2 + 1$

Given that $y(0) = 1$, find the first four terms in the Maclaurin expansion for $y(x)$, i.e., up to the term in x^3.

b. Hence, find an approximation for $y(1)$.

HL PAPER 3
TECHNOLOGY REQUIRED

Question 33 [29 marks]

This question will investigate multiple repeated factors of a polynomial.

a. Let $y = (x - 2)^2 (x^2 + 3)$

i. Find any intercepts on the x-axis.

ii. Find $\frac{dy}{dx}$ and factorize this expression as far as possible (with justification).

Hence, show that there is a minimum at the point where $x = 2$ (with justification).

b. Let $y = (x - 2)^3 (x^2 + 3)$

i. Find any intercepts on the x-axis.

ii. Find $\frac{dy}{dx}$ and factorize this expression as far as possible (with justification).

Hence, show that there is a horizontal point of inflexion at the point where $x = 2$ (with justification).

c. Let $p(x)$ be a polynomial that is exactly divisible by $(x - c)^2$

Show that the graph of $y = p(x)$ has a stationary point (a maximum, minimum or horizontal point of inflexion) at $(c, 0)$.

d. Let $p(x)$ be a polynomial that is exactly divisible by $(x - c)^3$

Show that $p''(c) = 0$

e. Let $p(x)$ be a polynomial. The highest power of $(x - c)$ that divides $p(x)$ exactly, is n, $n \geq 2$. So $p(x)$ is of the form $p(x) = (x - c)^n q(x)$, where $q(c) \neq 0$.

i. For $n = 2$, investigate the nature of the stationary point at $(c, 0)$ according to whether $q(c)$ is positive or negative.

ii. For $n \geq 3$, investigate the nature of the stationary point at $(c, 0)$ according to whether n is odd or even, and whether $q(c)$ is positive or negative.

THE INTERNAL ASSESSMENT: THE EXPLORATION

The exams at the end of the course in May or November are called the external assessment. The internally assessed component in these courses is a mathematical exploration and is an essential part of the course that is compulsory for both SL and HL students. It is worth noting that both the MAA and MAI complete the same type of internal assessment (IA).

As part of your MAA course, you need to write an exploration. The internal assessment allows you to work in school and at home, use your technology and internet research, communicate with friends and relatives, and it is worth 20% of your final grade. You can write the exploration by hand but very few students do this as there are so many tools available with the use of technology.

This section gives you advice on writing your exploration, and includes advice on choosing a topic, planning your work, satisfying the examination criteria, and suggestions to help you submit your best work to aim for a top grade.

What is the exploration?

This is a piece of written work that involves investigating an area of mathematics. It should be 12 to 20 pages long with double-line spacing, including diagrams and graphs, but excluding the bibliography. However, it is the quality of the mathematical writing that is important, not the length.

There is a lot of personal freedom for you here. You can choose any topic that you want to explore and use any area of mathematics from the course. This means that you can use only mathematics that you are good with in any situation that you want, and without the time limitations and other constraints that are associated with written examinations. It will be submitted to your teacher who will mark your work and then submit a sample from your school to the IB for moderation.

You may ask about the difference between an exploration in mathematics and an extended essay in mathematics. The criteria are completely different. It is envisioned that the exploration is to be a much less extensive piece of work than an extended essay. The exploration allows students to explore an idea, rather than have to do the formal research demanded in an extended essay.

Planning your exploration

A total of 10 to 15 hours should be set aside for the exploration work in class and another 10 to 15 hours outside of class time. In class, you should listen carefully to your teacher and note the deadlines for submissions of required work, such as a draft. Follow deadlines and do the work early.

Here is what you can expect for the 10 to 15 hours in class. Your teacher will go through the criteria with you, show you some marked explorations, help you with choosing a topic, be there as a guide for you to ask questions, set deadlines, monitor your progress and view a draft of your work.

Your 10 to 15 hours outside of class time should be used for choosing your title, researching that topic, deciding which mathematics to use, collecting data and writing your exploration. After the draft, you will have time to react to your teacher's comments before submitting a final exploration.

Assessment tip

Many good ideas for explorations are let down at the grading stage by work that has clearly been rushed. You should do your work early, read over it at least once and compare it to the criteria and not try to do everything on the day before the deadline.

Getting started

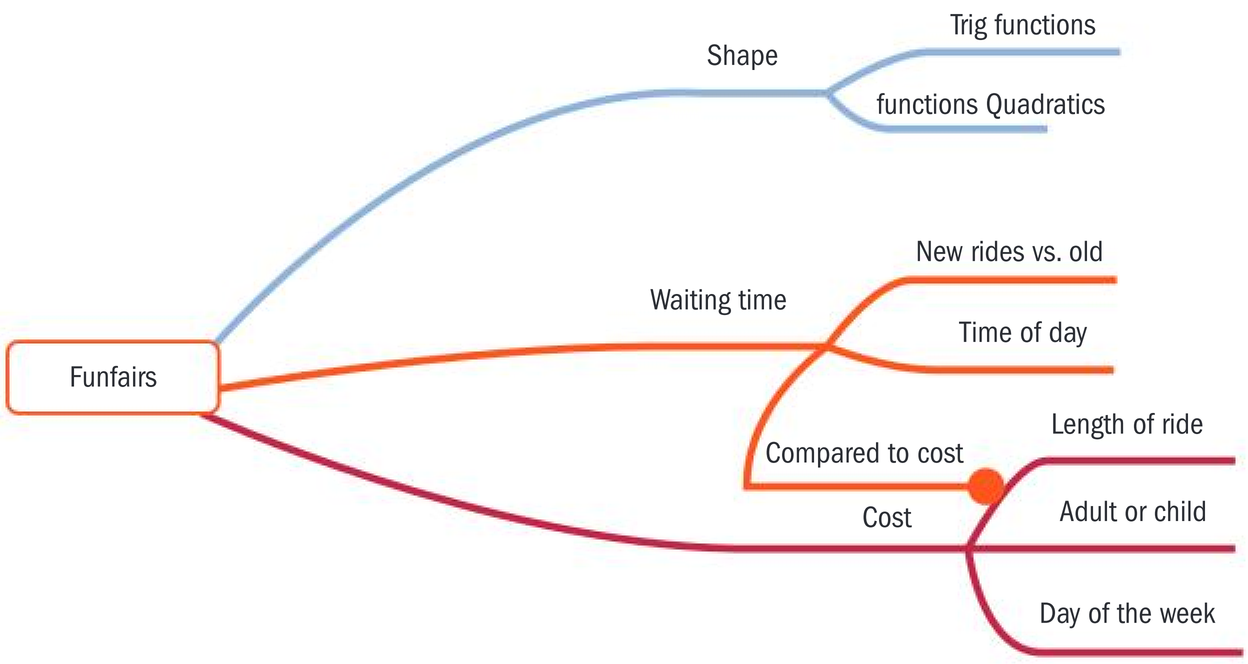
The exploration submitted for internal assessment must be your own work. However, it is not the intention that you should choose a title or topic and be left to work on the exploration without any help from your teacher. Your teacher will play an important role during both the planning stage and the period when you are working on the exploration.

