OXFORD IB DIPLOMA PROGRAMME

END OF CHAPTER TESTS

MATHEMATICS: ANALYSIS AND APPROACHES

STANDARD LEVEL

COURSE COMPANION

ENHANCED ONLINE

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OXFORD

1 From patterns to generalizations: sequences and series

Section A. A calculator is not allowed

1 The nth term of an arithmetic sequence is given by $u_n = 4 + 3n$

a Write down the first four terms.

b Find the common difference.

c Find the value of n if the n^{th} term is 109.

d Find the sum of the first 21 terms.

2 In an arithmetic sequence, the 40^{th} term is 144 and the sum of the first 40 terms is 2640. Find

a the first term

b the common difference.

3 Given the sequence $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, 2$:

a Find the common ratio.

b Find the eighth term.

4 A school theatre has 24 rows of seats. There are 18 seats in the first row and each subsequent row had two more seats than the previous row. What is the seating capacity of the theatre?

5 Find the first two terms of an arithmetic sequence where the sixth term is 21 and the sum of the first 17 terms is zero.

6 **a** How many terms are there in the expansion of $(a+b)^{10}$?

b Show the full expansions of $(3x+2y)^4$.

7 Evaluate

a $\sum_{n=1}^{5}\left(x^2+2\right)$ **b** $\sum_{i=2}^{6}2^i$

Section B. A calculator is allowed

8 A geometric sequence has a 2nd term of 6 and a 5th term of 162.

a Find the common ratio.

b Find the 10th term.

c Find the sum of the first 8 terms.

9 A rose bush is 1.67m tall when planted, and each week its height increases by 4%. How tall will it be after 10 weeks?

10 The Fibonacci sequence is named after Italian mathematician Leonardo of Pisa, who was known as Fibonacci.

In this Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, …,

a Find the 10th term of the Fibonacci sequence.

b Write a recursive formula for the Fibonacci sequence.

11 a Explain the condition for a geometric series to be convergent.

Given the series $10 - \frac{10}{3} + \frac{10}{9} - \dots$,

b Find the common ratio

c Find the sum to infinity.

12 Find the constant term in the expansion of $\left(x - \frac{2}{x^2}\right)^9$

13 Yosef drops a basketball from his bedroom window, which is 3m off the ground. After each bounce, the basketball comes back to 75% of its previous height. If it keeps on bouncing forever, what vertical distance, to the nearest metre, will it travel?

14 Tafari starts a job and deposits $500 from the first salary into a bank account producing 4% interest every month. Each month after that, Tafari deposits an additional $100.

a Calculate the amount Tafari has in the account at the end of each month for the first three months.

b Write a recursive formula for the amount of money in the account.

c Show that the formula for the amount of money in the account for year n is

$$500(1.04)^n + 2500\left((1.04)^n - 1\right)$$

d Use the formula to find the amount of money in Tafani's account after 24 months.

Answers

1 **a** 7, 10, 13, 16

b $d = 10 - 3 = 7$

c $4 + 3n = 109$

$3n = 105$

$n = 35$

d $S_n = \frac{n}{2}(u_1 + u_n)$

$u_1 = 7$, $n = 21$, $u_{21} = 4 + 3(21) = 67$

$S_{21} = \frac{21}{2}(7 + 67) = 777$

2 **a** $S_n = \frac{n}{2}(u_1 + u_n)$

$2640 = \frac{40}{2}(u_1 + 144)$

$132 = (u_1 + 144)$

$u_1 = -12$

b $u_n = u_1 + (n-1)d$

$144 = -12 + (40-1)d$

$39d = 156$

$d = 4$

3 Given the sequence $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, 2$

a $r = \frac{2/9}{2/27} = 3$

b $u_8 = \left(\frac{2}{27}\right)3^7 = \left(\frac{2}{3^3}\right)3^7 = 2 \times 3^4 = 162$

4 $s_n = \frac{n}{2}(2u_1 + (n-1)d)$

$s_{24} = \frac{24}{2}(2(18) + (24-1)2) = 984 \text{ seats}$

5 $u_n = u_1 + (n-1)d \qquad 21 = u_1 + 5d$

$s_n = \frac{n}{2}(2u_1 + (n-1)d) \qquad 0 = \frac{17}{2}(2u_1 + 16d)$

This gives the simultaneous equations

$u_1 + 5d = 21$

$u_1 + 8d = 0$

Solve to find $d = -7$ and $u_1 = 56$.

The first two terms are 56 and 49.

6 **a** 11

b Using the 4th row of Pascal's triangle, 1 4 6 4 1,

$1(3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)^1(2y)^3 + 1(2y)^4$

$= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

7 **a** $3 + 6 + 11 + 18 + 27 = 65$

b $4 + 8 + 16 + 32 + 64 = 124$

8 **a** $6r^3 = 162$

$r^3 = 27$

$r = 3$

b $u_n = (u_1)r^{n-1}$

$u_{10} = 2 \times 3^9 = 39366$

c $S_8 = \dfrac{2(3^8 - 1)}{3 - 1} = 6560$

9 $u_n = (u_1)r^{n-1}$

$u_{10} = (1.67)(1.04)^{10-1} = 2.38m$

10 **a** 55

b $u_{n+1} = u_n + u_{n-1}$

11 **a** $-1 < r < 1$

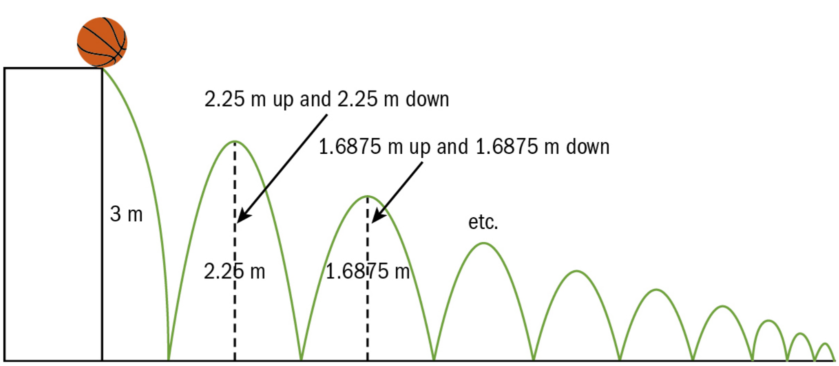
Given the series $10 - \dfrac{10}{3} + \dfrac{10}{9} - \ldots$

b $r = -\dfrac{10}{3} \div 10 = -\dfrac{1}{3}$

c $S_\infty = \dfrac{10}{1 - \left(-\dfrac{1}{3}\right)} = \dfrac{10}{4/3} = 7.5$

12 $\binom{9}{3} x^6 \left(\dfrac{-2}{x^2}\right)^3 = -672$

13



$u_1 = 3$, $r = 0.75$

$S_\infty = \dfrac{3}{1 - 0.75} = \dfrac{3}{0.25} = 12$

To account for upward and downward movement, multiply by 2, but the original drop of 3 only occurred once, so subtract 3 from the answer.

Vertical distance $= (12 \times 2) - 3 = 21$ metres

14 a Original investment = \$500

Value after 1 month = 1.04 (500) + 100 = \$620

Value after 2 months = 1.04 (1.04 (500) + 100) + 100 = \$744.80

Value after 3 months = 1.04 (1.04 (1.04 (500) + 100) + 100) + 100 ≈ \$874.59

b $u_n = u_{n-1}(1.04) + 100$

c Value after n years =

$$500(1.04)^n + 100(1.04)^{n-1} + 100(1.04)^{n-2} + 100(1.04)^{n-3} + \ldots + 100(1.04) + 100$$

$$= 500(1.04)^n + 100\left((1.04)^{n-1} + (1.04)^{n-2} + (1.04)^{n-3} + \ldots + (1.04) + 1\right)$$

$$= 500(1.04)^n + 100\left(\frac{(1.04)^n - 1}{1.04 - 1}\right) = 500(1.04)^n + 2500\left((1.04)^n - 1\right)$$

d $500(1.04)^{24} + 2500\left((1.04)^{24} - 1\right) = \$5190$

2 Representing relationships: introducing functions

Section A. A calculator is not allowed

1 Which of the following sets of ordered pairs are functions?

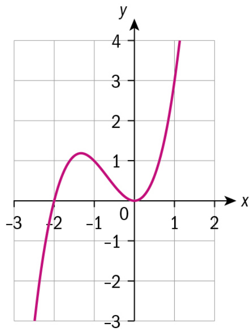
a {(1, 1), (2, 4), (3, 3), (3, 6), (4, 8)}

b {(−1, 3), (−2, 5), (−3, 7), (−4, 9), (−5, 11)}

c {(−2, −2), (−1, −1), (0, 0), (1, 1), (2, 2)}

2 Which of the following relations are functions?

a



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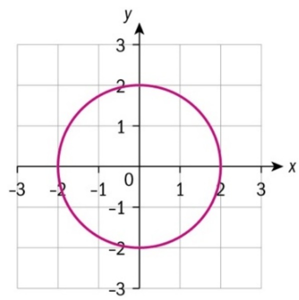
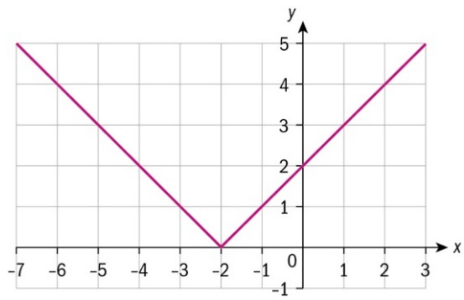
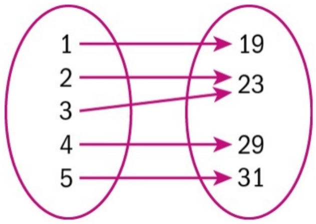


c



3 **a** Does this mapping diagram represent a function?



b State the domain and range.

4 $f(x) = 3x + 5$ and $g(x) = 2 - 2x$

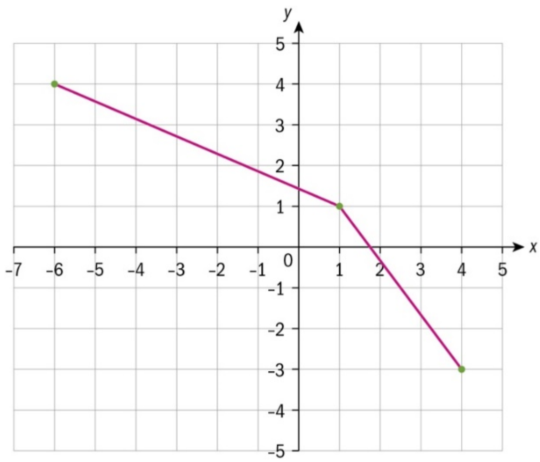
Find:

a $f(4)$ **b** $f^{-1}(2)$ **c** $f \circ g(x)$ **d** $g \circ f(-4)$

Solve:

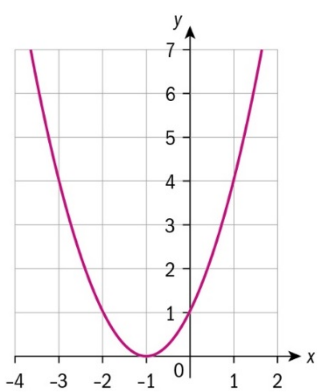
e $f(x) = -1$

5 Write down the domain and range of this function:

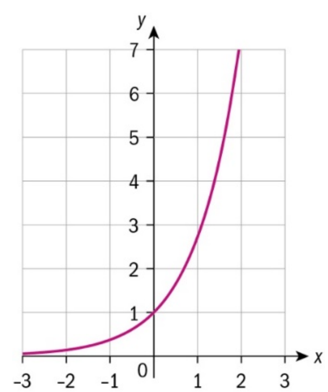


6 Which of these are one-to-one functions?

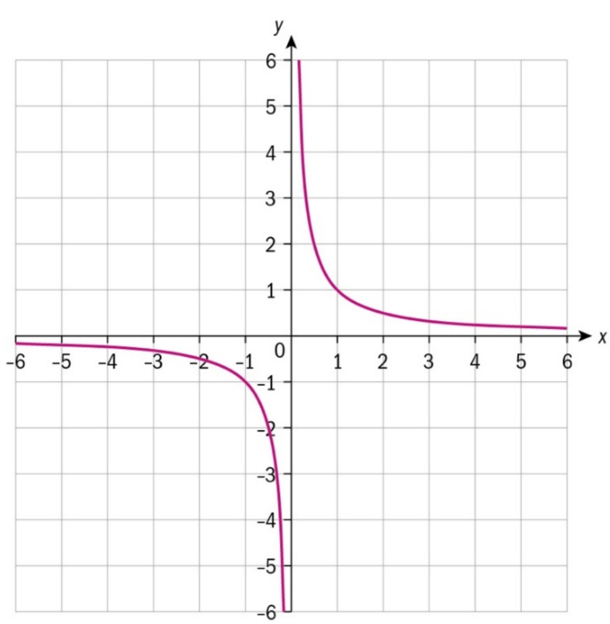
a



b



c



7 If $f(x) = \dfrac{x-1}{x-2}$, find $f^{-1}(x)$.

Section B. A calculator is allowed

8 Use your GDC to sketch these graphs. Write down the domain and range of each:

a $f(x) = x^2 + 2x - 3$ **b** $g(x) = \sqrt{x+2}$

9 If $l(x) = x^2$ and $m(x) = 2x - 3$:

a Find $(l \circ m)(x)$

b Sketch the function $(l \circ m)(x)$

c State its domain and range.

10 If $h(x) = 3x + 1$ and $p(x) = x^3$:

a Find x when $h(x) = 13$

b Find x when $p(x) = -27$

c Use a sketch to solve $h(x) = p(x)$.

11 Given $f(x) = 3x - 5$ and $g(x) = x - 2$,

a Find $f^{-1}(x)$

b Show that $(g^{-1} \circ f)(x) = 3x - 3$

Let $h(x) = \frac{f(x)}{g(x)}$ *and* $i(x) = x$

c Solve $h(x) = i(x)$ by sketching the functions.

12 Given $f(x) = x^3 - 3$:

a Find the inverse function f^{-1}

b If the point (a, b) lies on $f(x)$, what is the coordinate of its image on $f^{-1}(x)$?

c Sketch them both on the same axes.

d Solve $f(x) = f^{-1}(x)$.

Answers

1 **b** and **c**

2 **a** and **c**

3 **a** Yes

b The domain is 1, 2, 3, 4, 5. The range is 19, 23, 29, 31.

