OXFORD IB DIPLOMA PROGRAMME

IN COOPERATION WITH ib

ADDITIONAL EXERCISES

MATHEMATICS:
ANALYSIS AND APPROACHES

STANDARD LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1.1 Number patterns and sigma notation

1 For each of the following sequences,

i write down the next three terms in each sequence

ii find an expression for the n^{th} term

iii state whether the sequence is arithmetic, geometric, or neither.

a 6, 13, 20, 27, ...

b 400, 200, 100, 50, ...

c $\frac{1}{5}, \frac{3}{10}, \frac{5}{15}, \frac{7}{20}, \ldots$

d $3x, 6x^2, 9x^3, 12x^4, \ldots$

e 15, 11, 7, 3, ...

f 4, -12, 36, -108, ...

2 Write the first six terms of each recursive sequence.

a $u_n = u_{n-1} + 4,\ u_1 = -7$

b $u_n = 3(u_{n-1} - 1),\ u_1 = 5$

c $u_n = \dfrac{4}{u_{n-1} + 1},\ u_1 = 27$

d $u_n = (u_{n-1})^2,\ u_1 = -2$

3 Find a recursive formula for the general term of each sequence.

a 4, 16, 256, 65536, ...

b 5, 7, 11, 19, ...

c $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

d $x,\ 2x+1,\ 4x+3,\ 8x+7, \ldots$

4 For each of the following series in sigma notation, find the terms and calculate the sum.

a $\sum_{n=1}^{5} 2^{n-1}$

b $\sum_{k=0}^{6} (3k+8)$

c $\sum_{a=2}^{6} (a^2 - 3a)$

d $\sum_{n=1}^{4} n^3$

e $\sum_{x=1}^{4} \dfrac{2x}{x+1}$

f $\sum_{n=0}^{3} (-4)^n$

5 Write each of the following series in sigma notation.

a $4 + 7 + 10 + 13 + 16 + 19$

b $1 + 2 + 4 + 8 + 16$

c $\frac{1}{3} + \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 1$

d $3x + 5x + 7x + 9x + 11x + 13x + 15x$

e $-5 + 8 - 11 + 14 - 17 + 20 - 23$

f $3 + 9 + 27 + 81 + \ldots$

Answers

1 **a** **i** 34, 41, 48

ii $u_1 = 6$, and $d = 13 - 6 = 7$, so $u_n = 6 + 7(n-1) = 7n - 1$

iii The sequence is arithmetic, because there is a common difference between each of the terms.

b **i** 25, 12.5, 6.25

ii $u_1 = 400$, and $r = \frac{200}{400} = 2$, so $u_n = 400(2^{n-1})$

iii The sequence is geometric, because there is a common ratio between terms.

c **i** $\frac{9}{25}, \frac{11}{30}, \frac{13}{35}$

ii We can look at the numerator and denominator as two different sequences.

The numerator has a first term of 1 and a common difference of 2.

The denominator has a first term of 5 and a common difference of 5.

So $u_n = \frac{1 + 2(n-1)}{5 + 5(n-1)} = \frac{2n-1}{5n}$

iii The sequence is neither arithmetic nor geometric.

d **i** $15x^5, 18x^6, 21x^7$

ii We can look at the coefficient and the power as two different sequences.

The coefficient is arithmetic with a first term of 3 and a common difference of 3.

The power is geometric with a first term of x and a common ratio of x.

So $u_n = (3 + 3(n-1))(x(x^{n-1})) = 3nx^n$

iii The sequence is neither arithmetic nor geometric.

e **i** -1, -5, -9

ii $u_1 = 15$, and $d = 11 - 15 = -4$, so $u_n = 15 + (-4)(n-1) = 19 - 4n$

iii The sequence is arithmetic, because there is a common difference between each of the terms.

f **i** 324, -972, 2916

ii $u_1 = 4$, and $r = \frac{-12}{4} = -3$, so $u_n = 4 \cdot (-3)^{n-1}$

iii The sequence is geometric, because there is a common ratio between terms.

2 **a** The first term is -7, and each subsequent term is 4 more than the previous term, so the first six terms are 7, 11, 15, 19, 23, 30.

b The first term is 5. To find each subsequent term, we subtract 1 from the previous term and multiply the result by 3, so the first six terms are 5, 12, 33, 96, 285, 852.

c The first term is 27, and to find each subsequent term, we divide 4 by one more than the previous term. The first six terms are $27, \frac{1}{7}, \frac{7}{2}, \frac{8}{9}, \frac{36}{17}, \frac{68}{53}$.

d The first term is -2, and to find each subsequent term, we square the previous term. The first six terms are -2, 4, 16, 256, 65536, 4294967296.

3 **a** $u_n = (u_{n-1})^2,\ u_1 = 4$

b $u_n = u_{n-1} + 2^{n-1},\ u_1 = 5$

c $u_n = 2u_{n-1},\ u_1 = \frac{1}{3}$

d $u_n = 2(u_{n-1}) + 1,\ u_1 = x$

4 **a** $2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4$

$= 1 + 2 + 4 + 8 + 16 = 31$

b $(3(0)+8) + (3(1)+8) + (3(2)+8) + (3(3)+8) + (3(4)+8) + (3(5)+8) + (3(6)+8)$

$= 8 + 11 + 14 + 17 + 20 + 23 + 26 = 119$

c $(2^2 - 3(2)) + (3^2 - 3(3)) + (4^2 - 3(4)) + (5^2 - 3(5)) + (6^2 - 3(6))$

$= -2 + 0 + 4 + 10 + 18 = 30$

d $1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100$

e $\frac{2(1)}{1+1} + \frac{2(2)}{2+1} + \frac{2(3)}{3+1} + \frac{2(4)}{4+1} = 1 + \frac{4}{3} + \frac{3}{2} + \frac{8}{5} = \frac{163}{30}$

f $(-4)^0 + (-4)^1 + (-4)^2 + (-4)^3 = 1 + (-4) + 16 + (-64) = -51$

5 Note: In this question, other answers are possible, depending on the letter used for the index, and the values selected for the upper and lower limits.

a $\sum_{n=1}^{6} 3n + 1$

b $\sum_{x=1}^{5} 2^{x-1}$

c $\sum_{k=1}^{5} \frac{k+1}{6}$

d $\sum_{n=1}^{7} x(2n+1)$

e $\sum_{n=1}^{7} (-1)^n (3n+2)$

f $\sum_{k=1}^{\infty} 3^k$

1.2 Arithmetic and geometric sequences

1 For each arithmetic sequence, find the term indicated.

a 2, 6, 10, …, u_{16}

b 11, 8, 5, …, u_{12}

c 4, 4.7, 5.4, …, u_{21}

d 23.5, 22.8, 22.1, …, u_{25}

e $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots, u_9$

f $x-5,\ 2x-3,\ 3x-1,\ \ldots, u_{10}$

2 For each arithmetic sequence, find number of terms.

a 7, 10, 13, …, 40

b 80, 74, 68, …, 2

c 45, 44.6, 44.2, …, 35

d $4x, 12x, 20x, \ldots, 100x$

e 10.5, 12, 13.5, …, 48

f −21, −17, −13, …, 23

3 For each geometric sequence, find the term indicated.

a 2, 6, 18, …, u_9

b 64, 96, 144, …, u_7

c 3, -6, 12, …, u_{12}

d 5, 5.5, 6.05, …,

e $\frac{9}{4}, \frac{3}{2}, 1, \ldots, u_6$

f $m, m^3, m^5, \ldots, u_{10}$

4 An arithmetic sequence has a third term of 10 and a seventh term of 26.

Find the first term and the common difference.

5 An arithmetic sequence has a fourth term of 42 and a sixth term of 30.

Find the tenth term.

6 A geometric sequence has a second term of 12 and a fourth term of 108.

a Find the possible values of the common ratio.

b Find the possible values of the seventh term.

7 Consider a geometric sequence with $u_1 = 16$ and $r = 1.5$.

Find the least value of n such that $u_n > 900$.

8 An arithmetic sequence, u_n, has a first term of 1000 and a common difference of 15.

A geometric sequence, v_n, has a first term of 6 and a common ratio of $\frac{3}{2}$.

Find the least value of n for which $v_n > u_n$.

Answers

1 **a** $u_1 = 2$ and $d = 6 - 2 = 4$, so $u_{16} = 2 + 4(16-1) = 62$

b $u_1 = 11$ and $d = 8 - 11 = -3$, so $u_{12} = 11 + (-3)(12-1) = -22$

c $u_1 = 4$ and $d = 4.7 - 4 = 0.7$, so $u_{21} = 4 + 0.7(21-1) = 18$

d $u_1 = 23.5$ and $d = 22.8 - 23.5 = -0.7$, so $u_{25} = 23.5 + (-0.7)(25-1) = 6.7$

e $u_1 = \frac{1}{4}$ and $d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $u_9 = \frac{1}{4} + \frac{1}{4}(9-1) = \frac{9}{4}$

f $u_1 = x - 5$ and $d = (2x-3) - (x-5) = x + 2$, so $u_n = x - 5 + (x+2)(10-1) = 10x + 13$

2 **a** $u_1 = 7$ and $d = 3$

$7 + 3(n-1) = 40 \rightarrow 3n + 4 = 40 \rightarrow 3n = 36 \rightarrow n = 12$

b $u_1 = 80$ and $d = -6$

$80 + (-6)(n-1) = 2 \rightarrow -6n + 86 = 2 \rightarrow -6n = -84 \rightarrow n = 14$

c $u_1 = 45$ and $d = -0.4$

$45 + (-0.4)(n-1) = 35 \rightarrow -0.4n + 45.4 = 35 \rightarrow -0.4n = -10.4 \rightarrow n = 26$

d $u_1 = 4x$ and $d = 12x$

$4x + 12x(n-1) = 100x \rightarrow 12x(n-1) = 96x \rightarrow n - 1 = 8 \rightarrow n = 9$

e $u_1 = 10.5$ and $d = 1.5$

$10.5 + 1.5(n-1) = 48 \rightarrow 1.5n + 9 = 48 \rightarrow 1.5n = 39 \rightarrow n = 26$

f $u_1 = -21$ and $d = 4$

$-21 + 4(n-1) = 23 \rightarrow 4n - 25 = 23 \rightarrow 4n = 48 \rightarrow n = 12$

3 **a** $u_1 = 2$ and $r = \frac{6}{2} = 3$, so $u_9 = 2\left(3^{9-1}\right) = 2(6561) = 13122$

b $u_1 = 64$ and $r = \frac{96}{64} = \frac{3}{2}$, so $u_7 = 64\left(\left(\frac{3}{2}\right)^{7-1}\right) = 64\left(\frac{729}{64}\right) = 729$

c $u_1 = 3$ and $r = \frac{-6}{3} = -2$, so $u_{12} = 3\left((-2)^{12-1}\right) = 3(-2048) = -6144$

d $u_1 = 5$ and $r = \frac{5.5}{5} = 1.1$, so $u_8 = 5\left(1.1^{8-1}\right) = 5(2.14358881) = 10.71794405$

e $u_1 = \frac{9}{4}$ and $r = \frac{\left(\frac{3}{2}\right)}{\left(\frac{9}{4}\right)} = \frac{2}{3}$, so $u_6 = \frac{9}{4}\left(\left(\frac{2}{3}\right)^{6-1}\right) = \frac{9}{4}\left(\frac{32}{243}\right) = \frac{8}{27}$

f $u_1 = m$ and $r = \frac{m^3}{m} = m^2$, so $u_{10} = m\left(\left(m^2\right)^{10-1}\right) = m\left(m^{18}\right) = m^{19}$

4 $u_3 + 4d = u_7$, so $10 + 4d = 26 \rightarrow 4d = 16 \rightarrow d = 4$

$u_1 + 2d = u_3$, so $u_1 + 2(4) = 10 \rightarrow u_1 = 2$

5 $u_4 + 2d = u_6$, so $42 + 2d = 30 \rightarrow 2d = -12 \rightarrow d = -6$

$u_{10} = u_4 + 6d = 42 + 6(-6) = 42 - 36 = 6$

6 **a** $u_2(r^2) = u_4$, so $12(r^2) = 108 \rightarrow r^2 = 9 \rightarrow r = \pm 3$

b $u_7 = u_2(r^5)$, so $u_7 = 12(3^5) = 12(243) = 2916$,

or $u_7 = 12\left((-3)^5\right) = 12(-243) = -2916$

7 $u_n = 16\left(1.5^{n-1}\right)$

$u_{10} = 615.09375$, and $u_{11} = 922.640625$, so $n = 11$

(Remember to give both "crossover values", not just the final answer.)

8 $u_n = 1000 + 15(n-1)$ and $v_n = 6\left(\left(\frac{3}{2}\right)^{n-1}\right)$

When $n = 14$, $u_{14} = 1195$ and $v_{14} = 1167.7170...$, and

when $n = 15$, $u_{15} = 1210$ and $v_{15} = 1751.5755...$, so $n = 15$.

1.3 Arithmetic and geometric series

1 For each finite series below,

i decide whether the series is arithmetic or geometric

ii find the sum of the series.

a $17+19+21+\ldots+u_{12}$

b $36+28+20+\ldots+u_{16}$

c $5+10+20+\ldots+u_{14}$

d $76.3+75.5+74.7+\ldots+u_{10}$

e $64x+32x+16x+\ldots+u_7$

f $(a+3)+(4a+12)+(16a+48)+\ldots+u_6$

2 For each geometric series below, decide whether the series is converging or diverging.

If the series is converging, find S_∞.

a $1.6+2.4+3.6+\ldots$

b $81+54+36+\ldots$

c $\frac{8}{9}+\frac{2}{3}+\frac{1}{2}+\ldots$

d $\frac{4}{5}+1+\frac{5}{4}+\ldots$

3 Find the sum of the first 24 terms of the series 50 + 44 + 38 + …

4 In an arithmetic sequence, the fifth term is two times the second term, and the sum of the first ten terms is 390. Find the first term and the common difference.

5 For the series $2+2\sqrt{3}+6+6\sqrt{3}+\ldots$, find the least value of n for which $S_n > 5000$.

6 A geometric series has a first term of 2 and a common ratio of 3.

Find the value of n if $S_n = 59048$.

7 A series has the formula $S_n = 2^n - 1$.

a Find the values of S_1, S_2, and S_3.

b Find the values of u_1, u_2, and u_3.

c Write a general formula for u_n.

8 For the series 90 + 60 + 40 + …, find the value of r and the sum to infinity.

9 Consider the geometric sequence 2048, 1536, 1152, …

Find the least value of n such that the n^{th} term of the sequence is less than 4.

10 In a geometric sequence, the fourth term is 27 times the first term, and the sum of the first four terms is 68. Find the first term and the common ratio.

11 Consider a geometric series with $S_3 = 490$ and $S_\infty = 625$.

Find the first term and the common ratio.

Answers

1 **a** **i** arithmetic

ii $u_1 = 17$ and $d = 2$, so $S_{12} = \frac{12}{2}\left(2(17) + 2(12 - 1)\right) = 6(34 + 22) = 336$

b **i** arithmetic

ii $u_1 = 36$ and $d = -8$, so $S_{16} = \frac{16}{2}\left(2(36) + (-8)(16 - 1)\right) = 8(72 - 120) = -384$

c **i** geometric

ii $u_1 = 5$ and $r = 2$, so $S_{14} = \frac{5\left(2^{14} - 1\right)}{2 - 1} = 5(16383) = 81915$

d **i** arithmetic

ii $u_1 = 76.3$ and $d = -0.8$, so $S_{10} = \frac{10}{2}\left(2(76.3) + (-0.8)(10 - 1)\right) = 5(145.4) = 727$

e **i** geometric

ii $u_1 = 64x$ and $r = 0.5$, so $S_7 = \frac{64x\left(1 - 0.5^7\right)}{1 - 0.5} = \frac{64x(0.9921875)}{0.5} = 127x$

f **i** geometric

ii $u_1 = a + 3$ and $r = 4$, so

$$S_6 = \frac{(a+3)\left(4^6 - 1\right)}{4 - 1} = \frac{(a+3)(4095)}{3} = 1365(a+3) = 1365a + 4095$$

2 **a** diverging, because $r > 1$

b converging, because $r < 1$

$u_1 = 81$ and $r = \frac{2}{3}$, so $S_\infty = \frac{81}{\left(1 - \frac{2}{3}\right)} = 243$

c converging, because $r < 1$

$u_1 = \frac{8}{9}$ and $r = \frac{3}{4}$, so $S_\infty = \frac{\left(\frac{8}{9}\right)}{\left(1 - \frac{3}{4}\right)} = \frac{32}{9}$

d diverging, because $r > 1$

3 $u_1 = 50$ and $d = -6$, so $S_{24} = \frac{24}{2}\left(2(50) + (-6)(24 - 1)\right) = 12(100 - 138) = -456$

4 $u_5 = 2u_2$, so $u_1 + 4d = 2(u_1 + d) \rightarrow u_1 + 4d = 2u_1 + 2d \rightarrow u_1 = 2d$

$S_{10} = \frac{10}{2}(2u_1 + 9d) = 5\left(2(2d) + 9d\right) = 65d = 390 \rightarrow d = 6$

$u_1 = 2d = 2(6) = 12$

5 $u_1 = 2$ and $r = \sqrt{3}$, so $S_n = \dfrac{2\left(\left(\sqrt{3}\right)^n - 1\right)}{\sqrt{3} - 1}$

$S_{13} = 3446.9329\ldots$, and $S_{14} = 5972.263\ldots$, so $n = 14$

(Remember to give both "crossover values", not just the final answer.)

6 $u_1 = 2$ and $r = 3$, so $S_n = \frac{2\left((3)^n - 1\right)}{3-1}$

$S_{10} = 59048$, so $n = 10$

7 **a** $S_1 = 2^1 - 1 = 1$

$S_2 = 2^2 - 1 = 3$

$S_3 = 2^3 - 1 = 7$

b $u_1 = 1$

$u_2 = S_2 - S_1 = 3 - 1 = 2$

$u_3 = S_3 - S_2 = 7 - 3 = 4$

c $u_1 = 1$ and $r = 2$, so $u_n = 1\left(2^{n-1}\right) = 2^{n-1}$

8 $u_1 = 90$ and $r = \dfrac{60}{90} = \dfrac{2}{3}$, so $S_\infty = \dfrac{90}{\left(1 - \dfrac{2}{3}\right)} = 270$

9 $u_1 = 2048$ and $r = \dfrac{1536}{2048} = \dfrac{3}{4}$, so $u_n = 2048\left(\left(\dfrac{3}{4}\right)^{n-1}\right)$

$u_{22} = 4.87098\ldots$, and $u_{23} = 3.6532\ldots$, so $n = 23$

10 $u_4 = u_1\left(r^3\right) = 27u_1 \rightarrow r^3 = 27 \rightarrow r = 3$

$S_4 = \dfrac{u_1\left(r^4 - 1\right)}{r - 1} = \dfrac{u_1\left(3^4 - 1\right)}{3 - 1} = 40u_1 = 68 \rightarrow u_1 = \dfrac{68}{40} = 1.7$

11 $S_3 = \dfrac{u_1\left(1 - r^3\right)}{1 - r} = \dfrac{u_1}{1 - r}\left(1 - r^3\right) = 490$, and $S_\infty = \dfrac{u_1}{1 - r} = 625$

Substituting 625 for $\dfrac{u_1}{1 - r}$ in the first equation gives

$625\left(1 - r^3\right) = 490 \rightarrow 1 - r^3 = \dfrac{98}{125} \rightarrow r^3 = \dfrac{27}{125} \rightarrow r = \dfrac{3}{5}$

$S_\infty = \dfrac{u_1}{1 - \dfrac{3}{5}} = 625 \rightarrow u_1 = 250$

1.4 Applications of arithmetic and geometric patterns

1 A supermarket has a display of soup cans stacked in a pyramid.

The first row at the top of the pyramid has one can, the second row has three cans, the third row has five cans, and so on.

a Find the total number of soup cans in the top seven rows of the pyramid.

b Find the number of rows in the pyramid if the display contains 361 soup cans.

2 A restaurant has stacks of plates on a shelf. A stack of four plates is 4.1 cm high. A stack of twelve plates is 9.3 cm high.

a How much does each additional plate add to the height of a stack?

b How high is a stack of 16 plates?

c How many plates are in a stack 7.35 cm high?

3 On 01 January, 2015, the population of Franktown was 35,000 people, and it has been increasing at a steady rate of 1.7% annually.

a Estimate the population of Franktown on 01 January, 2019.

b During what year should the population of Georgetown first exceed 50000 people?

4 Ever since he turned ten years old, Bram's grandmother has given him money on his birthday. On his 10th birthday, she gave him €5. On his 11th birthday, she gave him €10. On his 12th birthday, she gave him €20.

If this pattern continues, how much will Bram's grandmother give him on his 18th birthday, and how much will she have given him in total?

5 Mina invests \$3000 in an account which pays 1.5% annual interest.

Assuming she makes no additional withdrawals or deposits, find the amount Mina will have in her account after ten years if the interest is compounded

a quarterly

b monthly

c weekly.

Answers

1 **a** $u_1 = 1$ and $d = 2$, so $S_7 = \frac{7}{2}\left(2(1) + 2(7-1)\right) = 3.5(14) = 49$

b $S_n = \frac{n}{2}\left(2(1) + 2(n-1)\right) = \frac{n}{2}(2n) = n^2 = 361 \rightarrow n = 19$

2 **a** $u_{12} = u_4 + 8d$, so $9.3 = 4.1 + 8d \rightarrow 8d = 5.2 \rightarrow d = 0.65$

Each additional plate adds 0.65 centimetres to the height of the stack.

b $u_4 = u_1 + 3d$, so $4.1 = u_1 + 3(0.65) \rightarrow u_1 = 2.15$

$u_{16} = u_1 + 15d$, so $u_{16} = 2.15 + 15(0.65) = 11.9$

A stack of 16 plates is 11.9 cm high.

c $u_n = 2.15 + 0.65(n-1) = 7.35 \rightarrow 0.65(n-1) = 5.2 \rightarrow n - 1 = 8 \rightarrow n = 9$

3 **a** $P = 35000\left(1.017^n\right)$, where n is the number of years since 01 January, 2015.

So on 01 January, 2019, $P = 35000\left(1.017^9\right) = 40733.958...$

The estimated population is 40734.

b $P = 35000\left(1.017^n\right)$

When $n = 21$ (on 01 January, 2036), $P = 49866.4045...$, and when $n = 22$ (on 01 January, 2037), $P = 50714.133...$, so the population will first exceed 50000 sometime during the year 2036.

4 $u_1 = 5$ and $r = 2$. On Bram's 18th birthday, $n = 9$.

$u_9 = 5\left(2^{9-1}\right) = 1280$, on Bram's 18th birthday his grandmother will give him €1280.

$S_9 = \dfrac{5\left(2^9 - 1\right)}{2 - 1} = 2555$, Bram's grandmother will have given him a total of €2555.

5 **a** $A = 3000\left(1 + \dfrac{0.015}{4}\right)^{10(4)} \approx \$3484.53$

b $A = 3000\left(1 + \dfrac{0.015}{12}\right)^{10(12)} \approx \$3485.18$

c $A = 3000\left(1 + \dfrac{0.015}{52}\right)^{10(52)} \approx \$3485.43$

1.5 The binomial theorem

1 Without using your GDC, calculate each value.

a ${}_6C_3$ **b** $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ **c** ${}^7C_4$

2 Use the binomial theorem to expand the expression $(x-2)^5$.

3 Use the binomial theorem to expand the expression $(2x+3)^4$.

4 Use the binomial theorem to expand the expression $\left(x+\dfrac{2}{x}\right)^5$.

5 Find the term in x^3 in the expansion of $(x+4)^5$.

6 In the expansion of $(2x-3)^8$, find the term in x^6.

7 Find the constant in the expansion of $\left(3x+\dfrac{1}{x}\right)^8$.

8 Find the constant term in the expansion of $\left(6x-\dfrac{2}{3x^2}\right)^9$.

9 In the expansion of $(4x+1)^n$, the coefficient of the term in x^2 is 1248.

Find the value of n.

10 In the expansion of $(5x+1)^n$, the coefficient of the term in x^3 is 7000.

Find the value of n.

11 Find the term in x^4 in the expansion of $3x(x-5)^6$.

12 In the expansion of $2x\left(4x+\dfrac{a}{x}\right)^5$, the constant term is 8640. Find the value of a.

Answers

1 **a** $_6C_3 = \frac{6!}{3!3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1} = \frac{6\times5\times4}{3\times2\times1} = 5\times4 = 20$

b $\binom{8}{2} = \frac{8!}{2!6!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times6\times5\times4\times3\times2\times1} = \frac{8\times7}{2\times1} = \frac{56}{2} = 28$

c $^7C_4 = \frac{7!}{4!3!} = \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times3\times2\times1} = \frac{7\times6\times5\times4}{4\times3\times2\times1} = 7\times5 = 35$

2 $(x-2)^5 =$

$\binom{5}{0}x^5(-2)^0 + \binom{5}{1}x^4(-2)^1 + \binom{5}{2}x^3(-2)^2 + \binom{5}{3}x^2(-2)^3 + \binom{5}{4}x^1(-2)^4 + \binom{5}{5}x^0(-2)^5$

$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

3 $(2x+3)^4 = \binom{4}{0}(2x)^4(3)^0 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)^1(3)^3 + \binom{4}{4}(2x)^0(3)^4$

$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

4 $\left(x+\frac{2}{x}\right)^5 = \binom{5}{0}x^5\left(\frac{2}{x}\right)^0 + \binom{5}{1}x^4\left(\frac{2}{x}\right)^1 + \binom{5}{2}x^3\left(\frac{2}{x}\right)^2 + \binom{5}{3}x^2\left(\frac{2}{x}\right)^3 + \binom{5}{4}x^1\left(\frac{2}{x}\right)^4 + \binom{5}{5}x^0\left(\frac{2}{x}\right)^5$

$= x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$

5 $\binom{5}{2}(x)^3(4)^2 = 160x^3$

6 $\binom{8}{2}(2x)^6(-3)^2 = 16128x^6$

7 $\binom{8}{4}(3x)^4\left(\frac{1}{x}\right)^4 = 5670$

8 $\binom{9}{3}(6x)^6\left(\frac{-2}{3x^2}\right)^3 = 84\left(46656x^6\right)\left(\frac{-8}{27x^6}\right) = -1161216$

9 $\binom{n}{2}(4x)^2(1)^{n-2} = \frac{n\times(n-1)\times(n-2)\times(n-3)\times\ldots}{2\times1\times(n-2)\times(n-3)\times\ldots}\left(16x^2\right)(1) = \frac{n\times(n-1)}{2\times1}\left(16x^2\right) = 1248x^2$

$= \frac{n^2-n}{2}(16) = 1248 \rightarrow n^2 - n = 156 \rightarrow n^2 - n - 156 = 0 \rightarrow n = -12$ or $n = 13$

but n must be a positive integer, so $n = 13$.

10 $\binom{n}{3}(5x)^3(1)^{n-3} = \frac{n\times(n-1)\times(n-2)\times(n-3)\times(n-4)\times\ldots}{3\times2\times1\times(n-3)\times(n-4)\times\ldots}\left(25x^3\right)(1)$

$= \frac{n\times(n-1)\times(n-2)}{3\times2\times1}\left(125x^3\right) = 7000x^3$

$= \frac{n^3-3n^2+2n}{6}(125) = 7000 \rightarrow n^3 - 3n^2 + 2n = 336 \rightarrow n^3 - 3n^2 + 2n - 336 = 0$

Use the polynomial root finder on your GDC to get $n = 8$.

11 $\binom{6}{3}(x)^3(-5)^3 = -2500x^3$

$3x\left(-2500x^3\right) = -7500x^4$

12 $\binom{5}{3}(4x)^2\left(\frac{a}{x}\right)^3 = \frac{160a^3}{x}$

$2x\left(\frac{160a^3}{x}\right) = 320a^3 = 8640 \rightarrow a^3 = 27 \rightarrow a = 3$

1.6 Proofs

1 Prove that $x(x+2)-2(3x-2)=(x-2)^2$.

2 **a** Show that $\dfrac{x^2+5x+6}{x+2}-3=x$.

b For what value of x does this statement not hold true?

3 Prove that $(x+2)^2-(x+3)^2=-(2x+5)$.

Answers

1 $x(x+2)-2(3x-2)=(x-2)^2$

$x^2+2x-6x+4=(x-2)^2$

$x^2-4x+4=(x-2)^2$

$(x-2)(x-2)=(x-2)^2$

$(x-2)^2=(x-2)^2$

2 **a** $\dfrac{x^2+5x+6}{x+2}-3=\dfrac{(x+3)(x+2)}{x+2}-3=x+3-3=x$

b The statement does not hold true when the denominator is equal to zero, when $x=-2$

3 $(x+2)^2-(x+3)^2=-(2x+5)$

$x^2+4x+4-(x^2+6x+9)=-(2x+5)$

$x^2+4x+4-x^2-6x-9=-(2x+5)$

$-2x-5=-(2x+5)$

$-(2x+5)=-(2x+5)$

2.1 What is a function?

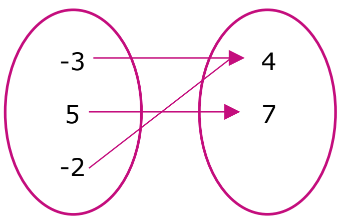
1 Decide if the following relations are functions:

a $\{(-5,1),(-2,7),(1,2),(4,3),(6,-5)\}$

b $\{(-5,1),(-2,7),(-2,2),(4,3),(6,-5)\}$

c $\{(-4,1),(-2,1),(1,1),(4,1),(6,1)\}$

d



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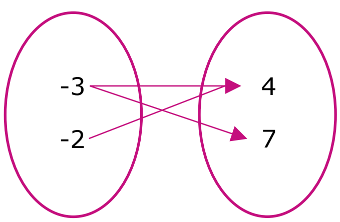
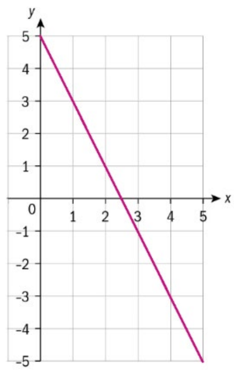


f $y = 3x + 1$

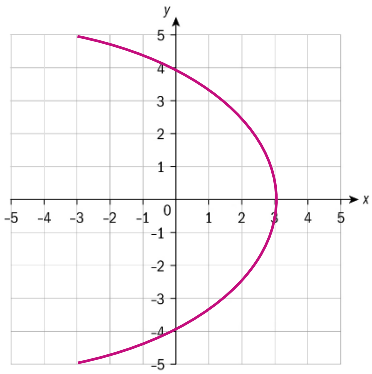
g $y = x^2 - 5$

2 Decide if the following graphs are functions:

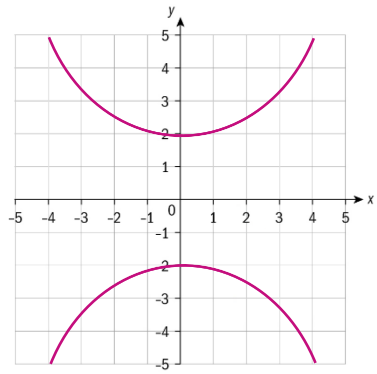
a



b



c



Answers

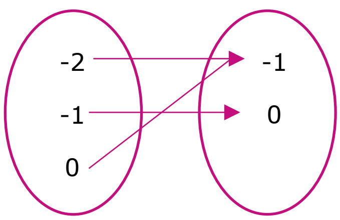
1	**a**	Yes	**b**	No	**c**	Yes	**d**	Yes	**e**	No	**f**	Yes	**g**	Yes
2	**a**	Yes	**b**	No	**c**	No								

2.2 Functional notation

1 Given the functions below, evaluate each for $f(-1)$ and $f(0)$

a $f(x) = \{(-6,0),(-1,1),(0,-4),(1,0)\}$.

b



c

x	-3	-2	-1	0
$f(x)$	0	-1	-2	-3

2 Given that $f(x) = -x^2 - 1$ and $g(x) = -3x + 1$, find:

a $f(-1)$

b $g(-2) + f(0)$

c $-2f(1) - g(3) + 1$

3 If $f : x \to -2x + 1$ and $g : x \to= 3x - 1$, given that $-2f(x) + g(x) = -2$, find the value of x.

4 Sanjeet wants to buy an annual membership to online grocery delivery service. The cost can be represented by the function $C(n) = 5n + 25$ where n is the number of time you place an order.

a Explain what the 25 in the function might represent.

b Explain what $C(12)$ means in the context of the question and find its value.

c Find the number of orders Sanjeet can place if we doesn't want to spend more than $100 this year for this membership.

Answers

1 **a** $f(-1)=1 \quad f(0)=-4$ **b** $f(-1)=0 \quad f(0)=-1$ **c** $f(-1)=-1 \quad f(0)=-3$

2 **a** $f(-1)=-(-1)^2-1=-1-1=-2$

b $g(-2)+f(0)=-3(-2)+1+-0^2-1=-6+1-1=-6$

c $-2f(1)-g(3)+1=-2\left(-(1)^2-1\right)-\left(-3(3)+1\right)+1=2(-2)-(-8)+1=-4+8+1=5$

3 $2f(x)+g(x)=-2$

$2(-2x+1)+3x-1=-2$

$-4x+2+3x-1=-2$

$-x+1=-2$

$-x=-3$

$x=3$

4 **a** The 25 might represent the annual fee before any orders have been placed.4b.

b $C(12)$ means the total cost of the membership after 12 orders have been places.