You could seek out marked explorations from your teacher and search for more online to become familiar with what a good exploration looks like.

Choosing a topic

Choosing a topic is one of the first things to do and probably your biggest hurdle to overcome. For many students, the first challenge is to *think of their own explorations* without some considerable help from the teacher. The best explorations always seem to be the ones that the students come up with by themselves. This might come from a hobby, family business or favourite subject. Yes, you could do an internal assessment in mathematics using a science experiment or economics topic like the Gini Coefficient, but your work must satisfy the mathematics' criteria. Descriptive pure mathematics or historical topics are not appropriate. You should not use difficult mathematics beyond the syllabus if this cannot lead to some creativity or personalized problem. Using difficult mathematics that goes well beyond the syllabus often results in a lack of thorough understanding; this in turn makes it difficult for students to demonstrate personal engagement or reflection.

You can choose from a wide variety of activities: for example, modelling, investigations and applications of mathematics. To assist students in the choice of a topic, a list of stimuli is available from your teacher. However, you are not restricted to this list. A stimulus might be an area of interest such as funfairs. A mind-map is a good way to jot down ideas related to an area of interest.



In the above example, you can see areas in a funfair where mathematics is used and this could develop into a question that you would like to ask, such as "Is there a connection between the length of a ride and the cost of a ticket?" This could be a title for an exploration.

Avoid *common textbook problems*. A number of explorations are seen based on common textbooks problems like the Birthday Paradox or Monty Hall Problem, and demonstrate little or superficial understanding of the mathematical concepts being explored.

The first page and the last page

It is important to lay out your thinking on the first page. You do not need a cover page and your school, your name or your teacher's name should not be there. A title and the number of pages is sufficient. Your title should not just be the stimulus. Rather, the title should give a better indication of where the stimulus has taken you. For example, rather than have the title "Funfairs", the title could be "Funfair rides—exploring the length of a ride and the cost of a ticket".

On this first page there should be a brief introduction to the topic and you should:

- State the reason that you chose to explore this area.
- Identify a clear aim for your exploration. During the coming pages you should stay focused on this aim and avoid going off topic.
- Give a plan of what you will do and why.
- Explain what mathematics will be used and the reason for employing them.

The final page, your conclusion, should relate closely to the first page by explaining the highlights of the mathematical findings and answering the aim. You need to see that there is far more that could be done than is possible in so few pages and ask questions such as "What if …" and look for extensions of your work.

Writing the exploration

You should write the exploration using the mathematical language of an educated teenager. This means something that your friends could read, follow and understand. Communication and understanding are very important. You can pretend that you are writing a chapter in a textbook and fully explain each step, remembering that your audience is another student from your course of average ability.

Assessment tip

All graphs, tables and charts should have a title.

Follow a loop

- Use future tense to explain what your next step will be and why you are doing it.
- Use the personal pronoun "I" in your work.
- Do the mathematics.
- Look back (past tense), like "I noticed …", and reflect on what the mathematics showed you with regards to your aim.
- Take pride in the look and feel of your work. Use creative formatting, clear tables, graphs and diagrams.

Now go and repeat the loop for your next mathematical strategy.

Marks are awarded for a student's development and contribution to their exploration, not for work found in literature or carried out by others.

Assessment tip

Instead of writing lines and lines of algebra, use written explanations of difficult or less obvious steps in work—they are a good way of demonstrating your understanding.

Assessment tip

You cannot assume that the examiner possesses knowledge beyond that of the relevant MAA syllabus (e.g., knowledge of physics). This is probably the area in which most marks are lost, when what is written is just not understandable and the examiner keeps wondering why you have included it.

Internal assessment criteria—SL and HL

The exploration is internally assessed by the teacher and externally moderated by the IB using assessment criteria that relate to the objectives for mathematics. Each exploration is assessed against five criteria:

- Criterion A: Presentation
- Criterion B: Mathematical communication
- Criterion C: Personal engagement
- Criterion D: Reflection
- Criterion E: Use of mathematics

Criteria A to D are the same for both SL and HL, while there is a slight difference in criterion E for each subject.

The final mark for each exploration is the sum of the scores for each criterion. The maximum possible final mark is 20. You will not receive a grade for your mathematics course if you have not submitted an exploration. The marking starts with the lowest level in each criterion and once your work meets that, it moves up to the next level until the correct mark is awarded.

The criteria are:

Criterion A: Presentation

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration has some coherence or some organization.
2	The exploration has some coherence and shows some organization.
3	The exploration is coherent and well organized.
4	The exploration is coherent, well organized, and concise.

This criterion assesses the organization and coherence of the exploration.

- A *coherent* exploration is logically developed, easy to follow and meets its aim. This refers to the overall structure or framework, including introduction, body, conclusion and how well the different parts link to each other.

- A *well-organized* exploration includes an introduction, describes the aim of the exploration and has a conclusion. Relevant graphs, tables and diagrams should accompany the work in the appropriate place and not be attached as appendices to the document. Appendices should be used to include information on large data sets, additional graphs, diagrams and tables. Graphs in colour that are uploaded in black and white lose integrity. Make sure to upload them in colour.
- A *concise* exploration does not show irrelevant or unnecessary repetitive calculations, graphs or descriptions.

The use of technology is not required but encouraged where appropriate. However, the use of analytic approaches rather than technological ones does not necessarily mean lack of conciseness and should not be penalized.

Assessment tip

- Think about the organization: the overall structure or framework, including introduction, body and conclusion.
- Maintain coherence. This means thinking about how well different parts link to each other; think about the flow of your argument.
- Ensure that key ideas and concepts are clearly explained (but do not include mathematical definitions and terminology that are standard).
- Structure your ideas in a logical manner. The reader should know what you are doing and why and what you are doing next and why. The reader should not have to keep referring back to previous pages.
- Focus on the aim and avoid irrelevance: the written work and conclusion should relate back to the aim.
- Use language appropriate for your peers.

Criterion B: Mathematical communication

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration contains some relevant mathematical communication which is partially appropriate.
2	The exploration contains some relevant appropriate mathematical communication.
3	The mathematical communication is relevant, appropriate and is mostly consistent.
4	The mathematical communication is relevant, appropriate and consistent throughout.

This criterion assesses to what extent the student has:

- Used appropriate mathematical language (notation, symbols, terminology). Calculator and computer notation are acceptable only if it is software generated. Otherwise it is expected that students use appropriate mathematical notation in their work.
- Defined key terms and variables, where required.
- Used multiple forms of mathematical representation, such as formulae, diagrams, tables, charts, graphs and models, where appropriate.

- Used a deductive method and set out proofs logically where appropriate.