4 **a** $f(4) = 3(4) + 5 = 17$

b $x = 3y + 5$

$x - 5 = 3y$

$y = \dfrac{x-5}{3}$

$f^{-1}(x) = \dfrac{x-5}{3}$

$f^{-1}(2) = \dfrac{2-5}{3} = -1$

c $f \circ g(x) = 3(2 - 2x) + 5 = 11 - 6x$

d $g \circ f(-4)$

$f(-4) = 3(-4) + 5 = -7$

$g(-7) = 2 - 2(-7) = 16$

e $3x + 5 = -1$

$3x = -6$

$x = -2$

5 The domain is $-6 \le x \le 4$ and the range is $-3 \le y \le 4$

6 **b**

7 $x = \dfrac{y-1}{y-2}$

$x(y-2) = y - 1$

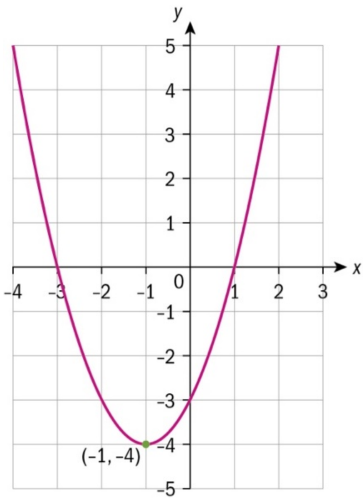
$xy - 2x = y - 1$

$xy - y = 2x - 1$

$y(x-1) = 2x - 1$

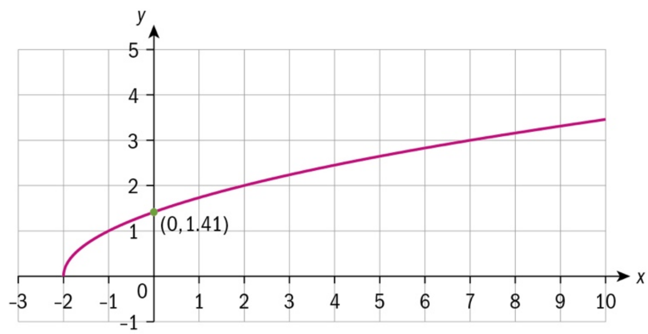
$f^{-1}(x) = \dfrac{2x-1}{x-1}$

8 **a** $f(x) = x^2 + 2x - 3$



Domain $x \in \mathbb{R}$ Range $y \ge -4$

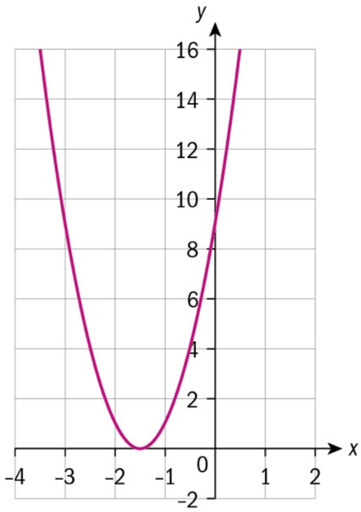
b $g(x) = \sqrt{x+2}$



Domain $x \ge -2$ Range $y \ge 0$

9 **a** $(l \circ m)(x) = (2x+3)^2 = 4x^2 + 12x + 9$

b



c Domain $x \in \mathbb{R}$ Range $y \geq 0$

10 **a** $3x + 1 = 13$

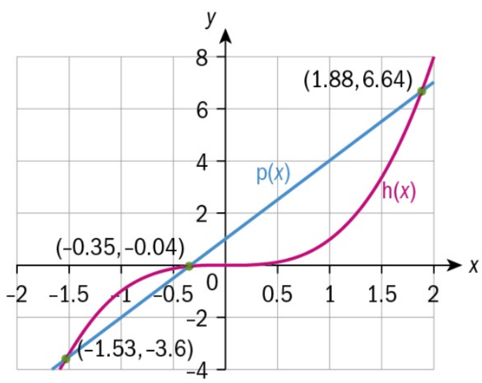
$3x = 12$

$x = 4$

b $x^3 = -27$

$x = -3$

c



$x = -1.53, -0.347, 1.88$

11 Given $f(x) = 3x - 5$ and $g(x) = x - 2$,

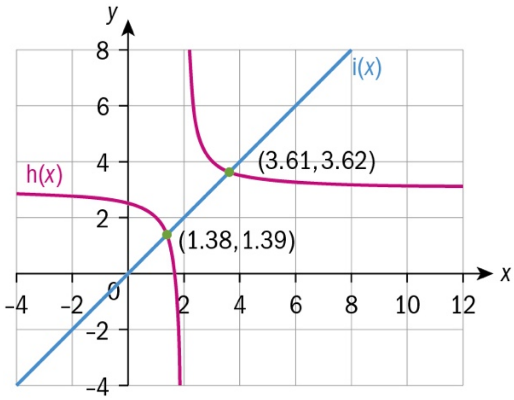
a $x = 3y - 5$

$x + 5 = 3y$

$f^{-1}(x) = \dfrac{x+5}{3}$

b $(g^{-1} \circ f)(x) = (3x - 5) + 2 = 3x - 3$

c



$x = 1.38, 3.61$

12 Given $f(x) = x^3 - 3$,

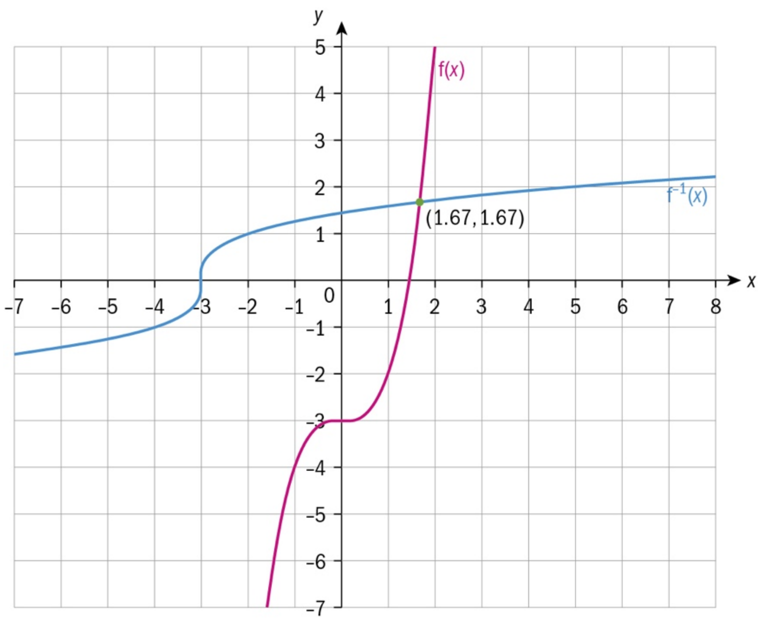
a $x = y^3 - 3$

$x + 3 = y^3$

$f^{-1}(x) = \sqrt[3]{x+3}$

b (b, a)

c



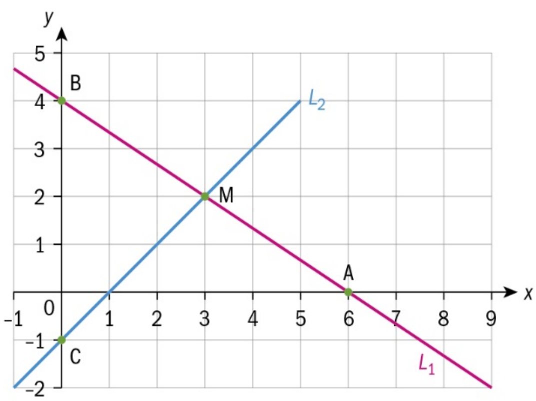
d $x = 1.67$

Modelling relationships: linear and quadratic functions

Section A. A calculator is not allowed

1 Find the equation of the line perpendicular to $y = 6 - 3x$ which passes through the point $(9, 10)$

2 The line L_1 shown on the set of axes below has equation $2x + 3y = 12$. L_1 cuts the x-axis at A and cuts the y-axis at B.



a Find the coordinates of A and B.

M is the midpoint of the line segment [AB].

b Find the coordinates of M.

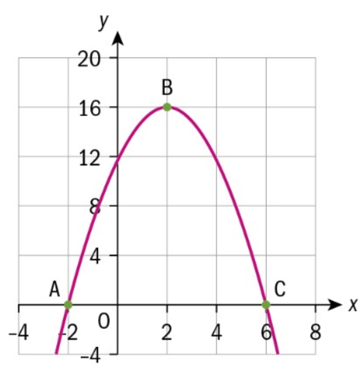
The line L_2 passes through the point M and the point C (0, −1).

c Find the equation of L_2.

d Find the equation of the line parallel to L_2 that passes through the point A

3 **a** Factor the expression $-x^2 + 4x + 12$

This is the graph of $-x^2 + 4x + 12$



b Find the coordinates of A, B and C.

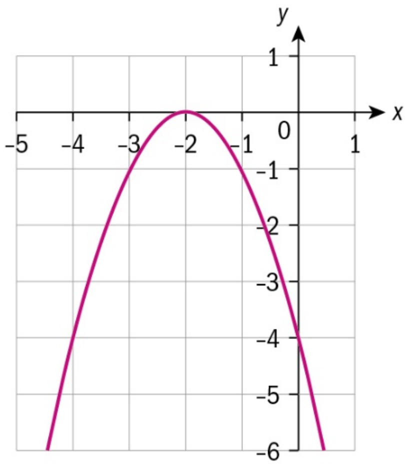
4 Consider $f(x) = x^2 + bx + c$ which has an axis of symmetry at $x = -\frac{1}{2}$, where the distance between the zeros is 7 units.

a Find the zeros of f.

b Find the values of b and c.

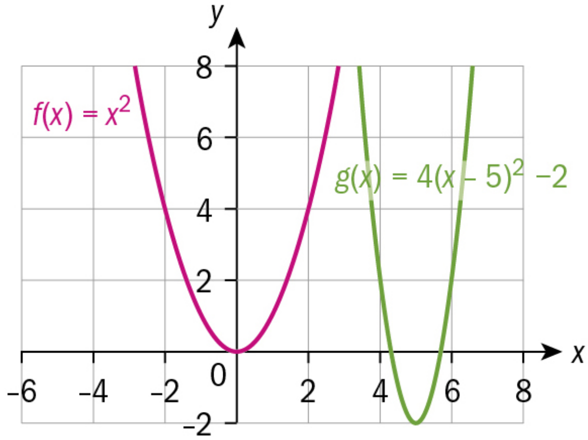
5 Copy and complete the table by indicating the correct box for this parabola with equation

$y = ax^2 + bx + c$



	Positive	**Zero**	**Negative**
a			
c			
$b^2 - 4ac$			
b			

6 The diagram shows parts of the graphs of $f(x) = x^2$ and $g(x) = 4(x - 5)^2 - 2$



The graph of *f* may be transformed into the graph of *g* by these transformations.

A vertical stretch with scale factor *s*, followed by a horizontal translation of *h* units and then a vertical translation of *v* units.

Write down the values of *s*, *h* and *v*.

Section B. A calculator is allowed

7 Find the value of *b* for which $3x^2 + bx + 5 = 0$ will have two different real roots.

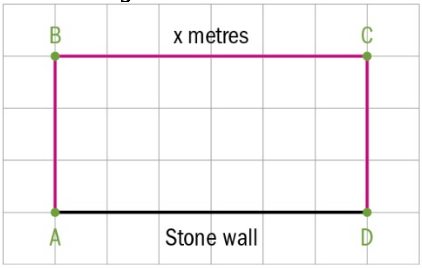
Give your answer correct to 2 decimal places.

8 **a** Draw a sketch showing the functions $f(x) = x^2$ and $g(x) = 2^x$.

b Label their intersection points

c Write down the solution to the equation $x^2 = 2^x$

9 Farmer Phil has 80 meters of chicken wire available for constructing a chicken enclosure against an existing stone wall. Phil wants the enclosure to be rectangular as shown in the diagram.



Let the length of the Phil's enclosure be x metres.

a Show that the length of side AB is $(40-\frac{1}{2}x)$

b Find the area of the enclosure in terms of x

c Sketch a graph of the function to find the maximum area that Phil can enclose and state the length, width and area.

10 The height (h metres) of a ball t seconds after it is thrown is given by the formula

$h(t)=10t-4.9t^2+1.5$

a Find the initial height of the ball.

b Sketch the graph of h.

Use your graph to write down the following

c The maximum height

d The time when the ball is at the maximum height.

e The time when the ball will hit the ground.

f Draw a line on your graph and find between what times the ball will be more than 5m off the ground.

Answers

1 If m = −3, the perpendicular slope is $\frac{1}{3}$

$y - y_1 = m(x - x_1)$

$y - 10 = \frac{1}{3}(x - 9)$

$y - 10 = \frac{1}{3}x - 3$

$y = \frac{1}{3}x + 7$

2 **a** For A, $y = 0$ therefore $2x = 12$ and $x = 6$. A is (6,0)

For B, $x = 0$ therefore $3y = 12$ and $y = 4$. B is (0,4)

b $M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) = \left(\frac{6+0}{2}, \frac{0+4}{2}\right) = (3,2)$

c $y = mx + b$

$y = mx - 1$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - -1}{3 - 0} = 1 \Rightarrow y = x - 1$

d $y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 6)$

$y = x - 6$

3 **a** $-x^2 + 4x + 12 = (-x + 6)(x + 2)$

b Find A and C by letting $(-x + 6)(x + 2) = 0$, which gives $x = -2, 6$

$A(-2,0), C(6,0)$

Point is the vertex where $x = -\frac{b}{2a} = -\frac{4}{-2} = 2$ and $y = -(2)^2 + 4(2) + 12 = 16$

$B(2,16)$

4 **a** The zeroes are equally spread around the axes.

zeros $= -\frac{1}{2} \pm \frac{7}{2} = -4$ and 3

b $(x + 4)(x - 3) = x^2 + x - 12 \quad b = 1, c = -3$

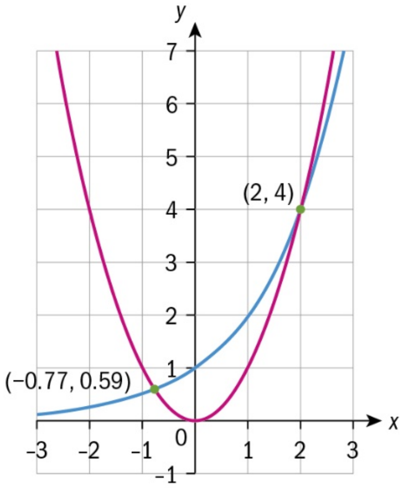
5

	Positive	**Zero**	**Negative**
a			X
c			X
$b^2 - 4ac$		X	
b	X		

6 $s = 4; h = 5; v = -2$

7 $b^2 - 4ac > 0 \Rightarrow b^2 - (4 \times 3 \times 5) > 0 \Rightarrow b^2 > 60 \Rightarrow b > 7.75, b < -7.75$

8 a,b



c $x = -0.766, 2$

9 a $AB + BC + CD = 80$

$AB = CD$ and $BC = x$

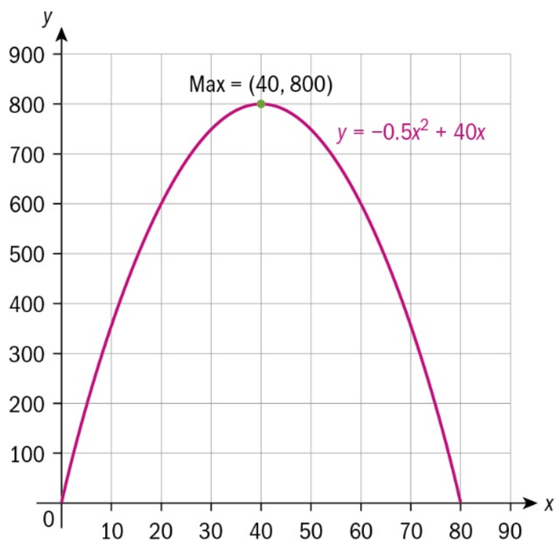
$2AB + x = 80$

$2AB = 80 - x$

$AB = \frac{80 - x}{2} = 40 - \frac{1}{2}x$

b Area $=$ $\text{AB} \times \text{BC} = \text{x}\left(40 - \frac{1}{2}x\right) = 40x - \frac{1}{2}x^2$

c

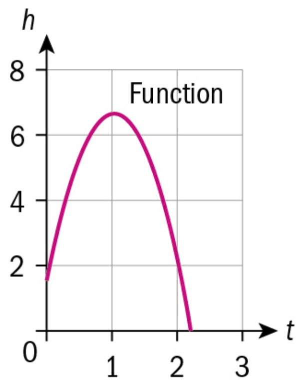


Length 40m, width 20m, area 800m

10 $h(t) = 10t - 4.9t^2 + 1.5$

a Initial height when $t = 0$. $h(0) = 10(0) - 4.9(0)^2 + 1.5 = 1.5m$

b

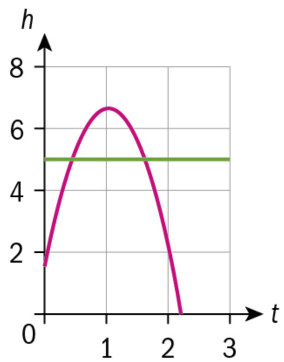


c The maximum height is the y value of the vertex which is 6.60m.

d The time when the ball is at the maximum height is the x value of the vertex which is 1.02secs

e The time when the ball will hit the ground is the x intercept of 2.18 secs.

f



The ball will be more than 5m off the ground between 0.449 secs and 1.59 secs.

