OXFORD IB DIPLOMA PROGRAMME

TEACHER NOTES

MATHEMATICS: ANALYSIS AND APPROACHES

STANDARD LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1 From patterns to generalizations: sequences and series

Essential understandings

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.
- The binomial theorem is a generalization which provides an efficient method for expanding binomial expressions

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Recognizing patterns helps you to find the general term or algebraic expression to describe them.	Investigation 1 & 2
A converging or diverging sequence can be distinguished by the value of the common ratio.	Investigation 4
The common difference or common ratio helps to distinguish between different arithmetic and geometric sequences and series.	Investigation 5
An infinite series may/can converge to a finite sum when the terms in the series form a converging sequence, while an infinite divergent series expands.	Investigation 7
Simple interest can be modelled by an arithmetic series.	Investigation 8
Compound interest can be modelled by a geometric series.	Investigation 8
Growth can follow different patterns and we can use these patterns to predict and compare outcomes.	Investigation 8 & 9
Approximating a growth rate using a geometric sequence allows us to model real-life growth.	Investigation 9
The nth row in Pascal's triangle represents the number of ways of choosing 0, 1, 2, 3, … items from n −1 items.	Investigation 12
The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.	Investigation 13
Proofs using algebra can prove a statement for all possible number values, allowing for generalization.	Investigation 14

Syllabus sections covered in this chapter:

- SL1.2*
- SL1.3*
- SL1.4*
- SL1.6
- SL1.8
- SL1.9

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 3: Solving linear equations, Fractions, Order of operations, Substituting into formulae	Page 12: Example 5 Page 26: Example 15 Page 41: Example 26 Page 53: Example 33	N/A	Pages 12, 21, 34, 43, 54, 56

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 56	Page 58	N/A

1.1 Number patterns and sigma notation

TOK

Where did numbers come from?

Answer: The history of number from Sumerians and its development to the present Arabic system is a fascinating development to trace. You might want to go back to the Ishango bone, evidence of counting from 20 000 years ago.

Investigation 1

1 Completed table:

Number of visits to gym	Total fees paid by Luis	Total fees paid by Elijah
0	\$100	\$0
1	\$105	\$15
2	\$110	\$30
3	\$115	\$45
4	\$120	\$60
5	\$125	\$75
6	\$130	\$90
7	\$135	\$115
8	\$140	\$130

2 The pattern for Luis is $100 + 5(n - 1)$.

The pattern for Elijah is $15(n - 1)$.

3 Elijah's pattern is increasing faster; it is adding \$15 more per line whereas Luis' is only adding \$5.

4 The students can solve a system of linear equations:

$100 + 5(n - 1) = 15(n - 1)$

$100 + 5n - 5 = 15n - 15$

$-10n = -110$

$n = 11$

After the 11th visit, the two men will pay the same fee.

Or students can continue the sequences until they reach the same value.

5 The limitation for the patterns is the number of times someone can reasonably visit a gym. In theory, this could be infinite and so the patterns will never reach a maximum value.

Investigation 1 continued

Conceptual understanding:
Recognizing patterns helps you to find the general term or algebraic expression to describe them.

1 100, 105, 110, …

2 The first term is 100 and 5 is added to each subsequent term.

3 5 is added to the first term (100) one less time than the number of terms.

4 0, 15, 30, …

5 The first term is 0 and 15 is added to each subsequent term.

6 Completed table:

Term number	Total fees paid by Luis	Pattern for Luis	Total fees paid by Elijah	Pattern for Elijah
u_1	\$100	100 + 5 × 0	\$0	15 × 0
u_2	\$105	100 + 5 × 1	\$15	15 × 1
u_3	\$110	100 + 5 × 2	\$30	15 × 2
u_4	\$115	100 + 5 × 3	\$45	15 × 3
u_5	\$120	100 + 5 × 4	\$60	15 × 4
u_n	---	$100 + 5(n - 1)$	---	$15(n - 1)$

7 $100 + 5(n - 1) = 100 + 5n - 5 = 5n + 95$

8 $u_n = 15(n - 1) = 15n - 15$

9 **(This is the conceptual understanding):** Recognizing patterns helps you to find the general term or algebraic expression to describe them.

TOK

Do the names that we give things impact how we understand them?

Answer: For instance, some large numbers are named, the google and the googolplex, while others are represented in this form?

Investigation 2

Conceptual understanding:
Recognizing patterns helps you to find the general term or algebraic expression to describe them.

1 Completed table:

Sequence 1	Sequence 2	Sequence 3
5	2	36
10	-4	12
20	8	4
40	-16	$\frac{4}{3}$
80	-32	$\frac{4}{9}$
160	64	$\frac{4}{27}$
320	-128	$\frac{4}{81}$

2 Each term is found by multiplying the previous term by a common value.

Completed table:

Term number	**Sequence 1**	**Pattern**	**Sequence 2**	**Pattern**	**Sequence 3**	**Pattern**
u_1	5	$5\times1=5\times2^0$	2	$2(1)=2(-2)^0$	36	$36\times1=36\left(\frac{1}{3}\right)^0$
u_2	10	$5\times2=5\times2^1$	-4	$2(-2)=2(-2)^1$	12	$36\times\frac{1}{3}=36\left(\frac{1}{3}\right)^1$
u_3	20	$5\times4=5\times2^2$	8	$2(4)=2(-2)^2$	4	$36\times\frac{1}{9}=36\left(\frac{1}{3}\right)^2$
u_4	40	$5\times8=5\times2^3$	-16	$2(-8)=2(-2)^3$	$\frac{4}{3}$	$36\times\frac{1}{27}=36\left(\frac{1}{3}\right)^3$
u_5	80	$5\times16=5\times2^4$	32	$2(16)=2(-2)^4$	$\frac{4}{9}$	$36\times\frac{1}{81}=36\left(\frac{1}{3}\right)^4$
u_6	---	$5\times2^{n-1}$	---	$2(-2)^{n-1}$	---	$36\left(\frac{1}{3}\right)^{n-1}$ or $36\frac{1}{3^{n-1}}$

3 The exponent of 2 is one less than the term number.

4 The general term is found by multiplying the first term of 5 by the value of 2, one less time than the term number.

5 See table above.

6 **Factual:** How do you find the general term of **this type** of sequence?

Answer: Follow the pattern and write a generalisation using the first term u_1 and the term number n. $u_n = u_1 \times r^{n-1}$

Note: The students may use a variable other than r.

7 **Conceptual:** How does looking at the pattern help you find the general term of any sequence?

Answer (this is the conceptual understanding): Recognizing patterns helps you to find the general term or algebraic expression to describe them.

TOK

Who would you call the founder of algebra?

Answer: Aryabhata is sometimes considered the "father of algebra" - compare with al-Khwarizmi.

Teachers will see an element of risk taking, reasoning and ethical decision making here. Whilst there may be no "right" answer it is a chance for students to present their decisions and understand the perspectives of others.

TOK

Is mathematics a language?

Answer: You will see the use of several alphabets in mathematical notation (e.g., the use of capital sigma for the sum). One point of view is that mathematics is not only a language but is the only language shared by humans around the world. For example, pi is 3.14159… regardless of what culture, language, nationality or religion you have.

A counterclaim might be whether or not we can communicate our ideas without the use of another spoken tongue.

Developing inquiry skills

Let's return to the chapter opening problem about the Koch snowflake.

i How many sides does the initial triangle have?

Answer: 3

ii How many sides does the second iteration have? What about the third iteration?

Answer: 12, 48

iii What kind of sequence do these numbers form?

Answer: Geometric, with an r value of 4.

iv Write the general term for the number of sides in any iteration.

Answer: $u_n = 3(4)^{n-1}$

v You can use your general term to find the number of sides for a given iteration by substituting the value for n. Find the number of sides the 12th iteration will have.

Answer: $u_n = 3(4)^{12-1} = 3(4)^{11} = 3(4194304) = 12582912$

1.2 Arithmetic and geometric sequences

Investigation 3

1 Completed table:

Number of tables	1	2	3	4	5	6	7	8
Number of seats	4	6	8	10	12	14	16	18

2 Two new seats are added.

3 The general term is $u_n = 2(n+1)$ or $u_n = 2n+2$.

4 $u_{20} = 2(20)+2 = 42$

TOK

Is all knowledge concerned with identification and use of patterns?

Answer: Consider Fibonacci numbers and connections with the Golden ratio.

An opportunity to use a TOK mantra "how do we know what we know?"

Questions might include:

To what extent do ways of knowing prevent us from deluding ourselves?

"Is a pattern only useful if it simplifies things?"

"What is the role of an anomaly in discovery?"

What does it take to know something?

Is it enough for knowledge to be shared by your teacher or do you need to discover it for yourself?

Investigation 4

Conceptual understanding: A converging or diverging sequence can be distinguished by the value of the common ratio.

1 3, 6, 12, 24, 48, 96

2 –4, 8, –16, 32, –64, 128

3 100, 50, 25, 12.5, 6.25, 3.125

4 –729, 243, –81, 27, –9, 3

5 10 000, 1000, 100, 10, 1, 0.1

6 1, 3, 9, 27, 81, 243

7 **Factual:** Sort the sequences into two sets: convergent or divergent.

Answer: Diverging: 1, 2 and 6. Converging: 3, 4 and 5

8 **Conceptual:** What do you notice about the value of r in each of the sets?

Answer: When a sequence is diverging, $r < -1$ or $r > 1$. When a sequence is converging, $-1 < r < 1,\ r \neq 0$.

9 **Conceptual:** How can you tell whether a sequence is converging or diverging?

Answer (this is the conceptual understanding): A converging or diverging sequence can be distinguished by the value of the common ratio.

Investigation 5

Conceptual understanding:

The common difference or common ratio helps to distinguish between different arithmetic and geometric sequences and series.

1 Geometric: a, c and f; Arithmetic: b, d and g; Neither: e and h

2 **a** $u_n = u_n + (n-1)d$

$u_n = 20 + (n-1)(-4)$

$u_n = 20 - 4n + 4$

$u_n = -4n + 24$

b $u_n = u_n r^{n-1}$

$u_n = 20\left(\frac{1}{2}\right)^{n-1}$

c $u_n = u_n r^{n-1}$

$u_n = -4(3)^{n-1}$

d $u_n = u_n + (n-1)d$

$u_n = -16 + (n-1)(-2)$

$u_n = -16 - 2n + 2$

$u_n = -2n - 14$

e not possible

f $u_n = u_n r^{n-1}$

$u_n = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$

$u_n = \left(\frac{1}{2}\right)^{n} = \frac{1}{2^n}$

g $u_n = u_n + (n-1)d$

$u_n = 2 + (n-1)(2)$

$u_n = 2 + 2n - 2$

$u_n = 2n$

h not possible

3 **Factual:** What is a common difference? What is a common ratio?

Answer: A common difference is what is used to create an arithmetic sequence. A common ratio is used to create a geometric sequence.

4 **Conceptual:** How can you determine if a sequence is geometric, arithmetic or neither?

Answer: You can determine if a sequence is arithmetic, geometric, or neither by examining the pattern from term to term. If you are adding the same value each time (positive or negative), it is an arithmetic sequence. If you are multiplying by the same number each time, it is a geometric sequence.

5 **Conceptual:** How do the common difference or common ratio help you describe and generalize a sequence?

Answer (this is the conceptual understanding): The common difference or common ratio helps to distinguish between different arithmetic and geometric sequences and series.

1.3 Arithmetic and geometric series

Investigation 6

1 Arithmetic

2 $u_n = u_1 + (n-1)d$

$u_n = 1 + (n-1)(1)$

$u_n = 1 + n - 1$

$u_n = n$

3 Completed table:

Number of rows	**Total number of pennies**
1	1
2	3
3	6
4	10
5	15

4 10 rows = $1+2+3+..+10=55$

20 rows = $1+2+3+..+20=210$

n rows = $1+2+3+..+n$

The proof of $S_n = u_1\left(\frac{1-r^n}{1-r}\right)$ is beyond the scope of the SL syllabus, but is included here included here for your reference and can be shared with students if desired.

The proof: $S_n = u_1 + u_1r + u_1r^2 + \ldots + u_1r^{n-2} + u_1r^{n-1}$

Factorize the common factor of u_1: $S_n = u_1\left(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}\right)$

Rearrange the terms: $S_n = u_1\left(r^{n-1} + r^{n-2} + \ldots + r^2 + r + 1\right)$

The expression in parenthesis is a polynomial of the form $x^{n-1} + \ldots + x^2 + x + 1$, which is the resultant from the quotient $\frac{x^n - 1}{x - 1}$, therefore: $S_n = u_1\left(\frac{r^n - 1}{r - 1}\right)$

Multiplying the right-hand side by $1 = \frac{-1}{-1}$: $S_n = u_1\left(\frac{r^n - 1}{1 - 2}\right)\left(\frac{-1}{-1}\right)$

$S_n = u_1\left(\frac{1-r^n}{1-r}\right)$ *QED*

TOK

How is intuition used in mathematics?

Answer: Gauss' method for adding up integers from 1 to 100. You might want to look at inductive and deductive methods of proof.

Is there a body of knowledge called intuitive mathematics? If so, how do these intuitions hinder or facilitate problem solving?

Try questions such as:

How many lines pass through any given two points?

A coin is tossed 12 times. The first 11 all come up heads. What would you expect the next toss to give? Why?

Which set has more members, the set of rational numbers or the set of irrational numbers?

Investigation 7

Conceptual understanding:

An infinite series may/can converge to a finite sum when the terms in the series form a converging sequence, while an infinite divergent series expands.

Part 1:

1 Completed table:

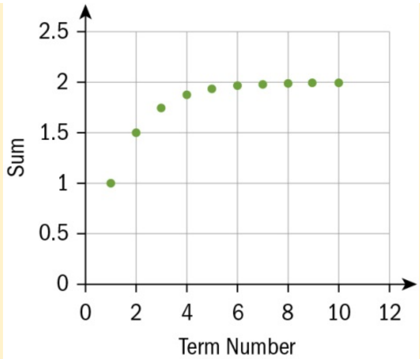
Term Number	Term	Sum
1	1	1
2	0.5	1.5
3	0.25	1.75
4	0.125	1.875
5	0.0625	1.9375
6	0.03125	1.96875
7	0.015625	1.984375
8	0.0078125	1.9921875
9	0.00390625	1.99609375
10	0.00195313	1.99804688
…	…	…

2 The value of the terms gets increasingly smaller.

3 The series is converging.

4 It will have a smaller and smaller effect on the sum.

5 The sum is approaching 2.

6 

7 Wherever the graphs start to level off, or become a horizontal line, is the sum of the sequence. Here, the graph levels off at 2.

Part 2:

1 a Completed table:

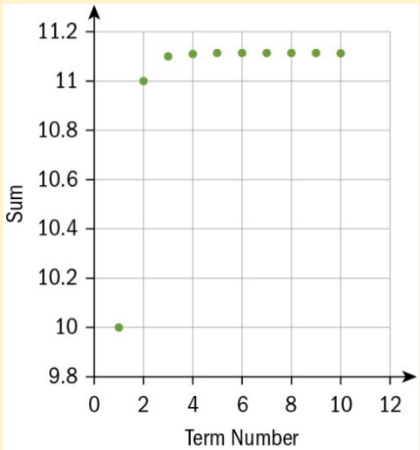
Term Number	**Term**	**Sum**
1	10	10
2	1	11
3	0.1	11.1
4	0.01	11.11
5	0.001	11.111
6	0.0001	11.1111
7	0.00001	11.11111
8	0.000001	11.111111
9	0.0000001	11.1111111
10	0.00000001	11.11111111

ii The value of the terms gets increasingly smaller.

iii The series is converging.

iv It will have a smaller and smaller effect on the sum.

v $11.\overline{1}$

vi 

vii $11.\overline{1}$

1 b i Completed table:

Term Number	Term	Sum
1	-3	-3
2	-6	-9
3	-12	-21
4	-24	-45
5	-48	-93
6	-96	-189
7	-192	-381
8	-384	-765
9	-768	-1533
10	-1536	-3069

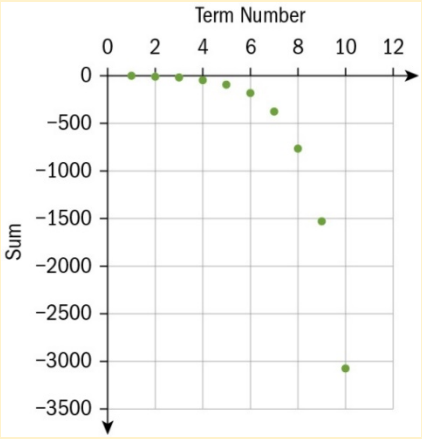
ii Increases negatively.

iii Diverging

iv It increases the sum by a larger and larger negative amount.

v $-\infty$

vi



vii $-\infty$

c i Completed table:

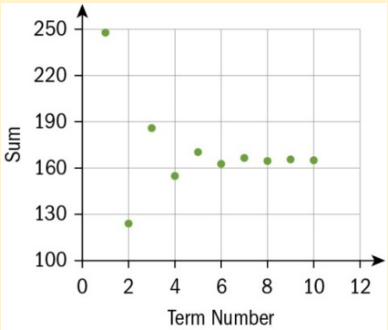
Term Number	Term	Sum
1	248	248
2	−124	124
3	62	186
4	−31	155
5	15.5	170.5
6	−7.75	162.75
7	3.875	166.625
8	−1.9375	164.6875
9	0.96875	165.65625
10	−0.484375	165.171875

ii The value of each term gets smaller and smaller.

iii Converging

iv It will have a smaller and smaller impact on the sum.

v 165

vi 

vii 165

1 d i Completed table:

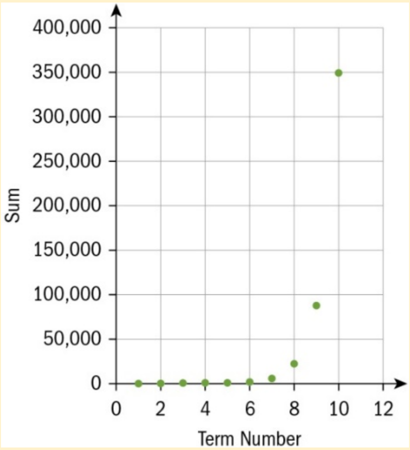
Term Number	Term	Sum
1	1	1
2	4	5
3	16	21
4	64	85
5	256	341
6	1024	1365
7	4096	5461
8	16384	21845
9	65536	87381
10	262144	349525

ii Increases negatively.

iii Diverging

iv It increases the sum by a larger and larger negative amount.

v ∞

vi 

vii ∞

2 **Conceptual:** When does an infinite geometric sequence have a finite sum?

Answer: An infinite geometric sequence has a finite sum only when it is converging.

3 **a** $10, 15, 20, 25, 30, 35$

b $10, 8, 6, 4, 2, 0$

c $10, 9.5, 9, 8.5, 8, 7.5$

d $10, 10.4, 10.8, 11.2, 11.6, 12$

4 **Conceptual:** Which of the arithmetic sequences are decreasing? Which are increasing?

Answer: Decreasing: $10, 8, 6, 4, 2, 0$ and $10, 9.5, 9, 8.5, 8, 7.5$

Increasing: $10, 15, 20, 25, 30, 35$ and $10, 10.4, 10.8, 11.2, 11.6, 12$

5 **Conceptual:** Does an infinite arithmetic sequence converge to a finite sum?

Answer: No

6 **Conceptual:** When does an infinite sequence have a finite sum?

Answer (this is the conceptual understanding): An infinite series may/can converge to a finite sum when the terms in the series form a converging sequence, while an infinite divergent series expands.

TOK

Is it possible to know things about which we can have no experience, such as infinity?

Answer: Consider the debate over the validity of the notion of "infinity".

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Answer: Consider for instance that a finite area can be bounded by an infinite perimeter.

Developing inquiry skills

Returning to the chapter opening investigation about the Koch snowflake, the enclosed area can be found using the sum of an infinite series.

In the second iteration, since the sides of the new triangles are $\frac{1}{3}$ the length of the sides of the original triangle, their areas must be $\left(\frac{1}{3}\right)^2 - \left(\frac{1}{9}\right)$ of its area.

If the area of the original triangle is 1 square unit, then the total area of the three new triangles is $3\left(\frac{1}{9}\right)$.

i Find the total area for the third and fourth iterations.

Answer: The total area for the third iteration is $12\left(\frac{1}{9}\right)^2$.

The total area for the fourth iteration is $48\left(\frac{1}{9}\right)^2$.

ii How can you use what you have learned in this section to find the total area of the Koch snowflake?

Answer: This makes the series: $1+3\left(\frac{1}{9}\right)+12\left(\frac{1}{9}\right)^2+48\left(\frac{1}{9}\right)^2+\ldots$

Since this is a converging geometric series with $r=\frac{1}{9}$,

$$S_\infty = 1+\frac{u_1}{1-r}$$

$$S_\infty = 1+\frac{\frac{1}{3}}{1-\frac{4}{9}}$$

$$S_\infty = 1+\frac{\frac{1}{3}}{\frac{5}{9}}$$

$$S_\infty = 1+\frac{3}{5}$$

$$S_\infty = \frac{8}{5}$$

iii How does the area of a Koch snowflake relate to the area of the initial triangle?

Answer: So, no matter the size of the initial triangle, the total area of the Koch snowflake is $\frac{8}{5}$ its area.

1.4 Applications of arithmetic and geometric patterns

Investigation 8

Conceptual understandings:

Simple interest can be modelled by an arithmetic series.

Compound interest can be modelled by a geometric series.

Growth can follow different patterns and we can use these patterns to predict and compare outcomes.

Part 1:

1 Amount of interest paid per month: $I=1000\left(\frac{0.05}{12}\right)=\$4.1\overline{6}\approx\$4.17$

Note: Most financial institutions round down, but we will follow normal IB rounding rules for all examples and questions.

2 \$1004.17, \$1008.33, \$1012.50

3 The sequence is arithmetic.

4 $u_n = u_1 + (n-1)d$

$u_n = 1004.17 + (n-1)(4.17)$

$u_n = 1004.17 + 4.17n - 4.17$

$u_n = 1000 + 4.17n$

5 $u_{24} = 1000 + 4.17(24) = \$1100.08$

Part 2:

1 Amount of interest paid per month $: I = 1000\left(\frac{0.05}{12}\right) = \$4.1\overline{6} \approx \$4.17$

2 $\$1000 + 4.17 = \$1004.17$

3 After the second month: $I = 1004.17\left(\frac{0.05}{12}\right) = \$4.18401\ldots \approx \$4.18$

$\$1004.17 + 4.18 = \$1008.35404\ldots \approx \$1008.35$

After the third month: $I = 1008.35\left(\frac{0.05}{12}\right) = \$4.201458\ldots\ldots \approx \$4.20$

$\$1008.35 + 4.20 = \$1012.551458\ldots\ldots \approx \$1012.55$

4 The sequence is geometric.

5 $u_n = u_1 r^{n-1}$

$u_n = 1000\left(1 + \frac{0.05}{12}\right)^{n-1}$

6 $u_{24} = 1000\left(1 + \frac{0.05}{12}\right)^{24-1} = 1000\left(1.0041\overline{6}\right)^{23} = 1100.356516\ldots \approx \$1100.36$

7 The investment with compound interest is worth more.

8 **Conceptual:** Which type of series models simple interest? Which type models compound interest?

Answer (this is the conceptual understanding): Simple interest can be modelled by an arithmetic series. Compound interest can be modelled by a geometric series.

9 **Conceptual:** How are outcomes and growth patterns related?
Answer (this is the conceptual understanding): Growth can follow different patterns and we can use these patterns to predict and compare outcomes.

TOK

Do all societies view investment and interest in the same way? What is your stance?

Answer: Students could research the reason as to why we charge interest on a loan and compare this with the perspectives in other societies such as where money in Islam is not regarded as an asset from which it is ethically permissible to earn a direct return. The Qur'an (2:279) sees interest as inequitable, as implied by the word "zulm" in Arabic which translates as oppression, exploitation, and the opposite of justice. There is no real loaning in Islam since lenders achieve ownership in the estates that they finance.

This allows students to view the perspectives of other societies and decide to what extent they agree with the charging of interest.

Investigation 9

Conceptual understandings:

Approximating a growth rate using a geometric sequence allows us to model real-life growth.

Growth can follow different patterns and we can use these patterns to predict and compare outcomes.

Part 1:

1 $\frac{35.535}{35.152} = 1.0108955\ldots \approx 1.0109$

$\frac{35.832}{35.535} = 1.0083579\ldots \approx 1.0084$

$\frac{36.264}{35.832} = 1.0120562\ldots \approx 1.0121$

$\frac{36.708}{36.264} = 1.0122435\ldots \approx 1.0122$

2 Average $= \frac{1.0109 + 1.0084 + 1.0121 + 1.0122}{4} = \frac{4.0436}{4} = 1.0109$

3 $u_n = u_1 r^{n-1}$

$u_n = (35.152)(1.0109)^{n-1}$, where n is the number of years after 2013.

4 $n = 2025 - 2013 = 12$

$u_{12} = (35.152)(1.0109)^{12-1} = (35.152)(1.0109)^{11} = 39.60410\ldots \approx 39.6041$

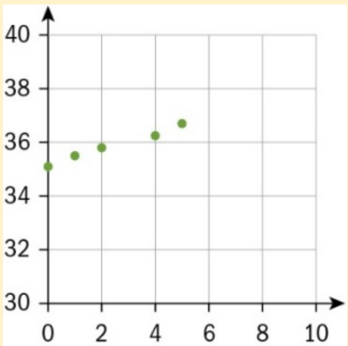
The population will be 39.6041 million people.

This value is only theoretical. Factors such as disease, natural disaster, medical advancement, etc. could affect the actual population.

Part 2:

The purpose of this part is to introduce the idea of modelling to the students.

1 & 2



3 Discuss how modelling is not perfect, but gives us an idea of the future.

4 **Conceptual:** How can you fit a model to real-life growth data?

Answer: Calculate the rate of change for each x value and find the average. Use the average rate to form a general term/formula.

(This is the conceptual understanding): Approximating a growth rate using a geometric sequence allows us to model real-life growth.

5 **Conceptual:** How can using sequences to model growth help us make predictions and comparisons?

> **Answer (this is the conceptual understanding):** Growth can follow different patterns and we can use these patterns to predict and compare outcomes.

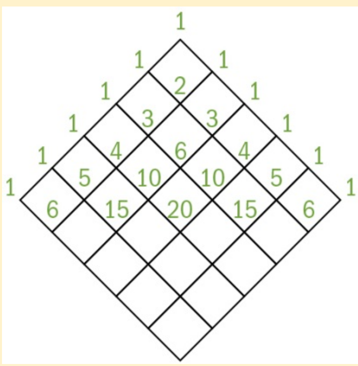
Extension:

Students could find data for a different country or region and model in the same way as above. This could be turned into a mini-exploration.

1.5 The binomial theorem

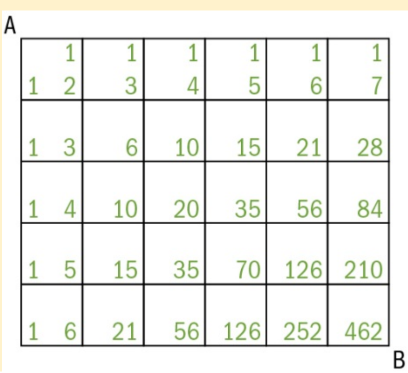
Investigation 10

3 Students should link the patterns to Pascal's triangle.



4 The rows can be completed since the grid has been turned 45° as each diagonal corresponds to the rows in Pascal's triangle.

5 There are 462 possible routes.



TOK

Why do we call this Pascal's triangle when it was in use before Pascal was born?

Are mathematical theories merely the collective opinions of different mathematicians, or do such theories give us genuine knowledge of the real world?

> **Answer:** Blaise Pascal is credited with Pascal's Triangle after he wrote about in a treatise called "The Arithmetic triangle", but the properties of "Pascal's Triangle" have been known in a number of different cultures long before Pascal. (e.g the Chinese mathematician
>
> Yang Hui, the Indian mathematician Pangala and Persian poet and mathematician Omar Khayyam).

Investigation 11

Part 1:

1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

2 Each term is the sum of the two terms directly above it.

3 Each row is symmetric.

Part 2:

1 The sums are 1, 2, 4, 8, 16, ….

2 These sums form a geometric sequence with $r = 2$.

3 The general term would be $u_n = 2^{n-1}$.

4 The sum of the 16th row would be $2^{16-1} = 2^{15} = 32768$.

TOK

How many different tickets are possible in a lottery? What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

Answer: You might want to use stimuli to start a discussion that might result in a class debate or a blog post such as:

Are lotteries marketed to people who don't have an emergency fund, are low on finances and are bad at math?

Is it ethical to sell hope if the odds are 1 in 45 million?

You are probably not going to win, but winning the lottery is not the point, it is the thrill.

Does the thrill generate and addiction to gambling?

Is gambling a tax on the less intelligent?

Investigation 12

Conceptual understanding:

The *n*th row in Pascal's triangle represents the number of ways of choosing 0, 1, 2, 3, … items from $n-1$ items.

Part 1:

1 **a** One way to choose none **b** A, B or C

c AB, AC, or BC **d** ABC

2 0 3 3 0

3 0 boxes: one way to choose none

1 box: A, B, C, or D

2 box: AB, AC, AD, BC, BD, CD

3 box: ABC, ABD, ACD, BDC

4 boxes: ABCD

1 4 6 4 1

4 Look back at the last investigation on Pascal's triangle.

5 Picking from three boxes corresponds to the 4th row.

Picking from four boxes corresponds to the 5th row.

Picking 0 boxes corresponds to the 1stterm in the row.

Picking 1 box corresponds to 2nd term in the row.

Picking 2 box corresponds to 3rd term in the row.

etc.

6 **(This is the conceptual understanding):** The *n*th row in Pascal's triangle represents the number of ways of choosing 0, 1, 2, 3, … items from $n-1$ items.

7

1	${}_0C_0$
1 1	${}_1C_0$ ${}_1C_1$
1 2 1	${}_2C_0$ ${}_2C_1$ ${}_2C_2$
1 3 3 1	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
1 4 6 4 1	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
1 5 10 10 5 1	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$

8 **Conceptual:** How would you use Pascal's triangle to explain why ${}_nC_r = {}_nC_{n-r}$?

Answer: ${}_nC_r = {}_nC_{n-r}$ due to the symmetric nature of the Pascal's triangle. For example, ${}_7C_4 = {}_7C_3$, ${}_9C_1 = {}_9C_8$, ${}_{12}C_5 = {}_{12}C_7$.

Investigation 13

Conceptual understanding:

The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.

1 **a** The number of terms is one more than the value of the exponent.

b They begin at n and decrease by one until 0 is reached.

c They begin at 0 and increase by one until n is reached.

d Each sum is equal to n.

e They are equal to the $n+1$ row in Pascal's triangle.

2 The coefficients will be the 7th row of Pascal's triangle or 1, 6, 15, 20, 15, 6, 1.

Therefore, the expansion must be: $1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1x^0y^6$

which simplifies to: $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

3 **Conceptual:** How can you explain the coefficients of terms in the binomial expansion using Pascal's triangle?

Answer: In ${}_nC_r$, n = row number – 1, r = term number – 1.

For example, to calculate ${}_5C_3$, you would go to the 6th row and use the 4th term, which is 10. To calculate ${}_3C_1$, you would go to the 4th row and use the 2nd term, which is 3.

4 **Conceptual:** How does the binomial theorem use combinations?

Answer (this is the conceptual understanding): The binomial theorem uses combinations to calculate the coefficients in the expansion and these coefficients display symmetry about the centre.

1.6 Proofs

TOK

Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting." -E. Mach

To what extent do you agree with this quote?

Answer: You might want to view the extraordinary links between Pascal's Triangle and the coefficients of polynomials and as if this is this just a coincidence.

Is the nature of mathematics more profound than we realise?

Investigation 14

Conceptual understanding:

Proofs using algebra can prove a statement for all possible number values, allowing for generalization.

1 $3(1-2)-4(3(1)+5)+2(1)-2=-7(1+4)$

$3(-1)-4(8)+2(1)-2=-7(5)$

$-3-32+2-2=-35$

$-35=-35$

2 $3(-2-2)-4(3(-2)+5)+2(-2)-2=-7(-2+4)$

$3(-4)-4(-1)+2(-2)-2=-7(2)$

$-12+4-4-2=-14$

$-14=-14$

3 **Factual:** If the proof holds true for two values of x, does this mean it will hold true for all values of x?

Answer: It does not mean it is true for all values, just the specific values that were tested.

4 Completed table:

LHS	**RHS**
$3(x-2)-4(3x+5)+2x-2$ Using distribution: $3x-6-12x-20+2x-2$ Collecting like terms: $-7x-28$ Now it is fully simplified, but we need it to look like the RHS. We can factorize the common factor of −7: $-7(x+4)$	$-7(x+4)$

5 **Conceptual:** Explain why using algebra is a better method than using substitution.

Answer (this is the conceptual understanding): Proofs using algebra can prove a statement for all possible number values, allowing for generalization.

TOK

What is the role of the mathematical community in determining the validity of a mathematical proof?

Answer: Knowledge claims in mathematics: Do proofs provide us with completely certain knowledge?

Can we talk about universal truth in mathematics?

Nature of mathematics and science: What is the difference between the Inductive method in Science and proof by induction in mathematics?

Developing inquiry skills

Return to the opening problem. How has your understanding of the Koch snowflake changed as you have worked through this chapter? What features of, for example, the ninth iteration can you now work out from what you have learned?

Answer: Discuss conclusions as a class.

Modelling and Investigation Activity: The Towers of Hanoi

Approaches to Learning: Thinking Skills, Communicating, Research

Exploration Criteria: Mathematical Communication (B); Personal Engagement (C); Use of Mathematics (E)

IB Topic: Sequences

Introduction

The Towers of Hanoi problem is a challenging ancient puzzle that prompts students to engage in problem solving. Students should understand that struggling with a problem, and possibly having to rethink their approach, is the nature of mathematics and a normal part of learning. Persevering with a challenging problem will grow new connections in the brain and, over time, makes difficult tasks easier.

Preparing for an exploration—for the teacher and student—should start from chapter 1. This chapter, on sequences and series, proof and binomial expansions, is a great base for explorations. The earlier the IA and the IA criteria are introduced the better, as this will encourage students to start to think of ideas and to make connections. The Towers of Hanoi problem is a "classic" mathematics problem. This could be an issue, as students could simply use what is already available online and in books. With this in mind, the problem is approached here by encouraging Personal Engagement with the problem (Criterion C) rather than seeking a solution online. Students should consider the possible approaches to answer a challenging mathematical problem like this. They are also required to consider how they communicate mathematically (Criterion B) and then are asked to think about possible extensions and to research other avenues of exploration (Criterion E: Use of Mathematics).

The Problem

The Towers of Hanoi problem, also called Tower of Hanoi or Towers of Brahma, involves three vertical pegs and a set of different sized disks with holes through their centres. The problem is widely believed to have been invented in 1883 by the French mathematician, Édouard Lucas (though his role in its invention has been disputed). Ever popular, the Towers of Hanoi, made of wood or plastic, can be found in toy shops around the world.

Lucas apparently spread the legend that helped popularize the game by including a written account in each of the toy boxes sold of the Brahmin monks moving 64 golden disks between 3 poles for many centuries with the legend saying that when they completed the puzzle the world would end! The legend varies over time and place, being set either in a temple or a monastery in Vietnam or India. In some versions of the legend the monks are only allowed to make one move per day.

The history of the problem is interesting, as it gives context and often a rationale for studying the problem.

To encourage students to engage with the problem, you could ask:

What is the history and legend behind the problem?

What is the significance of the 64 disks to the legend?

Why research the history of the problem?

Explore the Problem

The first thing for students to do is to have a "play" with the problem. From this, since 64 disks makes the problem too time consuming, one possible approach is to start with smaller numbers of disks and build up a sequence so as to develop a formula. This can lead to either finding a recursive formula, an explicit formula or a graphical solution.

Suggestions of simulations that could be used are:

Mathsisfun

Webgamesonline

Haubergs

If online simulations are not available, then a physical representation of the problem could be used.

When $n = 3$ the minimum number of moves required is 7.

When $n = 4$ the minimum number of moves required is 15.

To encourage discussion, you could ask:

How do you know these are minimum values?

This could be established through multiple students finding the same answer.

As part of the exploration, you could ask students to think about what representations (diagram, table or other form of representation) they could use to display the individual moves needed to solve the problem. They could then try to represent the moves made using their chosen method.

Possible methods of representation involve diagrams of the different moves, a table that represents the moves of individual disks, a graph theory approach, etc.

Try to Test a Rule

If the solution is arithmetic, then you could use the result for n = 4 and the common difference to find the result for n = 5.

The common difference between the number of moves for n = 3 and n = 4 is 15 − 7 = 8, so the common difference between n = 4 and n = 5 would also be 8.

The number of moves for n = 5 would be 15 + 8 = 23.

Using a simulator, the minimum number of moves when n = 5 is 31, not 23.

The solution does not follow an arithmetic sequence.

Find More Results

For n = 1 the minimum number of moves is 1.

For n = 2 the minimum number of moves is 3.

To put the results into context, you could encourage students to talk about the method being used to solve the problem.

The method is to move the disks so that all but the largest disk have been assembled in their correct order on peg B. The largest disk is then moved to peg C from peg A and then the remaining disks are assembled on top of this piece using the same number of moves as before.

n (the number of disks)	M_n (the minimum number of moves needed for n disks)
1	1
2	3
3	7
4	15
5	31

Some students may be able to spot the formula ($M_n = 2^n - 1$) from these data. Others may require more work.

As an **extension**, you could ask students to use a graphing package to graph the data from their table, with the number of disks, n, on the horizontal axis and the minimum number of moves, M, on the vertical axis.

You could ask:

How could you use the graph to find a formula?

Students may be able to spot the type of expression that could be used to fit through the points on the graph. However, this is a quadratic graph, which students have not yet covered in this course.

Try a Formula

To give students further guidance, you could ask:

What must happen before the largest disk can be moved to peg C?

Before the largest disk can be moved to peg C, the other disks need to be assembled in order on peg B.

It would take a minimum of seven moves to get the three pieces on peg B as shown.

It would then take one move to move the largest disk from peg A to peg C.

As the pieces need to reassemble as they are it would take another seven moves to move the three smaller disks to peg C.

Therefore, the total number of moves is 7 + 1 + 7 = 15.

This method will set up the recursive formula of the solutions:

$M_{n+1} = 2 \times M_n + 1$

where M_n is the minimum number of moves needed for n disks.

Note: Recursive formulae are **not** on the SL syllabus. However, they could provide interesting **extension** work.

Make sure that students carefully consider the notation in the formula and that the variables in their formula are well defined.

This is a recursive formula. It uses the minimum number of moves needed to solve an n disk puzzle to find the minimum number of moves needed for an (n + 1) disk puzzle.

To check that the formula works, students could try to solve n = 6 and check the result against the formula.

The problem with a recursive formula is that you need to have solved all previous iterations of the problem in order to solve the next one.

Try Another Formula: The Relationship Is Not Geometric Because There Is Not a Common Ratio Between Terms.

Some students may spot the relationship to 2^n. Others may not. Adding 1 to each term might help students to recognize the relationship:

2, 4, 8, 16, 32

The formula could be written as $M_n = 2^n - 1$

To emphasize the difference between a recursive formula and an explicit formula, you could ask:

How does an explicit formula differ from a recursive formula?

The explicit formula does not require you to know the previous terms. It is possible just to substitute in the value of n.

For n = 64, the minimum number of moves needed is:

$M_n = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$.

Even if the monks take one second per move this would take more than 584.9 billion years, which is longer than the history of our known universe (approximately 13.8 billion years).

Extension

The suggested extension activities look at different versions of the Towers of Hanoi problem, as well as exploring recursive formulae.

There are many classic puzzles like this involving sequences and series, and these can offer starting points for explorations. However, with this and other classic problems, it is important that students do not simply regurgitate what is already available, but that instead they engage with the problem. You could also consider extensions and additional research on top of the regular problem.

You could ask students what other "classic problems" in mathematics they know, and ask them to explore these problems.

2 Representing relationships: introducing functions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Functions represent mappings that assign to each value of the independent variable (input) one and only one dependent variable (output).

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
Creating different representations of functions, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.	Investigation 2
Different forms give you a different visualization of the function and the real-life situation.	Investigation 2
The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.	Investigation 3
Functional notation supports the different ways that real-life situations can be modelled using input and output variables (or independent and dependent variables).	Investigation 4
The domain includes all the possible input values for a function, and the range includes all the actual output values.	Investigation 5
Composition of functions may be formed using different simpler functions to illustrate the order or stages of a real-life problem.	Investigation 7
All composite functions may be decomposed to different, simpler functions and this allows real-life problems to be broken down by stages.	Investigation 8
The domain of the original function maps to the range of the inverse. The range of the original function maps to the domain of the inverse.	Investigation 9
The identity function maps each input to itself.	Investigation 10

Syllabus sections covered in this chapter:

- SL2.2*
- SL2.3*
- SL2.5

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 63: Coordinate geometry, Substitution into formulae, Solving linear equations, Graphing linear functions, Drawing a quadratic graph	Page 86: Example 13 Page 91: Example 17 Page 97: Example 21	N/A	Pages 68, 73, 76, 84, 92, 99

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 99	Page 101	N/A

2.1 What is a function?

Investigation 1

1 Students will give various answers.

2 The relations in the second column have no repeating x-values, whereas the relations in the first column do.

3 **Conceptual:** What is a function?