Examples of level 1 include graphs not being labelled or consistent use of computer notation with no other forms of correct mathematical communication. Level 4 can be achieved by using only one form of mathematical representation as long as this is appropriate to the topic being explored.

Assessment tip

- Computer notation should not be used. Common errors seen in criterion B include using unsuitable symbols such as ^ or *, for example writing 5^x instead of 5^x, or writing 5*x instead of $5x$. Other common errors include not defining variables, mixing Y and y, X and x (often a word-processing issue), changing limits of accuracy, like going from 1 to 2 to 3 significant figures without reason, and unlabelled axes on graphs.
- Mathematical definitions and terminology must be explained. Use contextual labels for graphs, such as time t or distance d instead of x and y.
- Use appropriate technology (graphic display calculators, screenshots, graphing, spreadsheets, databases, drawing, word-processing software, etc.) to enhance the paper; graphs should only be included where they service a purpose.
- Avoid making copies of your calculator screens, which do not meet the higher levels of criterion B.
- You should comment on and interpret results at the point at which these are used. This enhances communication and should be summarized in your conclusion.
- Use approximation signs where appropriate.
- Use a deductive method and set out proofs step by step.

Criterion C: Personal engagement

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of some personal engagement.
2	There is evidence of significant personal engagement.
3	There is evidence of outstanding personal engagement.

This criterion assesses the extent to which the student engages with the topic by exploring the mathematics and making it their own. It is not a measure of effort. Personal engagement may be recognized in different ways. These include:

- thinking independently or creatively
- presenting mathematical ideas in your own way
- exploring the topic from different perspectives
- making and testing predictions.

There must be evidence of personal engagement demonstrated in your work. Textbook-style explorations or reproduction of readily available mathematics without your own perspective are unlikely to achieve the higher levels.

Significant personal engagement is where the student demonstrates authentic personal engagement in the exploration on a few occasions. It is evident that these drive the exploration forward and help the reader to better understand the writer's intentions.

Outstanding personal engagement means the student demonstrates authentic personal engagement in the exploration in numerous instances and they are of a high quality. It is evident that these drive the exploration forward in a creative way. It leaves the impression that the student has developed, through their approach, a complete understanding.

Assessment tip

- It is not sufficient to show a personal interest in the topic, like "I have always enjoyed funfairs". You have to be engaged with the mathematics of the topic.
- Ask and answer questions: "I wonder if …", "What would happen if …" or "Why does that happen …"
- The exploration is intended to be an opportunity for you to use mathematics to develop an area of interest to you rather than merely to solve a problem set by someone else.
- Use your own experiment and collect primary data.
- Look for and create mathematical models for real-world situations.
- Consider historical and global perspectives.
- If you have learned a new technical tool or used some mathematics not yet taught in this class, tell the marker in your writing.
- Present mathematical ideas in your own way. This is vital if you are doing a pure mathematics topic like slope fields, for example. Choosing a historical or pure maths topic poses a much more difficult challenge in this criterion and might well mean that the top levels are not achieved in criterion C.
- Provide evidence that you "own" the work; it is not just simply a written report.
- Can the marker hear your voice when reading the exploration?

Criterion D: Reflection

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited reflection.
2	There is evidence of meaningful reflection.
3	There is substantial evidence of critical reflection.

This criterion assesses how the student reviews, analyses and evaluates the exploration. Although reflection may be seen in the conclusion to the exploration, it may also be found throughout the exploration. Simply describing results represents *limited reflection*. Further consideration is required to achieve the higher levels. Some ways of showing *meaningful reflection* include:

- linking to the aims of the exploration
- commenting on what you have learned
- considering some limitation or comparing different mathematical approaches.

Critical reflection is reflection that is crucial, deciding or deeply insightful. It will often develop the exploration by addressing the mathematical results and their impact on the student's understanding of the topic. Some ways of showing critical reflection are:

- considering what it is natural to do next
- discussing implications of results
- discussing strengths and weaknesses of approaches
- considering different perspectives—for instance, the problematical use of a statistical measure like the r value in isolation or the inaccuracy of working with grouped data.

Substantial evidence means that the critical reflection is present throughout the exploration. If it appears at the end of the exploration it must be of high quality and demonstrate how it developed the exploration in order to achieve a level 3.

Assessment tip

- Criterion D is all about looking backwards and explaining what you noticed and not just in the conclusion, but in the loops during your work.
- Although some reflections can be meaningful within the context of their tasks, it would be better if you could focus on the methods developed, the mathematical process applied, or the implication of the models utilized, since truly critical consideration of the implications or limitations are rare!
- Remember to make links to different fields and/or areas of mathematics.
- Many investigations that are submitted do not include enough reflection for the top marks to be awarded.

Criterion E: Use of mathematics—SL

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used.
2	Some relevant mathematics is used. Limited understanding is demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. Limited understanding is demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is mostly correct. Good knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Thorough knowledge and understanding are demonstrated.

In SL, this criterion assesses to what extent students use mathematics that is *relevant* to the exploration. *Relevant* refers to mathematics that supports the development of the exploration towards the completion of its aim. Overly complicated mathematics where simple mathematics would suffice is not relevant. Students are expected to produce work that is *commensurate with the level* of the course, which means it should not be completely based on mathematics listed in the prior learning. The mathematics explored should either be part of the syllabus or at a similar level.

A key word in the descriptor is *demonstrated*. The command term demonstrate means "to make clear by reasoning or evidence, illustrating with examples or practical application". Obtaining the correct answer is not sufficient to demonstrate understanding (even some understanding) in order to achieve level 2 or higher.

For knowledge and understanding to be *thorough*, it must be demonstrated throughout. The mathematics can be regarded as *correct* even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome.

The mathematics only needs to be what is required to support the development of the exploration. This could be a few small elements of mathematics or even a single topic (or sub-topic) from the syllabus. It is better to do a few things well than a lot of things not so well. If the mathematics used is relevant to the topic being explored, commensurate with the level of the course and understood by the student, then it can achieve a high level in this criterion.

Assessment tip

- You are encouraged to use technology to obtain results where appropriate, but *understanding must be demonstrated* to achieve higher than level 1, for example merely substituting values into a formula does not necessarily demonstrate understanding of the results.
- You must not use work only from the prior learning section, such as the Pythagorean Theorem, right triangle trigonometry, bar charts and simple probability. This would limit your maximum score to 2 in criterion E.
- You should explain why you are using a mathematical topic before it is used.
- Using only technology to find regression equations without showing any knowledge of how this is accomplished, or trying any type of analytical approach, is an issue. In such cases, technology often does the work and then the student comments on the superficial results. Few students go the extra mile and even attempt to explain the how or why behind these results.

Criterion E: Use of mathematics—HL

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used. Limited understanding is demonstrated.
2	Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding is demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Some knowledge and understanding are demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and demonstrates sophistication or rigour. Thorough knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and demonstrates sophistication and rigour. Thorough knowledge and understanding are demonstrated.