$C(12)=5(12)+25=60+25=\$28$

c $100=5n+25$

$75=5n$

n = 15 orders

2.3 Drawing graphs of functions

1 Use your GDC to sketch the graphs of the following functions:

a $f(x) = -0.3x^2 - 0.6x + 1.2$

b $g : x \to -2.5x + 6.3$

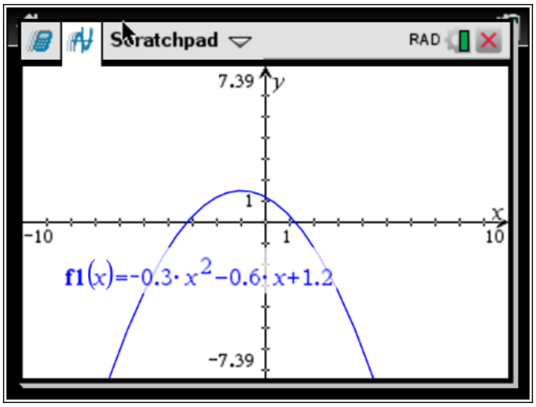
c $y = -2\cos(x - 1)$

d $f(x) = 2^{x-1} - 4$

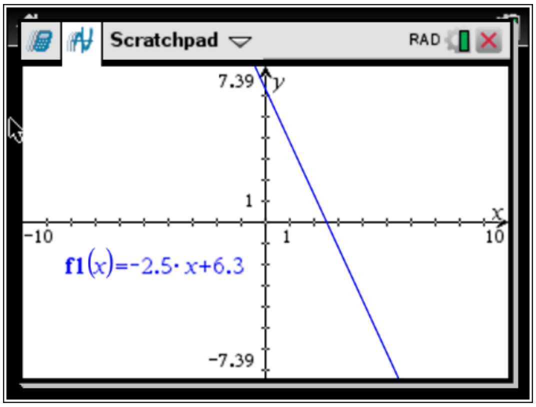
e $y = -0.25\sqrt{x} + 1.3$

Answers

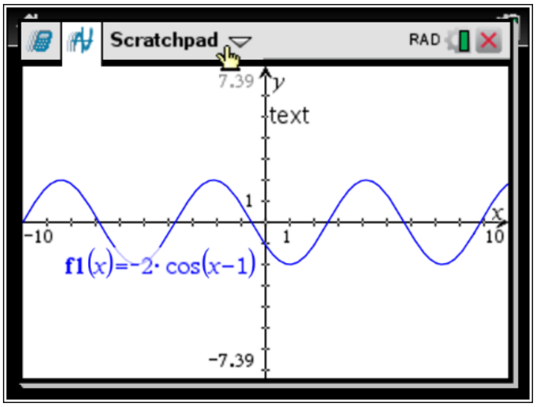
1 a



b



c



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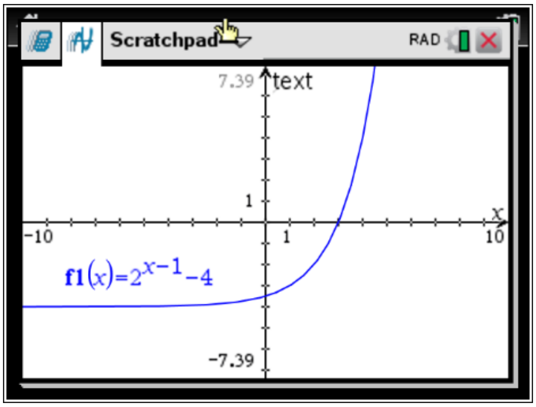

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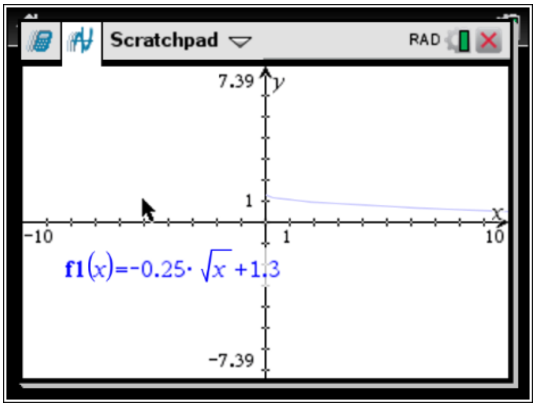


4.4 The domain and range of a function

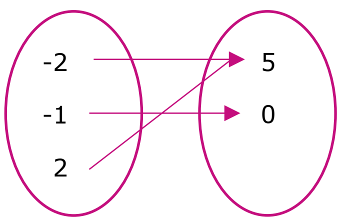
1 For each of the domains below, write the meaning in words.

a $(-3,4]$ **b** $x > -5$ **c** $]-\infty,\infty[$

2 For each function below, state the domain and range.

a $g : x \to \{(1,2),(5,6),(9,10),(13,14)\}$

b



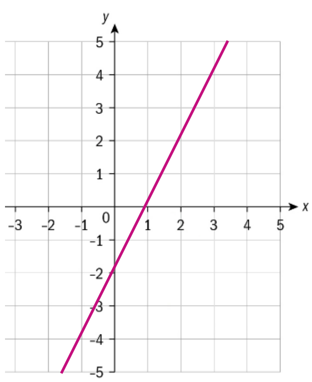
c

x	-10	-8	-6	-4
$f(x)$	-2	0	2	4

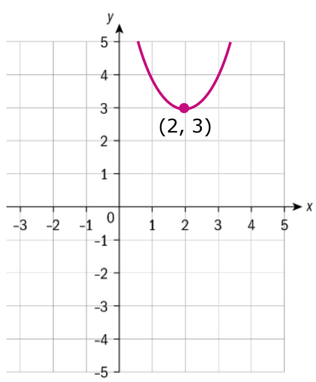
3 For each graph below:

i State if the graph is a function **ii** State the domain and range

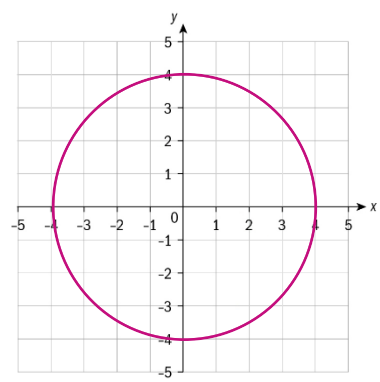
a $y = 2x - 2$



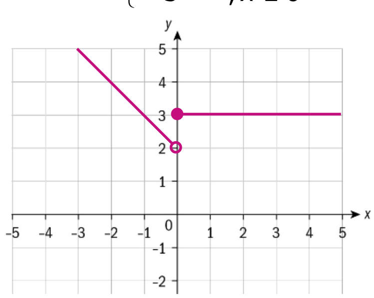
b $y = (x-2)^2 + 3$



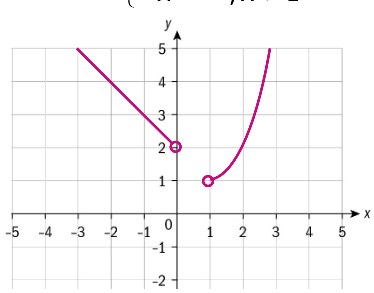
c $x^2 + y^2 = 16$



d $f(x) = \begin{cases} -x+2 & ,x < 0 \\ 3 & ,x \geq 0 \end{cases}$



e $f(x) = \begin{cases} -x+2 & ,x < 0 \\ x^2 & ,x > 1 \end{cases}$



4 For each domain and range below, draw a possible graph.

a Domain: $x \in \mathbb{R}$ Range : $y \in \mathbb{R}$

b Domain: $x \in \mathbb{R}$ Range : $y = 2$

c Domain: $[0, \infty[$ Range : $[0, \infty[$

d Domain: $-3 \le x < 2$ Range : $1 \le y < 4$

Answers

1 **a** All values from -3 non-inclusive to 4 inclusive

b All values greater than -5 inclusive

c All real numbers

2 **a** Domain = $\{1,5,9,12\}$, range = {2, 6, 10, 14}

b Domain = $\{-2,-1,2\}$, range = {0, 5}

c Domain = $\{-10,-8,-6,-4\}$, range = {-2, 0, 2, 4}

d Domain = $\{-4, -1, 0, 2, 3\}$, range = {-2, 0, 3, 5}

3 **a** **i** Yes

ii Domain: $x \in \mathbb{R}$ or $(-\infty,\infty)$ or $]-\infty,\infty[$, range: $y \in \mathbb{R}$ or $(-\infty,\infty)$ or $]-\infty,\infty[$

b **i** Yes

ii Domain: $x \in \mathbb{R}$ or $(-\infty,\infty)$ or $]-\infty,\infty[$, range: $y \geq 3$ or $[-3,\infty)$ or $[-3,\infty[$

c **i** No

ii Domain: $-4 \leq x \leq 4$ or $[-4,4]$, range: $-4 \leq y \leq 4$ or $[-4,4]$

d **i** Yes

ii Domain: $x \in \mathbb{R}$ or $(-\infty,\infty)$ or $]-\infty,\infty[$, range $y > 2$ or $(2,\infty)$ or $]-2,\infty[$

e **i** Yes

ii Domain: $x\langle 0, x\rangle 1$ or $(-\infty,0)\cup(1,\infty)$ or $]-\infty,0[\cup]1,\infty[$

4 There are many possible answers.

2.5 Composite functions

1 Given the functions

$f(x) = \{(-1,2),(2,4),(3,-5),(4,-6)\}$

$g(x) = \{(-6,-1),(-5,0),(-4,-1),(1,4)\}$

Find:

a $f(g(-4))$ **b** $g(f(3))$ **c** $f(f(2)$ **d** $g \circ f \circ g(1)$

2 Given $f(x) = -2x^2 - 2$, $g(x) = -3x + 2$ and $h(x) = -3$, find:

a $f \circ g(x)$ **b** $g(f(x))$ **c** $g(h(-2))$ **d** $f(f(-1))$ **e** $f \circ g \circ f \circ h(0)$

3 If $f(x) = -2x^2$ and $g(x) = 3\sqrt{x}$, state the domain of $f \circ g(x)$.

4 Michael works for a software company. He gets paid a bonus when his sales for a particular software program reach $1000. For any sales over $1000, he earns 2%. Consider two functions $f(x) = x - 1000$ and $g(x) = 0.02x$.

a Which composition of functions, $f(g(x))$ or $g(f(x))$ could be used to calculate the amount of Michael's bonus? Justify your choice.

b Calculate Michael's bonus if he sold $12350 worth of the software.

5 If $f \circ g(x) = 2x^2 + 2x + 2$ and $f(x) = 2x + 4$, find $g(x)$.

Answers

1 a $g(-4) = -1$

$f(g(-4)) = f(-1) = 2$

b $f(3) = -5$

$g(f(3)) = g(-5) = 0$

c $f(2) = 4$

$f(f(2)) = f(4) = -6$

d $g(1) = 4$

$f \circ g(1) = f(4) = -6$

$g \circ f \circ g(1) = g(-6) = -1$

2 a $f \circ g(x) = -2(-3x + 2)^2 - 2$

$f \circ g(x) = -2(9x^2 - 12x + 4) - 2$

$f \circ g(x) = -18x^2 + 24x - 8 - 2$

$f \circ g(x) = -18x^2 + 24x - 10$

b $g(f(x)) = -3(-2x^2 - 2) + 2$

$g(f(x)) = 6x^2 + 6 + 2$

$g(f(x)) = 6x^2 + 8$

c $h(-2) = -3$

$g(h(-2)) = g(-3) = -3(-3) + 2 = 9 + 2 = 11$

d $f(-1) = -2(-1)^2 - 2 = -2 - 2 = -4$

$f(f(-1)) = f(-4) = -2(-4)^2 - 2 = -32 - 2 = -34$

e $h(0) = -3$

$f \circ h(0) = f(-3) = -2(-3)^2 - 2 = -2(9) - 2 = -18 - 2 = -20$

$g \circ f \circ h(0) = g(-20) = -3(-20) + 2 = 60 + 2 = 62$

$f \circ g \circ f \circ h(0) = f(62) = -2(62)^2 - 2 = -2(3844) - 2 = -7688 - 2 = -7690$

3 $f \circ g(x) = -2\left(3\sqrt{x}\right)^2 = -2(9x) = -18x$. Due to the domain restriction on $g(x)$, the domain is $x \geq 0, or \left[0, \infty\right) or \left[0, \infty\right[$.

4 **a** The correct composition is $g\left(f(x)\right)$ and you need to calculate the sales over \$1000 before calculating the percent.

b $g\left(f(x)\right) = 0.02(x - 1000)$

$g\left(f(2350)\right) = 0.02(12350 - 1000) = 0.02(11350) = \$227$

5 Let $a = g(x)$.

$f\left(g(x)\right) = 2x^2 + 2x + 2 = f(a) = 2a + 4$

$2x^2 + 2x + 2 = 2a + 4$

$2x^2 + 2x - 2 = 2a$

$a = x^2 + x - 1$

$g(x) = x^2 + x - 1$

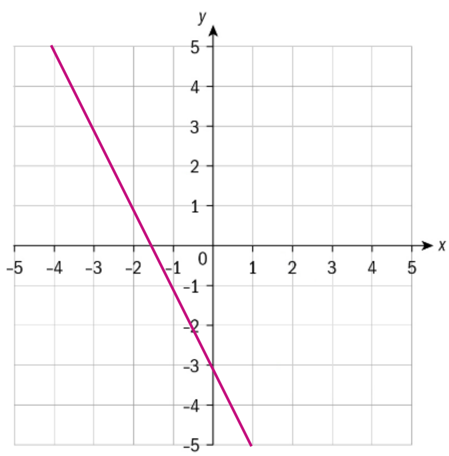
2.6 Inverse functions

1 Show that $f(x) - 2x + 4 =$ and $g(x) = -\frac{1}{2}x + 2$ are inverses.

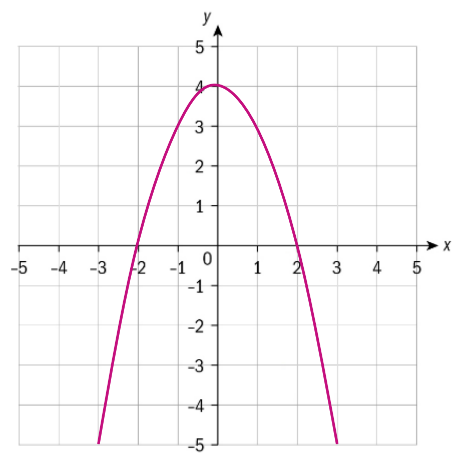
2 Explain what a one-to-one function is and give two different examples.

3 Given the graphs below, sketch the graph of their inverses. Which of the graphs below are one-to-one functions?

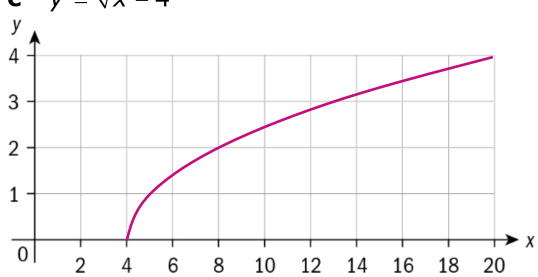
a $y = -2x - 3$



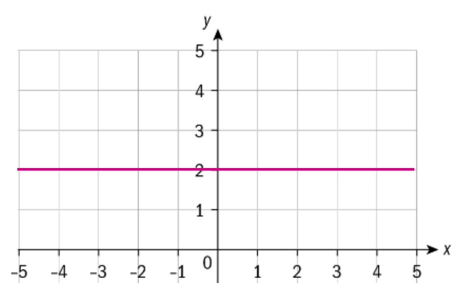
b $y = -x^2 + 4$



c $y = \sqrt{x - 4}$



d $y = 2$



4 Find the equation of the inverse of each of the functions below:

a $f : x \to -2x + 5$

b $g(x) = (x + 1)^2 - 3$

c $f(x) = -x^3 - 5$

d $y = 2\sqrt{x + 1} - 4$

5 Explain the relationship between the domain and range of a function and the domain and range of its inverse.

6 Let $f(x) = x^2 + 2x + 1$ and $g(x) = x - 5$, for $x \in \mathbb{R}$.

a Find $f(8)$.

b Find $(g \circ f)(x)$.

c Find $g(f(-1))$.

d Solve $g(f(x)) = 0$.

e State the domain and range of $f(x)$.

f State the domain and range of $f^{-1}(x)$.

g Draw the graph of $g(x)$.

h On the same set of axes, sketch the graph of $g^{-1}(x)$.

i Explain if $g(x)$ is a one-to-one function.

j Find the equation of $g^{-1}(x)$.

7 Show that $f(x) = \dfrac{3x - 5}{x - 3}$ is a self-inverse.

Answers

1 $f\left(g\left(x\right)\right) = -2\left(-\frac{1}{2}x+2\right)+4 = x-4+4 = x$

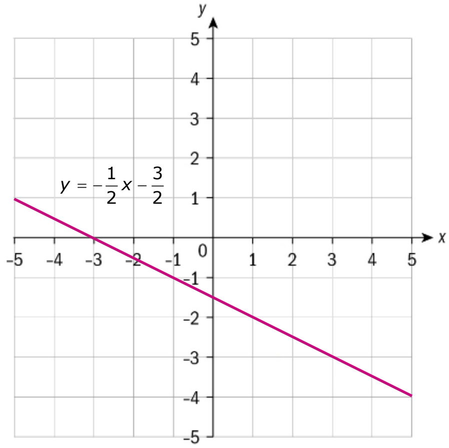
$g\left(f\left(x\right)\right) = -\frac{1}{2}\left(-2x+4\right)+2 = x-2+2 = x$

Since $f\left(g\left(x\right)\right) = g\left(f\left(x\right)\right) = x$, the two functions are inverses.

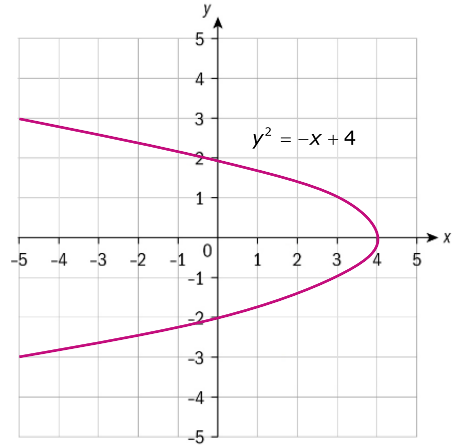
2 A one-to-one function is any functions whose inverse is also a function.

Two examples are $f\left(x\right) = x$ and $g\left(x\right) = \sqrt{x}$

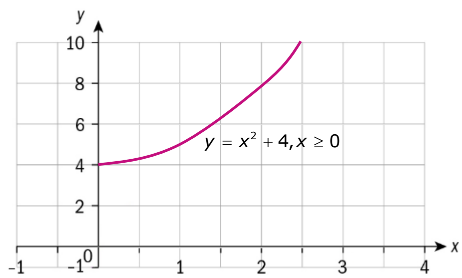
3 **a** Yes this is a one-to-one function.



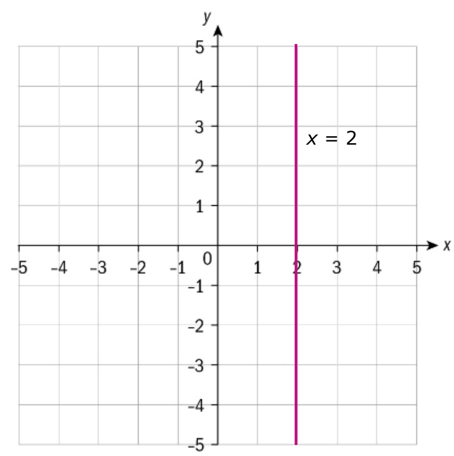
b No this is not a one-to-one function.



c Yes this is a one-to-one function.



d No this is not a one-to-one function.



4 **a** $f : x \to -2x+5$

$x = -2y+5$

$x-5 = -2y$

$y = -\frac{1}{2}x + \frac{5}{2}$

$f^{-1}x :\to -\frac{1}{2}x + \frac{5}{2}$

b $g(x) = (x+1)^2 - 3$

$x = (y+1)^2 - 3$

$x + 3 = (y+1)^2$

$\pm\sqrt{x+3} = y + 1$

$y = \pm\sqrt{x+3} - 1$

$g^{-1}(x) = \pm\sqrt{x+3} - 1$

c $f(x) = -x^3 - 5$

$x = -y^3 - 5$

$x + 5 = -y^3$

$-x - 5 = y^3$

$\sqrt[3]{-x-5} = y$

$f^{-1}(x) = \sqrt[3]{-x-5}$

d $y = 2\sqrt{x+1} - 4$

$x = 2\sqrt{y+1} - 4$

$x + 4 = 2\sqrt{y+1}$

$\frac{1}{2}x + 2 = \sqrt{y+1}$

$\left(\frac{1}{2}x + 2\right)^2 = y + 1$

$y = \left(\frac{1}{2}x + 2\right)^2 - 1$

$f^{-1}(x) = \left(\frac{1}{2}x + 2\right)^2 - 1, x \geq 0$

5 The domain of the inverse is the range of the original function. The range of the inverse is the domain of the original function.

6 **a** $f(8) = 8^2 + 2(8) + 1 = 64 + 16 + 1 = 81$

b $(g \circ f)(x) = (x^2 + 2x + 1) - 5 = x^2 + 2x - 4$

c $g(f(-1)) = (-1)^2 + 2(-1) - 4 = 1 - 2 - 4 = -5$

d $g(f(x)) = 0 \Rightarrow x^2 + 2x - 4 = 0$

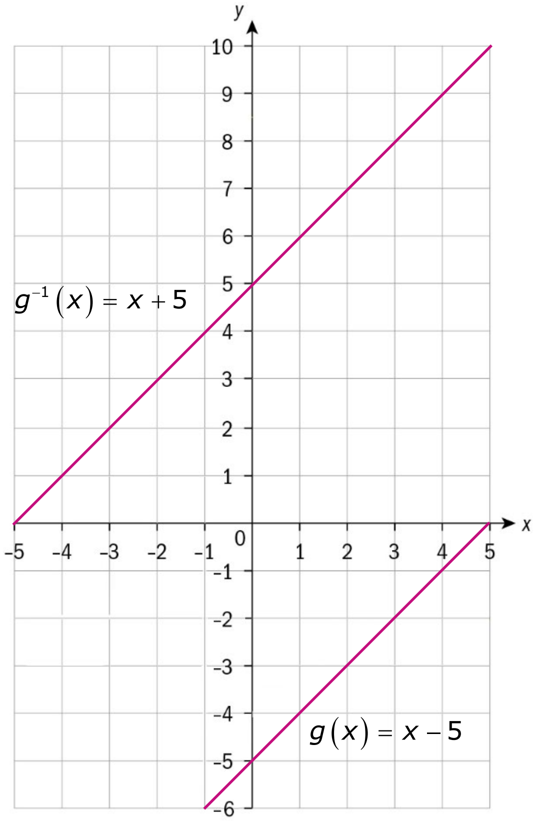
Using the quadratic formula: $x = -1 \pm \sqrt{5}$ or using GDC: $x \approx -3.23, 1.23$

e Since $f(x)$ is a parabola with vertex at (-1, 0),

Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$; Range: $y \geq 0$ or $[0, \infty)$ or $[0, \infty[$

f Domain: $x \geq 0$ or $[0, \infty)$ or $[0, \infty[$; Range: $y \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

g, h



Yes $g(x)$ is a one-to-one function since the inverse passes the vertical line test.

j $g^{-1}(x) = x + 5$

7 We need to show that $f\left(f\left(x\right)\right) = x$

$$\frac{3\left(\frac{3x-5}{x-3}\right)-5}{\left(\frac{3x-5}{x-3}\right)-3} = x$$

$$\frac{\frac{9x-15}{x-3}-\frac{5(x-3)}{x-3}}{\frac{3x-5}{x-3}-\frac{3(x-3)}{x-3}} = x$$

$$\frac{\frac{9x-15-5x+15}{x-3}}{\frac{3x-5-3x+9}{x-3}} = x$$

$$\frac{\frac{4x}{x-3}}{\frac{4}{x-3}} = x$$

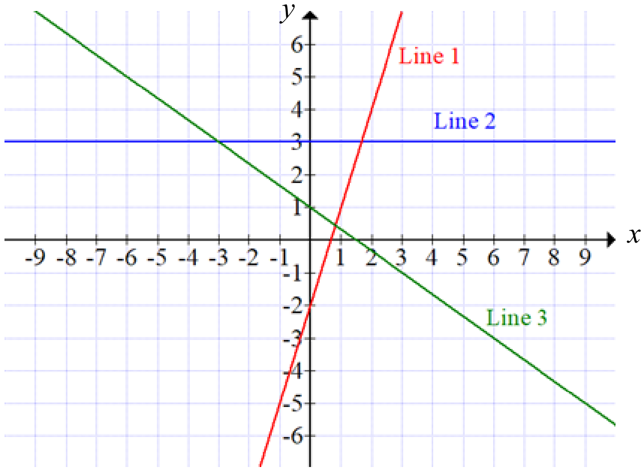
$$\frac{4x}{4} = x$$

$x = x$

$\therefore f\left(x\right)$ is a self-inverse.

3.1 Gradient of a linear function

1 Find the gradient of each line shown in the diagram below.



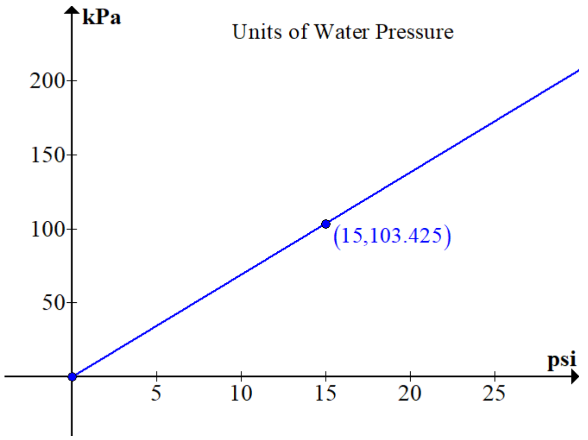
2 Find the gradient of the line passing through the given points.

a $(5,-2)$, $(1,4)$ **b** $(7,-2)$, $(4,-8)$ **c** $(5,-9)$, $(5,2)$

3 Line 1 has gradient $\frac{5}{3}$. Line 2 is parallel to line 1 and line 3 is perpendicular to line 1. Write down the gradient of line 2 and of line 3.

4 A line passes through the points $(6,-1)$ and $(3,y)$ and is perpendicular to a line with gradient $-\frac{3}{2}$. Find the value of y.

5 The relationship between two units of water pressure measurement, pounds per square inch (psi) and kilopascals (kPa) is modelled by the line shown in the diagram below.
The points $(0,0)$ and $(15,103.425)$ lie on the line.



a Find the gradient of the line.

b Explain the meaning of the gradient in context.

Answers

1 Line 1: From the point $(0,-2)$ you can move up 3 and right 1 to another point on the line, so $m=\frac{3}{1}=3$.

Line 2: Since the line is horizonal $m=0$.

Line 3: From the point $(0,1)$ you can move down 2 and right 3 to another point on the line, so $m=\frac{-2}{3}=-\frac{2}{3}$.

2 **a** $m=\frac{4-(-2)}{1-5}=\frac{6}{-4}=-\frac{3}{2}$

b $m=\frac{-8-(-2)}{4-7}=\frac{-6}{-3}=2$

c $m=\frac{2-(-9)}{5-5}=\frac{11}{0}\Rightarrow$ the gradient is undefined

3 Parallel lines have the same gradient, so the gradient of line 2 is $\frac{5}{3}$.

The product of the gradients of perpendicular lines is -1, so $\frac{5}{3}\times m=-1$ and so the gradient of line 3 is $-\frac{3}{5}$.

4 $-\frac{3}{2}\times m=-1\Rightarrow m=\frac{2}{3}$ **,** $\frac{y-(-1)}{3-6}=\frac{2}{3}\Rightarrow 3y+3=-6\Rightarrow y=-3$

5 **a** $m=\frac{103.425-0}{15-0}=6.895$

b For each increase of 6.895 kPa in pressure, there is an increase of 1 psi in pressure.

3.2 Linear functions

1 Draw the graph of each equation.

a $y = \frac{2}{3}x - 4$ **b** $x = -3$ **c** $y - 4 = -3(x + 2)$

d $y = 5$ **e** $y = \frac{4}{3}x$ **f** $4x - 2y + 6 = 0$

2 Find the equation of each line in either gradient-intercept form, $y = mx + c$, or point-gradient form, $y - y_1 = m(x - x_1)$. Then rewrite the equation in general form, $ax + by + d = 0$, where a, b and d are integers.

a the line with gradient 4 and y-intercept $(0,1)$

b the line with gradient 6 that passes through $(1,-3)$

c the line with y-intercept $(0,-2)$ and parallel to the line $y - 6 = -\frac{1}{3}(x + 1)$

d the line passes through $(-4,5)$ and is perpendicular to the line $y = -\frac{2}{3}x + 10$

3 Write an equation for each line.

a the horizontal line that passes through $(5,-6)$

b the vertical line that passes through $(-3,-2)$

c the line with gradient 0 that passes through $(6,8)$

d the line that passes through $(-4,5)$ and is perpendicular to the line $y = 3$

4 Write the equation in gradient-intercept form, $y = mx + c$. Then write down the gradient and the y-intercept of the line.

a $y + 5 = -2(x - 4)$ **b** $3x - 6y + 12 = 0$

5 Use a GDC to find the point of intersection of each pair of lines.

a $y = \frac{1}{2}x - 2$ and $y = 3x + 2$ **b** $y = 2.4x + 1.5$ and $y - 1.5 = -0.2(x + 3.4)$

6 Consider the linear functions $f(x) = 3x - 4$, $g(x) = \frac{1}{4}x + 2$, and $h(x) = -5$. Find the following.

a the domain and range of f **b** the domain and range of h

c $g(16)$ **d** $(f \circ g)(4)$ **e** $(f \circ g)(x)$ **f** $(g \circ f)(x)$

g $(f \circ h)(x)$ **h** $f^{-1}(x)$ **i** $g^{-1}(x)$

7 A certain banquet facility charges a room rental fee plus a meal cost per person. Their website shows the following examples of the total banquet cost for meals that do not include dessert.

Number of people	Cost of banquet in euros
20	800
60	1800
150	4050

a Find a linear model that relates the cost of the banquet in euros, C, to the number of people attending the banquet, n. Give your answer in the form $C(n) = mn + b$, where m and b are constants to be determined.

b Explain what the constants m and b represent in terms of the context.

c Use your model to find the cost of the banquet when 120 people attend.

d If dessert is included in the meal there is an additional charge of €5 per person. Find the function that gives the cost of the banquet when n people attend, and dessert is included.

Answers

1 **a** Plot the y -intercept $(0,-4)$ and use the gradient $\frac{2}{3}$ to plot other points on the line. Graph passes through points (3, -2) and (6, 0).

b Graph a vertical line with x -intercept $(-3,0)$.

c Plot the point $(-2,4)$ and use the gradient -3 to plot other points on the line. Graph passes through points (-2, 4) and (-1, 1).

d Graph a horizontal line with y -intercept $(0,5)$.

e Plot the y -intercept $(0,0)$ and use the gradient $\frac{4}{3}$ to plot other points on the line. Graph passes through (-3, -4) and (3, 4).

f Find and plot the intercepts: $4x-2(0)+6=0 \Rightarrow x=-\frac{3}{2}$ and $4(0)-2y+6=0 \Rightarrow y=3$.

2 **a** Gradient-intercept form: $y=4x+1$

General form: $4x-y+1=0$ or $-4x+y-1=0$

b Point-gradient form: $y+3=6(x-1)$

$y+3=6(x-1) \Rightarrow y+3=6x-6 \Rightarrow$ general form: $6x-y-9=0$ or $-6x+y+9=0$

c Parallel lines $\Rightarrow$ equal gradients, so $m=-\frac{1}{3}$ gradient-intercept form: $y=-\frac{1}{3}x-2$

$y=-\frac{1}{3}x-2 \Rightarrow \frac{1}{3}x+y+2=0 \Rightarrow 3\left(\frac{1}{3}x+y+2\right)=3(0) \Rightarrow$ general form: $x+3y+6=0$

or $-x-3y-6=0$

d Perpendicular lines $\Rightarrow -\frac{2}{3}\times m=-1$, so $m=\frac{3}{2}$ point-gradient form: $y-5=\frac{3}{2}(x+4)$

$y-5=\frac{3}{2}(x+4) \Rightarrow y-5=\frac{3}{2}x+6 \Rightarrow 2\left(\frac{3}{2}x-y+11\right)=2(0) \Rightarrow$ general form: $3x-2y+22=0$

or $-3x+2y-22=0$

3 **a** Since y remains constant on a horizontal line the equation is $y=-6$.

b Since x remains constant on a vertical line the equation is $x=-3$.

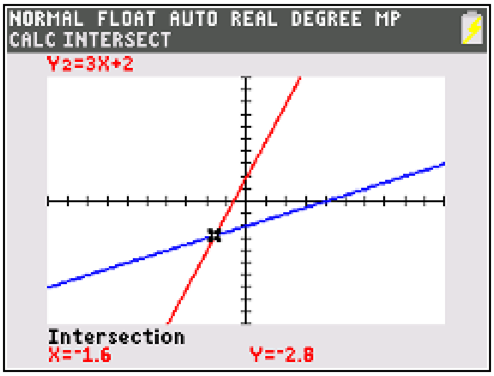
c A line with gradient 0 is horizontal, so the equation is $y=8$.

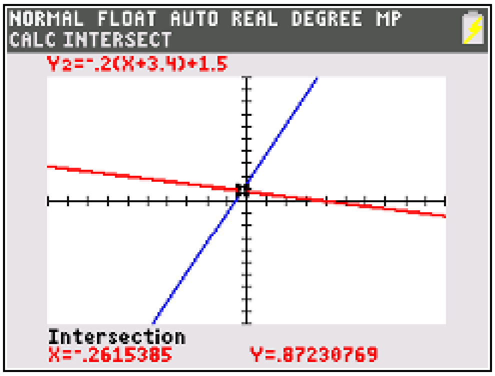
d The line $y=3$ is horizontal and so any line perpendicular to it is vertical. The equation of the vertical line through the point $(-4,5)$ is $x=-4$.

4 **a** $y+5=-2(x-4) \Rightarrow y+5=-2x+8 \Rightarrow y=-2x+3$; $m=-2$; $(0,3)$

b $3x-6y+12=0 \Rightarrow -6y=-3x-12 \Rightarrow y=\frac{1}{2}x+2$; $m=\frac{1}{2}$; $(0,2)$

5 **a** $(-1.6, -2.8)$ **b** $(-0.262, 0.872)$





6 **a** domain: all real numbers range: all real numbers

b domain: all real numbers range: $\{-5\}$

c $g(16) = \frac{1}{4}(16) + 2 = 4 + 2 = 6$

d $(f \circ g)(4) = f(g(4)) = f(3) = 5$

e $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{4}x + 2\right) = 3\left(\frac{1}{4}x + 2\right) - 4 = \left(\frac{3}{4}x + 6\right) - 4 = \frac{3}{4}x + 2$

f $(g \circ f)(x) = g(f(x)) = g(3x - 4) = \frac{1}{4}(3x - 4) + 2 = \left(\frac{3}{4}x - 1\right) + 2 = \frac{3}{4}x + 1$

g $(f \circ h)(x) = f(-5) = 3(-5) - 4 = -15 - 4 = -19$

h $x = 3y - 4 \Rightarrow x + 4 = 3y \Rightarrow \frac{1}{3}(x + 4) = y \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$

i $x = \frac{1}{4}y + 2 \Rightarrow x - 2 = \frac{1}{4}y \Rightarrow 4(x - 2) = y \Rightarrow g^{-1}(x) = 4x - 8$

7 **a** $m = \frac{4050 - 1800}{150 - 60} = \frac{1800 - 800}{60 - 20} = 25$; $C(20) = 25(20) + b \Rightarrow 800 = 500 + b \Rightarrow b = 300$;
$C(n) = 25n + 300$

b $m = 25$ represents a cost of 25 euros per meal; $b = 300$ represents a room rental fee of 300 euros

c $C(120) = 25(120) + 300 = 3300 \Rightarrow 3300$ euros

d Adding 5 euros for dessert per person makes the meal cost 30 euros per person and the total banquet cost when dessert is included is given by $C(n) = 30n + 300$.

3.3 Transformations of functions

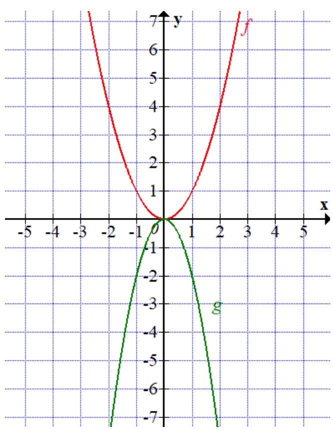
1 Describe the transformations of the graph of $f(x) = x^2$ that lead to the graph of g.

a $g(x) = x^2 - 5$ **b** $g(x) = (x-5)^2$ **c** $g(x) = -5x^2$

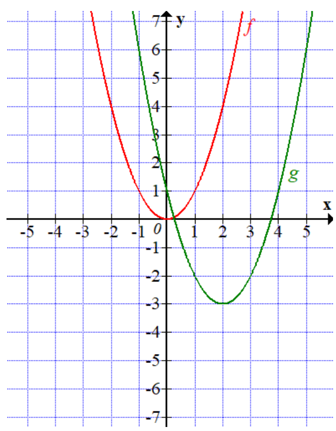
d $g(x) = \frac{1}{2}x^2 + 3$ **e** $g(x) = (x+6)^2 + 2$ **f** $g(x) = -(x-3)^2 + 1$

2 Describe the transformations of the graph of $f(x) = x^2$ that lead to the graph of g. Then write an equation for $y = g(x)$.

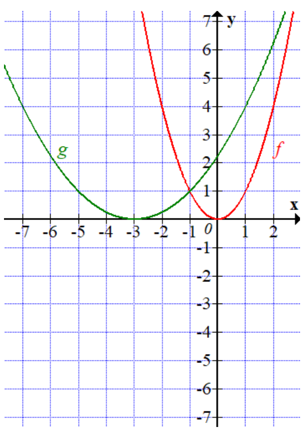
a



b



c

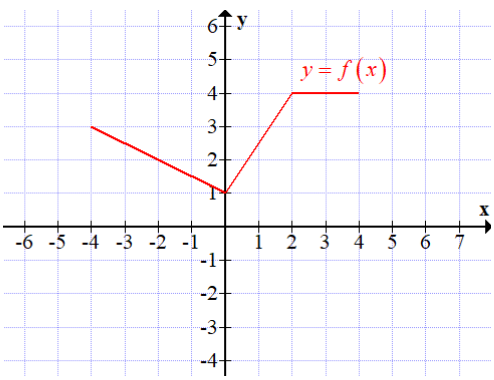


3 The graph of $y = f(x)$, for $-4 \le x \le 4$, is shown. Copy the graph of f and then draw the graph of $y = g(x)$ on the same axes.

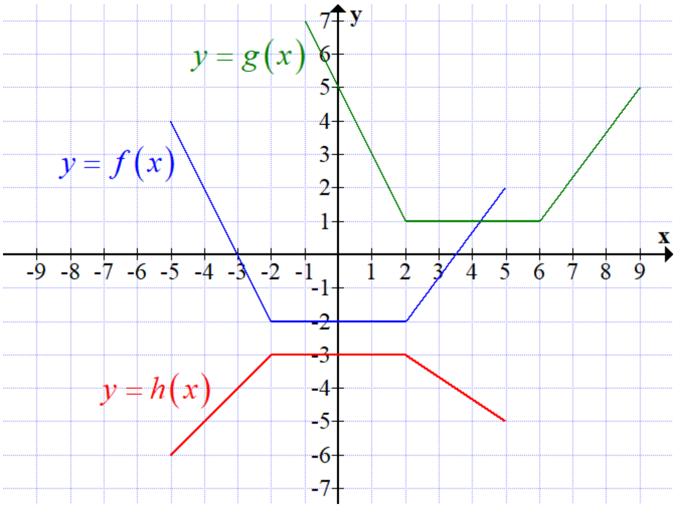
a $g(x) = -f(x)$ **b** $g(x) = f(-x)$

c $g(x) = f(x+2)$ **d** $g(x) = f(x) - 2$

e $g(x) = f\left(\frac{1}{2}x\right)$ **f** $g(x) = \frac{1}{2}f(x)$



4 The graph of $y = f(x)$, for $-5 \le x \le 5$, is shown. The graphs of g and h are transformations of the graph of f.



a Write down the range of f.

b Find the functions g and h in terms of f.

c Draw the graph of $y = g(-x)$.

The graph of a function s can be obtained by translating the graph of h. The domain of s is $-8 \le x \le 2$ and the range of s is $-1 \le y \le 2$.

d Find the function s in terms of h.

e Find the function s in terms of f.