Answer: A function represents a mapping or relation such that each input (x) has only one output (y).

4 **Conceptual:** How would you describe a function using mappings?

Answer: Using a mapping diagram.

5 Students will give varying examples.

6 **Factual:** What are functions used for?

Answer: Functions can be used to model, predict and interpret relationships between variables.

Investigation 2

Conceptual understandings:

Creating different representations of functions, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

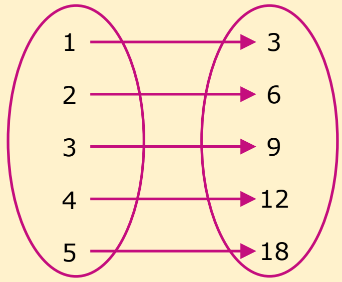
Different forms give you a different visualization of the function and the real-life situation.

1 **i** $\{(1,3),(2,6),(3,9),(4,12),(5,15),(6,18)\}$

ii Completed table:

x	1	2	3	4	5	6
y	3	6	9	12	15	18

iii



2 **i** You can tell if a set of ordered pairs is a function by looking for repeating y-values.

ii You can tell if a table of values is a function by looking for repeating y-values.

iii You can tell if a mapping diagram is a function by looking two arrows coming from one x-value.

3 It is difficult to express this function as a set or ordered pairs because the number of values is infinite.

4 $y = x^2$

5 **Factual:** What are the different ways to represent a function?

Answer: Words, set of ordered pairs, table of values, mapping diagram, equation (and graph, which will be introduced in the next investigation)

6 **Conceptual:** Hypothesize why there are so many ways to represent a relation or function.

Answer (this is the conceptual understanding): Creating different representations of functions, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

7 **Conceptual:** How do different forms of a function help you understand a real-life problem?

Answer (this is the conceptual understanding): Different forms give you a different visualization of the function and the real-life situation.

Investigation 3

Conceptual understanding:

The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.

1 It is not a function because an x-value has two corresponding y-values.

2 If the graph cuts the vertical line more than once, it is not a function.

3 **Conceptual:** How does the vertical line test help to identify functions?

Answer (this is the conceptual understanding): The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.

Developing inquiry skills

Let's return to the chapter opener about economics and supply and demand.

Look at the graph. Is this relation a function? Explain how you know. What does this mean in terms of this real-life situation?

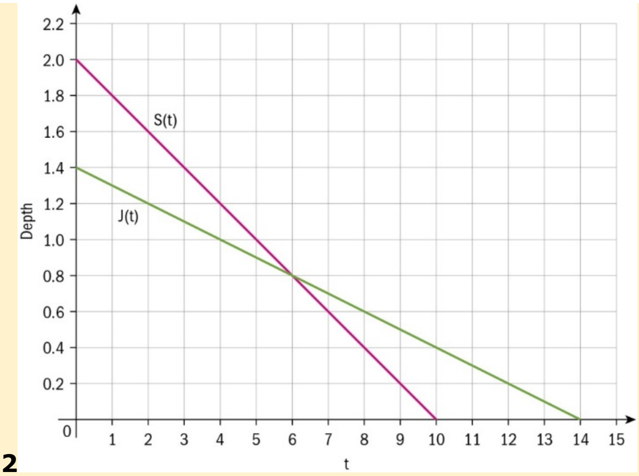
Answer: This relation is a function because it passes the vertical line test. This means that at each price, only one quantity will be demanded by consumers.

2.2 Functional notation

Investigation 4

Conceptual understanding:

Functional notation supports the different ways that real-life situations can be modelled using input and output variables (or independent and dependent variables).

1 & 2 

3 Some possible answers:

- Sophia's pool drains faster.

- Jacob's pool drains slower.

- The water levels are different.

- At around 6 hours, the water levels are the same.

The graphs are functions, because each value for time maps to a value for the depth of the pool.

4 $J(t) = -0.1t + 1.4$

$S(t) = \frac{-1}{5}t + 2$

5 **i** The time at which the two depths are equal.

ii The depth of Jacob's pool at 2 hours.

iii The depth of Sophia's pool at 8 hours.

iv The time at which the depth of Jacob's pool is 1 m.

v The time at which the depth of Sophia's pool greater than the depth of Jacob's pool.

6 **Conceptual:** What is the purpose of having different notation for different functions?

Answer (this is the conceptual understanding): Functional notation supports the different ways that real-life situations can be modelled using input and output variables (or independent and dependent variables).

TOK

What is the relationship between real-world problems and mathematical models?

Answer: You might want to look at the difference between a proof and a model.

Is a model personal knowledge?

Models often solve real world problems with mathematics as opposed to just problems in pure mathematics.

A good place to look at the connection between interpolation and extrapolation in terms of reliability.

Developing inquiry skills

Let's return to the chapter opener about economics and supply and demand. The equation of the line in the graph is $y = -10 + 2x$.

Rewrite this in functional notation and define the variables you choose.

Answer: One possible answer is $Q(P) = -10 + 2P$, where P represents price and Q represents the quantity supplied.

International-mindedness

The development of functions bridged many countries including Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland).

The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries, you can ask students "how did the notation we use today become internationally accepted?"

2.3 Drawing graphs of functions

TOK

Does a graph without labels have meaning?

Answer: What can you perceive from a graph with a labelled horizontal and/or vertical axis?

What about an unscaled axis?

Is it by intuition that we seek to have both axes the same scale?

How accurate is a visual representation of a mathematical concept?

Developing inquiry skills

Let's return to our opening problem about supply and demand.

The equation we generated in section 2.2 for supply was $Q(P) = -10 + 2P$, where P represents price and Q represents the quantity supplied by producers.

If the equation for the quantity of a product demanded by consumers is $D(P) = 30 - 3P$, use your GDC to sketch both graphs on the same set of axes and find the intersection point.

Explain what this intersection point means in terms of supply and demand.

Answer: Intersection point: (8,6)

This intersection point represents that at a price of \$8, 6 units of the product are both supplied and demanded. There will be no leftover products and no consumer who wanted one but didn't get one.

Note: A "sketch" of a graph need not include as much detail and information as a "drawing".

2.4 The domain and range of a function

Investigation 5

Conceptual understanding:

The domain includes all the possible input values for a function, and the range includes all the actual output values.

1 The independent variable is the number of chirps and the dependent is the temperature in degrees Celsius.

2 $D(c) = \frac{c}{3} + 4$, where D represents degrees Celsius and c is the number of cricket chirps in 25 seconds.

3

Number of chirps	−42	−12	−3	0	18	48	78
Temperature (°C)	−10°C	0°C	3°C	4°C	10°C	20°C	30°C

4 None of the negative chirp values makes sense.

5 The function will not work for temperatures less than 4°C.

6 Answers will vary.

7 Should follow from their answer above.

8 **Factual:** What do "domain" and "range" mean?

Answer (this is the conceptual understanding): The domain includes all the possible input values for a function, and the range includes all the actual output values.

9 **Conceptual:** How do the domain and range vary depending on the problem being modelled?

Answer: Most functions in real life will have some sort of restrictions on either variable.

TOK

Around the world you will off encounter different words for the same object, like trapezium and trapezoid or root and surd.

Sometimes more than one type of symbol might have the same meaning such as interval and set notation.

To what extent does the language we use shape the way we think?

Answer: Is it OK to only use one type of mathematical and/or national term?

Whose job is it to impart knowledge and understanding?

How has technology influenced the notation that we use?

You might want to research how many words that Eskimos have for snow.

How do you think that this use of language affects understanding of space, time, colours, and objects?

Investigation 6

1 -5

2 ∞

3 $[-5,\infty[$ or $[-5,\infty)$ $x \geq -5$

4 $-\infty$

5 5

6 $]-\infty,5]$ or $(-\infty,5\}$ $y \leq 5$

7 A graph helps you determine the domain and range because you can visually see the largest and smallest values for each variable.

Developing inquiry skills

Let's return to the chapter opener about supply and demand.

While $D(P)$ is a linear function, the domain is not.

$x \in \mathbb{R}$ because it is a real-life situation. What would the lower and upper limit be for the domain? What about the range?

Answer: The lower limit for the domain would be 0 because price cannot be negative. The upper limit would be a reasonable amount people would pay for this particular item.

The lower limit for the range would also be 0 because the quantity demanded also cannot be negative. The upper limit would be a reasonable number of items that could be produced.

2.5 Composite functions

TOK

The object of mathematical rigour is to sanction and legitimize the conquests of intuition - Jacques Hadamard

Do you think that studying the graph of a function contains the same level of mathematical rigour as studying the function algebraically?

Answer: A good opportunity to debate, in teams or pairs. first, define mathematical rigour.

Opening questions such as what can analysis offer us that graphing a function cannot?

What are the strengths of a function?

Investigation 7

Conceptual understanding:

Composition of functions may be formed using different simpler functions to illustrate the order or stages of a real-life problem.

1 **a** $13\ 500 - 1000 = \$12\ 500$ **b** $12\ 500 \times 0.05 = \$625$

2 $24\ 300 = 1000 = 23\ 300$

$23\ 300 \times 0.05 = 1165$

3 **a** $f(x) = x - 1000$ **b** $g(x) = f(x) \times 0.05$

4 $h(x) = (x - 1000) \times 0.05$

5 **Factual:** How do you write a composite function given two (or more) functions?

Answer: You insert the equation for the first step into the equation for the second step, and so on.

6 **Conceptual:** How can composition of functions help to represent the stages or order of a real-life problem?

Answer (this is the conceptual understanding): Composition of functions may be formed using different simpler functions to illustrate the order or stages of a real-life problem.

7 **Conceptual:** Does the order you perform the functions matter? Explain your answer.

Answer: Yes, the order matters. If you calculate the 5% commission first, then subtract \$1000, you will not get the same answer.

8 **a** $h(17\ 000) = (17\ 000 - 1000) \times 0.05 = \$800$

b $h(9550) = (9550 - 1000) \times 0.05 = \$427.50$

c $h(950) = (950 - 1000) \times 0.05 = -\$2.50$. It is impossible to earn a negative amount of money. Since Harper's sales were under the \$1000 minimum, she would receive no commission.

9 **Factual:** How do you calculate a value from a given composite function?

Answer: You substitute the value for the variable and follow order of operations.

Investigation 8

Conceptual understanding:

All composite functions may be decomposed to different, simpler functions and this allows real-life problems to be broken down by stages.

1 $g(x) = x - 1$

2 **Factual:** Explain how you can decompose a composite function into different, smaller component functions.

Answer: Set the inside function equal to $g(x)$. Replace that function with x to find $f(x)$.

3 **a** $g(x) = x - 5$, $f(x) = \sqrt{x}$

b $g(a) = a + 2$, $f(a) = a^2 - 3a$

c $g(x) = 2x - 2$, $f(x) = \dfrac{1}{x}$

d $g(x) = x - 4$, $f(x) = 4x - 5$

4 There could be three different functions:

$g(x) = x - 2$, $h(x) = x^4$, $j(x) = \sqrt{x}$

This makes the composition $f(x) = j(h(g(x)))$.

5 **(This is the conceptual understanding):** All composite functions may be decomposed to different, simpler functions and this allows real-life problems to be broken down by stages.

6 **Conceptual:** Hypothesize why one might need to decompose a composite function.

Answer: Decomposing functions allows real-life problems to be broken down by stages.

TOK

Is mathematics independent of culture?

Answer: Some believe that mathematics is its own language and needs no reference to applications.

Would mathematics exist without context?

If so, would this make mathematics an art form?

The culture of mathematics has been derived from many different cultures over millennia and has been transferred between different nationalities and religions.

Is mathematics a culture in itself?

2.6 Inverse functions

Investigation 9

Conceptual understanding:

The domain of the original function maps to the range of the inverse. The range of the original function maps to the domain of the inverse.

Part 1:

1 Each pair of functions consists of "opposites". The function in group 2 undoes what the function in group 1 does.

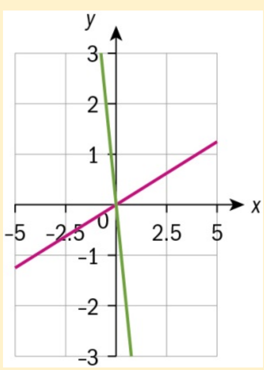
2 **Factual:** What is an inverse function? What does an inverse function map? What does the inverse map? Think of input (independent) and output (dependent).

Answer: An inverse function reverses a function. An inverse function maps the output (y) to its input (x).

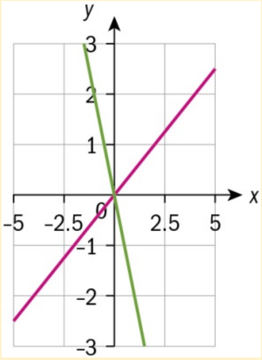
3 Exchange x with y and isolate y.

Part 2:

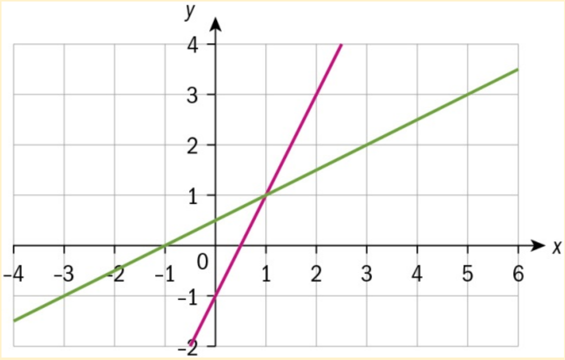
1 $y = -4x$ and $y = \frac{1}{4}x$



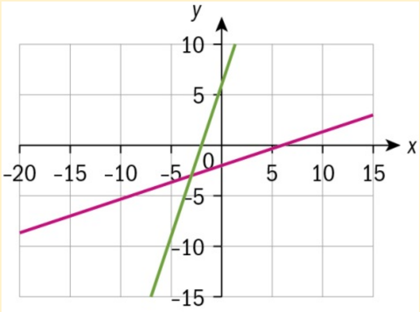
$y = \frac{1}{2}x$ and $y = -2x$



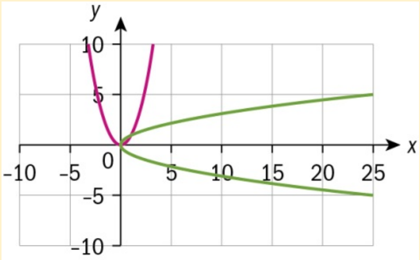
$y = 2x - 1$ and $y = \frac{x+1}{2}$



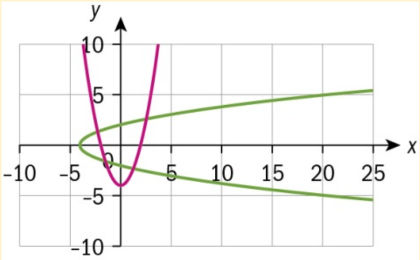
$y = \frac{1}{3}x - 2$ and $y = 3(x + 2)$



$y = x^2$ and $y = \pm\sqrt{x}$



$y = x^2 - 4$ and $y = \pm\sqrt{x+4}$



2 A function and its inverse are reflections over the line $y = x$.

3 **Conceptual:** How does a graph help you to determine whether two functions are inverses?

Answer: For each corresponding point on the function and its inverse, the x and y coordinates are switched. You can see this when looking at the functions graphically.

4

Group 1	Group 2
$y = x^2$ Domain: $x \in \mathbb{R}$ Range: $y \geq 0$	$y = \pm\sqrt{x}$ Domain: $x \geq 0$ Range: $y \in \mathbb{R}$
$y = x^2 - 4$ Domain: $x \in \mathbb{R}$ Range: $y \geq -4$	$y = \pm\sqrt{x+4}$ Domain: $x \geq -4$ Range: $y \in \mathbb{R}$

5 **Conceptual:** How are the domains and ranges of two inverse functions related?

Answer (this is the conceptual understanding): The domain of the original function maps to the range of the inverse. The range of the original function maps to the domain of the inverse.

6 The horizontal line test tells you if the inverse of a function is itself a function.

TOK

Do you think that mathematics is just the manipulation of symbols under a set of rules?

Answer: Questions that can be used to stimulate a discussion or a blog post about the nature of mathematics include -

What is mathematics?

Is it all shared knowledge?

How can you develop personal knowledge in mathematics?

Do you just use formulas to reason and evaluate?

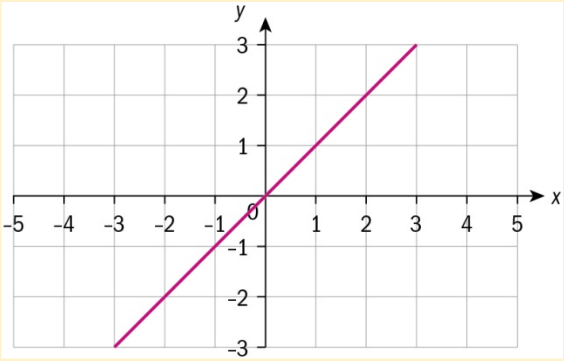
Does faith ever have a place in mathematics?

Investigation 10

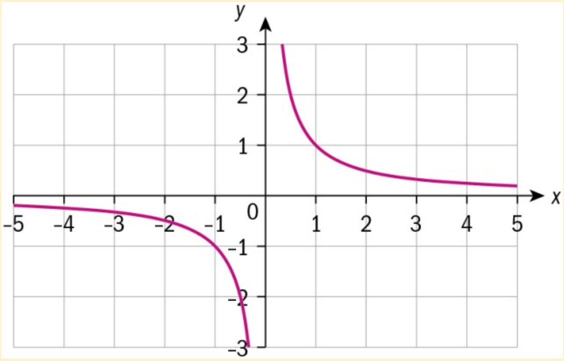
Conceptual understanding:

The identity function maps each input to itself.

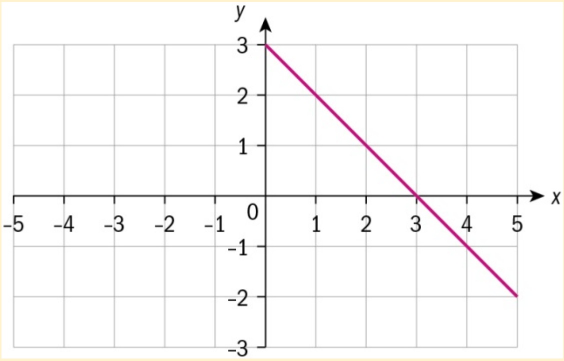
1 a



b



c



2 All of the inverses are the same as their original function.

3 a A self-inverse is any function where $f(x) = f^{-1}(x)$.

b $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ or, since $f(x) = f^{-1}(x)$, then $f\left(f(x)\right) = x$.

4 **Conceptual:** How does a graph help you identify a self-inverse function?

Answer: By drawing a graph, you can easily see that the reflection of $f(x)$ over the line $y = x$ will result in the same function.

5 The identify function is a line of all points where the x- and y-coordinates are identical.

6 **(This is the conceptual understanding):** The identity function maps each input to itself.

7 **Factual:** What do you think the identity line is?

Answer: The identity line is the function $y = x$.

International-mindedness

Which do you think is superior; the Bourbaki group analytical approach or the Mandelbrot visual approach to mathematics?

Answer: Nicolas Bourbaki was a pseudonym chosen by eight or nine young mathematicians in the 1930s in France. Their original aim was to write a rigorous textbook in analysis.

The ideas Benoît Mandelbrot, who died in 2018, gave a new, visual way to think about the world. The formal techniques of theorems and proofs didn't appeal to him. His approach to mathematics was highly visual and highly intuitive.

Allow students to research each and write a response.

Developing inquiry skills

Return to the opening problem. How is what you have learned in this chapter useful in real-life situations such as this? Why is understanding the behaviour of functions essential to successfully modelling real-life situations?

Answer: Discuss conclusions as a class.

Graphs of Functions: Describing the 'What' and Researching the 'Why'

Approaches to Learning: Thinking Skills, Communicating, Research

Exploration Criteria: Presentation (A); Mathematical Communication (B); Personal Engagement (C)

IB Topic: Graphs, Functions, Domain

Introduction

In this chapter, students have been looking at a large number of functions, their tables and their graphs. They should know that many real-life situations can be represented by an algebraic function, a table of values and also a graph of that function. A graph can provide a better visual representation of a data set than a function or table might. For example, on a graph it is easy to identify starting points, endpoints, rises, falls, maxima and minima, zeros, asymptotes, etc. However, it is important that students also appreciate that relying on a graph alone has the potential for error in some cases due to inaccuracies.

The purpose of this task is to consider graphs of real-life situations and to look at how to provide a verbal description of the graphs and, perhaps more importantly, to then research why the graph may have this particular shape. This is relevant to internal assessments, as students often describe the graph of data they have found without offering a critical explanation of why the graph may be as it is. More practice is then offered at the end.

Bulgaria Total Population

The table of data values for the graph is taken from www.gapminder.org/data/ (a fantastic source of world data).

To assist students when writing their paragraph ask, for example:

What exactly does the graph show?

This is a graph of the population of Bulgaria and how it has changed over time.

What do the axes show and what are the units?

The axes show the year and the population in millions.

What is the domain of the graph? What does this represent?

1800 to 2020.

Over what domain is the population rising? To what extent is it rising? When is its peak?

The population rises gently between 1800 and 1870. It then rises more rapidly and peaks at around 1989.

Over what domain is the population falling?

The population falls from 1989 onwards.

Add some mathematical details to these last statements—for example:

By how much did the population rise? By what percentage?

The population rises gently between 1800 and 1870 from a value of approximately 2 million to approximately 3 million (a rise of 50%). It then rises more rapidly between 1870 to peak at around 1989. Between these years the population rises from approximately 3 million to approximately 9 million—a 200% rise. It then begins to fall steadily from approximately 9 million to approximately 7 million—this is a fall of just over 20%.

Students should think carefully about the vocabulary that they use in their paragraph.

Here is some potentially useful vocabulary:

increased, grew, rose, went up, climbed
fell, decreased, declined, dropped
remained the same, remained constant, stabilized, levelled off
dramatically, significantly, considerably, enormously, rapidly, quickly, substantially, sharply, markedly, greatly, strongly, heavily, steeply
steadily, slightly, fractionally, gently, continuously, progressively, marginally

The interesting points and regions might be, for example, where the graph starts and ends; where the graph has a vertex (turning point); where the graph is rising (increasing) and falling (decreasing).

There is a point of inflexion at around 1870. There is a maximum at 1989. The population is rising before 1989 and falling after 1989.

Students might not know the above vocabulary yet. Encourage them to express their ideas clearly and concisely using vocabulary they do know. You could introduce new vocabulary if appropriate.

Possible reasons for rise: Low death rate/high birth rate, immigration

Possible reasons for decline: Emigration, high death rate/low birth rate, war/illness, natural disasters

Using Sources

Students should look for articles published in scholarly journals or websites and sources that require that certain standards or criteria are met before publication.

Students should compare several opinions by scholars and experts on the topic rather than accepting the first opinion they find.

This is an ideal opportunity to address the Academic Honesty policy of the school and the referencing system that will be expected in any assignments in mathematics.

Remind students that it is important to keep a record of any sources they use. In an Internal Assessment, these would be needed for academic honesty, citations and bibliography.

An important skill for students to develop is the ability to evaluate the reliability of any sources they use.

Encourage students to find websites/articles, etc. and to justify why they are reliable.

Here are some popular reasons for Bulgaria’s population decline:

Most of the population moved from the villages to the cities, where it is harder to raise children because the usual living space in the cities is not very large—a typical apartment is 40–80 square metres and has space for up to two children, or three if the second and third are twins. People usually own their apartments, so it is hard to move when the child arrives. This means that the quality of life decreases with each new family member.

All forms of child care—kindergartens, social security—are not well developed and do not help parents.

Pre-1989, the borders were closed and people were not allowed to leave.

The economy forced people to leave to seek employment and a better life.

Young people pursued further education abroad. Few of them came back.

The age at which people have their first child is increasing, due to cultural changes. Many women prefer to finish their education and build a career first.

For an extension, students could also try to find a function that best fits these data in order to make predictions of future values. This will be done in future chapters and tasks in this book.

Students can predict what will happen to the population of Bulgaria in the future, but should be aware of the dangers of extrapolation. This is covered later in the book.

Worldwide Wii Console Sales

Again, students could use the questions in the box at the start of this spread as guidance if needed.

Encourage students to Include some mathematical detail in their descriptions:

The domain is from 2006 to 2016.

The graph is increasing between 2006 and 2008.

The graph is then decreasing between 2008 and 2016.

Students should initially think of some possible reasons for the changes and trends and then research the actual reasons.

You could ask students:

Why do you think this is the domain of the data?

What are some interesting years and periods of time that it might be interesting to investigate?

The Wii was launched in September 2006 and was one of the biggest selling game consoles for a short time before declining from 2008. In 2016, Nintendo stopped producing and selling the Wii.

This type of sales shape is not unusual for console sales (and other technology)—there are similar patterns of peaking in the second/third year and then declining for PS2 and GameCube for example. Console hardware becomes outdated and potential buyers already own the devices.

The peak coincides with the iPhone release and “the death of TV-based devices” perhaps.

Global Mean Temperature Anomaly

Note: moving averages and linear trend lines are not in the current syllabus.

However, this is still a good topic for practising research and could also be the type of presentation technique that they could encounter or use in their Internal Assessment. This type of presentation is suitable for demonstrating and helping to illustrate and explain trends, and the mathematics involved is at a level commensurate with the SL course.

A moving average is a technique used to get an overview of the trends in a data set. It is an average of any subset of numbers.

A linear trend line is the straight line of best fit. It generally shows that something is increasing or decreasing at a steady rate.

Students could consider CO_2 emissions, volcanic eruptions, El Niño events, etc.

Here is an interesting article giving a possible reason for the drop in the 1940s:

www.newscientist.com/article/dn14006-buckets-to-blame-for-wartime-temperature-blip

Again, encourage students to think about the reliability of any sources they use.

You could ask:

Is this a reliable source?

Students may wish to compare these data with other data that can be found on the internet and try to explain any possible differences—different collection methods, slightly different data being collected, biases, etc.

Extension

This extension encourages students to do a number of things:

First, they need to look for an appropriate graph—appropriate here means interesting and relevant.

They also have to describe and explain the shape of the graph for themselves by researching.

Finally, by writing a series of questions they are thinking about the communication of the ideas behind the reason. By thinking about the questions they need to ask, they are also thinking about what areas they need to explain for themselves.

An example of the questions students could ask:

What explanations can you give for the sudden rise/fall in the data between … and …?

3 Modelling relationships: linear and quadratic functions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Linear relationships can be displayed by tables, graphs and various forms of equations and represent real-life situations with direct proportion and constant rates of change.	Investigation 1
The parameters of linear equations determine key features of the graph of the line.	Investigation 2
The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola.	Investigation 3
Reflecting the parent quadratic function in the vertical axis does not change the set of points that have been graphed for any horizontal stretch or compression.	Investigation 4
The parameters of a quadratic function in vertex form represent the transformation of the parent function and identify concavity, the equation of the axis of symmetry and the coordinates of the vertex.	Investigations 5 & 6
Different forms of the quadratic function allow easier identification of different key features of its graph, and can help with graph sketching or finding an equation of a function from its graph.	Investigation 6
Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the ‘missing’ area that would make a square.	Investigation 7
Factorization provides a method for finding the roots of some quadratic equations. Completing the square provides a method for finding the roots of any quadratic equation.	Investigation 7

Conceptual understandings (cont.)	Investigation
Completing the square for the general form of the quadratic equation extends results from a specific equation to the general quadratic equation and results in a formula (the quadratic formula) that can be used to find the roots of any quadratic equation.	Investigation 8
The quadratic formula contains the square root of the discriminant, and so its value determines the number and nature of the roots and the number of x-intercepts of the graph of the quadratic.	Investigation 9
Technology can be used to solve quadratic equations and inequalities.	Investigation 10
Quadratic relationships can be represented by tables, graphs and various forms of equations and can be used to model real world relationships such as projectile motion, kinematics, problems involving area, and maximizing income or minimizing cost.	Investigation 11
A certain form of a quadratic function may be a more suitable to model certain real-life problems. For example, when using a quadratic to find the maximum or minimum value of a real-life problem, the vertex form may be most suitable.	Investigation 11

Syllabus sections covered in this chapter:

- SL2.1*
- SL2.4*
- SL2.6
- SL2.7
- SL2.10
- SL2.11
- SL4.4*

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 109: Solving linear equations, Expanding brackets and factorizing, Factorizing quadratic expressions, Factorizing the difference of two squares	Page 135: Example 17 Page 146: Example 22 Page 156: Example 28 Page 163: Example 32 Page 175: Example 40	Page 124: Example 11 Page 142: Example 20 Page 158: Example 30	Pages 115, 129, 140, 153, 164, 172, 176

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 178	Page 179	N/A

3.1 Rate of change and gradient

Investigation 1

Conceptual understanding:

Linear relationships can be displayed by tables, graphs and various forms of equations and represent real-life situations with direct proportion and constant rates of change.

1 Days 2 and 3: $\frac{90.75-60.50}{3-2}=30.25$

Days 3 and 4: $\frac{121.00-90.75}{4-3}=30.25$

Days 4 and 5: $\frac{151.25-121.00}{5-4}=30.25$

2 The cost of renting a car increases 30.25 euros for each additional day.

3 Table 2: Each rate of change is $\frac{9}{5}$ or $\frac{1.8}{1}$.

Table 3: The rates of change are $\frac{20}{1}$, $\frac{5}{1}$, $-\frac{5}{1}$, and $-\frac{25}{1}$.

4 Table 2: For each increase in temperature of 9 degrees Fahrenheit, there will be an increase of 5 degrees Celsius.

Table 3: Each rate of change shows the change in height in metres over the given time interval in seconds.

5 Table 1:

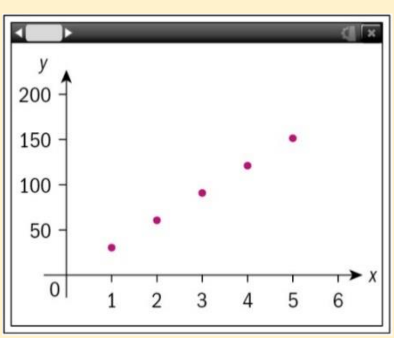


Table 2:

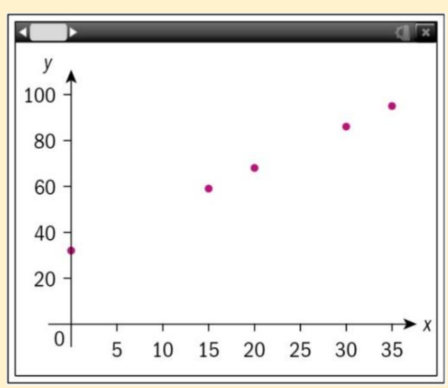
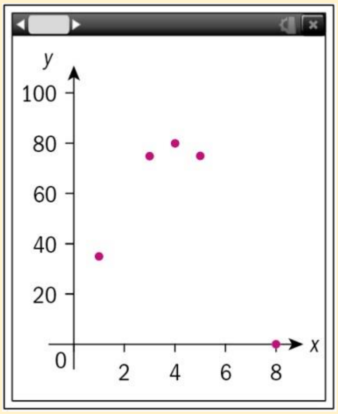


Table 3:



6 Possible observations:

The graph of the points in table 1 and in table 2 lie on a straight line.

The graph of the points in table 3 lie on a curve.

7 **Factual:** If you are given a graph which shows the relationship between two variables, how can you tell from the shape of the graph whether the rate of change between the variables is constant?

Answer: If the points of a graph lie on a straight line, the data has a constant rate of change.

8 **Factual:** If you are given a table of values which shows the relationship between two variables, how can you tell whether the rate of change between the variables is constant?

Answer: To tell from a table of values whether a rate of change is constant, you need to check to see if the ratio of $\frac{\Delta y}{\Delta x}$ is constant for any pair of points.

9 **Conceptual:** Why does a linear graph represent a constant rate of change?

Answer: Some students may say because each time x changes by a given amount, y also changes by a constant amount, so it makes a straight line.

10 The number of days and cost in euros having a linear relationship means the plotted data lies on a straight line and the rate of change is constant.

11 **Conceptual:** What is the relationship between real-life variables that have a constant rate of change, and how can this relationship be represented?

Answer (this is the conceptual understanding): Linear relationships can be displayed by tables, graphs and various forms of equations and represent real life situations with direct proportion and constant rates of change.

Reflect: What are some other real-life examples that show how the gradient of a line can be interpreted as a **rate** of change?

Answer: Distance traveled vs. time, when traveling at a constant rate.

Income vs. number of T-shirts sold, when T-shirts are sold at a fixed price.

Gallons of fuel remaining vs. distance traveled.

3.2 Linear functions

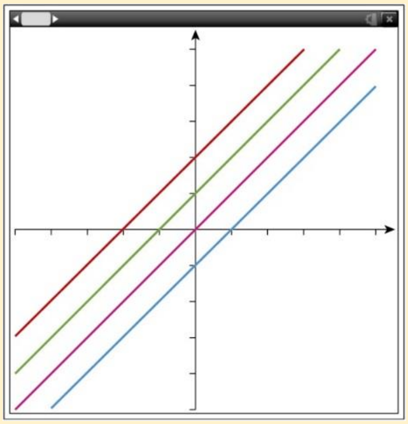
Investigation 2

Conceptual understanding:

The parameters of linear equations determine key features of the graph of the line.

1 **a** The gradient is 1.

b The coordinates of the y-intercept are (0,0).

2 

3 **Factual** Compare and contrast the graph of $y = x$ with the graphs of each of the other equations.

Answer: All the graphs have the same steepness or gradient. The lines have different y-intercepts.

(Some students may recognize that the graphs are vertical translations of the parent graph.)

4 **Conceptual** What feature of the graph is defined by the constant c?

Answer: The constant c determines the y-intercept of the lines.

5 **a, b** All the graphs are lines with a y-intercept of 0. The steepness or gradient of the lines vary.

c **Factual** What feature of the graph is defined by the constant m?

Answer: The constant m determines the gradient of the lines.

6 **a** The graph is a line with y-intercept -3 and gradient $\frac{1}{2}$.

b **Conceptual:** What are parameters?

Answer: Parameters are constants or letters that represent constants in a given function. The parameters determine key features of the graph of the function.

The variables in the equation $y = mx + c$ are x and y. The parameters are m and c.

c **Factual:** Which letters are generally used to represent parameters?

Answer: The letters a, b, c and m are often used as parameters.

d **Conceptual:** What do the parameters in the equation $y = mx + c$ tell you about the features of the graph of the line?

Answer: The parameter m is the gradient of the line and the parameter c is the y-intercept.

(This is the conceptual understanding): The parameters of linear equations determine key features of the graph of the line.

Reflect: Can you explain why this form of a straight line is called the gradient-intercept form?

Answer: The parameters of the equation tell you the gradient and intercept.

TOK

Descartes showed that geometric problems could be solved algebraically and vice versa.

What does this tell us about mathematical representation and mathematical knowledge?

Answer: Questions to start a discussion in class might include:

Is there a difference in accuracy between geometrical and algebraic representation?

Which forms of representation stay in your memory for longer? Formulas, diagrams, colours? Why?

Reflect: How could you verify that both answers shown for part **b** are equations of the same line?

Answer: You could rewrite each equation into gradient-intercept form and see that the equations are identical.

Reflect: What are the key features of the graph of a line?

Answer: The key features of the graph of a line the x-intercept, the y-intercept and the gradient.

Reflect: What are three different forms for the equation of a line?

Answer: The three different forms of the equation for a line are the gradient-intercept form, gradient-intercept form and general form.

Reflect: What is the relationship between the parameters in each of the forms of the equation of a line and key features of the graph of a line?

Answer: In the gradient-intercept form $y = mx + c$, the gradient is m and the y-intercept is $(0, c)$. In the gradient-intercept form $y - y_1 = m(x - x_1)$, the line passes through the point (x_1, y_1) and has gradient m. In the general form of the equation of a line is $ax + by + d = 0$, the x-intercept of the graph is $\left(-\frac{d}{a}, 0\right)$ and the y-intercept is $\left(0, -\frac{d}{b}\right)$.

Reflect: Which forms of the equations of a line have parameters giving a point on the line and the gradient?

Answer: The parameters of the gradient-intercept form give the gradient m and the point (x_1, y_1).

Reflect: Why are the gradients of a linear function and its inverse reciprocals?

Answer: If (x_1, y_1) and (x_2, y_2) are two points on a linear function the gradient is $\frac{y_2 - y_1}{x_2 - x_1}$. The inverse function would pass through the points (y_1, x_1) and (y_2, x_2) and have gradient $\frac{x_2 - x_1}{y_2 - y_1}$. To find the inverse you interchange x and y, so the gradient changes from a change in y over a change in x to a change in x over a change in y.

International-mindedness

The term "function" was introduced by the German mathematician Gottfried Wilhelm Leibniz in the 17th century. and the $f(x)$ notation was coined by Swiss Leonard Euler in the 18th century.

This international mindedness box and the next two, lead up to the TOK question on page 87.

Developing inquiry skills

In the opening scenario for this chapter you looked at how crates of emergency supplies were dropped from a plane.

The function $g(t) = -5t + 670$ gives the height of the crate when the parachute is open as a function of the number of seconds after the crate leaves the plane.

You were asked to explain what -5, the coefficient of t, represents. Do you agree with the answer you gave? If not, what is your answer now?

Answer: −5 is the gradient of the line which represents motion with the parachute open. The crate falls at a constant speed of 5 metres per second with the parachute open.

3.3 Transformations of functions

Investigation 3

Conceptual understandings:

The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola.

1 **a** Horizontal translation right 3 units

b Horizontal translation left 4 units

c Horizontal translation right h units for $h>0$ and left $|h|$ units for $h<0$.

2 **a** Vertical translation up 1 unit

b Vertical translation down 2 units

c Vertical translation up k units for $k>0$ and down $|k|$ units for $k<0$.

3 **a** Reflection in the x-axis

b Vertical stretch with scale factor 2

c Vertical compression with scale factor $\frac{1}{3}$

d Reflection in the x-axis and vertical stretch with scale factor 3

e Vertical stretch with scale factor $|a|$. Is $a<0$ there is also a reflection in the x-axis.

4 **a** Horizontal translation left 4; vertical translation down 2

b Vertical stretch with scale factor 2; horizontal translation right 3

c Reflection in the x-axis; vertical compression with scale factor $\frac{1}{2}$; vertical translation up 4

d Vertical stretch with scale factor 3; horizontal translation right 2; vertical translation down 4

e Vertical stretch with scale factor a; horizontal translation right h; vertical translation up k

5 **Factual:** What effect do a, h and k have in transforming the graph of $y = x^2$ to the graph of $y = a(x-h)^2 + k$?

Answer: The students' answers may be in their own words, but should convey the following information.

When $a < 0$ there is a reflection in the x-axis.

When $|a| > 1$ there is a vertical stretch with scale factor $|a|$.

When $|a| < 1$ there is a vertical compression with scale factor $|a|$.

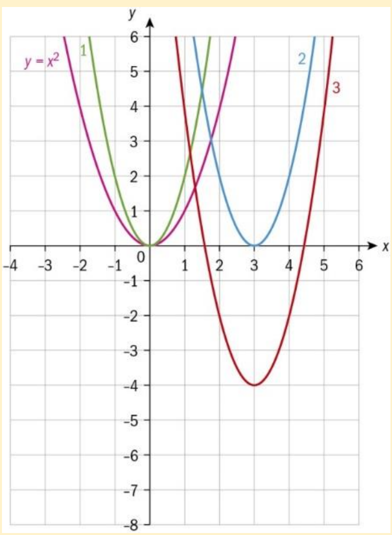
When $h > 0$ there is a horizontal translation right h.

When $h < 0$ there is a horizontal translation left $|h|$.

When $k > 0$ there is a vertical translation up k.

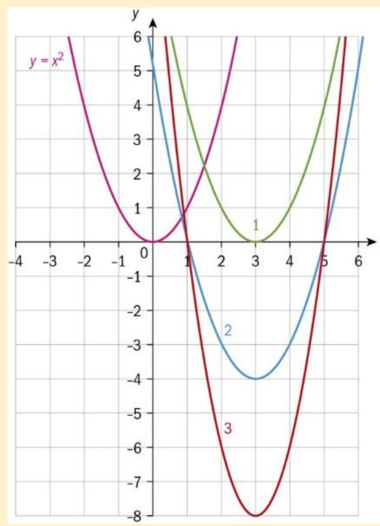
When $k < 0$ there is vertical translation down $|k|$.

6 **a**



$y = 2(x-3)^2 - 4$

b



$y = 2\left[(x-3)^2 - 4\right] = 2(x-3)^2 - 8$

7 **Conceptual:** Does the order in which the transformations of the graph of $y = x^2$ to the graph of $f(x) = a(x-h)^2 + k$ are completed matter? Explain why (or why not) this is the case.

Answer: Yes, it matters. The graphs in question 6 show that you do not get the same result when you perform the transformations in a different order. Students may not be able to articulate the reason until they have had practice with some of the exercises. The result in the first graph in question 6 is the graph of $y = 2(x-3)^2 - 4$. In the second graph, when the vertical stretch is applied last, the graph is of $y = 2\left[(x-3)^2 - 4\right]$, which is the same as $y = 2(x-3)^2 - 8$.