The previous comments for criterion E for SL also apply to HL, but the key differences between SL and HL for this criterion are the words *sophistication*, *rigour* and *precise*. HL students have higher expectations placed upon them when communicating mathematics used in their IA, as they are required to *demonstrate* in-depth understanding of mathematical content, and are expected to use advanced mathematical formulas and be able to explain it thoroughly *commensurate* with the level of the course. To be considered as *sophisticated,* the mathematics used should be commensurate with the HL syllabus or, if contained in the SL syllabus, the mathematics has been used in a complex way that is beyond what could reasonably be expected of an SL student.

*Sophisticatio*n in mathematics may include understanding and using challenging mathematical concepts, looking at a problem from different perspectives, and seeing underlying structures to link different areas of mathematics.

Rigour involves clarity of logic and language when making mathematical arguments and calculations. Mathematical claims relevant to the development of the exploration must be justified or proven.

HL students are also required to make accurate and *precise* calculations and check their work more rigorously throughout. *Precise* mathematics is error-free and uses an appropriate level of accuracy at all times.

Assessment tip

- HL (and SL) students sometimes use complicated mathematics taken from another source, which they evidently do not understood and do not properly explain, use or apply. Such work is basically reduced to substituting values into given formula and has little scope to show mathematical knowledge or understanding. It is better to take mathematics that you know and show how well you understand it in an exploration of a real-life situation.
- In general, the quantity of mathematics used is not the deciding factor as to the attainment level achieved, but rather the degree of understanding demonstrated by the candidate. The mathematics used need only be what is required to support the development of the exploration.

Academic honesty

There may be times when you find a piece of work, diagram or explanation online or in a text. How do you know if it is okay to place it in your exploration? Here are some definitions of what is and is not suitable. Your school might have an academic honesty policy that you can also refer to.

Plagiarism is defined as using, intentionally or unwittingly, the ideas, words or work of another person without proper citing. You should not take phrases from a source without using quotation marks, or find synonyms for the author's language while keeping to the same general structure and meaning of the original.

Collaboration is a good thing. You are expected to talk with others, discuss topic titles and aims, share ideas, view software, etc.

Assessment tip

There are so many plagiarism detectors available that even a 10-word phrase can be traced to its source; diagrams can also be traced easily. Always use your own work, write in your own voice and explain with your words. This counts well in criteria A, C and E.

> **Assessment tip**
>
> Clearly indicate which words, images and works are not your own, including graphs, charts, pictures and working. Give credit for copied, adapted and paraphrased material.
>
> When using text, make clear where the borrowed material starts and finishes.

Collusion is defined as working with others to intentionally deceive: for example, allowing your work to be copied or submitted for assessment by another.

Duplication of work is defined as the presentation of the same work for different assessment components and/or Diploma Programme requirements.

Your exploration should be your own work, but you are not expected to create new mathematics, so there will be times when you quote people, use their diagrams, definitions, etc. It is best to cite these instances at the bottom of the page *and* in a bibliography at the end of the exploration.

Any plagiarism will be taken very seriously by the IB and could mean that you will not gain your IB diploma. Your teacher has to sign that they believe that this is your own work.

Reflection

What if you knew of a single strategy that would improve your grade? This is the strategy of reflection or rereading. This is not the same *reflection* from criterion D, but your reflection on your work. Make sure that you finish your work well before the deadline to give yourself time to read through it. As you have probably already experienced with other school projects and essays, you will change and improve your work every time you go over it. You may see better ways to format your working by keeping the equals sign under the equals sign, changing the size, position and clarity of diagrams and spotting areas that can improve your score in each criterion.

When rereading your work, watch out for:

- any solutions to individual problems, tables or diagrams that may be cut in half accidentally by a page break
- comments like "see the graph below" when the graph was accidentally moved to the next page during editing
- graphs that occupy a whole page—they should be avoided if possible
- tables that are only two or three columns wide but many lines deep—consider making the table horizontal instead of vertical.

Some additional points:

- Consider using the text-wrap format to make graphs or charts an integrated part of the work.
- Don't have formulae and working inside a paragraph of writing. Look how working is formatted in your textbook.
- Excel graphs are generally not of the same mathematical quality as Geogebra or Desmos.
- Use online tools found from a search engine for better-looking statistical diagrams.
- It is okay to leave a little space at the bottom of a page! You don't have to fill the whole page with content!

Checklist

Here is a checklist to ensure that you have done your best to submit a good exploration.

Communication and mathematical presentation

- ☐ Do you have a title but not your name, candidate number, school or teacher's name?
- ☐ Did you start with an introduction?
- ☐ Have you stated a clear aim? What is the theme of your exploration? What do you hope it will do?
- ☐ Have you answered or responded to your aim in the conclusion?
- ☐ Do you have a clear rationale? Why did you choose this topic? Why is this topic of interest to you?
- ☐ Can an average student in your class read and follow your exploration? You may want to peer edit with a friend.
- ☐ Does the entire paper focus on the aim and avoid irrelevance?
- ☐ Does the writing flow nicely? Have you followed the loop?
- ☐ Did you include graphs, tables and diagrams with titles at appropriate places and not attach them all at the end?
- ☐ Have you had reread and improved your exploration?
- ☐ Did you cite all references in your bibliography and acknowledge direct quotes appropriately?
- ☐ Did you use appropriate mathematical language and representation? (No computer notation *, ^, etc.)
- ☐ Did you define key terms where necessary?
- ☐ Did you use appropriate technology?
- ☐ Did you think about the degree of accuracy? (For your topic, how many decimal places are relevant?)
- ☐ Did you end with a conclusion and relate it back to your aim and rationale?

Personal engagement

- ☐ Did you address why you think your topic is interesting or why it appealed to you?
- ☐ Did you use the personal pronoun?
- ☐ Did you ask and answer personal questions ("I wonder if …", "What if …")?
- ☐ Did you present mathematical ideas in your own way (as opposed to copying someone else's theory)?
- ☐ Did you explain any new mathematics or software that you had to employ?
- ☐ Did you try to add "your voice" to the work?
- ☐ Did you relate the results to your own life?

Reflection

- ☐ Did you explain what the results of your mathematics means after you have used a technique?
- ☐ Did you ask questions, make conjectures and investigate mathematical ideas?
- ☐ Did you consider the historical and global perspectives of your topic?
- ☐ Did you discuss the implications of your results? (What do they mean? Why are they important?)
- ☐ Did you look for possible limitations and/or extensions of your topic?
- ☐ Did you find any areas where your mathematics may be approximate or slightly inaccurate?
- ☐ Did you make links between your topic and different fields and/or areas of mathematics?

Use of mathematics

- ☐ Did you use mathematics from your course?
- ☐ Did you explain unfamiliar mathematics, or apply familiar mathematics to a new situation?
- ☐ Did you create mathematical models for real-world situations, if this applied to your topic?
- ☐ Did you apply problem-solving techniques?
- ☐ Did you look for and explain patterns, if this applied to your topic?