4 Equivalent representations: rational functions

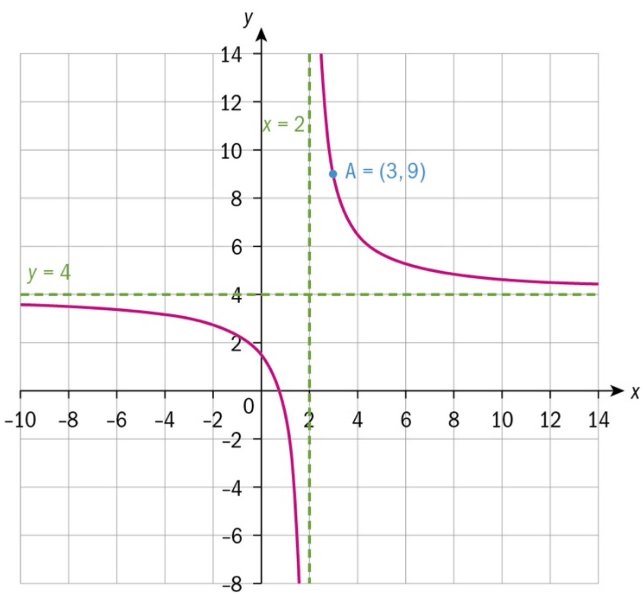
Section A. A calculator is not allowed

1 Find the reciprocals of

a 7 **b** -3 **c** $\frac{6}{11}$ **d** $-4\frac{2}{3}$

e $\frac{3\pi}{2}$ **f** Show that 3.5 and $\frac{2}{7}$ are reciprocals

2 This function is defined as $f(x) = a + \frac{b}{x-c}$



a Write down the values of a and c.

b Given that the curve passes through the point $(3,9)$, find the value of b.

3 Consider the function $y = \frac{4x-3}{2x+1}$:

a Find the equations of the horizontal and vertical asymptotes.

b State the domain and range of the function.

4 Consider the function $y = \frac{1}{x-2} - 3$:

a Write down the equations of the asymptotes of the function.

b Find the axes intercepts.

c Sketch the function $-2 \le x \le 4, -8 \le y \le 4$.

Show the asymptotes as broken lines.

Section B. A calculator is allowed

5 Let $f(x) = \frac{x-1}{x+1} - 3$

a Find $f^{-1}(x)$

b Write down the x-intercept and the y-intercept of the graph of f(x).

c Write down the equations of the asymptotes.

d Sketch the graph of f(x) for $-4 \le x \le 4$ and $-5 \le y \le 8$, including any asymptotes.

6 Travel expert Craig has a drive storing 60 Gb of holiday videos that he takes with him.

a Copy and complete this table, which shows the number of videos (x) and the size (y) that Craig can store on his drive.

x	1	2	3	4	5	6	8	10	12	15	30	60
y	60									4		

b Write down a function for y, the number of videos that Craig can take with him, in terms of the individual size x.

c Use your GDC to sketch a graph of the function.

Craig wants to have space to take 35 videos with him.

d Show the line x = 35 on your sketch and label the intersection point A.

e What is the average size for Craig to take 35 videos?

7 $f(x) = \frac{3-2x}{2-x}$ and $g(x) = e^{(x-3)} - 2$

a Sketch both functions on the same axes with $-1 \le x \le 6$ and $-7 \le y \le 10$.

b Write down the x intercepts of f and g.

c Write down the solution to the equation $f(x) = g(x)$.

d For what value of x is $f(x) > g(x)$?

8 The concentration, C (in mg/dl), of an antibiotic in a patient's bloodstream is modelled by $C(t) = \frac{20t}{t^2+5}$ where t is the time (in hours) after taking the antibiotic.

a Sketch the graph of $C(t)$ for the first 12 hours of medication.

b What is the concentration 4 hours after taking the antibiotic?

c The antibiotic must have 2 or more mg/dl in the bloodstream to be effective. If the first dose was taken at 8am, at what time must the second dose be taken after 12 noon?

Answers

1 **a** $\frac{1}{7}$ **b** $-\frac{1}{3}$ **c** $\frac{11}{6}$ **d** $-\frac{3}{14}$

e $\frac{2}{3\pi}$ **f** $3.5 = \frac{7}{2}\ \frac{7}{2} \times \frac{2}{7} = 1$

2 **a** $a = 4, c = 2$

b $9 = 4 + \frac{b}{3-2}$

$9 = 4 + b$

$b = 5$

3 **a** Horizontal asymptote when $y = \frac{4}{2}, y = 2$. Vertical asymptote when $2x + 1 = 0, x = -1$

b $x \in \mathbb{R}, x \neq -1, y \in \mathbb{R}, y \neq 2$

4 **a** $x = 2, y = -3$

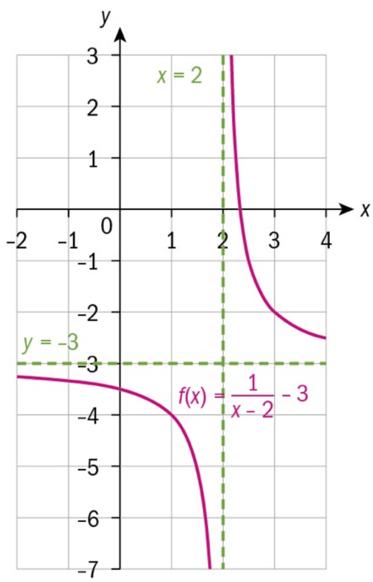
b x intercept when $y = 0$.

$\frac{1}{x-2} - 3 = 0\ \frac{1}{x-2} = 3\ x - 2 = \frac{1}{3}\ x = 2\frac{1}{3}$

y intercept when $x = 0$

$y = \frac{1}{0-2} - 3 = -\frac{1}{2} - 3 = -3\frac{1}{2}$

c



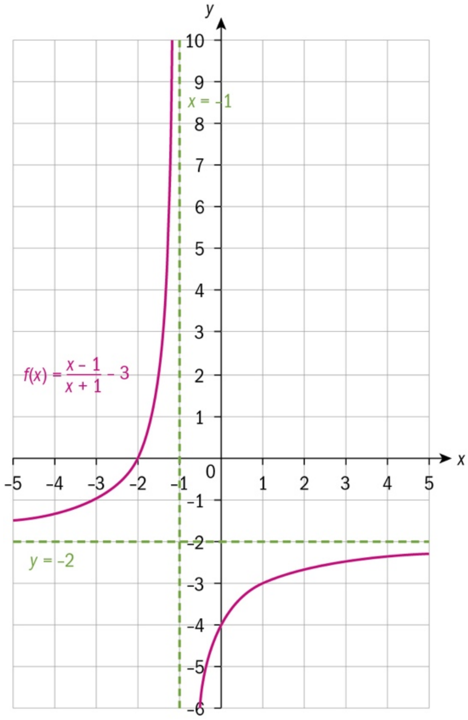
5 **a** $x = \frac{y-1}{y+1} - 3 \Rightarrow x + 3 = \frac{y-1}{y+1} \Rightarrow (x+3)(y+1) = (y-1) \Rightarrow xy + x + 3y + 3 = y - 1$

$\Rightarrow xy + 3y - y = -1 - x - 3 \Rightarrow xy + 2y = -x - 4 \Rightarrow y(x+2) = -x - 4$

$f^{-1}(x) = \frac{-x-4}{x+2}$

b x intercept at $(-2, 0)$ and y intercept at $(0, -4)$

c $x = -2$ and $y = -1$



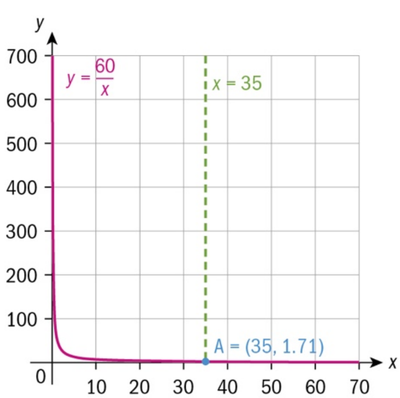
d

6 **a**

x	1	2	3	4	5	6	8	10	12	15	30	60
y	60	30	20	15	12	10	7.5	6	5	4	2	1

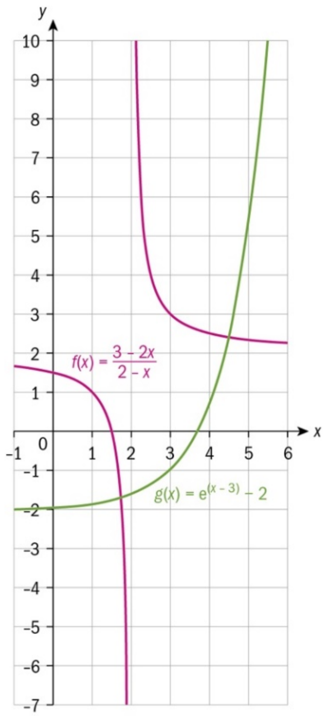
b $y = \frac{60}{x}$

c and d



e 1.71 Gb

7 a



b The x intercepts of f and g are 1.5 and 3.69

c $f(x) = g(x)$ at the x values of the intercepts. $x = 1.73$ and 4.48.

d $f(x) > g(x)$ where $f(x)$ is above $g(x)$. $x < 1.73$ and $2 < x < 4.48$.

8 a

b Draw the line $t = 4$ on the calculator and find the intersection point. $3.81\,\text{mg/dl}$

b and c

c Draw the line $C(t) = 2$ and find the t value of the second intersection point.

$t = 9.72\text{hrs} = 9\text{hrs } 43.2\text{mins}$

$8\text{am} + 9\text{hrs } 43\text{mins}$ makes the next dose at $5:43\text{pm}$

5 Measuring change: differentiation

Section A. A calculator is not allowed

1 Differentiate the following functions:

a $f(x) = x^3 - 4x^2 - 6x + 2$

b $g(x) = \frac{2}{x^3}$

c $h(x) = \sqrt{x}$

d $i(x) = \sqrt{x^2 + 5}$

2 **a** Express $\lim_{x \to 2} f(x) = 5$ in a sentence.

b Write down the limit of the sequence $0.6, 0.66, 0.666, 0.6666, \ldots$

3 Differentiate the following:

a $f(x) = \frac{5x + 3}{x^2 + 1}$

b $g(x) = (5x + 1)\sqrt{2x + 1}$

4 Consider the parabola $y = \frac{1}{2}x^2 - x - 4$

a Find $\frac{dy}{dx}$.

b Find the equation of the tangent to the curve where $x = 6$.

c Find the equation of the normal where $x = 6$ in the form $ax + by = c$, where a, b and c are positive integers.

5 The velocity, in ms^{-1}, of a train approaching a station is given by $v(t) = 20 - 4t$, where t is the time in seconds.

a What was the initial speed?

b When will the train be at rest in the station?

c What was its acceleration after 1.5secs?

6 Find the intervals for when the function $f(x) = x^3 - 12x - 1$ is increasing and decreasing.

Section B. A calculator is allowed

7 The equation parabola is $f(x) = ax^2 + bx$. The gradient of the tangent to the curve at point P (1,3) is 8. Find the values of a and b.

8 **a** Sketch the graph of $y = \frac{x + 2}{x - 2} + 3$.

b Write down the equations of any asymptotes.

c Find $\lim_{x \to \infty} \frac{x + 2}{x - 2} + 3$.

9 Consider the function $f(x) = 3x^4 - 4x^3 - 2$.

a Find all turning points, and determine their nature (you should justify your answers).

b Find the coordinates and nature of any inflexion points.

c Sketch the graph of *f*, with $-1 \le x \le 2$ and $-4 \le y \le 8$ labelling a horizontal inflexion A, minimum point B and a non-horizontal inflexion C.

10 The function $f(x) = \frac{10(x-1)}{x^2}$ has $f'(x) = -\frac{10(x-2)}{x^3}$ and $f''(x) = \frac{20(x-3)}{x^4}$

a Find the zeros of f(x).

b Find the coordinates of the local maximum point.

c Find the intervals where f(x) is concave up.

Given f(x) has the x and y axes as asymptotes,

d Sketch the function for $x \ge 0$.