Answers

1 **a** vertical translation down 5 units **b** horizontal translation right 5 units

c reflection in the x-axis and vertical stretch with scale factor 5

d vertical compression with scale factor $\frac{1}{2}$ and vertical translation up 3 units

e horizontal translation left 6 units and vertical translation up 2 units

f horizontal translation right 3 units, reflection in the x-axis and vertical translation up 1 unit

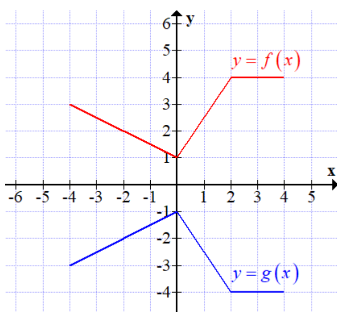
2 **a** reflection in the x-axis and vertical stretch with scale factor 2; $g(x) = -2x^2$

b horizontal translation right 2 units and vertical translation down 3 units; $g(x) = (x-2)^2 - 3$

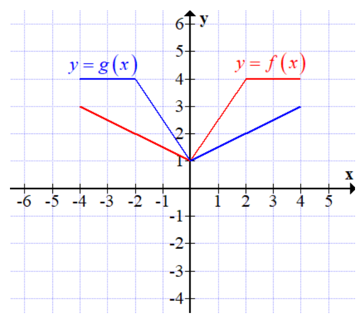
c horizontal translation left 3 units and vertical compression with scale factor $\frac{1}{4}$;

$g(x) = \frac{1}{4}(x+3)^2$

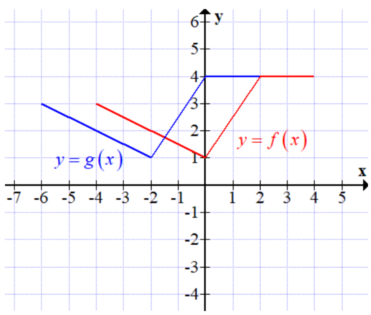
3 **a** reflect the graph of f in the x-axis



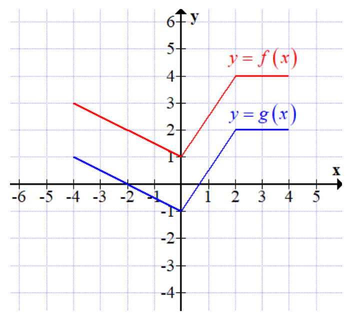
b reflect the graph of f in the y-axis



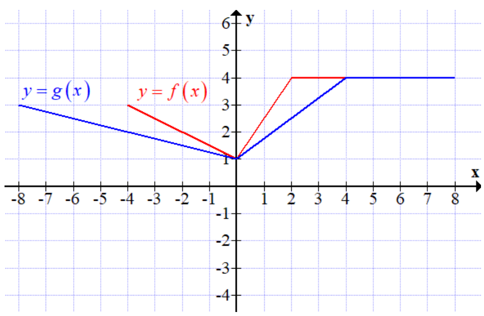
c shift the graph of f left 2 units



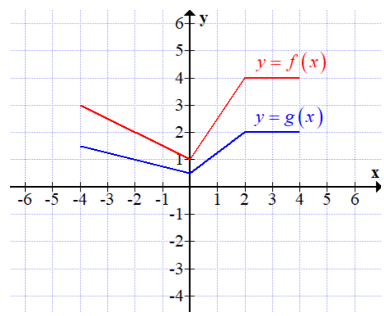
d shift the graph of f down 2 units



e horizontal dilation with scale factor $\frac{1}{\frac{1}{2}} = 2$



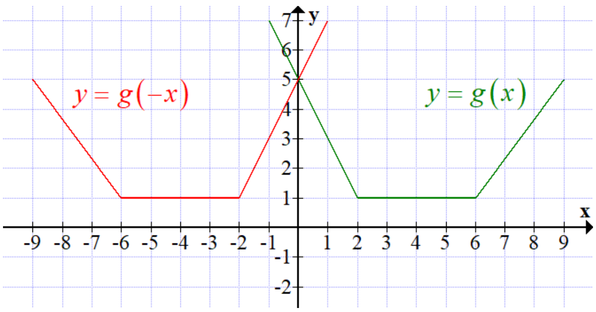
f vertical dilation with scale factor $\frac{1}{2}$



4 a $-2 \le y \le 4$

b The graph of g is obtained by translating the graph of f right 4 units and up 3 units, so $g(x) = f(x-4)+3$. The graph of h is obtained by compressing the graph of f by a scale factor of $\frac{1}{2}$, reflecting in the x-axis, and translating down 4 units, so $h(x) = -\frac{1}{2}f(x)-4$.

c reflect the graph of g in the y-axis



d $s(x) = h(x+3)+5$

e $s(x) = -\frac{1}{2}f(x+3)+1$

3.4 Graphing quadratic functions

For questions 1-2, use a GDC to plot the graph of each quadratic function. Then find: **a** the equation of the axis of symmetry, **b** the coordinates of the vertex, **c** the coordinates of the x and y-intercepts, and **d** the domain and range of the function.

1 $f(x) = 5x^2 - 6x - 8$

2 $f(x) = -6.5x^2 - 22.75x + 8$

For questions 3-4, use a GDC to help sketch the graph of each quadratic function over the given domain. Label the coordinates of key points of the graph. Then write down the range of the graph.

3 $f(x) = -2.8x^2 - 8.96x$, for $-3.5 \le x \le 0$

4 $f(x) = 3x^2 - 9x + 2$, for $0 \le x \le 3.5$

Answer the remaining questions <u>without</u> using a GDC.

5 Write down the equation of the axis of symmetry and the coordinates of the vertex for the graph of each function.

a $f(x) = (x-4)^2 + 7$ **b** $f(x) = -5(x+8)^2 + 6$ **c** $f(x) = 3(x-2)^2 - 1$

6 Find the coordinates of the y-intercept, the equation of the axis of symmetry and the coordinates of the vertex for the graph of each function.

a $f(x) = x^2 + 6x - 2$ **b** $f(x) = -4x^2 + 8x + 9$ **c** $f(x) = 2x^2 - 2x + 5$

7 Find the coordinates of the x-intercepts, the equation of the axis of symmetry and the coordinates of the vertex for the graph of each function.

a $f(x) = (x+4)(x-2)$ **b** $f(x) = 2(x-1)(x-3)$ **c** $f(x) = -(x+1)(x+2)$

8 Sketch a graph of each function and label key features of the graph.

a $f(x) = 2x^2 - 4x + 4$ **b** $f(x) = -(x+1)(x-5)$ **c** $f(x) = 3(x+4)^2 - 2$

9 The function $f(x) = -x^2 + 5x + 6$ can be written in the form $f(x) = a(x-p)(x-q)$, where $p > q$.

a Find the values of a, p and q.

b Write down the coordinates of the x and y-intercepts of the graph of f.

c Find the coordinates of the vertex of the graph of f.

10 Let $f(x) = (x+2)^2 - 3$.

a Write down the coordinates of the vertex of the graph of f.

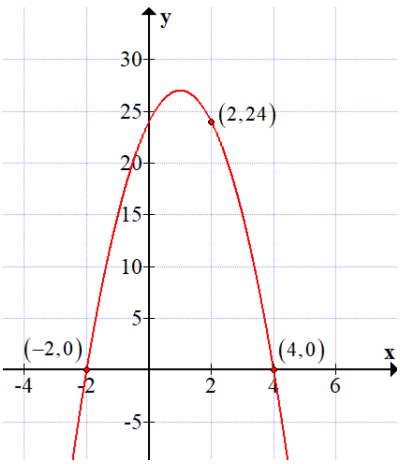
b Write the function f in the form $f(x) = ax^2 + bx + c$.

c Write down the coordinates of the y-intercept of the graph of f.

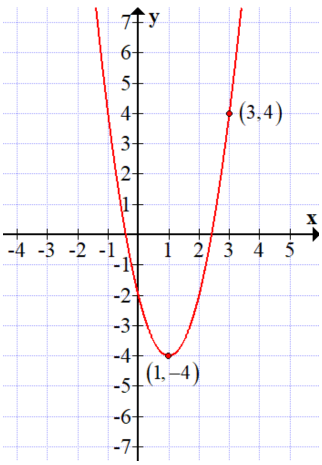
d Given that $h(x) = x - 1$, show that $(f \circ h)(x) = x^2 + 2x - 2$.

11 Use the information shown in the graph to find an expression for the quadratic function in the form $f(x) = a(x-h)^2 + k$ or the form $f(x) = a(x-p)(x-q)$. Then write the function in the form $f(x) = ax^2 + bx + c$.

a



b



Answers

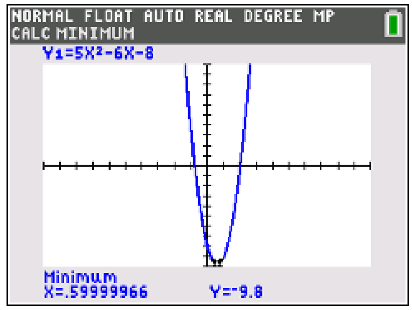
1 **a** $x = 0.6$

b $(0.6, -9.8)$

c $(-0.8, 0)$, $(2, 0)$, $(0, -8)$

d domain: all real numbers

range: $y \geq -9.8$



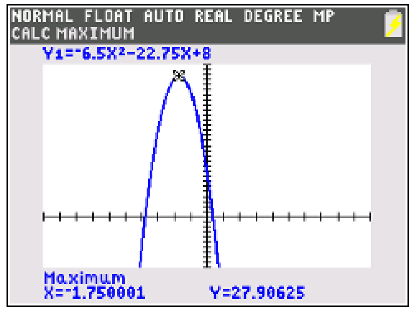
2 **a** $x = -1.75$

b $(-1.75, 27.90625)$

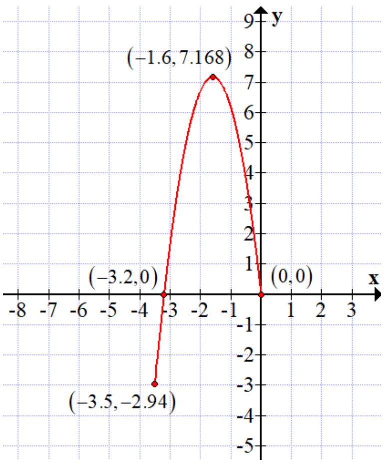
c $(-3.82, 0)$, $(0.322, 0)$, $(0, 8)$

d domain: all real numbers

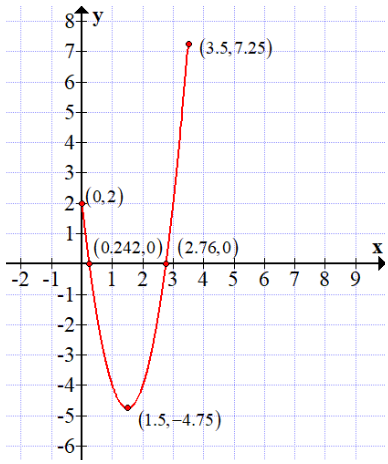
range: $y \leq 27.90625$



3 range: $-2.94 \leq y \leq 7.168$



4 range: $-4.75 \leq y \leq 7.5$



5 For the graph of $f(x) = a(x-h)^2 + k$, the axis of symmetry is $x = h$ and the vertex is (h, k).

a $x = 4$; $(4, 7)$ **b** $x = -8$; $(-8, 6)$ **c** $x = 2$; $(2, -1)$

6 **a** y-intercept: $c = -2 \Rightarrow (0, -2)$, axis of symmetry: $x = \dfrac{-6}{2(1)} \Rightarrow x = -3$

vertex: $f(-3) = (-3)^2 + 6(-3) - 2 = -11 \Rightarrow (-3, -11)$

b y-intercept: $c = 9 \Rightarrow (0, 9)$, axis of symmetry: $x = \dfrac{-8}{2(-4)} \Rightarrow x = 1$

vertex: $f(1) = -4(1)^2 + 8(1) + 9 = 13 \Rightarrow (1, 13)$

c y-intercept: $c = 5 \Rightarrow (0,5)$, axis of symmetry: $x = \frac{-(-2)}{2(2)} \Rightarrow x = \frac{1}{2}$

vertex: $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 5 = \frac{9}{2} \Rightarrow \left(\frac{1}{2}, \frac{9}{2}\right)$

7 For the graph of $f(x) = a(x-p)(x-q)$, the x-intercepts are $(p,0)$ and $(q,0)$.

a x-intercepts: $(-4,0)$, $(2,0)$, axis of symmetry: $x = \frac{-4+2}{2} \Rightarrow x = -1$

vertex: $f(-1) = (-1+4)(-1-2) = -9 \Rightarrow (-1,-9)$

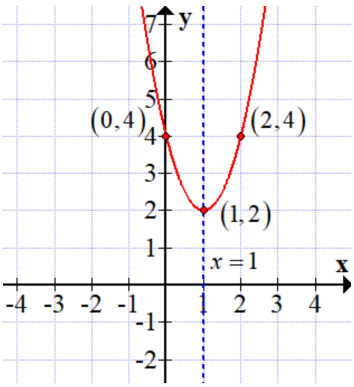
b x-intercepts: $(1,0)$, $(3,0)$, axis of symmetry: $x = \frac{1+3}{2} \Rightarrow x = 2$

vertex: $f(2) = 2(2-1)(2-3) = -2 \Rightarrow (2,-2)$

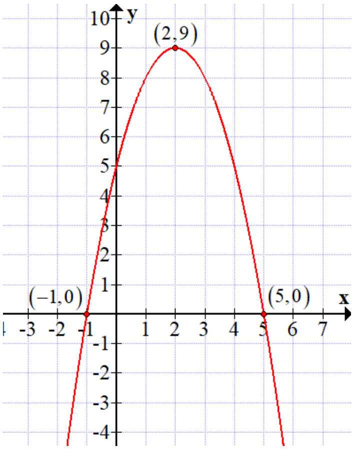
c x-intercepts: $(-1,0)$, $(-2,0)$, axis of symmetry: $x = \frac{-1+(-2)}{2} \Rightarrow x = -\frac{3}{2}$

vertex: $f\left(-\frac{3}{2}\right) = -\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}+2\right) = \frac{1}{4} \Rightarrow \left(-\frac{3}{2}, \frac{1}{4}\right)$

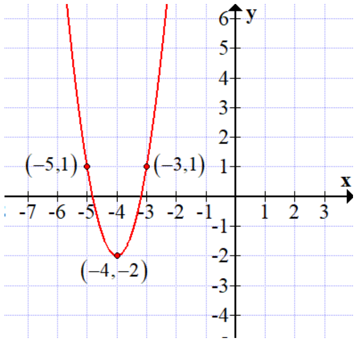
8 **a** Find and plot the vertex and y-intercept. Use the axis of symmetry to plot the point symmetric to the y-intercept



b Find and plot the vertex and x-intercepts.



c Find and plot the vertex. Apply a vertical stretch with scale factor 3 to $y = x^2$ to get points on each side of the vertex.



9 a $f(x) = -x^2 + 5x + 6 = -\left(x^2 - 5x - 6\right) = -(x-6)(x+1) \Rightarrow a = -1,\ p = 6,\ q = -1$

b $(6,0)$, $(-1,0)$; $(0,6)$

c $x = \dfrac{6+(-1)}{2} = \dfrac{5}{2}$ and $f\left(\dfrac{5}{2}\right) = -\left(\dfrac{5}{2} - 6\right)\left(\dfrac{5}{2} + 1\right) = \dfrac{49}{4} \Rightarrow \left(\dfrac{5}{2}, \dfrac{49}{4}\right)$

10 a $(-2,-3)$

b $f(x) = (x+2)^2 - 3 = \left(x^2 + 4x + 4\right) - 3 = x^2 + 4x + 1$

c $(0,1)$

d $(f \circ h)(x) = f(x-1) = (x-1)^2 + 4(x-1) + 1 = \left(x^2 - 2x + 1\right) + (4x - 4) + 1 = x^2 + 2x - 2$

11 a $p = -2,\ q = 4 \Rightarrow f(x) = a(x+2)(x-4)$ $(2,24) \Rightarrow 24 = a(2+2)(2-4) \Rightarrow 24 = -8a \Rightarrow a = -3$, so $f(x) = -3(x+2)(x-4)$

$f(x) = -3(x+2)(x-4) = -3\left(x^2 - 2x - 8\right)$, so $f(x) = -3x^2 + 6x + 24$

b $(-1,4) \Rightarrow f(x) = a(x-1)^2 - 4$

$(3,4) \Rightarrow 4 = a(3-1)^2 - 4 \Rightarrow 8 = 4a \Rightarrow a = 2$, so $f(x) = 2(x-1)^2 - 4$

$f(x) = 2(x-1)^2 - 4 = 2\left(x^2 - 2x + 1\right) - 4 = \left(2x^2 - 4x + 2\right) - 4$, so $f(x) = 2x^2 - 4x - 2$

3.5 Solving quadratic equations by factorization and completing the square

1 Solve the quadratic equation using the factorization method.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$ **c** $x^2 - 64 = 0$

d $3x^2 - 5x + 2 = 0$ **e** $4x^2 - 8x - 21 = 0$ **f** $25x^2 - 9 = 0$

2 Rewrite the quadratic equation in standard form and then solve using the factorization method.

a $x^2 - 4x - 4 = 2x + 12$ **b** $x(x-1) = 20$ **c** $2(x+3)(x-3) + 9x = 0$

d $x^2 + 12x = 2x - 25$ **e** $6x(x+1) = 5x + 2$ **f** $x(x+7) = 4(x^2+1)$

3 Find the exact solutions to the quadratic equation by factoring the perfect square on the left-hand side of the equation. Then check your work by using a GDC to approximate the solutions graphically.

a $x^2 - 6x + 9 = 7$ **b** $x^2 + 8x + 16 = 18$

4 Find the exact solutions to the quadratic equation by completing the square. You may check your work by using a GDC to approximate the solutions graphically.

a $x^2 - 8x = -12$ **b** $x^2 + 6x - 12 = 0$ **c** $x^2 + 3x - 1 = 0$

d $2x^2 - 10x - 4 = 0$ **e** $4x^2 - 20x - 5 = 0$ **f** $3x^2 + 6x = -2$

5 The market for a product is said to be in equilibrium when supply equals demand. For a certain product with unit price p dollars, the supply and demand functions are given below.

supply function: $s(p) = 2p + 4p^2$ demand function: $d(p) = 1536 - 30p$

a Write down an equation that can be solved to give the equilibrium unit price.

b Use completing the square to find the equilibrium unit price.

c Use a GDC to graphically check your solution to part **b**.

d Find the equilibrium quantity for the product.

Answers

1 **a** $x^2+7x+12=0 \Rightarrow (x+3)(x+4)=0 \Rightarrow x+3=0$ or $x+4=0 \Rightarrow x=-3,-4$

b $x^2+3x-10=0 \Rightarrow (x-2)(x+5)=0 \Rightarrow x-2=0$ or $x+5=0 \Rightarrow x=2,-5$

c $x^2-64=0 \Rightarrow (x-8)(x+8)=0 \Rightarrow x-8=0$ or $x+8=0 \Rightarrow x=8,-8$

d $3x^2-5x+2=0 \Rightarrow (3x-2)(x-1)=0 \Rightarrow 3x-2=0$ or $x-1=0 \Rightarrow x=\frac{2}{3},1$

e $4x^2-8x-21=0 \Rightarrow (2x-7)(2x+3)=0 \Rightarrow 2x-7=0$ or $2x+3=0 \Rightarrow x=\frac{7}{2},-\frac{3}{2}$

f $25x^2-9=0 \Rightarrow (5x-3)(5x+3)=0 \Rightarrow 5x-3=0$ or $5x+3=0 \Rightarrow x=\frac{3}{5},-\frac{3}{5}$

2 **a** $x^2-4x-4=2x+12 \Rightarrow x^2-6x-16=0 \Rightarrow (x-8)(x+2)=0 \Rightarrow x=8,-2$

b $x(x-1)=20 \Rightarrow x^2-x-20=0 \Rightarrow (x-5)(x+4)=0 \Rightarrow x=5,-4$

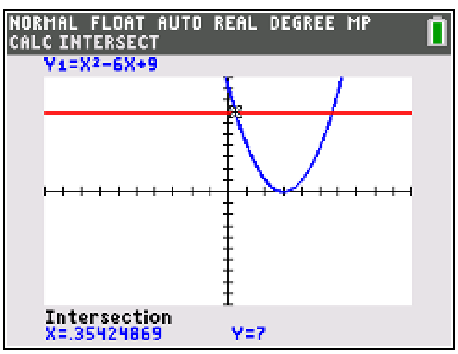
c $2(x+3)(x-3)+9x=0 \Rightarrow 2x^2+9x-18=0 \Rightarrow (2x-3)(x+6)=0 \Rightarrow x=\frac{3}{2},-6$

d $x^2+12x=2x-25 \Rightarrow x^2+10x+25=0 \Rightarrow (x+5)(x+5)=0 \Rightarrow x=-5$

e $6x(x+1)=5x+2 \Rightarrow 6x^2+x-2=0 \Rightarrow (3x+2)(2x-1)=0 \Rightarrow x=-\frac{2}{3},\frac{1}{2}$

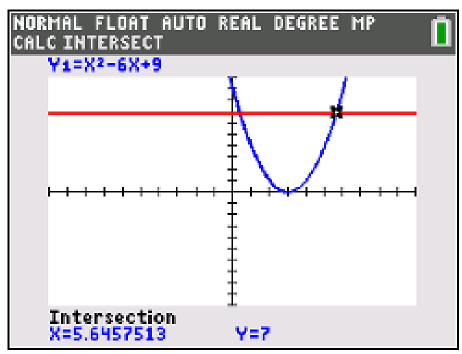
f $x(x+7)=4(x^2+1) \Rightarrow 3x^2-7x+4=0 \Rightarrow (3x-4)(x-1)=0 \Rightarrow x=\frac{4}{3},1$

3 **a** $x^2-6x+9=7 \Rightarrow (x-3)^2=7 \Rightarrow x-3=\pm\sqrt{7} \Rightarrow x=3\pm\sqrt{7}$



$x=0.354$

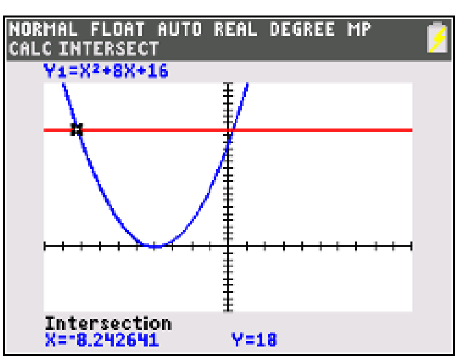
$3-\sqrt{7}\approx 0.354$



$x=5.65$

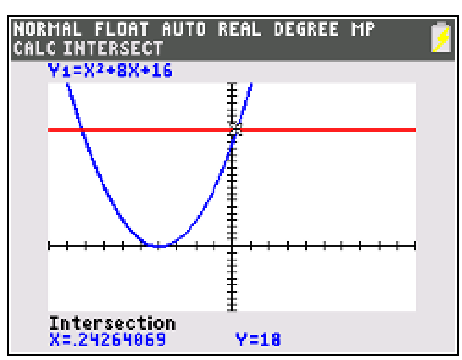
$3+\sqrt{7}\approx 5.65$

b $x^2+8x+16=18 \Rightarrow (x+4)^2=18 \Rightarrow x+4=\pm\sqrt{18} \Rightarrow x+4=\pm3\sqrt{2} \Rightarrow x=-4\pm3\sqrt{2}$



$x=-8.24$

$-4-3\sqrt{2}\approx -8.24$



$x=0.243$

$-4+3\sqrt{2}\approx 0.243$

4 a $x^2 - 8x = -12 \Rightarrow x^2 - 8x + \left(\frac{-8}{2}\right)^2 = -12 + \left(\frac{-8}{2}\right)^2 = x^2 - 8x + 16 = -12 + 16 \Rightarrow$

$(x-4)^2 = 4 \Rightarrow x - 4 = \pm\sqrt{4} \Rightarrow x - 4 = \pm 2 \Rightarrow x = 4 \pm 2 \Rightarrow x = 6, 2$

b $x^2 + 6x - 12 = 0 \Rightarrow x^2 + 6x = 12 \Rightarrow x^2 + 6x + \left(\frac{6}{2}\right)^2 = 12 + \left(\frac{6}{2}\right)^2 \Rightarrow x^2 + 6x + 9 = 12 + 9 \Rightarrow$

$(x+3)^2 = 21 \Rightarrow x + 3 = \pm\sqrt{21} \Rightarrow x = -3 \pm \sqrt{21}$

c $x^2 + 3x - 1 = 0 \Rightarrow x^2 + 3x = 1 \Rightarrow x^2 + 3x + \left(\frac{-3}{2}\right)^2 = 1 + \left(\frac{-3}{2}\right)^2 \Rightarrow x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4} \Rightarrow$

$\left(x + \frac{3}{2}\right)^2 = \frac{13}{4} \Rightarrow x + \frac{3}{2} = \pm\sqrt{\frac{13}{4}} \Rightarrow x = -\frac{3}{2} \pm \sqrt{\frac{13}{4}} = -\frac{3}{2} \pm \frac{\sqrt{13}}{2} = \frac{-3 \pm \sqrt{13}}{2}$

d $2x^2 - 10x - 4 = 0 \Rightarrow x^2 - 5x = 2 \Rightarrow x^2 - 5x + \left(\frac{-5}{2}\right)^2 = 2 + \left(\frac{-5}{2}\right)^2 \Rightarrow x^2 - 5x + \frac{25}{4} = 2 + \frac{25}{4} \Rightarrow$

$\left(x - \frac{5}{2}\right)^2 = \frac{33}{4} \Rightarrow x - \frac{5}{2} = \pm\sqrt{\frac{33}{4}} \Rightarrow x = \frac{5}{2} \pm \sqrt{\frac{33}{4}} = \frac{5}{2} \pm \frac{\sqrt{33}}{2} = \frac{5 \pm \sqrt{33}}{2}$

e $4x^2 - 20x - 5 = 0 \Rightarrow x^2 - 5x = \frac{5}{4} \Rightarrow x^2 - 5x + \left(\frac{-5}{2}\right)^2 = \frac{5}{4} + \left(\frac{-5}{2}\right)^2 \Rightarrow x^2 - 5x + \frac{25}{4} = \frac{5}{4} + \frac{25}{4} \Rightarrow$

$\left(x - \frac{5}{2}\right)^2 = \frac{30}{4} = \frac{15}{2} \Rightarrow x - \frac{5}{2} = \pm\sqrt{\frac{15}{2}} \Rightarrow x = \frac{5}{2} \pm \sqrt{\frac{15}{2}}$ or $\frac{5 \pm \sqrt{30}}{2}$

f $3x^2 + 6x = -2 \Rightarrow x^2 + 2x = -\frac{2}{3} \Rightarrow x^2 + 2x + \left(\frac{2}{2}\right)^2 = -\frac{2}{3} + \left(\frac{2}{2}\right)^2 \Rightarrow x^2 + 2x + 1 = -\frac{2}{3} + 1 \Rightarrow$

$(x+1)^2 = \frac{1}{3} \Rightarrow x + 1 = \pm\sqrt{\frac{1}{3}} \Rightarrow x = -1 \pm \sqrt{\frac{1}{3}}$ or $\frac{-3 \pm \sqrt{3}}{3}$

5 a $2p + 4p^2 = 1536 - 30p$

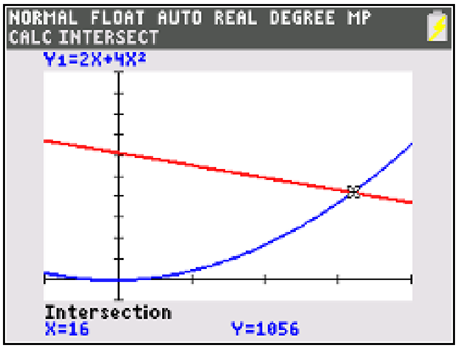
b $2p + 4p^2 = 1536 - 30p \Rightarrow$

$4p^2 + 32p = 1536 \Rightarrow p^2 + 8p = 384 \Rightarrow p^2 + 8p + \left(\frac{8}{2}\right)^2 = 384 + \left(\frac{8}{2}\right)^2 \Rightarrow$

$(p+4)^2 = 400 \Rightarrow p + 4 = \pm\sqrt{400} \Rightarrow p = -4 \pm 20 \Rightarrow p = -24, 16$

Since the price cannot be a negative value, the equilibrium price is $16.

c



d From the calculator check in part c, you can see that the equilibrium quantity is 1056. Alternately, you can find that either $s(16)$ or $d(16)$ equals 1056 units.

3.6 The quadratic formula and the discriminant

1 Use the quadratic formula to find the roots of each equation.

a $4x^2 - x - 2 = 0$ **b** $2x^2 + 3x + 1 = 0$ **c** $x^2 - 4x + 2 = 0$

d $5x^2 - x + 2 = 0$ **e** $3x - x^2 = -5$ **f** $2x^2 = -3x + 2$

g $x(x-2) = 6x - 5$ **h** $x(3x+4) = 1$ **i** $5x^2 - 9x + 10 = 2x + 4$

2 Identify which of the equation(s) in question **1** could be solved by factorization and then show the factorization.

3 The zeros of the function $f(x) = 2x^2 - 5x + c$, where c is a real number, are $\frac{5+\sqrt{57}}{4}$ and $\frac{5-\sqrt{57}}{4}$. Find c.

4 Find the value of the discriminant and then state the nature of the roots of each equation.

a $2x^2 + x + 4 = 0$ **b** $3x^2 + 4x - 4 = 0$ **c** $4x^2 + 12x + 9 = 0$

d $3x + 5 = x^2$ **e** $2x^2 - 8x + 5 = 0$ **f** $5x^2 - 2x = -3$

5 Identify which of the equation(s) in question **4** have rational roots. Justify your answer.

6 Find the value(s) of k such that the equation $2x^2 + 4x + k = 0$ has two distinct real roots.

7 Find the value(s)of p such that the graph of $f(x) = x^2 - 2px + 3p$ lies on the x-axis.

8 Find the value(s)of p such that the vertex of the graph of $f(x) = 2x^2 + 4x + m - 1$ has no x-intercepts.

9 Solve each inequality.

a $x^2 - 25 \le 0$ **b** $2x^2 - x - 6 > 0$ **c** $-x^2 + 2x \ge -5$

10 Find the value(s) of k that make the statement true.

a $x^2 - 3kx + 1 = 0$ has two distinct real roots

b the graph of $f(x) = kx^2 - 2kx + 3 + 2k$ has no x-intercepts

Answers

1 **a** $4x^2 - x - 2 = 0 \Rightarrow x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \dfrac{1 \pm \sqrt{33}}{8}$

b $2x^2 + 3x + 1 = 0 \Rightarrow x = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)} = \dfrac{-3 \pm 1}{4} = -1, -\dfrac{1}{2}$

c $x^2 - 4x + 2 = 0 \Rightarrow x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \dfrac{4 \pm \sqrt{8}}{2}$ or $2 \pm \sqrt{2}$

d $5x^2 - x + 2 = 0 \Rightarrow x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(5)(2)}}{2(5)} = \dfrac{1 \pm \sqrt{-39}}{10}$ $\Rightarrow$ no real roots

e $3x - x^2 = -5 \Rightarrow -x^2 + 3x + 5 = 0 \Rightarrow x = \dfrac{-3 \pm \sqrt{3^2 - 4(-1)(5)}}{2(-1)} = \dfrac{-3 \pm \sqrt{29}}{-2}$ or $\dfrac{3 \pm \sqrt{29}}{2}$

f $2x^2 = -3x + 2 \Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow x = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} = \dfrac{-3 \pm 5}{4} = -2, \dfrac{1}{2}$

g $x(x - 2) = 6x - 5 \Rightarrow x^2 - 8x + 5 = 0 \Rightarrow x = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(5)}}{2(1)} = \dfrac{8 \pm \sqrt{44}}{2}$ or $4 \pm \sqrt{11}$

h $x(3x + 4) = 1 \Rightarrow 3x^2 + 4x - 1 = 0 \Rightarrow x = \dfrac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} = \dfrac{-4 \pm \sqrt{28}}{6}$ or $\dfrac{-2 \pm \sqrt{7}}{3}$

i $5x^2 - 9x + 10 = 2x + 4 \Rightarrow 5x^2 - 11x + 6 = 0 \Rightarrow x = \dfrac{-(-11) \pm \sqrt{(-11)^2 - 4(5)(6)}}{2(5)} = \dfrac{11 \pm 1}{10} = 1, \dfrac{6}{5}$

2 **b** $2x^2 + 3x + 1 = 0 \Rightarrow (2x + 1)(x + 1) = 0 \Rightarrow x = -\dfrac{1}{2}, -1$

f $2x^2 = -3x + 2 \Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (2x - 1)(x + 2) = 0 \Rightarrow x = \dfrac{1}{2}, -2$

i $5x^2 - 9x + 10 = 2x + 4 \Rightarrow 5x^2 - 11x + 6 = 0 \Rightarrow (5x - 6)(x - 1) = 0 \Rightarrow x = \dfrac{6}{5}, 1$

3 $2x^2 - 5x + c = 0 \Rightarrow \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(c)}}{2(2)} = \dfrac{5 \pm \sqrt{25 - 8c}}{4}$

$\dfrac{5 \pm \sqrt{25 - 8c}}{4} = \dfrac{5 \pm \sqrt{57}}{4} \Rightarrow 25 - 8c = 57 \Rightarrow -8c = 32 \Rightarrow c = -4$

4 **a** $\Delta = 1^2 - 4(2)(4) = -31$; no real roots

b $\Delta = 4^2 - 4(3)(-4) = 64$; two distinct real roots

c $\Delta = 12^2 - 4(4)(9) = 0$; two equal real roots (one repeated root)

d $3x + 5 = x^2 \Rightarrow -x^2 + 3x + 5 = 0$; $\Delta = 3^2 - 4(-1)(5) = 29$; two distinct real roots

e $\Delta = (-8)^2 - 4(2)(5) = 24$; two distinct real roots

f $\Delta = (-2)^2 - 4(5)(3) = -56$; no real roots

5 **b** $3x^2 + 4x - 4 = 0$ has rational roots because 64 is a perfect square.

c $4x^2 + 19x + 9 = 0$ has rational roots because 0 is a perfect square.

6 $4^2-4(2)(k)>0 \Rightarrow 16-8k>0 \Rightarrow -8k>-16 \Rightarrow k<2$

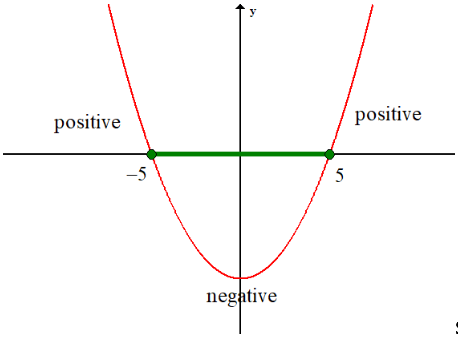
7 vertex on x-axis $\Rightarrow$ $x^2-2px+3p=0$ has one repeated real root, so $\Delta=0$.

$(-2p)^2-4(1)(3p)=0 \Rightarrow 4p^2-12p=0 \Rightarrow 4p(p-3)=0 \Rightarrow p=0,3$

8 no x-intercepts $\Rightarrow$ $2x^2+4x+m-1=2x^2+4x+(m-1)=0$ has no real roots, so $\Delta<0$.

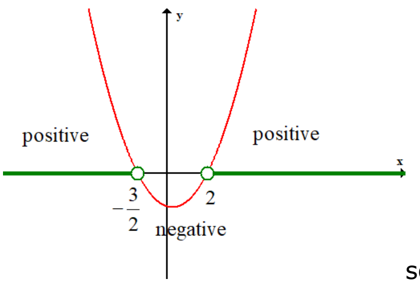
$4^2-4(2)(m-1)<0 \Rightarrow 16-8m+8<0 \Rightarrow -8m<-24 \Rightarrow m>3$

9 a $x^2-25=0 \Rightarrow (x-5)(x+5)=0 \Rightarrow x=-5,5$



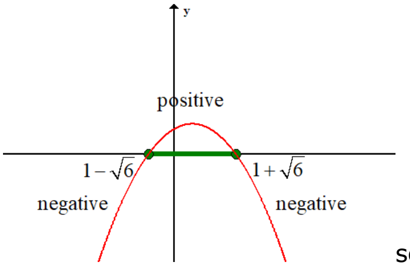
so $x^2-25 \le 0 \Rightarrow -5 \le x \le 5$

b $2x^2-x-6=0 \Rightarrow (2x+3)(x-2)=0 \Rightarrow x=-\dfrac{3}{2},2$



so $2x^2-x-6>0 \Rightarrow x<-\dfrac{3}{2}$ or $x>2$

c $-x^2+2x+5=0 \Rightarrow x=\dfrac{-2\pm\sqrt{2^2-4(-1)(5)}}{2(-1)}=1\pm\sqrt{6}$



so $-x^2+2x \ge -5 \Rightarrow 1-\sqrt{6} \le x \le 1+\sqrt{6}$

10 a $(-3k)^2-4(1)(1)>0 \Rightarrow 9k^2-4>0 \Rightarrow (3k+2)(3k-2)>0 \Rightarrow k<-\dfrac{2}{3}$ or $k>\dfrac{2}{3}$

b no x-intercepts $\Rightarrow$ $kx^2-2kx+(3+2k)=0$ has no real roots

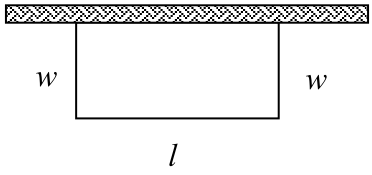
$(-2k)^2-4(k)(3+2k)<0 \Rightarrow 4k^2-12k-8k^2<0 \Rightarrow -4k(k+3)<0 \Rightarrow k<-3$ or $k>0$

3.7 Applications of quadratics

Answer questions 1-4 without the use of a GDC.

1 The base of a triangle is 8 more than twice its height. The area of the rectangle is 140 cm^2. Find the base and height of the triangle.

2 A rectangular lot of land is enclosed by 180 metres of fencing material on three sides. The fourth side of the plot is bordered by a stone wall. The diagram shows two sides of fencing of length w m and one side of fencing of length l m.



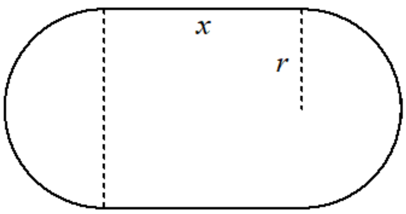
a Find an expression for A, the area of the plot in m^2, in terms of w.

b Find the dimensions of the fencing that yield an area of 3600 m^2.

c Find the dimensions of the fencing that maximize the area of the plot.

d Find the maximum area.

3 The diagram shows an athletic field which consists of a rectangle and two semicircles. The length of the rectangular region of the field is x m and the radius of each semicircle is r m. The perimeter of the athletic field is 336 m.



a Write an expression for the perimeter of the athletic field in terms of x and r.

b Find an expression for x in terms of r.

c Find an expression for A, the area of the rectangular portion of the field, in terms of r.

d Find the dimensions of the field, x and r, that maximize the area of the rectangular region.

4 A photographer has 600 photos of a celebrity event available for the event organizer to purchase. He sells the photo files for \$10 each.

a Find the amount of revenue generated from photo file sales if the photographer sells all 600 files.

Research has shown the photographer he will sell all 600 photo files at \$10 each and that for each \$0.50 increase per photo file, he will sell 10 fewer photo files. Suppose there have been x increases of \$0.50 in the price of a photo file.

b Find an expression for the price of a photo file in terms of x.

c Find an expression in terms of x for the number of photo files purchased.

d Find an expression in terms of x for the revenue generated by photo file sales.

e Find the revenue generated when the photo files are sold for \$14 each.

f Find the photo file price that maximizes revenue and the maximum revenue.

5 The height of a ball t seconds after it is thrown is modelled by the function $h(t) = -4.9t^2 + 24t + 2$, where h is the height of the ball in metres.

a Write down the initial height of the ball.

b Find the maximum height of the ball.

c Find the length of time the ball is at least 18 metres high.

d Find the amount of time elapsed when the ball hits the ground.

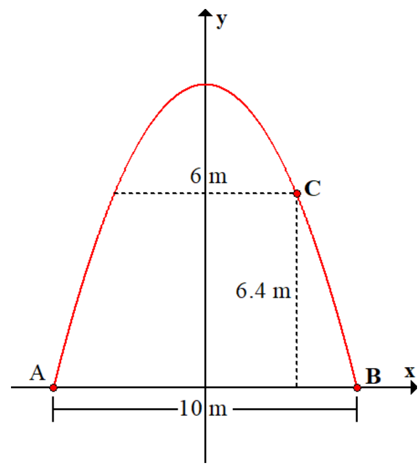
6 The diagram shows a parabolic archway with a base 10 metres wide. At a height of 6.4 metres the archway is 6 metres wide.

a Write down the coordinates of points A, B and C.

b Find an equation that models the archway in the form $y = a(x-p)(x-q)$.

c Find the height of the archway 1.5 m from either end.

d Find the maximum height of the archway.