PRACTICE EXAM PAPERS

Introduction

At this point you will have refamiliarized yourself with the contents of the IB Mathematics: Analysis and Approaches syllabus. Additionally you will have picked up some key techniques and skills to refine your exam approach. It is now time to put these skills to the test; in this section you will find practice examination papers: SL Papers 1 and 2, HL Papers 1, 2 and 3, with the same structure as the external assessment that you will complete at the end of the Diploma Programme course. Answers to these papers are available at **www.oxfordsecondary.com/ib-prepared-support**.

These answers are in a format that is similar to an IB markscheme. It shows how marks are to be gained in the different parts of a question. Examiners use annotations when marking and some of their abbreviations are shown here: M1 indicates a Method Mark, A1 an Achievement Mark and R1 a Reasoning Mark. If a mark is shown in brackets, e.g., (M1), it implies that the mark can be given even if the method has not been shown but is implied by subsequent student working.

The abbreviation AG indicates "As Given" and is used at the end of a "show that …" problem. This means no marks would be given for this line and examiners know that they have to check the student's work carefully here, to see that the student has not just written down the required answer.

You will require pens, pencil and a ruler for all exams. The use of correction fluid is not allowed. If you do make a mistake just put a single line through it. If it is a section that is wrong draw a line at the top and at the bottom and then a single line though the incorrect working.

A clean copy of *Mathematics: Analysis and Approaches formula booklet* is required for all papers. This will be provided by your school.

There will be 5 minutes of reading time at the start of paper. You may not write during this time but you can prepare your strategy for answering the questions.

All papers will have the instruction "Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures." Make sure that you understand what significant figures are and check that you have followed this instruction at the end of each question.

All questions in all sections and in all papers are to be attempted. Ensure that you divide your time wisely between the sections (which carry equal marks) and the questions. The number of marks allocated

Assessment tip

Always show your methods, then you will gain the Method Marks even if you make mistakes in your calculations.

to a question is a rough guide to how many minutes it should take to complete. If you become stuck on a question, do not waste time, be prepared to leave it and move on to the next question. Put a mark by it so that you can come back to it at the end of the exam if you have time.

Use of technology is not allowed for Paper 1. However, use of technology (for example, a graphic display calculator) is required for Papers 2 and 3.

Papers 1 and 2 will both be in two sections: Section A and Section B. Section A will consist of short-response questions and Section B will consist of extended-response questions. The order in which you do the questions does not matter. However there is a difficulty gradient going through Section A and then starting again with Section B, with the easier questions at the start of each section. Hence it is recommended that you start with the first questions of each section and methodically make your way through.

The Section A questions are to be answered on the examination paper in the specially prepared boxes under each question. If you run out of space, either because you are using an inefficient method or have had to cross out a mistake, then you can complete your answer in an answer booklet. Make sure that you indicate that you have done this in the box and make sure that the question is well labelled in the answer booklet.

The Section B questions are to be answered in the answer booklets provided. There will be an instruction to start each question on a fresh page. Obey this instruction; it will also give you space to go back if you have to complete a part in the question. Make sure that you clearly mark each question number and the subparts. In Section B on every page it will state "Do **not** write solutions on this page." This instruction is **very** important indeed. Anything written on these pages will not get scanned and sent to the examiner. This needs to be remembered in particular if there is a diagram in the question. It is no good putting extra symbols on the diagram since they will not be seen by the examiner. Redraw the diagram in your answer booklet and then add the extra symbols.

HL Paper 3 will consist of two compulsory, extended-response, problem-solving questions. Start each question on a fresh page in the answer booklet.

At the start of each paper there will be the advice "Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working." On papers that require the use of technology, there will also be the sentences "Solutions found from a graphical display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer." It should not be a surprise to you reading this on the exam. This is how you should have been answering all of the questions you have done in the two years leading up to the exam.

If you are a non-native English speaker, you are allowed to take a dual-language dictionary into the exam with you. This applies to all papers. It must be a physical, not an electronic, dictionary. Even if you are very confident in your English ability, it can still be helpful to have the dictionary available on your desk.

SL Practice paper 1

Time: 1 hour 30 minutes

Answer all the questions.

Technology not allowed.

You need the formula booklet for this paper.

Maximum mark: 80 marks

All numerical answers should be given exactly or correct to three significant figures, unless otherwise stated in the question.

Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working.

In the actual exam, answer boxes will be provided for Section A and all answers for Section B should be written in the answer booklets.

Section A (Short answers)

1. [Maximum mark: 6]

An arithmetic sequence has $u_5 = 18$ and $u_{10} = 38$.

Find

(a) the common difference, d

(b) the first term, u_1

(c) u_{101}. [6]

2. [Maximum mark: 8]

(a) Find the equation of the tangent to the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$. [5]

(b) Find the equation of the normal to the same curve at the same point. [3]

3. [Maximum mark: 8]

(a) Prove that $\frac{1}{x+1} + \frac{x}{x+2} \equiv \frac{x^2+2x+2}{(x+1)(x+2)}$, $x \neq -1$, $x \neq -2$. [2]

(b) Hence solve $\frac{1}{x+1} + \frac{x}{x+2} = \frac{10}{(x+1)(x+2)}$. [6]

4. [Maximum mark: 5]

(a) Find $\int \cos x(\sin x)^3 \,\mathrm{d}x$. [3]

(b) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos x(\sin x)^3 \,\mathrm{d}x$. [2]

5. [Maximum mark: 7]

Let $f(x) = 3\mathrm{e}^x + 4$

(a) State **(i)** the domain and **(ii)** the range of this function. [2]

(b) Find the inverse function $f^{-1}(x)$. [3]

(c) State **(i)** the domain and **(ii)** the range of the inverse function. [2]

6. [Maximum mark: 6]

James rolls two ordinary, fair dice. One of the dice is red and the other is blue.

Let the discrete random variables X and Y be the scores on the red dice and blue dice, respectively. A new random variable is defined by $D = |X - Y|$.

By constructing the probability distribution table for D, find $E(D)$. [6]

Section B (Long answers)

7. [Maximum mark: 13]

Let $f(x) = x^2 \cos x$.

(a) Find $f'(x)$. [3]

(b) Find and simplify $f''(x)$. [4]

(c) Evaluate

(i) $f(0)$

(ii) $f'(0)$

(iii) $f''(0)$. [3]

(d) Hence, investigate the nature of the graph of $f(x) = x^2 \cos x$ at the point where $x = 0$. [3]

8. [Maximum mark: 12]

Alun is a triathlete. Every morning in training, he either swims, cycles or runs with probabilities of $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{1}{3}$, respectively. He only does one type of training each day. These probabilities are independent of what training he did previously. If he swims in the morning, the probability that he has a sleep in the afternoon is $\frac{1}{2}$. If he cycles in the morning, he always has a sleep in the afternoon. If he runs in the morning, the probability that he has a sleep in the afternoon is $\frac{3}{4}$.