Answers

1 **a** $f'(x) = 3x^2 - 8x - 6$

b $g(x) = \frac{2}{x^3} = 2x^{-3}$

$g'(x) = -6x^{-4}$ or $-\frac{6}{x^4}$

c $h(x) = \sqrt{x} = x^{\frac{1}{2}} \ldots$

$h'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

d $u = x^2 + 5 \Rightarrow y = u^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 2x$

$\frac{dy}{dx} = x\left(x^2 + 5\right)^{-\frac{1}{2}}$

or $\frac{x}{\sqrt{x^2 + 5}}$

2 **a** The limit of $f(x)$, as x approaches positive 2, is 4.

b $\lim_{x \to \infty} u_n = \frac{2}{3}$

3 **a** $f(x) = \frac{5x + 3}{x^2 + 1}$

$u = 5x + 3,\ v = x^2 + 1$

$u' = 5,\ v' = 2x$

$f'(x) = \frac{vu' - uv'}{v^2}$

$f'(x) = \frac{\left(x^2 + 1\right)(5) - (5x + 3)(2x)}{\left(x^2 + 1\right)^2}$

$f'(x) = \frac{5x^2 + 5 - 10x^2 + 6x}{\left(x^2 + 1\right)^2}$

$f'(x) = \frac{-5x^2 + 6x + 5}{\left(x^2 + 1\right)^2}$

b $g(x) = (5x + 1)\sqrt{2x + 1}$

Use the chain rule for $v'(x)$

Let $y = \sqrt{2x + 1}$

$u = 2x + 1$

$y = u^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 2$

$\frac{dy}{dx} = (2x + 1)^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{2x + 1}}$

Using the product rule:

$u = (5x + 1)\ v = \sqrt{2x + 1}$

$u' = 5\ v' = (2x + 1)^{-\frac{1}{2}}$

$g'(x) = uv' + vu'$

$g'(x) = \frac{5x + 1}{\sqrt{2x + 1}} + 5\sqrt{2x + 1}$

4 **a** $\frac{dy}{dx} = x - 1$

b When $x = 6, y = \frac{1}{2}(6)^2 - 6 - 4 = 8$. The coordinate is $(6, 8)$

When $x = 6$, the gradient is $6 - 1 = 5$

$y - y_1 = m(x - x_1)$

$y - 8 = 5(x - 6)$

$y - 8 = 5x - 30$

$y = 5x - 22$

c When $x = 6$, the gradient of the normal is $-\frac{1}{5}$

$y - y_1 = m(x - x_1)$

$y - 8 = -\frac{1}{5}(x - 6)$

$y - 8 = -\frac{1}{5}x + \frac{6}{5}$

$y = -\frac{1}{5}x + \frac{46}{5}$

$\frac{1}{5}x + y = \frac{46}{5}$

$x + 5y = 46$

5 **a** Initial speed = 20 – 4(0) = 20 ms^{-1}

b $20 - 4t = 0, t = 5$

At rest after 5 seconds.

c Acceleration = $V'(t)$ = $-4ms^{-2}$

6 $f(x) = x^3 - 12x - 1$

$f'(x) = 3x^2 - 12$

$f'(x) = 3(x^2 - 4)$

$f'(x) = 3(x + 2)(x - 2)$

$f'(x) = 0$ when $x = -2, 2$

Choosing a value just below -2 such as -2.1

$f'(-2.1) = 3(-2.1)^2 - 12 = 1.23$

Choosing a value between -2 and 2 such as 0 ,

$f'(0) = 3(0)^2 - 12 = -12$

Choosing a value just above 2 such as 2.1

$f'(2.1) = 3(2.1)^2 - 12 = 1.23$

x	$x \le -2$	$-2 < x < 2$	$x \ge 2$
Sign of *f′*(*x*)	+	–	+
***f*(*x*)**	increasing	decreasing	increasing

$f(x)$ is increasing when $x \le -2$ and $x \ge 2$

$f(x)$ is decreasing when $-2 < x < 2$

7 Where $x = 1$ and $y = 3$, $3 = a + b$,

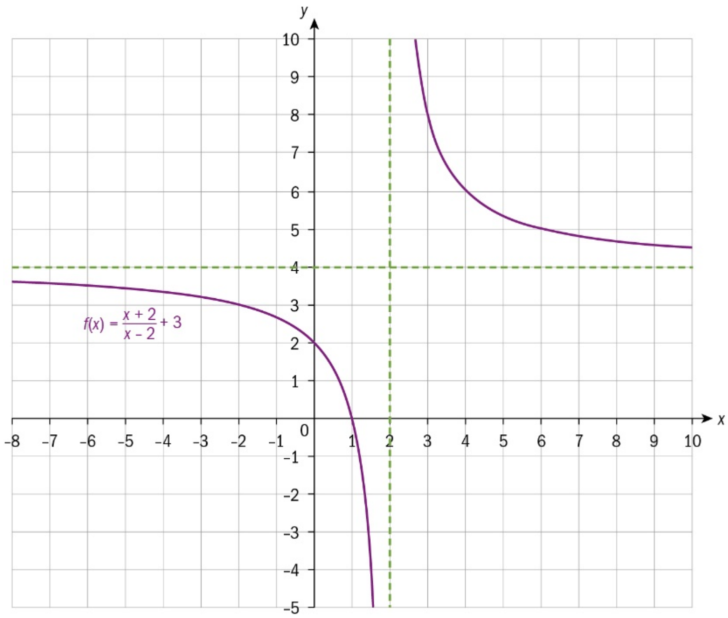
$f'(x) = 2ax + b$

$8 = 2a + b$

Solving simultaneously,

$a = 5$ and $q = -2$

8 a



b $x = 2, y = 4$

c $\lim\limits_{x\to\infty} \dfrac{x+2}{x-2} + 3 = 4$

9 a $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

$f'(x) = 0$, when $x = 0, 1$.

When $x = 0$

$f''(x) = 36x^2 - 24x$

$f''(0) = 36(0)^2 - 24(0) = 0$

The y value is $f(0) = 3(0)^4 - 4(0)^3 - 2 = -2$.

Horizontal inflexion point at $(0, -2)$

When $x = 1$

$f''(1) = 36(1)^2 - 24(1) = 12$

The y value is $f(1) = 3(1)^4 - 4(1)^3 - 2 = -3$.

Minimum point at (1, −3).

b $f''(x) = 0$ at an inflexion point.

$f''(x) = 36x^2 - 24x = 12x(3x-2) = 0$ when $x = 0, \frac{2}{3}$

$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^4 - 4\left(\frac{2}{3}\right)^3 - 2 = -\frac{70}{27}$ *or* -2.59.

There is a non-horizontal point of inflexion at $\left(\frac{2}{3}, 2.59\right)$.

c

10 a $10(x-1)=0$ when $x=1$.

b Maximum point where $f'(x)=0$, $-10(x-2)=0$, $x=2$

It is a maximum point when

$f''(x)<0$

$f''(2)=\dfrac{20(2-3)}{2^4}=-\dfrac{5}{4}$

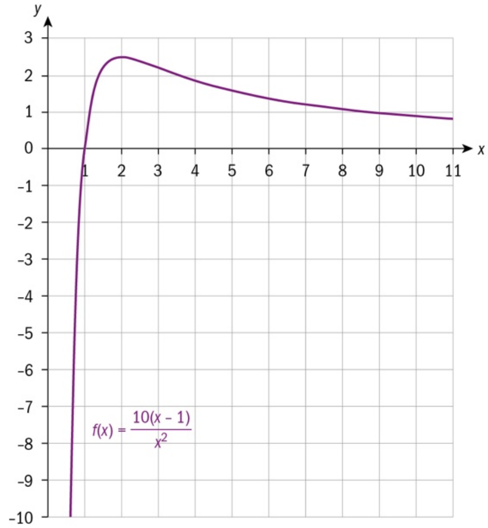
The y value is $f(2)=\dfrac{10(2-1)}{2^2}=2.5$.

The local maximum point is $(2, 2.5)$.

c $f(x)$ is concave up where $f''(x)>0$

$20(x-3)>0$, $x>3$

d



6 Representing data: statistics for univariate data

Section A. A calculator is not allowed

1 Classify the following data sets using the words

Qualitative. Quantitative. Discrete. Continuous.

a The number of children in your family.

b Your favourite color.

c The height of the students in your class.

d The time that it takes to go home from school.

2 Joao is studying the average weight of fish caught at a port.

Choose from convenience, simple random, systematic, stratified or quota to classify each of the following sampling techniques that Joao might use.

a Joao selects any 100 fish at random.

b A random fish is chosen and then every 20th fish after that until Joao has 100 fish.

c The fish consist of 80% round fish and 20% flat fish. Joao selects 80 round and 20 flat fish.

d A sample of 100 fish is taken by organizing the fish by 5 species and then taking 20 from each species

3 This table shows the ages of my relatives and the number of coffee drinkers.

Ages (x)	**Number of coffee drinkers (f)**
$20 \le x < 30$	5
$30 \le x < 40$	4
$40 \le x < 50$	3
$50 \le x < 60$	2
$60 \le x < 70$	3

a Find the mean coffee drinking age.

b Represent this data on a histogram.

4 Bilbo Air records the amount of time, in minutes. that its planes are delayed.

Delay time	Number of flights
$0 \le x < 30$	5
$30 \le x < 60$	15
$60 \le x < 90$	33
$90 \le x < 120$	21
$120 \le x < 150$	11
$150 \le x < 180$	7
$180 \le x < 210$	5
$210 \le x < 240$	3

a Construct a cumulative frequency table for Bilbo Air's data.

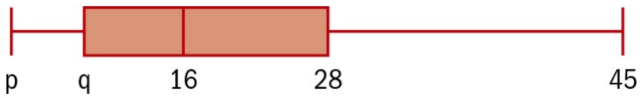
b Draw a cumulative frequency graph to represent the data.

Use your cumulative frequency graph to estimate

c The median delay time.

d The interquartile range of the delay times.

5 This diagram is a box plot for a data set.



a Write down the median.

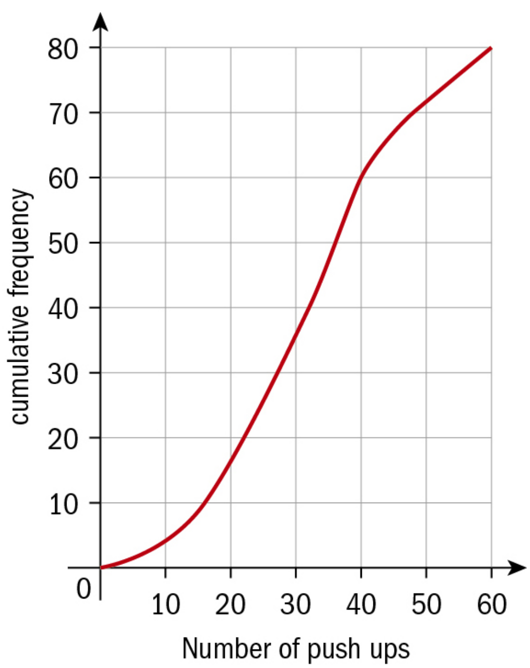
b If the range is 42 and the interquartile range is 16, find the values of p and q.

6 The cumulative frequency diagram shows the number of push-ups done in one minute by a group of 80 athletes.

a Write down the median.

b Find the interquartile range

c The top 10% of athletes completed at least how many push-ups?



Section B. A calculator is allowed

7 Yuyu has taken 5 class tests and her average is 88. 6%. How much must she score in the next test to finish with an average of 90%?

8 The table shows the numbers of bicycles that students have owned.

Bicycles	0	1	2	3	4
Students	28	24	20	17	11

a Is the data discrete or continuous?

b Write down the mode.

c Find the mean number of bicycles per student.

d Find the median.

9 Khabib has a papaya orchard. On a Sunday he harvests 72 green papaya and 28 yellow. The mean weight of a green papaya is 1.79kg and the mean weight of a yellow papaya is 1.62kg.

Find the mean weight of Khabib's 100 papayas.

10 Here is a group of university friends and their ages

Age (yrs)	19	20	21	22	23
f	6	8	7	11	9

Find

a the mean

b the standard deviation

c the variance

If the friends meet up for an alumni fund raiser after x years, what will be their

d mean

e standard deviation

Answers

1 **a** Quantitative. Discrete. **b** Qualitative.

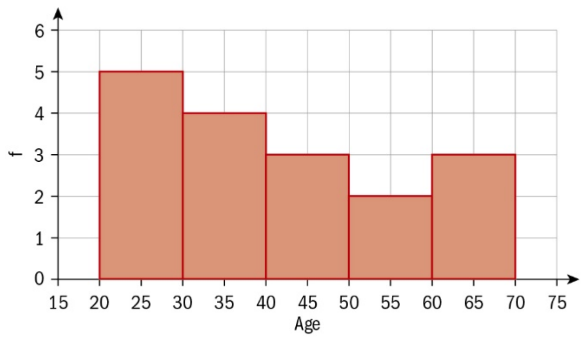
c Quantitative. Continuous. **d** Quantitative. Continuous.

2 **a** Simple random **b** Systematic

c Quota **d** Stratified.

3 **a** $\bar{x} = \frac{\sum fm}{\sum f} = \frac{(25\times 5)+(35\times 4)+(45\times 3)+(55\times 2)+(65\times 3)}{5+4+3+2+3} = \frac{705}{17} = 41.5$

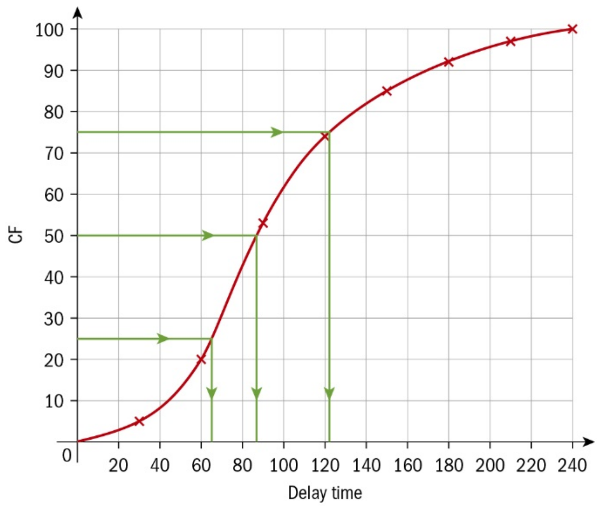
b



4 **a**

Delay time	Cumulative frequency
$x < 30$	5
$x < 60$	20
$x < 90$	53
$x < 120$	74
$x < 150$	85
$x < 180$	92
$x < 210$	97
$x < 240$	100

b



c The median delay time is 87 mins.

d $IQR = Q_3 - Q_1 = 123 - 65 = 58$ mins

5 **a** 16

b p = 45 − 42 = 3, q = 28 − 16 = 12

6 **a** The median is 32

b $IQR = Q_3 - Q_1 = 40 - 22 = 18$

c 10% of 80 = 8.

The 72nd athlete completed 48 push-ups.

7 Yuyu's total $= 88.6 \times 5 = 443$

To finish with 90%, Yuyu needs a total of $90 \times 6 = 540$

Yuyu must score $540 - 443 = 97\%$ in the next test.