7 From 2000 through 2020, $S(t) = -0.138t^2 + 2.76t + 45.8$ models the annual sales of an advertising company, in thousands of euros, where $t = 0$ represents the year 2000. Answer the following questions according to the model.

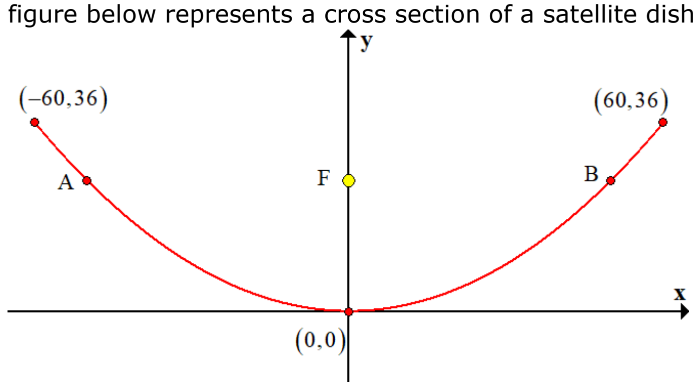
a Find the annual sales for the year 2006.

b Find the year in which the annual sales was the same as the sales in 2006.

c Use the model to show that the annual sales never reached €$60,000$ during the years 2000 through 2020.

8 A satellite dish antenna is constructed in the shape of a parabola.

The incoming signals reflect off the dish to a single point called the focus. The parabola in the figure below represents a cross section of a satellite dish with the focus point at F.



Given that the focus point F lies on segment AB and the x-coordinates of points A and B are -50 and 50, respectively, find the coordinates of F.

Answers

1 $b = 2h + 8;\ A = \frac{1}{2}(2h+8)(h)$

$\frac{1}{2}(2h+8)(h) = 140 \Rightarrow h^2 + 4h - 140 = 0 \Rightarrow (h+14)(h-10) = 0 \Rightarrow h = -14, 10$

$h = 10$ cm; $b = 2(10) + 8 = 28$ cm

2 **a** $2w + l = 180 \Rightarrow l = 180 - 2w$; $A = lw \Rightarrow A = (180-2w)w$ or $A = 180w - 2w^2$

b $3600 = 180w - 2w^2 \Rightarrow 2w^2 - 180w - 3600 = 0 \Rightarrow 2(w-30)(w-60) = 0 \Rightarrow w = 30, 60$

$l = 180 - 2(30) = 120$ or $l = 180 - 2(60) = 60$

So, $w = 30$ m and $l = 120$ m or $w = 60$ m and $l = 60$ m

c maximum occurs when $w = -\frac{b}{2a} = \frac{-180}{2(-2)} = 45$ m; $l = 180 - 2(45) = 90$ m

d $A(45) = 180(45) - 2(45^2) = 4050$ m^2

3 **a** perimeter $= 2x + 2\pi r$

b $2x + 2\pi r = 336 \Rightarrow x = 168 - \pi r$

c $A = (2r)(x) = 2r(168 - \pi r)$ or $336r - 2\pi r^2$

d $A = 336r - 2\pi r^2$ has a maximum when $r = -\frac{b}{2a} = \frac{-336}{2(-2\pi)} = \frac{84}{\pi}$ m and

$x = 168 - \pi r = 168 - \pi\left(\frac{84}{\pi}\right) = 84$ m

4 **a** $600(10) = \$6000$

b photo file price $= 10 + 0.5x$

c number of photo files purchased $= 600 - 10x$

d revenue $= (600 - 10x)(10 + 0.5x)$ or $6000 + 200x - 5x^2$

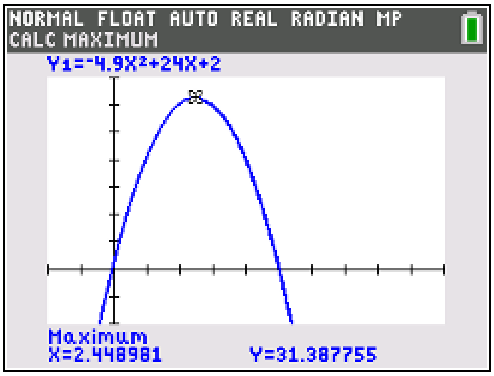
e $14 = 10 + 0.5x \Rightarrow x = 8$ price increase of \$0.50; revenue $= (600 - 10(8))(10 + 0.5(8)) = \$7280$

f maximum revenue occurs at $x = -\frac{b}{2a} = \frac{-200}{2(-5)} = 20$ price increases of \$0.50 each

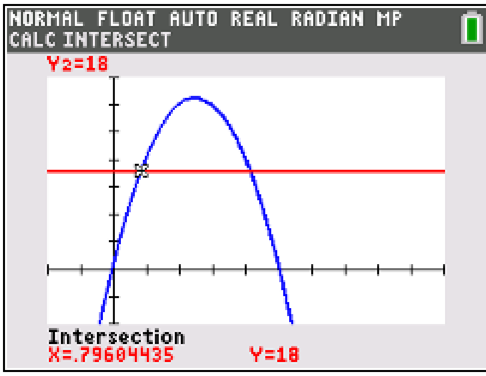
maximum revenue $= 6000 + 200(20) - 5(20^2) = \$8000$

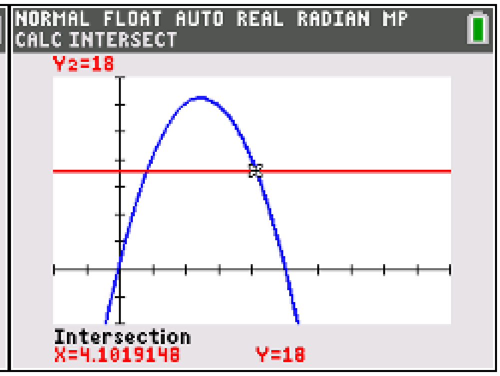
5 **a** $h(0) = 2$ m

b Use a GDC to find that the coordinates of the vertex are approximately $(2.45, 31.4)$ and so the maximum height of the ball is 31.4 m.

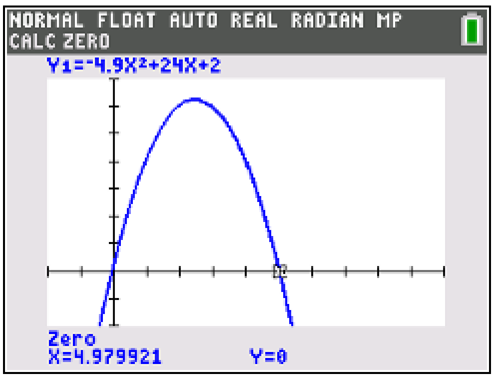


c Use a GDC to find when $h(t) = 18$. The ball is at least 18 metres high for $4.1010 - 0.7960 \approx 3.31$ seconds





d Use a GDC to find the t-intercept. The ball hits the ground after 4.98 seconds.



6 **a** $A(-5,0)$; $B(5,0)$; $C(3,6.4)$

b $p = -5$ and $q = 5 \Rightarrow y = a(x+5)(x-5)$; substitute the point $(3,6.4)$ to find a

$6.4 = a(3+5)(3-5) \Rightarrow a = -\frac{2}{5} \Rightarrow y = -\frac{2}{5}(x+5)(x-5)$

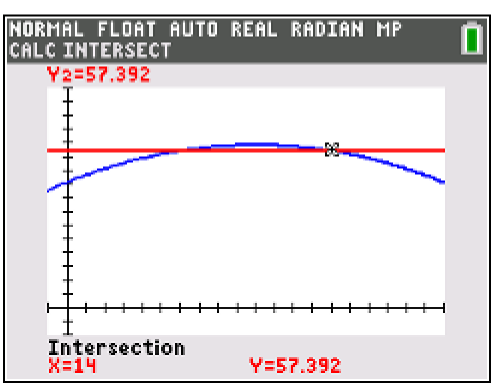
c A point 1.5 m from the right-hand end is $5 - 1.5 = 3.5$ m from the origin;

$y = -\frac{2}{5}(3.5+5)(3.5-5) \Rightarrow y = 5.1$ m

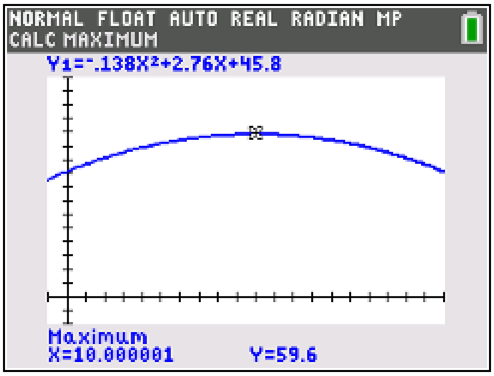
d the maximum height occurs at the vertex, when $x = 0$; $y = -\frac{2}{5}(0+5)(0-5) \Rightarrow y = 10$ m

7 **a** $2006 \Rightarrow t = 6$; $S(6) = 57.392 \Rightarrow$ €57,392

b $t = 14 \Rightarrow 2014$



c The maximum sales occur at the vertex and equal €59,600, so the sales never reach €60,000 during the years 2000 through 2020.



8 The vertex form of the equation of the parabola is $y = a(x-0)^2 + 0$. Substitute the point $(-60, 36)$ or $(60, 36)$ to find a. $36 = a(-60)^2 \Rightarrow a = 0.01 \Rightarrow y = 0.01x^2$

The y-coordinate of point F is the same as the y-coordinate of point A or B.
$y = 0.01(-50)^2 = 25 \Rightarrow$ the coordinates of F are $(0, 25)$.

4.1 The reciprocal function

1 Write down the reciprocal of each of these.

a 8 **b** -5 **c** $\frac{12}{7}$ **d** 4.5 **e** $2xy$ **f** $-\frac{2a}{b}$

2 Write down the answer when a value is multiplied by its reciprocal.

3 Sketch the graphs of these functions.

a $\frac{5}{x}$ **b** $-\frac{5}{x}$ **c** $\frac{5}{x}+2$ **d** $\frac{5}{x}-1$

4 **a** Sketch the curve $xy=7$.

b Write down the equations of the asymptotes.

c Write down the domain.

d Write down the range.

5 The relationship between the number of pipes (p) that are open and the time taken (t) to in hours to fill a swimming pool with water is modelled by the function $t=\frac{40}{p}$.

a How long will it take to fill the pool with 5 pipes open?

b How many pipes will need to be open to fill the pool 4 hours?

6 The amount (a), in kg, of food to feed the chickens (c) on a farm is modelled by $a=\frac{60}{c}$.

a Sketch a graph of the function for zero to sixteen chickens.

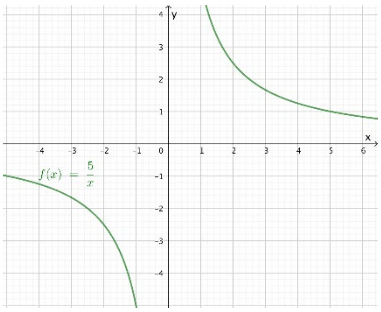
b Use your graph to find out how many kilograms of food will be needed for 10 chickens.

Answers

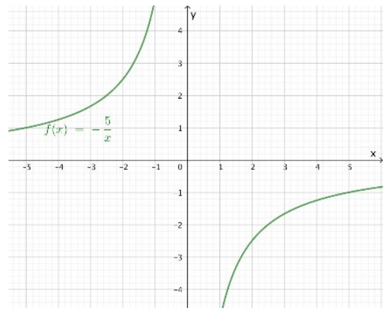
1 **a** $\frac{1}{8}$ **b** $-\frac{1}{5}$ **c** $\frac{7}{12}$ **d** $\frac{1}{4.5}=\frac{2}{9}$ **e** $\frac{1}{2xy}$ **f** $-\frac{b}{2a}$

2 1

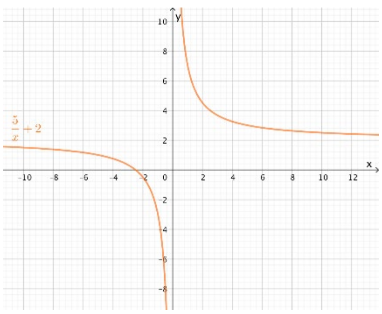
3 **a**



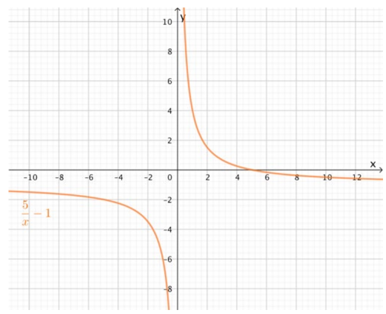
b



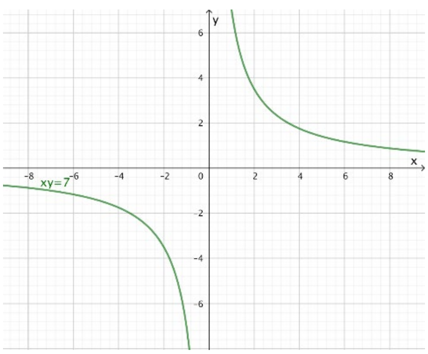
c



d



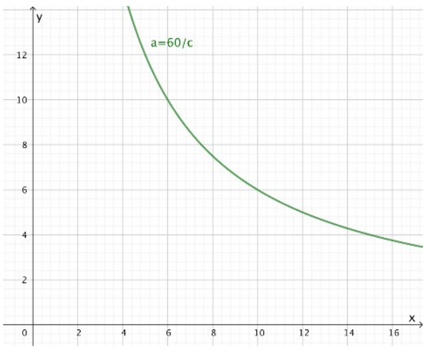
4 **a**



b $x = 0, y = 0$ **c** $x \in \mathbb{R}, x \neq 0$ **d** $y \in \mathbb{R}, y \neq 0$

5 **a** $t = \frac{40}{5} = 8$ hours **b** $4 = \frac{40}{p}, p = \frac{40}{4} = 4$ pipes

6 **a**



b 10kg

4.2 Transforming the reciprocal function

1 **a** Sketch the graph of $f(x)=\dfrac{6}{x}$ and the graph of $g(x)=\dfrac{6}{x-2}+3$ on the same axes.

b Describe the transformations that map *f* onto *g*.

2 Identify the horizontal and vertical asymptotes of these functions and state their range and domain.

a $y=\dfrac{2}{x+4}$ **b** $y=\dfrac{3}{x-2}$ **c** $y=\dfrac{5}{3x-2}$ **d** $y=\dfrac{8}{2x+5}+4$ **e** $y=\dfrac{7}{3x-1}-8$

3 Sketch each function and show the asymptotes on the graph.

a $f(x)=\dfrac{4}{x+1}+2$ **b** $f(x)=-\dfrac{4}{x+1}+2$ **c** $f(x)=\dfrac{3}{x+1}+2$

d $f(x)=\dfrac{3}{x+1}-2$ **e** $f(x)=\dfrac{6}{x}-2$ **f** $f(x)=\dfrac{6}{x-3}-2$

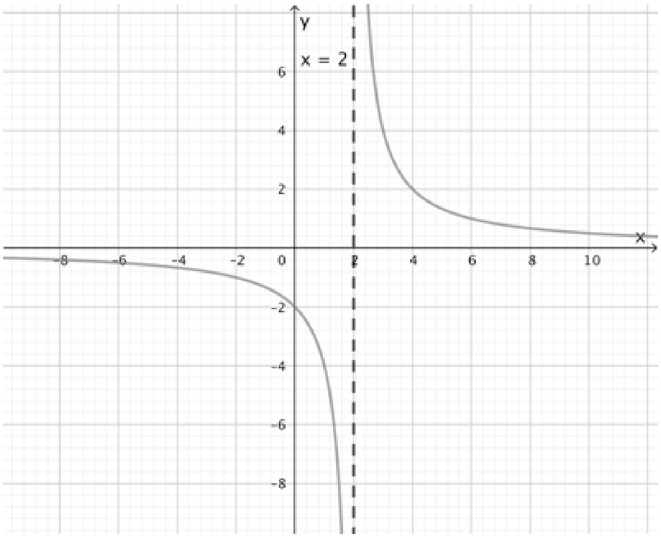
4 **a** Sketch the curve $f(x)=\dfrac{1}{x}$

The curve is now translated 2 units to the left and 3 units up to give $g(x)$.

b Sketch the curve of *g*, show and label any asymptotes.

c Write down the equation of *g*.

5 The graph shows the function $f(x)=\dfrac{p}{x-q}$. There is a vertical asymptote at $x=2$



a Write down the value of *q*.

The curve passes through the point $(6,1)$

b Find the value of *p*.

6 Have you ever noticed how the sound of the siren, like a police car or ambulance, changes as it passes you? The frequency of the sound of a siren (*f*), in tens of thousands of hertz, may be modelled by the equation $f = \frac{150}{750 - s}$, where s is the speed of the vehicle.

- **a** If the vehicle is travelling at 80 miles per hour, what will be the frequency of the sound that you hear?
- **b** What will happen to the frequency of the sound as the speed increases?

7 Let $f(x) = a + \frac{9}{x - b}$.

The line $x = 3$ is a vertical asymptote to the graph of f .

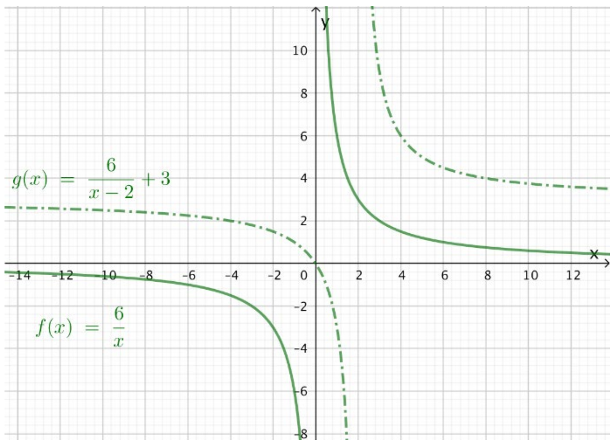
- **a** Write down the value of b.

The graph of f has a y-intercept at (0, 4).

- **b** Find the value of a.
- **c** Write down the equation of the horizontal asymptote of the graph of *f*.

Answers

1 a



b A horizontal translation of 2 units right and a vertical translation of 3 units up.

2 a Horizontal asymptote $y = 0$, vertical asymptote $x = -1$, $x \in \mathbb{R}, x \neq -1$. $y \in \mathbb{R}$

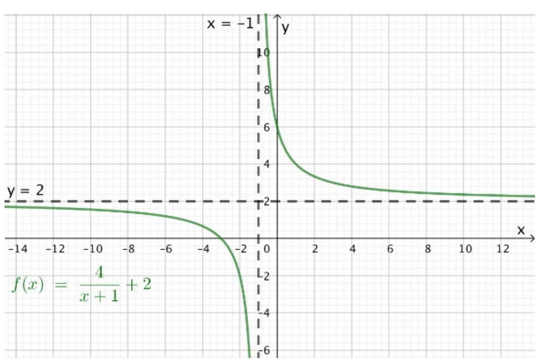
b Horizontal asymptote $y = 0$, vertical asymptote $x = 2$, $x \in \mathbb{R}, x \neq 2$. $y \in \mathbb{R}$

c Horizontal asymptote $y = 0$, vertical asymptote $x = \frac{2}{3}$, $x \in \mathbb{R}, x \neq \frac{2}{3}$. $y \in \mathbb{R}$

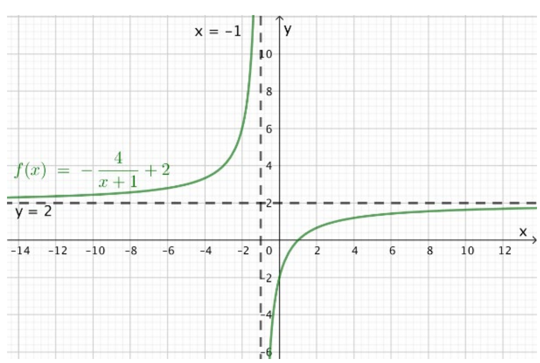
d Horizontal asymptote $y = 4$, vertical asymptote $x = -\frac{5}{2}$, $x \in \mathbb{R}, x \neq -\frac{5}{2}$. $y \in \mathbb{R}, y \neq 4$

e Horizontal asymptote $y = -8$, vertical asymptote $x = \frac{1}{3}$, $x \in \mathbb{R}, x \neq \frac{1}{3}$. $y \in \mathbb{R}, y \neq -8$

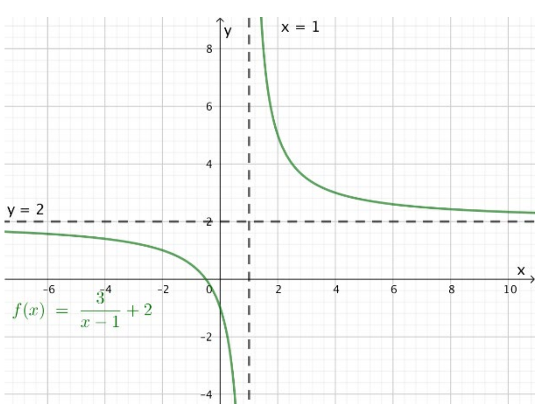
3 a



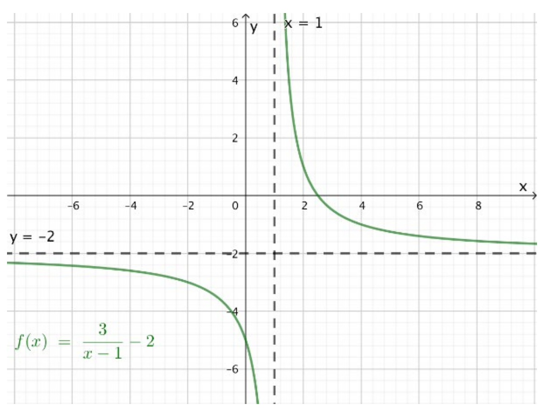
b



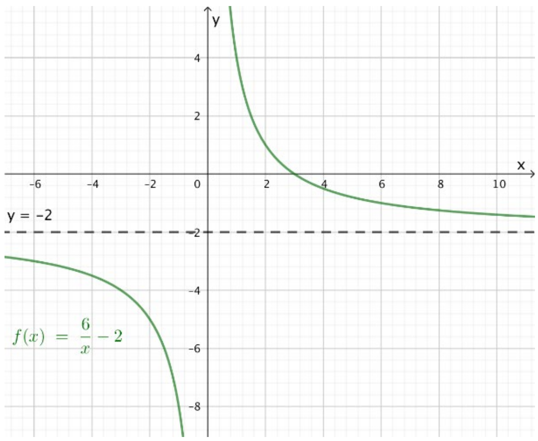
c



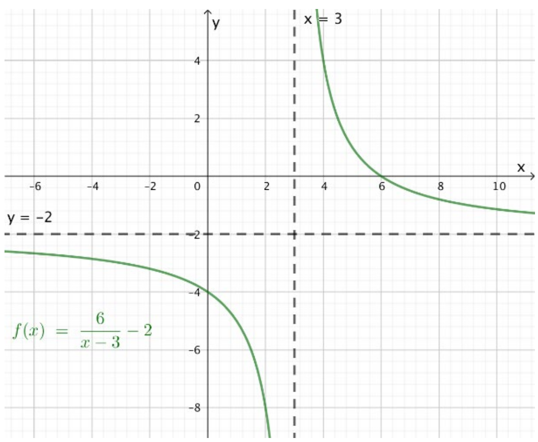
d



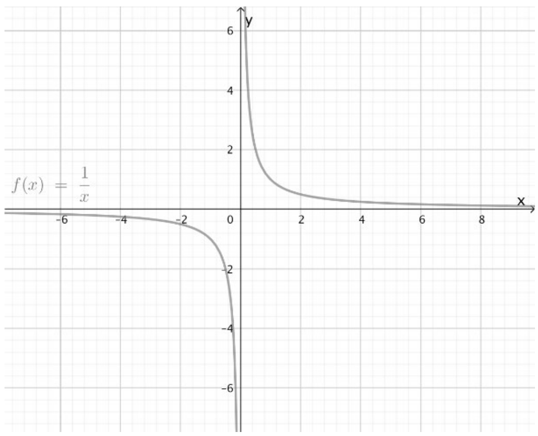
e



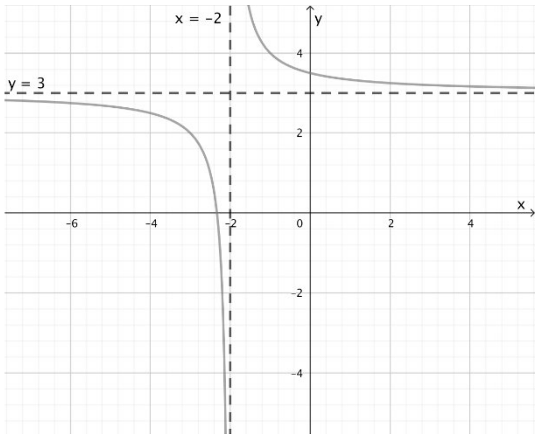
f



4 **a**



b



c $g(x) = \dfrac{1}{x+2} + 3$

5 **a** $q = 2$

b $1 = \dfrac{p}{6-2}$

$1 = \dfrac{p}{4}$

$p = 4$

6 **a** $f = \dfrac{150}{750-80} = \dfrac{150}{670} = 0.224$ or 2240 hertz

b The frequency will increase.

7 **a** $b=3$

b $f(x) = a + \dfrac{9}{x-3}$ $and\,at\,(0,4), x = 0\,and\,y = 4.$

$4 = a + \dfrac{9}{0-3}$

$4 = a - 3$

$a = 7$

c $y = 7$

4.3 Rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$

1 Identify the vertical and horizontal asymptotes of these functions and state their domain and range.

a $f(x) = \dfrac{5x}{x+2}$ **b** $f(x) = \dfrac{5+x}{5-x}$ **c** $f(x) = \dfrac{3-7x}{2x+3}$

2 Solve the equations

a $\dfrac{6}{5x} = 1 - \dfrac{1}{x}$ **b** $\dfrac{x}{15} + \dfrac{1}{3x} = \dfrac{2}{5}$ **c** $\dfrac{x}{x+3} + \dfrac{1}{4} = \dfrac{x}{2x+6}$

3 Sketch these rational functions. Label the asymptotes.

a $f(x) = \dfrac{6-x}{6+x}$ **b** $f(x) = \dfrac{3x-8}{x+3}$ **c** $f(x) = \dfrac{4x+3}{2x-5} - 3$

4 A water supply company removes particles from the water. The cost E (in Euros) of removing p % of the particles is given by $E = \dfrac{60000p}{100-p}$.

a How much will it cost to remove 60% of the particles.

b At the start of this year, the company had been removing 80% of the particles, but new regulations say that they must remove 90% of the particles before the end of the year. What will be the increase in costs to the water supply company?

5 The function f is given by

$$f(x) = \frac{2x+1}{x-3}$$

a Find $f^{-1}(x)$

b Show that $y = 2$ is an asymptote of the graph of $y = f(x)$.

c Find the vertical asymptote of the graph.

d Write down the coordinates at which the asymptotes intersect.

e Hence sketch the graph of $y = f(x)$, showing the asymptotes by dotted lines.

6 Use your GDC to solve $f(x) = g(x)$ when $f(x) = \dfrac{5-2x}{x+3}$ and $g(x) = 2x - 1$

Show your process by sketching the functions on the same axes, indicating the intersection points with their coordinates and writing the solution below your sketch.

Answers

1 **a** Horizontal asymptote $y=5$, vertical asymptote $x=-2$, $x \in \mathbb{R}, x \neq -2. y \in \mathbb{R}, y \neq 5$

b Horizontal asymptote $y=-1$, vertical asymptote $x=5$, $x \in \mathbb{R}, x \neq 5. y \in \mathbb{R}, y \neq -1$

c Horizontal asymptote $y=-\frac{7}{2}$, vertical asymptote $x=-\frac{3}{2}$, $x \in \mathbb{R}, x \neq -\frac{3}{2}. y \in \mathbb{R}, y \neq -\frac{7}{2}$

2 **a** $\frac{6}{5x}=1-\frac{1}{x}$

$6=5x-5$

$5x=6+5=11$

$x=\frac{11}{5}$

b $\frac{x}{15}+\frac{1}{3x}=\frac{2}{5}$

$(15x)\frac{x}{15}+(15x)\frac{1}{3x}=(15x)\frac{2}{5}$

$x^2+5=6x$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=1,5$

c $\frac{x}{x+3}+\frac{1}{4}=\frac{x}{2x+6}$

$\frac{x}{x+3}+\frac{1}{4}=\frac{x}{2(x+3)}$

$4(x+3)\frac{x}{x+3}+4(x+3)\frac{1}{4}=4(x+3)\frac{x}{2(x+3)}$

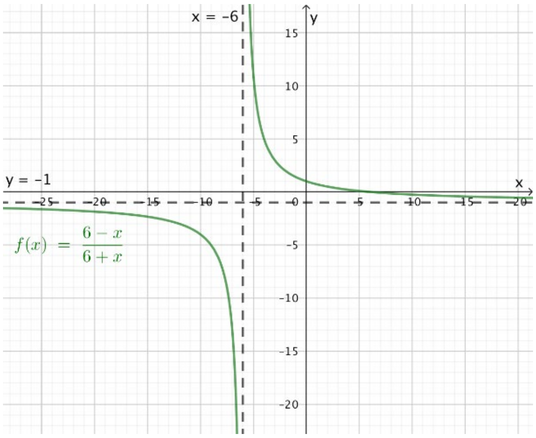
$4x+x+3=2x$

$5x+3=2x$

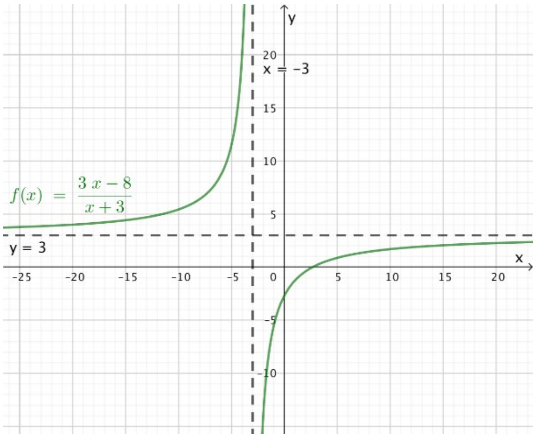
$3x=-3$

$x=-1$

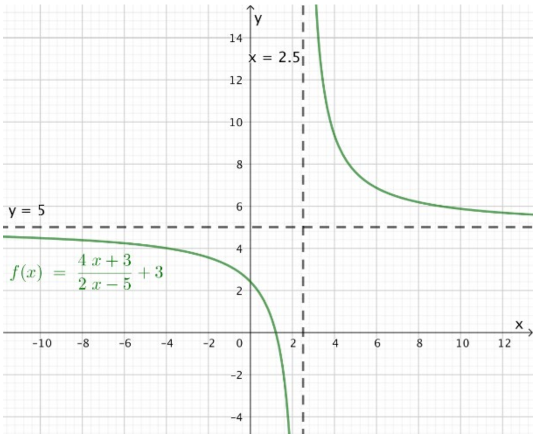
3 **a**



b



c



4 **a** $E = \dfrac{60000p}{100-p} = \dfrac{60000(60)}{100-60} = 90000\,Euros$

b Cost by the end of the year = cost later − cost now

Increase in cost $= \dfrac{60000(90)}{100-90} - \dfrac{60000(80)}{100-80} = 540000 - 24000 = 30000$ Euros

5 **a** $x = \dfrac{2y+1}{y-3}$

$x(y-3) = 2y+1$

$xy - 3x = 2y+1$

$xy - 2y = 3x+1$

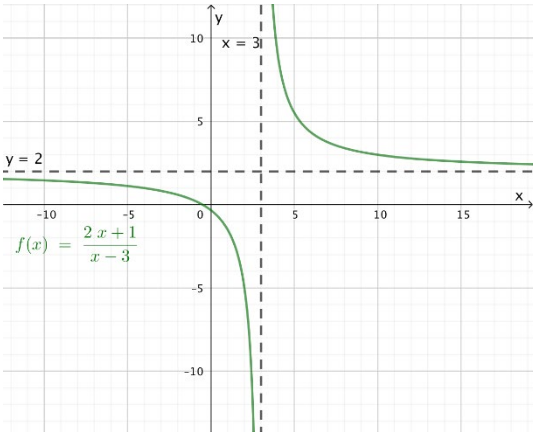
$y(x-2) = 3x+1$

$y = \dfrac{3x+1}{x-2}$

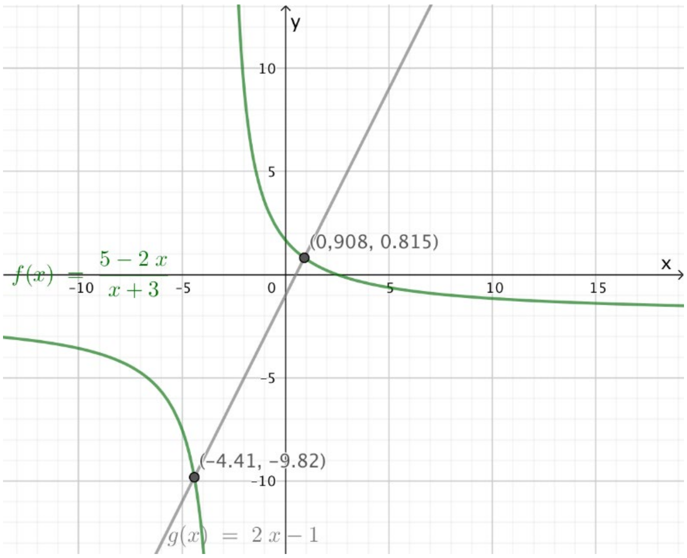
$f^{-1} = \dfrac{3x+1}{x-2}$

b $y = \dfrac{2}{1} = 2$ **c** $x-3=0$ when $x=3$ **d** $(3,2)$

e



6



x = −4.41,0.908

5.1 Limits and convergence

1 Use technology to examine each function numerically and graphically. Find the limit as x tends toward the given value, if it exists.

a $\lim_{x \to 2} \left(x^2 - 3\right)$

b $\lim_{x \to 3} \frac{x^2 + 5}{2x}$

2 State the equation of any vertical and horizontal asymptotes for each of the following functions.

a $f(x) = \frac{2x}{3x - 2}$

b $g(x) = -\frac{4x}{x^2 - 4}$

Answers

1 **a** 1 **b** $\frac{7}{3}$

2 **a** Since $3x - 2 = 0$ when $x = \frac{2}{3}$ there is a vertical asymptote of $x = \frac{2}{3}$.

Since $y \to \frac{2}{3}$ as $x \to \infty$ there is a horizontal asymptote of $y = \frac{2}{3}$.

b Since $x^2 - 4 = 0$ when $x = \pm 2$ there are vertical asymptotes of $x = 2$ and $x = -2$.
Since $y \to 0$ as $x \to \infty$ there is a horizontal asymptote of $y = 0$.

5.2 The derivative function

1 Find the derivative of each function.

a $f(x) = x^9$ **b** $f(x) = x^{-\frac{3}{2}}$ **c** $f(x) = \frac{1}{\sqrt[3]{x}}$

2 Find the gradient function of each of the following functions:

a $y = x^3 - \frac{1}{3}x$ **b** $p = -3q + 8$

3 Differentiate with respect to x:

a $f(x) = \frac{2}{5x^4}$ **b** $f(x) = (3x - 2)(x^3 + 1)$

4 Find $\frac{dy}{dx}$ for each of the following:

a $y = 2 + x\sqrt[3]{x}$ **b** $y = \sqrt{x} + \sqrt[5]{x}$

5 Find the gradient of the tangent to each function at the given value of x:

a $y = x^2 - 2x + 3$ at $x = 2$ **b** $f(x) = \sqrt[5]{x} + \frac{1}{x}$ at $x = 1$

6 Find the coordinates of the points on the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 2$ where the gradient is 1.

7 Find the equation of the normal to $y = x^3 + x^2 + 3$ at the point (1, 5).

8 Find the equation of any normals to the curve $y = x + \frac{2}{x}$ at points where the tangent is parallel to the line $y = -x$.

9 The gradient of the normal to the graph of $y = 2x^2 - kx + 3$ at $x = 2$ is $\frac{1}{4}$. Find the value of k.

10 Let $f(x) = \frac{12}{x} + 2$ and let $g(x) = 1 - 3x$. Find the values of x for which f and g have the same gradient.

Answers

1 **a** $f'(x) = 9x^8$ **b** $f'(x) = -\frac{3}{2}x^{-\frac{5}{2}}$ **c** $f'(x) = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x\sqrt[3]{x}}$

2 **a** $\frac{dy}{dx} = 3x^2 - \frac{1}{3}$ **b** $\frac{dp}{dq} = -3$

3 **a** $f'(x) = -\frac{8}{5x^5}$ **b** $f(x) = 3x^4 - 2x^3 + 3x - 2 \Rightarrow f'(x) = 12x^3 - 6x^2 + 3$

4 **a** $\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}} = \frac{4\sqrt[3]{x}}{3}$ **b** $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{5}x^{-\frac{4}{5}}$

5 **a** $\frac{dy}{dx} = 2x - 2$ so gradient $= 2$ **b** $\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}} - \frac{1}{x^2}$ so gradient $= -\frac{4}{5}$

6 $f'(x) = x^2 - x - 1$

$x^2 - x - 1 = 1 \Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x-2)(x+1) = 0$
$\Rightarrow x = -1$ or 2

So coordinates are $(-1, \frac{13}{6}), (2, \frac{2}{3})$

7 $\frac{dy}{dx} = 3x^2 + 2x = 5$ when $x = 1$

So gradient of normal $= -\frac{1}{5}$

So equation is $y = -\frac{1}{5}x + c$

$5 = -\frac{1}{5} + c \Rightarrow c = \frac{26}{5}$

$y = -\frac{1}{5}x + \frac{26}{5}$

8 $\frac{dy}{dx} = 1 - \frac{2}{x^2}$

$1 - \frac{2}{x^2} = -1 \Rightarrow x = \pm 1$
Gradient of normals is 1
Points are $(1,3)$, $(-1,-3)$
So equations are $y = x + 2$, $y = x - 2$

9 Gradient of normal $= \frac{1}{4} \Rightarrow$ gradient of tangent $= -4$

$\frac{dy}{dx} = 4x - k$
$8 - k = -4 \Rightarrow k = 12$

10 $f'(x) = -\frac{12}{x^2}, g'(x) = -3$

$-\frac{12}{x^2} = -3 \Rightarrow x^2 = 4$
$\Rightarrow x = \pm 2$

5.3 Differentiation rules

1 Use the chain rule to find $\frac{dy}{dx}$ for each function.

a $y = (3x - 1)^4$

b $y = 5\left(x^3 - \frac{1}{x^2}\right)^2$

2 Find the equation of the normal to the curve $y = \frac{2}{(1-x)^2}$ where $x = 2$.

3 Differentiate each function with respect to x:

a $y = x\sqrt{1-4x}$

b $y = \sqrt{x+1}(3-5x)$

4 Find the equation of the normal to the curve $y = \frac{2x}{1-3x}$ at the point (0, 0).

5 Differentiate each expression using the quotient rule.

a $y = \frac{\sqrt{x}}{3-4x}$

b $y = \frac{2-x}{x^3+1}$

6 Find the equation of the tangent and normal to the curve $f(x) = \frac{1-\sqrt{x}}{1+x}$ at the point $\left(4, -\frac{1}{5}\right)$.