(a) Draw a probability tree to represent this information. [3]

(b) Find the probability that he has a sleep in the afternoon. [2]

(c) Given that he had a sleep in the afternoon, find the probability that he cycled in the morning. [2]

(d) Given that he did not have a sleep in the afternoon, find the probability that he swam in the morning. [2]

(e) Find the probability that on two consecutive days he does a different type of training. [3]

9. [Maximum mark: 15]

Let $f(x) = \frac{x+1}{2x-3}$, $x \neq \frac{3}{2}$.

(a) Write down **(i)** the vertical asymptote and **(ii)** the horizontal asymptote of $f(x)$. [2]

(b) Find **(i)** the x-intercept and **(ii)** the y-intercept of $f(x)$. [2]

(c) Find and simplify $f'(x)$. [2]

(d) Hence, explain why $f(x)$ is always decreasing. [1]

(e) Find $f''(x)$. [2]

(f) Hence, describe the concavity of $f(x)$ for all values of the domain. [2]

(g) Show that $f(x)$ can be written as $f(x) = \frac{1}{2} + \frac{\frac{5}{2}}{2x-3}$. [2]

(h) Hence, find $\int f(x)\,dx$ for $x > \frac{3}{2}$. [2]

SL Practice paper 2

Time: 1 hour 30 minutes

Answer all the questions.

Technology required.

You need the formula booklet for this paper.

Maximum mark: 80 marks

All numerical answers should be given exactly or correct to three significant figures, unless otherwise stated in the question.

Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working.

Solutions found from a graphical display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer.

In the actual exam, answer boxes will be provided for Section A and all answers for Section B should be written in the answer booklets.

Section A (Short answers)

1. [Maximum mark: 6]

Consider the following data:

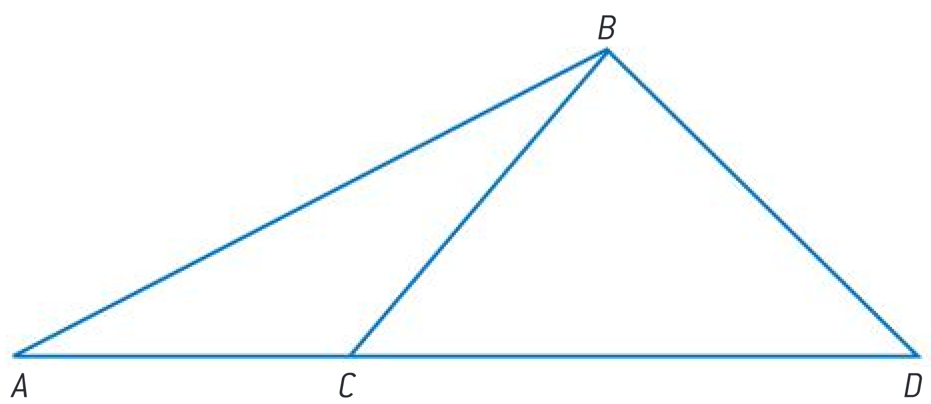
5, 7, 16, 8, 3, 2, 8.

(a) Find **(i)** the median and **(ii)** the mode. [2]

(b) Identify, with reasoning, any outliers. [4]

2. [Maximum mark: 7]

In the diagram below (not drawn to scale), $AB = 5, BC = BD = 4, B\hat{A}C = 40°$.



Find the length CD. [7]

3. [Maximum mark: 6]

Find the coefficient of the term in x^2 in the expansion of $\left(x^2 + \frac{1}{x}\right)^7$. [6]

4. [Maximum mark: 7]

Find the area enclosed between the curves of $y = 2x^2 - 5x + 10$ and $y = x^2 + 5x - 6$. [7]

5. [Maximum mark: 6]

In this question give all final monetary answers to two decimal places.

(a) Eva invests \$500 in a bank that gives an interest rate of 5% compounded yearly. Calculate how much money she will have saved at the end of 10 years. [2]

(b) Jessie puts $\$X$ into a bank that gives an interest rate of 3% compounded yearly. After 5 years she will withdraw \$200 and then leave the rest in for another 5 years. At this time she will withdraw \$300, which will leave the account empty. Find the value of X. [4]

6. [Maximum mark: 8]

Solve the equation $\log_4 x - 3\log_x 4 - 2 = 0$. [8]

Section B (Long answers)

7. [Maximum mark: 14]

The table below gives the number of points scored, x, and the number of points conceded, y, by the rugby team Wick Warriors during seven games.

x	24	17	3	27	0	48	3
y	0	3	24	3	33	0	29

(a) For this data calculate Pearson's product moment correlation coefficient. [2]

(b) Comment on what this value indicates. [1]

(c) Write down the regression line of y on x. [3]

(d) If in another match Wick Warriors scored 31 points, estimate the number of points they would have conceded (give your answer to the nearest integer). [2]

(e) If in yet another match Wick Warriors conceded 11 points, estimate the number of points they would have scored (give your answer to the nearest integer). Show clearly how this answer was obtained. [3]

(f) Find the point where the lines of best fit y on x and x on y must intersect. [3]

8. [Maximum mark: 16]

The times taken by riders in a 16 km bike time trial are normally distributed with a mean of 25 mins and a standard deviation of 2 mins.

(a) **(i)** Find the probability that a rider takes between 24 and 27 mins.

(ii) Find the probability that a rider takes over 26 mins.

(iii) The fastest 80% of the riders gain a medal. Find the time, to the nearest second, that a rider would have to complete the course in under, so that they would gain a medal. [7]

(b) Ten riders are chosen at random.

(i) Find the probability that exactly seven of them gained a medal.

(ii) Find the probability that at least seven of them gained a medal. [4]

(c) In another time trial over 40 km the riders' times are again normally distributed. This time the standard deviation is 4 mins. If 25% of the riders complete the course in under 1 hour, find the mean, μ, giving the answer to the nearest second. [5]

9. [Maximum mark: 10]

An object moving in a straight line has velocity given by $v = \sin\sqrt{t+1}$. Initially it starts at the origin. Time, t, is measured in seconds and distance is measured in metres.

(a) Write down **(i)** the maximum and **(ii)** the minimum velocity. [2]

(b) Calculate

(i) the displacement when $t = 10$

(ii) the total distance travelled in the first 10 seconds. [5]

(c) Find an expression for the object's acceleration. [3]

HL Practice paper 1

Time allowed: 2 hours

Answer all the questions.

Technology not allowed.

You need the formula booklet for this paper.

Maximum mark: 110 marks

All numerical answers should be given exactly or correct to three significant figures, unless otherwise stated in the question.

Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working.

In the actual exam, answer boxes will be provided for Section A and all answers for Section B should be written in the answer booklets.

Section A (Short answers)

1. [Maximum mark: 6]

An arithmetic sequence has $u_5 = 18$ and $u_{10} = 38$.