When graphing $y = a(x-h)^2 + k$ using order of operations, you would perform the horizontal shift h, the vertical stretch a, and the vertical shift d. You will get the same graph if you perform vertical dilations first and then follow with translations. Reflections in the x-axis and vertical dilations must be performed before vertical translations.

8 **Conceptual:** Do you think that the graph of any quadratic function can be obtained through transformations of the graph of $y = x^2$?

Answer: Yes, the correct shape of the graph can be obtained by the vertical stretches or compressions. The correct concavity can be obtained with a reflection in the x-axis. The correct position can be obtained by vertical and horizontal translations.

(This is the conceptual understanding): The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola.

Investigation 4

Conceptual understanding:

Reflecting the parent quadratic function in the vertical axis does not change the set of points that have been graphed for any horizontal stretch or compression.

1 Since the graph of $f(x)=(x)^2$ is symmetric about the y-axis, when you reflect the graph in the y-axis you get the same set of points. (Students might also say that both functions are equal to $y=x^2$.)

2 **a** $\frac{1}{4}$ **b** 4

3 **a** 2 **b** $\frac{1}{2}$

4 **Conceptual:** Why it is unnecessary to use reflections in the y-axis or horizontal stretches and compressions when obtaining graphs of quadratics through transformations of the graph of $f(x)=x^2$?

Answer: Reflecting the graph of $f(x)=x^2$ in the y-axis does not change the set of points graphed. For any horizontal stretch and compression of the graph of $f(x)=x^2$, there is a vertical stretch or compression that results in the same set of points.

(This is the conceptual understanding): Reflecting the parent quadratic function in the vertical axis does not change the set of points that have been graphed for any horizontal stretch or compression.

5 **a** Reflection in the y-axis **b** Reflection in the y-axis

6 A reflection in the y-axis changes the graph of $y=f(x)$ to the graph of $y=f(-x)$.

7 **a** No **b** Yes

Reflect: What is the relationship between the graphs of $y=f(x)$ and $y=f(-x)$?

Answer: Reflection in y-axis

Reflect: The graph of $y=f(qx)$ is a horizontal stretch or compression of the graph of $y=f(x)$. For which values of q is the transformation a stretch rather than a compression?

Answer: For q such that $0<|q|<1$.

Developing inquiry skills

In the opening scenario for this chapter you looked at how crates of emergency supplies were dropped from a plane.

The function $h(t)=-4.9t^2+720$ gives the height of the crate during free fall.

How could you transform the parent graph $h(t)=t^2$ to give the function $h(t)=-4.9t^2+720$?

What do these transformations tell you about the motion of the crate in this context?

Answer: Reflection in the t-axis, followed by a stretch in the y-direction scale factor 4.9, and then a translation of 720 upwards.

This tells you that the crate was dropped from a plane which was 720 metres high. The curve is concave down, so it accelerates towards the ground.

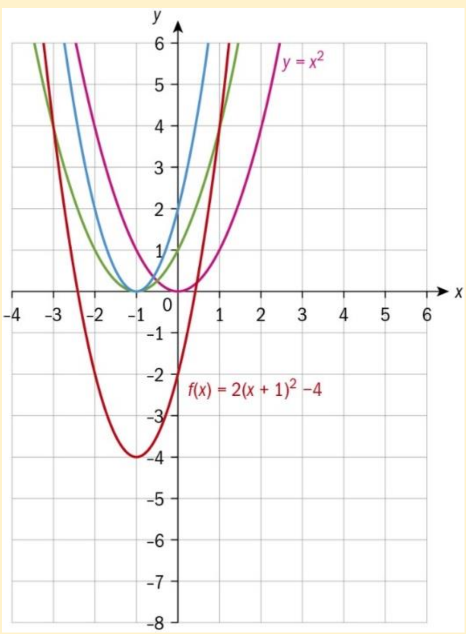
3.4 Graphing quadratic functions

Investigation 5

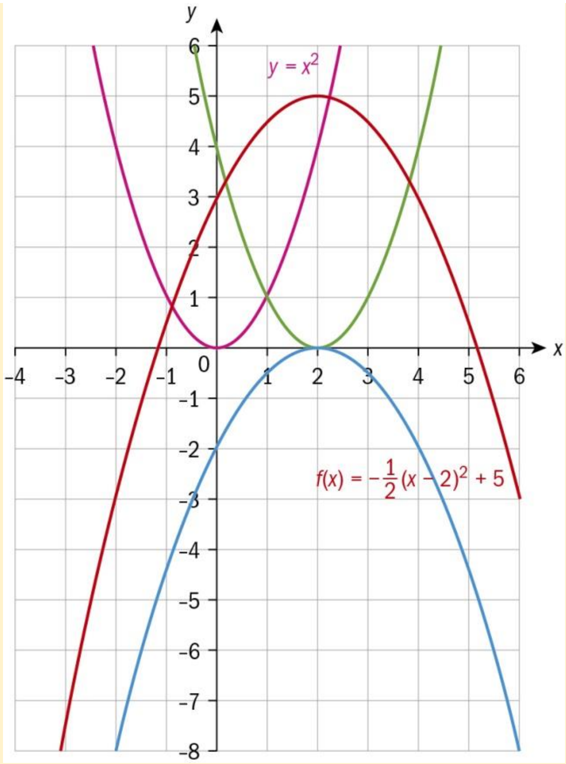
Conceptual understanding:

The parameters of a quadratic function in vertex form represent the transformation of the parent function and identify concavity, the equation of the axis of symmetry and the coordinates of the vertex.

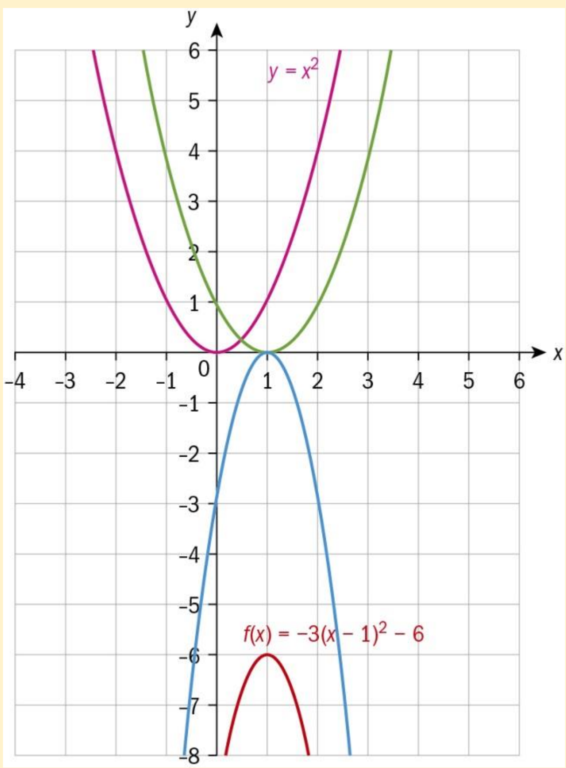
1 a



b



c



2 Completed table:

$f(x)=a(x-h)^2+k$	***a***	***h***	***k***	**Up/down**	**Axis of symmetry**	**Vertex**
a $f(x)=2(x+1)^2-4$	2	-1	-4	up	$x=-1$	$(-1,-4)$
b $f(x)=-\frac{1}{2}(x-2)^2+5$	$-\frac{1}{2}$	2	5	down	$x=2$	$(2,5)$
c $f(x)=-3(x-1)^2-6$	-3	1	-6	down	$x=1$	$(1,-6)$

3 **Conceptual:** Which parameter in the function $f(x)=a(x-h)^2+k$ determines whether the graph of the function is concave up or concave down? Explain how you determine whether it is up or down.

Answer: The parameter *a* determines the concavity. The graph is concave up when $a>0$ and concave down when $a<0$.

4 **Conceptual:** What is the equation of the axis of symmetry and the vertex of the graph of $f(x)=a(x-h)^2+k$?

Answer: Axis of symmetry is $x=h$; vertex is (h,k).

(This is the conceptual understanding): The parameters of a quadratic function in vertex form represent the transformation of the parent function and identify concavity, the equation of the axis of symmetry and the coordinates of the vertex.

5 Concave up; $x=3$; $(3,2)$

Investigation 6

Conceptual understandings:

The parameters of a quadratic function in vertex form represent the transformation of the parent function and identify concavity, the equation of the axis of symmetry and the coordinates of the vertex.

Different forms of the quadratic function allow easier identification of different key features of its graph, and can help with graph sketching or finding an equation of a function from its graph.

1 & 2 Completed table:

$f(x)=ax^2+bx+c$	$-\frac{b}{2a}$	$f\left(-\frac{b}{2a}\right)$	**Axis of symmetry**	**Vertex**
a $f(x)=2x^2+4x-2$	-1	-4	$x=-1$	$(-1,-4)$
b $f(x)=-\frac{1}{2}x^2+2x+3$	2	5	$x=2$	$(2,5)$
c $f(x)=-3x^2+6x-9$	1	-6	$x=1$	$(1,-6)$

3 **Conceptual:** What is the equation of the axis of symmetry of the graph $f(x)=ax^2+bx+c$?

Answer: Axis of symmetry: $x=\frac{-b}{2a}$

4 **Conceptual:** What are the coordinates of the vertex of the graph of $f(x) = ax^2 + bx + c$?

Answer: Vertex: $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

(This is the conceptual understanding): The parameters of a quadratic function in vertex form represent the transformation of the parent function and identify concavity, the equation of the axis of symmetry and the coordinates of the vertex.

5 **Conceptual:** What is the *x*-coordinate of the *y*-intercept of the graph of any function? Hence, find the *y*-intercept of the graph of $f(x) = ax^2 + bx + c$.

Answer: The *x*-coordinate of the *y*-intercept of the graph of any function is 0. The *y*-intercept of the graph of $f(x) = ax^2 + bx + c$ is $(0, c)$.

International-mindedness

How do you use the Babylonian method of multiplication?

Try 36 × 14

Answer: Using a table of squares and the formula $xy = \left[(x + y)^2 - (x - y)^2\right] \div 4$.

$36 \times 14 = \left[(36 + 14)^2 - (36 - 14)^2\right] \div 4$.

$36 \times 14 = \left[(50)^2 - (22)^2\right] \div 4$.

$36 \times 14 = \frac{2016}{5} = 504$

Reflect: What is the largest possible domain of a quadratic function?

Answer: All real numbers

Reflect: What is the range of the function $y = a(x-h)^2 + k$?

Answer: The range of the function $y = a(x-h)^2 + k$ with a domain of all real numbers is $y \geq k$ when $a > 0$ and $y \leq k$ when $a < 0$.

TOK

How would you choose which formula to use?

When is intuition helpful and harmful in mathematics?

Answer: The first question is factual but the second requires considering when in a more general form and the discussion might go outside of the quadratics content.

Reuben Hersh (1998) considers that "intuition is an essential part of mathematics". Explore and explain.

Reflect: What do the parameters in each of the different forms of an equation of a quadratic function tell you about the graph?

Answer: The parameters in each form of an equation of a quadratic function identify certain key features of the graph of the function.

Reflect: Why is it useful to have more than one form for representing a quadratic function?

Answer (this is a conceptual understanding): Different forms of the quadratic function allow easier identification of different key features of its graph, and can help with graph sketching or finding an equation of a function from its graph.

Reflect: What are three different forms of an equation of a quadratic function?

For each form, which features of the graph of the function help you determine the parameters in the equation?

Answer: Standard form: y-intercept determines c.

Vertex form: equation of axis of symmetry $x = h$ determines h; vertex (h, k) determines h and k.

Intercept form: x-intercepts determine p and q.

Developing inquiry skills

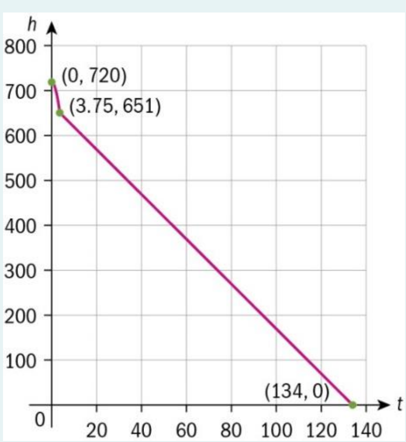
In the opening scenario for this chapter you looked at how crates of emergency supplies were dropped from a plane. The functions give the height of the crate.

During free fall: $h(t) = -4.9t^2 + 720$

With parachute open: $g(t) = -5t + 670$

Without using your GDC, sketch a graph of this piecewise function and label the key features.

Answer:



TOK

How can you deal with the ethical dilemma of using mathematics to plot the course of a missile?

Answer: Questions that might be used include:

Should mathematics be used to take lives?

What are the decisions to be made when developing the accuracy and stealth of weapons such as drones?

Consider the role of the scientists developing the atomic bomb.

3.5 Solving quadratic equations by factorization and completing the square

Investigation 7

Conceptual understandings:

Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the 'missing' area that would make a square.

Factorization provides a method for finding the roots of some quadratic equations. Completing the square provides a method for finding the roots of any quadratic equation.

1 9; $c = 9$; $x^2 + 6x + 9 = (x+3)^2$

2 4x; 4x; 16; $c = 16$; $x^2 + 8x + 16 = (x+4)^2$

3 x^2; 5x; 5x; 25; $c = 25$; $x^2 + 10x + 25 = (x+5)^2$

4 x^2; $\frac{b}{2}x$; $\frac{b}{2}x$; $\left(\frac{b}{2}\right)^2$; $c = \left(\frac{b}{2}\right)^2$; $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

5 $\left(\frac{b}{2}\right)^2$

6 **Conceptual:** How does the area model help you understand the process of completing the square and why is the process called completing the square?

Answer (this is the conceptual understanding): Completing the square converts a quadratic expression to a square plus another term and the process can be visualized by representing the terms in the expression as areas and finding the 'missing' area that would make a square.

Reflect: What can be said about the roots of a quadratic equation that can be solved by the factorization method?

Answer: The roots are rational.

Reflect: What can be said about the roots of a quadratic equation that can be solved by the completing the square?

Answer: The roots may be rational or irrational, i.e. real numbers.

Reflect: Can you use factorization to find the roots of any quadratic equation? Can you use completing the square to find the roots of any equation?

Answer (this is the conceptual understanding): Factorization provides a method for finding the roots of some quadratic equations. Completing the square provides a method for finding the roots of any quadratic equation.

Developing inquiry skills

In the opening scenario for this chapter you looked at how crates of emergency supplies were dropped from a plane. The functions $h(t)$ and $g(t)$ give the height of the crate:

During free fall: $h(t) = -4.9t^2 + 720$

With parachute open: $g(t) = -5t + 670$

How long after the crate leaves the plane does the parachute open? Write a quadratic equation you could solve to answer this question.

Answer: Opens when

$$\begin{aligned} h(t) &= g(t) \\ -4.9t^2 + 720 &= -5t + 670 \\ 0 &= 4.9t^2 - 5t - 50 \end{aligned}$$

How could you solve this equation?

Answer: This does not factorize, so solve either by completing the square, using quadratic formula or GDC to give $t = 3.75$ s. (Note: the negative root is meaningless.)

3.6 The quadratic formula and the discriminant

Investigation 8

Conceptual understanding:

Completing the square for the general form of the quadratic equation extends results from a specific equation to the general quadratic equation and results in a formula (the quadratic formula) that can be used to find the roots of any quadratic equation.

1 $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

2 $x^2 + \frac{b}{a}x = -\frac{c}{a}$

3 Complete the square by adding the square of half the coefficient of x to both sides of the equation.

4 Simplify $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and reorder the terms on the right-hand side of the equation.

5 Find a common denominator on the right-hand side of the equation.

6 $\left(x + \frac{b}{a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

7 Take the square roots of both sides of the equation.

8 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

9 **Conceptual:** What is the result of completing the square for the general equation $ax^2 + bx + c = 0$?

Answer: A formula for the two roots of a general quadratic equation.

10 Conceptual: Could you use this formula to find the roots of any quadratic equation?

Answer: Yes, because any quadratic can be rewritten in general form, and because it only needs the values *a*, *b* and *c* which occur in every quadratic. The values of *b* or *c* may be zero, but *a* has to be non-zero otherwise the equation is not quadratic.

11 Conceptual: How does completing the square for the general form allow you to find the roots of any quadratic equation?

Answer (this is the conceptual understanding): Completing the square for the general form of the quadratic equation extends results from a specific equation to the general quadratic equation and results in a formula (the quadratic formula) that can be used to find the roots of any quadratic equation.

Investigation 9

Conceptual understanding:

The quadratic formula contains the square root of the discriminant and its value determines the number and nature of the roots and the number of *x*-intercepts of the graph of the quadratic.

1 Completed table:

$ax^2+bx+c=0$	b^2-4ac	**Roots**	**Nature of roots**
a $-5x^2+6x+2=0$	76	$\frac{-6\pm\sqrt{76}}{-10}=\frac{3\pm\sqrt{19}}{5}$	two distinct real roots
b $3x^2+2x-1=0$	16	$\frac{-3\pm\sqrt{16}}{6}=-1,\frac{1}{3}$	two distinct real roots
c $x^2-4x-2=0$	24	$\frac{4\pm\sqrt{24}}{2}=2\pm\sqrt{6}$	two distinct real roots
d $-x^2-6x-9=0$	0	$\frac{6\pm\sqrt{0}}{-2}=-3$	two equal real roots (one repeated real root)
e $4x^2+12x+9=0$	0	$\frac{-12\pm\sqrt{0}}{8}=-\frac{3}{2}$	two equal real roots (one repeated real root)
f $-2x^2-4x-5=0$	–24	$\frac{4\pm\sqrt{-24}}{-4}\Rightarrow$ none	no real roots
g $4x^2-2x+3=0$	–44	$\frac{2\pm\sqrt{-44}}{8}\Rightarrow$ none	no real roots

2 The graphs have no intercepts on the *x*-axis.

3 Conceptual: What are the three types of roots that a quadratic equation can have?

Answer: Two distinct real roots, two equal real roots, or no real roots.

4 Factual: What is the relationship between the nature of the roots of the equation $ax^2+bx+c=0$ and the value of b^2-4ac?

Answer: For $b^2-4ac>0$, there are two distinct real roots. For $b^2-4ac=0$, there are two equal real roots (one repeated real root). For $b^2-4ac<0$, there are no real roots.

5 Factual: What is the relationship between the value of b^2-4ac and the number of *x*-intercepts of the graph of the function, $y=ax^2+bx+c$?

Answer: For $b^2-4ac>0$, the graph has two *x*-intercepts. For $b^2-4ac=0$, the graph has one *x*-intercept. For $b^2-4ac<0$, the graph has no *x*-intercepts.

Reflect: Why does a quadratic equation with a discriminant of 0 have two equal real roots?

Why does a quadratic equation with a discriminant greater than 0 have two distinct real roots?

Why does a quadratic equation with a discriminant less than 0 have no real roots?

How can you use the discriminant to determine when the roots of a quadratic equation will be rational?

Answer: When $b^2 - 4ac = 0$, both $x = \frac{-b+\sqrt{0}}{2a}$ and $x = \frac{-b-\sqrt{0}}{2a}$ equal $-\frac{b}{2a}$, so there are two equal real roots (or one repeated real root).

When $b^2 - 4ac > 0$ there are two distinct real roots, $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

For $b^2 - 4ac < 0$, there are no real roots because there is no real number whose square is negative.

When the discriminant is a perfect square, the roots of the equation will be rational.

Reflect: Why does the discriminant determine the nature of the roots?

Answer (this is the conceptual understanding): The quadratic formula contains the square root of the discriminant, and so its value determines the number and nature of the roots and the number of x-intercepts of the graph of the quadratic.

Investigation 10

Conceptual understanding:

Technology can be used to solve quadratic equations and inequalities.

1 **a** $x^2 + 10x + 25 = 0$ Perfect square; $x = -5, -5$

$x^2 + 5x + 6 = 0$ Factorize or use formula or GDC; $(x + 2)(x + 3) = 0$; $x = -2, -3$

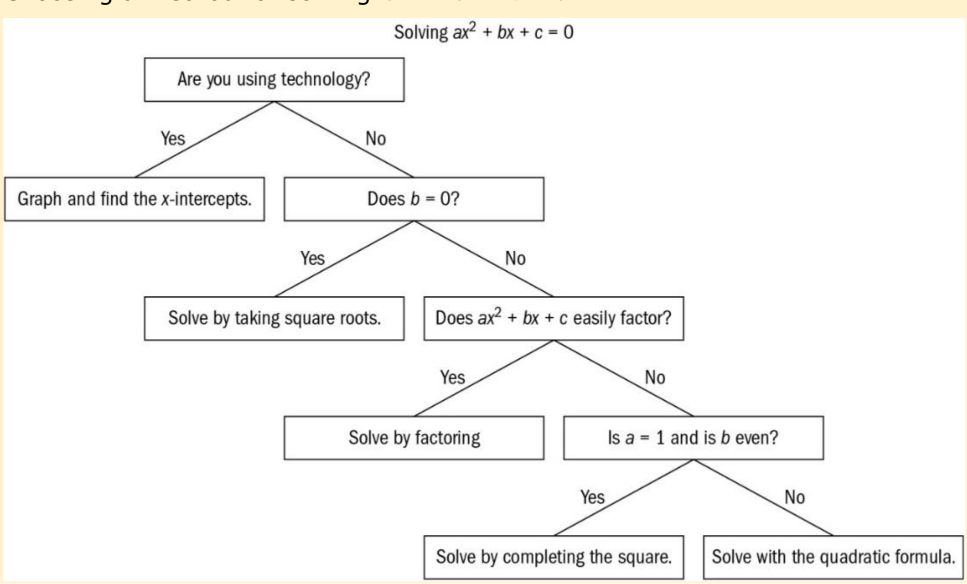
$2x^2 - 3x + 17 = 0$ Use GDC or quadratic formula; no real roots

$x^2 - 4x + 11 = 0$ Complete the square, use GDC or quadratic formula; $x = 2 + \sqrt{7}$, $x = 2 - \sqrt{7}$

$5x^2 - 6x - 9 = 0$ Use GDC or quadratic formula; $x = \frac{3+3\sqrt{6}}{5}, \frac{3-3\sqrt{6}}{5}$

b, c Students' own answers

2 Choosing a method for solving $ax^2 + bx + c = 0$:



3 **Conceptual:** How do you choose which method to use to solve a quadratic equation?

Answer: If the equation is factorizable, then factorize and solve.

If the equation is a perfect square, then take the square root of both sides and solve.

If $a = 1$ and b is even, complete the square.

Otherwise, use either the quadratic formula or complete the square.

4 **Conceptual:** Which methods of solving a quadratic equation work for any quadratic equation?

Answer: The quadratic formula and completing the square work for all equations that have real roots.

5 **Conceptual:** How is technology beneficial to help solve quadratic equations?

Answer (this is the conceptual understanding): Technology can be used to solve quadratic equations and inequalities.

Developing inquiry skills

The opening problem for this chapter considered the height of a crate dropped from a cargo plane. The functions h and g give the crate's height above the ground, measured in metres, t seconds after the crate leaves the plane.

During free fall: $h(t) = -4.9t^2 + 720$

With parachute open: $g(t) = -5t + 670$

Now that you have learned several methods for solving a quadratic equation you can answer the following questions.

How long after the crate leaves the plane does it reach the ground?

Answer: 134 s

What is the domain of h and g in the context of the problem?

Answer: h: $0 \le t \le 3.75$; g: $3.75 \le t \le 134$

3.7 Applications of quadratics

Investigation 11

Conceptual understandings:

Quadratic relationships can be represented by tables, graphs and various forms of equations and can be used to model real-world relationships such as projectile motion, kinematics, problems involving area, and maximizing income or minimizing cost.

A certain form of a quadratic function may be a more suitable to model certain real-life problems. For example, when using a quadratic to find the maximum or minimum value of a real-life problem, the vertex form may be most suitable.

1 $r = 26.2$; $s = 97$; $t = 194$

2 $y = a(x-97)^2 + 0$; $y = ax^2 + bx + 26.2$; $y = a(x-97)(x-97)$

3 $y = 0.00278(x-97)^2 + 0$ or $y = 0.00278x^2 - 0.540x + 26.2$ or $y = 0.00278(x-97)(x-97)$

4 **Factual:** Which form of the quadratic function did you use and why?

Answer: Answers will vary, but it is hoped students will choose either $y = a(x-97)^2 + 0$ or $y = a(x-97)(x-97)$. The reason should be that it is easier to solve for one missing parameter than two.

5 x is the horizontal distance in metres from the left-hand tower and y is the distance between the road deck and the supporting cable.

6 **Conceptual:** How can the quadratic function help you answer questions about the supporting cable of the Clifton Suspension Bridge?

Answer: Answers will vary, but may include the fact that you could find the height of the cable from the road deck at any given distance from a tower.

7 The height of supporting cable 24 m from a tower is 14.8 m.

8 The cables are 65.6 m and 128 m from the left-hand tower.

Reflect: How can quadratic relationships be represented? What type of real-world relationships can be modelled by quadratic equations?

Answer (this is the conceptual understanding): Quadratic relationships can be represented by tables, graphs and various forms of equations and can be used to model real world relationships such as projectile motion, kinematics, problems involving area, and maximizing income or minimizing cost.

Reflect: How do you decide which form of quadratic function to use for your model?

Answer (this is the conceptual understanding): A certain form of a quadratic function may be a more suitable to model certain real-life problems. For example, when using a quadratic to find the maximum or minimum value of a real-life problem, the vertex form may be most suitable.

Reflect: When solving a real-life problem modelled by a quadratic equation, how do you determine which method of solution to use?

Answer: In many real-life problems you will need to use technology to solve quadratic equations because the numbers will not be friendly. If you cannot use technology, use the flow chart developed in Investigation 10 to determine the solution method.

TOK

How accurate is a visual representation of a mathematical concept?

Answer: A chance to explore the limitations of graphs and charts in delivering information about functions and phenomena in general, relevance of modes of representation.

A good question to ask for a response to is “Should we accept simplicity over accuracy to relay information”.

Developing inquiry skills

In the opening scenario for this chapter you looked at how crates of emergency supplies were dropped from a plane. The functions $h(t)$ and $g(t)$ give the height of the crate:

During free fall: $h(t) = -4.9t^2 + 720$

With parachute open: $g(t) = -5t + 670$

How realistic is this model in representing this real-life situation? State at least two advantages of the model, and give at least one criticism.

Advantage: The idea of a piecewise model is good, because the motion is likely to be modelled by two different functions as different forces act on the crate before and after the parachute is open.

Advantage: The initial quadratic function models the situation well, as the crate will accelerate due to gravity when it leaves the plane. A linear function for height against time would show only constant speed.

Disadvantage: There is likely to be a deceleration as soon as the parachute opens. This is not shown on this model.

Suppose a heavier crate was dropped, but its parachute was the same. How would the model be different for this heavier crate?

Answer: The initial freefall would be the same, because acceleration due to gravity is independent of mass. However, the speed of the object once the parachute is open would be greater (the object has to travel faster for drag to be equal to weight).

Suggest suitable functions to model the path of a javelin through the air, or the path of a basketball as it leaves a player's hands and passes through a hoop. What do you need to consider in finding a model for each? How could each model help to make predictions in a real-life scenario?

Answer: Students' own answers, which should be clearly thought through and justified.

Hanging Around

Approaches to Learning: Thinking Skills: Creating, Generating, Planning, Producing

Exploration Criteria: Presentation (A); Personal Engagement (C); Reflection (D)

IB Topic: Quadratic Modelling, Using Technology

Introduction

This task introduces students to the possibility of importing images into graphing packages in order to find a curve to best fit the object in the image. This is something that a lot of students do or could be advised to do in their explorations.

Students will find a curve and reflect on why the curve may not fit the image exactly, which will lead them to consider their choice of curve.

The task therefore encourages Personal Engagement (Criterion C) by considering using actual data provided by the student and using technology creatively to fit a model. The use of the graph and the equation to represent the model will add to Criterion A: Presentation. A consideration of the suitable fit (or otherwise) of the model will lead to critical reflection for Criterion D: Reflection.

Investigate

For this task, students will need:

A piece of rope or chain

Access to a graphing package such as Desmos (which is used in these notes) or Autograph, GeoGebra, Logger Pro, Geometer's Sketchpad, etc

A piece of rope or chain

Access to a graphing package such as Desmos (which is used in these notes) or Autograph, GeoGebra, Logger Pro, Geometer's Sketchpad, etc.

If a piece of rope or chain is not available, then students could use a picture of a hanging rope or chain from the internet.

Students will probably suggest that the shape is a parabola (quadratic curve), since they have studied this in chapter 3. They could also suggest a cosine curve if this has been studied:

$y = ax^2 + bx + c$ or $y = \cos x$

Students could test this by trying to fit a curve that is quadratic (or cosine).

Import the Curve Into a Graphing Package

The aim here is to use a graphing package to try to find the equation of the quadratic that best fits the shape of the hanging chain in the form $y = ax^2 + bx + c$.

When taking their photograph, students should realize that it is important to get a clear image with the camera looking straight at the rope/chain rather than from above/below or from the side.

Make sure that students follow the instructions correctly to import their image into the graphing package Desmos or equivalent.

For example, to import the image into the Desmos graphing package, the steps are:

Open Desmos.

Click on the  on the top left.

Click on "image" and locate your image to import.

The image should appear in the graphing screen.

Fit and Equation to Three Points on the Curve

While it does not really matter which three points students select, different points may result in slightly different equations.

Two points would not be sufficient, as there is not a unique quadratic curve going through two points.

For extension, you could also ask:

Could you use more than three points?

What difference would this make?

You could use as many points as you want. Three is the minimum required to fit a unique parabola. More than three points may not produce an R-squared value of 1, but may produce a curve that more closely fits the shape of the image.

For students who need further guidance on how to enter their three chosen points into a table, the points $(-2.9, 0)$, $(-0.1, -4.7)$ and $(3, 1)$ could be used as an example.

Make sure that students follow the instructions correctly for the graphing package Desmos or equivalent.

The steps for the Desmos graphing package are:

Type $y_1 \sim ax_1^2 + bx_1 + c$

If appropriate, you could tell students that the ~ symbol (called a tilde) means that Desmos will provide the best fit equation for the provided form.

Explain to students that y_1 and x_1 refer to the headings on the table.

The values of a, b and c are given under the equation. These are the values of the coefficients of the quadratic function that best fits the points provided—for example, for the coordinates used above the best fit equation is $y = 0.596x^2 + 0.110x - 4.69$

to three significant figures.

Desmos also provides the R2 (or R-squared) value. For the example used, this value is 1.

If appropriate for extension work, or if students ask what it is, you could introduce the coefficient of determination, R2.

R2, also known as the coefficient of determination, is a statistical measure of how close the data are to the fitted curve. A value of 1 means that the curve passes through the three points exactly.

You could ask:

Why is the R2 value 1 here?

Try using four or more points. Does the value of R2 change?

The R2 value is 1 because Desmos has provided a curve that exactly fits through the three points. This may not be the case if more than three points are used because the curve may no longer fit exactly through all the points.

Test the Fit of Your Curve

No, the curve does not fit exactly.

It does not overlap entirely with the curve, but it is close.

Possible reasons:

The choice of curve might not be correct.

Inaccuracy in selecting and marking points—for example, if the rope or chain is thick and the points in the middle of the rope/chain cannot easily be selected.

It could be the image is of poor quality, has a shadow that distorts the picture or is taken from a difficult angle and not straight on to the object.

You could also discuss how moving, expanding or contracting the image would affect the equation. The resulting equation students would be working out would differ depending on the image, location and size. However, if the distance between the two points from which the rope is hanging is known, students could work out the scale of the graph.

If students were going to consider areas or geometrical findings based on their curve, they would need to have a sense of scale. They could find this by knowing, for example, the physical distance between the two points where the rope is hooked.

For **extension**, students could try following the same process and fitting a cosine curve.

They could then investigate the limitations of this approach to finding a curve of best fit.

The catenary and the parabola are two different curves. They look generally quite similar in that they are symmetrical and U-shaped, going up infinitely on either side of a minimum. However, they do have a different shape.

Note: the catenary curve is not part of the SL syllabus.

A catenary is slightly "flatter" at the bottom and rises faster than a parabola for large values of x.

For **extension** work, you could ask:

What else could you use this process to model?

What could the resulting equations be used to find?

It would be possible to import any photo like this and students could then fit any curve of which they know the general form of the equation.

One of the key points from this task is the importance of finding the correct type of curve for a particular given situation—this may involve doing some research first.

Once students have an equation it would be possible to perhaps find the area under a curve or a volume of revolution based on the curve—see Calculus chapter.

Students may be able to make predictions for the position of other points on the curve that perhaps cannot be seen in a photo/diagram.

It may be possible to find the highest or lowest point of a travelling object, or where that object might end up/land.

For extension work, you could also ask:

What is the minimum number of points you would need to find a cubic model?

Is it different from a quadratic? Why?

A cubic model would require four points to fit a unique curve, as a general cubic curve

$$y = ax^3 + bx^2 + cx + d$$

has four parameters.

In fact, a polynomial of degree n would require $n + 1$ points.

Extension

These are just examples that may be modelled using quadratics—students' IAs often include many other types of functions that they model; the important point is that they can justify the choice of the model that they select or can suggest why the model does not fit exactly if it does not.

For additional interest, watch this from Matt Parker (Stand Up Mathematician!)—Sydney: The Unsuccessful Hunt for Parabola.

4 Equivalent representations: rational functions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
In a reciprocal function, as the independent variable approaches zero the dependent variable becomes infinitely large.	Investigation 2
The parent positive or negative reciprocal graph follows a distinct shape symmetrical about the positive/ negative identity line with values either in the second and fourth quadrant or values in the first and third quadrants, and contains asymptotes for undefined values of the denominator.	Investigation 4
In a table of values for a real life situation, pairs of independent and dependent variables with constant product fit a reciprocal model that can be used for prediction.	Investigation 5
The vertical translations of a reciprocal function have an effect on the asymptotes of the function and its range and domain.	Investigation 6

Syllabus sections covered in this chapter:

- SL2.3*
- SL2.4*
- SL2.8
- SL2.10
- SL2.11

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 185: Solving linear equations, Finding the gradient of a line given two points	Page 203: Example 4 Page 205: Example 7	Page 198: Example 3 Page 205: Example 7	Pages 195, 198, 206

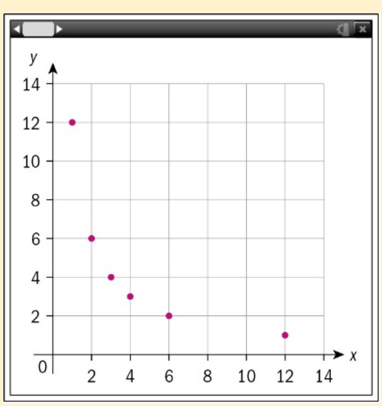
Assessment opportunities

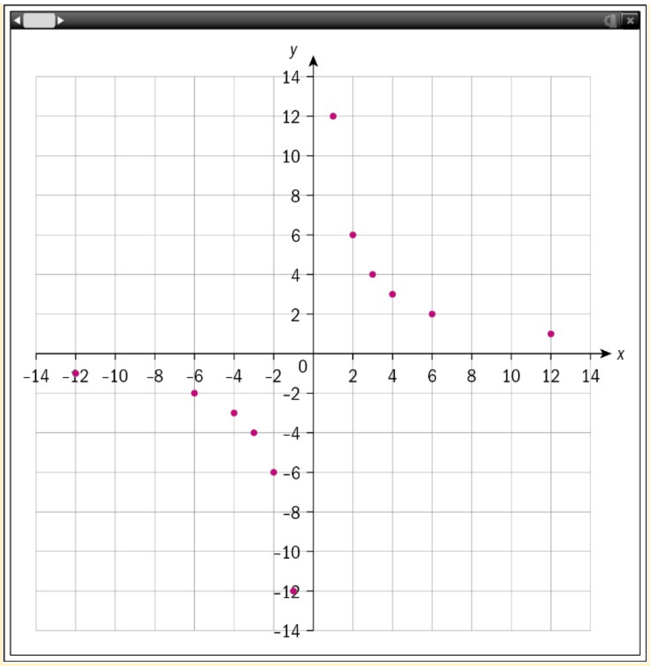
End of chapter test	Mixed review exercise	Exam practice
Page 208	Page 209	Page 209

4.1 The reciprocal function

Investigation 1

1 Student will give various answers.

2 



3

4 **a** x gets smaller as y gets bigger.

b y gets smaller as x gets bigger.

5 The graph approaches zero in both directions.

Reflect: Why does zero not have a reciprocal?

Answer: $\frac{1}{0}$ is undefined

TOK

Is zero the same as nothing?

Answer: A good debate for pairs or a small group to then present their opinions to the rest of the class. You might like to reference "Coke zero" or "Ground Zero" juxtaposed with zero money in the bank. You might like to conclude with 'zero' is a number while 'nothing' is a concept.

Reflect: Are there any values of x for which the reciprocal function is not defined?

Answer: $x = 0$ and $y = 0$.

Investigation 2

Conceptual understanding:

In a reciprocal function, as the independent variable approaches zero the dependent variable becomes infinitely large.

1 **a** **i** 0.5 **ii** 0.05 **iii** 0.005

iv -0.5 **v** -0.05 **vi** -0.005

b y gets smaller as x gets larger.

c **Factual:** Is there a value of *x* which would make the value of *y* zero?

Answer: No

d **Conceptual:** What can you say about the value of *y* when the absolute value of *x* becomes infinitely large?

Answer: The value of *y* approaches zero

2 **a** **i** 120 **ii** 12000 **iii** 1200000 **iv** -120 **v** -12000 **vi** -1200000

b **Conceptual:** Explain what is happening to *y* as the absolute value of *x* approaches zero.

Answer (this is the conceptual understanding): In a reciprocal function, as the independent variable approaches zero the dependent variable becomes infinitely large.

c **Factual:** Given any large value of *y*, is it possible to find a value of *x* for which $y = \frac{1}{x}$?

What does this tell you about the range of f(*x*)?

Answer: Yes. The limit of *y* as *x* approaches zero is infinity.

TOK

"Film is one of the three universal languages, the other two: mathematics and music", is a quote from movie director, Frank Capra

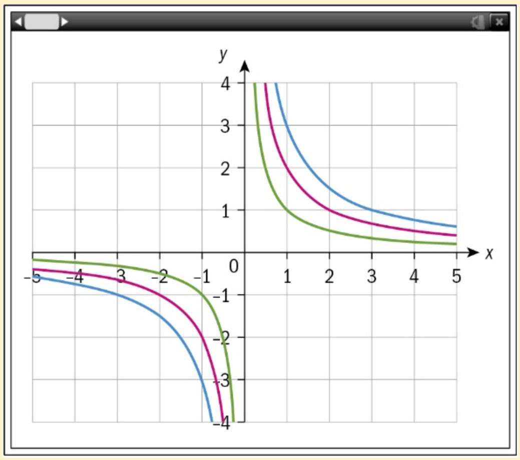
To what extent do you agree?

Answer: An opportunity to get the views of students with expertise in each of those areas.

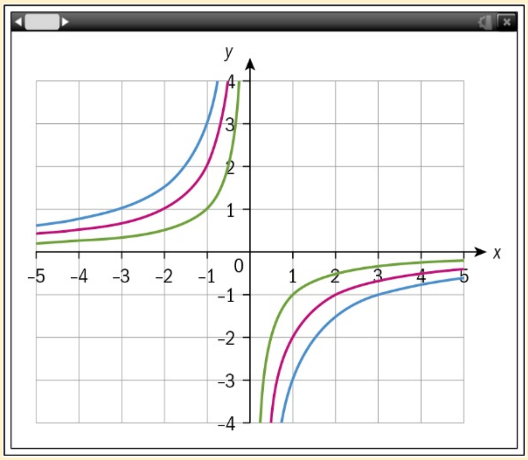
Investigation 3

1 Sketch each of these functions with the help of technology. You may use you calculator or graphing software.

a $f(x) = \frac{1}{x}$ **b** $g(x) = \frac{2}{x}$ **c** $h(x) = \frac{3}{x}$



d $p(x) = -\frac{1}{x}$ **e** $q(x) = -\frac{2}{x}$ **f** $r(x) = -\frac{3}{x}$



2 **Factual:** What features of each graph in question 1 are:

a the same for each graph?

Answer: The graphs have the same general shape and the same asymptotes.

b different for each of the three graphs?

Answer: They are not in the same place

3 **Conceptual:** What effect does changing the magnitude of the parameter k in the function $f(x) = \frac{k}{x}$ have on the graph of y = f(x)?

Answer: The k is the scale for a vertical stretch.

4 A reflection in the x-axis.

5 A reflection in the y-axis.

6 **Factual:** State what you notice about the graphs of p(x) and q(x). Can you explain why this is the case?

Answer: The graphs are the same. Both graphs are y = -f(x).

7 **Conceptual:** What effect does the sign of the parameter have on the graph of a reciprocal function?

Answer: A positive parameter has a reciprocal graph with values in the second and fourth quadrant. A negative parameter has a reciprocal graph in with values in the first and third quadrants

8 **Reflect:** What are the equations of the asymptotes of $y = \frac{1}{x}$?