Answers

1 **a** $12(3x-1)^3$ **b** $10\left(x^3-\frac{1}{x^2}\right)^3\left(3x^2+\frac{2}{x^3}\right)$

2 $\frac{dy}{dx}=\frac{4}{(1-x)^3}=-4$ when $x=2$

So gradient of normal $=\frac{1}{4}$

So equation is $y=\frac{1}{4}x+c$

When $x=2,\ y=2$ so $c=\frac{3}{2}$

So equation is $y=\frac{1}{4}x+\frac{3}{2}$

3 **a** $\frac{dy}{dx}=\sqrt{1-4x}+x\times\frac{1}{2}(1-4x)^{-\frac{1}{2}}\times-4=\sqrt{1-4x}-\frac{2x}{\sqrt{1-4x}}$

b $\frac{dy}{dx}=\frac{1}{2}(x+1)^{-\frac{1}{2}}(3-5x)-5\sqrt{x+1}=\frac{3-5x}{2\sqrt{x+1}}-5\sqrt{x+1}$

4 $\frac{dy}{dx}=\frac{2(1-3x)+6x}{(1-3x)^2}=2$ when $x=0$

So equation of normal is $y=-\frac{1}{2}x$

5 **a** $\frac{dy}{dx}=\frac{\frac{1}{2}x^{-\frac{1}{2}}\times(3-4x)-x^{\frac{1}{2}}\times-4}{(3-4x)^2}=\frac{\frac{1}{2}(3-4x)+4x}{\sqrt{x}(3-4x)^2}=\frac{3+4x}{2\sqrt{x}(3-4x)^2}$

b $\frac{dy}{dx}=\frac{-1(x^3+1)-(2-x)3x^2}{(3-4x)^2}=\frac{2x^3-6x^2-1}{(x^3+1)^2}$

6 $f'(x)=\frac{-\frac{1}{2}x^{-\frac{1}{2}}(1+x)-(1-\sqrt{x})}{(1+x)^2}$

So when $x=4,\ f'(x)=\frac{-\frac{1}{2}4^{-\frac{1}{2}}(1+4)-(1-\sqrt{4})}{(1+4)^2}=\frac{-\frac{5}{4}-3}{25}=-0.17$

So equation of tangent is $y=-0.17x+0.48$
and equation of normal is $y=5.88x-23.7$

5.4 Graphical interpretation of first and second derivatives

1 Find the intervals where $f(x)$ is increasing or decreasing.

a $f(x) = 3x^2 - 1$

b $f(x) = 2\sqrt{x}$

2 For the function $f(x) = x^3 - 2x^2 + x$, find and classify any turning points.

3 The cubic function $f(x) = x^3 + bx + c$ has a local maximum at $(-1, 3)$. Find the coordinates of the local minimum.

4 Find the second derivative of $f(x) = 3x^{\frac{2}{3}}$.

5 Given that $y = (x + a)^3$ and $\frac{d^2y}{dx^2} = 6x - 3$, find the value of a.

6 For each function find

i any points of inflection

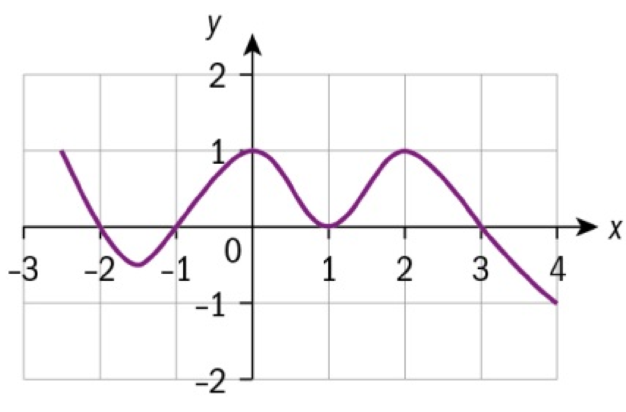
ii intervals where it is concave up

iii intervals where it is concave down.

a $y = 2x^4 - 3x^2$

b $y = x^4 - 4x^3 - 18x^2 + 2x - 1$

7 Copy the graph and on the same axes sketch the graphs of the first and second derivatives.



Answers

1 **a** Increasing when $x > 0$, decreasing when $x < 0$.

b Always increasing

2 $f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1)$

So stationary points at $x = 1, \ \frac{1}{3}$

$f''(x) = 6x - 4$ so >0 at $x = 1, \ <0$ at $x = \frac{1}{3}$

So maximum at $\left(\frac{1}{3}, \frac{4}{27}\right)$, minimum at $(1, 0)$

3 $f(-1) = 3 \Rightarrow -1 - b + c = 3 \Rightarrow b - c = -4$

$f'(-1) = 0 \Rightarrow 3 + b = 0 \Rightarrow b = -3, c = 1$
So $f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3$
So other stationary point is at $(1, -1)$ and it is a minimum since $f''(1) > 0$.

4 $f'(x) = 2x^{-\frac{1}{3}} \Rightarrow f''(x) = -\frac{2}{3}x^{-\frac{4}{3}}$

5 $\frac{dy}{dx} = 3(x+a)^2 \Rightarrow \frac{d^2y}{dx^2} = 6(x+a) \Rightarrow a = -\frac{1}{2}$

6 **a** **i** $\frac{dy}{dx} = 8x^3 - 6x \Rightarrow \frac{d^2y}{dx^2} = 24x^2 - 6 = 6(x^2 - 1)$

So points of inflexion at $(-1, -1), \ (1, -1)$

ii Concave up when $x < -1, \ x > 1$

iii Concave down when $-1 < x < 1$

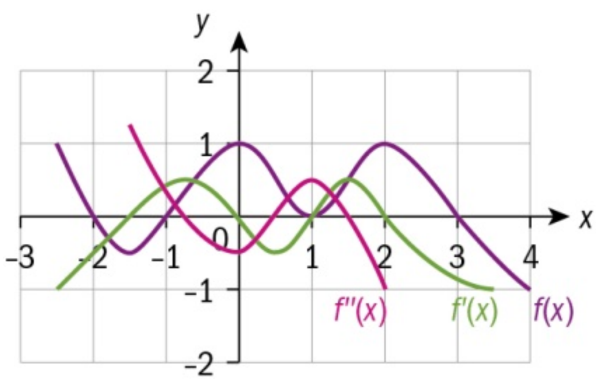
b **i** $\frac{dy}{dx} = 4x^3 - 12x^2 - 36x + 2 \Rightarrow \frac{d^2y}{dx^2} = 12x^2 - 24x - 36 = 12(x^2 - 2x - 3) = 12(x-3)(x+1)$

So points of inflexion at $(-1, -16), \ (3, -184)$

ii Concave up when $x < -1, \ x > 3$

iii Concave down when $-1 < x < 3$

7



5.5 Application of differential calculus: optimization and kinematics

1 A rectangle has perimeter 60 cm. Find the dimensions of the rectangle if the area is to be as large as possible.

2 An open cylindrical container is to be built so that the sum of the radius and the height is 10 m. Find the maximum volume of the cylinder.

3 A ball is thrown vertically so that its height above the ground (in metres) is given by $s(t) = 20t - 5t^2 + 2$. Calculate the maximum height reached and the time when this happens.

4 A particle travels in a straight line so that at time t its displacement is given by

$s(t) = \frac{2}{3}t^{\frac{3}{2}} - 2t + 4$.

Find

a an expression for the velocity of the particle

b the time and position of the particle when it changes direction

c an expression for the acceleration of the particle.

Answers

1 Dimensions are $x, 30 - x$

Area is $x(30 - x) = 30x - x^2$

$\frac{dA}{dx} = 30 - 2x = 0$ when $x = 15$

This is a maximum because $\frac{d^2A}{dx^2} = -2$.

So the dimensions are 15 by 15 which is a square.

2 $V = \pi r^2 \left(10 - r\right) \Rightarrow \frac{dV}{dr} = 20\pi r - 3\pi r^2$

$= \pi r\left(20 - 3r\right)$

So $\frac{dV}{dr} = 0$ when $r = \frac{20}{3}$

$\frac{d^2V}{dr^2} = 20\pi - 6\pi r < 0$ when $r = \frac{20}{3}$

So it is a maximum

and maximum volume = $465.4 \approx 465\text{ m}^3$

3 $s'(t) = 20 - 10t = 0$ when $t = 2$

and maximum height = 22 m.

4 **a** $v = t^{\frac{1}{2}} - 2$

b $t^{\frac{1}{2}} - 2 = 0 \Rightarrow t = 4$

$s = \frac{2}{3} \times 4^{\frac{3}{2}} - 4 + 4 = \frac{8}{3}$

c $a = \frac{1}{2}t^{-\frac{1}{2}}$

6.1 Sampling

1 Classify the following data as categorical or numerical:

a Favourite colour

b Annual salary

c Number of cars in the car park

d Height of students in your class

e Genre of music

2 Classify the following numerical data as discrete or continuous.

a Shoe size

b Time taken to get home from school

c The number of relatives that you have

d The lengths of the rooms in your home

e The number of computers that your classmates have owned

3 Identify the sampling method being used.

a Ask every tenth student on the school role starting from A and finishing with Z.

b Choose a sample of 20 players in a local sports' day by putting names in a hat and drawing out 20.

c In a school, you choose 50 students from each grade.

d To find out which social media is the best, you ask 20 students from each age group.

e Ask all of the students in the canteen between 1 p.m. and 1:30 p.m. about the quality of the food.

f A researcher asks for a sample of 100 females between the ages of 15 and 25.

g To predict who will win the next election, a researcher asks 500 people at a railway station.

Answers

1 **a** Categorical

b Numerical

c Numerical

d Numerical

e Categorical

2 **a** Discrete

b Continuous

c Discrete

d Continuous

e Discrete

3 **a** Systematic sampling

b Random sampling

c Stratified sampling

d Stratified sampling

e Convenience sampling

f Quota sampling

g Random sampling

6.2 Presentation of data

1 Thirty students in the school blood drive are given a blood test to determine their blood type. These are the results

O, B, A, O, O, AB, B, A, O, A, B, B, AB, O, A, A, O, O, O, B, AB, AB, O, B, O, O, A, B, B, A

a Copy and complete the following table.

Blood types	Frequency
A	
B	
O	
AB	
Total	30

b Construct a bar chart for this data.

2 The masses, in kg, of the 26 students in the football squad is given below.

55, 83, 51.4, 63, 73.4, 66.3, 78, 43.2, 84.3, 76, 61, 73, 74.6, 88, 62, 72, 54.6, 71, 82, 65, 71.9, 47, 59, 74.2, 81, 65.8

a Is the data discrete or continuous?

b Copy and complete the following table.

Mass (kg)	Frequency
$40 \le m < 50$	
$50 \le m < 60$	
$60 \le m < 70$	
$70 \le m < 80$	
$80 \le m < 90$	

c Represent this data on a histogram.

3 The time taken to complete a test is shown below.

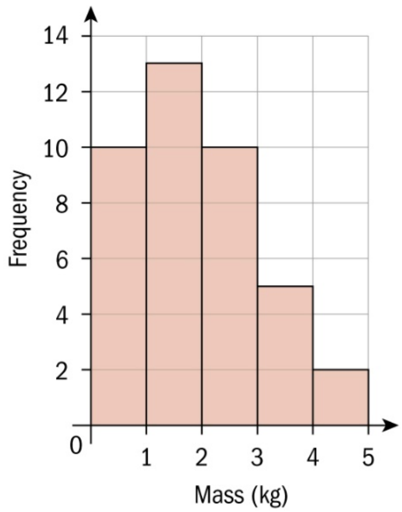
Time (mins)	Frequency
$0 \le m < 16$	3
$16 \le m < 32$	5
$32 \le m < 48$	8
$48 \le m < 64$	12

a Is the data discrete or continuous?

b Show this data on a histogram.

c Describe the skew.

4 The mass of fish caught by a group of fishermen on a pier is shown in this histogram.



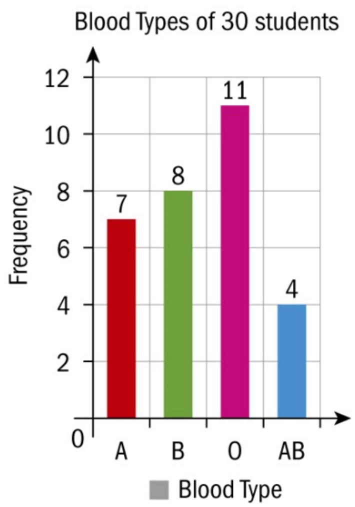
Show this as a frequency table with classes $0 \le m < 1$, $1 \le m < 2$, and so on.

Answers

1 a

Blood types	**Frequency**
A	7
B	8
O	11
AB	4
Total	30

b

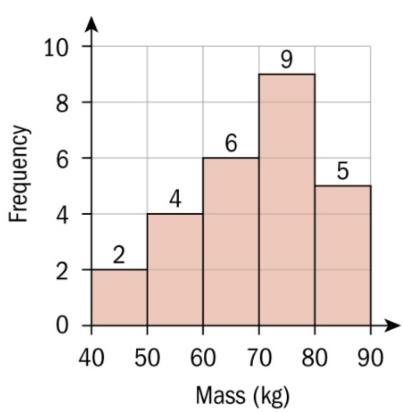


2 a Continuous

b

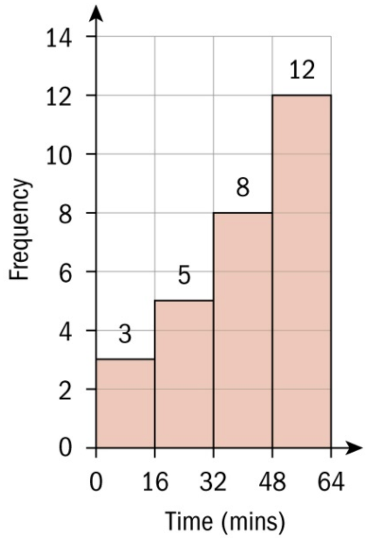
Mass (kg)	**Frequency**
$40 \le m < 50$	2
$50 \le m < 60$	4
$60 \le m < 70$	6
$70 \le m < 80$	9
$80 \le m < 90$	5

c



3 **a** Continuous

b



c Negatively skewed.

4

Mass(kg)	**Frequency**
$0 \le m < 1$	10
$1 \le m < 2$	12
$2 \le m < 3$	9
$3 \le m < 4$	5
$4 \le m < 5$	2

6.3 Measures of central tendency

1 Copy and complete this table using your GDC to write down the answers.

Values	**Mode**	**Median**	**Mean**
151, 158, 171, 163, 184, 148, 171			
25, 33, 18, 17, 44, 81, 27, 25			
53, 61, 58, 82, 45, 72, 82			

2 The mean age of a group of university graduates is 22.4 and the median is 22.9. If they all attend a reunion after 10 years, what will the mean and median be then?

3 In the following ordered data, the mean is 6.5 and the median is 6.

2, 3, 4, *a*, 7, 9, 10, *b*

Find each of the following:

a the value of *a*

b the value of *b*.

4 Luca has received the following grades this term: 75, 87, 90, 88, 78. If he wishes to earn an 85 average, what must he score on his final test?

5 The ages of students at a playground are:

9, 7, 8, 11, 9, 6, 10, 8, 12, 6, 8, 13, 7, 9, 10, 9, 10, 11, 12, 8, 7, 13, 10, 7, 7

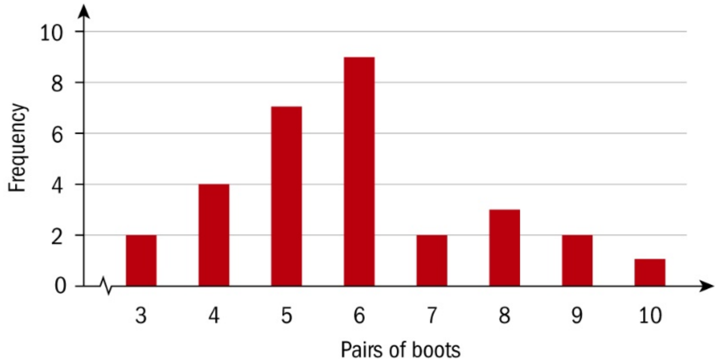
a Show the data in a frequency table

b Calculate the mean age of the students.

c What is the modal age?

d Write down the median age.

6 The bar chart below shows the pairs of boots owned by members of a rugby team.



a How many players are represented?

b Write down the mode of the distribution.

c Find, correct to the nearest whole number, the mean number boots owned by the players.

7 The table below shows the number of pets for a group of 25 children.

Number of pets	0	1	2	3	4	5
Frequency	4	3	8	q	4	1

a Find the value of q.

Use your GDC to find the following, but show working for each one.

b the mean

c the median

d the mode

8 The scores, out of 50, for a math test are shown below.

Score	***f***
$0 < x \le 10$	2
$10 < x \le 20$	8
$20 < x \le 30$	14
$30 < x \le 40$	24
$40 < x \le 50$	13

Write down

a the mean score

b the median score.

Answers

1

Values	Mode	Median	Mean
151, 158, 171, 163, 184, 148, 171	171	163	164
25, 33, 18, 17, 44, 81, 27, 25	25	26	33.75
53, 61, 58, 82, 45, 72, 82	82	61	64.7

2 The mean 32.4 and the median 32.9 as all of their ages will be the graduating age plus 10.

3 **a** The median of 6 is half-way between *a* and 7.

$$\frac{a+7}{2}=6$$

$$a=5$$

b $\text{mean}=\dfrac{2+3+4+5+7+9+10+b}{8}=6.5$

$$40+b=52$$

$$b=12$$

4 Let Luca's next score be x

$$\frac{75+87+90+88+78+x}{6}=85$$

$$\frac{418+x}{6}=85$$

$$418+x=510$$

$$x=92$$

Luca will need to score a 92 on his last test to earn an average of 85 for the term.

5 **a**

Ages	Frequency
6	2
7	5
8	4
9	4
10	4
11	2
12	2
13	2

b $\text{mean}=\dfrac{\text{sum of the ages}}{\text{total frequency}}=\dfrac{227}{25}=9.08$

c The mode is the age with the highest frequency, which is 7.

d 9

6 **a** $2+4+7+9+2+3+2+1=30$

b 6

c mean $= \dfrac{6+16+35+54+14+24+18+10}{30} = \dfrac{177}{30} = 5.9$

To the nearest whole number, the mean number of boots owned by members of the rugby team is 6

7 **a** $4+3+8+q+4+1=25$

$q=5$

b mean $= \dfrac{\text{total number of pets}}{\text{total frequency}} = \dfrac{55}{25} = 2.2$

c Median $=$ the middle member $= \left(\dfrac{25+1}{2}\right)$th $=$ 13th $= 2$

d The mode is the number of pets with the highest frequency, which is 2.

8 **a** 31.2

b 35

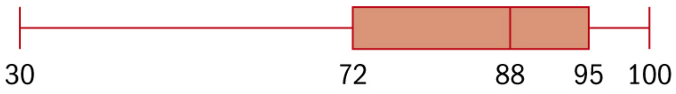
6.4 Measures of dispersion

1 Andre kept a record of how many times a month that he visited the gym after he bought a two-year membership.

11, 10, 3, 3, 5, 2, 1, 5, 0, 8, 3, 15, 3, 6, 6, 4, 3, 0, 8, 9, 7, 0, 2, 9

Show this data on a box plot.

2 The temperatures of teachers' coffee in °C is recorded in the box plot.

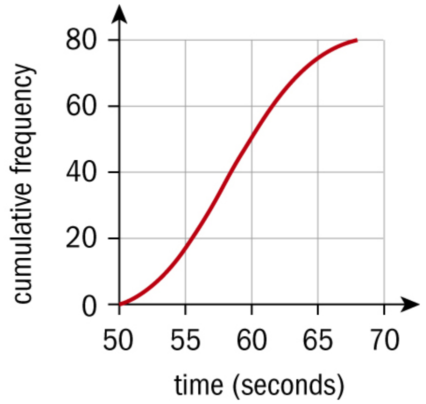


a Write down the median temperature for their coffee?

b Find the interquartile range.

c Show that 30°C is an outlier.

3 The students in the swimming club were timed for one lap of the pool and the results are displayed on this cumulative frequency graph.



a How many students were in the swimming club?

b What could have been the fastest time to swim one lap?

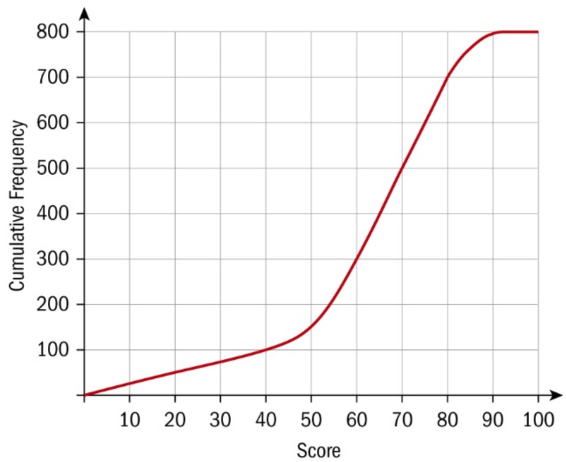
c What might have been the longest time?

d Estimate the median time.

e Find the interquartile range.

f Estimate the time corresponding to the fastest 10% of the swimmers.

4 The results of a mathematics test are represented in this cumulative frequency diagram.



a How many students took the exam?

b Estimate the median score.

c Find the interquartile range.

d Estimate the number of students who scored 60 marks or less for the test.

e What score is needed to score an A, if an A grade is awarded to the top 12.5% of students?

5 A survey was conducted of the number of bags that students brought to school with them. The results are shown in the following table.

Number of bags	1	2	3	4	5	6
Number of students	41	60	52	32	15	8

a Is the data discrete or continuous?

b Write down the mean number of bags per student.

c Write down the standard deviation of the number of bags.

d Find the variance.

e Find how many students have a number of bags greater than one standard deviation above the mean.

6 The table below shows the widths, W, cm, of the tables in the school cafeteria.

Width (W)	$80<W\le85$	$85<W\le90$	$90<W\le95$	$95<W\le100$	$100<W\le105$	$105<W\le110$	$110<W\le115$
Number of tables	5	10	15	26	13	7	4

Use the midpoint of each interval to find an estimate for the standard deviation of the weights.

7 A data set has a mean of 20 and a standard deviation of 6.

Each value in the data set has 10 added to it. Write down the value of

a the new mean

b the new standard deviation.

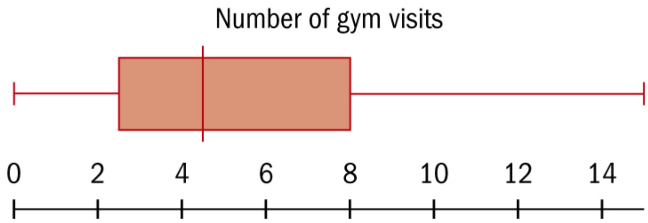
Each value in the original data set is multiplied by 10.

c Write down the value of the new mean.

d Find the value of the new variance.

Answers

1



2 **a** 88°C

b IQR $= Q_3 - Q_1 = 95 - 72 = 23$°C

c An outlier would be below $Q_1 - 1.5\text{IQR} = 72 - (1.5 \times 23) = 73 - 34.5 = 38.5$

Therefore 30°C is an outlier.

3 **a** 80 students

b 50 seconds

c 70 seconds

d The median $=$ the 40th swimmer $=$ 58.5 seconds

e IQR $= Q_3 - Q_1 = 61.5 - 55.5 = 6$ seconds

f The fastest 10% would be a line drawn across from 8 and then down to the *x* axis to give 53 seconds.

4 **a** 800

b 65

c IQR $= Q_3 - Q_1 = 75 - 54 = 21$ marks

d 300

e 80

5 **a** Discrete

b 2.73

c 1.34

d variance $= (\text{standard deviation})^2 = (1.34)^2 = 1.80$

e $2.73 + 1.34 = 4.07$

Greater than 4.07 is $15 + 8$, 23 students

6 7.41 cm

7 **a** 30

b 6

c 200

d variance $= (\text{standard deviation})^2 = (6 \times 10)^2 = (60)^2 = 3600$

7.1 Scatter diagrams

1 Match the diagrams with the description of their correlation.

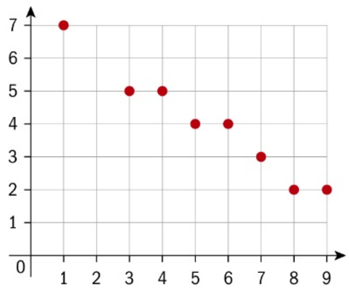
a positive correlation

b negative correlation

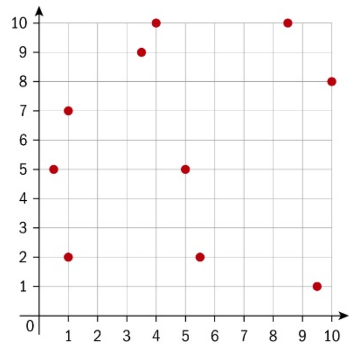
c no correlation

d nonlinear

i



ii



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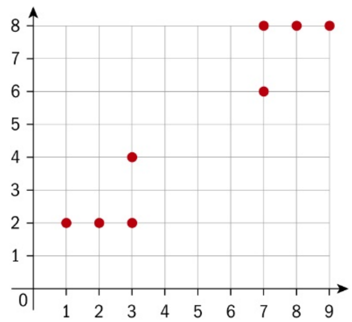

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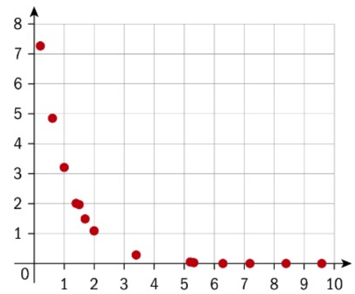
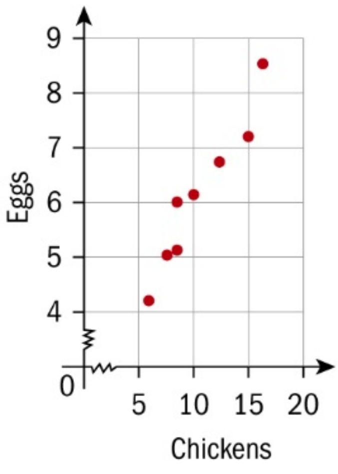


2 A farmer has 8 henhouses with different numbers of chickens in each. The scatterplot shows the number of chickens in each henhouse and the average number of eggs laid per day.

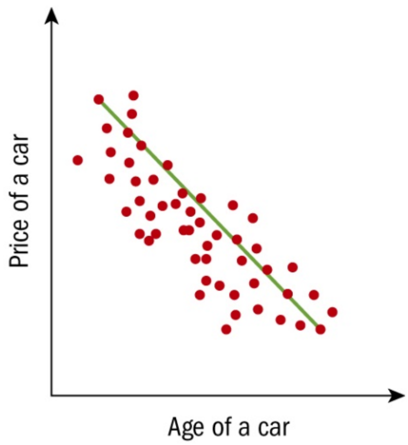


a Copy and complete the following sentence.

As the number of chickens ....,

the .... increases.

b Now write a similar sentence for this scatterplot.



3 Do the following data sets have a positive, negative, or no correlation?

a The height of a teenager and the year

b The temperature and the day of the week

c The amount of bacteria on your hands and the number of times you wash your hands

4 The height of a plant and the distance they are planted apart are shown in this table.

Distance apart (cm)	Height (cm)
62	120
63	125
67	155
68	135
69	175

a Show this data on a scatterplot.

b Describe the correlation.

c Describe the situation with a suitable sentence.

Answers

1 **a** iii **b** i **c** ii **d** iv

2 **a** As the number of chickens increases, the number of eggs increases.

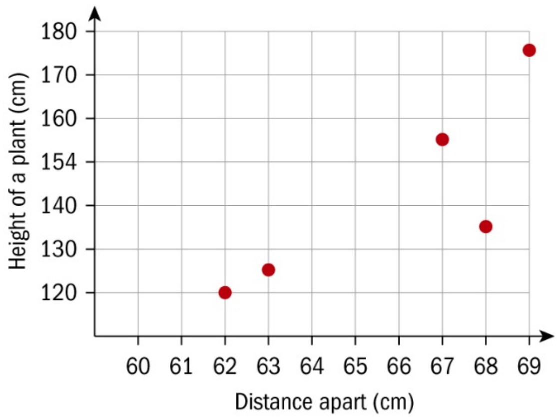
b As the age of a car increases, the price of a car decreases.

3 **a** Positive

b No correlation

c Negative

4 **a**



b Positive correlation.

c As the distance apart increases, the height of a plant increases.

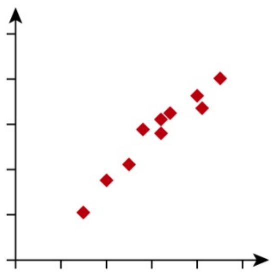
7.2 Measuring correlation

1 Match the scatterplot with the description.

a no correlation **b** strong positive **c** strong negative

d weak positive **e** weak negative

i



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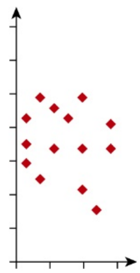

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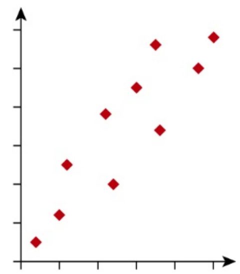

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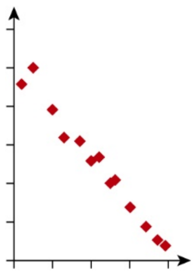

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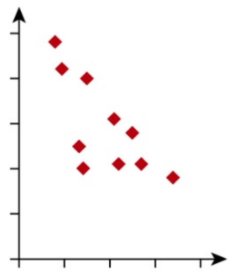
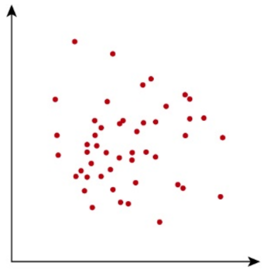


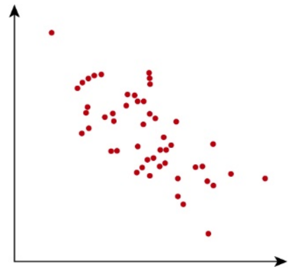
2 Match the diagram with the *r* value

−0.95, −0.65, −0.25, 0, 0.5, 0.9

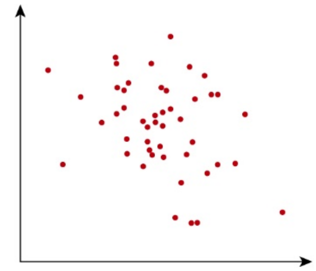
a



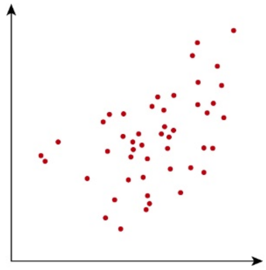
b



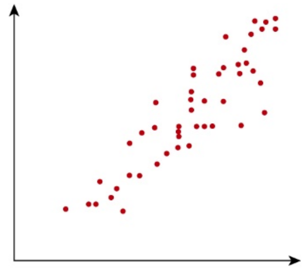
c



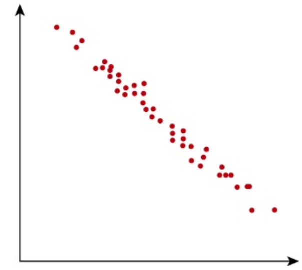
d



e



f



3 Which of the following values cannot represent a correlation coefficient? Explain your answer.

a $r = 1.06$ **b** $r = -0.95$ **c** $r = 0$

d $r = -1$ **e** $r = 0.999$

4 Lina had always wondered if the number of musical instruments (x) in an orchestra was related to the number of views (y) on social media. She collected the data displayed below.

x	10	14	15	22	26	32	32	38	40	42
y	450	493	525	710	730	894	854	900	800	1050

a Display the data on a scatterplot.

b Write down the r value.

c Interpret the correlation between the number of musical instruments in an orchestra and the number of views on social media.

5 The temperature (x) in 16 greenhouses and the number of tomatoes (y) produced is shown in this table.

x	72	73	75	76	77	78	79	80	80	81	82	83	84	85	86	88
y	45	38	41	35	31	40	25	32	36	29	34	38	26	32	28	27

a Show this on a scatterplot.

b Write down the r value.

c Interpret the r value for the temperature in 16 greenhouses and the number of tomatoes produced.

Answers

1 **a** ii **b** i **c** iv

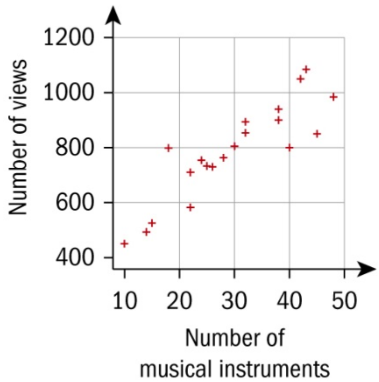
d iii **e** v

2 **a** 0 **b** –0.65 **c** –0.25

d 0.5 **e** 0.9 **f** –0.95

3 $r \neq 1.06$, because r must lie between −1 and 1, inclusive.

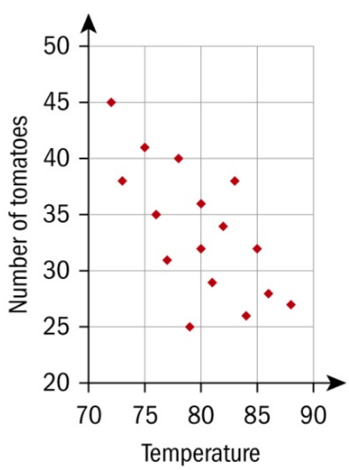
4 **a**



b $r = 0.882$

c There is a strong positive correlation between the number of musical instruments in an orchestra and the number of views on social media. This means that as the number of musical instruments in an orchestra increases, the number of views on social media increases.

5 **a**



b $r = –0.680$

c There is a moderate negative correlation between the temperature in 16 greenhouses and the number of tomatoes produced. This means that as the temperature in the greenhouses increases, the number of tomatoes produced decreases.

7.3 The line of best fit

1 Miss Lily recorded the number of hours (x) that her students in her tutor group studied for an exam and their exam grade (y).

x	0	0.5	1	1.5	2	2.5
y	20	20	30	60	70	70

a Write down the mean number of hours and the mean exam grade.

b Write down the equation of the line of best fit.

c Construct a scatterplot and draw the line of best fit through the mean point.

d Use your diagram to find how long a student should study to aim for 100%.

2 Yuyu measures the height of the classroom plant every week for six weeks

Age (*x* weeks)	0	1	2	3	4	5	6
Height (*y* cm)	2.4	4.3	5.0	6.9	9.1	11.4	13.5

a Find the mean age and mean height.

b Write down the equation of the regression line.

c Show the data and line of best fit on a scatterplot.

d Find the height of the plant after 3.5 weeks.

e Find the height of the plant after 18 weeks. Is this a sensible answer?

3 David's tech store sells used computers. The age (x) in years and the selling price (y) in hundreds of dollars is show in the table.

x	1	1	2	3	4	4	5	5.5	7	7
y	6	7	5	8	4	6	6	4	2	4

a Find the r value. What does this tell you about the age and price of David's computers.

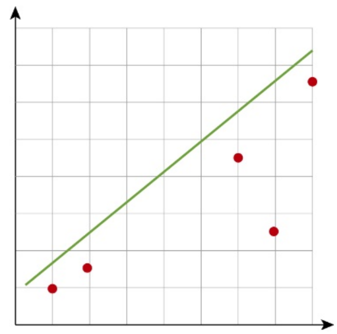
b Write down the equation of the regression line.

c Show the data and line of best fit on a scatterplot.

d Find the price of a 2.5-year-old computer.

e Why can't you use the regression equation to predict the price of a 20-year-old computer?

4 The graph shows one student's approximation of the best-fitting line for the data in the scatterplot. Describe an the error in the student's work.