Find

(a) the common difference, d

(b) the first term, u_1

(c) u_{101}. [6]

2. [Maximum mark: 8]

(a) Find the equation of the tangent to the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$. [5]

(b) Find the equation of the normal to the same curve at the same point. [3]

3. [Maximum mark: 5]

(a) Find $\int \cos x \sin^3 x \, dx$. [3]

(b) Hence, find the value of $\int_0^{\frac{\pi}{2}} \cos x \sin^3 x \, dx$. [2]

4. [Maximum mark: 7]

Let $f(x) = 3e^x + 4$.

(a) State **(i)** the domain and **(ii)** the range of $f(x)$. [2]

(b) Find the inverse function $f^{-1}(x)$. [3]

(c) State **(i)** the domain and **(ii)** the range of $f^{-1}(x)$. [2]

5. [Maximum mark: 6]

James rolls two ordinary, fair dice. One of the dice is red and the other is blue.

Let the discrete random variables X and Y be the scores on the red dice and blue dice, respectively. A new random variable is defined by $D = |X - Y|$.

By constructing the probability distribution table for D, find $E(D)$. [6]

6. [Maximum mark: 5]

Let $z = 2 + 3i$ and $w^* = 1 + 4i$. Find $\frac{z}{w}$, giving your answer in the form $\frac{a}{b}$ where $a, b \in \mathbb{Q}$. [5]

7. [Maximum mark: 5]

Differentiate $f(x) = x^2 + x + 2$ from first principles. [5]

8. [Maximum mark: 6]

Let $p(x) = x^3 + x^2 + ax + b$.

When $p(x)$ is divided by $x - 1$ the remainder is 7.

When $p(x)$ is divided by $x + 2$ the remainder is -8.

Find the values of a and b. [6]

9. [Maximum mark: 7]

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 2 \\ l \end{pmatrix}$.

It is known that **a** and **b** are perpendicular and that **c** and **b** are parallel.

(a) Find the value of k. [3]

(b) Find the value of l. [2]

(c) Write down the value of $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$, giving a reason. [2]

Section B (Long answers)

10. [Maximum mark: 12]

Alun is a triathlete. Every morning in training, he either swims, cycles or runs with probabilities of $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{1}{3}$, respectively. He only does one type of training each day. These probabilities are independent of what training he did previously. If he swims in the morning, the probability that he has a sleep in the afternoon is $\frac{1}{2}$. If he cycles in the morning, he always has a sleep in the afternoon. If he runs in the morning, the probability that he has a sleep in the afternoon is $\frac{3}{4}$.

(a) Draw a probability tree to represent this information. [3]

(b) Find the probability that he has a sleep in the afternoon. [2]

(c) Given that he had a sleep in the afternoon, find the probability that he cycled in the morning. [2]

(d) Given that he did not have a sleep in the afternoon, find the probability that he swam in the morning. [2]

(e) Find the probability that on two consecutive days he does a different type of training. [3]

11. [Maximum mark: 11]

Let $f(x) = \dfrac{3x+4}{x^2+x-6}$, $x \neq -3$ and $x \neq 2$.

(a) By first writing $f(x)$ in partial fractions, find $\int f(x)\,dx$ for $x > 2$. [8]

(b) Using the partial fraction representation you found in part **(a)**, find $f'(x)$ and hence explain why $f(x)$ is always decreasing on its domain. [3]

12. [Maximum mark: 16]

(a) Prove by induction that $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$ for $n \in \mathbb{Z}^+$. [9]

(b) Use your answer to part **(a)** to show that $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$. [2]

(c) Use your answer to part **(a)** to find and simplify an expression for the sum of the cubes of the first n positive odd integers. (*Hint: Use the fact that all integers are either odd or even.*) [5]

13. [Maximum mark: 16]

Consider the function $y = x^x$ for $x > 0$.

(a) By first taking logarithms to base of both sides, find $\frac{dy}{dx}$. [5]

(b) Hence, **(i)** find the exact coordinates of any maximum of minimum points of the function and **(ii)** classify their nature. [5]

(c) Use L'Hôpital's rule to find $\lim_{x \to 0} (\ln y)$. [4]

(d) Hence, find the value of $\lim_{x \to 0} x^x$. [2]

HL Practice paper 2

Time allowed: 2 hours

Answer all the questions.

Technology required.

You need the formula booklet for this paper.

Maximum mark: 110 marks

All numerical answers should be given exactly or correct to three significant figures, unless otherwise stated in the question.

Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working.

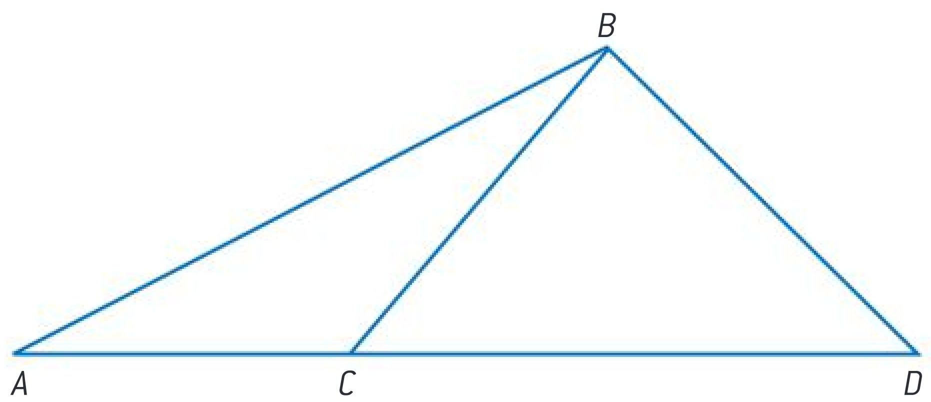
Solutions found from a graphical display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer.

In the actual exam, answer boxes will be provided for Section A and all answers for Section B should be written in the answer booklets.

Section A (Short answers)

1. [Maximum mark: 7]

In the diagram below (not drawn to scale), $AB = 5$, $BC = BD = 4$ and $B\hat{A}C = 40°$.



Find the length CD. [7]

2. [Maximum mark: 6]

Find the coefficient of the term in x^2 in the expansion of $\left(x^2 + \frac{1}{x}\right)^7$ [6]

3. [Maximum mark: 6]

(a) Eva invests \$500 in a bank that pays an interest rate of 5% compounded yearly. Calculate how much money she will have saved at the end of 10 years. [2]

(b) Jessie puts \$$X$ into a bank that pays an interest rate of 3% compounded yearly. After 5 years she will withdraw \$200 and then leave the rest in for another 5 years. At this time she will withdraw \$300, which will leave the account empty. Find the value of X. [4]

4. [Maximum mark: 8]

Solve the equation $\log_4 x - 3\log_x 4 - 2 = 0$. [8]

5. [Maximum mark: 7]

The portion of the curve $y = \sin\sqrt{x}$ from $x = 0$ to $x = \pi^2$ is rotated through 2π about the x-axis to create a solid of revolution.