8 **a** Discrete

b 1 bicycle.

c $\text{Mean} = \dfrac{\text{Total number of bicycles}}{\text{Total number of students}} = \dfrac{163}{100} = 1.63\text{bicycles}$

d $\text{Median} = \left(\dfrac{n+1}{2}\right)^{th} = \left(\dfrac{100+1}{2}\right)^{th} = 50.5^{th} = 1\text{bicycle}$

9 $\text{Mean} = \dfrac{(72 \times 1.79) + (28 \times 1.62)}{100} = 1.74\text{kg}$

10 **a** $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{870}{41} = 21.2$

b 1.37

c the variance$= (1.370636\ldots)^2 = 1.88$

d 21.2 + x

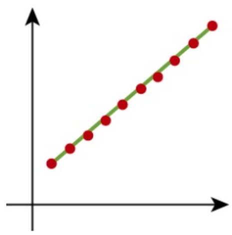
e 1.37

7 Modelling relationships between two data sets: statistics for bivariate data

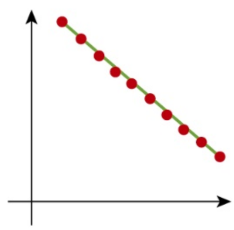
Section A. A calculator is not allowed

1 Match the diagram with the correlation coefficient

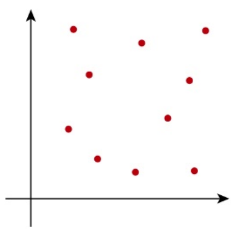
A



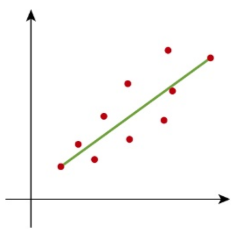
B



C



D

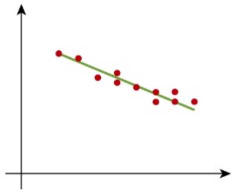


i -1 **ii** 0 **iii** 0.8 **iv** 1

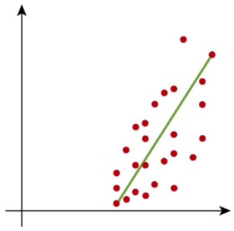
2 Describe the correlation for each diagram using the words

Strong, moderate weak, positive, negative, linear, non linear, strong, moderate weak, no correlation.

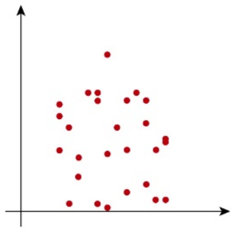
A



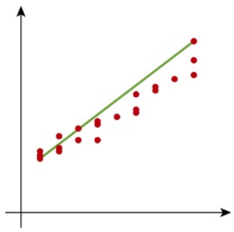
B



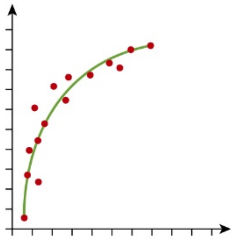
C



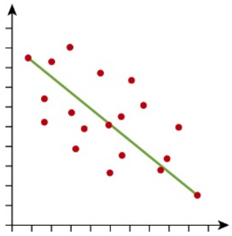
D



E



F



3 The scatter diagram shows the age of a family of pet rabbits and their weights in kg.

a Describe the correlation.

b What should you do with the outlier? Explain your reasons.

4 The production cost at Luigi's pizzeria is modelled by

$c(p) = 6p + 50$

where c is the production cost in dollars and p is the number of pizzas made.

a How much does it cost Luigi if the shop does not produce any pizzas?

b Interpret the meaning of the value of the gradient, 6.

5 A basketball coach recorded the average number of training hours per week for him team and the average change in the number of points scored in a season.

Hours (x)	0	1	2	3	4	5
Points change (y)	-5	0	1	10	15	18

a Show the data on a scatter plot

b Find the mean point and indicate this on your scatter plot by the label M.

c Draw a line of best fit through your data.

d Describe the correlation.

e What can you say about the number of hours trained and the number of points scored?

Section B. A calculator is allowed

6 Ten students recorded how many hours of exercise they have at the weekend and their weights in kg.

Hours (x)	6	2	7	1	0	3	10	8	9	4
Weight (y)	80	60	70	50	90	80	70	100	55	60

Write down:

a the equation of the regression line

b the r value.

c Is it appropriate to estimate the weight of a student who does 5 hrs of exercise at the weekend? Give reasons.

7 Here are the scores for 10 of the 12 students in my class for mathematics and science.

Mathematics (x)	90	66	84	75	90	88	69	95	73	81
Science (y)	73	60	79	67	78	67	55	82	59	80

a Write down the r value.

b What does this tell you about the correlation?

c Write down the equation of the y on x regression line.

Sara was absent for the science test but scored 80 in mathematics.

d Estimate her score using the equation of the regression line.

e Is this a valid estimate? Give reasons.

f Can the regression line be used to estimate the score of a student who scored 10% in mathematics? Give reasons.

g Find the equation of the regression line of x on y.

h What can this equation be used for?

8 I asked the first ten people that I saw this morning “How many pairs of shoes do you own”

Age (x)	15	24	42	13	56	16	14	20	6	12
Pairs of shoes (y)	2	7	5	4	6	8	4	8	2	6

a What type of sampling method did I use?

b Write down the r value.

c Explain what a positive value for the coefficient of correlation indicates.

d Write down the linear regression equation of y on x in the form $y = ax + b$

e Use your equation to determine the number of pairs of shoes that an 18 yr old would have.

f Can your answer in part e be considered reliable? Give a reason for your answer.

9 Vin thinks that price of a car is related to its age and collects the following data where the age of the car is in years and the cost is in thousands of dollars.

Age of the car (x)	0	5	10	15	20	25	30	35
Cost (\$1000) (y)	20	15	12	8	1	6	13	18

a Show this data on a scatter plot.

b Can you draw a single line of best fit to represent this data? Explain your answer.

c Find two piecewise linear functions that best represent the data.

Answers

1 A 1 **B** –1 **C** 0 **D** 0.8

2 **A** Strong negative linear **B** Weak positive linear

C No correlation **D** Moderate positive linear

E Non linear **F** Weak, negative linear

3 **a** Moderate positive correlation

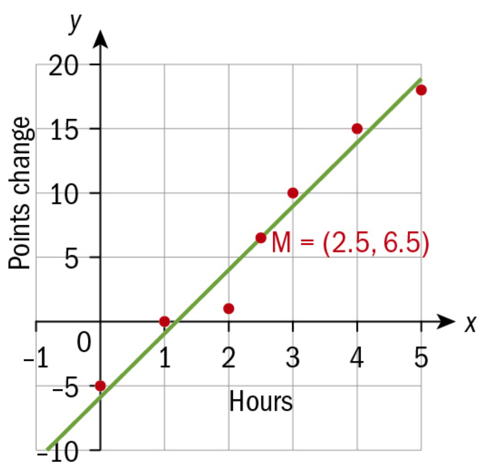
b Discard the outlier. It appears to be an error.

4a $50

b Each pizza costs $6 to produce.

5a, b, c on the graph.

b $(\bar{x}, \bar{y}) = \left(\frac{0+1+2+3+4+5}{6}, \frac{-5+0+1+10+15+18}{6}\right) = (2.5, 6.5)$



d There is a strong, positive linear correlation.

e As the number of hours of practice increases the number of points scored increases.

6 **a** $y = 0.364x + 69.7$

b 0.0795

c Is it not appropriate to estimate the weight of a student who does 5 hrs of exercise at the weekend because the r value is weak.

7 **a** 0.812

b Strong, positive, linear correlation?

c $y = 0.801x + 5.02$

d $y = 0.801(80) + 5.02 = 69\%$

e This is a valid estimate as it has a strong r value and we are using interpolation.

f The regression line cannot be used to estimate the score of a student who scored 10% in mathematics as the score is outside of the given domain and extrapolation is unreliable.

g $x = 0.823y + 23.5$

h This equation be used to estimate a mathematics score, given a science score.

8 **a** Random sampling

b 0.295

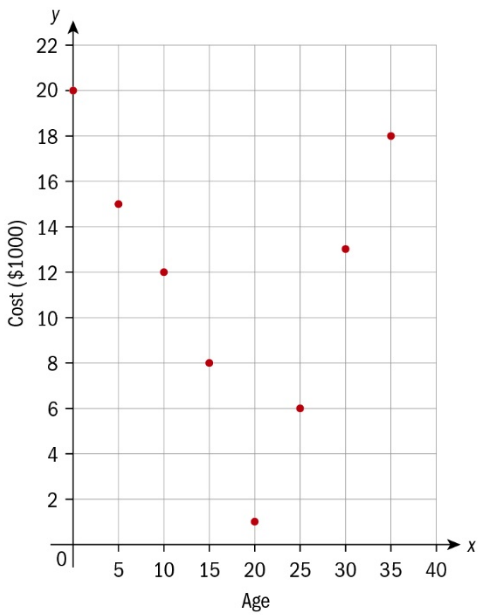
c As the age of a person increases, so does the number of shoes they have.

d $y = 0.0421x + 4.28$

e $y = 0.0421(18) + 4.28 = 4.62 \text{ or } 5 \text{ pairs of shoes}$

f No. The r value is weak.

9 a



b No. The data cannot be represented by one line but two lines would be appropriate.

c Using the calculator to find a function from $x = 0$ to $x = 20$ gives $y = -0.9x + 20.2$

and then using the calculator to find a function from $x = 20$ to $x = 35$ gives $y = 1.16x - 22.4$

8 Quantifying randomness: probability

A calculator is allowed

1 If the probability of rain on any day this week is $\frac{1}{3}$, find the probability that,

a It will not rain today.

b It will rain on Monday and Tuesday

c It will rain on Friday and Saturday, but not Sunday.

d What is the expected number of dry days in a six-day period?

2 In a class of 20 students, each student chooses a favorite colour from red, blue and purple.

The results are shown in this table.

	Red	**Blue**	**Purple**
Female	5	3	3
Male	4	2	3

Find the probability that the student was:

a a male who chose red

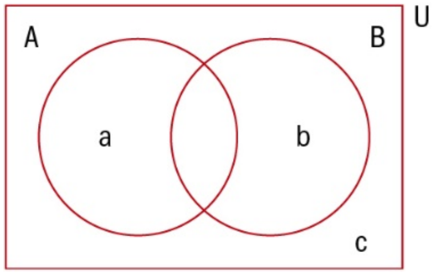
b a male or a student who chose red, but not both.

c Given that a student is female, calculate the probability that she did not choose red.

d Two students are chosen at random Find the probability that neither chose red.

3 If $P(A) = 0.6, P(B) = 0.5$ and $P(A \cap B) = 0.2$,

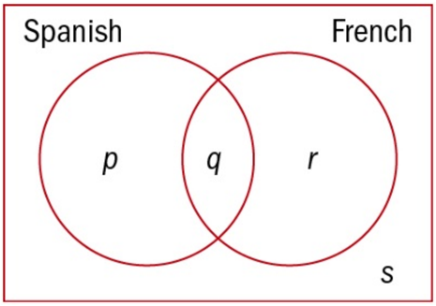
a Show this data on a Venn diagram like the one shown.



b Find the values of *a, b* and *c.*

c Find $P(A' \cap B)$.

4 In a class of 16 students, 12 speak Spanish, 8 speak French and one speaks neither Spanish nor French. The class is shown on the Venn diagram where p, q, r and s represent the number of students.



a Write down the value of s.

b Find the value of q.

c Write down the values of p and r.

A student is selected at random from the class.

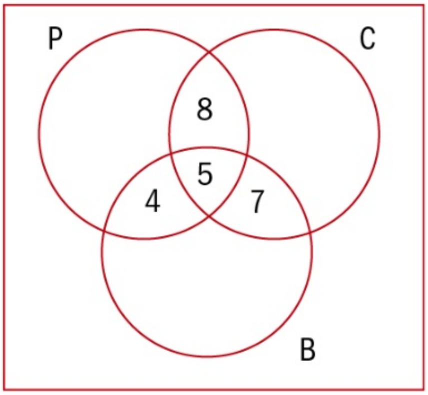
d Given that the student speaks French, write down the probability that the student speaks Spanish.

e Hence, show that speaking Spanish and speaking French are not independent events.

One student is selected at random from the class and asked to stand outside, then another is selected.

f Find the probability of choosing two students where the first student speaks only French and the second speaks only Spanish.

5 There are 120 students in an IB group. 40 take physics (P), 35 take chemistry (C) and 30 take biology (B). This is shown in the Venn diagram.



Find the probability that a student chosen at random will

a Take all three of these sciences.

b Study exactly two of these sciences.

c Study only chemistry.

d Not study physics, chemistry or biology.

6 Roz recorded the number of siblings that students in his class had in school.

a Copy the table and complete the relative frequency column.

Siblings	Frequency	Relative frequency
0	8	
1	4	
2	3	
3	3	
4	2	

What is the probability that a student:

b had 3 siblings in school

c had at least 2 siblings in school.

7 The probability that Serene wins her first tennis match is $\frac{7}{8}$.

If she wins the first match, the probability that she wins the second is $\frac{3}{4}$.

If she loses the first match, the probability that she loses the second match is $\frac{3}{5}$.

a Show the possible results for Serene's first two matches on a tree diagram.

Calculate the probability that Serene will

b Win her first two matches.

c Not lose her first two matches.

d Win only one of her first two matches.

e Given that Serene loses her second match, what is the probability that she won her first?

8 The probability of two events is given as $P(A) = 0.2$ and $P(B) = 0.5$.

Find $P(A \cup B)$ when

a A and B are Independent.

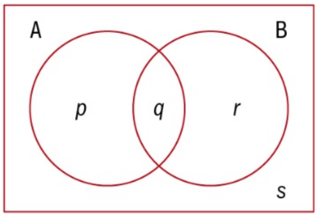
b A and B are mutually exclusive.

c Find $P(A \mid B)$ when $P(A \cup B) = 0.6$

9 When $P(A) = \frac{1}{6}, P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{12}$,

a Find $P(A \cap B)$

The sets A and B can be represented in a Venn diagram as shown.



b Find the values of p, q, r, s.

c Find $P(A'|B')$.