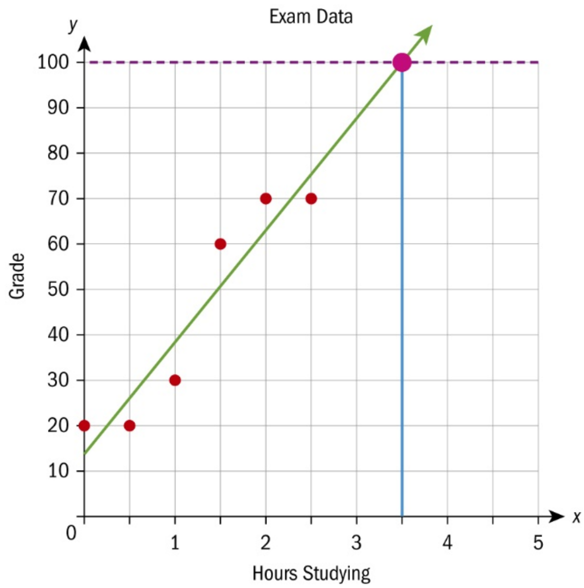


Answers

1 **a** Mean number of hours = 1.65 and the mean exam grade = 55.9.

b $y = 23.8x + 16.7$

c

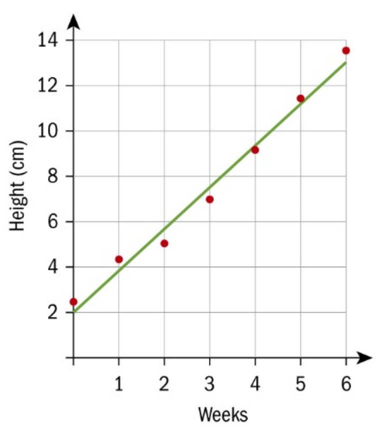


d 3.5 hours

2 **a** Mean age = 3 weeks, mean height = 7.51cm.

b $y = 1.99 + 1.84x$.

c



d The height after 3.5 weeks: $y = 1.985 + (1.843 \times 3.5) = 8.44\,\text{cm}$

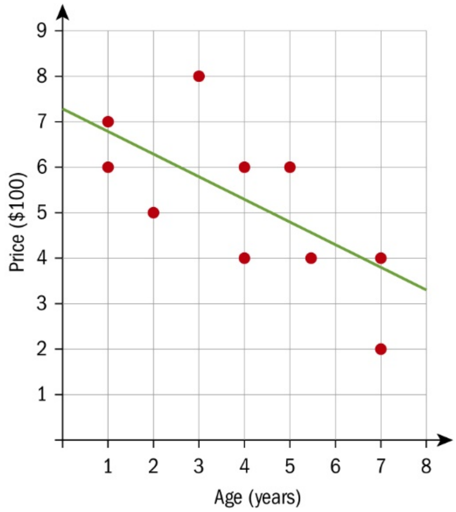
e $y = 1.985 + (1.843 \times 18) = 35.2\,\text{cm}$

It would not be sensible to predict the height after 18 weeks from this equation – we don't know whether the relationship will continue to be linear. The process of trying to predict a value from outside the range of your data is called extrapolation.

3 **a** $r = -0.608$. There is a moderate, neagtive correlation between the age of a computer and the price. As the age of a computer increases, the price decreases.

b $y = 7.37 - 0.471x$

c



d $y = 7.37 - (0.471 \times 2.5) = 6.19$. A 2.5-year-old computer costs $619.

e Predicting outside of the known domain is called extrapolation and is unreliable.

4 The line is drawn too low. The sum of the residuals above the line should be close to the sum of residuals below the line.

7.4 Least squares regression

1 When C is the cost in dollars of producing p pizzas made at Paolo's Pizza shop, interpret the meaning of the gradient and the constant term in the regression line $C = 15 + 5p$.

2 The height of a shrub (h cm) and the number of weeks (w) after it was planted are modelled by $h = 20 + 8w$. Interpret the meaning of the gradient and the constant term.

3 The number (n) of electric vehicles in Venilan is modelled by $n = 120 + 45y$, where y is the number of years after 2014.

a How many electric vehicles were there in 2014?

b How many vehicles do you estimate there will be in 2021?

c What does the gradient of 45 indicate?

4 The heights (x) in metres and weights (y) in kilograms of the soccer team are as shown.

Height (*x* metres)	1.36	1.47	1.54	1.56	1.59	1.63	1.66	1.67	1.69	1.74	1.81
Weight (*y* kg)	52	50	67	62	69	74	59	87	77	73	67

a Find the mean height and weight.

b Write down the equation of the regression line.

c Use the regression line to estimate the weight of someone whose height is 1.6 m.

d Interpret the meaning of the gradient of the regression line.

5 The number of banks (b) and the number of traffic lights (t) in eight towns is shown below.

b	8	9	7	6	13	7	11	12
t	35	49	27	33	60	21	45	51

a Sketch the scatterplot for this data.

b Copy and complete the following sentence.

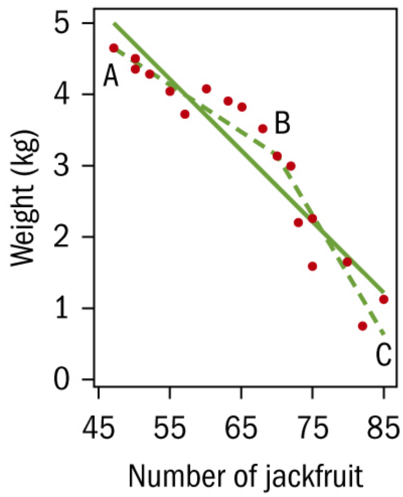
As the number of banks ...., the number of traffic lights ....

c Write down and interpret the r value.

d Write down the equation of the regression line.

e Interpret the meaning of the gradient of the regression line.

6 The diagram displays the number (n) of jackfruit grown on a tree and the average weight (w) of each jackfruit on Lee's farm.



There is one linear regression line and also a piecewise linear function drawn on the scatterplot.

a Which do you think best represents the data?

AB has equation $w = 8 - 0.07n$, and BC has equation $w = 15 - 0.17n$.

b Find the coordinates of point B.

c If Lee wants to grow 60 jackfruit, what average weight can be expected?

7 The mathematics (m) and physics (p) scores for eight students are shown in this table.

m	88	94	78	60	92	86	87	73
p	78	90	66	44	81	78	80	65

a Find the regression line of p on m.

b Steve scored 90 in mathematics but was absent for the physics. Estimate his physics score.

c Find the regression line of m on p.

d Cass scored 75 in physics; estimate the mathematics score.

Answers

1 The gradient of 5 means it costs Paolo \$5 to make each pizza and the constant of 15 means that it costs Paolo \$15 before he starts making pizzas.

2 The gradient shows an increase of 8 cm per week and the constant term shows that the shrub was 20 cm when it was first planted.

3 **a** $n = 120 + 45(0) = 120$ **b** $n = 120 + 45(7) = 435$

c The increase in the number of electric vehicles every year.

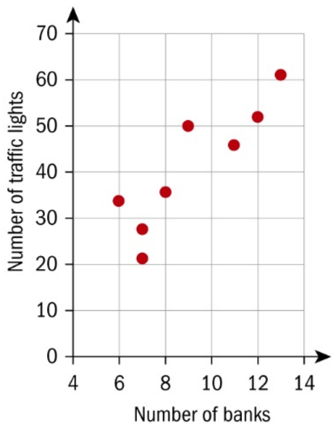
4 **a** $\bar{x} = \frac{\sum x}{n} = \frac{17.72}{11} = 1.61\,\text{m}$, $\bar{y} = \frac{\sum y}{n} = \frac{737}{11} = 67\,\text{kg}$

b $y = 55x - 22.4$

c $y = (55.5 \times 1.6) - 22.4 = 66.4\,\text{kg}$

d An increase of 1 m in height results in an increase of 55 kg in weight.

5 **a**



b As the number of banks increases, the number of traffic lights increases.

c $r = 0.886$. There is a strong positive linear association between the number of banks and the number of traffic lights.

d $t = 4.54b - 1.31$ **e** Every extra bank gives 4.54 new traffic lights.

6 **a** The piecewise function.

b $15 - 0.17n = 8 - 0.07n$

$-0.1n = -7$

$n = 70$

Substitute $n = 70$ into either equation to find *w:* $w = 8 - 0.07(70) = 3.1$

B is point $(70, 3.1)$

c Using the first equation: $w = 8 - 0.07(60) = 3.8$

Lee can expect the average jackfruit to weigh 3.8 kg.

7 **a** $p = 1.22m - 27.4$

b $p = 1.22(90) - 27.4 = 82.3$. Steve's scores is estimated to be 82.3

c $m = 0.787p + 25.0$

d $m = 0.787(75) + 25.0 = 84.0$ Cass' score is estimated to be 84.0

8.1 Theoretical and experimental probability

1 A bag contains 24 balls numbered 1 through 24. A ball is selected at random.

Find the probability that the number on the ball is

a an even number

b a multiple of 5

c a factor of 24

d a prime number.

2 A school has students in grades 9–12. The number of students in each grade is given in the table below.

Grade	Number of students
9	246
10	209
11	215
12	187

A student at the school is selected at random. Find the probability that

a the student is in grade 9

b the student is in grade 11 or higher

c the student is not in grade 12.

3 A card is selected at random from a regular deck of 52 playing cards.

Find the probability that the card

a is an Ace

b is a face card

c is a heart

d a black card

e shows a number less than 7.

Answers

1 a There are 12 even numbers (2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24).

$$\text{P}(\textit{even}) = \frac{12}{24} = \frac{1}{2}$$

b There are 4 multiples of 5 (5, 10, 15, 20).

$$\text{P}(\textit{multiple of } 5) = \frac{4}{24} = \frac{1}{6}$$

c There are 8 factors of 24 (1, 2, 3, 4, 6, 8, 12, 24).

$$\text{P}(\textit{factor of } 24) = \frac{8}{24} = \frac{1}{3}$$

d There are 9 prime numbers (2, 3, 5, 7, 11, 13, 17, 19, 23).

$$\text{P}(\textit{prime}) = \frac{9}{24} = \frac{3}{8}$$

2 There are 857 students total.

a There are 246 students in grade 9.

$$\text{P}(\textit{grade } 9) = \frac{246}{857} \approx 0.287$$

b There are $215 + 187 = 402$ students in grade 11 or higher.

$$\text{P}(\textit{grade 11 or higher}) = \frac{402}{857} \approx 0.469$$

c There are $857 - 187 = 670$ students not in grade 12.

$$\text{P}(\textit{not in grade } 12) = \frac{670}{857} \approx 0.782$$

3 a There are 4 Aces in a regular deck of cards.

$$\text{P}(\textit{Ace}) = \frac{4}{52} = \frac{1}{13}$$

b There are 12 face cards (4 Jacks, 4 Queens, 4 Kings) in a regular deck of cards.

$$\text{P}(\textit{face card}) = \frac{12}{52} = \frac{3}{13}$$

c There are 26 black cards (13 clubs, 13 spades) in a regular deck of cards.

$$\text{P}(\textit{black}) = \frac{26}{52} = \frac{1}{2}$$

d There are 20 cards showing numbers less than 7 (four 2s, four 3s, four 4s, four 5s, four 6s) in a regular deck of cards.

$$\text{P}(\textit{number less than } 7) = \frac{20}{52} = \frac{5}{13}$$

8.2 Representing probabilities: Venn diagrams and sample spaces

1 At a particular school, boys and girls may participate in a variety of extracurricular sports. 80% of the students participate in sports, and 45% of the students at the school are boys. One third of the boys at the school do not play sports.

a Make a Venn diagram to represent this situation.

A student is selected at random. Find the probability that the student

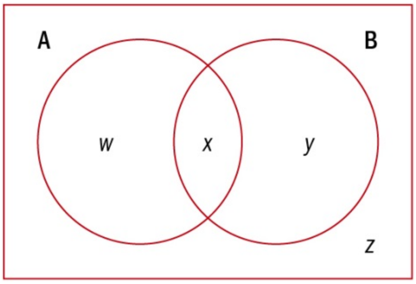
b does not participate in sports

c is a girl

d is a boy who plays sports

e is a girl who does not play sports.

2 The Venn diagram below shows events A and B where $P(A) = 0.5$, $P(A \cup B) = 0.8$ and $P(A \cap B) = 0.15$. The values w, x, y and z are probabilities.



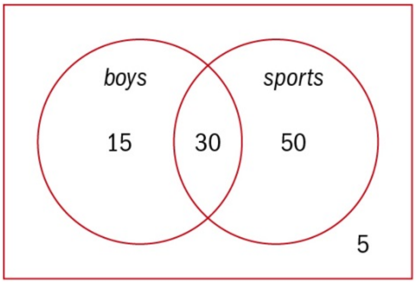
a Find the values of w, x, y and z.

b Find $P(B)$.

c Find $P(A \cap B')$.

Answers

1 a



b 20% of the students do not participate in sports

$\mathrm{P}(\textit{student does not play sports}) = 0.2$

c 55% of the students are girls

$\mathrm{P}(\textit{girl}) = 0.55$

d 30% of the students are boys who play sports

$\mathrm{P}(\textit{boy who plays sports}) = 0.3$

e 5% of the students are girls who do not play sports

$\mathrm{P}(\textit{girl who does not play sports}) = 0.05$

2 a $w = 0.35$, $x = 0.15$, $y = 0.3$ and $z = 0.2$

b $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$0.8 = 0.5 + \mathrm{P}(B) - 0.15 \rightarrow \mathrm{P}(B) = 0.45$

c $\mathrm{P}(A \cap B') = 0.35$

8.3 Independent and dependent events and conditional probability

1 A bag contains ten marbles; six are red and four are blue. One marble is randomly selected from the bag, its colour noted, then returned to the bag. Then a second marble is randomly selected from the bag. Find the probability that

a both marbles are blue

b the first marble is blue, and the second marble is red

c at least one of the marbles is blue.

2 Consider the independent events A and B. Given that $P(A) = 2P(B)$,

and $P(A \cup B) = 0.72$, find $P(B)$.

3 A bag contains eight marbles; five are green and three are white.

Two marbles are selected at random without replacement.

a Find the probability that

i the first marble is green

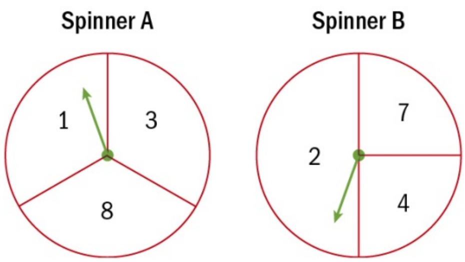
ii both marbles are green

iii at least one of the marbles is white

iv the second marble is green.

b Given that the second marble is green, find the probability that the first marble is white.

4 You play a game in which each of the following spinners is spun once.



a Find the probability of spinning an even number on spinner A.

b Find the probability of spinning an even number on spinner B.

c Find the probability of spinning two even numbers.

d Find the probability of spinning one even number and one odd number.

e Find the probability of spinning two numbers with a sum of 10.

5 Let A and B be independent events, where P(A) = 0.5, and P(B) = x.

a Write down an expression for $\mathrm{P}(A \cap B)$ in terms of x.

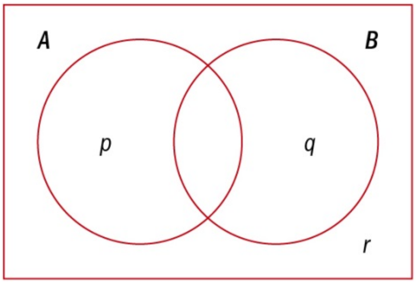
b Given that $\mathrm{P}(A \cup B) = 0.65$,

i find x

ii find $\mathrm{P}(A \cap B)$.

6 Consider the events A and B, where P(A) = 0.7, P(B) = 0.4 and $\mathrm{P}(A \cap B) = 0.3$.

The Venn diagram below shows the events A and B, and the probabilities p, q and r.



a Find the values of p, q and r.

b Are A and B independent events? Justify your answer.

Answers

1 **a** $P(\text{both blue}) = P(\text{blue first}) \times P(\text{blue second}) = \frac{4}{10} \times \frac{4}{10} = \frac{16}{100} = \frac{4}{25}$

b $P(\text{blue, then red}) = P(\text{blue first}) \times P(\text{red second}) = \frac{4}{10} \times \frac{6}{10} = \frac{24}{100} = \frac{6}{25}$

c $P(\text{at least one blue}) = 1 - P(\text{both red}) = 1 - (P(\text{red first}) \times P(\text{red second}))$

$P(\text{at least one blue}) = 1 - \left(\frac{6}{10} \times \frac{6}{10}\right) = 1 - \frac{36}{100} = \frac{64}{100} = \frac{16}{25}$

2 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - (P(A) \times P(B))$

$0.72 = 2P(B) + P(B) - (2P(B) \times P(B))$

$2(P(B))^2 - 3P(B) + 0.72 = 0$ (use polynomial root finder on your GDC)

$P(B) = 0.3$

3 **a** **i** $P(\text{first green}) = \frac{5}{8}$

ii $(\text{both green}) = P(\text{green first}) \times P(\text{green second}) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}$

iii $P(\text{at least one white}) = 1 - P(\text{both green}) = 1 - \frac{5}{14} = \frac{9}{14}$

iv $P(\text{green second}) = (P(\text{white first}) \times P(\text{green second})) + P(\text{both green})$

$P(\text{green second}) = \left(\frac{3}{8} \times \frac{5}{7}\right) + \frac{20}{56} = \frac{15}{36} + \frac{20}{56} = \frac{35}{56} = \frac{5}{8}$

b $P(\text{first white} \mid \text{green second}) = \frac{P(\text{white first}) \times P(\text{green second})}{P(\text{green second})} = \frac{\left(\frac{15}{56}\right)}{\left(\frac{5}{8}\right)} = \frac{3}{7}$

4 **a** $P(\text{A even}) = \frac{1}{3}$

b $P(\text{B even}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

c $P(\text{both even}) = P(\text{A even}) \times P(\text{B even}) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

d $P(\text{one even and one odd}) = (P(\text{A even}) \times P(\text{B odd})) + (P(\text{A odd}) \times P(\text{B even}))$

$P(\text{one even and one odd}) = \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{2}{3} \times \frac{3}{4}\right) = \frac{1}{12} + \frac{6}{12} = \frac{7}{12}$

e $P(\text{sum of 10}) = P(\text{3 on A and 7 on B}) + P(\text{8 on A and 2 on B})$

$P(\text{sum of 10}) = \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$

5 **a** $P(A \cap B) = P(A) \times P(B) = 0.5x$

b **i** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.65 = 0.5 + x - 0.5x \rightarrow 0.5x = 0.15 \rightarrow x = \dfrac{0.15}{0.5} = 0.3$

ii $P(A \cap B) = 0.5x = 0.5(0.3) = 0.15$

6 **a** $p = 0.4$, $q = 0.1$, and $r = 0.2$

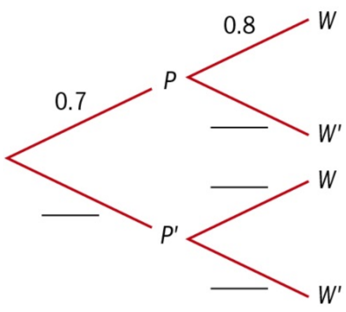
b If A and B are independent events, then $P(A) \times P(B) = P(A \cap B)$.

In this case, $0.7 \times 0.4 = 0.28 \neq 0.3$, so the events are NOT independent.

8.4 Probability tree diagrams

1 There is a 70% chance that George will play in the basketball game on Friday. If he plays in the game, there is an 80% chance that his team will win. If he does NOT play, there is a 40% chance that his team will NOT win. Let P be the event "George plays in the game", and W be the event "George's team wins the game".

a Complete the tree diagram below by filling in the missing probabilities.

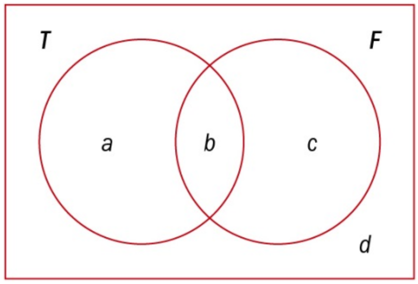


b What is the probability that George plays in the game and his team wins?

c What is the probability that George's team wins the game?

d If it is known that George's team DID win the game, what is the probability that George did NOT play?

2 In a group of 80 students, 35 are on the track team, and 42 are on the football team. 30 of the students are not on either team. The Venn diagram below shows T (a student is on the track team) and F (a student is on the football team). The values a, b, c and d represent numbers of students.



a Find the values of a, b, c and d.

b Write down the probability that a student selected at random is on both the track team and the football team.

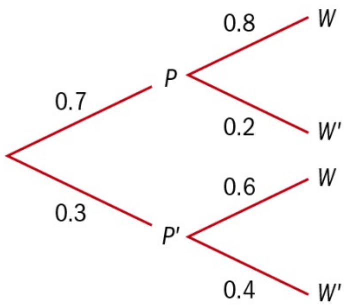
c A student is selected at random. Given that the student is on the track team, write down the probability the student is also on the football team.

d Show that T and F are not independent events.

e Two students are selected at random, one after the other. Find the probability that both students are on the track team.

Answers

1 a



b $\mathrm{P}(P \cap W) = 0.7 \times 0.8 = 0.56$

c $\mathrm{P}(W) = \mathrm{P}(P \cap W) + \mathrm{P}(P' \cap W) = 0.56 + (0.3 \times 0.6) = 0.56 + 0.18 = 0.74$

d $\mathrm{P}(P' | W) = \dfrac{\mathrm{P}(P' \cap W)}{\mathrm{P}(W)} = \dfrac{0.18}{0.74} = \dfrac{9}{37} \approx 0.243$

2 a $n(T \cup F) = n(T) + n(F) - n(T \cap F)$

$50 = 35 + 42 - n(T \cap F) \rightarrow n(T \cap F) = 27$

$a = 8$, $b = 27$, $c = 15$ and $d = 30$

b $\mathrm{P}(T \cap F) = \dfrac{27}{80}$

c $\mathrm{P}(F | T) = \dfrac{\mathrm{P}(F \cap T)}{\mathrm{P}(T)} = \dfrac{\left(\frac{27}{80}\right)}{\left(\frac{35}{80}\right)} = \dfrac{27}{35} \approx 0.771$

d If T and F are independent events, then $\mathrm{P}(T) \times \mathrm{P}(F) = \mathrm{P}(T \cap F)$.

In this case, $\dfrac{35}{80} \times \dfrac{42}{80} = \dfrac{1470}{6400} = \dfrac{147}{640} \neq \dfrac{27}{80}$, so the events are NOT independent.

e $(\text{both T}) = \mathrm{P}(\text{T first}) \times \mathrm{P}(\text{T second}) = \dfrac{35}{80} \times \dfrac{34}{79} = \dfrac{1190}{6320} = \dfrac{119}{632} \approx 0.188$

9.1 Exponents

1 Simplify

a $e^3 \times e^7$ **b** $(x^2y)(x^3y^5)$ **c** $\dfrac{x^2y^5}{xy^3}$ **d** $(4e^2)^3$

e $\left(\dfrac{2x}{3y^3}\right)^3$

2 Find all values of x that solve the following equations. For many problems, you may want to write both sides with the same base.

a $3^x = 27$ **b** $3(2^x) = 48$ **c** $2^x = \dfrac{1}{4}$ **d** $4^x + 9 = 73$

e $2^x \times 2^{x-2} = \sqrt{2}$ **f** $2^x = \dfrac{4}{\sqrt{2}}$ **g** $2^x = \dfrac{1}{\sqrt{8}}$ **h** $3^x = 9^5$

i $\left(\dfrac{1}{6}\right)^x = 36$ **j** $e^x = e\sqrt{e}$ **k** $e^x = \dfrac{1}{\sqrt[3]{e}}$ **l** $4^{2x} = 8^{x+3}$

m $3^x = \left(\dfrac{1}{9}\right)^{4-x}$ **n** $\left(\sqrt{2}\right)^x = 8$ **o** $\left(2^{x+1}\right)^3 = \dfrac{1}{8}$

3 Consider the function $y = 2^{x-2}$.

a **i** Find the y-intercept.

ii Write down the equation of the asymptote.

b State the domain and range of the function.

c Using your GDC, draw a sketch and solve the equation $2^{x-2} = e^{-x}$.

4 Han invests \$1000 at a fixed rate of 7.5% per annum. If the interest is compounded annually, how much is his investment worth after 10 years

Answers

1 **a** e^{10} **b** x^5y^6 **c** xy^2 **d** $64\,e^6$ **e** $\dfrac{(2x)^3}{(3y^3)^3}=\dfrac{8x^3}{27y^6}$

2 **a** $3^x=3^3,\ x=3$

b $2^x=16=2^4,\ x=4$

c $2^x=\dfrac{1}{2^2}=2^{-2},\ x=-2$

d $4^x=64,\ x=3$

e $2^{x+x-2}=2^{2x-2}=2^{\frac{1}{2}},\ 2x-2=\dfrac{1}{2},\ 2x=\dfrac{5}{2},\ \ x=\dfrac{5}{4}$

f $2^x=\dfrac{2^2}{2^{\frac{1}{2}}}=2^{\frac{3}{2}},\ x=\dfrac{3}{2}$

g $2^x=\dfrac{1}{\sqrt{8}}=\dfrac{1}{2^{\frac{3}{2}}}=2^{-\frac{3}{2}},\ x=-\dfrac{3}{2}$

h $3^x=9^5=\left(3^2\right)^5=3^{10},\ x=10$

i $\left(\dfrac{1}{6}\right)^x=36,\ 6^{-x}=6^2,\ \ x=-2$

j $e^x=e\sqrt{e}=e\times e^{\frac{1}{2}}=e^{\frac{3}{2}},\ x=\dfrac{3}{2}$

k $e^x=\dfrac{1}{\sqrt[3]{e}}=e^{-\frac{1}{3}},\ x=-\dfrac{1}{3}$

l $4^{2x}=8^{x+3},\ \left(2^2\right)^{2x}=\left(2^3\right)^{x+3},\ 2^{4x}=2^{3x+9},\ 4x=3x+9,\ x=9$

m $3^{3x}=\left(\dfrac{1}{9}\right)^{4-x}=\left(3^{-2}\right)^{4-x}=3^{2x-8},\ 3x=2x-8,\ x=-8$

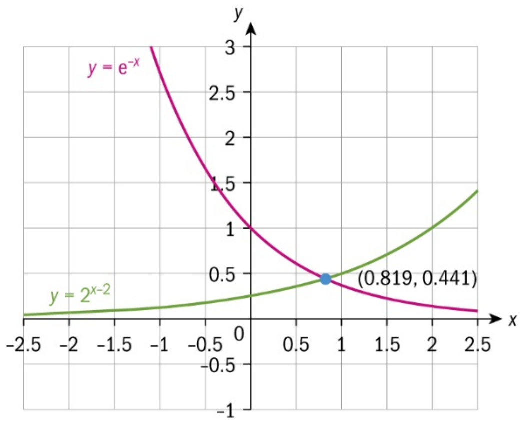
n $2^{\frac{1}{2}x}=2^3,\ \dfrac{1}{2}x=3,\ x=6$

o $\left(2^{x+1}\right)^3=\dfrac{1}{8},\ 2^{3x+3}=2^{-\frac{1}{3}},\ 3x+3=-\dfrac{1}{3},\ 3x=-\dfrac{10}{3},\ x=-\dfrac{10}{9}$

3 **a** **i** y-intercept when $x=0$. So $y=2^{0-2}=2^{-2}=0.25$ **ii** $y=0$

b $x\in\mathbb{R},\ y>0$

c



$x=0.819$

4 $\$1000\times1.075^{10}=\$2061$

9.2 Logarithms

1 **a** Write down the asymptote of $f(x) = 1 + \ln(x+3)$.

b If $g(x) = e^x - 2$, find the solution to $f(x) = g(x)$, showing a sketch to explain your process.

2 Write each equation in exponential form:

a $\log_5 25 = x$ **b** $\log_4 36 = y$ **c** $\log_p 79 = q$ **d** $\log_s r = t$

Solve for x:

e $\log_2 x = 3$ **f** $\log_x 16 = 4$ **g** $\log_2 64 = x$ **h** $\log_x 0.25 = -2$

3 Write each equation in logarithmic form.

a $3^x = 21$ **b** $x^9 = 33$ **c** $p^6 = q$ **d** $l^m = n$

4 Evaluate

a $\log_3 3$ **b** $\log_3 1$ **c** $\log_3 243$ **d** $\log_3 \frac{1}{81}$

e $\log \sqrt[3]{100}$ **f** $\log_5 \frac{1}{125}$ **g** $\ln \sqrt[5]{e^3}$ **h** $e^{2\ln x}$

5 Write as a single log:

a $\log x + \log y + \log z$ **b** $2\log x + \log y$ **c** $\log x - \log y - \log z$

d $2\log x - 2\log y$ **e** $3\log x + 2\log y - \log z$ **f** $\frac{1}{2}\log x + \frac{2}{3}\log y - 4\log z$

6 Given that $\log 2 = x$, $\log 3 = y$ and $\log 7 = z$, write the following expressions in terms of x, y and z.

a $\log 12$ **b** $\log 200$ **c** $\log \frac{14}{3}$

d $\log 0.3$ **e** $\log 1.5$ **f** $\log 15$

7 Simplify:

a $\ln e^5$ **b** $\ln \frac{1}{e^x}$ **c** $e^{\ln p}$

8 Evaluate $\log_2 20$ using the change of base method.

9 Solve:

a $\log_2(10x+2) = 5$ **b** $3^x - 6 = 8$ **c** $3^x = 5^{x+1}$

d $5^{2x} - 5^x - 12 = 0$ **e** $e^{2x} - 3e^x - 10 = 0$

10 Toya invests \$10000 into a bank account with an interest rate of 6% per annum, compounded monthly, to save up for a car costing \$15000.

a Write down an expression for the value of the account after t months.

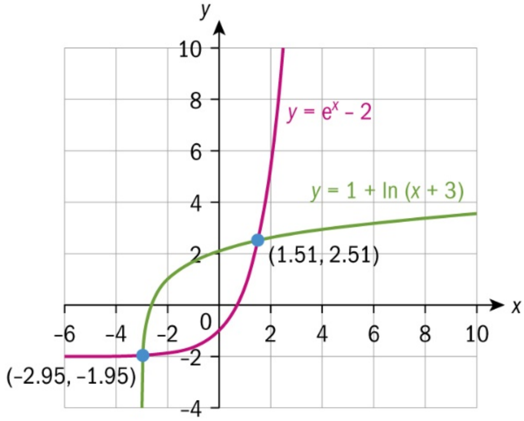
b Find the value of Toya's investment after 2 years to the nearest dollar.

c How long will it take Toya to have \$18000 for the car?

Answers

1 **a** $x = -3$

b $x = -2.95,\ 1.51$



2 **a** $5^x = 25$ **b** $4^y = 36$ **c** $p^q = 79$ **d** $s^t = r$

e $x = 2^3 = 8$ **f** $x^4 = 16,\ x = 2$ **g** $2^x = 64,\ x = 6$ **h** $x^{-2} = \frac{1}{4},\ x = 2$

3 **a** $\log_3 21 = x$ **b** $\log_x 33 = 9$ **c** $\log_p q = 6$ **d** $\log_l n = m$

4 **a** 1 **b** 0 **c** 5 **d** –4

e $\log \sqrt[3]{100} = \log 10^{\frac{2}{3}} = \frac{2}{3}$

f $\log_5 \frac{1}{125} = \log_5 5^{-3} = -3$

g $\ln \sqrt[5]{e^3} = \ln e^{\frac{3}{5}} = \frac{3}{5}$

h $e^{2\ln x} = e^{\ln x^2} = x^2$

5 **a** $\log xyz$

b $2\log x + \log y = \log x^2 + \log y = \log x^2 y$

c $\log x - \log y - \log z = \log x - (\log y + \log z) = \log \frac{x}{yz}$

d $2\log x - 2\log y = \log x^2 - \log y^2 = \log \frac{x^2}{y^2} = \log\left(\frac{x}{y}\right)^2$

e $3\log x + 2\log y - \log z = \log x^3 + \log y^2 - \log z\ \log \frac{x^3 y^2}{z}$

f $\frac{1}{2}\log x + \frac{2}{3}\log y - 4\log z = \log\sqrt{x} + \log\sqrt[3]{y^2} - \log z^4 = \log \frac{\sqrt{x}\sqrt[3]{y^2}}{z^4}$

6 **a** $\log 12 = \log 2^2 + \log 3 = 2x + y$

b $\log 200 = \log 2 \times \log 100 = x + 2$

c $\log \frac{14}{3} = \log \frac{2 \times 7}{3} = \log 2 + \log 7 - \log 3 = x + z - y$

d $\log 0.3 = \log \frac{3}{10} = \log 3 - \log 10 = y - 1$

e $\log 1.5 = \log \frac{3}{2} = \log 3 - \log 2 = y - x$

f $\log 15 = \log \frac{10 \times 3}{2} = \log 10 + \log 3 - \log 2 = 1 + y - x$

7 a $\ln e^5 = 5\ln e = 5$ **b** $\ln\frac{1}{e^x} = \ln e^{-x} = -x\ln e = -x$ **c** p

8 $\log_2 20 = \frac{\log 20}{\log 2} = 4.32$

9 a $10x + 2 = 2^5$

$10x + 2 = 32$

$10x = 30$

$x = 3$

b $3^x - 6 = 8$

$3^x = 14$

$\log 3^x = \log 14$

$x\log 3 = \log 14$

$x = \frac{\log 14}{\log 3} = 2.40$

c $3^x = 5^{x+1}$

$\log 3^x = \log 5^{x+1}$

$x\log 3 = (x+1)\log 5$

$x\log 3 = x\log 5 + \log 5$

$x\log 3 - x\log 5 = \log 5$

$x(\log 3 - \log 5) = \log 5$

$x = \frac{\log 5}{\log 3 - \log 5} = -3.15$

d $5^{2x} - 5^x - 10 = 0$

$(5^x)^2 \ -5^x - = 0$

$(5^x + 3)((5^x - 4) = 0$

Either

$5^x = -3$ or $5^x = 4$

5^x cannot equal -3,

so $5^x = 4$

$\log 5^x = \log 4$

$x\log 5 = \log 4$

$x = \frac{\log 4}{\log 5} = 0.861$

e $e^{2x} - 3e^x - 10 = 0$

$(e^x)^2 - 3e^x - 10 = 0$

$(e^x + 2)((e^x - 5) = 0$

Either $e^x = -2$ or $e^x = 5$

e^x cannot equal -2, so
$e^x = 5$

$\ln e^x = \ln 5$

$x\ln e = \ln 5$

$x = \ln 5 = 1.61$

10 a $FV = PV(1+r)^t = 10000\left(1 + \frac{0.06}{12}\right)^t$

b $10000\left(1 + \frac{0.06}{12}\right)^{24} = \$11272$

c $10000\left(1 + \frac{0.06}{12}\right)^t = 15000$

$10000(1.005)^t = 15000$

$(1.005)^t = 1.5$

$\log(1.005)^t = \log 1.5$

$t\log 1.005 = \log 1.5$

$t = \frac{\log 1.5}{\log 1.005} = 81.2955....$ months

Toya will be able to afford the car after 82 months.

9.3 Derivatives of exponential functions and the natural logarithmic function

1 Differentiate

a e^{11x} **b** e^{x^2+2x} **c** $\ln(3x+1)$ **d** $\ln(x^2+3x-5)$

e x^2e^x **f** $2x\ln 2x$ **g** $\frac{e^x}{e^x+1}$ **h** $\frac{e^{2x}}{\ln x}$

2 Find the turning point of the curve $f(x)=(\ln x)^2$.

3 The radioactive decay of carbon-14 is modelled by $A = Me^{-kt}$, where M is the original mass and A is the amount of carbon-14 remaining after t years.

A half-life is how long it takes for an element to have its mass reduced to 50%.

If the half-life of carbon-14 is 5700 years, find the value of k.

4 The number of snails (s) in a rice field can be modelled by $s(t) = s_0e^{kt}$, where t is the time (in days) and s_0 is the initial number of snails.

a If there are 200 snails when t = 0, and 30 days later there are 900 snails, find k.

b Find the rate of change of snails per day at 10 days.

Answers

1 **a** $u = 11x,\ y = e^u$

$$\frac{dy}{du} = e^u = e^{11x},\ \frac{du}{dx} = 11$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 11e^{11x}$$

b $u = x^2 + 2x,\ y = e^u$

$$\frac{dy}{du} = e^u = e^{x^2+2x},\ \frac{du}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2x+2)e^{x^2+2x}$$

c $u = 3x + 1,\ y = \ln u$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{3x+1},\ \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{3}{3x+1}$$

d $u = x^2 + 3x - 5,\ y = \ln u$

$$\frac{dy}{du} = 2x + 3,\ \frac{du}{dx} = \frac{1}{u} = \frac{1}{x^2+3x-5}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2x+3}{x^2+3x-5}$$

e $u = x^2,\ v = e^x$

$$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \left(e^x \times 2x\right) + \left(x^2 \times e^x\right) = xe^x(x+2)$$

f $u = 2x,\ v = \ln x$

$$\frac{du}{dx} = 2,\ \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (2 \times \ln 2x) + \left(2x \times \frac{1}{x}\right) = 2\ln 2x + 2 = 2(\ln 2x + 1)$$

g $u = e^x,\ v = e^x + 1$

$$\frac{du}{dx} = e^x,\ \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{e^x\left(e^x+1\right) - \left(e^x \times e^x\right)}{\left(e^x+1\right)^2} = \frac{e^x}{\left(e^x+1\right)^2}$$

h $u = e^{2x},\ v = \ln x$

$\frac{du}{dx} = 2e^{2x},\ \frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\ln x\left(2e^{2x}\right) - (e^{2x} \times \frac{1}{x})}{(\ln x)^2} = \frac{e^{2x}\left(2\ln x - \frac{1}{x}\right)}{(\ln x)^2}$$

2 $u = \ln x,\ y = u^2$

$\frac{dy}{du} = 2u = 2\ln x,\ \frac{du}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2\ln x}{x}$

At a turning point, $\frac{dy}{dx} = 0$

$\frac{2\ln x}{x} = 0$ when $\ln x = 0,\ x = 1$

$y = \frac{2\ln 1}{1} = 0$

The turning point is at $(1, 0)$

3 a $\frac{M}{2} = Me^{-5700k}$

Dividing both sides by M, $\ln\frac{1}{2} = \ln e^{-5700k}$

$\ln\frac{1}{2} = -5700k \ln e$

$\ln\frac{1}{2} = -5700k$

$k = \frac{\ln\frac{1}{2}}{-5700} = 0.000122$

4 a $900 = 200e^{30k}$

$4.5 = e^{30k}$

$\ln 4.5 = \ln e^{30k}$

$30k = \ln 4.5$

$k = \frac{\ln 4.5}{30} = 0.0501$

b $u = 0.0501t,\ y = 200e^{u}$

$\frac{dy}{du} = 200e^{u} = 200e^{0.0501t},\ \frac{du}{dx} = 0.0501$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 0.0501\left(200e^{0.0501t}\right) = 10.0e^{0.0501t}$

$10.0e^{0.0501(10)} = 16.5$ snails per day

10.1 Antiderivatives and the indefinite integral

1 Find the antiderivative of each function

a $f(x) = x^7$

b $f(x) = x^{-3}$

c $f(x) = x^{-\frac{2}{3}}$

d $f(x) = \dfrac{1}{x^4}$

e $f(x) = \sqrt[5]{x^4}$

2 Find the indefinite integrals

a $\int x^5 dx$

b $\int 3dx$

c $\int \dfrac{2}{x^3} dx$

d $\int \sqrt[7]{m^4} dm$

3 Given that $\int (ax + b) dx = x^2 + 4x + C$, find the values of a and b.

4 Find the indefinite integrals

a $\int \dfrac{1}{3x} dx$

b $\int \dfrac{e^x}{4} dx$

c $\int s^2 (s - 1) ds$

d $\int e^{\ln(2x)} dx$

Answers

1 **a** $\frac{1}{8}x^8 + C$

b $-\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$

c $3x^{\frac{1}{3}} + C$

d $-\frac{1}{3}x^{-3} + C = -\frac{1}{3x^3} + C$

e $\frac{5}{9}x^{\frac{9}{5}} + C$

2 **a** $\frac{1}{6}x^6 + C$

b $3x + C$

c $-\frac{1}{x^2} + C$

d $\frac{7}{11}m^{\frac{11}{7}} + C$

3 $\frac{1}{2}ax^2 + bx = x^2 + 4x \Rightarrow a = 2, b = 4$

4 **a** $\frac{1}{3}\ln|x| + C$ or $\frac{1}{3}\ln|3x| + C$

b $\frac{1}{4}e^x + C$

c $\frac{1}{4}s^4 - \frac{1}{3}s^3 + C$

d $\int 2x dx = x^2 + C$

10.2 More on indefinite integrals

1 Find the indefinite integrals

a $\int (2x-3)^5 dx$

b $\int 2e^{3-5x} dx$

c $\int \frac{1}{(3x-5)^2} dx$

d $\int \frac{1}{\sqrt{2x+7}} dx$

2 Given that $f(x) = \frac{1}{2x+1}$, find

a $f'(x)$

b $\int f(x) dx$

3 Let $g'(t) = 3t^2 - 3$. Given $g(1) = 1$, find $g(t)$.

4 A particle moving in a straight line has velocity v m s^{-1} and displacement s m at time t. The particle's velocity is given by $v(t) = 4t - t^2$

a Find the acceleration when $t = 2$.