(a) **(i)** Write down an integral for the volume of the solid generated.

(ii) Find the value of this integral. [4]

(b) **(i)** Find the greatest radius of this solid of revolution.

(ii) State the x-value of the point where this greatest radius occurs. [3]

6. [Maximum mark: 9]

A square of side length l lies symmetrically inside a circle of radius r.

l is increasing at a rate of $0.1\,\text{m s}^{-1}$ and r is increasing at a rate of $0.2\,\text{m s}^{-1}$.

Let A be the area of the shape between the circle and the square.

(a) Find the rate of change of A at the instant when $r = 2$ and $l = 1$. [7]

(b) Given that the rate of change of A is $10\,\text{m}^2\,\text{s}^{-1}$, find l in terms of r. [2]

7. [Maximum mark: 8]

A function $f(x)$ satisfies the differential equation $f'(x) = (x + 1)f(x) + 3$.

Given that $f(0) = 2$, find the first four terms in the Maclaurin expansion of $f(x)$. [8]

Section B (Long answers)

8. [Maximum mark: 14]

The table below gives the number of points scored, x, and the number of points conceded, y, by the rugby team Wick Warriors during seven games.

x	24	17	3	27	0	48	3
y	0	3	24	3	33	0	29

(a) Calculate Pearson's product moment correlation coefficient for this data. [2]

(b) Comment on what this value indicates. [1]

(c) Write down the regression line of y on x. [3]

(d) If, in another match, Wick Warriors scored 31 points, estimate the number of points they would have conceded (give your answer to the nearest integer). [2]

(e) If, in yet another match, Wick Warriors conceded 11 points, estimate the number of points they would have scored (give your answer to the nearest integer). Show clearly how this answer was obtained. [3]

(f) Find the point where the lines of best fit of y-on-x and x-on-y must intersect. [3]

9. [Maximum mark: 15]

Solve the differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ for $x > 0$, given that $y = 1$ when $x = 1$.

Give your answer in the form $\arctan\frac{y}{x} = f(x, y)$ for some function of x and y to be found. [15]

10. [Maximum mark: 12]

A continuous random variable X has a probability distribution function given by

$$f(x) = \begin{cases} a\mathrm{e}^{x}, 0 \le x \le 1 \\ 0, \text{elsewhere} \end{cases}$$

(a) Show that the exact value of the constant a is $\frac{1}{\mathrm{e}-1}$. [2]

(b) Calculate

(i) the mean

(ii) the variance of X. [6]

(c) Find the median of X. [4]

11. [Maximum mark: 18]

A point is given by $P = (2, 1, 1)$, a line is given by $\mathrm{L}: \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1}$ and a plane is given by $\Pi: 2x + 2y + z = 9$.

(a) Calculate, in degrees, the acute angle between the line and the plane. [7]

(b) Explain why the line and the plane must intersect, and find the point of intersection. [5]

(c) Find the shortest distance from the point P to the plane Π. [6]

HL Practice paper 3

Time allowed: 1 hour

Answer all the questions.

Technology required.

You need the formula booklet for this paper.

Maximum mark: 55 marks

All numerical answers should be given exactly or correct to three significant figures, unless otherwise stated in the question.

Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working. You are therefore advised to show all working.

Solutions found from a graphical display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer.

In the actual exam, all answers should be written in the answer booklets.

1. [Maximum mark: 28]

This question investigates higher derivatives of the product of two functions.

(a) Let $f(x) = \mathrm{e}^{x} \sin x$.

Find and simplify

(i) $f'(x)$

(ii) $f''(x)$

(iii) $f'''(x)$

(iv) $f^{(4)}(x)$.

Hence for $n \in \mathbb{Z}^+$, deduce

(v) $f^{(4n)}(x)$

(vi) $f^{(4n+1)}(x)$

(vii) $f^{(4n+2)}(x)$

(viii) $f^{(4n+3)}(x)$. [13]

(b) Let $f(x) = g(x) \times h(x)$.

Write down simplified expressions, involving the derivatives of $g(x)$ and $h(x)$, for

(i) $f'(x)$

(ii) $f''(x)$

(iii) $f'''(x)$. [5]

(c) For $f(x) = g(x) \times h(x)$ you are told that

$$f^{(n)}(x) = \sum_{i=0}^{n} a_i g^{(n-i)}(x) \times h^{(i)}(x)$$

where a_i are constants that are independent of the functions $g(x)$ and $h(x)$.

By putting $g(x) = \mathrm{e}^{\lambda x}$ and $h(x) = \mathrm{e}^x$ into the equation above, find the values of a_i. [10]

2. [Maximum mark: 27]

In this question, complex numbers will be used in order to establish trigonometrical identities.

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Write down the modulus of z. [1]

(b) Show that

(i) $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(ii) $z^n + \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$. [5]

(c) **(i)** By expanding $\left(z + \dfrac{1}{z}\right)^2$ show that $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

(ii) By expanding $\left(z - \dfrac{1}{z}\right)^2$ show that $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$. [5]

(d) **(i)** By expanding $\left(z + \dfrac{1}{z}\right)^3$ using the binomial theorem, show that

$\cos^3\theta = a\cos 3\theta + b\cos\theta$,

where a and b are constants to be determined.

(ii) Find a similar expression for $\cos^4\theta$ in terms of cosines of multiples of θ. [7]

(e) **(i)** By expanding $\left(z - \dfrac{1}{z}\right)^3$ by the binomial theorem, show that

$\sin^3\theta = c\sin 3\theta + d\sin\theta$

where c and d are constants to be determined.

(ii) Find a similar expression for $\sin^4\theta$ in terms of cosines of multiples of θ. [6]

(f) By factorizing show that $\cos^4\theta - \sin^4\theta = \cos 2\theta$ and further show that this result is consistent with your answers to parts **(d)(ii)** and **(e)(ii)**. [3]

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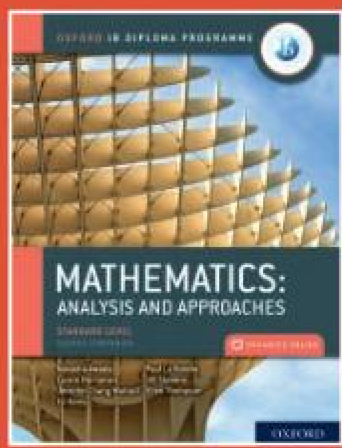

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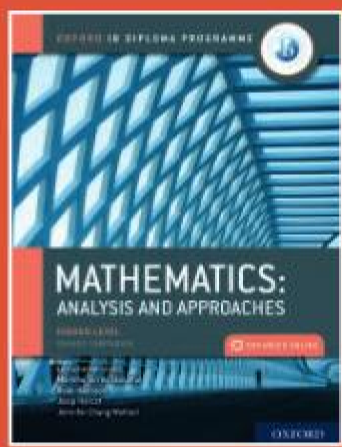

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