Answer: x = 0 and y = 0.

9 No never.

10 **Conceptual:** How does the graph of $y = \frac{k}{x}$ differ from the graph of $y = \frac{1}{x}$?

Answer: Discuss as a class.

Investigation 4

Conceptual understanding:

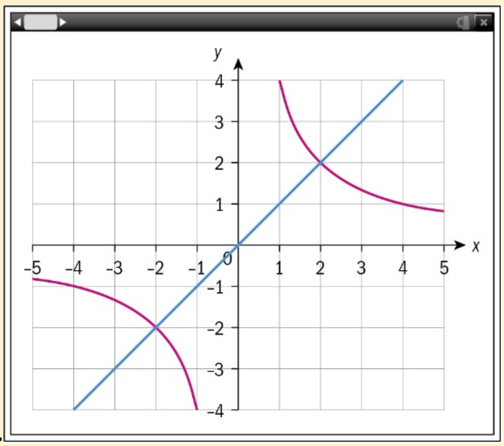
The parent positive or negative reciprocal graph follows a distinct shape symmetrical about the positive/ negative identity line with values either in the second and fourth quadrant or values in the first and third quadrants, and contains asymptotes for undefined values of the denominator.

1 a Competed table:

x	0.25	0.4	0.5	1	2	4	8	10	16
f(x)	16	10	8	4	2	1	0.5	0.4	0.25

2 **Factual:** What do you notice about the top and bottom rows of the table?

Answer: Their product is 4.

3, 4 

5 The refection in y=x reflects $f(x)=\frac{4}{x}$ onto itself

6 **Factual:** What does this tell you about the inverse function $f^{-1}(x)$?

Answer: Reciprocal functions are self-inverses.

7 $(f \circ f)(x)=\dfrac{k}{\frac{k}{x}}=x$

8 **Conceptual:** What does this tell you about the inverse of the reciprocal function?

Answer: Reciprocal functions are self-inverses.

Investigation 5

Conceptual understanding:

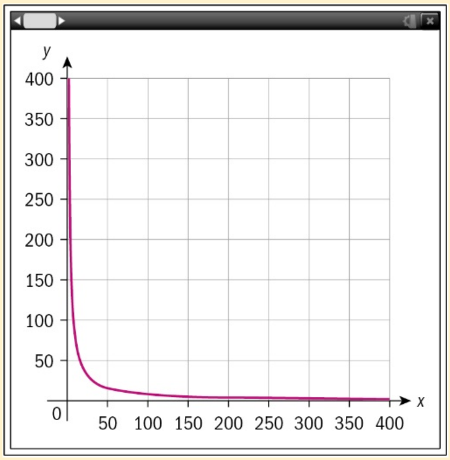
In a table of values for a real life situation, pairs of independent and dependent variables with constant product fit a reciprocal model that can be used for prediction.

1 $40\times 20, 80\times 10$ etc.

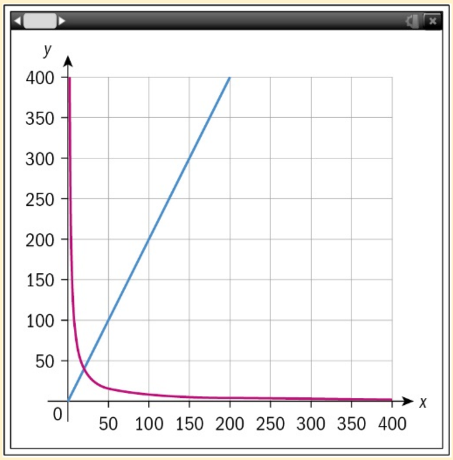
2 **Factual:** What type of relationship is this? How can you be sure? Can you write an equation linking the width (x metres) and the length (y metres) of the car park?

Answer: Reciprocal. The product of the *x* and *y* values are the same number. The function is $y=\frac{800}{x}$

3



4 $y = 2x$



5 Width 20m, length 40m

6 **Factual:** Are the values you found in question 5 the only possible values for the length and width, given the restrictions placed on the students? How can you be sure?

Answer: Yes. There is only point of intersection.

7 **Conceptual:** How can you tell from a table of values that a real-life situation could be represented by a reciprocal function?

Answer (this is the conceptual understanding): In a table of values for a real life situation, pairs of independent and dependent variables with constant product fit a reciprocal model that can be used for prediction.

TOK

How can a mathematical model give us knowledge even if it does not yield accurate predictions?

Answer: Students may well be using words like estimation and approximation. A good example is the way that weather is forecast where Meteorologists use tools to measure atmospheric conditions that occurred in the past and present, and they apply this information to create educated guesses about the future weather.

Developing inquiry skills

Look back at the opening problem for this chapter, in which a water park was designing a water slide.

Using what you have learned about reciprocal functions, what type of function could you use to model the shape of the waterslide?

How would the parameter of the function affect the shape of the waterslide? What parameter might you choose to ensure the slide is both exciting, and safe? What other things might you need to take into consideration when modelling the shape of the slide?

Answer: Discuss as a class.

4.2 Transforming the reciprocal function

Investigation 6

Conceptual understanding:

The vertical translations of a reciprocal function have an effect on the asymptotes of the function and its range and domain

1 Check students' GDC graphs.
2 Completed table:

Rational function	*Vertical asymptote*	*Horizontal asymptote*	*Domain*	*Range*
$y=\frac{1}{x}$	$x=0$	$y=0$	$x\in\mathbb{R}, x\neq 0$	$y\in\mathbb{R}, y\neq 0$
$y=\frac{1}{x}+2$	$x=0$	$y=2$	$x\in\mathbb{R}, x\neq 0$	$y\in\mathbb{R}, y\neq 2$
$y=\frac{1}{x}-3$	$x=0$	$y=-3$	$x\in\mathbb{R}, x\neq 0$	$y\in\mathbb{R}, y\neq -3$

3 **a** **i** Vertical translation of 2.

ii Vertical translation of 2. As x becomes large, $\frac{1}{x}\to 0$ so $\frac{1}{x}+2\to 2$

iii No change.

iv (-0.5, 0)

$\frac{1}{x}+2=0$

$\frac{1}{x}=-2$

$1=-2x$

$x=-0.5$

b **i** Vertical translation of -3.

ii Vertical translation of -3. As x becomes large, $\frac{1}{x}\to 0$ so $\frac{1}{x}-3\to -3$

iii No change.

iv $\left(\frac{1}{3}, 0\right)$

$$\frac{1}{x} - 3 = 0$$
$$\frac{1}{x} = 3$$
$$1 = 3x$$
$$x = \frac{1}{3}$$

4 **Conceptual:** What are the asymptotes of a rational function of the form $y = \frac{p}{x} + d$?

Answer: $x = 0$, $y = \text{d}$

5 Completed table:

Rational function	*Vertical asymptote*	*Horizontal asymptote*	*Domain*	*Range*
$y = \frac{1}{x}$	$x = 0$	$y = 0$	$x \in \mathbb{R}, x \neq 0$	$y \in \mathbb{R}, y \neq 0$
$y = \frac{1}{x+2}$	$x = -2$	$y = 0$	$x \in \mathbb{R}, x \neq -2$	$y \in \mathbb{R}, y \neq 0$
$y = \frac{1}{x-3}$	$x = 3$	$y = 0$	$x \in \mathbb{R}, x \neq 3$	$y \in \mathbb{R}, y \neq 0$
$y = \frac{4}{x-3}$	$x = 3$	$y = 0$	$x \in \mathbb{R}, x \neq 3$	$y \in \mathbb{R}, y \neq 0$
$y = \frac{4}{2x-3}$	$x = \frac{3}{2}$	$y = 0$	$x \in \mathbb{R}, x \neq \frac{3}{2}$	$y \in \mathbb{R}, y \neq 0$

6 **a** Horizontal stretch with scale factor $\frac{1}{c}$, horizontal translation -d, vertical stretch with scale factor p.

b Horizontal asymptote is $y = 0$

c Vertical asymptote is $x = -\frac{d}{c}$

d y-axis at point $\left(\frac{p-d}{c}, 0\right)$

$$0 = \frac{p}{cx+d}$$
$$cx + d = p$$
$$cx = p - d$$
$$x = \frac{p-d}{c}$$

7 **Conceptual:** Using what you have learned about transformations of the graph $y = \frac{1}{x}$, can you find the equations of the asymptotes of the function $y = \frac{p}{cx+d} + q$, and hence state the domain and range?

Answer: $x = -\frac{d}{c}$, $y = q$, $x \in \mathbb{R}, x \neq -\frac{d}{c}$, $y \in \mathbb{R}, y \neq q$

8 **i** Vertical stretch with scale factor k.

ii Vertical stretch with scale factor k, reflection in x-axis.

TOK

When students see a familiar equation with a transformation, they will often get a "gut feeling" about what the function looks like.

Respond to this question.

Is intuition helpful or harmful in mathematics?

Answer: Questions that a teacher might like to pose include:

Is it a bad idea to try to solve problems using our "gut feeling", intuition, or use reason and evidence to make a decision?

Where does that gut feeling come from?

When has it helped you?

When has it misled you?

Developing inquiry skills

Look back at the opening problem for this chapter, in which a water park was designing a water slide. The floor and wall of the swimming pool building are to be modelled by the x- and y-axes respectively.

The designer decides that he wants to move the point where a person sets off—at the top of the slide—away from the wall in order to avoid injuries. How could he develop his mathematical model of the slide to do this?

Answer: Transform $y = \frac{a}{x}$ into $y = \frac{a}{x-b}$, where b is the distance to be moved

The designer also decides that he wants to move the end of the slide up a little, so that a person has a short vertical drop between when they leave the end of the slide and when they hit the water. How could he develop his mathematical model of the slide to do this?

Answer: Transform $y = \frac{a}{x-b}$ into $y = \frac{a}{x-b} + c$, where c is the distance to be raised.

4.3 Rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$

International-mindedness

The first Chinese Abacus was invented around 500 B.C.

The Moroccan scholar Ibn al-Bannāʾ al-Marrākush used a lattice multiplication in the 14th century that Joh Napier, a Scottish mathematician, improved in the 17th century. Edmund Gunter of Oxford then produced the first slide rule before Blaise Pascal, started to develop a mechanical calculator in France in the 18th century which led to our present day GDC.

Answer: An opportunity to see where mathematical knowledge transcends national, cultural and religious borders

Investigation 7

1 **Factual:** Look again at the function $f(x) = \dfrac{p}{cx+d} + q$ which you studied at the end of section 4.2. Is this a rational function?

Answer: Yes, it is.

2 Make *p* into the *a* fraction by multiplying by *cx* + *d* over *cx* + *d* and add the fractions.

3 $\dfrac{p}{cx+d} + \dfrac{q(cx+d)}{cx+d} = \dfrac{p+qcx+qd}{cx+d} = \dfrac{(p+qd)+qcx}{cx+d}$. $a = p+qd,\ b = qc$

4 $x = \ x = -\dfrac{d}{c}, y = \dfrac{a}{c}$

5 Completed table:

Rational function	*Vertical asymptote*	*Horizontal asymptote*	*Domain*	*Range*
$y = \dfrac{x}{x+2}$	$x = -2$	$y = 1$	$x \in \mathbb{R}, x \neq -2$	$y \in \mathbb{R}, y \neq 1$
$y = \dfrac{x+1}{x+2}$	$x = -2$	$y = 1$	$x \in \mathbb{R}, x \neq -2$	$y \in \mathbb{R}, y \neq 1$
$y = \dfrac{2x}{x+2}$	$x = -2$	$y = 2$	$x \in \mathbb{R}, x \neq -2$	$y \in \mathbb{R}, y \neq 2$
$y = \dfrac{2x-1}{x+2}$	$x = -2$	$y = 2$	$x \in \mathbb{R}, x \neq -2$	$y \in \mathbb{R}, y \neq 2$

6 Check answers as a class.

7 The vertical asymptote occurs when the denominator is zero.

8 $cx + d = 0 \Rightarrow cx = -d \Rightarrow x = -\dfrac{d}{c}$

TOK

Look at the development of calculating technology on P21, and then respond to this question

What are the ethical considerations when sharing mathematical knowledge?

Answer: Questions that might help to produce debate include:

Is it right to keep mathematical knowledge to yourself?

Could this harm the progress of the world?

Should you pass on mathematical knowledge to friends in a test?

TOK

What is the biggest number that you know? Million? Billion? What about the prefixes used in storage like giga and tera? What are they?

What is a googol? What is larger?

In this chapter we have seen the symbol ∞ used for infinity.

Research Hilbert's paradox of the Grand Hotel.

What do you think is meant by infinity?

Answer: Infinity - the concept of not having an end.

Potential and actual infinity might be researched, as might Zeno's paradoxes.

Developing inquiry skills

Look back at the designer's progress with the waterslide at the end of section 4.2. How could you model the shape of this waterslide as a rational function of the form $f(x)=\frac{ax+b}{ac+d}$?

Answer: Finding the vertical asymptote allows us to find the denominator. The horizontal asymptote then gives *a*, and substituting in one point will find *b*.

If the swimming pool is in a building 20 metres high and there is 15 metres of space between the wall and the edge of the pool, suggest suitable values for the parameters *a*, *b*, *c* and *d* and suggest a suitable domain for the model.

Answer: $f(x)=\frac{x}{x-1}+1$

To Infinity and ...

Approaches to Learning: Thinking Skills, Communication, Research, Collaboration

Exploration Criteria: Mathematical Communication (B); Personal Engagement (C)

IB Topic: Linking Different Areas

Introduction

In the coming chapter, students will consider the crucial role that infinity plays in calculus, where the idea of an infinitely small quantity is used. It has already come up in chapter 1 as well as in this chapter. Rather than simply preparing students for an exploration, this RLT is designed to address the curiosity that students often feel regarding infinity as a concept. There are obviously some clear Theory-of-Knowledge ideas that can be addressed as well. It is not actually expected that this will lead to an exploration topic, because often explorations based on something like this will not be successful due to their being low scoring in Criterion E: Use of Mathematics.

However, the RLT may prompt students to think about researching and finding appropriate sites and books. The suggestion in the extension box of a presentation to the class will hopefully encourage students to communicate complex ideas in a manner that their peers will understand. Conducting research, choosing a demonstrable area of interest and seeing the links between areas of mathematics all contribute towards Criterion C: Personal Engagement. Students could be given homework to research and prepare the presentation and be prepared to present for five minutes per group in the next lesson.

What Is Infinity?

In these early discussions it is important to address the misconception that infinity is an ordinary number — it is not.

Discussions may also lead towards thinking about different kinds of infinity, rather than there being one single entity that is infinity.

As will be seen, the understanding and study of the concept of infinity has led to advances in mathematics and science.

It is also very intriguing, and a little disturbing at times. (The ancient Greeks, for example, viewed it with suspicion and hostility, and some refused to recognize it.)

As **extension** work, you could encourage students to research the history of the concept of infinity.

Thinking about the concept of infinity is quite complicated and produces some quite challenging questions and some surprising results and paradoxes (and a few headaches)! The introduction of the concept of infinity is credited to a philosopher named Zeno of Elea, who lived from 490 BC. Since that time, mathematicians, physicists, theologians, philosophers and even artists have grappled with the concept of infinity.

Four chapters into this book, and students have already encountered infinity a number of times.

In this chapter, students have been introduced to the idea of a limit and considered the “limit as x tends towards **infinity**”, from which they were able to identify asymptotes of rational functions.

Students will encounter limits again when they study calculus (differentiation and then integration) in the next chapters. Infinity plays a crucial role in calculus, where the idea of an infinitely small quantity is used.

In chapter 1, students were introduced to a geometric sequence where they had to find a “sum to **infinity**”.

For example, students will have considered the sum of an **infinite** geometric series such as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ and discovered that, although there are an infinite number of terms, it actually has a **finite** sum of 1.

If you think about it, this is really quite surprising!

Other examples where students have met the concept of infinity could include:

Lines (with infinite length)

Sets of numbers (real, rational, integers, etc.)

Irrational numbers (π, *e*, etc.)

Non-terminating rational numbers

Recurrence relations

Sequences and series

Limits in differentiation

Sums of areas under a curve leading to integration

Art—perspective point

If appropriate, you could ask students to make a short presentation on one of the examples they give.

Let's Think Further About Infinity

The following three games each look at infinity in a slightly different way.

Before each game starts, you should randomly select the order of players in the class.

Game 1: The winner is the person who names the biggest positive natural number.

The last player will win.

Game 2: The winner is the person who names the closest rational number to 0.

The last player will win.

Game 3: The winner is the person who names the closest real number to 1.

The last player will win.

To prompt discussion, you could ask:

What if, in these games, no one goes last?

What if you never stopped playing but just kept on going around the class?

If no one went last and they never stopped playing the games, no one would ever win and the games would go on forever.

In Game 1, the natural numbers chosen can be infinitely large. They can all be listed; however, this is clearly impracticable—if you counted forever, you would be sure not to miss any out. Therefore, they are countably infinite.

In Game 2, the numbers need to be infinitely small. Students are only allowed rational numbers (those that can be written as fractions). There are seemingly more rational numbers than natural numbers but, as it is still possible to count them all and to map them on to the natural numbers, there is actually the same number!

You could use this table to convince students of this. Every possible number $\frac{p}{q}$ is represented (it is on the p^{th} row and q^{th} column).

$\frac{1}{1}$	1	$\frac{1}{2}$	2	$\frac{1}{3}$	6	$\frac{1}{4}$	7	$\frac{1}{5}$	15	$\frac{1}{6}$	16	…
$\frac{2}{1}$	3	$\frac{2}{2}$	5	$\frac{2}{3}$	8	$\frac{2}{4}$	14	$\frac{2}{5}$	17	…		
$\frac{3}{1}$	4	$\frac{3}{2}$	9	$\frac{3}{3}$	13	$\frac{3}{4}$	18	…				
$\frac{4}{1}$	10	$\frac{4}{2}$	12	$\frac{4}{3}$	19	…						
$\frac{5}{1}$	11	$\frac{5}{2}$	20	…								
$\frac{6}{1}$	21	…										
…												

You can count these numbers, not by counting along the rows (as you would never finish the first row), but rather by following the order numbers in the second columns. In this way you have counted every rational once (as long as you ignore those rational numbers that are not in their simplest form).

In Game 3, the set of numbers concerns the real numbers. The real numbers are not actually countable (they are uncountably infinite). The proof of this involves Cantors Diagonal Proof, which is one of the possible presentations in the **extension** box.

Conclusion

The answer is no. Two sets that contain an infinite number of numbers are not necessarily the same size.

One infinity can be larger than another because one infinity can be countable, whereas another infinity may not be countable.

Extension Presentation

Students could also research another concept that is not in the list if preferred.

Students could work individually, or you could pair/group students with shared interests.

You could provide advice regarding good research techniques, using the internet, recording sources, going to the library, etc. The idea here is that students are grappling with some complicated concepts about how best to present these to a class of their peers—this can be challenging but perhaps is a timely opportunity to suggest to students that they will be submitting an IA that has an "audience of their peers".

Make sure that students use appropriate language in their presentations. Researching the concepts of infinity is a good introduction to looking at the concept of infinity and limits in the next chapter, which is on calculus.

5 Measuring change: differentiation

Essential understandings

Calculus describes rates of change between two variables. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function
- Examining rates of change close to turning points helps you identify intervals where the function increases/decreases, and identify the concavity of the function
- Mathematical modeling can provide effective solutions to real-life problems in optimization by maximising or minimising a quantity, such as cost or profit
- Derivatives and integrals describe real-world kinematics problems in two and three dimensional space by examining displacement, velocity and acceleration

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
A limit describes the output of a function as the input approaches a certain value from the left and from the right. For the limit of a function to exist, as the input of the function approaches a certain value from the left and from the right, the outputs must approach the same real number.	Investigation 2
Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.	Investigation 3
An average rate of change describes the ratio between the total change in the output values and the total change in the input values.	Investigation 4
Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.	Investigation 5
The power rule provides an efficient method for finding the derivative of a polynomial function.	Investigation 6
The derivative of a linear function in gradient intercept form provides the gradient of the graph. A constant function represents a horizontal line, with a gradient of zero.	Investigation 7
The chain rule may be used to find the derivative of composite functions.	Investigation 8
The product rule provides an efficient method for finding the derivative of a product of functions without expanding them.	Investigation 9

Conceptual understandings (cont.)	**Investigation**
The quotient rule provides an efficient method of differentiating rational functions without having to change the expression into a product form.	Investigation 10
Examining the signs of the gradients of a function identifies the increasing/decreasing intervals of the function.	Investigation 11
Examining the signs of the gradients to the left and right of stationary points help determine the nature of stationary points, and identify increasing/ decreasing intervals of the function.	Investigation 12
The 2nd derivative describes the rate of change of the gradient function (1st derivative); therefore at a stationary point an increasing first derivative function represents a minimum while a decreasing first derivative function represents a maximum. When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.	Investigation 13
Examining rate of change close to turning points and inflexion points helps you identify intervals where the function increases/decreases, and examining the rate of the rate of change of the intervals helps you identify the concavity of the function that is when the first derivative is increasing or decreasing.	Investigation 14
A horizontal point of inflexion satisfies the conditions that the first derivative and second derivative equal zero and a change in concavity on either side of the stationary point.	Investigation 15
Identifying turning points, points of inflexion, intervals where the function increases/decreases, and concavity, facilitates sketching the derivatives of the function.	Investigation 16
Identifying the zeros of a derivative function, examining positive/ negative intervals of the range of the function and examining concavity facilitates sketching the function from the graph of its derivative.	Investigation 17
The zeros of f' occur where f'' has stationary points. Where the graph of f'' goes from positive to negative, f' has a maximum, and where the graph of f'' goes from negative to positive f' has a minimum.	Investigation 18
Optimization in Calculus uses mathematical models, or functions, to provide largest and least-value solutions to real-life problems.	Investigation 19
Finding a mathematical model for a general case allows for multiple applications to specific cases.	Investigation 20
An object moving in a positive direction represents positive velocity while an object moving in a negative direction represents negative velocity.	Investigation 21
Both positive or both negative velocity and acceleration indicate an increasing speed of the object since they describe the same direction. Velocity and acceleration with different signs indicate a slowing down since they exhibit opposite directions.	Investigation 22

Syllabus sections covered in this chapter:

- SL5.1*
- SL5.2*
- SL5.3*
- SL5.4*
- SL5.6
- SL5.7
- SL5.8
- SL5.9

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 215: Finding the gradient of a line given two points	Page 231: Example 8 Page 235: Example 11 Page 238: Example 13 Page 250: Example 20 Page 266: Example 28	Page 218: Example 1 Page 232: Example 9 Page 245: Example 16	Pages 221, 233, 240, 259, 267

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 268	Page 271	N/A

5.1 Limits and convergence

Investigation 1

1 **a** 10 m **b** 1 m **c** 0.1 m

2 0.01

3 0.001

4 10 + 1 + 0.1 + 0.01 + 0.001; $a = 10$, $r = 0.1$

5 $|r| < 1$

6 Yes, since $r = 0.1 < 1$; $S\infty = \frac{a}{1-r} = \frac{10}{1-0.1} = 11.11$ m

7 Yes, after 11.11 m.

TOK

Should paradoxes change the way that mathematics is viewed as an area of knowledge?

Answer: Paradoxes might change your opinion about how mathematics is seen as an area of knowledge. Zeno's paradox would be mathematically correct; however, this is clearly not an accurate representation. Achilles would be able to overtake the tortoise quite quickly, because although the tortoise is moving, Achilles still runs faster. Thus, this mathematical paradox would be illogical. Does that mean that mathematics is an unreliable area of knowledge?

You might want students to try to write their own liar paradoxes such as "I am lying".

Investigation 2

Conceptual understanding:

A limit describes the output of a function as the input approaches a certain value from the left and from the right.

For the limit of a function to exist, as the input of the function approaches a certain value from the left and from the right, the outputs must approach the same real number.

2

x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$f(x) = \frac{x^2-1}{x-1}$	1.6	1.7	1.8	1.9	Undefined	2.1	2.2	2.3	2.4

3 2

4 **Conceptual:** What is meant by the limit of a function at a particular point?

Answer (this is the conceptual understanding): A limit describes the output of a function as the input approaches a certain value from both the negative and positive directions.

5 The function approaches the same value of 2 as x approaches 1 from the negative direction and from the positive direction, although the function itself is not defined at $x = 1$.

6 **Conceptual:** What condition is placed on the left and right limits of f at $x = a$ for the limit of f to exist at a?

Answer (this is the conceptual understanding): For the limit of a function to exist, as the input of the function approaches a certain value from the negative and positive directions, the outputs must approach the same real number.

7

x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$g(x) = x + 1$	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4

8 **Conceptual:** Does a function need to be defined at $x = c$ in order to have a limit at $x = c$?

Answer: No.

9 **Factual:** Write down graphical similarities and differences between f and g, and explain. Can you see the graphical difference between f and g on your GDC or graphing software? If not, explain why not.

Answer: The graphs of f and g are identical except at $x = 1$, as f is undefined at this value, and g is defined at this value.

TOK

What value does the knowledge of limits have?

Answer: Is infinitesimal behaviour applicable to real life?

Are intuition and imagination valid ways of knowing in mathematics?

Investigation 3

Conceptual understanding:

Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.

1 **a** $y = -2$ **b** $y = 0$ **c** None

d $y = 0$ **e** $y = \frac{1}{4}$ **f** None

g $y = \frac{2}{3}$ **h** $y = 0$ **i** None

2 Completed table:

	Degree(f(x))	Leading coefficient	degree(g(x))	Leading coefficient	Horizontal asymptote
a	1	-2	1	1	$y = -2$
b	1	0	2	-3	$y = 0$
c	3	1	2	1	none
d	1	0	2	1	$y = 0$
e	3	1	3	4	$y = \frac{1}{4}$
f	2	1	1	1	none
g	1	2	1	3	$y = \frac{2}{3}$
h	2	1	5	1	$y = 0$
i	5	1	3	1	none

3 **Conceptual:** What do the degree of the numerator and the degree of the denominator tell you about the asymptotes of a function?

Answer (this is the conceptual understanding): Comparing the degree of the numerator in relationship to the degree of the denominator can help determine if the rational function has a horizontal asymptote.

Some students may be able to go further and formulate the following results:

If Deg $(f(x))$>Deg$(g(x))$ $\Rightarrow$ no horizontal asymptote.

If Deg$(f(x)$ =Deg$(g(x))$ $\Rightarrow$ horizontal asymptote is given by

$$y = \frac{\text{leading coefficient of } f(x)}{\text{leading coefficient of } g(x)}$$

Deg$(f(x))$<Deg $(g(x))$ $\Rightarrow$ horizontal asymptote is $y = 0$

4 $y = 0$

5 $x = 0$

5.2 The derivative function

Investigation 4

Conceptual understanding:

An average rate of change describes the ratio between the total change in the output values and the total change in the input values.

1 In 2009, Bolt was slower to cover the first 30 m than in 2008, but in 2009 he completed the final 70 m quicker than he did in 2008.

2 2008: 10.3 m s–1; 2009: 10.4 m s–1

3 2009; 12.35 m s–1 between 60 m and 70 m

4 **Factual:** Compare Bolt's fastest speed with his average speed in that race. If you were to examine Bolt's speed at any point in that race, which of the two speeds you found would be likely be closest to the speed that he was running at that instant?

Answer: Average speed; this gives a good indication of the speed he was running at over the whole race. He only achieved the highest speed at one point in the race.

5 **Factual:** From the given information, is it possible to find his fastest speed at any particular moment in time, e.g., at 3.25s?

Answer: No, the best we can do is find his average speed over each 10 m interval. This is not the same as his instantaneous speed.

6 **Conceptual:** What is an average rate of change between output and input values of a function?

Answer (this is the conceptual understanding): An average rate of change describes the ratio between the total change in the output values and the total change in the input values.

Investigation 5

Conceptual understanding:

Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.

1 Completed table:

x	B(x, f(x))	Gradient of [AB]
2	(2,4)	$\frac{4-1}{2-1} = 3$
1.5	(1.5, 2.25)	2.5
1.1	(1.1, 1.21)	2.1
1.01	(1.01,1.0201)	2.01
1.001	(1.001,1.002)	2.001

2 The gradient of [AB] approaches 2

3 Completed table:

x	B(x, f(x))	Gradient of [AB]
0	(0,0)	1
0.8	(0.8, 2.25)	1.8
0.9	(0.9, 1.21)	1.9
0.999	(0.999,1.0201)	1.999

From the left, the gradient of [AB] also approaches 2.

4 $\dfrac{(x+h)^2 - x^2}{(x-(x+h))} = 2x + h$

5 **Conceptual:** How is the gradient of a tangent to the curve at a point related to the gradient of a secant line passing through the same point?

Answer: The gradient of the secant line between two points approaches the gradient of the tangent at one of these two points as the difference in the x-coordinates between the two points approaches 0.

6 As $h \to 0$, we see that the gradient of the tangent to $y = x^2$ at any point x is $y = 2x$

7 2

8 **Conceptual:** What does the gradient of the tangent to the curve $y = f(x)$ at any point tell you about the instantaneous rate of change between x and y at that point?

Answer (this is the conceptual understanding): Tangents to a curve at a point identify the instantaneous rate of change of its function at specific values of the independent variable.

Investigation 6

Conceptual understanding:

The power rule provides an efficient method for finding the derivative of a polynomial function.

1 Completed table:

x	-2	-1	0	1	2	3
$\frac{dy}{dx}$	-4	-2	0	2	4	6

3 $\frac{dy}{dx} = 2x$

4 Completed table:

Function	Gradient function
$y = x^2$	$2x$
$y = x^3$	$3x^2$
$y = x^3$	$4x^3$

5 $y = nx^{n-1}$

6 **Conceptual:** How is the power rule useful in differentiating a polynomial function?

Answer (this is the conceptual understanding): The power rule provides an efficient method for finding the derivative of a polynomial function.

Investigation 7

Conceptual understanding:

The derivative of a linear function in gradient intercept form provides the gradient of the graph.

A constant function represents a horizontal line, with a gradient of zero.

The parameter m represents the gradient of a straight line in the formula $y = mx + c$, hence m represents the derivative of the linear function.

1 **Factual:** What is the gradient of a straight line parallel to the x-axis?

Answer: zero

2 **a-d** The derivative of each function is zero.

3 **Conceptual:** What is the derivative of a constant function?

Answer (this is the conceptual understanding): A constant function, i.e., a line parallel to the x-axis, has derivative 0.

4 **Factual:** For a straight line with equation $y = mx + c$, which parameter tells you the gradient of the line?

Answer: m

5 **a** -1 **b** 2 **c** -0.5

6 **Conceptual:** How is the derivative of a linear function related to the gradient of its graph?

Answer (this is the conceptual understanding): The parameter m represents the gradient of a straight line in the formula $y = mx + c$, hence m represents the derivative of the linear function at every point.

TOK

Mathematics and the real world: the seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?

Answer: To explore the relation between mathematical models and reality, the essential issue of mathematical modelling is dependent on social and personal construction processes where absolute agreement cannot be expected.

A prime pragmatic function of such models is to enable calculations and predictions of physical phenomena. Mathematical models are useful also in representing aspects of reality that are hard to visualize. Models serve as conceptual frameworks that can lead to important physical discoveries.

TOK

Mathematics - invented or discovered?

If mathematics is created by people, why do we sometimes feel that mathematical truths are objective facts about the world rather than something constructed by human beings?

Answer: Ask students to write down three things that were invented and three things that were discovered.

Inventions might include the airplane, lightbulbs, computer and discoveries might include dinosaur fossils, magnets, the source of the Nile.

Ask students to create and share their own definitions.

We are now approaching one of the big TOK questions; is mathematics discovered or invented?

An invention is something that was not previously there. A discovery concerns something that already exists at the time of discovery but was previously unknown. As a result of the discovery, nothing has changed apart from an associated increase in knowledge.

Is there something that is in the intersection? What about music?

Developing inquiry skills

Is it possible to find Usain Bolt's speed at any particular moment during a race? How could you use what you have learned in this section to suggest a method to find his instantaneous speed?

Answer: Consider a graph of distance against time for Bolt's race. You can find an average speed by considering $\frac{s(t+h)-s(t)}{(t+h)-t}$. As h gets very small, you will have an expression for his instantaneous speed at t.

5.3 Differentiation rules

TOK

What is the difference between inductive and deductive reasoning?

Answer: Deductive reasoning usually follows steps. A suitable definition is that deductive reasoning is the process by which a person makes conclusions based on previously known facts.

You might want to give examples such as "all dolphins are mammals, all mammals have kidneys; therefore, all dolphins have kidneys", or $x = y$ and $y = z$, therefore, $x = z$.

Inductive reasoning is the opposite of deductive reasoning. Inductive reasoning draws conclusions based on a set of observations. This might not be a valid method of proof. Just because a you observe a number of situations in which a pattern exists doesn't mean that that pattern is true for all situations.

Examples such as "60% of the students in your class like strawberries, so 60% of people in the world like strawberries", or "Ali's mother and father are doctors, Ali is a doctor, so Ali's brother must be a doctor".

Investigation 8

Conceptual understanding:

The chain rule may be used to find the derivative of composite functions.

1 $(1+x)^2 = 1+2x+x^2$

$\frac{dy}{dx} = 2+2x$

$\frac{dy}{dx} = 2(1+x)$

2 For example, let $g(x) = (1+x)$ and $f(x) = x^2$

3 $\frac{du}{dx} = \frac{d}{dx}(1+x) = 1$

4 $y = f(u) = u^2$

$\frac{dy}{du} = 2u$

5 $\frac{dy}{du}\cdot\frac{du}{dx} = 2u\cdot 1 = 2\left(1+x\right)\cdot 1 = 2\left(1+x\right)$

6 **Factual:** Compare the expression you obtained for $\frac{dy}{du}\cdot\frac{du}{dx}$ to the expression you found for $\frac{dy}{dx}$ in question 1.

Answer: The expressions are the same.

7 Repeat questions 1 – 6 for each of these functions.

Question	$y = (1-2x)^2$	$y = (3x-1)^2$	$y = (1+ax)^2$
1	$\frac{dy}{dx} = -4(1-2x)$	$\frac{dy}{dx} = 6(3x-1)$	$\frac{dy}{dx} = 2a(1+ax)$
2	$g(x) = (1-2x)$ and $f(x) = x^2$	$g(x) = (3x-1)$ and $f(x) = x^2$	$g(x) = (1+ax)$ and $f(x) = x^2$
3	$\frac{du}{dx} = \frac{d}{dx}(1-2x) = -2$	$\frac{du}{dx} = \frac{d}{dx}(3x-1) = 3$	$\frac{du}{dx} = \frac{d}{dx}(1+ax) = a$
4	$\frac{dy}{du} = 2u$	$\frac{dy}{du} = 2u$	$\frac{dy}{du} = 2u$
5	$\frac{dy}{du}\cdot\frac{du}{dx} = 2u(-2)$ $= 2(1-2x)(-2)$ $= -4\left(1-2x\right)$	$\frac{dy}{du}\cdot\frac{du}{dx} = 2u(3)$ $= 2(3x-1)(3)$ $= 6(3x-1)$	$\frac{dy}{du}\cdot\frac{du}{dx} = 2u(a)$ $= 2(1+ax)(a)$ $= 2a(1+ax)$
6	same expression	same expression	same expression

8 **Conceptual:** How can you find the derivative of a composite function $y = f\left(g\left(x\right)\right)$?

Answer (this is the conceptual understanding): The derivative of a composite function $y = f\left(g\left(x\right)\right)$ is: $\frac{dy}{dx} = f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$

Investigation 9

Conceptual understanding:

The product rule provides an efficient method for finding the derivative of a product of functions without expanding them.

1

f(x)	Expand	f′(x)
x^3	x^3	$3x^2$
$x(x+1)$	x^2+x	$2x+1$
$(x-1)(x+1)$	x^2-1	$2x$
$(2-x)(3-x^2)$	$6-2x^2-3x+x^3$	$3x^2-4x-3$

2

f(x)	u(x)	u′(x)	v(x)	v′(x)	u(x)v′(x)+v(x)u′(x)
x^3	x	1	x^2	$2x$	$2x^2+x^2=3x^2$
$x(x+1)$	x	1	$(x+1)$	1	$x+x+1=2x+1$
$(x-1)(x+1)$	$(x-1)$	1	$(x+1)$	1	$x-1+x+1=2x$
$(2-x)(3-x^2)$	$(2-x)$	-1	$(3-x^2)$	$-2x$	$3x^2-4x-3$

3 **Factual:** For each function $f(x)$, compare the expression you found for $f'(x)$ in question 1 with the expression you found for u(x)v′(x)+v(x)u′(x) in question 2.

Answer: The expression for $f'(x)$ is the same as the expression for u(x)v′(x)+v(x)u′(x).

4 **Conceptual:** How is the product rule useful to help you differentiate the product of two polynomial functions?

Answer (this is the conceptual understanding): The product rule provides an efficient method for finding the derivative of a product of functions without expanding them.

TOK

Who do you think should be considered the discoverer of calculus?

Answer: The debate over whether Newton or Leibnitz discovered certain calculus concepts.

An opportunity to have students research and present a document/wall display.

Instructions to students might include:

- I think it was Newton or Leibniz
- You find one more person from a calculus timeline.
- Take on the role of their representative and in one paragraph, state their case with reasons and evidence.
- Now take on the role of the judge and in one paragraph, stet who you think should be called the discoverer of calculus and why.

You might want a picture of the mathematicians with their paragraphs and a picture of the student with their summary. This will see students taking ownership of their decisions that might even be called “risk taking”.

Investigation 10

Conceptual understanding:

The quotient rule provides an efficient method of differentiating rational functions without having to change the expression into a product form.

1 $u(x) = Q(x)v(x)$

2 $u'(x) = v(x)Q'(x) + Q(x)v'(x)$

3 $Q'(x) = \dfrac{u'(x) - Q(x)v'(x)}{v(x)}$

4 $Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

5 **Conceptual:** Which method (the product rule or the quotient rule) would be more efficient to use for differentiating rational functions?

Answer (this is the conceptual understanding): The quotient rule provides an efficient method of differentiating rational functions without having to change the expression into a product form.

Reflect: If you are asked to differentiate an expression in an examination, how could you determine which of the three differentiation rules (chain, product, and quotient rules) to use?

Answer: If the expression contains a composite function ($u\left(v\left(x\right)\right)$) you will always need to use the chain rule.

If the expression contains a product ($f\left(x\right)\cdot g\left(x\right)$) you will always need to use the product rule.

If the expression contains a quotient $\dfrac{f\left(x\right)}{g\left(x\right)}$ you can choose whether to use the quotient rule, or to write as $f\left(x\right)\left(g\left(x\right)\right)^{-1}$ and use the product rule (unless you're specifically told to use one or the other).

When using the product and quotient rules, if either of the functions f and g contain a composite function (e.g., $f\left(x\right) = u\left(v\left(x\right)\right)$), you will need to use the chain rule too.

5.4 Graphical interpretation of first and second derivatives

Investigation 11

Conceptual understanding:

Examining the signs of the gradients of a function identifies the increasing/decreasing intervals of the function.

1

x	-2	-1	0	1	3	4	5
$f'\left(x\right)$	36	15	0	-9	-9	0	15

2 Positive; function is increasing.

3 Negative; function is decreasing.

4 Positive; function is increasing.

5 **Conceptual:** What do the signs of the gradient of $f(x)$ tell you about whether $f(x)$ is increasing or decreasing on a given interval?

Answer (this is the conceptual understanding): Examining the signs of the gradients of a function identifies the intervals where the function is increasing/decreasing.

$f'(x) > 0 \Rightarrow f(x)$ is increasing

$f'(x) < 0 \Rightarrow f(x)$ is decreasing

Investigation 12

Conceptual understanding:

Examining the signs of the gradients to the left and right of stationary points help determine the nature of stationary points, and identify increasing/ decreasing intervals of the function.

1 0

2 negative

3 positive

4 **i** $x > 0$ **ii** $x < 0$

5 **Conceptual:** What is the nature of a stationary point if the gradient of the parabola changes from negative to positive in going through the turning point?

Answer (this is the conceptual understanding): The gradient of a parabola at points close to a minimum point change from negative to positive in going through the point.

6 0

7 positive

8 negative

9 **Conceptual:** What is the nature of a stationary point if the gradient of the parabola changes from positive to negative in going through the stationary point?

Answer (this is the conceptual understanding): The gradient of a parabola at points close to a maximum point change from positive to negative in going through the point.

10 **Factual:** In the interval where the gradients are negative, is the function increasing or decreasing?

Answer: decreasing

11 **Factual:** In the interval where the gradients are positive, is the function increasing or decreasing?

Answer: increasing

12 **i** $x<0$ **ii** $x>0$

13 **Conceptual:** How can you identify the intervals on which a function is increasing/decreasing using the first derivative test?

Answer (this is the conceptual understanding): Examining rates of change close to stationary points helps you identify intervals where the function increases or decreases.

14 **Conceptual:** How does the 1st derivative test help in classifying local extrema and identifying intervals where a function is increasing/decreasing?

Answer (this is the conceptual understanding): Examining the signs of the gradients to the left and right of extrema help determine their nature and identify the intervals where the function is increasing/decreasing.