When $t = 0$, the displacement s of the particle is 0.

b Find an expression for the displacement of the particle in terms of t.

5 The derivative of the function g is given by $g'(x) = \frac{1}{3x+1}$. Given that $g(0) = 2$, find $g(x)$.

Answers

1 **a** $\frac{1}{12}(2x-3)^6+C$

b $-\frac{2}{5}e^{3-5x}+C$

c $-\frac{1}{3(3x-5)}+C$

d $\sqrt{2x+7}+C$

2 **a** $-\frac{1}{(2x+1)^2}$

b $\frac{1}{2}\ln|2x+1|+C$

3 $g(t)=t^3-3t+C$

$1-3+C=1 \Rightarrow C=3$
$g(t)=t^3-3t+3$

4 **a** $a(2)=v'(2)=4-2(2)=0\ \text{ms}^{-2}$

b $s(t)=2t^2-\frac{1}{3}t^3+C$

$s(0)=0 \Rightarrow C=0$
$s(t)=2t^2-\frac{1}{3}t^3$

5 $g(x)=\frac{1}{3}\ln|3x+1|+C$

Since $g(0)=2,\ C=2$
$g(x)=\frac{1}{3}\ln|3x+1|+2$

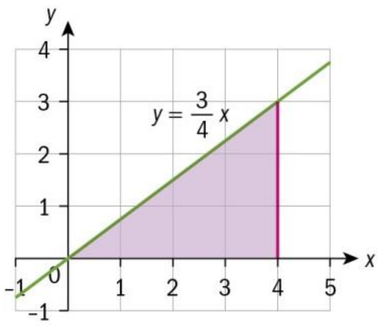
10.3 Area and definite integrals

1 For the following diagrams

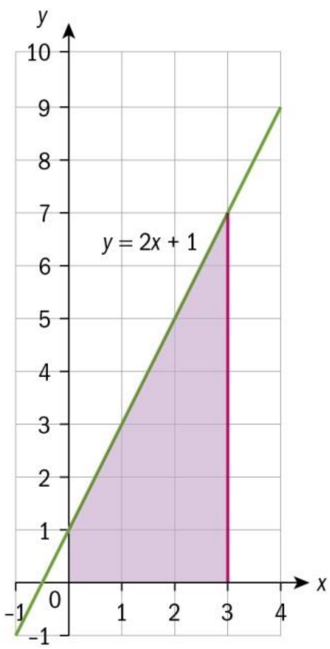
i write down a definite integral that gives the area of the shaded region and evaluate the integral on your GDC.

ii find the area of the shaded region using geometric formulas to verify your answer for part **i**.

a



b



2 Given that $\int_0^3 f(x)dx = 7$ and $\int_0^8 f(x)dx = 4$, find

a $\int_3^8 f(x)dx$ **b** $\int_2^5 f(x-2)dx$

Answers

1 **a** **i** $\int_0^4 \frac{3}{4}x\,dx = 6$ **ii** $\frac{1}{2} \times 4 \times 3 = 6$

b **i** $\int_0^3 (2x+1)\,dx = 12$ **ii** $\frac{1}{2} \times (1+7) \times 3 = 12$

2 **a** $4 - 7 = -3$

b 7

10.4 Fundamental theorem of calculus

1 Evaluate the following definite integrals without using a GDC.

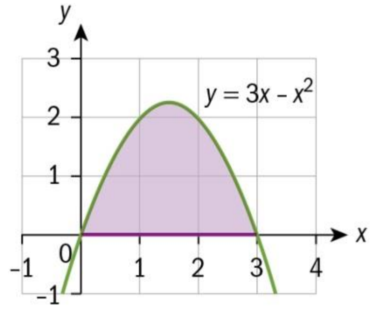
a $\int_{-1}^{4} 5x dx$

b $\int_{2}^{4} \frac{1}{x^2} dx$

c $\int_{-2}^{3} \left(x^2 + x + 1\right) dx$

d $\int_{e}^{e^2} \frac{1}{x} dx$

2 Write down a definite integral that represents the area of the shaded region and find the area, without using a GDC.



3 Evaluate the following definite integrals without using a GDC.

a $\int_{1}^{3} \frac{1}{2x-1} dx$

b $\int_{0}^{3} (x+1)(x-2) dx$

c $\int_{1}^{4} \frac{2\sqrt{s}+1}{\sqrt{s}} ds$

Answers

1 **a** $\int_{-1}^{4} 5x dx = \left[\frac{5}{2}x^2\right]_{-1}^{4}$

$= 40 - \frac{5}{2} = 37\frac{1}{2}$

b $\int_{2}^{4} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{2}^{4}$

$= -\frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{1}{4}$

c $\int_{-2}^{3} \left(x^2 + x + 1\right) dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_{-2}^{3}$

$= \left(9 + \frac{9}{2} + 3\right) - \left(-\frac{8}{3} + 2 - 2\right)$

$= \frac{33}{2} + \frac{8}{3} = \frac{115}{6}$

d $\int_{e}^{e^2} \frac{1}{x} dx = \left[\ln x\right]_{e}^{e^2}$

$= 2 - 1 = 1$

2 $\int_{0}^{3} (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{3} = \frac{27}{2} - 9 = \frac{9}{2}$

3 **a** $\int_{1}^{3} \frac{1}{2x-1} dx = \left[\frac{1}{2}\ln\left(2x-1\right)\right]_{1}^{3} = \frac{\ln 5}{2}$

b $\int_{0}^{3} \left(x+1\right)\left(x-2\right) dx = \int_{0}^{3} (x^2 - x - 2) dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{0}^{3}$

$= 9 - \frac{9}{2} - 6 = -\frac{3}{2}$

c $\int_{1}^{4} \frac{2\sqrt{s}+1}{\sqrt{s}} dx = \left[2s + 2s^{\frac{1}{2}}\right]_{1}^{4}$

$= \left(8 + 4\right) - \left(2 + 2\right)$

$= 8$

10.5 Area between two curves

1 In each question, sketch the two graphs and find the area enclosed by the two curves. Do not a GDC.

a $y = x^2,\ y = 2x$

b $y = x^2 - 4x + 3,\ y = 3$

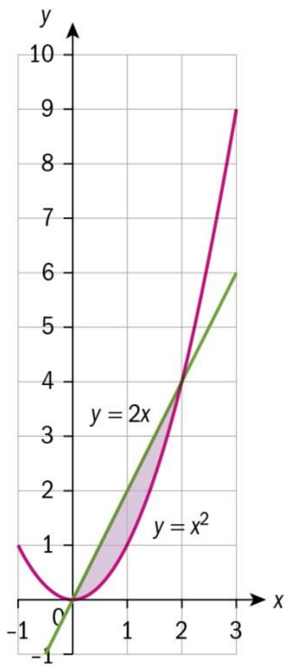
2 In each question, write an integral expression to find the area of the region bounded by the two curves. Use your GDC to examine the graphs and evaluate any definite integrals.

a $f(x) = x^2 - x,\ y = x + x^2 - 2x^3$

b $y = x^4 - 6x^2 + 8,\ y = -x^4 + 4x^2$

Answers

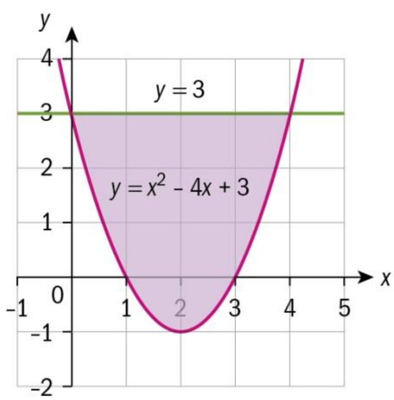
1 a



Area $= \int_0^2 (2x - x^2)\,dx = [x^2 - \frac{1}{3}x^3]_0^2$

$= 4 - \frac{8}{3} = \frac{4}{3}$

b



Area $= \int_0^4 \left(3 - \left(x^2 - 4x + 3\right)\right)dx$

$= \int_0^4 \left(4x - x^2\right)dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= 32 - \frac{64}{3} = \frac{32}{3}$

2 **a** From GDC, area =

$$\int_{-1}^{0}\left(\left(x^2-x\right)-\left(x+x^2-2x^3\right)\right)dx+\int_{0}^{1}\left(\left(x+x^2-2x^3\right)-\left(x^2-x\right)\right)dx=1$$

b From GDC, area =

$$\int_{-2}^{-1}\left(\left(-x^4+4x^2\right)-\left(x^4-6x^2+8\right)\right)dx+\int_{-1}^{1}\left(\left(x^4-6x^2+8\right)-\left(-x^4+4x^2\right)\right)dx+\int_{1}^{2}\left(\left(-x^4+4x^2\right)-\left(x^4-6x^2+8\right)\right)dx=16$$

11.1 The geometry of 3D shapes

1 Find the mid-point of

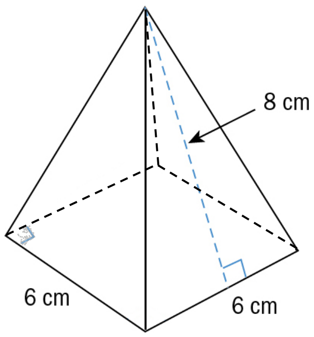
a $(-2, 1, 4)$, $(7, 3, -1)$ **b** $(3, 0, 6)$, $(-3, 5, 5)$

2 Find the distance between these points

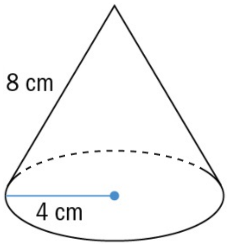
a $(1, 2, 4)$ and $(3, 6, 9)$ **b** $(2, 1, -7)$ and $(-4, 2, 6)$

3 Find the volume and surface area of these shapes:

a

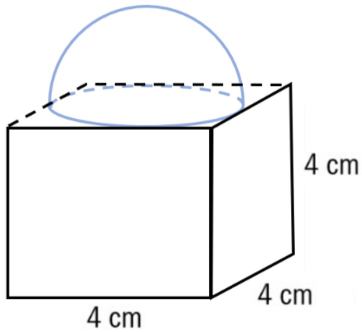


b



4 Find the volume and surface area of a sphere of radius 7 m.

5 A solid consists of a cube of side 4 m with a hemisphere on top.



Calculate the volume and surface area of the solid.

Answers

1 **a** $\left(2\frac{1}{2}, 2, 1\frac{1}{2}\right)$ **b** $\left(0, 2\frac{1}{2}, 5\frac{1}{2}\right)$

2 **a** $\sqrt{45} \approx 6.71$ **b** $\sqrt{206} \approx 14.4$

3 **a** Volume $= \frac{1}{3} \times 36 \times \sqrt{64-9} \approx 87.4\,\text{cm}^3$

Surface area $= 36+4 \times 24=132\,\text{cm}^2$

b Volume $= \frac{1}{3} \times \pi \times 16 \times \sqrt{64-16} \approx 116\,\text{cm}^3$

Surface area $= 16\pi + \pi \times 4 \times 8 \approx 151\,\text{cm}^2$

4 Volume $= \frac{4}{3}\pi \times 7^3 \approx 1440\,\text{m}^3$

Surface area $= 4\pi \times 7^2 \approx 616\,\text{m}^2$

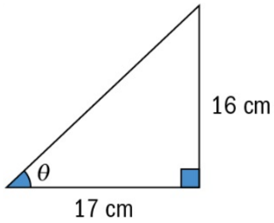
5 Volume $= 4^3 + \frac{1}{2} \times \frac{4}{3}\pi \times 2^3 \approx 80.8\,\text{m}^3$

Surface area $= 5 \times 16 + 2 \times \pi \times 2^2 \approx 105\,\text{m}^2$

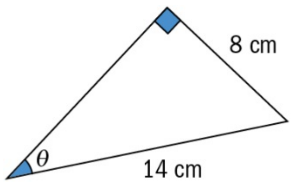
11.2 Right-angled triangle trigonometry

1 Find the angle θ in each triangle.

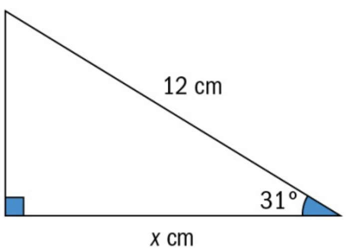
a



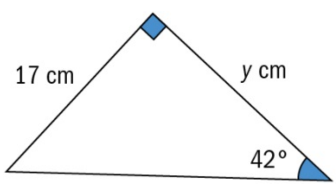
b



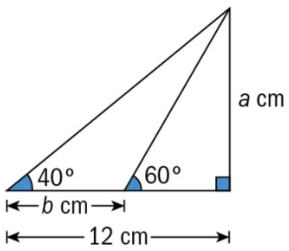
2 **a** Find the side x.



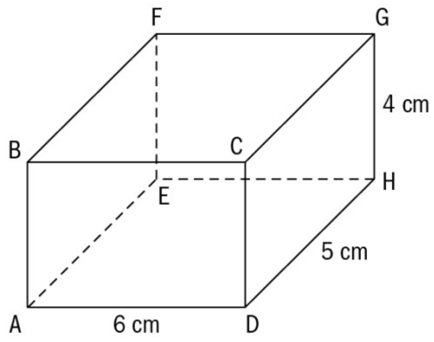
b Find the side y.



3 Find a and b.



4 The diagram shows a cuboid with sides 4 cm, 5 cm and 6 cm.



a Calculate the length AC.

b Calculate the length AG

c Find angle GAH.

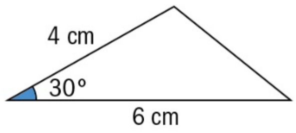
5 A tree casts a 7 m shadow when the angle of elevation of the Sun is 32°. How tall is the tree?

6 A woman in a tall building looking out of a window 50 m above ground level sees an object on the ground. The angle of depression when she is looking at the object is 20°. How far away is the object from the foot of the building?

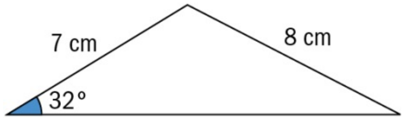
7 A boat is 50 km east of a port. It then sails for 40 km on a bearing of 072°. What is the bearing and distance of the ship from the port?

8 Find the area of these triangles.

a



b



9 Without using a calculator, find an exact value for the area of a regular hexagon of side 2 cm.

Answers

1 **a** $\arctan\left(\frac{16}{17}\right) \approx 43.3°$

b $\arcsin\left(\frac{8}{14}\right) \approx 34.8°$

2 **a** $12\sin 31° \approx 6.18$ cm

b $\frac{17}{\tan 42°} \approx 18.9$ cm

3 $a = 12\sin 40° = 7.713\ldots \approx 7.71$ cm

$b = 12 - \frac{7.713}{\tan 60°} \approx 7.55$ cm

4 **a** $\text{AC} = \sqrt{6^2 + 4^2} \approx 7.21$ cm

b $\text{AG} = \sqrt{6^2 + 5^2 + 4^2} \approx 8.77$ cm

c $\text{AH} = \sqrt{61}$

Angle $\text{GAH} = \arcsin\frac{4}{\sqrt{77}} \approx 27.1°$

5 $7\tan 32° \approx 4.37$ m

6 $\frac{50}{\tan 20°} \approx 137$ m

7 Distance east = $50 + 50\sin 72° = 97.55\ldots$ km

Distance north $= 50\cos 72° = 15.45\ldots$ km

so distance $= \sqrt{97.55\ldots^2 + 15.45\ldots^2} \approx 98.8$ km

Bearing = $\arctan\left(\frac{97.55\ldots}{15.45\ldots}\right) \approx 081°$

8 **a** $\frac{1}{2} \times 4 \times 6 \times \sin 30° = 6\text{ cm}^2$

b Call the angle opposite 7 cm, x

$\frac{\sin x}{7} = \frac{\sin 32°}{8} \Rightarrow x = 27.6\ldots°$

so 3rd angle $= 120.3\ldots°$

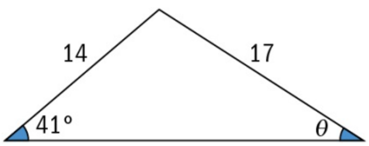
area = $\frac{1}{2} \times 7 \times 8 \times \sin 120.3\ldots° = 24.15 \approx 24.2\text{ cm}^2$

9 $6 \times \frac{1}{2} \times 2 \times 2 \times \sin 60° = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}\text{ cm}^2$

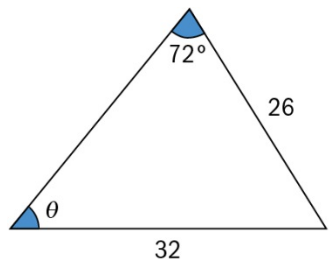
11.3 The sine rule

1 Find the angle θ in these triangles.

a

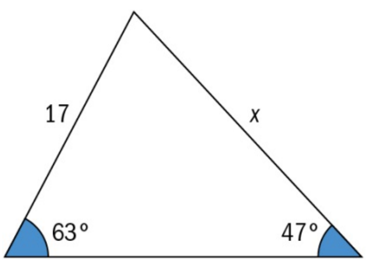


b

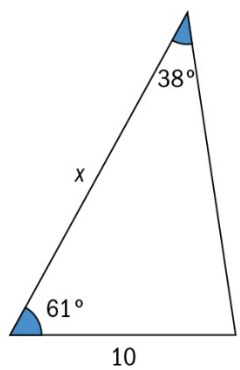


2 Find the length x in these triangles.

a

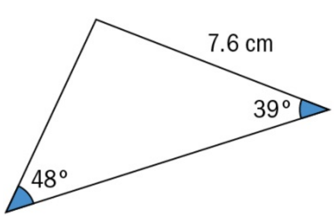


b



3 Lighthouses A and B are located 100 km apart. Lighthouse B is due north of lighthouse A. Someone at lighthouse A spots a boat on a bearing of 070° from the lighthouse, while someone else at lighthouse B spots the same boat on a bearing 123°. How far from lighthouse A is the boat?

4 Find the area of this triangle.



5 Find the two possible values for $\angle B$ in a triangle ABC when $\angle C = 32°$, $c = 4.7$ and $a = 6.1$.

Answers

1 **a** $\dfrac{\sin\theta}{14}=\dfrac{\sin 41^\circ}{17} \Rightarrow \theta \approx 32.7^\circ$

b $\dfrac{\sin\theta}{26}=\dfrac{\sin 72^\circ}{32} \Rightarrow \theta \approx 50.6^\circ$

2 **a** $\dfrac{x}{\sin 63^\circ}=\dfrac{17}{\sin 47^\circ} \Rightarrow x \approx 20.7$

b $\dfrac{x}{\sin 81^\circ}=\dfrac{10}{\sin 38^\circ} \Rightarrow x \approx 16.0$

3 $\dfrac{x}{\sin 57^\circ}=\dfrac{100}{\sin 53^\circ} \Rightarrow x \approx 105\text{ km}$

4 Other angle is 93°

Side opposite that angle is given by $\dfrac{x}{\sin 93^\circ}=\dfrac{7.6}{\sin 48^\circ}$

$\Rightarrow x = 10.21....$

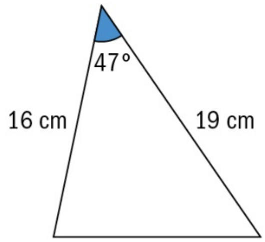
So area = $\dfrac{1}{2}\times 7.6\times 10.21...\times \sin 39^\circ \approx 24.4\text{ cm}^2$

5 $\dfrac{\sin A}{6.1}=\dfrac{\sin 32^\circ}{4.7} \Rightarrow A = 43.4...^\circ$ or $180^\circ - 43.4...^\circ \Rightarrow B \approx 105^\circ$ or 11.5°

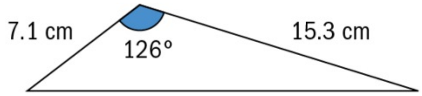
11.4 The cosine rule

1 Find the missing sides in these triangles.

a

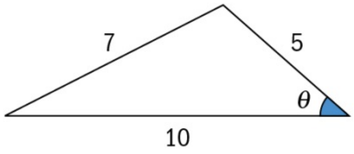


b

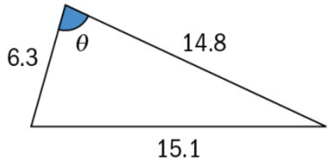


2 Find the angle θ in these triangles.

a



b



3 The lengths of the sides of a triangle are 7 cm, 10 cm and 12 cm.

a Find the largest angle in the triangle.

b Find the area of the triangle.

4 A boat sails 70 km on a bearing of 080°. It then changes course and sails 80 km on a bearing of 150°. How far is it from its starting point?

Answers

1 **a** $\sqrt{19^2 + 16^2 - 2 \times 19 \times 16 \times \cos 47°} \approx 14.2$ cm

b $\sqrt{7.1^2 + 15.3^2 - 2 \times 7.1 \times 15.3 \times \cos 126°} \approx 20.3$ cm

2 **a** $\arccos\left(\dfrac{5^2 + 10^2 - 7^2}{2 \times 5 \times 10}\right) \approx 40.5°$

b $\arccos\left(\dfrac{6.3^2 + 14.8^2 - 15.1^2}{2 \times 6.3 \times 14.8}\right) \approx 80.5°$

3 **a** $\arccos\left(\dfrac{7^2 + 10^2 - 12^2}{2 \times 7 \times 10}\right) = 87.95...° \approx 88.0°$

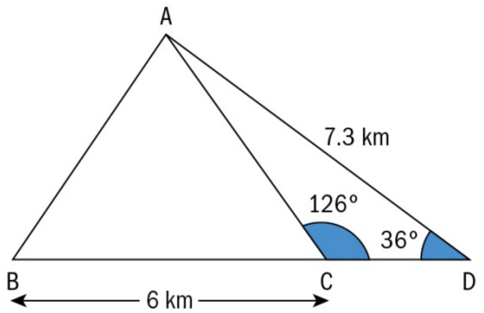
b $\dfrac{1}{2} \times 7 \times 10 \times \sin 87.95...° \approx 35.0$ cm^2

4 $\sqrt{70^2 + 80^2 - 2 \times 70 \times 80 \times \cos 110°} \approx 123$ km

11.5 Applications of right- and non-right-angled trigonometry

1 Find the area of a triangle with sides 7 cm, 8 cm and 9 cm.

2



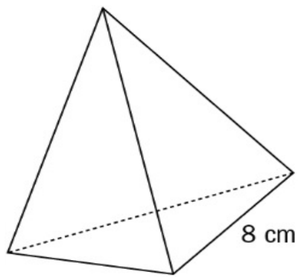
a Find the length AC.

b Find the length AB.

c Find the area of triangle ABD.

3 B is 60 km from A on a bearing of 086°. C is 80 km from B on a bearing of 147°. Find the distance and bearing of C from A.

4 This regular tetrahedron has edges of length 8 cm.



Calculate the angle between two faces.

5 In triangle ABC, AB = 7 cm, AC = 8 cm and BC = 10 cm.

a Without using a calculator, find the exact value of the cosine of angle BAC.

b Without any further calculation, explain how you know that this angle is acute.

Answers

1 Angle between the sides of length 7 cm and 8 cm is given by

$\arccos\left(\dfrac{7^2+8^2-9^2}{2\times7\times8}\right)=73.3\ldots°$

So area = $\dfrac{1}{2}\times7\times8\times\sin73.3\ldots°\approx26.8\text{ cm}^2$

2 **a** $\dfrac{AC}{\sin36°}=\dfrac{7.3}{\sin126°}\Rightarrow AC=5.303\ldots\text{ km}\approx5.30\text{ km}$

b $AB=\sqrt{5.303\ldots^2+6.1^2-2\times5.303\ldots\times6.1\times\cos54°}=5.225\ldots\text{ km}\approx5.23\text{ km}$

c $\dfrac{CD}{\sin18°}=\dfrac{7.3}{\sin126°}\Rightarrow CD=2.788\ldots\text{ km}$

So area of ABD $=\dfrac{1}{2}\times7.3\times(6.1+2.788\ldots)\times\sin36°$

$\approx19.1\text{ km}^2$

3 Distance = $\sqrt{60^2+80^2-2\times60\times80\cos119°}$

$=121.0\ldots\text{ km}\approx121\text{ km}$

$\angle BAC$ is given by $\dfrac{\sin\angle BAC}{80}=\dfrac{\sin119°}{121.0\ldots}$

$\Rightarrow\angle BAC=35.3\ldots°$

So bearing $\approx 86+35=121°$

4 The perpendicular height of each triangular face $=8\sin60°$. So the angle between each face is given by $\arccos\left(\dfrac{(8\sin60°)^2+(8\sin60°)^2-8^2}{2\times8\sin60°\times8\sin60°}\right)\approx70.5°$.

5 **a** $\cos\angle BAC=\dfrac{7^2+8^2-10^2}{2\times7\times8}=\dfrac{13}{112}$

b Because the value of the cosine is positive, the angle must be acute.

12.1 Radian measure, arcs, sectors and segments

1 Convert each angle measure from degrees to radians. Give exact answers in terms of π.

a $30°$ **b** $120°$ **c** $330°$ **d** $300°$ **e** $90°$ **f** $150°$

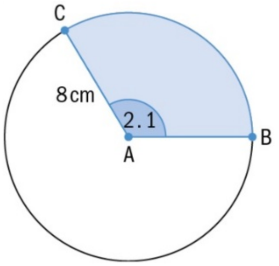
2 Convert each angle measure from radians to degrees.

a $\frac{\pi}{3}$ **b** $\frac{3\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** $\frac{5\pi}{4}$ **e** $\frac{7\pi}{8}$ **f** 4π

3 Convert each angle measure from degrees to radians. Give answers to 3 s.f.

a $28°$ **b** $137°$ **c** $250°$ **d** $100°$ **e** $275°$ **f** $146°$

4 The following diagram shows a circle with centre A and radius 8 cm.



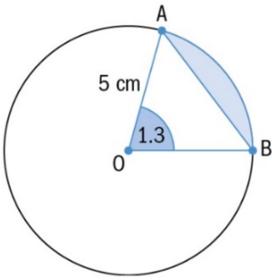
Points B and C lie on the circumference of the circle, and $\angle BAC = 2.1$ radians.

a Find the length of arc BC.

b Find the perimeter of the shaded sector.

c Find the area of the shaded sector.

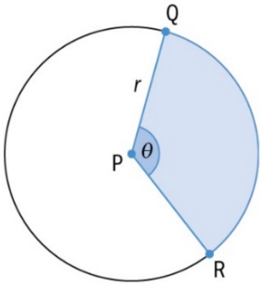
5 The following diagram shows a circle with centre O and radius 5 cm.



Points A and B lie on the circumference of the circle, and $\angle AOB = 1.3$ radians.

Find the area of the shaded segment.

6 The following diagram shows a circle with centre P and radius r cm. Points Q and R lie on the circumference of the circle, and $\angle QPR = \theta$. Given that the shaded sector has a perimeter of 42 cm and an area of 110 cm^2, find θ and r.



Answers

1 **a** $30°\left(\frac{\pi}{180°}\right)=\frac{30\pi}{180}=\frac{\pi}{6}$

b $120°\left(\frac{\pi}{180°}\right)=\frac{120\pi}{180}=\frac{2\pi}{3}$

c $330°\left(\frac{\pi}{180°}\right)=\frac{330\pi}{180}=\frac{11\pi}{6}$

d $300°\left(\frac{\pi}{180°}\right)=\frac{300\pi}{180}=\frac{5\pi}{3}$

e $90°\left(\frac{\pi}{180°}\right)=\frac{90\pi}{180}=\frac{\pi}{2}$

f $150°\left(\frac{\pi}{180°}\right)=\frac{150\pi}{180}=\frac{5\pi}{6}$

2 **a** $\frac{\pi}{3}\left(\frac{180°}{\pi}\right)=\frac{180°}{3}=60°$

b $\frac{3\pi}{4}\left(\frac{180°}{\pi}\right)=\frac{540°}{4}=135°$

c $\frac{5\pi}{6}\left(\frac{180°}{\pi}\right)=\frac{900°}{6}=150°$

d $\frac{5\pi}{4}\left(\frac{180°}{\pi}\right)=\frac{900°}{4}=225°$

e $\frac{7\pi}{8}\left(\frac{180°}{\pi}\right)=\frac{1260°}{8}=157.5°$

f $4\pi\left(\frac{180°}{\pi}\right)=720°$

3 **a** $28°\left(\frac{\pi}{180°}\right)=0.489$

b $137°\left(\frac{\pi}{180°}\right)=2.39$

c $250°\left(\frac{\pi}{180°}\right)=4.36$

d $100°\left(\frac{\pi}{180°}\right)=1.75$

e $275°\left(\frac{\pi}{180°}\right)=4.80$

f $146°\left(\frac{\pi}{180°}\right)=2.55$

4 **a** $2.1(8)=16.8$

b $16.8+8+8=32.8$

c $\frac{1}{2}(2.1)(8^2)=67.2$

5 sector area = $\frac{1}{2}\times 1.3\times(5^2)=16.25$

triangle area = $\frac{1}{2}(5)(5)(\sin 1.3)=12.044\ldots$ (be sure your GDC is in radian mode)

segment area = $16.25-12.044\ldots\approx 4.21\text{ cm}^2$

6 sector area = $\frac{1}{2}\theta(r^2)=110\text{ cm}^2$

perimeter of sector = $\theta r+r+r=42\to\theta=\frac{42-2r}{r}=\frac{42}{r}-2$

substitution gives $\frac{1}{2}\left(\frac{42}{r}-2\right)(r^2)=110\to\left(\frac{21}{r}-1\right)(r^2)=110\to 21r-r^2=110$

$r^2-21r+110=0\to(r-10)(r-11)=0$

$r=10\text{ cm}$ or $r=11\text{ cm}$

If $r=10\text{ cm}$, $\theta=\frac{42}{10}-2=2.2$ rad.

If $r=11\text{ cm}$, $\theta=\frac{42}{11}-2=\frac{20}{11}$ rad.

12.2 Trigonometric ratios in the unit circle

1 Name three angles between -2π and 2π with the same cosine as $\frac{5\pi}{3}$.

2 Name three angles between $-360°$ and $360°$ with the same sine as $210°$.

3 Name three angles between -2π and 2π with the same tangent as $\frac{\pi}{4}$.

Answers

1 $-\frac{5\pi}{3}, \pm\frac{\pi}{3}$

2 $-30°, -150°, 330°$

3 $\frac{-7\pi}{4}, \frac{-3\pi}{4}, \frac{5\pi}{4}$

12.3 Trigonometric identities and equations

1 Without using your GDC, solve each equation for $-\pi \le \theta \le 2\pi$.

a $\sin 2\theta = \cos\theta$

b $\sin^2\theta = 3\cos^2\theta$

c $(\sin\theta + \cos\theta)^2 = \frac{1}{2}$

2 Without using your GDC, solve each equation for $-360° \le \theta \le 360°$.

a $\cos 2\theta = \sin\theta$

b $\cos\left(\frac{\theta}{3}\right) = \frac{\sqrt{3}}{2}$

c $2\cos^2\theta - 3\cos\theta + 1 = 0$

3 Given that $\sin\theta = \frac{1}{4}$ and $0 \le \theta \le \frac{\pi}{2}$, find each value. Do not use your GDC.

a $\cos\theta$

b $\tan\theta$

c $\sin(2\theta)$

d $\cos(2\theta)$

e $\sin(\theta + \pi)$

f $\cos(-\theta)$

4 Given that $\cos\theta = -\frac{2}{3}$ and $\frac{\pi}{2} \le \theta \le \pi$, find each value. Do not use your GDC.

a $\sin\theta$

b $\tan\theta$

c $\sin(2\theta)$

d $\cos(2\theta)$

e $\sin(-\theta)$

f $\cos(\theta + \pi)$

Answers

1 **a** $\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ or $2\sin\theta - 1 = 0 \to \sin\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

b $\sin^2\theta = 3\cos^2\theta$

$\frac{\sin^2\theta}{\cos^2\theta} = 3$

$\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$

$\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

c $(\sin\theta + \cos\theta)^2 = \frac{1}{2}$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{2}$

$1 + 2\sin\theta\cos\theta = \frac{1}{2}$

$1 + \sin 2\theta = \frac{1}{2}$

$\sin 2\theta = -\frac{1}{2}$

$2\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\theta = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

2 **a** $\cos 2\theta = \sin\theta$

$1 - 2\sin^2\theta = \sin\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(\sin\theta + 1)(2\sin\theta - 1) = 0$

$\sin\theta + 1 = 0 \to \sin\theta = -1$ or $2\sin\theta - 1 = 0 \to \sin\theta = \frac{1}{2}$

$\theta = -90°, 270°, -210°, -330°, 30°, 150°$

b $\cos\left(\frac{\theta}{3}\right) = \frac{\sqrt{3}}{2}$

$\frac{\theta}{3} = \pm 30°$

$\theta = \pm 90°$

c $2\cos^2\theta - 3\cos\theta + 1 = 0$

$(\cos\theta - 1)(2\cos\theta - 1) = 0$

$\cos\theta - 1 = 0 \rightarrow \cos\theta = 1$ or $2\cos\theta - 1 = 0 \rightarrow \cos\theta = \frac{1}{2}$

$\theta = 0°, \pm 360°, \pm 60°$

3 a $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \rightarrow \frac{1}{16} + \cos^2\theta = 1 \rightarrow \cos^2\theta = 1 - \frac{1}{16} \rightarrow \cos^2\theta = \frac{15}{16}$

$\cos\theta = \frac{\sqrt{15}}{4}$

b $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\tan\theta = \frac{\left(\frac{1}{4}\right)}{\left(\frac{\sqrt{15}}{4}\right)} = \frac{1}{\sqrt{15}}$

c $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{1}{4}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{2\sqrt{15}}{16} = \frac{\sqrt{15}}{8}$

d $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{1}{4}\right)^2 = 1 - \frac{2}{16} = \frac{14}{16} = \frac{7}{8}$

e $\sin(\theta + \pi) = -\sin\theta = -\frac{1}{4}$

f $\cos(-\theta) = \cos\theta = \frac{\sqrt{15}}{4}$

4 a $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(-\frac{2}{3}\right)^2 = 1 \rightarrow \sin^2\theta + \frac{4}{9} = 1 \rightarrow \sin^2\theta = 1 - \frac{4}{9} \rightarrow \sin^2\theta = \frac{5}{9}$

$\sin\theta = \frac{\sqrt{5}}{3}$

b $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\tan\theta = \frac{\left(\frac{\sqrt{5}}{3}\right)}{\left(-\frac{2}{3}\right)} = -\frac{\sqrt{5}}{2}$

c $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$

d $\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{2}{3}\right)^2 - 1 = 2\left(\frac{4}{9}\right) - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$

e $\sin(-\theta) = -\sin\theta = -\frac{\sqrt{5}}{3}$

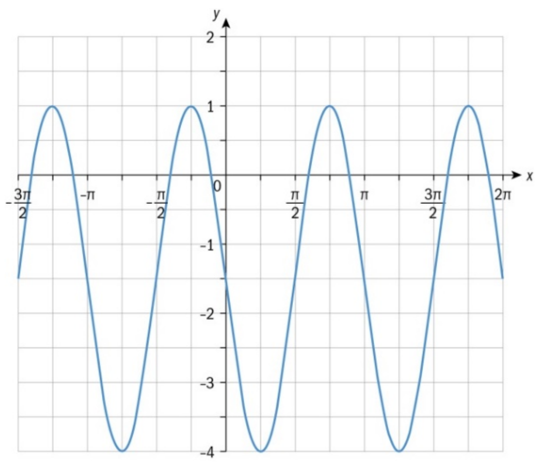
f $\cos(\theta + \pi) = -\cos\theta = \frac{2}{3}$

12.4 Trigonometric functions

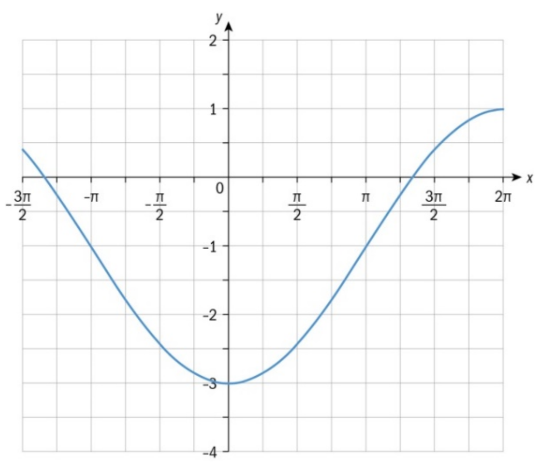
1 For each function shown in the three graphs below, write down the period and amplitude.