Answers

1 **a** $1-\frac{1}{3}=\frac{2}{3}$ **b** $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$ **c** $\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}=\frac{2}{27}$ **d** $\frac{2}{3}\times 6=4$

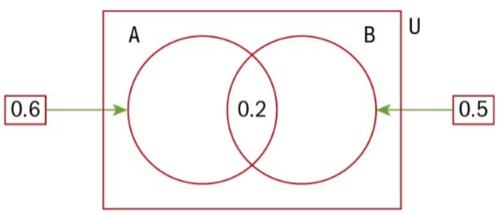
2 **a** $\frac{4}{20}=\frac{1}{5}$

b $\frac{5+3+2}{20}=\frac{10}{20}=\frac{1}{2}$

c $\frac{\text{Number of females who did not choose red}}{\text{Number of females}}=\frac{6}{11}$

d $\frac{11}{20}\times\frac{10}{19}=\frac{11}{38}$

3 **a**



b $a=0.6-0.2=0.4$

$b=0.5-0.2=0.3$

$c=1-(0.4+0.3+0.2)=0.1$

c $\mathrm{P}(A'\cap B)=0.3$

4 **a** $s=1$

b $q=21-16=5$

c $p=7, r=3$

d $\frac{5}{8}$

e For independent events

$P(S)\times P(F)=P(S\cap F)$

$\frac{12}{16}\times\frac{8}{16}\neq\frac{5}{16}$

f $\frac{3}{16}\times\frac{7}{15}=\frac{7}{80}$

5 **a** $\frac{5}{120}=\frac{1}{24}$

b $\frac{8+4+7}{120}=\frac{19}{120}$

c $\frac{35-(8+5+7)}{120}=\frac{15}{120}=\frac{1}{8}$

d Only physics $= 40 - (8 + 5 + 4) = 23$

Only biology $= 30 - (7 + 5 + 4) = 14$

Not studying science $= 120 - (15 + 23 + 14 + 8 + 5 + 4 + 7) = 44$

$P(\text{no science}) = \frac{44}{120} = \frac{11}{30}$

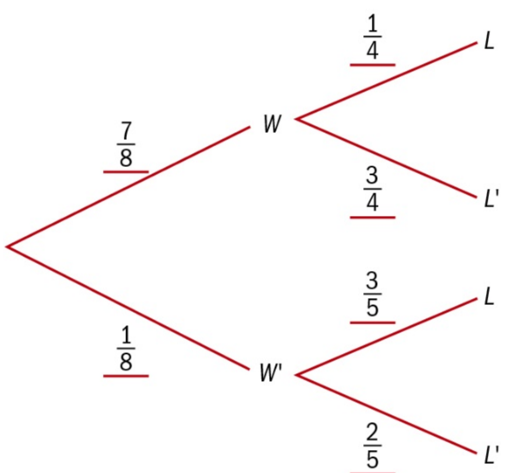
6 **a**

Siblings	Frequency	Relative frequency
0	8	0.4
1	4	0.2
2	3	0.15
3	3	0.15
4	2	0.1

b 0.15

c $0.15 + 0.15 + 0.1 = 0.4$

7 **a**



b $\frac{7}{8} \times \frac{3}{4} = \frac{21}{32}$

c $1 - \left(\frac{1}{8} \times \frac{3}{5}\right) = \frac{37}{40}$

d $\left(\frac{7}{8} \times \frac{1}{4}\right) + \left(\frac{1}{8} \times \frac{2}{5}\right) = \frac{7}{32} + \frac{1}{20} = \frac{43}{160}$

e $P(W \mid L) = \frac{P(W \cap L)}{P(L)} = \frac{\frac{7}{8} \times \frac{1}{4}}{\frac{47}{160}} = \frac{35}{47}$

8 **a** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A)(P(B)$

$P(A \cup B) = 0.2 + 0.5 - 0.1 = 0.4$

b $P(A \cup B) = P(A) + P(B) = 0.2 + 0.5 = 0.7$

c $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$P(A \cap B) = 0.2 + 0.5 - 0.6 = 0.1$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.1}{0.5} = \dfrac{1}{5}$

9 **a** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \dfrac{1}{6} + \dfrac{1}{3} - \dfrac{5}{12} = \dfrac{1}{12}$

b $q = \dfrac{1}{12}$

$p = \dfrac{1}{6} - \dfrac{1}{12} = \dfrac{1}{12}$

$r = \dfrac{1}{3} - \dfrac{1}{12} = \dfrac{1}{4}$

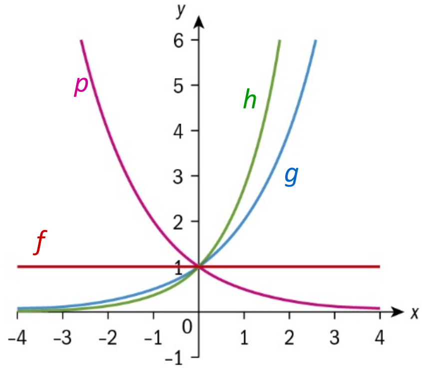
$s = 1 - \left(\dfrac{1}{12} + \dfrac{1}{12} + \dfrac{1}{4}\right) = \dfrac{7}{12}$

c $P(A'|B') = \dfrac{P(A' \cap B')}{P(B')} = \dfrac{\frac{7}{12}}{\frac{8}{12}} = \dfrac{7}{8}$

9 Representing equivalent quantities: exponentials and logarithms

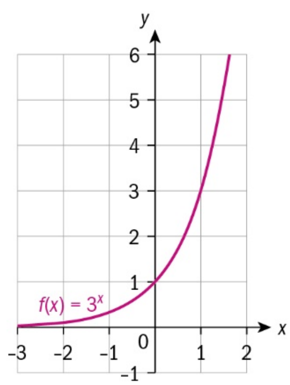
Section A. A calculator is not allowed

1 Match the curves f, g, h and p with the functions $y = e^x, y = 1^x, y = \frac{1}{2}^x, y = 2^x$



2 Copy the graph below with the function $f(x) = 3^x$ and add the curves

$g(x) = 1.5^x, h(x) = 3^x + 2, p(x) = 3^{-x} + 2 \text{ and } q(x) = 3^{x+2}$.



3 Evaluate the log of each number

a 10000 **b** $\frac{1}{100}$ **c** 0.001 **d** $10^{2.5}$

e $\sqrt{10}$ **f** $10\sqrt[3]{10}$ **g** 0

4 Solve for x

a $3^{x^2+2x} = 27$ **b** $7^{5-x} = \frac{1}{49}$ **c** $5^{3x} = 25^{x-1}$ **d** $2^{2-3x} = 8^{2x-3}$

5 If $\log_x 2 = p$ and $\log_x 5 = q$, find expressions in terms of p and q for

a $\log_x 2.5$ **b** $\log_x 40$ **c** $\log_2 5$

6 If $a = \ln x, b = \ln y, \text{and } c = \ln z$, write $\ln \frac{x}{y^2\sqrt{z}}$ in terms of a, b and c.

Section B. A calculator is allowed

7 Solve the following equations to find the value of x to 3 significant figures

a $4^x = 9$ **b** $e^{3x} - 5^{1-x} = 0$

c Solve $6^x = 3^{x+1}$ giving your answer in the form $\frac{\ln a}{\ln b}$ where a, b are integers

8 Solve $\log_2 x = 1 - \log_2 (x - 6)$

9 Find the equation of the tangent to the curve $f(x) = e^{2x}$ at the point where $x = 1$.

10 Given $f(x) = x ln(4 - x^2)$

a Find $f'(x)$

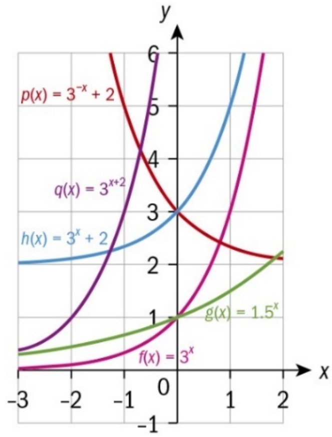
b Sketch the curve within the domain $-2 \le x \le 2$.

c Hence write down the solutions of $f'(x) = 0$.

Answers

1 f is $y = 1^x$, g is $y = e^x$, h is $y = 2^x$ and p is $y = \frac{1}{2}^{x}$.

2



3 **a** $\log 1000 = \log 10^4 = 4\log 10 = 4$ **b** $\log \frac{1}{100} = \log 10^{-2} = -2$

c $\log 0.001 = \log 10^{-3} = -3$ **d** $\log 10^{2.5} = 2.5$

e $\log\sqrt{10} = \log 10^{0.5} = 0.5$ **f** $\log 10\sqrt[3]{10} = \log 10^{\frac{4}{3}} = \frac{4}{3}$ **g** $\log 0 = 1$

4 **a**
$$3^{x^2+2x} = 27$$
$$3^{x^2+2x} = 3^3$$
$$x^2 + 2x = 3$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = -3, 1$$

b
$$7^{5-x} = \frac{1}{49}$$
$$7^{5-x} = 7^{-2}$$
$$5 - x = -2$$
$$x = 7$$

c
$$5^{3x} = 25^{x-1}$$
$$5^{3x} = \left(5^2\right)^{x-1}$$
$$3x = 2x - 2$$
$$x = -2$$

d
$$2^{2-3x} = 8^{2x-3}$$
$$2^{2-3x} = \left(2^3\right)^{2x-3}$$
$$2 - 3x = 6x - 9$$
$$9x = 11$$
$$x = \frac{11}{9}$$

5 **a** $\log_x 2.5 = \log_x 5 - \log_x 2 = q - p$

b $\log_x 40 = \log_x 8 + \log_x 5 = \log_x 2^3 + \log_x 5 = 3\log_x 2 + \log_x 5 = 3p + q$

c $\log_2 5 = \frac{\log_x 5}{\log_x 2} = \frac{q}{p}$

6
$$\ln\frac{x}{y^2\sqrt{z}} = \ln x - \ln y^2 - \ln\sqrt{z}$$
$$\ln\frac{x}{y^2\sqrt{z}} = \ln x - 2\ln y - \frac{1}{2}\ln z = a - 2b - \frac{1}{2}c$$

7 **a**
$$\log 4^x = \log 9$$
$$x\log 4 = \log 9$$
$$x = \frac{\log 9}{\log 4} = 1.58$$

b
$$e^{3x} - 5^{1-x} = 0$$
$$e^{3x} = 5^{1-x}$$
$$\ln e^{3x} = \ln 5^{1-x}$$
$$3x\ln e = (1-x)\ln 5$$
$$3x = \ln 5 - x\ln 5$$
$$3x + x\ln 5 = \ln 5$$
$$x(3 + \ln 5) = \ln 5$$
$$x = \frac{\ln 5}{3 + \ln 5} = 0.349$$

c
$$\ln 6^x = \ln 3^{x+1}$$
$$x\ln 6 = (x+1)\ln 3$$
$$x\ln 6 = x\ln 3 + \ln 3$$
$$x\ln 6 - x\ln 3 = \ln 3$$
$$x(\ln 6 - \ln 3) = \ln 3$$
$$x = \frac{\ln 3}{(\ln 6 - \ln 3)}$$
$$x = \frac{\ln 3}{\ln 2}$$

8 $\log_2 x + \log_2(x-6) = \log_2 2$

$\log_2(x^2 - 6x) = 2$

$x^2 - 6x = 2$

$x^2 - 6x - 2 = 0$

$x = -0.316, 6.32$

$x = -0.316$ is not a valid answer as it leaves the log of a negative.

Therefore $x = 6.32$

9 $f(1) = e^2 \text{ or } 7.39$

It is better to keep your answer as e^2 for accuracy and simplicity.

$f'(x) = 2e^{2x}$

$f'(1) = 2e^2 \text{ or } 14.8$

$y - y_1 = m(x - x_1)$

$y - e^2 = 2e^2(x-1)$

$y = 2e^2 x - e^2$

10 Given $f(x) = x\ln(4 - x^2)$

a $f'(x) = uv' + vu'$

Differentiate $\ln(4-x^2)$ using the chain rule

$u = 4 - x^2 \quad y = \ln u$

$$\frac{du}{dx} = -2x, \ \frac{dy}{du} = \frac{1}{u} = \frac{1}{4-x^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-2x}{4-x^2}$$

$u = x \ v = \ln(4-x^2)$

$$u' = 1 v' = \frac{-2x}{4-x^2}$$

$$f'(x) = \left(x \times \frac{-2x}{4-x^2}\right) + \ln(4-x^2)$$

$$f'(x) = \left(\frac{-2x^2}{4-x^2}\right) + \ln(4-x^2)$$

b Graph of $y = \left(\frac{-2x^2}{4-x^2}\right) + \ln(4-x^2)$

c $x = -1.15, 1.15$

10 From approximation to generalization: integration

Section A. A calculator is not allowed

1 Integrate the following

a $4x^7-9x^2-4$ **b** $\sqrt[3]{x^2}$ **c** $\dfrac{4x^5-x^2-10x}{2x}$

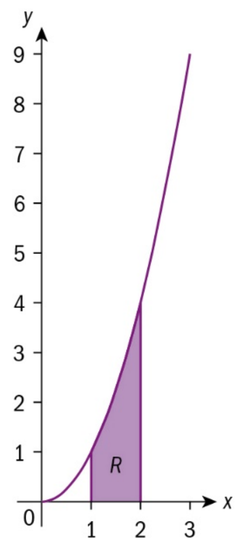
d $3x^2\sqrt{x^3+3}$ **e** $x\left(x^2+6\right)^5$ **f** $\dfrac{4}{x}+5e^x$

g $2\left(3x-7\right)^6$ **h** $\dfrac{3}{5x-1}$ **i** $\dfrac{1}{\left(2x+1\right)^2}$

2 Evaluate

a $\int_0^1 (4x^2-4x+1)dx$ **b** $\int_1^5 \left(4x+\dfrac{3}{x}\right)dx$

3 This diagram shows part of the graph of $y=x^2$. Find the area of region R.



4 When $\int_0^4 f\left(x\right)dx=9$ and $\int_0^4 g\left(x\right)=4$, evaluate $\int_0^4 \left(3f\left(x\right)+g\left(x\right)\right)dx$

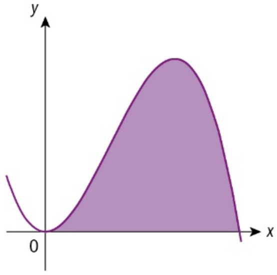
5 Given that $f'\left(x\right)=10x^4+6x$ and $f\left(1\right)=7$, find $f\left(x\right)$

6 The gradient of $f\left(x\right)$ is $e^{-3x}+\dfrac{1}{1-x}$, $x<1$ and $f\left(x\right)$ has a y intercept of 4.

Find $f\left(x\right)$.

Section B. A calculator is allowed

7 This is the graph pf $f\left(x\right)=6x^2\left(1-x\right)$.

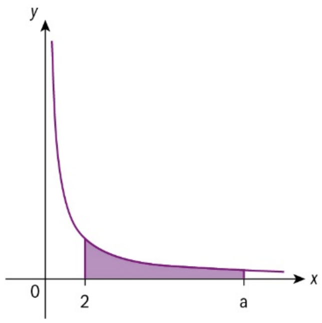


a Find the value of the x-intercepts.

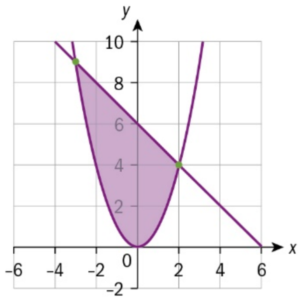
b Write down an integral which represents the shaded area.

c Write down the area of the shaded region.

8 The shaded region under the curve of $y=\frac{1}{x}$ has an area of 2 square units, find the value of a.



9 The diagram shows $f\left(x\right)=x^2$ and $g\left(x\right)=6-x$



a Find the x values of their intersection points.

b Write down an integral for the area of the shaded region.

c Write down the area of the shaded region.

10 a Sketch the graph of $f\left(x\right)=x^3-x^2-6x$ with $-3\le x\le 4$ and $-10\le y\le 5$

Write down

b the values of the x-intercepts.

c An integral to represent the area between $f\left(x\right)$ and the x axis.

d The area between the curve and the x axis.