TOK

The Nature of Mathematics: Does the fact that Leibnitz and Newton came across the Calculus at similar times support the argument of Platonists over Constructivists

Answer: Platonists would say that mathematical objects exist independently of the human mind and are, thus, discovered. This is often claimed to be the view most people have of numbers.

This objective existence, however, does not mean an empirical existence but, rather, an abstract existence, hence its 'Platonic' label.

The main problem with a Platonistic view of mathematics is an epistemic one. If, indeed, mathematical objects are abstract objects and objectively exist - then how do we know anything about them?

Constructive mathematics requires that proof be algorithmic. The emphasis in constructive theory is placed on hands-on provability, instead of on an abstract notion of truth.

Note that these are not the only two philosophies of mathematics.

Investigation 13

Conceptual understanding:

The 2nd derivative describes the rate of change of the gradient function (1st derivative); therefore at a stationary point an increasing first derivative function represents a minimum while a decreasing first derivative function represents a maximum.

When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.

1 The gradients go from negative (at points which are left of the minimum point), to 0 (at the minimum), to positive (at points which are right of the minimum). The graph of f′(x) is below the x-axis where the gradients of f are negative, and above the x-axis where the gradients of f are positive.

2 positive

3 increasing

4 The gradients go from positive (at points which are left of the minimum point), to 0 (at the minimum), to negative (at points which are right of the minimum). The graph of f′(x) is above the x-axis where the gradients of f are positive, and below the x-axis where the gradients of f are negative.

The sign of the gradient of f′(x) at any value of x is negative.

5 decreasing

6 f′′(x) is positive at a local minimum and negative at a local maximum.

7 Students should find that their results from question 6 hold true for the function

$$f(x) = 3 + x + \frac{1}{x}.$$

8 **Conceptual:** How is the second derivative useful in classifying local extrema?

Answer (this is the conceptual understanding): The second derivative describes the rate of change of the gradient function (1st derivative); therefore at a stationary point an increasing first derivative function represents a minimum while a decreasing first derivative function represents a maximum.

9 No, since the 2nd derivative is 0, but the stationary point is a minimum point.

10 **Conceptual:** How can you determine the nature of a turning point when the second derivative test is inconclusive?

Answer (this is the conceptual understanding): When the 2nd derivative test fails to determine the nature of a stationary point, revert to testing the gradient on each side of the stationary point to test for a point of inflexion.

Investigation 14

Conceptual understanding:

Examining rate of change close to turning points and inflexion points helps you identify intervals where the function increases/decreases, and examining the rate of the rate of change of the intervals helps you identify the concavity of the function that is when the first derivative is increasing or decreasing.

1 $f'(x) = 3x^2 - 3$; $f''(x) = 6x$

2 $x = 0$; gradient of f′ at $x = 0$ is 0

3 No

4 $f''(0) = 0$

5 Concave down to the left of $x = 0$ and concave up to the right of $x = 0$.

6 negative to the left of $x = 0$ and then positive to the right of $x = 0$.

7 Students should conclude that to the left of $x = 0$, $y = -f(x)$ is concave up and $f''(x)$ is positive; whilst to the right of $x = 0$, $y = -f(x)$ is concave down and $f''(x)$ is negative.

8 **Conceptual:** Why does the sign of a function's second derivative at any point indicate the concavity of the function at that point?

Answer (this is the conceptual understanding): Examining the signs of the 2nd derivative at a stationary point of f' can help identify the intervals where the function is concave up/down since the 2nd derivatives indicates if the first derivative is increasing/decreasing.

9 $f''(c) = 0$ at the point of inflexion.

10 No. Using their results from question 9, students should conjecture that $f''(x) = 0$ at each point of inflexion. For $f(x) = x^4$, it is true that $f''(0) = 0$, but this is a minimum point, not a point of inflexion.

11 **Conceptual:** Analyse the concavity of $f(x) = x^3 - 3x + 1$ on both sides of the point of inflexion, and the concavity of $f(x) = x^4$ on both sides of its stationary point. Can you determine the additional condition necessary for f to have a point of inflection at x=c.

Answer: In addition to $f''(c) = 0$, $f''(x)$ must change sign as x passes through c. This indicates a change in concavity.

Investigation 15

Conceptual understanding:

A horizontal point of inflexion satisfies the conditions that the first derivative and second derivative equal zero and a change in concavity on either side of the stationary point.

1 The concavity changes at $x = 0$ and $f''(0) = 0$.

2 $f'(0) = 0$

3 Positive

The signs of the gradients would change in going through a turning point.

A point of inflexion whose gradient is parallel to the x-axis.

6 **Conceptual:** What three conditions are necessary for a function $f(x)$ to have a horizontal point of inflexion at $x = c$

Answer (this is the conceptual understanding): At a horizontal point of inflexion *c*, $f'(x) = f''(x) = 0$ and the concavity of f changes as *x* passes through *c*.

International-mindedness

The Greeks' mistrust of zero meant that Archimedes' work did not lead to calculus.

Answer: Archimedes, one of the greatest ancient Greek mathematicians of all times. Archimedes was a Greek ancient mathematician, astronomer, physicist, inventor, and engineer. He is credited with introducing infinitesimals, the foundation of calculus.

His work was not considered valid as the ancient Greeks did not have zero in their counting system and had a general mistrust of the number.

Investigation 16

Conceptual understanding:

Identifying turning points, points of inflexion, intervals where the function increases/decreases, and concavity, facilitates sketching the derivatives of the function.

3 **i** $-2 < x < -1$; $1 < x < 2$, f' is positive

ii $-\infty < x < -2$; $-1 < x < 1$; $2 < x < +\infty$, f' is negative.

5 Points of inflexion

6 Concave up: $(-\infty,-1)$, $(0,1)$, $f'' > 0$; concave down: $(-1,0)$, $(1,2)$, $f'' < 0$

8 **Conceptual:** Describe how you can sketch the graphs of $y = f'(x)$ and $y = f''(x)$ from the graph of $y = f(x)$.

Answer (this is the conceptual understanding): Identifying turning points, points of inflexion, intervals where the function increases/decreases, and concavity, facilitates sketching the derivatives of the function.

TOK

Mathematics and Knowledge Claims: Euler was able to make important advances in mathematical analysis before Calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work.

What does this suggest regarding intuition and imagination in Mathematics?

Answer: You might want to consider a few different ways of knowing with statements such as:

Reason and imagination are equally important to understanding mathematics. If one can only visualize it in the head but cannot express it satisfactorily, then the conclusion is flawed and inconsistent.

Much work can also be done using intuition if one has an ingenious insight of the material. For instance, Einstein knew the big ideas of his general theory of relativity but he just lacked the necessary mathematical language to present it until he was helped by mathematicians such as Sir Arthur Eddington.

Investigation 17

Conceptual understanding:

Identifying the zeros of a derivative function, examining positive/ negative intervals of the range of the function and examining concavity facilitates sketching the function from the graph of its derivative.

2 Stationary points of f occur when $f'(x) = 0$; happens when x = 0, 3

3 $f(x)$ is increasing when $f'(x) > 0$, and $f(x)$ is decreasing when $f'(x) < 0$;

i]3, ∞[**ii**]-∞, 0[∪]0,3[

4 $f(x)$ is concave up when $f'(x)$ is increasing (that is equivalent to when $f''(x)$ is positive) and $f(x)$ is concave down when $f'(x)$ is decreasing (that is equivalent to when $f''(x)$ is negative).

i $]-\infty, 0[$ and $]2, \infty[$ **ii** $]0, 2[$

7 They are all vertical translations of one another.

8 **Conceptual:** Given the graph $y = f'(x)$, describe how can you sketch a possible graph of $y = f(x)$

Answer (this is the conceptual understanding): Identifying the zeros of a derivative function, and examining intervals where the function is above/below the x-axis, and where it is concave up/down all facilitate sketching the function from the graph of its derivative.

Investigation 18

Conceptual understanding:

The zeros of f' occur where f'' has stationary points. Where the graph of f'' goes from positive to negative, f' has a maximum, and where the graph of f'' goes from negative to positive f' has a minimum.

1 f has possible points of inflexion where the graph of f'' is 0.

2 **Conceptual:** What information can you deduce about $f'(x)$ from the graph of $y = f''(x)$.

Answer (this is the conceptual understanding): The zeros of f' occur where f'' has stationary points. Where the graph of f'' goes from positive to negative, f' has a maximum, and where the graph of f'' goes from negative to positive f' has a minimum.

5 Discuss results as a class.

5.5 Application of differential calculus: optimization and kinematics

Investigation 19

Conceptual understanding:

Optimization in Calculus uses mathematical models, or functions, to provide largest and least-value solutions to real-life problems.

1 $A = 2\pi r^2 + 2\pi rh$; radius and height of the cylinder

2 $V = \pi r^2 h = 330$

3 $h = \dfrac{330}{\pi r^2} \Rightarrow A = 2\pi r^2 + \dfrac{660}{r}$; $r > 0$

4 $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - \dfrac{660}{r^2} = 0$ for min

Solving gives r = 3.74 cm; h = 7.49 cm; A = 264 cm^2

6 **Conceptual:** What is optimization in Calculus?

Answer (this is the conceptual understanding): Optimization in Calculus uses mathematical models, or functions, to provide largest and least-value solutions to real-life problems.

7 E.g., cola can: r = 6.63 cm; h = 11.5 cm; A = 376 cm^2

8 Some considerations might be: average hand-size of soft drink consumers, stacking costs on shelves, cost of buying aluminium in bulk, desired aesthetics of can, etc.

Investigation 20

Conceptual understanding:

Finding a mathematical model for a general case allows for multiple applications to specific cases.

2 $A = x(10 - x)$; domain $0 < x < 10$. $x > 0$ because width cannot be negative, and $x < 10$ in order that the length of the enclosure is non-zero.

3 length = width = 5m; Area = 25m^2

4 The largest enclosure is a square.

5 If width = x, then length = $\dfrac{P - 2x}{2}$ and hence $A = \dfrac{1}{2}x(P - 2x)$.

Differentiating this: $\dfrac{dA}{dx} = \dfrac{P}{2} - 2x$.

Setting equal to zero: $\dfrac{dA}{dx} = 0 \Rightarrow x = \dfrac{P}{4}$. Hence, all sides of the rectangle are equal, therefore the rectangle is a square.

6 Letting x be the base of the triangle gives 19.2 cm^2; equilateral triangle.

7 **Conceptual:** How does finding a mathematical model for a general case help you apply solutions to particular cases?

Answer (this is the conceptual understanding): Finding a mathematical model for a general case allows for multiple applications to specific cases.

TOK

How can you justify the raise in tax for plastic containers e.g. plastic bags, plastic bottles etc. using optimization?

Answer: An important environmental concern that is well document in the media where students might research and write a report using a mathematical model which would allow them to access the skills of mathematical presentation, communication and personal engagement with areas such as the countries where customers have to pay for plastic supermarket bags or a tax increase on water sold in plastic bottles.

Investigation 21

Conceptual understanding:

An object moving in a positive direction represents positive velocity while an object moving in a negative direction represents negative velocity.

Speed is the absolute value of velocity.

2 **a** $0 < t < 1$, Ben is moving from A to B; $3 < t < 4$, Ben is again moving from A to B

b $1 < t < 3$, Ben is moving from B to A

3 **Conceptual:** What does a positive or negative value for velocity represent?

Answer (this is the conceptual understanding): An object moving in a positive direction (when Ben is running from A to B in this case) has positive velocity, and an object moving in a negative direction (when Ben is running from B to A in this case) has negative velocity.

4 **Conceptual:** What is the connection between speed and velocity?

Answer (this is the conceptual understanding): Speed is the absolute value of velocity.

5 At $t = 1$ Ben has reached B and so is momentarily stopped as he turns around; at $t = 3$ Ben has reached A and so is momentarily stopped as he turns around.

6 At $t = 0$ and $t = 4$ Ben reaches maximum velocity of 9 m s^{-1}. Ben was already running at 9 m s^{-1} when he passed point A for the first time, and he is also running at 9 m s^{-1} when he passes point B for the final time.

Investigation 22

Conceptual understanding:

Both positive or both negative velocity and acceleration indicate an increasing speed of the object since they describe the same direction. Velocity and acceleration with different signs indicate a slowing down since they exhibit opposite directions.

1 $v(t) = 3t^2 - 14t + 11; a(t) = 6t - 14$

2 **a** $t = 1, 3.67$; the graph of speed against time cuts the x-axis at these points.

b $1 < t < 2.33$; both acceleration and velocity are negative in this time interval, so speed is increasing in negative direction.

$t > 3.67$; both acceleration and velocity are positive in this time interval, so speed is increasing in positive direction.

c $0 < t < 1$; velocity is positive but acceleration is negative, so the particle's speed is decreasing as it travels in the positive direction.

$2.33 < t < 3.67$; velocity is negative but acceleration is positive, so the particle's speed is decreasing as it travels in a negative direction.

3 **Conceptual:** What must be true about the signs of the velocity and acceleration in order for a particle to speed up, and in order for a particle to slow down?

Answer (this is the conceptual understanding): Both positive or both negative velocity and acceleration indicate an increasing speed of the object since they describe the same direction. Velocity and acceleration with different signs indicate a slowing down since they exhibit opposite directions.

4 $t = 1$, $t = 3.67$; the particle's speed changes sign at these points.

Developing inquiry skills

Going back to the opener problem, the distance the sprinter has travelled at various times is given by the function $s(t) = -0.102t^3 + 1.98t^2 + 0.70t - 0.65$ for $0 \le t \le 10$ seconds.

1 What is the fastest instantaneous speed of the sprinter in the race?

Answer:

$v(t) = s'(t) = -0.306t^2 + 3.96t + 0.7$

$v'(t) = 0$

$t = 6.47$ seconds

$v(6.47) \simeq 39\ \text{ms}^{-1}$

2 Draw the graph of v(t) and state whether or not the sprinter is running at a constant speed throughout the race.

Answer:

The graph of v(t) is not linear, so the speed of the sprinter is not constant.

3 How could you determine the sprinter's speed at any particular instant $t = c$?

Answer: Find the value of $s'(c)$.

River Crossing

Approaches to Learning: Thinking Skills: Evaluate, Critiquing, Applying

Exploration Criteria: Personal Engagement (C); Reflection (D); Use of Mathematics (E)

IB Topic: Differentiation, Optimization

Introduction

This task introduces students to the idea that their inspiration for an exploration can come from many different sources—one of which, as in this case, may be examples and questions in textbooks. However, if they use these then they are advised to contextualize the idea in real life in order to score well in Criterion C: Personal Engagement. They also need to reflect on the occasional necessary simplifications, assumptions, guesses or estimates that are required in order to model and solve a complicated situation with multiple, and often unknowable, variables (Criterion D: Reflection). The task also gives students the opportunity to practise another optimization problem and demonstrate their understanding of the mathematics of this (Criterion E: Use of Mathematics).

It is possible to use textbook examples and questions from exercises as inspiration or springboards for an exploration idea. If a question resonates with a student or reminds them of a particular experience or can be adapted, then it may be possible to develop it into a workable exploration idea.

The Problem

To start, you could discuss the problem as a whole class.

Students should start by answering the "textbook" question given, using the optimization techniques learnt in this chapter.

Visualize the Problem

Encourage students to always sketch a diagram to visualize a written mathematical problem. Remind them of the importance of labelling their diagram carefully.

You start at *A*.

B is directly opposite you, on the other bank of the river.

C is a point on the opposite bank x km from *B*.

D is the campground.

Solve the Problem

Using Pythagoras' theorem $AC = \sqrt{x^2 + 1}$

Remind students of the kinematic formulae that they met in the chapter:

If travel is at a constant rate of speed then

$$\text{Time taken} = \frac{\text{Distance travelled}}{\text{Speed}}$$

Time taken to swim from A to C = $\dfrac{\sqrt{x^2+1}}{3}$

Time taken to run from C to D = $\dfrac{2-x}{8}$

Total time taken T = $\dfrac{\sqrt{x^2+1}}{3} + \dfrac{2-x}{8}$

To find $\dfrac{dT}{dx}$, students should first simplify the expression for T:

$$T = \frac{\sqrt{x^2+1}}{3} + \frac{2-x}{8} = \frac{1}{3}\left(x^2+1\right)^{\frac{1}{2}} + \frac{1}{4} - \frac{x}{8}$$

They can then differentiate:

$$\frac{dT}{dx} = \frac{1}{3} \times \left(x^2+1\right)^{-\frac{1}{2}} \times 2x - \frac{1}{8} = \frac{x}{3\sqrt{x^2+1}} - \frac{1}{8}$$

They can find the minimum time taken by setting $\dfrac{dT}{dx}$ equal to zero:

$$\frac{x}{3\sqrt{x^2+1}} - \frac{1}{8} = 0$$

$$\frac{x}{3\sqrt{x^2+1}} = \frac{1}{8}$$

$$8x = 3\sqrt{x^2+1}$$

$$64x^2 = 9\left(x^2+1\right)$$

$$64x^2 = 9x^2 + 9$$

$$55x^2 = 9$$

$$x^2 = \frac{9}{55}$$

$$x = \pm\frac{3}{\sqrt{55}}$$

Now $x \neq -\dfrac{3}{\sqrt{55}}$ because x is a length and so cannot be negative.

Therefore $x = \dfrac{3}{\sqrt{55}} = 0.405$ km

To evaluate the validity of the value—the value is less than 2 and it is positive.

For **extension**, students could also check that it is a minimum value by using the second derivative test:

For x = 2:

$$T = \frac{\sqrt{\left(\frac{3}{\sqrt{55}}\right)^2 + 1}}{3} + \frac{\left(2 - \frac{3}{\sqrt{55}}\right)}{8} = 0.360 + 0.199 = 0.559 \text{ (3 sf)}$$

Therefore, the minimum time is 0.559 hours (= 33.5 minutes)

You would swim from point *A* to point *C*, which is 0.405 km from *B*. From there you would run the remaining 1.595 km to the campground. The swim would take 0.360 hours (21.6 minutes) and the run would take 0.199 hours (11.9 minutes).

Assumptions Made in the Problem

Assumptions could include:

The river is exactly 1 km wide.

The campground is exactly 2 km along the bank on the over side of the river.

It is possible to swim at a constant speed of 3 km/h.

The river flow does not push you downstream or make it more difficult to maintain speed in the middle.

There is no time taken to enter and leave the river.

It is possible for you to run at a constant speed of 8 km/h—over any terrain—for the remaining distance.

The banks of the river are perfectly parallel and straight.

When you are standing at the edge of the river, you are unlikely to know:

The width of the river

The distance to the campground on the other side

The speed you can swim or run.

Encourage students to explore and discuss the problem and their answers to the questions.

You could ask:

What methods could you use to find this information?

How accurate would any distance measurements be?

What assumptions would you need to make when calculating any speeds?

For finding distances, it is possible to use trigonometry or similar triangles if you have instruments to measure distances and angles.

Note: Some students may already be familiar with using right-angled triangles and trigonometry. Similar triangles are presumed knowledge.

As an **extension**, you could guide students to research these methods.

It may also be possible to measure on a map if it is a reliable scale, or use GPS if it is available.

Another method is to guess or estimate, based on a known distance.

The answers will have varying degrees of accuracy depending on the accuracy of the measuring instruments used or the ability to guess accurately.

Calculations of speed are most likely to be based on a known running/swimming time for a given distance.

The assumptions are that this time remains constant regardless of river flow or terrain.

Additional information you would need to know to determine the shortest time possible:

The effect of the flow of the river.

What you are carrying and whether this will have any effect.

Emphasize to students that in order to make it possible to answer the question in "real life", certain assumptions, guesses and estimates do have to be made, otherwise the question would be too complicated to answer, with too many unknowns and variables.

However, if this were an exploration it would be important to reflect critically on any assumptions made and the subsequent significance and limitations of the results.

Extension

In this chapter students have been introduced to some classic optimization problems in the examples and exercises. For example, there is the “open-box problem” on page 275 and the “volume of a cylindrical can” on page 260.

One problem with the open-box example is that it does not provide the box with any extra material to build stability into the construction. A box built in such a way would probably not be strong and so it would be wise to have tabs on the sides being folded that could be used to add this stability.

In both the open-box problem and the can problem, the question of aesthetics is not considered, and neither is functionality. Cans, for example, are not always built in such a way as to provide the maximum volume for the minimum use of materials but also with design, packaging and functionality (e.g. considering average grip size or the need for can openers, etc) in mind.

This could produce an interesting discussion on design and mathematics, etc.

6 Representing data: statistics for univariate data

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Organising, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Modelling through statistics can be reliable, but may have limitations.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
Different sampling techniques can be used to give a better representation of the population as a whole.	Investigation 1
An extreme value distorts the mean so that it is not very representative of the population as a whole.	Investigation 2
Comparison of different data sets may involve comparing measures of central tendency and measures of dispersion.	Investigation 4
Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation. Multiplying every value in a data set by a constant results in the mean and standard deviation being multiplied by that constant. Standard deviation represents a particular measure of dispersion which compares each data point with the mean.	Investigation 5

Syllabus sections covered in this chapter:

- SL4.1*
- SL4.2*
- SL4.3*

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 277: Statistical graphs, Measures of central tendency, Exponential expressions	Page 291: Example 7 Page 302: Example 11 Page 309: Example 12	Page 283: Example 1 Page 291: Example 6 Page 299: Example 10	Pages 282, 286, 294, 309

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 310	Page 311	N/A

6.1 Sampling

International-mindedness

Ronald Fisher (1890-1962) lived in the UK and Australia and has been described as "a genius who almost single-handedly created the foundations for modern statistical science". He used statistics to analyse problems in medicine, agriculture and social sciences.

Who else could be considered “the father of statistics”?

Reflect: Is the taxiing speed of an airplane discrete or continuous?

Answer: Taxiing speed: continuous

Reflect: Is the number of airplanes waiting to take off discrete or continuous?

Answer: Number of airplanes: discrete

Reflect: Could data ever be classified as both discrete *and* continuous?

Why is it important to consider the nature of the variable, rather than just the data values themselves, when classifying whether data is discrete or continuous?

Answer: It is the nature of the variable, not the data values themselves, that defines whether data is discrete or continuous. Continuous data, such as measurements of length and time, might be recorded to the nearest integer value, but it is still continuous data because length and time are continuous variables.

Reflect: How many students would you need to survey to find a good estimate of the average time students at your school spend doing homework?

Answer: This depends on the number of students in the school, and the age ranges etc. Students could discuss this. It also leads onto *how* to conduct the survey for it to be best representative of the population. This will be covered next.

Investigation 1

Conceptual understanding:

Different sampling techniques can be used to give a better representation of the population as a whole.

1 Completed table:

Sampling technique	Advantages	Disadvantages
Beth's suggestion: interview their friends	very easy to conduct	only takes students from one year group, so may not be representative
Emily's suggestion: interview two people from each year group	chooses a good spread from across the different year-groups	Only gives a very small sample size, so may not be big enough to be representative
Natasha's suggestion: pick 10 boys and 10 girls	gives an equal spread across the sexes in case there is a difference between males and females.	May not choose equal amounts of people from different year-groups.
Amanda's suggestion: Use a random number generator to choose a sample	Very fair and systematic	Might not get a good split between males & females, or across the year-groups
Greg's suggestion: choose every 20^{th} person from an alphabetical list	Very fair and systematic Would give a sample of size 75, which is about the right size.	Might not get a good split between males & females, or across the year-groups

2 Answers may vary. Perhaps the fairest way would be to divide students by year group, and then again by gender. Within those subgroups (boys in year 7, girls in year 7, etc.), you could choose every 20^{th} person.

This would give a good sample size, and an equal proportion of males and females across all year groups within the school.

3 Students' own answers. Alternative methods may include asking everybody in school, or asking the students' council.

4 **Conceptual:** Why must you consider the context of a scenario in order to choose an appropriate sampling technique?

Answer (this is the conceptual understanding): Different sampling techniques can be used to give a better representation of the population as a whole.

Reflect: In investigation 1, five students suggested sampling techniques to help investigate the mean average time that each student in their school spend doing homework.

Decide which category of sampling technique each of the five suggestions fits into.

Answer:

Beth's suggestion: interview their friends	Convenience
Emily's suggestion: interview two people from each year group	Stratified
Natasha's suggestion: pick 10 boys and 10 girls	Stratified
Amanda's suggestion: Use a random number generator to choose a sample	Simple random
Greg's suggestion: choose every 20th person from an alphabetical list	Systematic

TOK

The nature of mathematics: why have mathematics and statistics sometimes been treated as separate subjects?

Answer: Teacher's might want to consider the history and relevant "newness" of statistics. The amount of numeracy, tables, formulas and charts put statistics into most school mathematics curricula, but as statistics has become more important, its connections with everyday human sciences, natural science, arts and languages suggest teaching statistics across the curriculum might be more appropriate.

This is a good place for teachers to view the knowledge framework for mathematics.

Developing inquiry skills

In the opening problem for the chapter, you were given the test scores, out of 10, of 32 students.

Are the test scores an example of discrete or continuous data?

Before marking every student's test paper, the teacher wishes to choose a sample of 8 papers to mark first that will give her an estimate of the mean average mark for the class. Describe a suitable sampling method the teacher could use.

Answer: Scores are discrete data.

It would be most appropriate to use quota sampling. She should divide students into strata depending on ability, and select students from each ability group in proportion to the size of that group.

6.2 Presentation of data

TOK

Can you justify using statistics to mislead others?

How easy is it to be misled by statistics?

Answer: Misleading statistics; examples of problems caused by absence of representative samples, e.g. Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston "pot-hole".

Reflect: What do the frequency table and bar chart show you about the test scores that you didn't see from the raw data?

Answer: Answers will vary, but students should pick up on the shape / distribution of the data, which is easier to see from a bar chart than from a table. The majority of scores are grouped around the median.

For those students who have met types of distribution before, they may notice that scores are fairly normally distributed.

TOK

The nature of knowing: is there a difference between information and data?

Answer: A common misconception is to consider them to mean the same thing.

Data is raw, unorganized facts or numbers.

When you organise data or present it in a given context so as to make it useful, it is called information.

For example: the individual students' scores on a test are data but the class mean is information.

Reflect: Write a question for your classmates that will require them to collect continuous data. Show this as a frequency table and draw a histogram for your data.

Answer: Students answer will vary.

6.3 Measures of central tendency

Reflect: Write down what you now know about the mode. Share your answer with a classmate and discuss.

Answer: The mode is the most popular, the highest bar in a bar chart, the biggest sector in a pie chart.

Investigation 2

Conceptual understanding:

An extreme value distorts the mean so that it is not very representative of the population as a whole.

1 285 smiles

2 685 smiles

3 34.3 smiles per day

4 **Conceptual:** What happens to the mean when we add one extreme value to the data set?

Answer (this is the conceptual understanding): An extreme value distorts the mean so that it is not very representative of the population as a whole.

5 The mode (or median, which we will come onto next). These are not affected by extreme values.

TOK

Do different measures of central tendency express different properties of the data?

How reliable are mathematical measures?

Answer: In question 4 and the associated discussion box, you might want to look at number sets where the mean, mode and median are different, and ask which the best measure is to use.

Consider the responses in terms of the perspectives of the different people.

You might want to consider the readings on reliability in statistics.

Reflect: Copy and complete the following table. You should use your GDC to calculate the mean, mode and median each time.

Answer: Completed table:

	Data	Mean	Mode	Median
Data Set	6, 7, 8, 10, 12, 14, 14, 15, 16, 20	12.2	14	13
Add 4 to each data set		16.2	18	17
Multiply the original data set by 2		24.4	28	26

a If you add 4 to each data value the mean, mode and median go up 4.

b If you multiply each data value by 2 the mean, mode and median are multiplied by 2.

Developing inquiry skills

1. Find the mean, mode, and median of the class scores from the start of this chapter.
2. Which of mean, median, or mode gives the best indication of the 'average' score? Are any two averages roughly equal?

Answers:

1. mean 5.25, mode 7, median 5.
2. Mean and median are roughly similar, but for this data, the mean is probably the correct measure to use as there are no outliers and the data is normally distributed.

6.4 Measures of dispersion

Reflect: The values from Q_1 to Q_3 are sometimes called the range of the "middle half" of the data. Can you explain why is this the case?

Answer: Q1 is the first quarter, Q3 is the third quarter $\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$

TOK

To what extent can we rely on technology to produce our results?

Answer: The median, and the upper quartile divide the ordered data into four groups with approximately the same number of observations in each group.

How can you do this for a small sample size like 2.5, 3.1, 6?

For small samples, the is no obvious way to do this, and concessions of some sort must be made.

Do the quartiles have any meaning for samples of this size?

There are different algorithms in technology simply because different people have different ideas how to make the compromises. They may have slightly different objectives in mind how to use the quartiles in practice.

Respond to the question "To what extent can we rely on technology to produce our results?".

Investigation 3

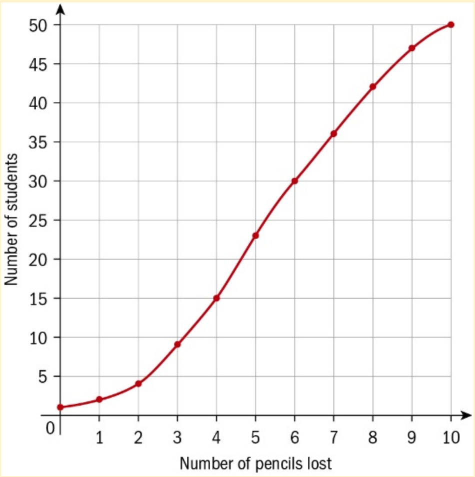
1 Completed table:

Pencils	0	1	2	3	4	5	6	7	8	9	10
Number of students who lost this many	1	1	2	5	6	8	7	6	6	5	3

2 Completed table:

Pencils	0	1	2	3	4	5	6	7	8	9	10
Number of students who lost this many or less.	1	2	4	9	15	23	30	36	42	47	50

3



4 The number of students losing *x* pencils increases at a fairly constant rate in the middle of the graph (when $3 \le x \le 9$, say) but starts off and finishes at a lower rate (for $x = 1, 2, 10$).

5 The number of students and pencils always increases.

International-mindedness

Why are there different formulae for the same statistical measures like mean and standard deviation?

Answer: The sample and population are at work here. You might want to look into when we should use Greek letters like μ and when we should use the symbols from the modern English alphabet like $\bar{x}$.

TOK

Is standard deviation a mathematical discovery or a creation of the human mind?

Answer: A typical teacher led discussion might go something like this:

Over the centuries people have debated whether mathematics is discovered, or if it is simply invented by the minds of great mathematicians.

What do you think?

If you think it is discovered, where are you looking?

If you think it is invented, why can't a mathematician say that he has 4 times 2 =10?

Now, what about the standard deviation?

Investigation 4

Conceptual understanding:

Comparison of different data sets may involve comparing measures of central tendency and measures of dispersion.

1 Completed table:

	Total points	Mean	Median	Range	Standard Deviation
Player A	150	15	15	5	1.41
Player B	150	15	15	39	11.5

2 Situation 1. The coach would be better choosing player A. Whilst the players have the same mean and median, the lower standard deviation and range indicate that he is more reliable.

3 Situation 2. The coach would be better choosing player B. Whilst the players have the same mean and median, the higher standard deviation and range indicate that he is more likely to be the one to score a large number of points.

4 **Conceptual:** What factors do you need to take into account to compare two data sets effectively?

Answer (this is the conceptual understanding): Comparison of different data sets may involve comparing measures of central tendency and measures of dispersion

Investigation 5

Conceptual understanding:

Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation.

Multiplying every value in a data set by a constant results in the mean and standard deviation being multiplied by that constant.

Standard deviation represents a particular measure of dispersion which compares each data point with the mean.

1 mean: 3.9; standard deviation: 2.47

2 104, 102, 100, 109, 103, 105, 105, 101, 104, 106.

3 **Factual:** Find the mean of data set B. How does this differ from the mean of data set A? Can you explain why this is the case?

> **Answer:** Mean of set B is 103.9. This is because each of the values has increased by 100, so the mean average of them will have increased by 100 too.

4 **Factual:** Find the standard deviation of data set B. How does this differ from the standard deviation of data set A? Can you explain why this is the case?

> **Answer:** Standard deviation of set B is 2.47. It is the same as that of set A. The standard deviation is unchanged because the spread of the numbers remains the same.

5 **Conceptual:** Using your answers to questions 3 and 4, state what happens to the mean and standard deviation of a data set which has a constant c added to each value.

> **Answer (this is the conceptual understanding):** Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation.

6 8, 4, 0, 18, 6, 10, 10, 2, 8, 12

7 **Factual:** Find the mean of data set C. How does this differ from the mean of data set A? Can you explain why this is the case?

> **Answer:** Mean of set C is 7.8. Since each value has been multiplied by 2, the mean has also been multiplied by 2.

8 **Factual:** Find the standard deviation of data set C. How does this differ from the standard deviation of data set C? Can you explain why this is the case?

> **Answer:** The standard deviation of data set C is 4.94. Since the average distance of all numbers from the mean is now doubled, the standard deviation will be doubled.

9 **Conceptual:** Using your answers to questions 7 and 8, state what happens to the mean and standard deviation of a data set when each value is multiplied by a constant d.

> **Answer (this is the conceptual understanding):** Multiplying every value in a data set by a constant results in the mean and standard deviation being multiplied by that constant.

Reflect: How will the variance of a data set be affected either by adding c to every value, or by multiplying each value by d? Explain why this is the case.

> **Answer:** Adding a constant value, c, to a random variable does not change the variance, because the spread remains constant. Multiplying a random variable by d increases the variance by the square of d.

Developing inquiry skills

In the opening problem for the chapter, you were given the test scores, out of 10, of 32 students.

Represent the scores on a cumulative frequency graph

Write a five-number summary for the data from the opening problem on class test scores.

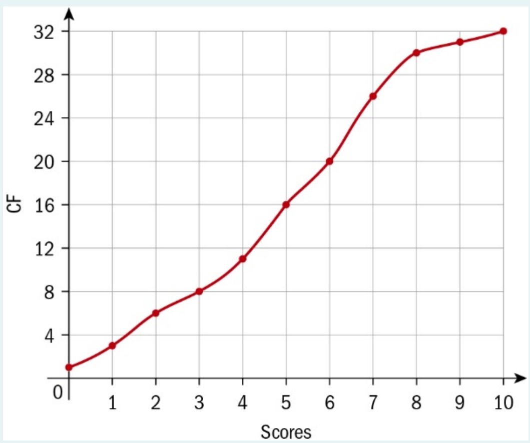
Draw a boxplot for the data.

What does this tell you about the class? How would you allocate grades A, B, C and D?

What would happen to the mean, median and standard deviation if the teacher decided to multiply all of the scores by 10 to show them as a percentage?

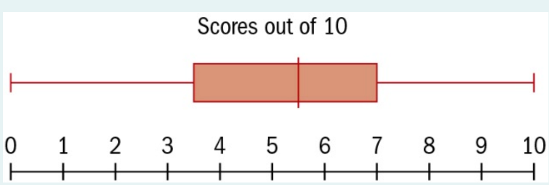
Answer:

1



2 Min = 0, $Q_1 = 3.5$ Median = 5 $Q_3 = 7$ max = 10

3



4 There is a wide range of abilities. Answer to the grades may vary. A possible answer is grade A for the to 24%, B for the second 25%, C for the third, D for the fourth.

5 The mean, median and standard deviation would all be multiplied by 10.

What's the Difference?

Approaches to Learning: Thinking Skills, Communicating, Collaborating, Research

Exploration Criteria: Presentation (A); Mathematical Communication (B) Personal Engagement (C); Reflection (D); Use of Mathematics (E)

IB Topic: Statistics, Mean, Median, Mode, Range, Standard Deviation, Box Plots, Histograms

Introduction

Many IAs use statistics. The basis of such explorations is often students making and testing a hypothesis by collecting data. The data are analysed using some of the techniques from this chapter. Many of these explorations take the form of scientific enquiry, and so this is a good opportunity for you to discuss the use of mathematics as a tool in other subjects. In this task these skills are developed further.

Students are required to make decisions about the data that they need to collect to meet the aim of the question they pose themselves. They can then analyse the data that are collected, and decide whether the analysis supports their hypothesis.

Students could prepare a mini-exploration (a couple of pages only), poster or presentation to demonstrate what they have found out. Students could then discuss what methods of data collection, analysis and representation are appropriate and effective. This will obviously require a longer amount of time allocated to it in class but, given that this is a common exploration topic, it may be worthwhile. It may be necessary, and indeed desirable, depending on different class situations, for students to work in pairs, groups or as a whole class. It is certainly a good opportunity for collaboration between classmates as they discuss appropriate methods, analysis, conclusions and limitations.

The task could be assessed against a shortened form of the real IA assessment criteria (rather than all of it, as that can be a little daunting—a suggestion is given below), so this task offers a good opportunity to introduce or revisit the criteria.

Criterion A: Presentation (2)

Your writing should be well-organized, coherent, logically developed and easy to follow. It should include an aim (hypothesis) and conclusion.

Criterion B: Mathematical Communication (2)

Use appropriate mathematical language and representation, and define key terms.

Criterion C: Personal Engagement (2)

Present mathematical ideas in your own way. Run experiments to collect data.

Criterion D: Reflection (2)

Review, analyse and evaluate. Considering the significance of your findings and results. Stating possible limitations and/or extensions. Explain why you chose this method rather than another.

Criterion E: Use of Mathematics (2)

Demonstrate that you fully understand the mathematics used in your exploration.

TOTAL (10)

Example Experiment

To help to introduce the experiment, you could demonstrate a reaction-time test (or similar) to the class. Ask students to discuss in pairs, groups or as a class:

What could possibly make your performance better or worse if you do it again?

Why?

How you could test this?

Students might think of these changes to Raghu's experiment:

Participants use their non-dominant hand instead. Performance is likely to get worse.

Participants do the test at a different time of day (for example, early in the morning or late at night). Performance might be better in the morning, and worse late at night.

Participants are now allowed to practise for a while and are then asked to do it again. After practice, performance is likely to improve.

Participants are observed while doing the test a second time. Performance could get worse if participants feel nervous.

Students might think of these different groups:

A younger or older group—you might expect younger people to have faster reactions than older people.

A different gender group—this would probably have no effect on the results.

A group who have been given different instructions—depending on the instructions, this could improve the performance or make it worse, or it could have no effect.

A group that has been given some sort of motivation/incentive—a group with motivation is likely to perform better.

The discussions should hopefully make the next part of the task more straightforward and help students to come up with ideas of their own. They can of course also use one of these suggestions.

Your Experiment

In the example given, reaction times have been used. There are many different sites that can be used to test reaction times. However, there are also good online sites that test different skills and abilities, for example estimating angle sizes or times. There are also others that test ability with respect to multitasking or spatial awareness.

Students can choose to base their hypotheses on the example experiment and on one of the given suggestions, or they can devise their own experiment and hypothesis.

Here are some sites that can be used to collect data:

Reaction timer	nrich
Multitasking	notdoppler
Reaction timer	faculty.washington.edu
Reaction timer	faculty.washington.edu
Reaction timer	faculty.washington.edu
Reaction timer	faculty.washington.edu
Reaction timer	faculty.washington.edu
Eye-balling game	woodgears
The sheep dash game	BBC
Subitizing	BBC
Estimating angles	nrich
Estimating time	nrich

It is possible to conduct experiments without technology as well.

For example, for the reaction-time test, students could stand on one foot for the second test, or they could do the test without, and then with, eyes closed or they could do the test before and then after exercise.

Step 1: What you are going to test? State your aim and hypothesis.

You could refer back to the example of Raghu's experiment to help students to answer the questions:

Raghu could write the following (note these are brief responses only at this stage—in a real exploration it would be possible and necessary to expand on these with more detail):

> *The aim of my exploration is to investigate the effect of the time of day on the reaction time of a group of students at my school.*
>
> *My hypothesis is that if students do the same reaction test in the morning and then again in the afternoon, the general performance would be worse in the afternoon because students are likely to be more tired.*
>
> *This is important as it has implications to when students are given or asked to perform particular tasks that require reactions and concentration.*

Make sure it is clear for what students are testing. Are they testing, for example, for:

a change in the average?

a change in the spread of the data?

some combination of the above?

Step 2: How you are going to collect the data? Write a plan.

Students should share their aims, hypotheses, plans and responses to the questions with another student in the class or with a group or the whole class before they begin collecting any data. They can then discuss what is good and what might need improving.

Step 3: Do the experiment and collect the data.

Students should ensure that everyone knows what they are doing before starting the experiment. It may be advisable that only a couple of tests are chosen for the class to conduct, otherwise there could be too many tests going on at once and it could get very confusing. Students could vote somehow on which of the experiments they would like to run and work collectively as a class or in small groups to overcome any of the difficulties of data collection that might arise. It may be necessary, if possible, to arrange for each group to use a different classroom to do their experiment, since by discussing the question itself they may be biasing the results in their own group. This is also interesting for discussion.

It might not be possible for this to be done in a single lesson, so homework/pre-planning may be required.

Step 4: Present the data for comparison and analysis.

Students could use some of the ideas in this chapter, for example, tables, box plots or histograms, to present their data.

Students will need to be aware of the difference between the mean and standard deviation of a population and that of a sample.

Note: this is not covered in this chapter, but will be an important distinction to make in an exploration.

Step 5: Comparison and analyse.

You could ask:

Which summary statistics are appropriate for your aim and hypothesis?