Then write a cosine equation for each function.

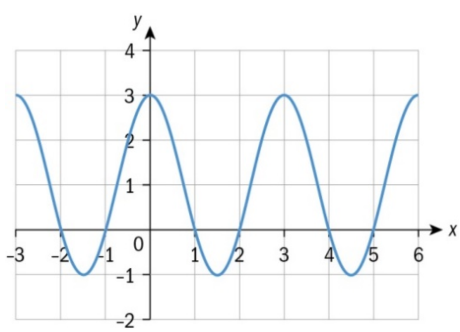
a



b



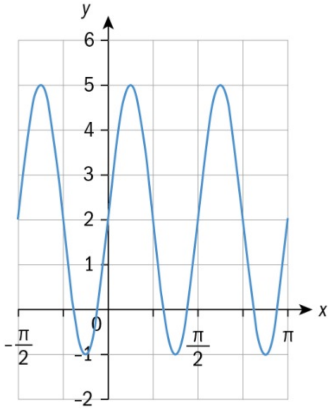
c



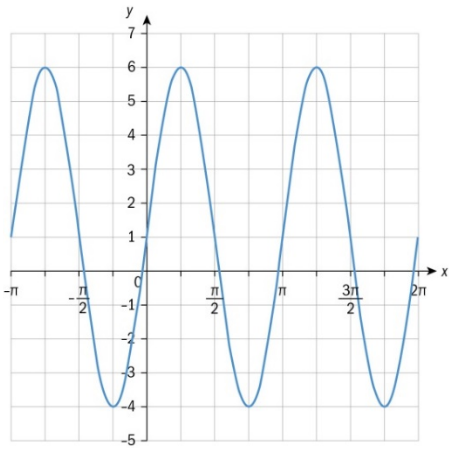
2 For each of the three graphs shown below, write down the period and amplitude.

Then write a sine equation for each function.

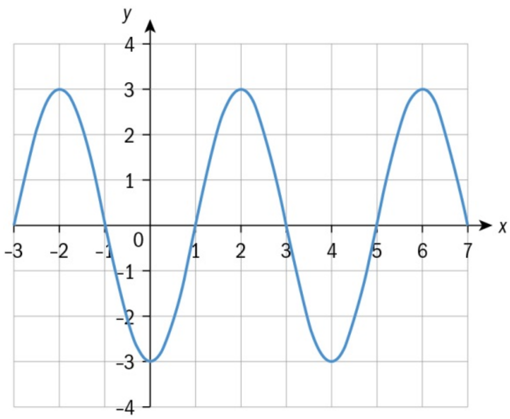
a



b



c

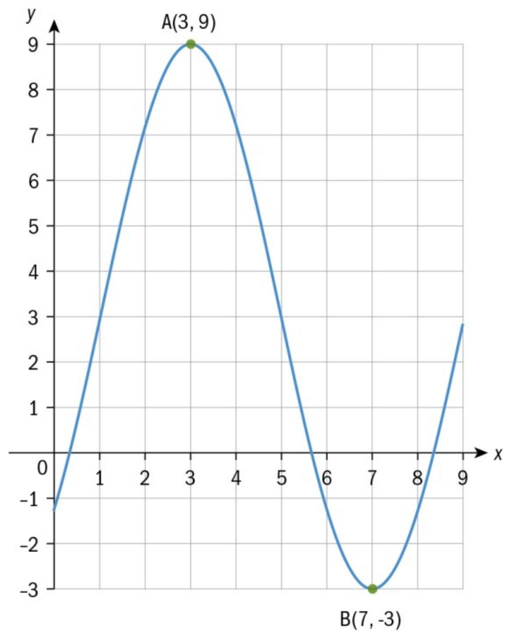


3 Sketch the graph of each function for $-\pi \le x \le 2\pi$.

a $y = 4\sin\left(x - \frac{\pi}{2}\right) + 1$

b $y = -3\cos(2x) - 2$

4 The following diagram shows part of the graph of the function $y = a\cos(b(x-c))+d$. The graph has a local maximum at point A(3, 9), and it has a local minimum at the point B(7, −3), as shown. Find the values of a, b, c and d.



5 The depth of the water at the end of a pier can be estimated by the function $D(t) = 6.7\sin(0.515(t-3))+12$, where D is the depth of the water in metres and t is the number of hours after midnight.

a Find the period of the function.

b Find the amplitude of the function.

c Find the depth of the water at 7:00 a.m.

d At what time does the water first reach its greatest depth?

e During the first 12 hours after midnight, find the length of time the depth of the water is greater than 15 m.

Answers

1 Note that, in each part of this question, there are many correct cosine equations for the function shown in each graph. However, the period, amplitude and vertical shift will be the same for each of these correct equations.

a period = $\frac{7\pi}{4}-\frac{3\pi}{4}=\pi$ (maximum to maximum)

amplitude = $\frac{1-(-4)}{2}=2.5$

$y=2.5\cos\left(2\left(x+\frac{\pi}{4}\right)\right)-1.5$

b period = $2(2\pi-0)=4\pi$ (maximum to maximum)

amplitude = $\frac{3-(-1)}{4}=2$

$y=-2\cos\left(\frac{1}{2}x\right)-1$

c period = $3-(-2)=5$ (minimum to maximum is half a period)

amplitude = $\frac{1-(-3)}{2}=2$

$y=2\cos\left(\frac{2\pi}{5}(x-3)\right)+1$

2 Note that, in each part of this question, there are many correct sine equations for the function shown in each graph. However, the period, amplitude and vertical shift will be the same for each of these correct equations.

a period = $\frac{\pi}{2}-0=\frac{\pi}{2}$

amplitude = $\frac{5-(-1)}{2}=3$

$y=3\sin(4x)+2$

b period = $\pi-0=\pi$

amplitude = $\frac{6-(-4)}{2}=5$

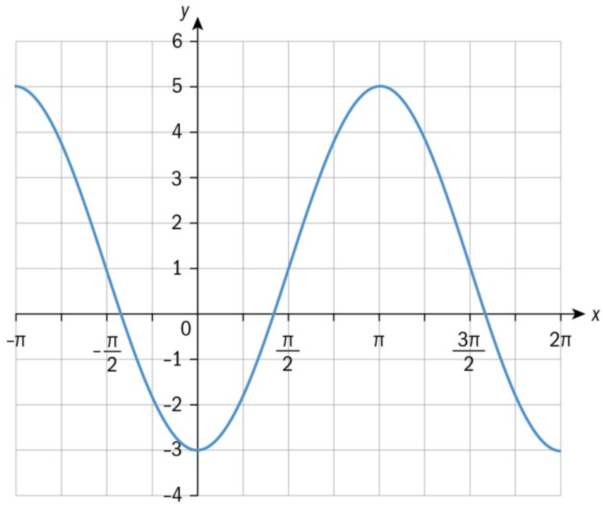
$y=5\sin(2x)+1$

c period = $5-1=4$ (minimum to maximum is half a period)

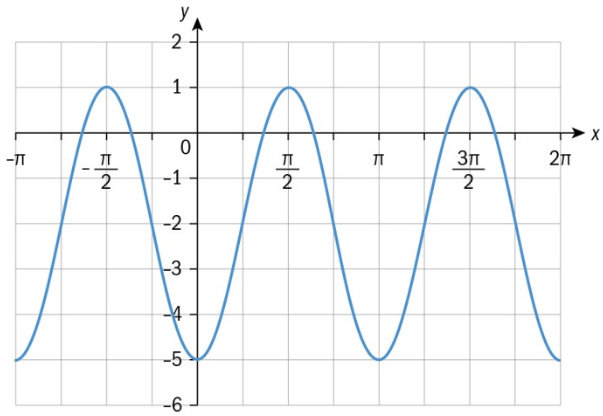
amplitude = $\frac{3-(-3)}{2}=3$

$y=3\sin\left(\frac{\pi}{2}(x-1)\right)$

3 **a**



b



4 a = amplitude $= \dfrac{9-(-3)}{2} = 6$

period $= 2(7-3) = 8$, so $b = \dfrac{2\pi}{\text{period}} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$

c = horizontal shift = 3

d = vertical shift $= \dfrac{9+(-3)}{2} = 3$

5 **a** period $= \dfrac{2\pi}{0.515} \approx 12.2$ hours

b amplitude = 6.7

c $D(7) = 6.7\sin\big(0.515(7-3)\big) + 12 \approx 17.9$ metres

d maximum point on graph of D at (6.05…, 18.7), so $t \approx 6.05$, the time is 6:03

e The depth is 15 metres when $t \approx 3.90147\ldots$ and when $t \approx 8.1987\ldots$, so the depth is greater than 15 metres for $8.1987 - 3.90147 \approx 4.30$ hours .

13.1 Derivatives with sine and cosine

1 Find the derivative of each function.

a $f(x) = 2\sin x + 5\cos x$

b $y = \sin\frac{3}{x}$

c $g(x) = \sin^3 x - \cos x$

d $y = \sin 3x \cos 3x$

2 Let $g(x) = \cos 2x$.

a Find the following:

i $f'(x)$ **ii** $f''(x)$ **iii** $f'''(x)$ **iv** $f^{(iv)}(x)$

b Suppose that $f^{(n)}(x) = 2^n \cos 2x$. Write down all the possible values of *n* that are less than 20.

c Find

i $f^{(40)}(x)$ **ii** $f^{(27)}(x)$

3 Without using a calculator, find the equation of the tangent and normal to the curve $y = \sin x - 2\cos x$ when $x = \pi$.

4 Without using a GDC, consider the function $f(x) = 3\sin x$ for $0 \le x \le 2\pi$. Find the values of *x* where $f'(x) = \frac{3}{2}$ and find the equation of the tangents at these points.

5 Find the derivative of each function.

a $f(x) = 2\sin(4x - 1) + 2x$

b $y = \frac{\cos x}{1 + \sin x}$

c $f(x) = \sin(e^{x^3})$

d $f(x) = e^{2x}\cos 3x$

e $y = \ln(\cos 2x)$

6 Consider the function $f(x) = x - \sin x,\ 0 \le x \le 2\pi$

a Find $f'(x)$ and $f''(x)$.

b Find the intervals where the graph of the function is concave up and concave down.

c Find the coordinates of any points of inflexion.

Answers

1 **a** $f'(x) = 2\cos x - 5\sin x$ **b** $\dfrac{dy}{dx} = -\dfrac{3}{x^2}\cos\dfrac{3}{x}$

c $g'(x) = 3\sin^2 x\cos x + \sin x$ **d** $\dfrac{dy}{dx} = 3\cos^2 3x - 3\sin^2 3x$

2 **a** **i** $-2\sin 2x$ **ii** $-4\cos 2x$ **iii** $8\sin 2x$ **iv** $16\cos 2x$

b n = 4, 8, 12, 16

c **i** $2^{40}\cos 2x$ **ii** $2^{27}\sin 2x$

3 $y = 2$ when $x = \pi$

$\dfrac{dy}{dx} = \cos x + 2\sin x = -1$ when $x = \pi$

So equation of tangent is $y = -x + 2 + \pi$

and equation of normal is $y = x + 2 - \pi$

4 $f'(x) = 3\cos x = \dfrac{3}{2}$ when $\cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f\left(\dfrac{\pi}{3}\right) = \dfrac{3\sqrt{3}}{2}$

$f\left(\dfrac{5\pi}{3}\right) = -\dfrac{3\sqrt{3}}{2}$

So the equations are: $y = 3x - \dfrac{\pi - 3\sqrt{3}}{2}$, $y = 3x - \dfrac{5\pi + 3\sqrt{3}}{2}$

5 **a** $f'(x) = 8\cos(4x - 1) + 2$

b $\dfrac{dy}{dx} = \dfrac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \dfrac{-\sin x - (\cos^2 x + \sin^2 x)}{(1+\sin x)^2} = \dfrac{-(1+\sin x)}{(1+\sin x)^2} = -\dfrac{1}{1+\sin x}$

c $f'(x) = \cos\left(e^{x^3}\right) \times 3x^2 e^{x^3} = 3x^2 e^{x^3}\cos\left(e^{x^3}\right)$

d $f'(x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x = e^{2x}(2\cos 3x - 3\sin 3x)$

e $\dfrac{dy}{dx} = \dfrac{1}{\cos 2x} \times -2\sin 2x = -2\tan 2x$

6 **a** $f'(x) = 1 - \cos x,\ f''(x) = \sin x$

b Concave up when $\sin x > 0$

Concave down when $\sin x > 0$

Concave up when 0< $x < \pi$

Concave down when $\pi < x < 2\pi$

c When $x = \pi,\ y = \pi$ so point of inflexion at (π, π)

Chapter 13

13.2 Applications of derivatives

1 The displacement of an object moving in a straight line is given by $s(t) = \dfrac{t-1}{e^t} + 1$. Without using a GDC:

a Verify that the displacement is given by $v(t) = \dfrac{2-t}{e^t}$.

b Find the velocity and speed when $t = 4$.

c Find the time and position when the particle changes direction.

2 The height in meters of the tide at a certain point is given by $h(t) = 10 + 5\sin\left(\dfrac{\pi t}{12}\right) + 4\cos\left(\dfrac{\pi t}{12}\right)$ where t is the hours after midnight. Without using a GDC, find the height of the tide and the rate at which it is decreasing at 4 a.m.

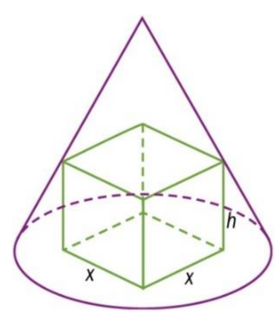
3 The profit in dollars from the sale of an item is modelled by $P(x) = -0.7x^3 + 400000x - 300000$ where x is the number of items sold.

a Find $P'(200)$, the marginal profit when $x = 200$.

b The revenue $R(x) = 600000x - 300000$. Find the cost function $C(x)$.

4 The cost in euros to produce x items is $C(x) = 4x + \dfrac{200}{x}$. Find the value of x that will minimise the cost.

5 A cone has radius 5 cm and height 8 cm. A square prism with sides x cm, x cm and h cm is inscribed in the cone as shown in the diagram.



a Find an expression for h in terms of x.

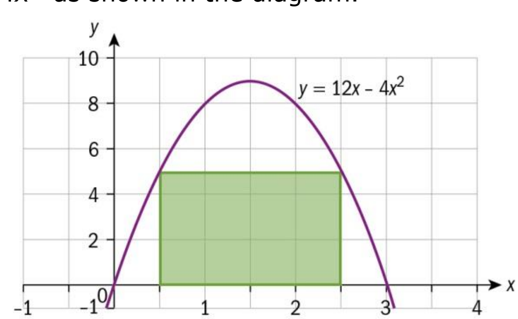
b Show that the volume of the prism is given by

$$V = 8x^2 - \frac{8x^3}{5\sqrt{2}}.$$

c Find the maximum volume of the prism.

6 A rectangle is to be drawn under the curve $y = 12x - 4x^2$ as shown in the diagram.

Find the maximum volume of the rectangle



Answers

1 **a** $\frac{d}{dt}\left(\frac{t-1}{e^t}+1\right)=\frac{1\times e^t-(t-1)e^t}{e^{2t}}=\frac{1-(t-1)}{e^t}=\frac{2-t}{e^t}$

and when $t = 0$, $s=\frac{-1}{e^0}+1=0$ so verified.

b When $t = 4, v=\frac{2-4}{e^4}=-2e^{-4}$ $\therefore$ speed $= 2e^{-4}$

c Changes direction when velocity changes sign, so $t = 2$

Position $=\frac{1}{e^2}+1$

2 $h(4)=10+5\sin\frac{\pi}{3}+4\cos\frac{\pi}{3}=10+\frac{5\sqrt{3}}{2}+2=12+\frac{5\sqrt{3}}{2}$

So, the height at 4 a.m. is $12+\frac{5\sqrt{3}}{2}$ m.

$h'(t)=\frac{5\pi}{12}\cos\frac{\pi}{12}t-\frac{4\pi}{12}\sin\frac{\pi}{12}t$

$h'(4)=\frac{5\pi}{12}\cos\frac{\pi}{3}-\frac{4\pi}{12}\sin\frac{\pi}{3}=\frac{5\pi-4\pi\sqrt{3}}{24}<0\Rightarrow$ decreasing

So, at 4 a.m. the height is changing at a rate of $\frac{5\pi-4\pi\sqrt{3}}{24}$ m h^{-1}.

3 **a** $P'(x)=-2.1x^2+400000$ so $P'(200)=316000$

b $P(x)=R(x)-C(x)$

$\Rightarrow C(x)=(600000x-3000000)-(-0.7x^3+400000x-300000)=0.7x^3+200000x$

4 $C'(x)=4-\frac{200}{x^2}$

$C'(x)=0$ when $x^2=50$

$C''(x)=\frac{400}{x^3}>0$ when $x=\sqrt{50}$ so a minimum

So 7 items will minimise the cost

5 **a** using similar triangles $\frac{8}{5}=\frac{8-h}{\frac{x}{\sqrt{2}}}\Rightarrow h=8-\frac{8x}{5\sqrt{2}}$

b $V=x^2h=x^2\left(8-\frac{8x}{5\sqrt{2}}\right)=8x^2-\frac{8x^3}{5\sqrt{2}}$

c $\frac{dV}{dx}=16x-\frac{24x^2}{5\sqrt{2}}$

$\frac{dV}{dx}=0$, when $x=0$ or $x=\frac{10\sqrt{2}}{3}=4.71404\ldots$

$\frac{d^2V}{dx^2}=16-\frac{48x}{5\sqrt{2}}<0$ when $x=\frac{10\sqrt{2}}{3}\approx 4.71\Rightarrow$ maximum

maximum volume $=V\left(\frac{10\sqrt{2}}{3}\right)=\frac{1600}{27}=59.2592\ldots\approx 59.3$

6 Area $= ((3-x)-x)\times(12x-4x^2) = 4x(3-2x)(3-x) = 8x^3 - 36x^2 + 36x$

So $\dfrac{dA}{dx} = 24x^2 - 72x + 36 = 12(2x^2 - 6x + 3)$

So $\dfrac{dA}{dx} = 0 \Rightarrow x = \dfrac{6 \pm \sqrt{12}}{4} = 0.6339\ldots,\ 2.366\ldots$

$\dfrac{d^2A}{dx^2} = 48x - 72 < 0$ so maximum when $x = 0.6339\ldots$

maximum area = $10.3923\ldots \approx 10.4$

13.3 Integration with sine, cosine and substitution

1 Find each indefinite integral

a $\int 3\cos x \, dx$

b $\int 5\sin(3x) \, dx$

c $\int \left(x^2 - \cos(3x)\right) dx$

d $\int \sin\left(\frac{\pi x}{3}\right) dx$

2 Find each indefinite integral

a $\int \left(2x^2 - 3x\right)^3 (4x - 3) \, dx$

b $\int (4x - 1)\sin(2x^2 - x) \, dx$

c $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}} \, dx$

d $\int \cos x \sin^3(x) \, dx$

3 $f'(x) = (x - 1)(x^2 - 2x)^3$. Given that $f(1) = 3$, find $f(x)$.

4 Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{5\pi}{3}} 3\cos x \, dx$.

5 Evaluate

a $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \sin^2 x \, dx$

b $\int_{1}^{3} \frac{4x}{\sqrt{2x^2 + 3}} \, dx$

6 Use a GDC to find the area bounded by the curves $y = \cos x$ and $y = x^2$.

Answers

1 **a** $3\sin x + C$ **b** $-\frac{5}{3}\cos(3x) + C$ **c** $\frac{1}{3}x^3 - \frac{1}{3}\sin(3x) + C$ **d** $-\frac{3}{\pi}\cos\frac{\pi x}{3} + C$

2 **a** $u = 2x^2 - 3x \Rightarrow \frac{du}{dx} = 4x - 3 \Rightarrow du = (4x-3)dx$

So $\int (2x^2 - 3x)^3 (4x-3)dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}(2x^2 - 3x)^4 + C$

b $u = 2x^2 - x \Rightarrow \frac{du}{dx} = 4x - 1 \Rightarrow du = (4x-1)dx$

So $\int (4x-1)\sin(2x^2 - x)dx = \int \sin u \, du = -\cos u + C = -\cos(2x^2 - x) + C$

c $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

So $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx = \int 2\cos u \, du = 2\sin u + C = 2\sin(\sqrt{x}) + C$

d $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

So $\int \cos x \sin^3(x)dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$

3 $u = x^2 - 2x \Rightarrow \frac{du}{dx} = 2x - 2 \Rightarrow du = 2(x-1)dx$

$\int (x-1)(x^2-2x)^3 dx = \frac{1}{2}\int u^3 du = \frac{1}{8}u^4 + C = \frac{1}{8}(x^2 - 2x)^4 + C$

$f(1) = 3 \Rightarrow -\frac{1}{8} + C = 3 \Rightarrow C = \frac{23}{8}$

$f(x) = \frac{1}{8}(x^2 - 2x)^3 + \frac{23}{8}$

4 $\int_{\frac{\pi}{6}}^{\frac{5\pi}{3}} 3\cos x \, dx = [3\sin x]_{\frac{\pi}{6}}^{\frac{5\pi}{3}} = 3\sin\frac{5\pi}{3} - 3\sin\frac{\pi}{6} = 3 \times \left(-\frac{\sqrt{3}}{2}\right) - 3 \times \frac{1}{2} = -\frac{3\sqrt{3}+3}{2}$

5 **a** $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

So $\int \cos x \sin^2(x)dx = \int u^2 du$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{6}} \cos x \sin^2 x \, dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^2 du = \left[\frac{1}{3}u^3\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}-1}{24}$

b $u = 2x^2 + 3 \Rightarrow \frac{du}{dx} = 4x \Rightarrow du = 4x \, dx$

So $\int_1^3 \frac{4x}{\sqrt{2x^2+3}}dx = \int_5^{21} u^{-\frac{1}{2}} du = \left[2u^{\frac{1}{2}}\right]_5^{21} = 2(\sqrt{21} - \sqrt{5})$

6 From GDC, the curves cross when $x = \pm 0.8241...$

So the required area is $\int_{-0.8241...}^{0.8241...} (\cos x - x^2)dx = 1.094...$ (from GDC) ≈ 1.09

13.4 Kinematics and accumulating change

1 The displacement of a particle is given by the function $s(t) = t^2 - 5t + 6,\ 0 \le t \le 5$.

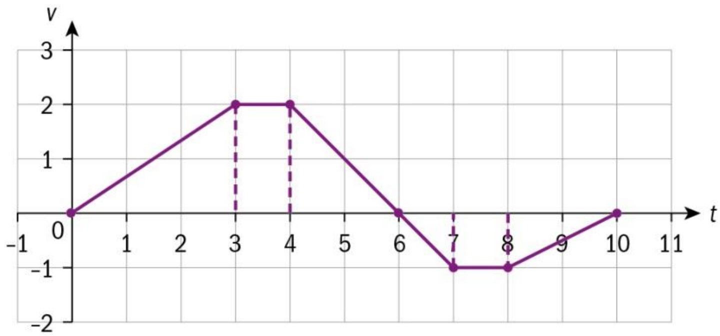
a Find the velocity of the particle at time t.

b Write down and evaluate a definite integral for the particle's displacement over the interval.

c Write down and evaluate a definite integral for the particle's distance travelled over the interval.

2 **Do not use a calculator for this question.**

The velocity v m s^{-1} of a particle moving along a line for $0 \le t \ge 10$ is shown in the diagram.



a Find the acceleration when t = 6.

b Find the displacement over the interval.

c Find the total distance travelled over the interval.

3 **Do not use a calculator for this question.**

A particle moving along a straight line has displacement in metres given by $s(t) = 2 + 3\cos t$ where t is in seconds.

a Calculate the velocity when $t = \frac{\pi}{3}$.

b Calculate the acceleration when $t = \frac{\pi}{2}$.

c Calculate the total distance travelled during the interval $0 \le t \le 2\pi$.

4 A particle moves along a straight line with velocity $v(t) = (t+1)\cos\frac{t^3}{3},\ 0 \le t \le 10$.

a **i** Find the acceleration when t = 2.7.

ii Find the acceleration of the particle at the point where it is first at rest.

b Find the displacement of the particle when t = 10, given that the displacement is 0 when t = 0.

c Find the total distance travelled during the time $0 \le t \le 10$.

5 A swimming pool is being filled with water at a rate given by $r(t) = 100 - 1.07^t$ litres per minute. How much water is in the pool after 30 minutes, given that it is initially empty?

Answers

1 **a** $v = 2t - 5$

b $\int_0^5 (2t-5)dt = 0$

c $\int_0^5 |2t-5|\, dt = 3.44146\ldots \approx 3.44$

2 **a** Acceleration = gradient = −1

b Using area of trapezoids, displacement = $\frac{1}{2} \times 2 \times (6+1) - \frac{1}{2} \times 1 \times (4+1) = \frac{9}{2}$

c Using area of trapezoids, distance travelled = $\frac{1}{2} \times 2 \times (6+1) + \frac{1}{2} \times 1 \times (4+1) = \frac{19}{2}$

3 **a** $v\left(\frac{\pi}{3}\right) = -3\sin\left(\frac{\pi}{3}\right) = -3 \times \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{3}$ m s^{-1}

b $a(0) = -3\cos\left(\frac{\pi}{2}\right) = -3 \times 0 = 0$ m s^{-1}

c Distance travelled = $\int_0^{2\pi} |-3\sin t|\, dt$

$$= \int_0^{\pi} (3\sin t)\, dt + \int_{\pi}^{2\pi} (-3\sin t)\, dt$$
$$= \left[-3\cos t\right]_0^{\pi} + \left[3\cos t\right]_{\pi}^{2\pi}$$
$$= (3-(-3)) + (3-(-3))$$
$$= 12\,\text{m}$$

4 **a** **i** From GDC −6.435… ≈ -6.44

ii From GDC, velocity first = 0 at t = 1.676…

So from GDC acceleration = −7.52316… ≈ -7.52

b Displacement = $\int_0^{10} (t+1)\cos\left(\frac{t^3}{3}\right) dt = 1.619\ldots \approx 1.62$ from GDC

c Distance travelled = $\int_0^{10} \left|(t+1)\cos\left(\frac{t^3}{3}\right)\right| dt = 38.65\ldots \approx 38.7$ from GDC

5 Volume = $\int_0^{30} \left(100 - 1.07^t\right) dt = 2902.2\ldots \approx 2900$ litres from GDC

14.1 Random variables

1 A discrete random variable X has the following probability distribution.

x	0	1	2	3
P($X = x$)	0.05	0.25	0.4	p

a Find p.

b Find $E(X)$.

2 Consider the following probability distribution for the discrete random variable Y.

y	1	2	4	8
P($Y = y$)	0.1	0.3	a	b

a Given that $E(Y) = 4.9$, find the values of a and of b.

b Find $P(Y \leq 4)$.

3 A discrete random variable X has a probability distribution given by

$P(x) = \frac{c}{x}$, for $x = 1, 2, 3, 4$.

a Complete the table below to show this probability distribution.

x	1	2	3	4
P($X = x$)				

b Find the exact value of c.

c Find $P(X \geq 2)$.

4 A bag contains eight balls; three are green and five are white.

Balls are drawn at random from the bag one at a time, without replacement.

Let W be the number of white balls drawn before the first green ball is drawn.

a Explain why the only possible values of W are 0, 1, 2, 3, 4, 5.

b Complete the probability distribution for W.

w	0	1	2	3	4	5
P($W = w$)						

c What is the most likely value of W?

d Find the expected value of W.

Answers

1 **a** $0.05 + 0.25 + 0.4 + p = 1$

$p = 0.3$

b $E(X) = 0(0.05) + 1(0.25) + 2(0.4) + 3(0.3) = 1.95$

2 **a** $0.1 + 0.3 + a + b = 1$

$a + b = 0.6 \rightarrow b = 0.6 - a$

$E(X) = 1(0.1) + 2(0.3) + 4(a) + 8(b) = 4.9$

Using substitution, $E(X) = 1(0.1) + 2(0.3) + 4(a) + 8(0.6 - a) = 4.9$

$0.1 + 0.6 + 4a + 4.8 - 8a = 4.9 \rightarrow -4a = -0.6 \rightarrow a = 0.15$

$b = 0.6 - a = 0.6 - 0.15 = 0.45$

b $P(Y \leq 4) = 1 - P(Y = 8) = 1 - 0.45 = 0.55$

3 **a**

x	1	2	3	4
P(*X* = *x*)	c	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$

b $c + \frac{c}{2} + \frac{c}{3} + \frac{c}{4} = 1$

Multiplying both sides by 12 gives $12c + 6c + 4c + 3c = 12 \rightarrow 25c = 12 \rightarrow c = \frac{12}{25} = 0.48$

c $P(X \geq 2) = P(X = 3) + P(X = 4)$

$P(X \geq 2) = \frac{0.48}{3} + \frac{0.48}{4} = 0.16 + 0.12 = 0.28$

4 **a** If the first ball drawn is green, then $W = 0$, and if all the white balls are drawn before the first green ball is drawn, then $W = 5$.

b

w	0	1	2	3	4	5
P(*W* = *w*)	$\frac{3}{8}$	$\frac{15}{56}$	$\frac{5}{28}$	$\frac{3}{28}$	$\frac{3}{56}$	$\frac{1}{56}$

c The most likely value of W is 0, since $\frac{3}{8}$ is the greatest probability.

d $E(W) = 0\left(\frac{3}{8}\right) + 1\left(\frac{15}{56}\right) + 2\left(\frac{5}{28}\right) + 3\left(\frac{3}{28}\right) + 4\left(\frac{3}{56}\right) + 5\left(\frac{1}{56}\right) = 1.25$

14.2 The binomial distribution

1 If $X \sim \mathrm{B}\left(5, \frac{1}{4}\right)$, find

a $\mathrm{P}(X = 2)$

b $\mathrm{P}(X \geq 3)$

2 The probability that George makes it to school on time on any given day is 0.7.

Let X represent the number of days George makes it to school on time in a five-day school week.

a Find $\mathrm{P}(X = 4)$.

b Find $\mathrm{P}(X \geq 4)$.

3 Let H be the random variable that represents the number of heads obtained when a fair coin is flipped four times. Complete the probability distribution table for H.

h	0	1	2	3	4
P(*H* = *h*)					

4 Three fair dice are rolled. Let T be the number of 3s that appear. The results are shown in the probability distribution table below.

t	0	1	2	3
P(*T* = *t*)	$\frac{125}{216}$	a	$\frac{15}{216}$	b

a Find the values of a and b, giving exact answers.

b Find the probability of rolling at least one 3.

5 A hockey player makes 30% of the goals he attempts.

a If the player attempts six goals in a game, find the probability that he will make exactly two of them.

b Find the least number of goals the player must attempt so that the probability that he makes at least one goal is greater than 99%.

6 Pierre is playing a game in which he tosses two fair coins.

If both coins land on tails, he wins a prize.

a Find the probability that Pierre will win a prize on his first turn.

b Pierre decides to keep playing the game until he wins two prizes.

Find the probability that he wins his second prize on his eighth turn.

7 A fair coin is tossed 100 times. Let T be the number of times the coin lands on tails.

Find the probability that $46 < T < 55$.

Answers

1 **a** $\mathrm{P}(X=2)=\binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3=\frac{135}{512}\approx 0.264$

b $\mathrm{P}(X\geq 3)=1-\mathrm{P}(X\leq 2)=1-\frac{459}{512}=\frac{53}{512}\approx 0.104$

2 **a** $\mathrm{P}(X=4)=\binom{5}{4}(0.7)^4(0.3)^1=0.36015\approx 0.360$

b $\mathrm{P}(X\geq 4)=1-\mathrm{P}(X\leq 3)=1-0.47178=0.52822\approx 0.528$

3

h	0	1	2	3	4
P(*H* = *h*)	0.0625	0.25	0.375	0.25	0.0625

4 **a** $a=\mathrm{P}(T=1)=\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2=\frac{25}{72}$

$b=\mathrm{P}(T=3)=\binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0=\frac{1}{216}$

b $\mathrm{P}(T\geq 1)=1-\mathrm{P}(T=0)=1-\frac{125}{216}=\frac{91}{216}\approx 0.421$

5 **a** $\mathrm{P}(X=2)=\binom{6}{2}(0.3)^2(0.7)^4=0.324135\approx 0.324$

b Let n be the number of goals attempted.

$\mathrm{P}(X\geq 1)=1-\mathrm{P}(X=0)=1-0.7^n>0.99$

$0.7^n<0.01\rightarrow n>12.91139\ldots\rightarrow n=13$

6 **a** $\mathrm{P}(H=2)=\binom{2}{2}(0.5)^2(0.5)^0=0.25$

b In order to win his second prize on his eighth turn, Pierre must win one prize in his first seven turns, then win on his eighth turn.

$\binom{7}{1}(0.25)^1(0.75)^6\times 0.25=0.0778656\ldots\approx 0.0779$

7 $\mathrm{P}(46<T<55)=\mathrm{P}(T\leq 54)-\mathrm{P}(T\leq 46)=0.815899\ldots-0.242059\ldots=0.5738\ldots\approx 0.574$

14.3 The normal distribution

1 If $X \sim \mathrm{N}(28,5)$, find

a $\mathrm{P}(X > 32)$

b $\mathrm{P}(X > 32 \mid X > 28)$

2 The heights of women in a certain country are normally distributed with a mean of 163 cm and a standard deviation of 8 cm.

a A woman is selected at random.

Find the probability that she is taller than 172 cm.

b Three women are selected at random.

Find the probability that at least one of them is taller than 172 cm.

3 A machine manufactures a large number of bottles. The height, H mm, of a bottle is normally distributed, with a mean of 160 mm and a standard deviation of 1.3 mm.

a Find the probability that a bottle selected at random has a height less than 158 mm.

b The probability that the height of a bottle is greater than x mm is 0.1. Find x.

c If the height of a bottle is within 2 standard deviations of the mean, it is said to be acceptable. What percentage of the bottles are acceptable?

d Eight bottles are selected at random.

Find the probability that at least six of these bottles are acceptable.

4 The masses of onions grown on a large farm are normally distributed with a mean

of 110 grams and a standard deviation of 15 grams.

a Find the probability that the mass of an onion grown on this farm is

i between 100 and 120 grams

ii greater than 130 grams.

b One percent of the onions grown on this farm have masses greater than x kg. Find x.

5 Scores on a university admissions test are normally distributed with a mean of 22 and a standard deviation σ.

a If 95% of the scores are less than 30, find the value of σ.

b Find the probability that a student selected at random earns a score of 27 or greater on the admissions test.

c If it is known that a student scored at least 27 on the test, find the probability that she scored at least 30.

6 The lengths of fish in a pond are normally distributed with a mean of μ and a standard deviation of σ.

a If 15% of the fish have lengths greater than 54 cm, and 10% of the fish have lengths less than 25 cm, find μ and σ.

b Ten fish are selected at random.

i Find the probability that exactly three of them have lengths greater than 48 cm.

ii Find the probability that at least three of them have lengths greater than 48 cm.

Answers

1 **a** $\mathrm{P}(X > 32) = 0.036819\ldots \approx 0.0368$

b $\mathrm{P}(X > 32 \mid X > 28) = \dfrac{\mathrm{P}(X > 32)}{\mathrm{P}(X > 28)} = \dfrac{0.036819\ldots}{0.5} = 0.073638\ldots \approx 0.0736$

2 **a** Let H be the woman's height: $\mathrm{P}(H > 172) = 0.13029\ldots \approx 0.130$

b Let X be the number of women (out of the three selected) who are taller than 172 cm, so $X \sim \mathrm{B}(3, 0.13029\ldots)$

$\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.65783\ldots = 0.34216\ldots \approx 0.342$

3 **a** $\mathrm{P}(H < 158) = 0.061967\ldots \approx 0.0620$

b Using the inverse normal function on the GDC, $x = 163.024\ldots \approx 163$

c $\mathrm{P}(157.4 \leq H \leq 162.6) = 0.95449\ldots \approx 0.954$

d Let X be the number of bottles (out of the eight selected) that are acceptable,

so $X \sim \mathrm{B}(8, 0.95449\ldots)$

$\mathrm{P}(X \geq 6) = 1 - \mathrm{P}(X \leq 5) = 1 - 0.004438\ldots = 0.99556\ldots \approx 0.996$

4 **a** **i** $\mathrm{P}(100 \leq M \leq 120) = 0.495015\ldots \approx 0.495$ **ii** $\mathrm{P}(M > 130) = 0.091211\ldots \approx 0.0912$

b Using the inverse normal function on the GDC, $x = 144.895\ldots \approx 145$

5 **a** $\mathrm{P}(Z > z) = 0.95 \rightarrow z = 1.64485\ldots$

$1.64485\ldots = \dfrac{30 - 22}{\sigma} \rightarrow \sigma = 4.86365\ldots \approx 4.86$

b $\mathrm{P}(X \geq 27) = 0.151967\ldots \approx 0.152$

c $\mathrm{P}(X \geq 30 \mid X \geq 27) = \dfrac{\mathrm{P}(X \geq 30)}{\mathrm{P}(X \geq 27)} = \dfrac{0.04999\ldots}{0.15196\ldots} = 0.329018\ldots \approx 0.329$

6 **a** $\mathrm{P}(Z > z_1) = 0.85 \rightarrow z_1 = 1.03643\ldots$

$1.03643\ldots = \dfrac{54 - \mu}{\sigma} \rightarrow \mu + 1.03643\ldots\sigma = 54$

$\mathrm{P}(Z > z_2) = 0.10 \rightarrow z_2 = -1.28155\ldots$

$-1.28155\ldots = \dfrac{25 - \mu}{\sigma} \rightarrow \mu - 1.28155\ldots\sigma = 25$

Using the simultaneous equation solver on the GDC,

$\mu = 41.0333\ldots \approx 41.0$ and $\sigma = 12.51086\ldots \approx 12.5$

b First, find the probability that a fish has a length greater than 48 cm

$\mathrm{P}(L > 48) = 0.288814\ldots$

Let X be the number of fish (out of the ten selected) that have lengths greater than 48 cm, so $X \sim \mathrm{B}(10, 0.288814\ldots)$

i $\mathrm{P}(X = 3) = 0.26602\ldots \approx 0.266$

ii $\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2) = 1 - 0.10991\ldots = 0.89008\ldots \approx 0.890$