11 The acceleration of a particle is given modelled by $12t^2-6$.

a Find the velocity after 10 seconds if the initial velocity was $2\,\text{ms}^{-1}$.

b Find the displacement after 3 seconds if the displacement after 1 second was -3 m.

Answers

1 **a** $\frac{1}{2}x^8 - 3x^3 - 4x + c$ **b** $\frac{3}{5}x^{\frac{5}{3}}$ **c** $\frac{2}{5}x^5 - \frac{x^2}{4} - 5x + c$

d $\frac{2}{3}\left(x^3+3\right)^{\frac{3}{2}} + c$ **e** $\frac{1}{10}\left(x^2+6\right)^5 + c$ **f** $4\ln|x| + 5\mathrm{e}^x + c$

g $\frac{2}{21}(3x-7)^7 + c$ **h** $\frac{3}{5}\ln|5x-1| + c$ **i** $-\frac{1}{6(2x+1)^3}$

2 **a** $\int_0^1 (4x^2 - 4x + 1)dx = \left[\frac{4}{3}x^3 - 2x^2 + x\right]_0^1 = \left(\frac{4}{3}(1)^3 - 2(1)^2 + (1)\right) - \left(\frac{4}{3}(0)^3 - 2(0)^2 + (0)\right) = \frac{1}{3}$

b $\int_1^2 \left(4x + \frac{3}{x}\right)dx = \left[2x^2 + 3lnx\right]_1^2 = (8 + 3ln2) - (2 + 3ln1) = 6 + 3ln2$

3 Area$R = \int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \left(\frac{1}{3}2^3\right) - \left(\frac{1}{3}1^3\right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ square units

4 $\int_0^4 \left(3f(x) + g(x)\right)dx = 3(9) + 4 = 31$

5 $\int f'(x)dx = f(x)$

$\int \left(10x^4 + 6x\right)dx = 2x^5 + 3x^2 + c$

$f(1) = 2(1)^5 + 3(1)^2 + c = 5 + c = 7, c = 2$

$f(x) = 2x^5 + 3x^2 + 2$

6 $\int \left(e^{-3x} + \frac{1}{1-x}\right)dx = -\frac{1}{3}e^{-3x} - \ln(1-x) + c$

At $(0,4)$ $4 = -\frac{1}{3}e^{-2(0)} - \ln\left(1-(0)\right) + c$

$4 = -\frac{1}{3} - \ln 1 + c$

$c = 4\frac{1}{3}$

$f(x) == -\frac{1}{3}\mathrm{e}^{-3x} - \ln(1-x) + 4\frac{1}{3}$

7 **a** $f(x) = 6x^2(1-x) = 0$ when $x = 0, 1$

The x intercepts are 0 and 1

b $\int_0^1 6x^2(1-x)dx$

c 0.5 square units

8 $\int_2^a \frac{1}{x}dx = \left[\ln x\right]_2^a = \ln a - \ln 2 = 2$

$\ln a = 2 + \ln 2$

$a = \mathrm{e}^{2+\ln 2} = 14.8$

9 a $x^2 = 6 - x$

$x^2 + x - 6 = 0$

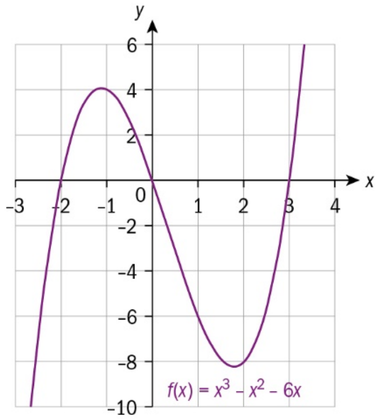
$(x+3)(x-2) = 0$

$x = -3, 2$

b $\int_{-3}^{2} \left(6 - x - x^2\right) dx$

c $\frac{125}{6} \approx 20.8$ square units

10 a



b $-2, 0, 1$

c $\int_{3}^{0} \left(x^3 - x^2 - 6x\right) dx + \int_{-2}^{0} \left(x^3 - x^2 - 6x\right) dx$

d $\frac{253}{12} \approx 21.1$ square units

11 a $v(t) = \int \left(12t^2 - 6\right) dt = 4t^3 - 6t + c$

An initial velocity of $2\,\text{ms}^{-1}$ gives $(0, 2)$ which makes

$2 = 4(0)^3 - 6(0) + c, c = 2$

$v(t) = 4t^3 - 6t + 2$

At 10 secs, $t = 10$

$v(10) = 4(3)^3 - 6(3) + 2 = 92\,\text{ms}^{-1}$

b $s(t) = \int \left(4t^3 - 6t + 2\right) dt = t^4 - 3t^2 + 2t + c$

A displacement after 1 second of $-3\,\text{m}$ gives $(1, -3)$ which makes

$-3 = (1)^4 - 3(1)^2 + 2(1) + c,\ c = -3$

$s(t) = t^4 - 3t^2 + 2t - 3$

After 3 secs, $t = 3$

$s(3) = (3)^4 - 3(3)^2 + 2(3) - 3 = 57\,\text{m}$

11 Relationships in space: geometry and trigonometry in 2D and 3D

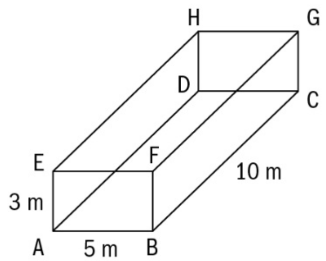
A calculator is allowed

1 Find the midpoint of and the distance between the following points

a $(2,6)$ and $(-3,-5)$

b $(3,5,4)$ and $(9,6,2)$.

2 The shape shown is a cuboid.



a Find the volume of the cuboid.

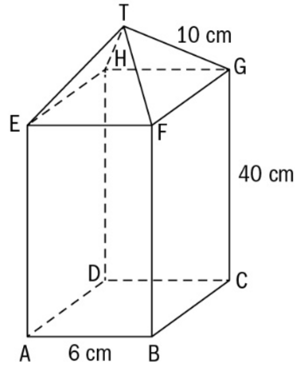
b Find the length AC.

c Find the length AG.

d Find the angle that AG makes with the plane ABCD.

3 Coco is standing 20m away from a tree at the end of its shadow and measures the angle of elevation of the sun as $37°$. How tall is the tree?

4 The body of a juice carton is the shape of a cuboid with a square base. The top of the carton is a square based pyramid.

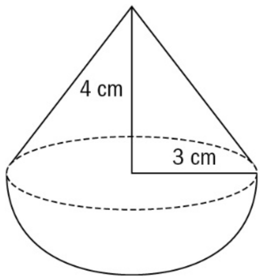


a The size of the angle between TF and FG.

b The total surface area of the four triangular sections of the roof.

c The volume of juice in the carton.

5 A fishing weight is made by combining a hemisphere of radius 3 cm and a right circular cone of height 4cm as shown on the diagram.

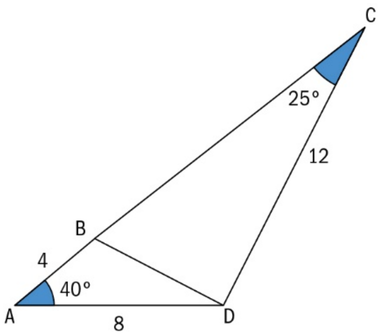


a Find the volume of the fishing weight

b Find the total surface area.

6 Find the smallest angle in a triangle of sides 6cm, 7cm, 8cm.

7 Grumpy and Happy have a garden each that is joined by fence BD



a Find the length of the fence.

b Find two possible values for angle CBD.

c If angle CBD is acute, find the perimeter, ABCDA, of their gardens.

8 An airplane leaves Manpool on a bearing of 030° to Doondee, a distance of 250km. They take off from Doondee and head to Bernich on a bearing of 100° for 400km.

a Show this on a diagram.

b What is the bearing and distance that the pilot must use to return home to Manpool?

Answers

1 **a** Midpoint $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{2-3}{2},\frac{6-5}{2}\right)=\left(-\frac{1}{2},\frac{1}{2}\right)$

Distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(-3-2)^2+(-5-6)^2}=\sqrt{146}=12.1$

b Midpoint $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)=\left(\frac{3+9}{2},\frac{5+6}{2},\frac{4+2}{2}\right)=\left(6,\frac{11}{2},3\right)$

Distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=\sqrt{(9-3)^2+(6-5)^2+(2-4)^2}=\sqrt{41}=6.40$

2 **a** $V=lbh=3\times5\times10=150\,\text{m}^3$

b $AC^2=5^2+10^2=125$

$AC=\sqrt{125}=11.2\,\text{m}$

Remember to use $\sqrt{125}$ and not 11.2 in any further calculations!

c $AG=3^2+\left(\sqrt{125}\right)^2=134$

$AG=\sqrt{134}=11.6\,\text{m}$

d $\sin CAG=\frac{CG}{AG}=\frac{3}{\sqrt{134}}$

$\angle CAG=\sin^{-1}15.0°$

3 $\tan 37°=\frac{h}{20}$

$h=20\tan 37°=15.1\,\text{m}$

4 **a** $\cos TFG=\frac{3}{10}=0.3$

$\angle TFG=\cos^{-1}0.3=72.5°$

b Surface area $=4\times\frac{1}{2}\times TF\times FG\times\sin TFG=2\times10\times6\times\sin72.5°=114\,\text{cm}^2$

c The volume of juice in the carton=Volume of the cuboid+volume of the pyramid

$V=lbh+\left(\frac{1}{3}\text{base area}\times\text{height}\right)$

The height of the pyramid

Find EG

$EG^2=6^2+6^2=72$

$EG=\sqrt{72}$

Height $=\sqrt{10^2-\left(\frac{1}{2}\sqrt{72}\right)^2}=9.06\,\text{cm}$

$V=(6\times6\times40)+\left(\frac{1}{3}\times6\times6\times9.06\right)=1550\,\text{cm}^3$

5 **a** Total volume = volume of the hemisphere + volume of the cone

$$V=\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2h=\frac{2}{3}\pi(3)^3+\frac{1}{3}\pi(3)^2(4)=30\pi=94.2\,\text{cm}^3$$

b The slant height (l) of the cone is 5cm

Total surface area (SA) = SA of the hemisphere + Curved SA of the cone

$$SA=\frac{1}{2}\times 4\pi r^2+\pi rl=2\pi(3)^2+(\pi\times 3\times 5)=33\pi=104\,\text{cm}^2$$

6 Find the smallest angle in a triangle of sides 6cm, 7cm, 8cm.

$$\cos\theta=\frac{7^2+8^2-6^2}{2\times 7\times 8}=\frac{77}{112}$$

$$\theta=\cos^{-1}\frac{77}{112}=46.6^\circ$$

7 **a** $BD^2=4^2+8^2-(2\times 4\times 8\times\cos 40^\circ)$

$BD=5.57$

b $\dfrac{\sin CBD}{12}=\dfrac{\sin 25^\circ}{5.57}$

$$CBD=\sin^{-1}\frac{12\sin 25^\circ}{5.57}=65.7^\circ\text{ or }114^\circ$$

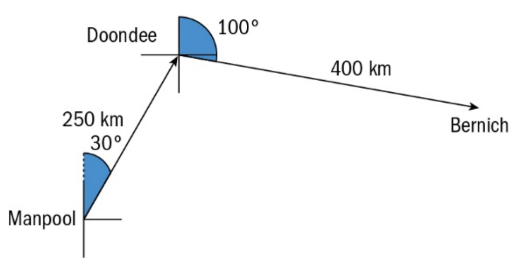
c $\angle BDC=(180-65.7-25)^\circ=89.3^\circ$

$$\frac{BC}{\sin 89.3}=\frac{5.57}{\sin 25}$$

$BC=13.2$

Perimeter $=13.2+4+8+12=37.2$

8 **a**



b $\angle MDB=360-100-150=110^\circ$

$BM=250^2+400^2-(2\times 250\times 400\times\cos 110)$

$BM=539km$

$$\cos BMD=\frac{400^2+250^2-539^2}{2\times 400\times 250}$$

$\angle BMD=44.3^\circ$

Bearing M to B = 074

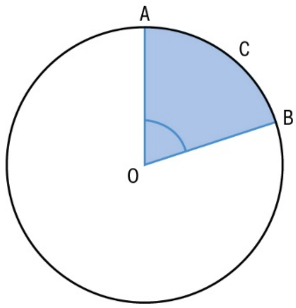
Bearing B to M $=360-74=286^\circ$

12 Periodic relationships: trigonometric functions

Section A. A calculator is not allowed

1 If $\sin\theta = \frac{3}{8}$ and $0 < \theta < 90°$, find the exact value of $\cos\theta$.

2 The circle with centre O has a radius of 4cm.



If $\angle AOB = 1.5\,\text{radians}$, find

a the length of arc ACB

b the shaded area.

3 Solve, giving your answer in radians as an exact value with $0 \le x \le 2\pi$.

a $\sin x + 4 = 5$

b $\sqrt{3}\tan x + 2 = 1$

c $10\cos x - 5\sqrt{2} = 0$

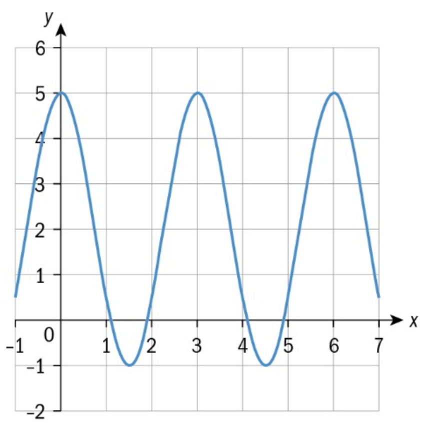
4 Solve $\sin 2x = 2\cos x$, for $0 \le x \le 3\pi$

5 **a** Write $\sin x + 2\cos^2 x$ in terms of $\sin x$.

b Hence solve $\sin x + 2\cos^2 x - 2 = 0$ in the range $0 \le x \le \pi$

6 Show that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

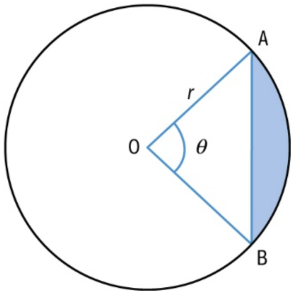
7 This is the graph of $f(x) = a\cos(bx) + c$. Find the values of a, b and c.



Section B. A calculator is allowed

8 The diagram shows a sector of a circle centre O, with radii OA and OB.

Find the shaded area when $OA = 15\,\text{cm}$ and $\theta = 2$ radians



9 **a** Write $4\cos x + 3\sin^2 x$ in the form $a\cos^2 x + b\cos x + c$

b Hence solve $4\cos x + 3\sin^2 x = 4$, with $0 \le x \le \frac{\pi}{2}$

10 The depth (d) of water at the end of a pier on Tuesday is modelled by

$$d(h) = 4\sin h\left(\frac{\pi}{6}\right) + 8$$

where h is the number of hours after midnight.

a Find the maximum depth of water.

b Sketch the graph showing the height of water for Tuesday and draw a line on the sketch to show the times between which there is 10m or more of water at the end of the pier. State these times (in the form am and/or pm) under your sketch.