Emphasize to students that they should only calculate what is needed. For example, if they are not interested in the spread of data then there is no need to calculate the standard deviation; however, it might be useful when discussing the comparison of means, in which case they would need to find it.

Make sure that students only calculate what is needed, and that their conclusion is relevant.

Step 6: Conclusions and implications

Discussing the implications and scope of a student's results is essential to this task. They will need to understand the nature of inductive reasoning here and that their conclusions are only valid for their particular experiment. Although it may be possible to speculate beyond the scope of their particular investigation, this is not going to be conclusive.

You could ask:

Are your conclusions limited to this group?

Extension

There are statistical tests that can be found for testing whether the spread of the class data has changed significantly rather than just the average performance.

Students could think about what other methods of representation there are to analyse changes in individual results rather than whole-class changes. Students may devise questions that require bivariate data analysis, which is covered in the next chapter.

It is possible to test data using a more "mathematical" test.

Note: the "difference in means" test is beyond the SL curriculum but could be accessible to some students.

7 Modelling relationships between two data sets: statistics for bivariate data

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Organizing, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Some techniques of statistical analysis, such as regression, standardization or formulae, can be applied in a practical context to apply to general cases.
- Modelling through statistics can be reliable, but may have limitations.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
The line of best fit by eye represents a trend line and must pass through the mean of each data set.	Investigation 1
Linear regression analyses bivariate data and finds the line of best fit using the sum of the least squares approach.	Investigation 2
Piecewise linear models display situations where one single equation of a line does not fit the entire set of data points.	Investigation 3

Syllabus sections covered in this chapter:

- SL4.4*
- SL4.10

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 319: Exponential expressions, Equations of lines, Using the gradient formula to find the equation of a line	Page 334: Example 5 Page 338: Example 6	Page 320: Example 1 Page 327: Example 3 Page 334: Example 5	Pages 324, 327, 332, 342

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 342	Page 344	N/A

7.1 Scatter diagrams

International-mindedness

In 1956, Australian statistician, Oliver Lancaster made the first convincing case for a link between exposure to sunlight and skin cancer using statistical tools including correlation and regression.

The correlation between smoking and lung cancer was "discovered" using mathematics.

Science had to justify the cause.

Reflect: Can you suggest two variables which have

a positive correlation?

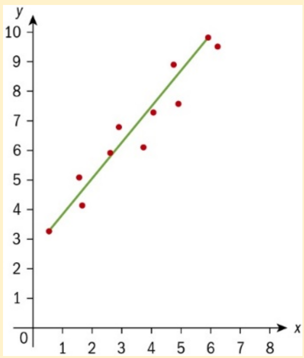
a negative correlation?

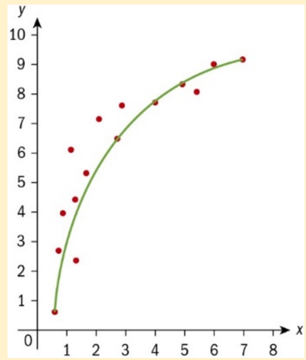
no correlation?

Answer: Some examples might be hours studied and a student's grade in mathematics or the length of a person's arm and their height. A negative correlation might be the price of a car and its age for the first twenty years or the increase in speed of a train and the length of time to get to the final destination. Examples of no correlation could include the number of pets that a student has and their ability to speak Spanish or the GDP of a country and the amount of rainfall received.

Reflect: How would you describe the correlation between variables *x* and *y* in these two scatter diagrams?

Answer:





strong, positive, linear

Strong, positive, non-linear. The points on this graph form a curve.

TOK

What is the difference between correlation and causation?

To what extent do these different processes affect the validity of the knowledge obtained?

Answer: Correlation is the idea of modelling a pattern based on data. Causation is using data as proof that one thing causes the other.

Does correlation need causation?

What is a cause and effect relationship?

Is making a model for a given situation valid as personal knowledge?

Developing inquiry skills

In the opening problem for the chapter, you saw how a rice farmer recorded the data of rainfall (in cm) and rice yield (in tons) from his farm for the last eleven years.

Rainfall (cm)	*130*	*150*	*140*	*180*	*350*	*210*	*190*	*100*	*160*	*175*	*145*
Rice yield (tonnes)	*94*	*100*	*90*	*120*	*20*	*150*	*135*	*80*	*130*	*110*	*105*

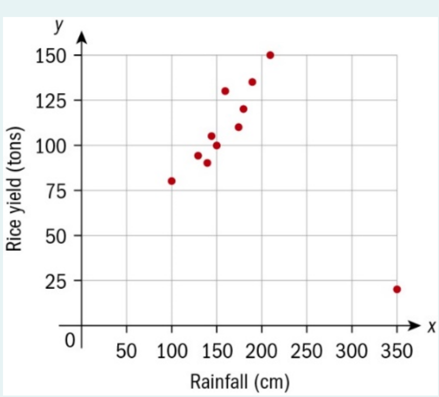
Using what you have learned in this chapter

a Draw a scatter diagram for the data

b Describe the correlation

Answer:

a



b Strong, positive, linear.

7.2 Measuring Correlation

TOK

We can often use mathematics to model everyday processes.

Do you think that this is because we create mathematics to emulate real life situations or because the world is fundamentally mathematical?

Answer: Mathematics has been called the rational mind at work.

When most abstracted from the world, mathematics stands apart from other areas of knowledge, concerned only with its own internal workings.

In its practical form it is an integral part of many other fields—engineering, medicine, marketing, architecture.

There are also many everyday life uses in the home from house maintenance to recipes.

TOK

Can all data be modelled by a known mathematical function?

Answer: Consider the reliability and validity of mathematical models in describing real-life phenomena.

Developing inquiry skills

In the opening problem for the chapter, you saw how a rice farmer recorded the data of rainfall (in cm) and rice yield (in tons) from his farm for the last eleven years.

Rainfall (cm)	130	150	140	180	350	210	190	100	160	175	145
Rice yield (tonnes)	94	100	90	120	20	150	135	80	130	110	105

Using what you have learned in this chapter

1 Plot a scatter diagram for this data, and describe the correlation.

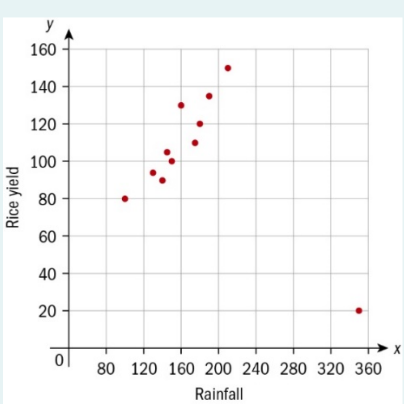
2 Use technology to find the r-value for this bivariate data.

3 Observe the point at (350, 20). This distorts the data. remove the outlier and redraw your scatter diagram.

4 Find the r-value without the (350, 20). Is the correlation stronger or weaker when this point is removed?

5 Give a reason why the outlier may have occurred.

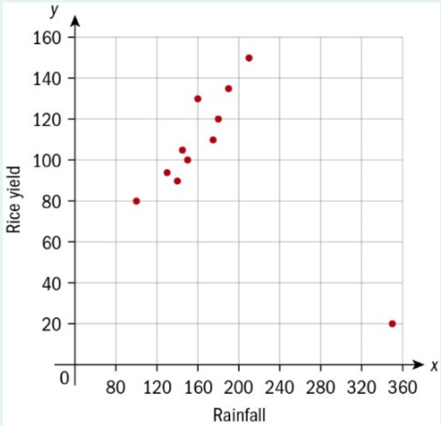
Answer:

1 

Strong, positive, linear, except for the one point at (350, 20)

2 -0.447; this describes a negative correlation. The point at (350, 20) is distorting the data.

3



4 0.918; this is a strong positive correlation. The *r*-value is much more accurate with the point (350, 20) removed.

5 The rainfall is very high, the rice fields were flooded.

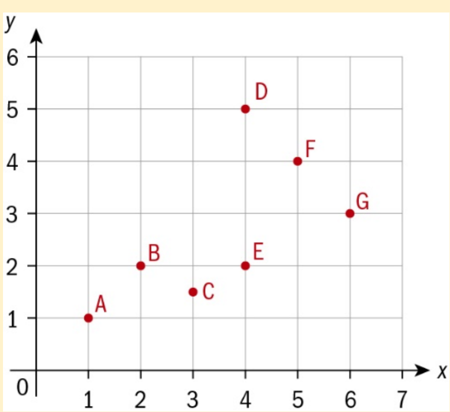
7.3 The line of best fit

Investigation 1

Conceptual understanding:

The line of best fit by eye represents a trend line and must pass through the mean of each data set.

1



2 Positive, linear correlation

3 Lines may vary, and may or may not pass through the origin. You might discuss how data in a given context might necessarily pass through the origin – e.g. a best fit line showing height against weight will pass through the origin.

Students also might elicit that a best fit line should balance the number of points above and below the line.

4 $\bar{x} = 3.57$, $\bar{y} = 2.64$

5 Lines should be adjusted accordingly

6 **Conceptual:** Can you explain why your line of best fit should pass through the mean point $(\bar{x}, \bar{y})$?

Answer: $(\bar{x}, \bar{y})$ gives you an average of the data. The best-fit line tells you the average relationship which the data satisfies. Hence, the best fit line should pass through the mean point $(\bar{x}, \bar{y})$.

7 **Conceptual:** What does the line of best fit represent, and what point should it pass through?

Answer (this is the conceptual understanding): The line of best fit by eye represents a trend line and must pass through the mean of each data set.

TOK

Is extrapolation knowledge gained using intuition and, possibly, emotion?

If so, how would you describe interpolation in terms of ways of knowing?

Answer: An opportunity to write a blog post or have a class debate using the ways of knowing.

7.4 Least squares regression

Investigation 2

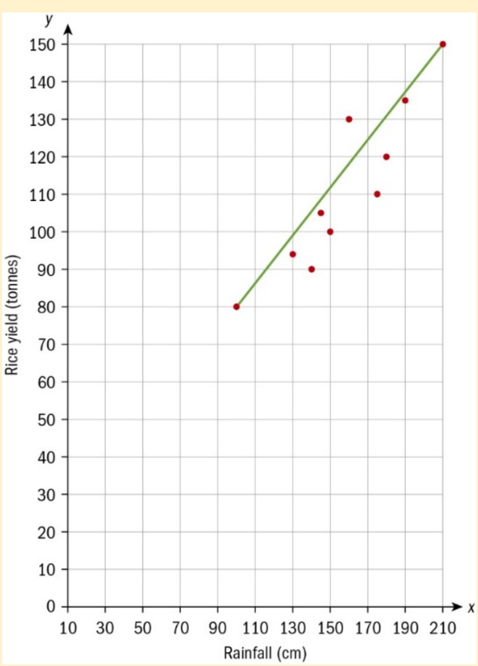
Conceptual understanding:

Linear regression analyses bivariate data and finds the line of best fit using the sum of the least squares approach.

1 Mean rice yield: 158

mean rainfall: 111.4

2



An approximate line of best fit is $y = 0.8x - 18$

3 Students should draw in the residuals, measure them and square them, and record their results.

4 Students should reach the conclusion that the lower the sum of squares of residuals, the more accurate their best fit line is.

5 Students will require access to the internet or Geogebra, either individually or in groups.

6 Calculator uses least sum of squares of residuals to calculate the equation of the line of regression.

7 **Conceptual:** How does the regression line estimate the line of best fit?

Answer (this is the conceptual understanding): Linear regression analyses bivariate data and finds the line of best fit using the sum of the least squares approach.

TOK

To what extent can you reliably use the equation of the regression line to make predictions?

Answer: If you were able to make predictions about something important to you, you'd probably feel more comfortable. It's even better if you know that your predictions are sound.

TOK

"Everything that can be counted does not count. Everything that counts cannot be counted" (Albert Einstein).

What counts as understanding in mathematics?

Answer: A class discussion might follow by showing the TOK box to the class and following up with questions like -

Is it sufficient to get the right answer to a mathematical problem to say that one understands the relevant mathematics?

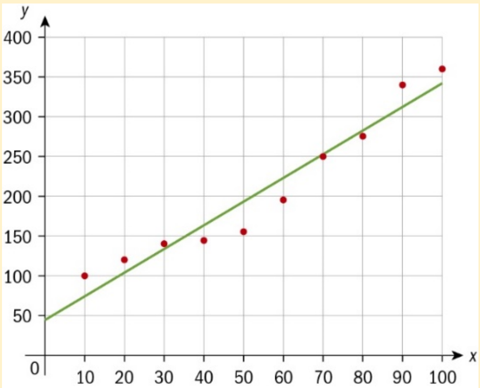
Demonstrating understanding is seen in criterion E in the exploration.

Investigation 3

Conceptual understanding:

Piecewise linear models display situations where one single equation of a line does not fit the entire set of data points.

1, 2



3 **Factual:** Comment on how well the best fit-line fits the data, and how accurately you could use it to make predictions.

Answer: Earlier points are below and later points are above the line. Not a good fit for prediction.

4 **Factual:** Explain how this information might cause you to conclude that a single regression line is not enough to best represent the relationship between number of rods and production cost.

Answer: When Robin produces less than 50 rods, he uses one machine and one regression line models the relationship. When he produces more than 50 rods, he uses two machines and so a different regression line is needed.

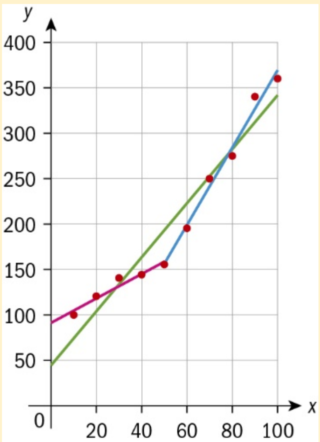
5 **Factual:** What is a piecewise linear model?

Answer: A graph which has two different straight lines for different values of *x.*

6 **Conceptual:** When can piecewise models be used to represent bivariate data?

Answer (this is the conceptual understanding): Piecewise linear models display situations where one single equation of a line does not fit the entire set of data points.

7



8 The piecewise model is a better fit.

Developing inquiry skills

Look back at the scatter graph you produced for the rice farmer's data on the amount of rainfall in cm and rice yield in tonnes from the last eleven years.

1 Find the mean point.

2 Draw a line of best fit by eye through the mean point.

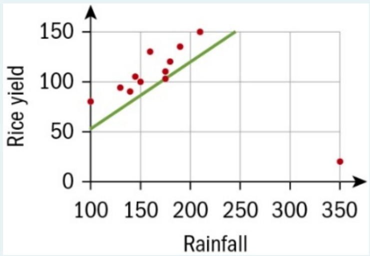
3 Now find the regression line in the form y = ax + b from your GDC and draw this on the same graph.

4 Comment on which is the best line to represent the data and why.

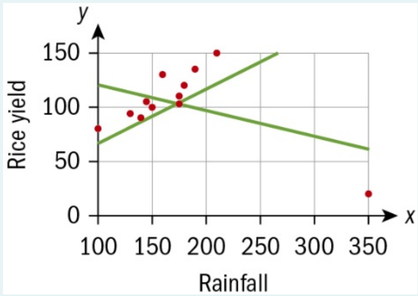
Answer:

1 (175, 103)

2



3 $y = -0.237 + 145$



4 The line of best fit by eye is better as the GDC graph is distorted by the outlier.

Rank My Maths

Approaches to Learning/Learner Profile: Collaboration, Communication

Exploration Criteria: Personal Engagement (C); Use of Mathematics (E)

IB Topic: Bivariate Data, Correlation, Spearman's Rank

Introduction

This task asks students to consider Spearman's rank as a valid alternative to Pearson's. In explorations, students often automatically consider Pearson's correlation because that is the correlation method they have been taught. They will then blindly apply a linear best fit line which is entirely inappropriate given the data under discussion. This shows a lack of mathematical understanding and prevents students from reaching the higher levels in Criterion E: Use of mathematics. By researching and using Spearman's rank as a possibly valid alternative, students can also gain credit in Criterion C: Personal Engagement, as they will be exploring new mathematics.

There is also the opportunity to design an experiment that looks at student's rankings and to compare these.

Pearson Product Moment Correlation Coefficient

Remind students that Pearson's product moment correlation (PMCC) evaluates the **linear** relationship between two continuous variables. A relationship is **linear** when a change in one variable is associated with a proportional change in the other variable.

In the example, there is clearly a strong relationship between x and y. This is not linear (perhaps exponential).

PMCC is 0.726

This suggests there is a moderate linear correlation.

The relationship is not actually linear, so the PMCC does not show a strong correlation even though a strong relationship exists.

Spearman's Rank Correlation

A more complete definition of the Spearman's rank correlation is:

> *The Spearman correlation evaluates the **monotonic** relationship between two **variables**. In a monotonic relationship, the variables tend to change together, but not necessarily at a **constant rate**. The Spearman correlation coefficient is based on the **ranked values** for each variable rather than the **raw data**.*

Check that students understand all of the bold words in the given definition.

Note: Spearman's rank correlation is not on the SL syllabus, but it shows an alternative method.

Monotonic: A monotonic relationship is a relationship that does one of the following:

1) as the value of one variable increases, so does the value of the other variable

2) as the value of one variable increases, the other variable value decreases.

This is a good app that allows students to play around with points and see the effect on both the Pearson and Spearman correlations:

www.economicsnetwork.ac.uk/statistics/pearson_spearman.htm

As **extension** work, students could investigate how the formula for Spearman's correlation is derived.

Spearman's rank correlation can be used to see the strength and direction of the monotonic relationship between two variables and can therefore be useful in explorations.

This table will help you interpret Spearman's rank correlation coefficient.

For the given data, $r_s = 1 - 0 = 1$

Students may need help to evaluate the value of r_s.

This table shows how to interpret Spearman's rank correlation coefficient.

r_s	Correlation
$r_s = 1$	perfect positive correlation
$0.7 \le r_s < 1$	strong positive correlation
$0.4 \le r_s < 0.7$	moderate positive correlation
$0 < r_s < 0.4$	weak positive correlation
$r_s = 0$	no correlation
$0 > r_s > -0.4$	weak negative correlation
$-0.4 \ge r_s > -0.7$	moderate correlation
$-0.7 \ge r_s < -1$	strong negative correlation
$r_s = -1$	perfect negative correlation

This calculated value indicates perfect Spearman's rank correlation.

The answer will always be between −1 and 1.

As with Pearson, a value close to −1 represents a strong negative rank correlation and near to +1 represents a strong positive rank correlation. A value close to 0 suggests no rank correlation. 1 is a perfect positive monotonic relationship and −1 is a perfect monotonic negative relationship.

Activity 1

It is important that students appreciate that if a scatterplot shows that the relationship between two variables looks monotonic, they should run a Spearman's correlation, because this will then measure the strength and direction of this monotonic relationship. However, if the relationship appears linear, they should run a Pearson's correlation, because this would measure the strength and direction of any linear relationship. If students are not able to check visually whether they have a monotonic relationship, they might run a Spearman's correlation anyway.

Pearson's correlation is not appropriate because the relationship is non-linear, negative and strong.

If needed, assist students in following the steps to calculate the Spearman's rank correlation coefficient for these data.

For **extension** work, students could also get the value by putting the **ranks** (rather than the raw data) into a graphical calculator and finding the value of *r* as described in this chapter. This is the value of r_s for the data:

The value is −0.721.

There appears to be a strong negative correlation between; however, it is not linear.

Activity 2

Introduce this task by explaining that Spearman's rank can also be useful in explorations, as it gives you the opportunity to design an experiment that could compare the rankings given to something by two people (or two sets of people) to determine how similar they are and what agreement there is.

Arrange students in groups of three if possible (pairs and fours would also work, but would mean fewer/more results to analyse). Given the nature of the task it may be worth using randomly picked groups (although using groups that are not random and allowing students to choose groups also gives interesting discussions).

Instruct the groups to select between six and ten different pieces of music.

Students could perhaps choose a random playlist or the first few random songs in someone's music collection, or they could deliberately choose particular songs if they wish to test a particular hypothesis.

Students should rank their songs by number, from favourite to least favourite, with 1 being the favourite.

Here is an example of a table that could be used:

Song number	**Student 1 rank**	**Student 2 rank**	**Student 3 rank**
1			
2			
3			
Etc...			

To prompt discussion, you could ask:

Do you expect the rankings within your group to be the same, similar or completely different?

Could you predict who might have similar tastes (where the strongest correlation would be)?

Check that the students calculate the Spearman's rank correlations correctly.

When evaluating the correlations, students should consider the size and direction of the values.

Check that students write clear, concise and relevant conclusions.

Extension

Students could think about designing a similar experiment to compare ranks in students' taste in films, art, hobbies, food, etc.

The experimenter should be careful to not give any indication of bias or preference themselves.

It is important also that there is no discussion between students, so that they do not come to an agreement/disagreement but rather choose independently and without undue influence.

Also encourage students to prepare the table/form/etc. that they will use to records their results before they conduct their experiment. This will also ensure that they have considered the nature of the results that they are going to obtain.

8 Quantifying randomness: probability

Essential understandings

Probability enables us to quantity the likelihood of events occurring and so evaluate risk. It provides important representations which enable us to make predictions, valid comparisons and informed decisions.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Difficult probabilities can be modelled with simulation and so allow us to make predictions even in complex situations The expected number of occurrences represents a prediction of how many times an event will likely occur in a given number of trials and can help to evaluate risk.	Investigation 1
Short-run and long-run experimental probabilities differ so the number of trials of an experiment may alter the accuracy of predictions. As the number of trials increases the relative frequency approaches an estimate for the probability.	Investigation 2
Venn diagrams are useful in illustrating relationships and obtaining results in probability.	Investigation 3
Understanding that moving from probability diagrams to formulae helps develop the ability to make generalised predictions.	Investigation 4
‘With’ or ‘without’ replacement categorizes the analysis and representation of probability events since the outcome of the previous event can sometimes be used to make predictions about future events.	Investigation 8

Syllabus sections covered in this chapter:

- SL4.5*
- SL4.6*
- SL4.11

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students’ understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 533: Fractions, orders of operations, percentages	Page 376: Example 7 Page 378: Example 8 Page 384: Example 10	N/A	Pages 361, 373, 379, 385

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 386	Page 388	N/A

8.1 Theoretical and experimental probability

Investigation 1

Conceptual understanding:

Difficult probabilities can be modelled with simulation and so allow us to make predictions even in complex situations

The expected number of occurrences represents a prediction of how many times an event will likely occur in a given number of trials and can help to evaluate risk.

Factual: What is a simulation?

Answer: A probability simulation is a way to model random events

Why does this choice of outcomes represent Andy being better at chess than Bexultan?

Answer: Andy has a $\frac{2}{3}$ chance of winning. Bexultan has $\frac{1}{3}$ chance.

How much better is Andy than Bexultan in this situation?

Answer: Andy is therefore 'twice as likely to win' than Bexultan

What other choice of outcomes would also suggest that Andy is better than Bexultan?

Answer: Could use 1,2,3,4,5 for Andy and 6 for Bexultan. 1-6 for Andy would mean that Bexultan never wins.

What is you had a dice with 12 sides? What different combinations would represent Andy being better than Bexultan?

Answer:

Andy	Bexultan
1-7	8-12
1-8	9-12
1-9	10-12
1-10	11-12
1-11	12

1-4 Class results will be different. Discuss

5 Unpack the word 'expect' – this is the theoretical number based on the probability.

6 These numbers would change as the numbers changed. If Andy has 1-5 then we would expect him to win more often.

7 We would need to know the long-term average number of wins for A and B. We could use a dice with more sides so that the probability could be closer to the observed probability over time. We could use a computer simulation.

8 **Conceptual:** How can a simulation be helpful in making predictions?

Answer (this is the conceptual understanding): Difficult probabilities can be modelled with simulation and so allow us to make predictions even in complex situations

9 **Conceptual:** How can we calculate an expected number of occurrences?

Answer: If the probability of an event is P, in *n* trials you would expect the event to occur $n \times P$ times.

(This is the conceptual understanding): The expected number of occurrences represents a prediction of how many times an event will likely occur in a given number of trials and can help to evaluate risk.

TOK

Play the game of the St Petersburg Paradox and decide how much would you pay to play the game.

Answer: The game in the St Petersburg paradox follows these rules. You flip a coin, if it lands tails up then you lose and the game is over. If the coin lands heads up then you win one ruble and the game continues. The coin is tossed again. If it is tails, then the game ends and you keep the money you have won.

If it is heads, then you win an additional two rubles.

For each successive head you double your winnings from the previous round, but, at the first tail, the game is over.

Investigation 2

Conceptual understanding:

Short-run and long-run experimental probabilities differ so the number of trials of an experiment may alter the accuracy of predictions.

As the number of trials increases the relative frequency approaches an estimate for the probability.

1 Students should be encouraged to record their results systematically.

2 The more times you roll the more accurate the estimate will be.

3 Use and combine class results for even more accurate prediction.

4 Probably not.

5 **Conceptual question:** Why does the number of trials of an experiment affect the accuracy of predictions?

Answer: The more trials run the better the estimate becomes as we are approaching an infinite number of trials.

(This is the conceptual understanding): Short-run and long-run experimental probabilities differ so the number of trials of an experiment may alter the accuracy of predictions

6 There is a balance between accuracy and lengthy experiments. Enough is when the results begin to ‘settle down’

7 **Conceptual:** How many trials of an experiment before we can be sure of the theoretical probability?

Answer: When the estimated probability begins to approach a value.

8 Student graphs will vary.

9 The graph is approaching a value.

10 **Factual:** What happens to the relative frequency of a square face as the number of rolls increases?

Answer: It becomes closer to the true value.

11 It may not be exact but it is a good estimate.

12 **Conceptual:** What happens to the relative frequency as the number of trials increases?

Answer (this is the conceptual understanding): As the number of trials increases the relative frequency approaches an estimate for the probability.

TOK

When watching crime series, reading a book or listening to the news, the evidence of DNA often closes a case. If it was that simple, no further detection or investigation would be needed. Research the “Prosecutor’s fallacy”.

How will reason contrast with emotion in making a decision based solely on DNA evidence?

Answer: First ask for research to answer the question “What is a fallacy?”

This is an opportunity for a debate or a prosecution and defence style trial scene using the prosecutor’s fallacy.

Thinking about what we have studied here what method would be best to use to predict the probability of:

a obtaining a run of 5 heads on a fair coin

We could work this out theoretically

b obtaining a run of 5 heads on a damaged coin

This would require an experiment

c rain tomorrow

We would need to base this on previous observations and simulate

d winning the lottery

Theoretical probability

Discuss: How can you predict the probability that an event occurs?

Answer: Probabilities can be predicted by theoretical methods where outcomes are equally likely, and otherwise by experimental methods.

Developing inquiry skills

In order to find a solution to the Monty Hall problem you could try to run a simulation in class.

You will need a partner and three cups.

- In your pair, decide which of you will be the host and which the contestant. The host should place an object under one of the cups while the second person (the contestant) closes their eyes.
- When ready, the contestant should choose a cup, but not lift it.
- The host then lifts one of the other two cups which is empty.
- The contestant either switches or does not switch their choice of cup.
- Record whether the player has won or lost.
- Calculate the win percentage for switching and the win percentage for not switching.
- What is the best strategy?

Answer: Switching is the best strategy

	SWITCH	DO NOT SWITCH
NUMBER WIN		
NUMBER LOSE		
WIN PERCENTAGE		

What do your results suggest is the probability of winning if you switch or do not switch?

Answer: Encourage students to be systematic to test a hypothesis.

How many trials do you think will be necessary for this simulation to see more reliable results?

When will you know when you have done enough?

Answer: The results will settle down to a true value

It is also possible to use an online simulator for this problem. There are many available if you search for them. Here is one http://www.mathwarehouse.com/monty-hall-simulation-online/ Alternatively, you could write a programme for a spreadsheet or a code to do the same.

What is the advantage of running a simulation using technology?

Answer: Technology will speed up the process!

8.2 Representing probabilities: Venn diagrams and sample spaces

Investigation 3

Conceptual understanding:

Venn diagrams are useful in illustrating relationships and obtaining results in probability.

1 **a** R **b** Neither R or T

c T and sometimes (but not always) R **d** Both R and T

2 We can find something that goes in each area

3 Only R – A,H,J Only T – B, E, G, I; R and T – C, D; Neither R and T – K, F

4 & 5a Any shape with right angles that is not a triangle.

b Any shape that does not contain a rectangle and is not a triangle

c A right-angled triangle

6 $\frac{2}{11}$ (there are 11 shapes and 2 are right angled triangles

7 **Conceptual:** What are Venn diagrams useful for?

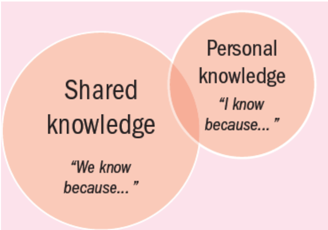
Answer (this is the conceptual understanding): Venn diagrams are useful in illustrating relationships and obtaining results in probability.

Extension problem:

In this problem students may wish to list all the subjects that students in their class take and see if there are any combinations that work. Other possibilities could be activities, interests/hobbies, physical attributes etc.

TOK

In TOK it can be useful to draw a distinction between shared knowledge and personal knowledge. The IB use a Venn diagram to represent these two types of knowledge. If you are to think about Mathematics (or any subject, in fact) what could go in the three regions illustrated in the diagram?



Answer: The difference between personal knowledge and shared knowledge offers you the chance to consider the difference between "what I know" and "what we know".

Shared knowledge may come from texts, teachers, media etc, personal knowledge is gained through the experience of the individual, such as ice is cold or rabbits are fluffy.

Somebody who studies computing might view their laptop differently because of their academic knowledge. Their personal knowledge had been affected by the shared knowledge they had gained in class. This would be an intersection of the two types of knowledge.

Reflect: What does 'and' and 'or' mean in mathematics? How do they differ from regular usage in English (inclusive and exclusive or)?

Answer: Notice that 'or' in mathematics includes the possibility of 'both' (Hannah would answer yes to the question above). So A or B means either in A or in B or in both A and B. This is known as the 'inclusive' or.

Investigation 4

Conceptual understanding:

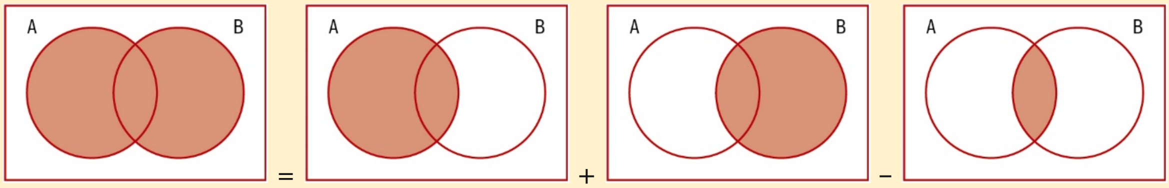
Understanding that moving from probability diagrams to formulae helps develop the ability to make generalised predictions.

1 $P(A) = 0.36$, $P(B) = 0.4$, $P(A \cap B) = 0.12$, $P(A \cup B) = 0.64$

2 Because some of the students will have been counted twice

3 **Factual:** Use the values of $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$ to determine the correct rule for $P(A \cup B)$ in terms of $P(A)$, $P(B)$ and $P(A \cap B)$.

Answer: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4 

5 **Conceptual:** Why might this rule be useful?

Answer: It allows us to calculate one of these probabilities if we know the others and to make predictions based on these probabilities

(This is the conceptual understanding): Understanding that moving from probability diagrams to formulae helps develop the ability to make generalised predictions

Reflect: When might a formula or a diagram be more appropriate?

Answer: This will depend on the question. Often questions where probabilities have been given are better with the formula and those with lists or numbers of objects in different groups may benefit from a Venn diagram.

Which is the most efficient method? What do we mean by efficient here?

Answer: Efficient or appropriate may be the method which is the quickest and/or easiest to understand.

TOK

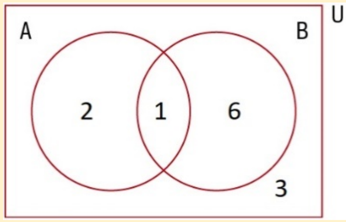
Do ethics play a role in the use of mathematics?

Answer: A good question for a debate or blog post would be - Ethics is an area of knowledge in its own right. Where do you see an intersection of the AOKs of mathematics and ethics?

When taking a chance decision in your life, which skills do you rely and in which order? Intuition (your gut feeling)? Reason? Emotion? Memory? Faith? Imagination?

Investigation 5

First Problem

1 

2 Completed table:

	Archery	Not Archery	Totals
Badminton	1	6	7
Not Badminton	2	3	5
Totals	3	9	12

3 $\frac{6}{12} = \frac{1}{2}$

4 Discuss student preferences

Second Problem

1 36 outcomes

2 It can be very time consuming

3 Completed table:

	Dice 2						
Dice 1		1	2	3	4	5	6
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

4 $\frac{10}{36}$

5 It can be less time consuming to answer a series of questions based on the same starting point

6 For small sets or if there are more than 2 ‘dice’.

Third Problem

1 It would not be easy to do this

2 111 211 311

112 212 312

113 213 313

121 221 321

122 222 322

123 223 323

131 231 331

132 232 332

133 233 333

3 27

4 111 211 311

112 212 312

113 **213** 313

121 221 321

122 222 322

123 **223** 323

131 231 331

132 232 332

133 233 333

There are 5

5&6 The sample space would need to be 3 dimensional

7 $\frac{5}{27}$

8 List, probability space diagram, Venn diagram

Developing inquiry skills

If we return to the Monty Hall problem, can any of these methods of representing sample spaces be used?

Copy and complete the following table for all the possible outcomes.

Answer:

Contestant chooses	**Prize door**	**Door Monty opens**	**Result if Stay**	**Result if Switch**
1	3	2	LOSE	WIN
1	2	3	LOSE	WIN
1	1	2 or 3	WIN	LOSE
2	3	1	LOSE	WIN
2	2	1 or 3	WIN	LOSE
2	1	3	LOSE	WIN
3	3	1 or 2	WIN	LOSE
3	2	1	LOSE	WIN
3	1	2	LOSE	WIN

How many different options are there in total.

Answer: 9

In how many of them do you win the prize when you switch?

Answer: 6

In how many of them do you win a prize if you don't switch?

Answer: 3

What is the probability of winning a prize if you switch?

Answer: $\frac{2}{3}$

What is the probability of winning a prize if you don't switch?

Answer: $\frac{1}{3}$

Should you switch or stay?

Answer: You should switch

8.3 Independent and dependent events and conditional probability

International-mindedness

During the mid-1600s, mathematicians Blaise Pascal, Pierre de Fermat and Antoine Gombaud puzzled over this simple gambling problem:

Which is more likely: rolling at least one six on four throws of one dice or rolling at least one double six on 24 throws with two dice?

Answer: Which option do you think is more likely? Why? We will return to this question at the end of this section.

Consider the ethics of gambling.

Investigation 6

1, 3 & 4

	1	2	3	4	5	6
H	**(H,1)**	**(H,2)**	**(H,3)**	**(H,4)**	**(H,5)**	**(H,6)**
T	**(T,1)**	**(T,2)**	**(T,3)**	**(T,4)**	**(T,5)**	(T,6)

2 12 possible outcomes

3 $\frac{6}{12} = \frac{1}{2}$

4 $\frac{10}{12} = \frac{5}{6}$

5 $\frac{5}{12}$

6 $P(H \cap L) = P(H) \times P(L)$ so $\frac{5}{12} = \frac{1}{2} \times \frac{5}{6}$

7 It doesn't

TOK

Do you think that mathematics is a useful way to measure risks?

To what extent do emotion and faith play a part in taking risks?

Answer: Can calculation of gambling probabilities be considered as an application of mathematics?

This is a good opportunity to generate a debate on the nature, role and ethics of mathematics regarding its applications. Bayesian v classical statisticians.

Investigation 7

If we know that a particular student studies biology, how does this affect the probability that they also study art?

Answer: It will reduce the sample space to students who do Biology

1 40, in circle for B.

2 12, the intersection of A and B, $A \cap B$

3 $\frac{12}{40} = \frac{3}{10}$

4 $P(A \mid B) = \frac{n(A \cap B)}{n(B)}$

5 $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$, $P(A \mid B') = \frac{P(A \cap B')}{P(B')}$

6 $P(A \cap B) = P(A \mid B) \times P(B)$

7 **Conceptual:** What does this imply about $P(A \mid B)$ and $P(A)$ for independent events?

Answer: $P(A \mid B) = P(A)$

8 **Conceptual:** Which situations describe conditional probability?

Answer: Conditional probability can be used to describe 'given that' situations

9 **Factual:** What is conditional probability and how may it be represented?

Answer (This is the conceptual understanding): Conditional probability reflects situations for a probability of an event given another event has occurred and may be represented by tree and Venn diagrams.

Reflect: How can conditional probability be used to make predictions?

Answer: By knowing the probabilities of events happening (e.g $A \cap B$ and B) means we can predict the likelihood of event A given that B has happened by using the formula

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Developing inquiry skills

The Monty Hall problem: conditional probability solution

Let's take a typical situation in the game. Suppose the contestant has chosen door 3 and Monty Hall reveals that there is an unwanted prize behind door 2.

What is the conditional probability that the prize is behind door 1?

Let A be the condition that there is a car behind door 1 and the contestant has chosen door 3.

Let B be the condition that there is a dud behind door 2 given that the choice was door 3

What is $P(A \cap B)$? The probability that there is a car behind door 1, the contestant has chosen door 3 and there is a dud behind door 2.

Here the car is behind door 1 (probability $\frac{1}{3}$) and the contestant has chosen door 3 (probability $\frac{1}{3}$) Monty Hall has to show what is behind door 2. (probability 1)

so $P(A \cap B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

What is the probability of being shown a dud behind door 2 given that the choice was door 3.

This situation can arise in two ways:

1. When the car is behind door 1

Probability is $\frac{1}{9}$ as shown above

2. When the car is behind door 3.

Here Monty could reveal either what is behind door 1 or door2. He is equally likely to choose either of these doors so the probability of showing what is behind door 2 is $\frac{1}{2} \times \frac{1}{9} = \frac{1}{18}$

Therefore the probability of there being revealed an unwanted prize behind door 2 given that the contestant has chosen door 3 is the sum of these $\frac{1}{18} + \frac{1}{9} = \frac{3}{18}$

This is $P(B)$.

We want the conditional probability $P(A \mid B)$ as this relates to the contestant winning the prize.

This is given by $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{9}}{\frac{3}{18}} = \frac{2}{3}$

This means that the conditional probability that the car is behind door 3 given that the contestant has chosen door 3 and has been shown that there is an unwanted prize behind door 2 is only $\frac{1}{3}$.

It is therefore worthwhile switching!

Talk through solution with students

8.4 Probability tree diagrams

Investigation 8

Conceptual understanding:

'With' or 'without' replacement categorizes the analysis and representation of probability events since the outcome of the previous event can sometimes be used to make predictions about future events.

3 balls are to be picked, one at a time, from a bag which contains 5 blue balls and 2 green ball

Consider two different situations

1) when the balls are replaced after each time that a ball is selected and the colour noted.

2) when the balls are not replaced after each time that a ball is selected and the colour.

A single trial in these experiments consists of selecting a sample of 3 balls, one at a time, from the bag.

What is the main difference between these two scenarios and what effect will it have on the probabilities involved?

Answer: In the first the ball is replaced. In the second it is not.

In the two situations how many balls are you choosing from on each go?

1 Completed table

	Situation 1	Situation 2
How many balls to choose from in first pick?	7	7
How many balls to choose from in second pick?	7	6
How many balls to choose from in third pick?	7	5

Make a list of all the possible outcomes in each situation.

How many different possible outcomes are there?

e.g. you could get BBG (a Blue followed by a Blue followed by a Green) in each case

1 Completed table

	Situation 1	Situation 2
List of possible outcomes	BBB, BBG, BGG, BGB, GBB, GBG, GGB, GGG	BBB, BBG, BGG, BGB, GBB, GBG, GGB,
Number of possible outcomes	8	5

2 GGG, because there are only 2 green balls

3-7

	Situation 1	Situation 2
Probability of blue on first pick	$\frac{5}{7}$	$\frac{5}{7}$
Probability of blue on second pick	$\frac{5}{7}$	$\frac{4}{6}$
Probability of blue on third pick	$\frac{5}{7}$	$\frac{3}{5}$
Dependent or independent?	Independent	Dependent

8 Situation 1

9

	Situation 1	Situation 2
Probability of blue followed by blue followed by blue (BBB)	0.364 (More likely)	0.357
Probability of BBG	0.146	0.19 (more likely)

10

BBB	0.364	0.357
BBG	0.146	0.190
BGG	0.058	0.048
BGB	0.146	0.190
GBB	0.146	0.190
GGB	0.058	0.048
GGG	0.023	0
GBG	0.058	0.048

11 Conceptual: How does "with" or "without" replacement affect the analysis and representation of probability?

Answer: 'With' or 'without' replacement categorizes the analysis and representation of probability events since the outcome of the previous event can sometimes be used to make predictions about future events.

TOK

What do we mean by a "fair" game? Is it fair that casinos should make a profit?

Answer: How does a knowledge of probability theory affect decisions we make?

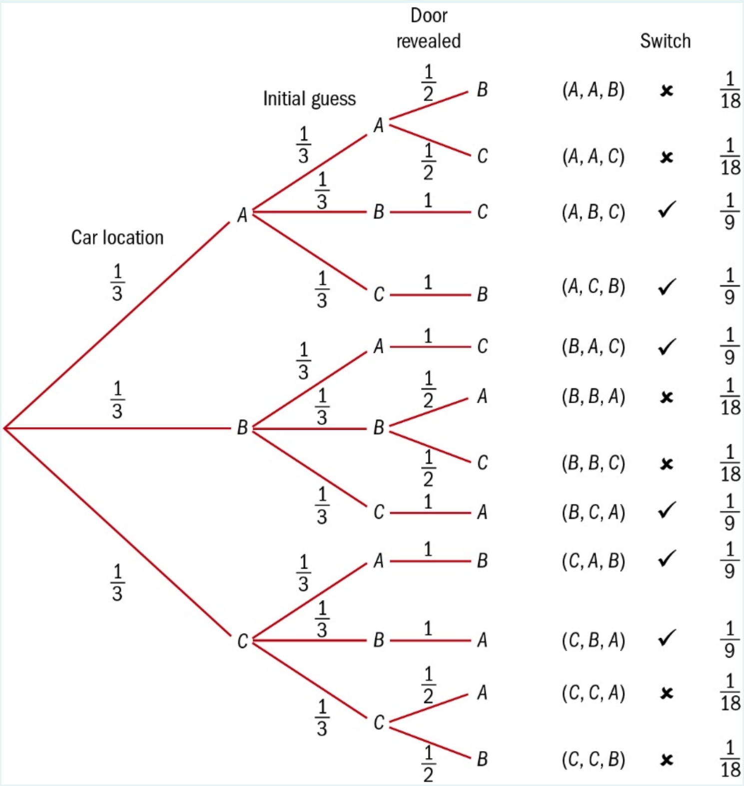
Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (e.g. economics)?

Developing inquiry skills

The Monty Hall problem: tree diagram solution

Represent this problem in a tree diagram.

Once the tree diagram has been completed you can find the probability of winning when switching (or losing when not switching) and losing when switching (or winning when not switching) by adding up the relevant results in the last column.



What is the probability of winning when switching?

Answer: $\frac{4}{9} = \frac{2}{3}$

Is it worthwhile switching?

Answer: Yes!

We have seen four different solutions to the Monty Hall problem provided in this chapter. There are many other possible methods of solving the problem.

How could you extend this problem for an exploration?

Answer: You could change the number of doors, 4, 5, n etc, so for example consider n doors and k revealed objects. You could change the number of winnable objects, you could change the rules of the game

What about changing the number of doors? What effect will this have on the situation?

Answer: Discuss as a class.

Random Walking

Approaches to Learning/learner profile: Critical Thinking

Exploration Criteria: Mathematical Communication (B); Personal Engagement (C); Use of Mathematics (E)

IB Topic: Probability, Discrete Distributions

Introduction

This problem is designed to encourage students to think of simulation as a reasonable and acceptable approach to probability problems that may be too difficult to approach theoretically as they develop. This problem has a clearly stated aim and is accessible at first using Mathematical Communication (Criterion B) and methods familiar to students from the chapter. The mathematics required to prove the result is quite difficult for some, but this should not restrict students from accessing it and using the tools available to them.

At the beginning of the problem students use basic coin-tossing simulations and collect results as a class to produce more results and a hopefully a more accurate answer. At the end of the task, students are asked to consider using computer simulation. Coding for this is accessible not only for a computer science student or experienced programmer but can be learnt with a little effort and Personal Engagement (Criterion C). Further, Personal Engagement can be shown by extending the problem once the code has been mastered.

The proof of the result requires some understanding of probability distributions, so this task may be better covered after that.

The Problem

The problem is adapted from a famous problem in a branch of mathematical problems involving "Random Walks". Study of this branch has contributed to many different areas in physics and chemistry (Brownian motion and diffusion), biology (genetics, animal movements, population dynamics), economics (modelling share prices) and computer science (social media suggestions), among others.

Explore the Problem

Since the man moves left or right with equal probability, a coin toss can be used to simulate this.

If appropriate, ask:

Why is a coin toss a suitable simulation?

Students play the game ten times and find the average number of steps taken. Discuss why this may not be an accurate result.

Ask:

What could you do to improve the accuracy of the average?

Discuss the improved result based on a larger sample size.

The average may be getting closer, but you do not know if this is the actual number.

You can only be certain by proving the result theoretically.

Ask:

What have you noticed so far?

(For example, always an odd number of steps, theoretically, could go on forever, etc.).

Calculate Probabilities

Students may need help when constructing the tree diagram.

Ask:

What are the limitations of using a tree diagram in this case?

The tree diagram is very large. It becomes impossible to draw after six or seven tosses.

Remind students how to use a tree diagram to find probabilities.

If needed, to help students find the probability that the man falls into the ditch after a total of exactly five steps, ask:

If the man moves Left (L) then Left again (L) and then Right (R) and then Left (L) and then Left (L), then he will be in the ditch. Where is this scenario on your tree diagram?

What is the probability that the man takes this particular sequence of steps? In other words what is the probability of TTHTT?

The probability is $(0.5)^5 = 0.03125 = (1/32)$

What other sequences of coin tosses will lead to the man falling into the ditch after exactly five steps?

TTHTT, THTTT, THHHH, HTTTT, HTHHH, HHTHH

What are the probabilities associated with each of these sequences?

They are all $(0.5)^5 = 0.03125 = 1/32$

What is the probability that the man falls into the ditch after a total of exactly five steps?

$6(0.5)^5 = 6(0.03125) = 3/16$

The minimum number of steps to fall into the ditch is three.

The maximum is infinite.

The probability that the man falls into the ditch after a total of exactly three steps is

$2(0.5)^3 = 2(0.125) = 1/4$

To explain why all the paths have an odd number of steps:

From the centre, after the first step, the man will always be one step away from the centre (two steps away from the ditch on that side).

From here, after two steps, he will either be in the ditch on the same side, back to the same position, or one step from the centre (two steps away from the ditch) on the other side. This will repeat.

This gives 1 + a multiple of 2, which is odd.

You could use a diagram to demonstrate this, with the centre shown by a black dot, one step from the centre on either side shown by a red dot, two steps away from the centre on each side shown by a black dot, and the ditches shown by blue dots.

The probabilities are:

x	1	2	3	4	5	6	7	8	9	10	11	12	….
$P(X = x)$	0	0	1/4	0	3/16	0	9/64	0	27/256	0	81/1024	0	

The table alternates between 0 and a value.

The next value is calculated by multiplying by ¾.

Ask:

Can you explain where the value of ¾ comes from in this situation?

Again, you could use diagrams to demonstrate this.

Simulation

Students will study expectation in chapter 14.

For now, if appropriate, you could ask:

What is an expected value?

You could also give students the formula for calculating the expected number of steps that would be required: $E(X) = \sum_{1}^{\infty} xP(X = x)$

E(X) is the expected number of steps.

This will give you the exact theoretical answer to the problem posed.

Help them to understand and use this formula if needed.

The formula is to multiply each probability by its related value and sum the result.

Students may note here that this problem will be complicated because there are an infinite number of values of x that will result in falling into the ditch.

It is *possible* to calculate the expected value, but it requires mathematics that will be beyond the SL and HL syllabus.

As extension, students could perhaps try to do this when they have completed chapter 14 and fancy a challenge!

Here is the solution written out:

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x * P(X = x) \\
&= (1*0) + (2*0) + (3*\frac{1}{4}) + (4*0) + (5*\frac{3}{16}) + (6*0) + (7*\frac{9}{64}) + ... \\
&= (3*\frac{1}{4}) + (5*\frac{3}{16}) + (7*\frac{9}{64}) + (9*\frac{27}{264}) + ... \\
&= \frac{1}{4}(3 + 5*\frac{3}{4} + 7*\frac{9}{16} + 9*\frac{27}{64} + ...) \\
&= \frac{1}{4}(3 + 5*\frac{3}{4} + 7*(\frac{3}{4})^2 + 9*(\frac{3}{4})^3 + ...)
\end{aligned}$$

This is an infinite Aritmetico-Geometric series, its sum can be found neatly as follows...

$$\begin{aligned}
E(X) &= \frac{1}{4}[3 + 5*(\frac{3}{4}) + 7*(\frac{3}{4})^2 + 9*(\frac{3}{4})^3 + ...] \\
\frac{3}{4}*E(X) &= \frac{1}{4}*[3*(\frac{3}{4}) + 5*(\frac{3}{4})^2 + 7*(\frac{3}{4})^3 + 9*(\frac{3}{4})^4 + ...]
\end{aligned}$$

Subtracting,

$$\begin{aligned}
\frac{1}{4}E(X) &= \frac{1}{4}[3 + 2*(\frac{3}{4}) + 2*(\frac{3}{4})^2 + 2*(\frac{3}{4})^3 + ...] \\
E(X) &= 3 + 2*(\frac{3}{4}) + 2*(\frac{3}{4})^2 + 2*(\frac{3}{4})^3 + ... \\
E(X) &= 3 + 2*\frac{3}{4}*[1 + (\frac{3}{4}) + (\frac{3}{4})^2 + (\frac{3}{4})^3 + ...] \\
&= 3 + 2*\frac{3}{4}*\frac{1}{1-\frac{3}{4}} \\
&= 3 + 2*\frac{3}{4}*4 \\
&= 9
\end{aligned}$$

This shows that the theoretical result is 9. Compare this with the result from the earlier simulation by the class.

At this stage, you could tell the class that the expected number of steps is 9.

Ask:

Why do you think simulations are used?

Collecting and recording large numbers of results by hand is very time consuming and can be very expensive.

Here are some examples of computer coding that could be copied or shown to the students. It would be possible to replicate these results on most computer coding systems. (If there are computer science students in the class, then they could be encouraged to help the class, although simple coding should be accessible to all.)

Note: The idea here is that *simple* computer coding is accessible to all students with a little work, and then more complicated problems can be solved. This is considerably more efficient than more manual methods. It is also worth pointing out that this is actually an exceptionally real process in many, many fields of work, such as meteorology, disaster management, economics and finance, sports predictions, etc. This is a real avenue of work for the modern mathematician.

Extension

Suggestions of how students could vary the problem:

Change the start position.

Use a bias coin.

Change the number of steps from the centre to the ditch.

Change the problem to 2 dimensions.

Students may also be able to devise their own probability question, which they could answer using simulation.

9 Representing equivalent quantities: exponentials and logarithms

Essential understandings

Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems. Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Numbers and formulas can appear in different, but equivalent, forms, or representations, which can help us to establish identities
- Logarithm laws provide the means to find inverses of exponential functions which model real-life situations.
- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving
- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
Exponential functions can model real-life situations between two variables that exhibit constant percentage change.	Investigation 2
The parameters of exponential functions transform the function by altering the asymptotes, the range, the rate of growth or decay, and its intercepts.	Investigation 3
Logarithms represent inverse functions of the exponential functions and can solve problems with unknown bases.	Investigation 5, 6
Different bases can be used to represent equivalent values in different ways.	Investigation 7
Two or more combinations of logarithms with the same base may be reduced to a single logarithm.	Investigation 8
The gradient function of the exponential function with base e maps onto itself.	Investigation 9

Syllabus sections covered in this chapter:

- SL1.5*
- SL1.7
- SL2.9
- SL2.10
- SL5.6

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 395: Exponential expressions	Page 400: Example 4 Page 408: Example 7 Page 420: Example 17 Page 423: Example 18	Page 402: Example 5 Page 403: Example 6	Pages 406, 420, 424

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 425	Page 426	Page 426

9.1 Exponents

Investigation 1

1 $2^1 = 2$ $2^2 = 4$ $2^3 = 8$ $2^4 = 16$ $2^5 = 32$ $2^6 = 64$

$2^7 = 128$ $2^8 = 256$ $2^9 = 512$ $2^{10} = 1024$ $2^{11} = 2048$ $2^{12} = 4096$

2 The last digit repeats in the pattern 2, 4, 8, 6 as the power of 2 increases.

3 8192, 16384, 32768, 65536.

4 Find the remainder when n is divided by 4.

When the remainder is 1, the last digit of 2^n is 2.

When the remainder is 2, the last digit of 2^n is 4.

When the remainder is 3, the last digit of 2^n is 8.

When the remainder is 0, the last digit of 2^n is 6.

5 **a** 6 **b** 2 **c** 8

The size of each of these numbers will be too big for most calculators to show the whole number. You may be able to use computer software to check your answer!

6 An exponent, or power, tells you how many times a number is multiplied by itself.

7 You can write a number in exponential form if the number is equal to a certain base, raised to a power. For example, you can write 8 in exponential form as 2^3.

Reflect: How could you use both $\frac{a^m}{a^n} = a^{m-n}$ the result above to write x^{-p} in terms of x^p .

Answer: You could write:

$$\begin{aligned} x^{-p} &= x^{0-p} \\ &= \frac{x^0}{x^p} && \text{since } \frac{a^m}{a^n} = a^{m-n} \\ &= \frac{1}{x^p} && \text{since } x^0 = 1 \end{aligned}$$

TOK

The phrase "exponential growth" is used popularly to describe a number of phenomena.

Do you think that using mathematical language can distort understanding?

Answer: How does exponential growth in mathematics differ from its use in English?

Is language an inadequate vehicle for expressing everything we can experience and think?

Investigation 2

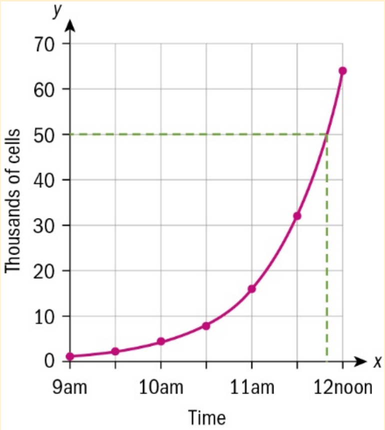
Conceptual understanding:

Exponential functions can model real-life situations between two variables that exhibit constant percentage change.

1

Time (x)	9am	9:30am	10am	10:30am	11am	11:30am	12noon
Thousands of cells (y)	1	2	4	8	16	32	64

2 64 000



3

4 There are 50 000 cells at approximately 11:50.

5 **Conceptual:** What is exponential growth or decay?

Answer: Exponential growth (or decay) is the increase (or decrease) between two variables that exhibit constant percentage change.

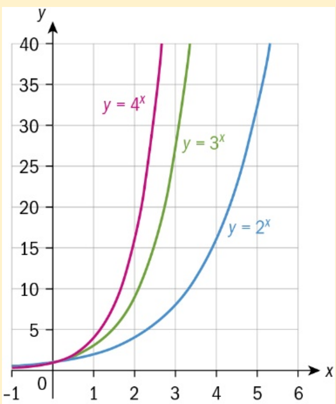
6 **Conceptual:** How do exponential functions model real-world problems and their solutions?

Answer (this is the conceptual understanding): Exponential functions can model real-life situations between two variables that exhibit constant percentage change.

Investigation 3

Conceptual understanding:

The parameters of exponential functions transform the function by altering the asymptotes, the range, the rate of growth or decay, and its intercepts.

1 

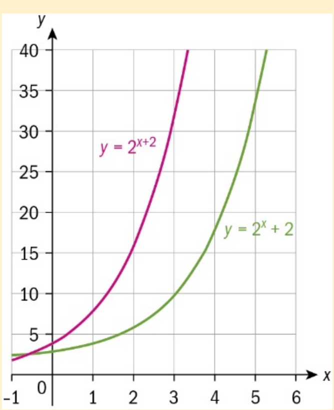
2 All three have asymptote $y = 0$

3 All three intersect at y = 1, because any base to the power of zero is 1.

4 For all three: $x \in \mathbb{R}, y > 0$

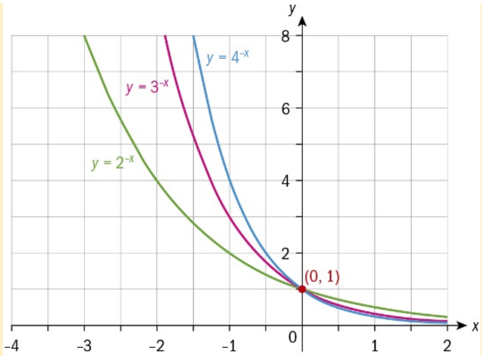
5 **Conceptual:** What effect does the value of the base have on the graph of an exponential function?

Answer: For an exponential function, the value of the base determines the scale factor of the vertical stretch.

6 

7 **a** $y = 2^x$ is translated vertically up 2 units to map onto $y = 2^x + 2$.

b $y = 2^x$ is translated horizontally, 2 units to the left, to map onto $y = 2^{x+2}$.

8 

9 They have the same domain, range, y-intercept and asymptote.

10 $y = a^{-x}$ and $y = a^{x}$ are reflections of one another in the y-axis.

11 Factual: If $y = 2^{x}$ is the parent function, copy and complete this table to summarize your findings about the parameters of an exponential function.

Answer:

Function	Transformation of the parent function
$y = a2^{x}$	Vertical stretch; scale factor a
$y = 2^{ax}$	Horizontal stretch; scale factor $\frac{1}{a}$
$y = 2^{x} + b$	Vertical translation; b units up
$y = 2^{x+c}$	Horizontal translation; c units left
$y = 2^{-x}$	Reflection in the line $x = 0$ (the y-axis)
$y = -2^{x}$	Reflection in the line $y = 0$ (the x-axis)

12 Conceptual: How do the parameters of exponential functions affect the graph of the function?

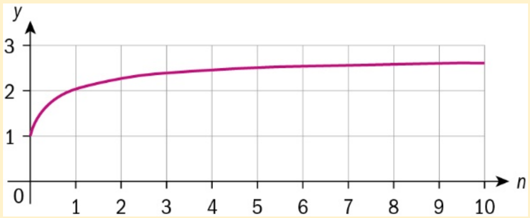
Answer (this is the conceptual understanding): The parameters of exponential functions transform the function by altering the asymptotes, the range, the rate of growth or decay, and its intercepts

Investigation 4

1 Completed table:

1	2
2	2.25
3	2.37037
4	2.44140
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827

2



3 $y = 2.71827$

4 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 2.718$ (to 3 d.p.)

5 0

6 $a = 2.718...$

Developing inquiry skills

In Investigation 2 earlier in the chapter you looked at bacteria growing in a Petri dish.

At 9 am there were 1000 bacteria present.

Using the mathematics you have learned in this chapter, write down an expression for N, the number of bacteria cells present t hours after 9 am.

Use this to find the time at which the number of bacteria cells reaches 64 000.

Answer: $N = 10 \times 2^t$

$64000 = 10000 \times 2^t \Rightarrow 2^6 = 2^t$

When $t = 6$ hours, i.e. at 15:00

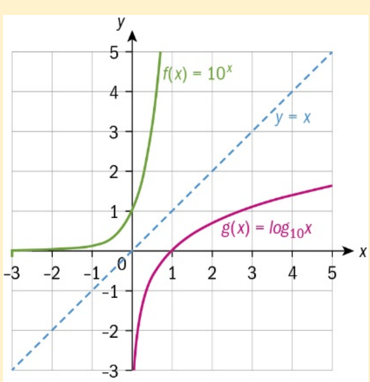
9.2 Logarithms

Investigation 5

Conceptual understanding:

Logarithms represent inverse functions of the exponential functions and can solve problems with unknown bases.

1



2 $x \in \mathbb{R}, y > 0;\ y = 0$

3 $x > 0, y \in \mathbb{R};\ x = 0$

4 **Factual:** What does the line y = x tell you about the functions f and g? Explain your reasoning.

Answer: The functions are inverses of each other, as they are reflections in the line y = x.

5 **Factual:** What is the relationship between exponents and logarithms?

Answer: Exponents and logarithms are inverses of each other.

6 **Conceptual:** What is the relationship between exponents and logarithms?

Answer (this is the conceptual understanding): Logarithms represent inverse functions of the exponential functions and can solve problems with unknown bases.

7 **Conceptual:** How could you use the domain and range of an exponential function to state the domain and range of any log function?

Answer: The domain of a log function is the range of an exponential function, i.e. x > 0

The range of a log function is the domain of an exponential function, i.e. $y \in \mathbb{R}$

8 **Factual:** What graph do you get when you reflect the graph of $f(x) = a^x$ in the line y = x?

Answer: You obtain the graph of $g(x) = log_a x$

Investigation 6

Conceptual understanding:

Logarithms represent inverse functions of the exponential functions and can solve problems with unknown bases.

1

$10^0 = 1$	$1og_{10}1 = 0$
$10^1 = 10$	$1og_{10}10 = 1$
$10^2 = 100$	$1og_{10}100 = 2$
$10^3 = 1000$	$1og_{10}1000 = 3$
$10^4 = 10000$	$1og_{10}10000 = 4$
$10^5 = 100000$	$1og_{10}100000 = 5$
$10^6 = 1000000$	$1og_{10}1000000 = 6$

2 **Conceptual:** Using the table above, describe how the same information can be conveyed in both exponential form and log form.

Answer: A logarithm is an exponent which, when raised to the base, tells you the number.

3 $1og_2 8 = 3$

4 $1og_a c = b$

5 $5^2 = 25$

6 **Conceptual:** How are logarithms and exponents different representations of the same quantity? why is it useful to be able to move from one form to another?

Answer (this is the conceptual understanding): Logarithms represent inverse functions of the exponential functions and can solve problems with unknown bases.

TOK

Mathematics is all around us in patterns, shapes, time and space.

What does this tell you about mathematical knowledge?

Answer: Some mathematical constants like π, e and the Fibonacci numbers appear consistently in nature. Research where these may be found and consider if they are natural occurrences or are we applying the mathematics that we know to these instances.

Investigation 7

Conceptual understanding:

Different bases can be used to represent equivalent values in different ways.

1 Given that $1og_a c = b$ is equivalent to $a^b = c$, copy and complete the table.

Log form	Exponent form	Solve for x
$\log_2 2 = x$	$2^x = 2$	$x = 1$
$\log_3 3 = x$	$3^x = 3$	$x = 1$
$\log_4 4 = x$	$4^x = 4$	$x = 1$
$\log_5 5 = x$	$5^x = 5$	$x = 1$
$\log_a a = x$	$a^x = a$	$x = 1$

2 The exponent is 1.

3 **a** $x = 1$

b $p = 5$

c $q = 3$

4 **a** 0

b 0

c 0

5 The log of one equals zero in any base.

6 **Factual:** How can you write 1 in logarithm form?

Answer: $\log_a a = 1$ in any base a

7 **Conceptual:** How can equivalent values be represented in different ways by using logarithms?

Answer (this is the conceptual understanding): Different bases can be used to represent equivalent values in different ways.

TOK

How does mathematical proof differ from reasoning in everyday life?

Answer: Is mathematical reasoning different from scientific reasoning?

What is the difference between deductive and inductive methods?

Reflect: You can extend this result to give $\log_a(a^x) = x$. Using the result above, and the definition of a logarithm, can you explain why this is the case?

Answer: Let $\log_a a^x = b$. We will try to show that $b = x$

By changing to exponent form, $\log_a(a^x) = b \Rightarrow a^b = (a^x)$

Since the base is the same on the left and right-hand sides, we can conclude that $a = x$

Investigation 8

Conceptual understanding:

Two or more combinations of logarithms with the same base may be reduced to a single logarithm.

Part 1

1 $\log 2 + \log 3 = 0.778$

$\log 3 + \log 4 = 1.08$

$\log 5 + \log 6 = 1.48$

$\log 5 + \log 3 = 1.18$

$\log 6 + \log 3 = 1.26$

$\log 4 + \log 5 = 1.30$

$\log 6 = 0.778$

$\log 12 = 1.08$

$\log 15 = 1.18$

$\log 18 = 1.26$

$\log 20 = 1.30$

$\log 30 = 1.48$

2 Students should notice common answers and a pattern such as $\log 2 + \log 3 = log 6$

3 **Factual:** Can you work together to write a general rule for $\log A + \log B$ as a single logarithm, for all positive values of A and B?

Answer: $\log A + \log B = \log AB$

4 Students will be able to apply the rules to further values.

Part 2

5 $\log 6 - \log 3 = 0.301$

$\log 16 - \log 4 = 0.602$

$\log 15 - \log 5 = 0.477$

$\log 100 - \log 10 = 1$

$\log 15 - \log 3 = 0.699$

$\log 2 = 0.301$

$\log 3 = 0.477$

$\log 4 = 0.602$

$\log 5 = 0.699$

$\log 10 = 1$

6 Students should notice common answers and a pattern such as $\log 6 - \log 3 = \log 2$

7 **Factual:** Can you work together to suggest a general rule for $\log A - \log B$ as a single logarithm, for all positive values of A and B?

Answer: $\log A - \log B = \log \frac{A}{B}$

8 Students will be able to apply the rules to further values.

9 **Conceptual:** How do properties of logarithms help to simplify expressions?

Answer (this is the conceptual understanding): Two or more combinations of logarithms with the same base may be reduced to a single logarithm.

Reflect: Can you use the same approach to show, step-by-step, that $log_a \frac{x}{y} = log_a x - log_a y$

Answer:

$\log_a\left(\frac{x}{y}\right) = \log_a\left(\frac{a^m}{a^n}\right)$ By substitution

$= \log_a\left(a^{m-n}\right)$ By properties of exponents from section 1

$= m - n$ Because $\log_a\left(a^b\right) = b$

$= \log_a x - \log_a y$ By substitution

TOK

One reason why mathematics enjoys special esteem, above all other sciences, is that its propositions are absolutely certain and indisputable - Albert Einstein.

How can mathematics, being after all a product of human thought which is independent of experience, be appropriate to the objects of the real world?

Answer: Are logarithms a natural occurrence or are they a human invention?

Is mathematics created to solve real world problems?

Do you need imagination to create new mathematics?

Does faith have a role to play in the careers of mathematicians?

The natural logarithm appears in physics, biology, sociology, economics and more. Students of physics know that many of the calculations, for example in electrodynamics and quantum mechanics, would be impossible if it were not for the natural logarithm. The universal applicability of the natural logarithm suggests that it is something that exists in the world in which we live and therefore it is a characteristic of the natural world.

Reflect: Can you use what you have learned about logarithms so far to write down the values of $\ln 1$ and $\ln e$. Justify your answers.

Answer: $\ln 1 = 0$ since $\log_a 1 = 0$ for any base a

$\ln e \equiv \log_e e = 1$ since $log_a a = 1$ for any base a

TOK

Is mathematics invented or discovered?

Answer: Consider the number e or logarithms, did they already exist before people defined them? This topic is an opportunity for teachers to generate reflection on “the nature of mathematics”.

Reflect: In deriving the change of base formula, you saw an example of how using logarithms in an equation allowed you to isolate the variable. Look back at this, and explain what rule of logarithms we used in order to isolate the variable.

Answer: After taking logarithms, to base b, of both sides, we had $log_b a^y = log_b x$.

We then used $log_a\left(x^p\right) = plog_a x$ in order to write $ylog_b a = log_b x$.

Developing enquiry skills

In Investigation 2 earlier in the chapter you looked at bacteria growing in a Petri dish.

Using the mathematics you have learned in this chapter, and your results from the end of section 9.1, use logarithms to determine the time at which the number of bacteria cells equals 10 000. Give your answer to the nearest minute.

Answer:

$$\begin{aligned}10000 &= 1000 \times 2^t \\ 10 &= 2^t \\ \log 10 &= \log 2^t \\ \log 10 &= t \log 2 \\ t &= \frac{\log 10}{\log 2} = 3.3219\ldots\end{aligned}$$

This is equivalent to 3 hours and 19 minutes, i.e. at 12:19

9.3 Derivatives of exponential functions and the natural logarithmic function

TOK

Why is proof important in mathematics?

Answer: We need proofs in mathematics, first, because we want to be sure that what we

do is correct.

Proof in essential shows you whether a statement is true or not.

In math, unlike any other area of knowledge, you can prove that what we do is perfectly correct, because mathematics is not dependent on most ways of knowing but simply on reason.

As a counterclaim, you might want to have students research the link between proof and intuition linked to Ramanajuan.

Investigation 9

Conceptual understanding:

The gradient function of the exponential function with base e maps onto itself.

1

$f(x)$	$f(1)$	$f'(1)$
2^x	2	1.386
$(2.1)^x$	2.1	1.558
$(2.2)^x$	2.2	1.734
$(2.3)^x$	2.3	1.916
$(2.4)^x$	2.4	2.101
$(2.5)^x$	2.5	2.291
$(2.6)^x$	2.6	2.484
$(2.7)^x$	2.7	2.682
e^x	2.718	2.718
$(2.8)^x$	2.8	2.883
$(2.9)^x$	2.9	3.088
3^x	3	3.296

2 The values of $f'(1)$ got closer and closer to the value of $f(1)$, and they were the same when $f(x) = e^x$.

3 (Note: Students may have deduced this in the question above, but it is an important point and needs to be emphasized by this question)

$f'(1) = 2.718$ or e; that is, $f'(1) = f(1)$ when $f(x) = e^x$

4

x	$f(x)$	$f'(x)$
1	2.718	2.718
2	7.389	7.389
3	20.086	20.086
4	54.598	54.598
5	148.413	148.413

5 **Conceptual:** Use your results from question 4. What can you learn about the slope of the graph of e^x by finding the value of it's derivative at any point?

Answer: The slope is the same as the function value for all points on the graph. For example, the slope of $f(x) = e^x$, where $x = 2$ is $e^2 \, or \, 7.39$

6 **Conceptual:** With a classmate, discuss what your results from questions 4 and 5 tell you about the derivative of $f(x) = e^x$.

Answer (this is the conceptual understanding): The gradient function of the exponential function with base e maps onto itself.

7 **Factual:** How is the derivative of e^x different to the derivative of other functions?

Answer: For $f(x) = e^x$, $f(x) = f'(x)$ for all x. There is no other function that has this property.

Investigation 10

1

x	1	2	3	4	5	10	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$f'(x)$	1	0.5	0.333	0.25	0.2	0.1	2	3	4

2 **Conceptual:** With a classmate, look at your results from question 1 and suggest a function which is the derivative of $f(x) = \ln x$.

Answer: The derivative of lnx is the reciprocal of x, i.e. $f'(x) = \frac{1}{x}$.

TOK

In what ways might a pragmatist view the differentiation of exponentials and logs?

Answer: Pragmatism - something is knowledge if and only if the proposed bit of knowledge works in real life settings. We do not know anything until we see that it works.

Is pure mathematics ever useful unless it has an application?

Developing inquiry skills

In investigation 2 earlier in the chapter you looked at bacteria multiplying in a Petri dish.

The number of bacteria cells in a second Petri dish after 9 am that day is modelled by the equation $B = 24e^{2t-3} + \ln(3t)$

Using the mathematics you have learned in this chapter, find the rate at which the number of bacteria in the second dish are increasing at 10 pm that same day.

Answer: $\frac{dB}{dt} = 24\left(2e^{2t-3}\right) + \frac{1}{3t} \times 3 = 48e^{2t-3} + \frac{1}{t}$

At 22:00, $t = 13$ hence $\frac{dB}{dt} = 48e^{2\times13-3} + \frac{1}{13} = 4.68 \times 10^{11}$

This means that at 22:00 the number of bacteria is increasing at a rate of 4.68×10^{11} bacteria per hour.

A Passing Fad?

Approaches to Learning: Communication, Research

Exploration Criteria: Mathematical Presentation; Reflection (D); Use of Mathematics (E)

IB Topic: Exponentials and Logarithms

Introduction

This task gives further practice to students in finding and using data to model (in this case, exponential functions) and then reflecting on the usefulness of the model for prediction. Students could model by hand or use of technology (or both). Students are not penalized for using technology if they can demonstrate understanding of the process that is being used.

At the time of writing, Fortnite is a massive phenomenon and the task here is to see whether this is a passing fad or if the exponential growth will continue. Similar discussions could be had regarding "the latest fad" and, if data could be found, then this could form the basis of the discussions for the task at the end.

Depending on where this is covered, during the chapter and previously, you may wish to work through the calculations of the "by hand" model and the model using technology. Students can then use this for their own data or, if they are able to, they could then devise these models themselves.

There are opportunities in the task for discussing the importance for consistent notation in an IA as well as Reflection (Criterion D) on the reliability of data that they can find.

Look at the Data

The data are taken from the press releases of the developers, Epic Games, but will supposedly be subject to checks and scrutiny by rival companies.

The data are to the nearest million, so are not particularly accurate.

The data relate to users, but some of these users may play frequently and some may play only once in the time period being measured.

The dates are not very accurate—you assume the data are all released at the same point in the month.

Daily Average Users (DAU) or Monthly Average Users (MAU) or the amount of time spent on the game may be more interesting information to collect, if it is possible to find.

The growth looks exponential. This could be due to word of mouth, with people suggesting to friends, etc, to play the game.

Model the Data

The model could be useful in terms of predicting future advertising prices and revenues for the company or for rival companies to consider when a game may be reaching saturation point.

Emphasize the importance of making sure that the variables in the model are clearly defined.

Look at these models (or equivalent using the available software) with students:

By hand	By technology (using TI-NSPIRE)	By technology (using Desmos)
Choose two points (t, P) For example, (1, 1) and (6, 45). Substitute into the equation: $P = a \cdot b^t$ $1 = a \cdot b^1$ $\therefore a \cdot b = 1$ $a = \frac{1}{b}$ and $45 = a \cdot b^6$ So $45 = b^5$ $b = \sqrt[5]{45}$ $b = 2.14$ (3sf) $a = \frac{1}{2.14}$ $a = 0.467$ (3 sf) $\therefore P = 0.487(2.14)^t$ Draw the curve. Note: if you choose two different points you would get a different curve.	Enter the data into a table. Label x list t and y list P. Menu > Statistics > A:Exponential Regression. Select x list as t and y list as P. Press OK. Therefore, the equation of the exponential curve that best fits the data according to the GDC is $P = 1.81 \cdot 1.56^t$.	Follow the procedure from the Rich Task in chapter 3. The equation of the exponential curve that best fits the data according to Desmos is $P = 1.81 \times 1.56^t$. Make sure that Desmos is in "<u>log mode</u>" to obtain the same result. If log mode is not ticked then a different equation will be found. $P = 8.65 \times 1.27^t$ (this may be an opportunity to discuss/explore residuals. Clicking residuals "plot" gives a good visual measure of how accurate the model is).

The "by hand" model uses only two points. The curve will naturally pass through these two points but is not necessarily close to any of the other points.

You could also choose a different function of a similar form, say $P = a \cdot b^t + c$, and find parameters a, b and c for the model. This is not a function that is available as an option on the GDC but you could use Desmos, for example, to find it.

If you were to find a model $P = a \cdot b^t + c$ by hand you would require three points to find the three variables.

You need to calculate how many months it is since July 2107. Substitute a value. The result is likely to be very large and discussion will be around the fact that the game will have reached saturation or a new game will have come along, etc.

Students could research the number of Fortnite players there are in the current month.

You could ask:

Is Fortnite still a popular game?

Students could compare this figure with their prediction based on their model. How big is the error? What does this tell you about the reliability of your previous model?

This will hopefully support the above.

Plot a new graph with the updated data you have found and try to fit another function to these data.

Will a modified exponential model be a good fit?

If not, what other function would be a better model that could be used to predict the number of users now?

This could be a good opportunity for a discussion around a logistic model that may be more appropriate. This from Khan Academy (https://goo.gl/KMmFbC) is a good summary of what happens when an exponential model is constrained by real life.

Extension

Hopefully there are numerous examples of “the next big thing”. Good sources will be social media sites, games, technology uptake, etc.

Students should be encouraged to find their own data to display and model.

This task could be written up as a mini-exploration, perhaps assessed against a smaller number of criteria.

Here is a possible structure for this:

Mini-Exploration

Write a brief exploration on what you find out.

This exploration should be between one and two pages, depending on the number of diagrams/graphs that you use.

This is not an exercise in being able to copy from Wikipedia or other websites, but rather to find out relevant information and to rewrite it into an exploration.

Students could be marked against parts of the Criteria of the Real Mathematics Exploration:

Criterion A: Presentation (3)

Your writing should be well-organized, coherent, logically developed and easy to follow. It should include an introduction, aim and conclusion.

Criterion B: Mathematical Communication (3)

Use appropriate mathematical language and representation, and define key terms.

Criterion D: Reflection (3)

You should review, analyse and evaluate your exploration. You should consider the significance of your findings, state possible limitations and/or extensions and make links to different fields and/or areas of mathematics.

Criterion E: Use of Mathematics (1)

Demonstrate that you fully understand the mathematics used in your exploration.

TOTAL (10)

10 From approximation to generalization: integration

Essential understandings

Calculus describes the accumulation of limiting areas. Understanding these accumulations allow us to model, interpret and analyze real-world problems and situations. Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Areas under curves can be can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Numerical integration can be used to approximate areas in the physical world.
- (Derivatives and) integrals describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
The process of finding the anti-derivatives of a function leads to a family of functions that differ by a constant.	Investigation 1
The reverse chain rule helps to integrate composite functions which contain linear functions.	Investigation 3
Finding a particular solution for an indefinite integral requires a boundary condition.	Investigation 4
The definite integral represents the area under the curve for positive functions.	Investigation 5
The definite integral represents the area under the curve for positive functions.	Investigation 6
Applying limits to the process of summing areas of rectangles, gives the exact value of the area under a curve and of the definite integral representing that area.	Investigation 7
Modification of the representation of the height of a rectangle in the definite integral for the area under a curve leads the definite integral for the of area between two curves.	Investigation 8

Syllabus sections covered in this chapter:

- SL5.5*
- SL5.9
- SL5.10
- SL5.11

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students’ understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 433: Area Expanding brackets and factorization	Page 422: Example 6 Page 448: Example 9 Page 456: Example 12	Page 457: Example 13 Page 459: Example 14	Pages 439, 443, 449, 454, 459

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 460	Page 460	N/A

10.1 Antiderivatives and the indefinite integral

Investigation 1

Conceptual understanding:

The process of finding the anti-derivatives of a function leads to a family of functions that differ by a constant.

1 $F(x) = 2x^2 - x$

2 **a** $4x - 1$

b $4x - 1$

c $4x - 1$

3 $F(x) = 2x^2 - x + C$, where C is any constant. (Accept any representation of the constant.)

4 **Conceptual:** The process of finding the anti-derivatives of a function leads to a family of functions. What can you say about this family of functions?

Answer (this is the conceptual understanding): The process of finding the anti-derivatives of a function leads to a family of functions that differ by a constant.

Students will most likely answer in their own words, for example, 'The functions are the same except for a constant term'.

Another possible answer, that can lead to a good class discussion, is that 'the graphs of the anti-derivative functions are vertical translations of each other'.

5

$f(x)$	*Anti-derivatives of f (where C represents an arbitrary constant)*
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
x^3	$\frac{1}{4}x^4 + C$
x^4	$\frac{1}{5}x^5 + C$

6 **Factual:** What is a general expression (rule) for the anti-derivatives of x^n ?

Answer: $\frac{1}{n+1}x^{n+1} + C$

7 The rule does apply.

x^{-4} : $\frac{1}{-4+1}x^{-4+1} + C = -\frac{1}{3}x^{-3} + C$ and $\frac{d}{dx}\left(-\frac{1}{3}x^{-3} + C\right) = -3 \cdot -\frac{1}{3}x^{-4} + 0 = x^{-4}$

$x^{\frac{1}{2}}$: $\frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C = \frac{2}{3}x^{\frac{3}{2}} + C$ and $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} + C\right) = \frac{3}{2} \cdot \frac{2}{3}x^{\frac{3}{2}-1} + 0 = x^{\frac{1}{2}}$

8 Yes, the rule does not apply when $n = -1$. Applying the rule when $n = -1$ leads to division by zero, which is undefined.

TOK

Where does mathematics come from?

Galileo said that the universe is a grand book written in the language of mathematics

Does it start in our brains or is it part of the universe?

Answer: Mathematics may not be the language of the universe, but rather the language/logical system the brain uses to analyze and respond to the universe. Ask students what they think about this and to consider where the real foundations of mathematics originate.

You might want to use facts such as The Higgs Boson was predicted with the same tool as the planet Neptune and the radio wave: with mathematics, which might mean that our universe isn't just described by mathematics, but that it is mathematics in the sense that we're all parts of a giant mathematical object.

Max Tegmark and Brian Butterworth provide some interesting insights and contrasting opinions on YouTube.

Investigation 2

This investigation introduces the integrals $\int \frac{1}{x}\,dx$ and $\int e^{x}dx$. It does not have a TU.

1 The value of n is -1

2 $\frac{1}{x}$ is the derivative of $\ln x$

3 $\int x^{-1}dx = \int \frac{1}{x}dx = \ln x + C$. (The need to express the rule as $\int \frac{1}{x}dx = \ln|x| + C$ is explained in text that follows.)

4 The function is $f(x) = e^{x}$.

Developing inquiry skills

In the opening scenario for this chapter, the function $g(t) = \frac{1}{50}e^{-\frac{t}{10}+3.09}$ models the rate of growth of the hydrangea bush in meters per day for $t \ge 20$ days.

Find the antiderivatives of the function g.

Answer: $-\frac{1}{5}e^{-\frac{t}{10}+3.09} + C$

10.2 More on indefinite integrals

Investigation 3

Conceptual understanding:

The reverse chain rule helps to integrate composite functions which contain linear functions.