Answers

1 $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{3}{8}\right)^2 = \frac{55}{64}$

$\cos\theta = \frac{\sqrt{55}}{8}$

2 **a** $s = r\theta = 4 \times 1.5 = 6\,\text{cm}$

b $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 16 \times 1.5 = 12\,\text{cm}^2$

3 **a** $\sin x = 1, \quad x = \frac{\pi}{2}$

b $\tan x = -\frac{1}{\sqrt{3}}, \quad x = \frac{5\pi}{6}, \frac{11\pi}{6}$

c $\cos x = \frac{\sqrt{2}}{2}, \quad x = \frac{\pi}{4}, \frac{7\pi}{4}$

4 $2\sin x\cos x = 2\cos x$

$2\sin x\cos x - 2\cos x = 0$

$2\cos x(\sin x - 1) = 0$

$\cos x = 0 \quad \sin x = 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

5 **a** $\sin x + 2\cos^2 x = \sin x + 2(1 - \sin^2 x) = -2\sin^2 x + \sin x + 2$

b $\sin x + 2\cos^2 x - 2 = 0$

$-2\sin^2 x + \sin x = 0$

$\sin x(1 - 2\sin x) = 0$

$\sin x = 0, \sin x = \frac{1}{2}$

$x = 0, \frac{\pi}{6}, \pi, \frac{5\pi}{6}$

6 $\frac{\sin 2x}{1 + \cos 2x} = \frac{2\sin x\cos x}{1 + 2\cos^2 x - 1} = \frac{\sin x}{\cos x} = \tan x$

7 amplitude $(a) = \frac{1}{2}(5 - -1) = 3$

Period $= \frac{2\pi}{b} = 3 \; b = \frac{2\pi}{3}$

Vertical shift$(c) = 5 - 3 = 2$

8 Shaded area $= \frac{1}{2}r^2\theta - \frac{1}{2} \times OA \times OB \times \sin\theta$

Shaded area $= \frac{1}{2}(15)^2(2) - \frac{1}{2} \times 15 \times 15 \times \sin 2 = 123\,\text{cm}^2$

9 **a** $4\cos x + 3\sin^2 x = 4\cos x + 3(1 - \cos^2 x) = -3\cos^2 x + 4\cos x + 3$

b $4\cos x + 3\sin^2 x = 4$

$-3\cos^2 x + 4\cos x + 3 = 4$

$3\cos^2 x - 4\cos x + 1 = 0$

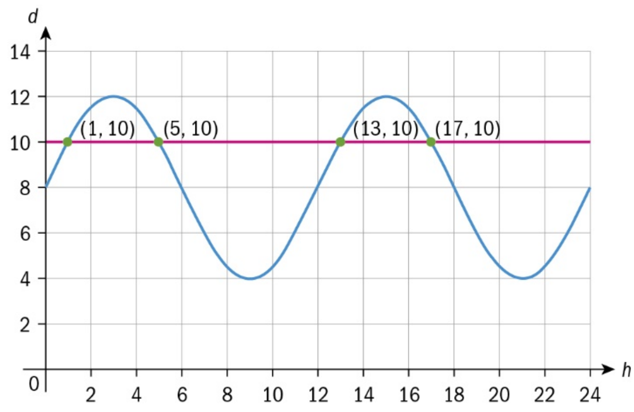
$(3\cos x - 1)(\cos x - 1) + 0$

$\cos x = \frac{1}{3}, 1$

$x = 0, 1.23$

10 a Maximum depth when $h\left(\frac{\pi}{6}\right) = 1$, $d(h) = 4(1) + 8 = 12m$

b



1am to 5am and 1pm to 5pm

13 Modelling change: more calculus

Section A. A calculator is not allowed

1 Find $\int \sin 5x \cos 5x \, dx$

2 Find the value of *a* for the curve $f(x) = 5x + a\sin x$, which has a gradient of 7 at the point where $x = \frac{\pi}{3}$.

3 Find $f'(\pi)$ when $f(x) = \frac{3x}{\cos x}$.

4 Let $f(x) = 2e^x \cos x$. Find the gradient of the normal to the curve where the curve crosses the y axis.

5 Find $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} dx$.

Section B. A calculator is allowed

6 **a** Given $f(x) = e^x(\cos x + \sin x)$, find $f'(x)$

b The curve has a maximum point between 0 and 3. Find the coordinates of the maximum point.

c Sketch $f(x) = e^x (\cos x + \sin x)$ with $-2 \le x \le 3$ and shade the area bounded by the curve, the positive x axis and the y axis.

d Write down an integral for the shaded area.

e Find the shaded area.

7 The velocity, $v \text{ ms}^{-1}$, of a particle at time *t* seconds, is given by

$v(t) = 2t + \cos 2t$, for $0 \le t \le 2$.

Find

a The initial velocity of the particle.

b An expression for the acceleration of the particle.

c The time when the acceleration is zero.

d An integral to represent the distance travelled during the first second.

e Write down the distance travelled in the first second.

Answers

1 Let $u = \sin 5x$

$$\frac{du}{dx} = 5\cos 5x$$

$$\frac{1}{5}\int u\,du = \frac{1}{5}\times\frac{u^2}{2} + c$$

$$\int \sin 5x \cos 5x\,dx = \frac{1}{10}\sin^2 3x + C$$

2 $f'(x) = 5 + a\cos x$

$$5 + a\cos\frac{\pi}{3} = 7$$

$$\frac{1}{2}a = 2 \quad a = 4$$

3 $f'(x) = \dfrac{3\cos x - (3x)(-\sin x)}{\cos^2 x}$

$$f'(x) = \frac{3\cos x + 3x\sin x}{\cos^2 x}$$

$$f'(\pi) = \frac{3\cos(\pi) + 3(\pi)\sin(\pi)}{\cos^2(\pi)}$$

$$f'(\pi) = \frac{-3}{1} = -3$$

4 $f'(x) = \left(2e^x \times (-\sin x)\right) + \left(\cos x \times 2e^x\right) = 2e^x\cos x - 2e^x\sin x = 2e^x(\cos x - \sin x)$

$$f'(0) = e^0\left(\cos(0) - \sin(0)\right) = 1$$

Gradient of the normal is -1

5 Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\int \frac{\sin x}{\sqrt{\cos x}}dx = \int -u^{-\frac{1}{2}}du = -2u^{\frac{1}{2}} = -2\sqrt{\cos x}$$

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}dx = \left[-2\sqrt{\cos x}\right]_0^{\frac{\pi}{3}} = \left(-2\sqrt{\cos\frac{\pi}{3}}\right) - \left(-2\sqrt{\cos 0}\right) = 2 - \frac{2}{\sqrt{2}}$$

6 **a** $f(x) = e^x(\cos x + \sin x)$

$$f'(x) = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)$$

$$= 2e^x\cos x$$

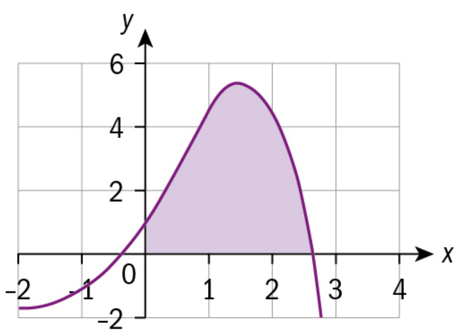
b $2e^{x}\cos x = 0$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$

$y = e^{\frac{\pi}{2}}\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}$

The coordinate of the maximum point is $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$

c



d $\int_{0}^{2.36} e^{x}(\cos x + \sin x)\,dx$

e 7.46 square units

7 **a** $v(0) = 2(0) + \cos 2(0) = 1\,\text{ms}^{-1}$

b $\text{Acc}(t) = v'(t) = 2 - 2\sin 2t$

c $2 - 2\sin 2t = 0$

$\sin 2t = 1$

$t = \frac{\pi}{4}$

d $\int_{0}^{1}(2t + \cos 2t)\,dt$

e 1.45m

14 Valid comparisons and informed decisions: probability distributions

Section A. A calculator is not allowed

1 The random variable, X, has the probability distribution

x	1	2	3	4	5
P(*X* = *x*)	0.05	0.4	a	0.2	0.1

a Find the value of a

b Find P(X > 3)

2 The discrete random variable, X, can only take the values 0,1, 2, 3 and

$$P(X = x) = k\left(\frac{1}{2}\right)^{x-1}.$$

Find the value of k.

3 A standard die (the singular of dice) is thrown and the score is squared.

a Copy and complete the table

x	1	4	9		25	36
P(*X* = *x*)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	

b Find the expected score.

4 A fund raising game at the school fair sells lottery tickets for $4. The possible prizes are $4, $5, $10, $100 or you could not win anything. W is the amount won and is represented in this distribution.

w	0	4	5	10	100
P(*W* = *w*)	k	0.2	0.2	0.1	0.002

a Find the value of k

b Find how much money the fundraiser would expect to make if it sold 500 tickets.

Section B. A calculator is allowed

5 Find the probability of obtaining exactly 3 heads in 5 tosses of a fair coin.

6 The probability that Blake scores from a penalty in hockey is 0.8.

Blake takes 6 penalties, what is the probability that Blake,

a does not score on his first shot

b scores exactly 3

c scores at least once.

7 In a baseball game, the probability that Eden will get on base safely when she comes to bat is 0.6 .

a What is the probability that she will get on base safely at least 3 out of 4 times?

If Eden gets to bat 50 times in a season

b What is the expected number of times that she will make first base?

c What is the variance of her number of bases?

8 The weights of Tuna sent for sushi follow a normal distribution with a mean of 150kg and a standard deviation of 5kg.

a Find the probability that a tuna weighs less than 142kg.

b Fuji restaurant only takes tuna from the top 20%. How heavy is their lightest tuna?

9 The rice harvest in Thaibodia yields an average of 341kg per farmer and is normally distributed with a standard deviation of 57kg.

It is known that 2.5% of the farmers had a weak season. Find how many kg of rice make a weak season.

10 On a mathematics test, the scores were normally distributed with a mean of 74 and a standard deviation of 7.

a Sketch this on a normal curve and shade the proportion of the class would be expected to score between 60 and 80 points.

b Find the shaded area and state the percentage of the class expected to score between 60 and 80.

11 After Sage sprays the garden with a weed killer, the survival time of weeds in a field is normally distributed where 15 % of the weeds survive greater than 9 days and 12 % of them last less than 4 days. Find the mean and standard deviation for the survival time of the weeds in sage's garden.

12 A pack of coffee is sold as weighing 230gms but the manufacturer maintains a weight of 231 gms as the average so as not to cheat customers. Coffee packs are normally distributed with a mean of 231 grams and a standard deviation of 1.5 grams.

A pack of coffee is considered to be underweight if it weighs less than 228 grams.

a What is the probability that a pack of coffee is underweight?

The manufacturer decides that the probability of a pack being underweight must be reduced to 0.002. He gives this problem to two junior executives, Eden and Haven

b Eden's suggestion is to increase the mean and leave the standard deviation unchanged. Find the value of the new mean.

c Haven's suggestion is to reduce the standard deviation and leave the mean unchanged. Find the value of the new standard deviation.

d After the probability of a pack of coffee being underweight has been reduced to 0.002, the store sells 100 packs. Find the probability that at least two of the boxes are underweight.

Answers

1 **a** $1-(0.05+0.4+0.2+0.1)=0.25$

b $0.2+0.1=0.3$

2 $P(X=0)=k\left(\frac{1}{2}\right)^{0-1}=2k$.

$P(X=1)=k\left(\frac{1}{2}\right)^{1-1}=k$.

$P(X=x)=k\left(\frac{1}{2}\right)^{2-1}=\frac{1}{2}k$.

$P(X=x)=k\left(\frac{1}{2}\right)^{3-1}=\frac{1}{4}k$.

$2k+k+\frac{1}{2}k+\frac{1}{4}k=1$

$\frac{15}{4}k=1$

$k=\frac{4}{15}$

3 **a**

x	1	4	9	16	25	36
P(*X* = *x*)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b $E(X)=\left(1\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(9\times\frac{1}{6}\right)+\left(16\times\frac{1}{6}\right)+\left(25\times\frac{1}{6}\right)+\left(36\times\frac{1}{6}\right)=\frac{91}{6}=15\frac{1}{6}$

4 **a** $k=1-(0.2+0.2+0.1+0.002)=0.498$

b $E(W)=(0\times 0.498)+(4\times 0.2)+(5\times 0.2)+(10\times 0.1)+(100\times 0.002)=\$3$

Profit $=500\times\$1=\$500$

5 $P(3\text{heads in}5\text{ tosses})=\binom{5}{3}(0.5)^3(0.5)^2=0.3125$

6 **a** $1-0.8=0.2$ **b** $\binom{6}{3}(0.8)^3(0.2)^3=0.08192$ **c** $1-(0.2)^6=0.999936$

7 **a** $P(\text{On base safely in at least }3\text{ out of }4\text{ at bats})=\text{P}(3)+\text{P}(4)$

$=\binom{4}{3}(0.6)^3(0.4)^1+\binom{4}{4}(0.6)^4(0.4)^0=0.4752$

b $50\times 0.6=30$ **c** $\text{Var}=npq=50\times 0.6\times 0.4=12$

8 **a** $P(W<142)=0.0548$

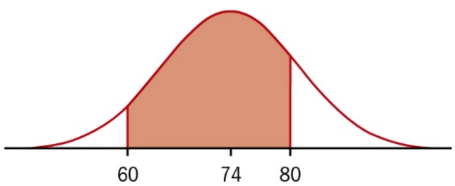
b $P(V>w)=0.2, x=154.2$. Their lightest tuna weighs 154.2kg

9 Using the GDC for inverse normal calculations, $z = -1.96$

$$-1.96 = \frac{x-\mu}{\sigma} = \frac{x-341}{57}$$

$x = 229\,\text{kg}$

10 a



b Using the GDC $P(60 \le s \le 80) = 78.2\%$

11 $P(X>9)=0.15$ and $P(X<4)=0.12$

Find the z scores from the GDC as 1.036 and -1.175

$$1.04 = \frac{9-\mu}{\sigma}, -1.18 = \frac{4-\mu}{\sigma}$$

Solve simultaneously to give
$\mu = 6.66$, $\sigma = 2.26$

12 a $X \sim N(231, 1.5^2)$

P(X < 228) = 0.0228

b $X \sim \text{N}(\mu,\ 1.5^2)$
P(X < 228) = 0.002

$$-2.88 = \frac{228-\mu}{1.5}.$$

$\mu = 232$ grams

c $X \sim N(231, \sigma^2)$

$$-2.88 = \frac{228-231}{\sigma}.$$

$\sigma = 1.04$ grams

d $X \sim B(100, 0.002)$

$P(X \le 1) = 0.982\ldots$

$P(X \le 2) = 1 - P(X \le 1) = 0.0174$

The probability that 2 packs are underweight is 0.0174