1 $\frac{d}{dx}\left(\frac{1}{5}x^{5} + C\right) = 5\cdot\frac{1}{5}x^{4} + 0 = x^{4}$

2 $\frac{d}{dx}\left[\frac{1}{5}(2x+3)^{5} + C\right] = 5\cdot\frac{1}{5}(2x+3)^{4}\cdot 2 + 0 = 2(2x+3)^{4} \ne (2x+3)^{4}$

3 **a** $\int (2x+3)^{4}dx = \frac{1}{5}(2x+3)^{5}\cdot\frac{1}{2} + C = \frac{1}{10}(2x+3)^{5} + C$

$\frac{d}{dx}\left(\frac{1}{10}(2x+3)^{5} + C\right) = 5\cdot\frac{1}{10}(2x+3)^{4}\cdot 2 + C = (2x+3)^{4}$

b $\int (8x+4)^{3}dx = \frac{1}{4}(8x+4)^{4}\cdot\frac{1}{8} + C = \frac{1}{32}(8x+4)^{4} + C$

$\frac{d}{dx}\left(\frac{1}{32}(8x+4)^{4} + C\right) = 4\cdot\frac{1}{32}(8x+4)^{4}\cdot 8 + 0 = (8x+4)^{4}$

c $\int (-5x+7)^{9}dx = \frac{1}{10}(-5x+7)^{10}\cdot -\frac{1}{5} + C = -\frac{1}{50}(-5x+7)^{10} + C$

$\frac{d}{dx}\left(-\frac{1}{50}(-5x+7)^{10} + C\right) = 10\cdot -\frac{1}{50}(-5x+7)^{10}\cdot -5 + 0 = (-5x+7)^{10}$

4 a **Factual:** What is $\int (ax+b)^n dx$?

Answer: $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$

b $\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1} + C\right) = (n+1)\cdot\frac{1}{a(n+1)}(ax+b)^n \cdot a + 0 = (ax+b)^n$

5 a **Factual:** What is $\int e^{ax+b} dx$?

Answer: $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$

b $\frac{d}{dx}\left(\frac{1}{a}e^{ax+b} + C\right) = \frac{1}{a}e^{ax+b}\cdot a + 0 = e^{ax+b}$

6 a **Factual:** What is $\int \frac{1}{ax+b} dx$?

Answer: $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C$

b $\frac{d}{dx}\left(\frac{1}{a}\ln|ax+b| + C\right) = \frac{1}{a}\cdot\frac{1}{ax+b}\cdot a + 0 = \frac{1}{ax+b}$

7 **Conceptual:** What rule of differentiation can be reversed to help you integrate composite functions of the form $f(ax+b)$?

Answer (this is the conceptual understanding): The reverse chain rule helps to integrate composite functions of the form $f(ax+b)$?

TOK

Consider $f(x) = \frac{1}{x}, 1 \le x \le \infty$

An infinite area sweeps out a finite volume. Can this be

reconciled with our intuition? What does this tell us about mathematical knowledge?

Answer: The clash between infinite and finite may be explored here. Is this a contradiction?

You might also want to explore Gabriel's Horn and the Von Koch Snowflake.

Investigation 4

Conceptual understanding:

Finding a particular solution for an indefinite integral requires a boundary condition.

1 You find the velocity function by differentiating the displacement function.

2 **Factual:** Given the velocity function, how would you find the displacement function?

Answer: To find the displacement function you would integrate the velocity function.

3 $s(t) = \int (6t+3)\,dt = 3t^2 + 3t + C$

(The students may or may not be concerned that they cannot find C at this point.)

4 No, knowing the derivative is not enough because you cannot find the constant, C.

5 $4 = s(2) = 3(2)^2 + 3(2) + C$

$C = 4 - 18 = -14$

$\therefore s(t) = 3t^2 + 3t - 14$

6 **Conceptual:** In general, given a derivative of a certain function, what else do you need to know to find that particular function?

Answer (this is the conceptual understanding): Finding a particular solution for an indefinite integral requires a boundary condition.

(The students will not know the term boundary condition and so may answer, 'Finding a particular answer for an indefinite integral requires that you know the coordinates of a point on the graph of the function', or similar.)

Developing inquiry skills

In the opening scenario for this chapter, you were asked to find the function f that models the height or the hydrangea bush for $t \geq 20$ and to show that the height of the hydrangea bush does not exceed 1.5 m. You should now be able to answer these questions.

Answer: $f(t) = -\frac{1}{5}e^{-\frac{t}{10}+3.09} + 1.1956878$; as t approaches infinity, $f(t)$ approaches 1.1956878 which is less than 1.5

10.3 Area and definite integrals

Investigation 5

Conceptual understanding:

The definite integral represents the area under the curve for positive functions.

Students may say, 'It is the area under the curve.'

1 **a** $5(4) = 20$

b $\frac{1}{2}(3 \times 6) = 9$

c $\frac{1}{2}\left(\frac{5}{2}+1\right)(3) = \frac{21}{4} = 5.25$

d $\frac{1}{2}\left(\pi \times 4^2\right) = 8\pi \approx 25.1$

2 **a** 20

b 9

c 5.25

d 25.1

3 The area of the region is equal to the value of the definite integral between the upper and lower limits of x.

4 **Conceptual:** What does a definite integral represent?

Answer (this is the conceptual understanding): The definite integral represents the area under the curve for positive functions. Students may say, 'It is the area under the curve.'

(Note that the students will not yet know to restrict their answers to positive functions, but they will revisit this in the next investigation.)

Investigation 6

Conceptual understanding:

The definite integral represents the area under the curve for positive functions.

1 **a** 8

b 8

2 **a** $\frac{1}{2}(2)(2)=2$

b $\int_{-3}^{-1}(x+1)dx=-2$

c The value of the definite integral is the negative value of the area found by a geometric method.

3 **Conceptual:** What does a definite integral represent?'

Answer (this is the conceptual understanding): The definite integral represents the area under the curve for positive functions.

Students may also note that the definite integral is the opposite of the area for negative functions.

4 **a** $\int_{-3}^{3}(x+1)dx=6$

b It is the sum. $\int_{-3}^{3}(x+1)dx=\int_{-3}^{-1}(x+1)dx+\int_{-1}^{3}(x+1)dx=8+(-2)=6$

c **Factual:** How do you think $\int_{a}^{c}f(x)dx+\int_{c}^{b}f(x)dx$ can be written as a single integral?

Answer: $\int_{a}^{c}f(x)dx+\int_{c}^{b}f(x)dx=\int_{a}^{b}f(x)dx$

5 **Factual:** If you are asked to find the total area under the graph f(x) = x + 1 between x = −3 and x = 3, why should you not find the value of $\int_{-3}^{3}(x+1)\,dx$? How would you calculate this total area?

Answer: Because $f(x)=x+1$ lies below the x-axis for $-3<x<-1$, $\int_{-3}^{-1}(x+1)dx$ gives a negative value. To find the total area, you should find
$\int_{-3}^{3}(x+1)dx=\left|\int_{-3}^{-1}(x+1)dx\right|+\left|\int_{-1}^{3}(x+1)dx\right|=8+2=10$

6 **Conceptual:** How could you find the total area under a curve $f(x)$ between two limits $x = a$ and $x = b$?

Answer: Find any point c where $a < c < b$ such that $f(c) = 0$ (i.e. the points where $f(x)$ crosses the x-axis.

Then total area = $\left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$

TOK

Is imagination more important than knowledge?

Answer: Have students express their thoughts on what is imagination.

"Imagination is the beginning of creation. You imagine what you desire, you will what you imagine and at last you create what you will." George Bernard Shaw

Albert Einstein thought so. He said: "I'm enough of an artist to draw freely on my imagination, which I think is more important than knowledge.

Knowledge is limited. Imagination encircles the world."

When you see your students share the knowledge they have learned from you, don't you feel proud?

Now, when you see their imagination use that knowledge and take a step further, that's amazing.

Have students write about which they think is the more important, and why.

Reflect: The total area under the curve $y = f(x)$ between $x = 0$ and $x = 12$ is not zero. How could you sum three definite integrals to find this total area?

Answer: Total area $= \left|\int_0^5 f(x)dx\right| + \left|\int_5^{10} f(x)dx\right| + \left|\int_{10}^{12} f(x)dx\right| = 10 + 14 + 4 = 28$

Investigation 7

Conceptual understanding:

Applying limits to the process of summing areas of rectangles, gives the exact value of the area under a curve and of the definite integral representing that area.

1 **a** 0.5

b 1; 1.25; 2; 3.25

c 3.75

2 5.75

3 **a** $\int_0^2 (x^2 + 1)dx$

b 4.67 (to 3 significant figures) and $3.75 < 4.67 < 5.75$

4 You could get better approximations by using more rectangles (or rectangles with smaller widths). Students may also suggest other answers, such as using trapeziums rather than rectangles.

5

# Rectangles	Lower sum	Upper sum
4	3.75	5.75
10	4.28	5.08
50	4.5872	4.7472
100	4.6268	4.7068

Note: If students do not have access to technology for finding Riemann sums, provide them with the answer to question 5.

6 **Conceptual:** What major concept of calculus can be applied to the process of summing areas of rectangles, to give the exact value of the area under a curve and of the definite integral representing that area?

Answer (this is the conceptual understanding): Applying limits to the process of summing areas of rectangles, gives the exact value of the area under a curve and of the definite integral representing that area.

10.5 Area between two curves

TOK

Why do we study mathematics? What's the point? Can we do without it?

Answer: At this level of complex mathematics, some students may well be asking this question!

Ask students what words come to mind when they think about the word mathematics.

Ask them where they see mathematics in everyday life.

Galileo said that we find mathematics everywhere in nature. Ask students to illustrate some examples.

Ask students where they benefit from mathematics.

You could create a wall or board or website display of the answers.

Maybe you could ask them to describe the scenario of a world without mathematics, highlighting the positives and negatives.

Investigation 8

Conceptual understanding:

Modification of the representation of the height of a rectangle in the definite integral for the area under a curve leads the definite integral for the of area between two curves.

1 R2: $h = g(1.5) - f(1.5) = 1 - (-2.05) = 3.05$

R3: $h = f(5) - g(5) = 1.92 - (-1.33) = 3.25$

R4: $h = g(8) - f(8) = -3.33 - (-5.28) = 1.95$

(The purpose of this question is to help students see that the height of a rectangle can be found by taking a difference of the two functions, regardless of the relative position of the rectangle to the x-axis.)

2 **Factual:** When using rectangles to approximate the area of a region between the curves $y = f(x)$ and $y = g(x)$, how do you choose whether to represent the height of the rectangle as $f(x) - g(x)$ or $g(x) - f(x)$?

Answer: Use $f(x) - g(x)$ when $f(x) \geq g(x)$ and $g(x) - f(x)$ when $g(x) \geq f(x)$. Students may also answer less formally. For instance, use $f(x) \geq g(x)$ when f is the top curve and use $g(x) - f(x)$ when g is the top curve.

3 Use $f(x) - g(x)$, because $f(x) \geq g(x)$ from $x = -0.75$ to $x = 1.75$. (because f is the top curve for $x = -0.75$ to $x = 1.75$)

4

interval	width	height	area
$-0.75 \leq x \leq -0.25$	0.5	$f(-0.5) - g(-0.5) = 0.25 - (-7.25) = 7.5$	$0.5(7.5) = 3.75$
$-0.25 \leq x \leq 0.25$	0.5	$f(0) - g(0) = -1 - (-7) = 6$	$0.5(6) = 3$
$0.25 \leq x \leq 0.75$	0.5	$f(0.5) - g(0.5) = -1.75 - (-6.75) = 5$	$0.5(5) = 2.5$
$0.75 \leq x \leq 1.25$	0.5	$f(1) - g(1) = -2 - (-6.5) = 4.5$	$0.5(4.5) = 2.25$
$1.25 \leq x \leq 1.75$	0.5	$f(1.5) - g(1.5) = 2 - (-6.25) = 4.5$	$0.5(4.5) = 2.25$

5 Area $\approx 3.75 + 3 + 2.5 + 2.25 + 2.25 = 13.75$

6 **Conceptual:** How can you modify the formula for the area under a curve to find the area between two curves? Explain your reasoning.

Answer (this is the conceptual understanding): Modification of the representation of the height of a rectangle in the definite integral for the area under a curve leads the definite integral for the of area between two curves.

Explanation of reasoning: For area under a curve, you can think of the y in $A = \int_a^b |y|\,dx$ as representing the height of a rectangle used in the approximation of the area under the curve. For the area between two curves y_1 and y_2, where $y_1 \geq y_2$ the height of a rectangle is equal to $y_1 - y_2$ and so the formula becomes $A = \int_a^b (y_2 - y_1)dx$.

7 **Factual:** Which part(s) of the expression $\int_a^b (y_1 - y_2)dx$ represent(s) each of the following?

a Height of a rectangle

b Width of a rectangle

c Area of a rectangle

d Sum of the areas of an infinite number of rectangles from $x = a$ to $x = b$.

Answer:

a Height of a rectangle $= y_1 - y_2$

b Width of a rectangle $= dx$

c Area of a rectangle $= (y_1 - y_2)dx$

d Sum of the areas $= \int_a^b (y_1 - y_2)dx$

8 Area $= \int_{-0.75}^{1.75} \left[\left(x^2 - 2x - 1\right) - \left(\frac{1}{2}x - 7\right) \right] dx = 13.8$. This is close to the approximation of 13.75.

Developing inquiry skills

Return to the opening problem. How have the skills you have learned in this chapter helped you to solve the problem?

Answer: To find the function for the height of the plant for $t > 20$. you needed to be able to integrate the composite function e^{ax+b}. This was a skill learned in the chapter.

In the Footsteps of Archimedes

Approaches to Learning/Learner profile: Research, Critical Thinking

Exploration Criteria: Mathematical Communication (B); Personal Engagement (C); Use of Mathematics (E)

IB Topic: Integration, Proof, Coordinate Geometry

Introduction

This task looks at a topic in the vast history of mathematics. Many students in fact write their explorations on the history of mathematics. Clearly, this demonstrates that students are interested. This is fine, but a merely historical account copied or summarized from the internet is unlikely to score well on most criteria, especially if there is little or no mathematics actually used. There is often very little or no opportunity for Reflection or Personal Engagement in this type of exploration. Having said that, it is still worthwhile including aspects of mathematical history in class, to give the subject or a particular topic some context. It will also provide students with examples that they may incorporate into their TOK lessons and assessments.

This particular task looks at Archimedes' method for finding the area of a parabolic segment and is a remarkable result. This is interesting, as he worked on it almost 2,000 years before Newton and Leibniz had formalized differential and integral calculus. Archimedes, in his time and in his own way, already had a good grasp of the basics of calculus, but despite this he is not really part of the debate about the Father of Calculus.

The task is set up so that there are opportunities to discover aspects of his proof and consider ways that Personal Engagement (Criterion C), Mathematical Communication (Criterion B) and Use of Mathematics (Criterion E) can be demonstrated. There are also clear links between different areas of mathematics covered in the syllabus which, depending on the approach to the discussion, can also be credited in Criterion C and Criterion D.

The Area of a Parabolic Segment

This task looks at Archimedes' work on finding the area of a parabolic segment.

To set the scene, you could ask students what they know about Archimedes.

Archimedes was a famous Greek mathematician working 2,300 years ago. He has many theorems and discoveries credited to him and is generally regarded as one of the greatest mathematicians of all time.

You could also talk about the history of calculus, and who is credited for any breakthroughs.

Newton and Leibniz are credited with the invention of calculus and the methods used in this chapter (although there is still some controversy over exactly who developed it first between these two), but the most remarkable thing about Archimedes' work is that it predates their work by around 2,000 years.

Archimedes would not have used the coordinate system as we know it today, since it was not invented until the 17th century by Descartes, but his method would have been similar.

The x-value is 0.5, as this is halfway between -2 and 3.

The corresponding y-value is $0.5^2 = 0.25$.

So, $C(0.5, 0.25)$.

Students can verify the result in this case by calculating the area of the triangle and the area of the shaded area.

In the given example, the line y = x + 6 intersects the parabola at A(-2, 4) and B(3, 9) and the other vertex of the triangle on $y = x^2$ is at C(0.5, 0.25).

Methods for calculating the area of the triangle include:

Working out the length of *AC* and the perpendicular height from *AC* extended to *B* and using the formula area = $\frac{1}{2}bh$.

Using the formula for calculating the area that uses the three vertices of the triangle:

Area = $[A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)] / 2$

where A_x and A_y are the *x- and y*-coordinates of the point *A*, etc.

Using the bound box method. This is where a box is drawn around the triangle and the area of around the edge of the required triangle area is calculated and subtracted from the area of the box.

Finding the length of the three sides and then using Heron's formula.

Whichever method is used, area $\Delta ABC = 15.625 = 15\frac{5}{8}$ unit^2

Students can use integration to calculate the required area between the two curves. The upper curve is the line $y_2 = x + 6$ and the lower curve is $y_1 = x^2$. The limits of integration are $x = -2$ and $x = 3$.

$$\int_{-2}^{3}\left[(x+6) - x^2\right]\mathrm{d}x = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = 20\frac{5}{6}$$

So, the area between the two curves is $20\frac{5}{6}$ unit^2.

If needed, assist students with the above integration.

So according to Archimedes, the area of the parabolic segment will be:

$\frac{4}{3} \times 15\frac{5}{8} = 20\frac{5}{6}$ unit^2 as required.

You could discuss how Archimedes proved his result:

The way Archimedes achieved this result was by **inscribing** successively smaller polygons (triangles in this case) until the area was filled. You can calculate the sum of the area of these (infinitely many) triangles and hence the area of the curved shape.

D (-0.75, 0.5625)

Students could use one of the above methods to calculate the area of triangle *ACD*.

Area of triangle *ACD* = 1.953125 unit^2

E (1.75, 3.0625)

Area of triangle *BCE* = 1.953125 unit^2

Ratio is 1.953125/15.625 = 1/8

The area of each new triangle is 1/8 the area of the big triangle. (The sum of the two triangles is ¼ the area of the big triangle).

This can be **proved** generally to always be the case, because the new triangles have 1/2 the width of the original triangle and 1/4 of the height.

Note that with the seven triangles, as before, the new triangles will be 1/8 the area of the previous triangles. In fact, it can be shown that this is true of any triangles inscribed in such a way within the parabola.

Again, students could be encouraged to work on this or research reasons why.

You could improve the approximation by using more triangles.

It is possible to speed up calculations by constructing a spreadsheet. This should allow you to easily add increasingly more triangles. However, this will not be a proof.

Students could be encouraged to do this or at least note that this is a possibility in this sort of task and could be a good approach if it is an exploration to demonstrate Personal Engagement.

Generalize the Problem

Archimedes originally used a geometric proof. The method shown here, however, uses mathematics covered earlier in the course—the sum of an infinite geometric sequence.

Total area of the next two triangles = $2(X/8) = X/4$

Total area of the next four triangles = $4(X/64) = X/16$

Total area of the next eight triangles = $4(X/512) = X/64$

$X + 2(X/8) + 4(X/64) + 8(X/512) + \ldots$

$= X + X/4 + X/16 + X/64 + \ldots$

This is a geometric series, with common ratio $r = 1/4$ and first term $U_1 = X$

The sum of this series is given by: Sum = $\dfrac{U_1}{1-r} = \dfrac{X}{1-\frac{1}{4}} = \dfrac{X}{3/4} = \dfrac{4X}{3}$

The sum of the areas of all the triangles gives the area of the parabolic segment and this is $\dfrac{4X}{3}$, which is $\dfrac{4}{3}$ the area of the original triangle, just as Archimedes claimed!

Archimedes' thinking behind this solution is very similar to the ideas behind the development of calculus as seen in this chapter.

Extension

Studying the history of mathematics helps students develop a deeper understanding of the mathematics they have already studied by seeing how it was developed over time and, importantly, in various places across the world. It also encourages creative and flexible thinking by allowing students to see historical evidence that there are different and perfectly valid ways to view concepts and to carry out computations.

There is a clear link to TOK too.

Consider this from the knowledge framework in the IB TOK guide for mathematics

Historical development

How has our understanding and perception of mathematics changed over time?

How has the role of mathematics within society developed?

To what extent has the nature of mathematics (for example, the different forms of mathematics) changed?

What relationship does today's mathematical understanding have with that of the past? (to paraphrase Newton, does it "stand on the shoulders of giants"?)

Here are a couple of interesting websites to explore regarding the history of mathematics:

www.storyofmathematics.com/

www-groups.dcs.st-and.ac.uk/~history/

Again, it is important to counsel students on writing explorations purely on the history of a mathematical topic. Use of Mathematics (Criterion E) must be clearly demonstrated and Personal Engagement (Criterion C) should be considered beyond just "this is interesting" or "this is important", and Reflection (Criterion D) beyond just "this is hard".

11 Relationships in space: geometry and trigonometry in 2D and 3D

Essential understandings

Geometry and trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The properties of shapes depend on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
3D coordinates represent positions in 3D space and can be used to calculate midpoints of lines. The Pythagorean theorem can be used to find distance in 3D space.	Investigation 1
The relationship between the sides and angles of triangles leads to the exploration of trigonometric functions.	Investigation 4
Sine and Cosine ratios represent complementary functions with one being the 90-degree horizontal translation of the other. In a right triangle there is a right angle and two other angles that add up to 900. These are called complementary angles.	Investigation 6
The Sine rule expresses the equivalence of the ratios of the length of sides with the sine of the opposite angle for any triangle. The sine rule can be used to solve for a missing side or angle in a triangle, given the measures of one side and two angles, or the measures of two sides and a non-contained angle.	Investigation 7
The symmetry and periodic nature of trigonometric functions implies that different angles may lead to the same ratios.	Investigation 9
The cosine rule generalizes the Pythagorean theorem by considering acute, right and obtuse angles and can be used to solve a triangle given the measures of all three sides or the measures of two sides and the included angle.	Investigation 10

Syllabus sections covered in this chapter:

- SL3.1*
- SL3.2*
- SL3.3*

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 467: Pythagoras' theorem	Page 485: Example 9 Page 493: Example 12	Page 478: Example 4 Page 483: Example 8 Page 485: Example 9 Page 490: Example 11	Pages 473, 483, 489, 491, 493

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 496	Page 499	N/A

11.1 The geometry of 3D shapes

Investigation 1

Conceptual understanding:

3D coordinates represent positions in 3D space and can be used to calculate midpoints of lines.

The Pythagorean theorem can be used to find distance in 3D space.

1 C(0, 4, 0), D(0, 0, 4), E(4, 0, 4) and G(0, 4, 4)

2 Students' answers will vary.

3 **Conceptual:** How are 3D coordinates used to find measures?

Answer (this is the conceptual understanding): 3D coordinates represent positions in 3D space and can be used to calculate midpoints of lines.

4 **Conceptual:** How is the Pythagorean Theorem related to the distance formula?

Answer (this is the conceptual understanding): The Pythagorean theorem can be used to find distance in 3D space

5 Half of (4,4,4)

6 $4\sqrt{2}$, $4\sqrt{3}$

Investigation 2

1 **a** Students' answers will vary.

b Prisms have a cross section.

Faces, edges and vertices are visible in a prism.

A cube is a prism.

The volume of a prims is the area of the cross section multiplied by the length.

The surface area of a prism is the sum of the area of all of the faces.

2 **a** Check students' drawings.

b A cylinder is a prism.

There is a radius.

It has a curved surface of area $2\pi rh$.

The volume is $\pi r^2 h$.

Cans are often the shape of a cylinder.

3 Students' answers will vary.

4 **Conceptual:** How can you classify or describe a three-dimensional shape?

Answer: Discuss ideas as a class.

Investigation 3

1 **Conceptual:** How do you find the surface area of 3D shapes?

Answer: Sum the areas of all of the faces.

2 **Factual:** What is a net?

Answer: A net is a 2D pattern that you can cut and fold to make a model of a 3D shape.

3 **c** x^2

d $2xl$

4 **Factual:** What is the surface area of a right-angled pyramid?

Answer: $\frac{1}{2}xl$

5 $x^2 + 2xl$

Developing inquiry skills

The Louvre pyramid reaches a height of 21.6 m and its square base has sides of 35 m. Find the total volume of the pyramid.

Answer: 8820 m^3

11.2 Right-angled triangle trigonometry

Investigation 4

Conceptual understanding:

The relationship between the sides and angles of triangles leads to the exploration of trigonometric functions.

1 Students' answers will vary.

2 Completed table:

$\triangle EAD$	$\triangle FAC$	$\triangle GAB$
$\frac{EA}{AD} \approx 1.06$	$\frac{FA}{AC} \approx 1.06$	$\frac{GA}{AB} \approx 1.06$
$\frac{AD}{EA} \approx 0.940$	$\frac{AC}{FA} \approx 0.940$	$\frac{AB}{GA} \approx 0.940$
$\frac{EA}{DE} \approx 2.92$	$\frac{FA}{CF} \approx 2.92$	$\frac{GA}{BG} \approx 2.92$
$\frac{AD}{DE} \approx 2.74$	$\frac{AC}{CF} \approx 2.74$	$\frac{AB}{BG} \approx 2.74$
$\frac{DE}{EA} \approx 0.342$	$\frac{CF}{FA} \approx 0.342$	$\frac{BG}{GA} \approx 0.342$
$\frac{DE}{AD} \approx 0.364$	$\frac{CF}{AC} \approx 0.364$	$\frac{BG}{AB} \approx 0.364$

3 Answers will vary but students should have approximately the same value as the sides are all on corresponding sides of similar triangles.

4 Each row contains the same values. There is a relationship between the ratios formed between the sides of triangle.

5 The teacher might have to explain the meaning of "hypothesis". There is a relationship resulting in a ratio between the sides and angles in a right triangle that can be used to find other sides or angles in a right triangle.

6 The opposite divided by the hypotenuse is the sine ratio. The adjacent divided by the hypotenuse is the cosine ratio and the opposite divided by the adjacent is the tangent ratio.

7 **Factual:** What are the three trigonometric ratios?

Answer: Sine, cosine and tangent.

8 **Factual:** What are sine, cosine and tangent?

Answer: $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ and $\tan\theta = \frac{O}{A}$

9 **Conceptual:** How do the trigonometric ratios help to find the lengths and angles in right angled triangles?

Answer: We can select an appropriate trig ratio to reduce the problem to a simple equation.

Investigation 5

1 Completed table:

H =	2	3	4	5
O =	$2\sin\theta$	$3\sin\theta$	$4\sin\theta$	$5\sin\theta$
A =	$2\cos\theta$	$3\cos\theta$	$4\cos\theta$	$5\cos\theta$

2 $\sin\theta = \dfrac{O}{H}$

3 O = $H\sin\theta$

4 $\cos\theta = \dfrac{A}{H}$

5 A = $H\cos\theta$

6 A formula to find the side of a triangle.

7 **Conceptual:** How do trigonometric ratios allow us to solve problems involving right-angled triangles?

Answer: Trigonometric ratios can be used as a formula to solve for a side or an angle.

Investigation 6

Conceptual understanding:

Sine and Cosine ratios represent complementary functions with one being the 90-degree horizontal translation of the other.

In a right triangle there is a right angle and two other angles that add up to 900. These are called complementary angles.

1 Completed table:

θ	$sin\theta$	$\cos(90-\theta)$	$cos\theta$	$\sin(90-\theta)$
30	0.5	0.5	0.866	0.866
50	0.766	0.766	0.643	0.643
60	0.866	0.866	0.5	0.5
75	0.966	0.966	0.259	0.259
80	0.984	0.984	0.174	0.174

2-4 Responses may include that $\sin\theta = \cos(90-\theta)$ and $\cos\theta = \sin(90-\theta)$

5 The sine of 40° is equal to the cosine of 50°

The cosine of 70° is equal to the sine of 20°

sin A = cos B when A + B = 90°.

Developing inquiry skills

Find the angle between a face and a base of the Louvre pyramid.

Answer: 51°

Find the angle between an edge and the base.

Answer: 27°

11.3 The sine rule

Investigation 7

Conceptual understanding:

The Sine rule expresses the equivalence of the ratios of the length of sides with the sine of the opposite angle for any triangle.

The sine rule can be used to solve for a missing side or angle in a triangle, given the measures of one side and two angles, or the measures of two sides and a non-contained angle.

1 $\sin B = \dfrac{h}{a}$

2 $h = a\sin B$

3 $\sin A = \dfrac{h}{b}$

4 $\sin A = \dfrac{a\sin B}{b}$

5 $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

6 **Factual:** What is the sine rule?

Answer: The sine rule is an equation relating the lengths of the sides of any triangle to the sines of its angles.

Investigation 8

The ambiguous case of the sine rule considers two possible triangles once given two sides and one non-included angle, with the unknown angle positioned opposite the longer of the two sides.

You will need a compass, protractor, pencil and ruler for this.

Teachers might also want to use the applet at https://www.geogebra.org/m/MJYSH8q7

1 In a triangle, the biggest side is opposite the biggest angle.

The shortest distance from a point to a line is a perpendicular path

The angles in a triangle add up to 180^0.

In an acute triangle, all of the angles are less than 90^0.

In an obtuse triangle, one angle is greater than 90^0 and the other two are less than 90^0

2-7 Check students' constructions.

8 When you are given two sides and an angle not in between those sides, you need to be aware that this could lead to two different triangles.

Investigation 9

Conceptual understanding:

The symmetry and periodic nature of trigonometric functions implies that different angles may lead to the same ratios.

1 Completed table:

θ	$\sin\theta$	$\sin(180-\theta)$
10	0.174	0.174
30	0.5	0.5
45	0.707	0.707
60	0.866	0.866
80	0.985	0.985

2 The two columns have the same answers.

3 $\sin\theta = \sin(180-\theta)$

4 **Factual:** What is the relationship between the sines of supplementary angles?

Answer: They are equal.

5 The sine of 50° is equal to the sine of 130°.

The cosine of 120° is equal to the sine of 60°.

sin A = sin B when $A + B = 180°$

6 **a** $\sin^{-1} 0.5 = 30°$ and $150°$

b $\sin^{-1} 0.7 = 44.4°$ and $136°$

11.4 The cosine rule

Investigation 10

Conceptual understanding:

The cosine rule generalizes the Pythagorean theorem by considering acute, right and obtuse angles and can be used to solve a triangle given the measures of all three sides or the measures of two sides and the included angle.

1 **Conceptual:** How does the relationship between the sides and the angles of triangles lead you to the cosine rule?

Answer (this is the conceptual understanding): The cosine rule generalizes the Pythagorean theorem by considering acute, right and obtuse angles and can be used to solve a triangle given the measures of all three sides or the measures of two sides and the included angle.

2 $b^2 = x^2 + h^2$

3 $\cos A = \frac{x}{b}$, $x = b\cos A$

4 $a^2 = h^2 + (c-x)^2$

5 $(c-x)^2 = c^2 - 2cx + x^2$

6 $a^2 = c^2 - 2cx + b^2$

7 $a^2 = c^2 - 2bc\cos A + b^2$

8 $a^2 = b^2 + c^2 - 2bc\cos A$

11.5 Applications of right and non-right-angled trigonometry

Developing inquiry skills

Chicky is a glass cleaner half way up of one of the edges of the Louvre pyramid. How far is it to the opposite corner of the base?

Answer: 41.7 m

Three Squares

Approaches to Learning/Learner Profile: Research, Critical Thinking

Exploration Criteria: Personal Engagement (C); Use of Mathematics (E)

IB Topic: Proof, Geometry, Trigonometry

Introduction

This is a good opportunity to revisit Proof, which was introduced at the beginning of the book.

Proof is a difficult topic to pursue for an exploration, as it will not always score highly in all criteria. Some suggestions are given at the end on how possibly to extend this and other topics that give the possibility of scoring better on Personal Engagement (Criterion C) and even Reflection (Criterion D).

The Problem: Some History

This problem first appeared in the "Mathematical Games" column of *Scientific American* in 1970. It was submitted by Martin Gardner—he had been introduced to the problem by a friend, Lyber Katz, who had had to do it for extra credit in Moscow when he was in Grade 4. The problem was published again in *Journal of Recreational Math* in 1971, where readers could try to solve it and send back their solutions. Many people, ranging from high school students to undergraduates to people in mathematical professions, sent back many different solutions. Charles W. Trigg gathered all these solutions and published 54 of them under the title "A Three-Square Geometry Problem" in the following volume of *Journal of Recreational Math*.

Emphasize to students that these 54 solutions, and others found since, lead to exactly the same answer. In a mathematics class, students tend to focus on just finding the correct solutions, rather than concentrating on how to find that answer. To find a solution, one way may be easy, but to find another, alternative solution to the same problem perhaps requires a deeper understanding of it, using different skills and mathematical concepts. The idea of finding multiple solutions also links to the concept of divergent thinking, which is a crucial thought process used by mathematicians, engineers, architects and other problem solvers.

This is an interesting problem—not because the proofs of the answer are particularly difficult, but because there are so many of them.

Exploring the Problem

This is an interesting crossover with TOK that looks at inductive and deductive reasoning and the concept of proof in mathematics and all other areas of knowledge.

The answer is 90, and students will probably get something close to this if they measure the angles or just guess.

Direct Proof

$\alpha = 45°$

$\alpha = \beta + \varphi = 45°$

AC = $\sqrt{2}$, AD = $\sqrt{5}$, AE = $\sqrt{10}$

To help students explain why the triangles are similar, ask:

Which sides correspond?

SSS Similar triangle theorem:

In ΔACD, the ratio of $CD : AC : AD = 1 : \sqrt{2} : \sqrt{5}$

In ΔACE, the ratio of $AC : CE : AE = \sqrt{2} : 2 : \sqrt{10} = 1(\sqrt{2}) : \sqrt{2}(\sqrt{2}) : \sqrt{5}(\sqrt{2})$

So $CD : AC : AD = AC : CE : AE$

Hence $\angle CAD = \angle CEA$, using the properties of two similar triangles.

Using exterior angle theorem:

$\angle ACB = \angle CAD + \angle ADC$

So $\alpha = \beta + \phi$

and $\beta + \phi = 45°$

Thus $\alpha + \beta + \phi = 90°$

Proof Using an Auxiliary Line

$\angle BAC = \alpha$: Angle of isosceles right triangle.

$\angle EAF = \phi$: Alternate interior angles.

This will complete the proof because it will show that ∠BAC + ∠EAF + ∠GAC = 90° since ∠BAF is a right angle because it is an angle of a square.

To help students show that ΔGAC and ΔABD are similar, you could hint:

Calculate the lengths of sides CG and AC.

$CG : AC = AB : BD$ since $\sqrt{2}/2 : \sqrt{2} = 1 : 2$

and $\angle GCA = \angle ABD = 90°$

$\angle GAC = \angle BDA = \beta$

These are equivalent angles in similar triangles.

Completing the proof:

$\angle BAC + \angle EAF + \angle GAC = 90°$

and so $\alpha + \beta + \phi = 90°$

Adding an auxiliary line is often a good method in a geometrical proof.

Proof Using the Cosine Rule

$\angle XEY = \beta$ because it is a 2 by 1 triangle like ΔABD.

$AE = \sqrt{10}$ and $AY = \sqrt{5}$

$5^2 = (\sqrt{10})^2 + (\sqrt{5})^2 - 2(\sqrt{10})(\sqrt{5})\cos\theta$

$\cos\theta = -1/\sqrt{2}$

$\theta = 135°$

$\beta + \phi = 180 - 135 = 45°$

To complete the proof:

$\alpha = 45°$

So $\alpha + \beta + \phi = 90°$

You could ask:

Which proof do you think is the best?

Which do you like the most?

What criteria are you using to make these judgements?

Again, this is a good TOK point where the concept of "best" can be discussed.

Students might consider "best" to be, for example:

quickest or easiest

most efficient or most elegant

most surprising or using unusual mathematics

hardest to understand, etc.

Questions involving proof are a good starting point for an exploration, but would rarely score highly if that were all they involve as there is limited chance for Personal Engagement beyond possibly engaging in mathematics that is new to the student or beyond the scope of the course.

Extension

You could direct students towards problems in the textbook or problems on sites such as Nrich, Brilliant or various mathematical competition sites.

Students could combine this with the extension task in chapter 7 on Spearman's rank and provide their classmates with different proofs, asking them to rank them.

The problems that students choose may also give them a chance to extend the problems and ask them to consider "what if…?" to demonstrate Personal Engagement (Criterion C).

For the problem in this task, they could discuss:

How else could the problem be extended?

What about four squares, five squares, n squares?

Students could then work on generalizing the problem.

This is good advice whenever a student tackles a problem where solutions are readily available by internet search.

12 Periodic relationships: trigonometric functions

Essential understandings

Trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.
- Different representations of the values of trigonometric relationships, such as exact or approximate, may not be equivalent to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically represent the periodic or symmetric nature of their values.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Trigonometric identity formulas that hold true in the first quadrant, can be extended into all four quadrants using the unit circle.	Investigation 1
Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle. Different measurement systems can be used for angles to facilitate ease of calculation.	Investigation 2
The symmetry and periodicity nature of trigonometric functions implies that different angles may lead to the same ratios. The unit circle in the coordinate plane enables the extension of trigonometric ratios of angles in a right triangle to all the angles measured counter clockwise around the unit circle. Special right triangles with angles of 30/60 or 45/45 degrees build understandings of the relationships in the unit circle.	Investigation 3
The symmetry and periodic nature of trigonometric functions implies that different angles may lead to the same ratios and with the appropriate use of trigonometric identities algebraic manipulation the solution of a trigonometric equation can be found. The use of reference angles, appropriate trigonometric identities and algebraic manipulation allows to find the solution of a trigonometric equation.	Investigation 4
Transformations of the parent trigonometric function displayed by the parameters may identify the zeros, amplitude and intercepts to help sketch trigonometric functions.	Investigation 5
Transformations of the parent trigonometric function displayed by the parameters may identify the zeros, amplitude and intercepts to help sketch trigonometric functions.	Investigation 6
Trigonometric periodic functions and their transformations may be used to solve equations and also applied to real life situations, such as tides, positions on a Ferris wheel etc.	Investigation 8

Syllabus sections covered in this chapter:

- SL3.4
- SL3.5
- SL3.6
- SL3.7
- SL3.8

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 507: Finding a zero	Page 519: Example 6 Page 521: Example 8 Page 531: Example 12	Page 510: Example 4 Page 525: Example 12	Pages 512, 517, 522, 532

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 534	Page 536	N/A

12.1 Radian measure, arcs, sectors and segments

Investigation 1

Conceptual understanding:

Trigonometric identity formulas that hold true in the first quadrant, can be extended into all four quadrants using the unit circle.

8 The sine function is periodic. It will continue repeating its shape if you extend the graph in either direction, horizontally.

9 **Factual:** What is a periodic function?

Answer: A periodic function repeats itself.

10 What shape is the graph of a periodic function?

Answer: A periodic function repeats itself in an identical pattern at regular intervals.

11 Yes.

12 **Conceptual:** How does the shape of a periodic function show there will always be multiple x values that give the same values of y?

Answer: A horizontal line will cross the sine curve many times.

13 The line y= ½ crosses the sine curve at 60^0 *and* 120^0

14 **Conceptual:** How do we extend trigonometric ratios beyond right-angled triangles?

Answer (this is the conceptual understanding): Trigonometric identity formulas that hold true in the first quadrant, can be extended into all four quadrants using the unit circle.

Investigation 2

Conceptual understanding:

Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle.

Different measurement systems can be used for angles to facilitate ease of calculation.

1-11 This investigation will need big, clear diagrams. Starting out with the biggest paper possible will help. It might even be suitable for an activity on "butcher" paper.

12 Just over 3.1. The value is close to pi.

13 **Factual:** What is the relationship between radians and degrees?

Answer: pi radians = $180°$

14 **Factual:** What are some uses of the radian measure?

Answer: Calculus, arc length and sector area, angular velocity, pulley systems and many other Physics applications.

(This is the conceptual understanding): Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle.

15 **Conceptual:** Why are radians dimensionless?

Answer: Although the radian is a unit of measure, it is a dimensionless quantity. This can be seen from the definition of the angle being equal to the ratio of the length of the enclosed arc to the length of the circle's radius. Since the units of measurement cancel, this ratio is dimensionless.

(This is the conceptual understanding): Different measurement systems can be used for angles to facilitate ease of calculation.

16 Student discussions will vary.

Developing inquiry skills

How would you describe the shape of the function in the opening problem?

Answer: A sine curve.

Is this periodic function? Explain your reasoning.

Answer: Yes, after one second the curve will be repeated.

12.2 Trigonometric ratios in the unit circle

Investigation 3

Conceptual understanding:

The symmetry and periodicity nature of trigonometric functions implies that different angles may lead to the same ratios.

The unit circle in the coordinate plane enables the extension of trigonometric ratios of angles in a right triangle to all the angles measured counter clockwise around the unit circle.

Special right triangles with angles of 30/60 or 45/45 degrees build understandings of the relationships in the unit circle.

1-5 A$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

6 θ = 60°: $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, θ = 120°: $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, θ = 150°: $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, θ = 210°: $\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$,

θ = 240°: $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, θ = -30°: $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$, θ = -60°: $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, θ = -120°: $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

7 sin 30° = sin 150°

sin 60° = sin 120°

cos 30° = cos −30°

cos 60° = cos −60°

cos 120° = cos −120°

tan 30° = tan 210°

tan 60° = tan 240°

8 x = …, -330°, -210°, 30°, 150°, 390°, 510°. The solutions will repeat every 360°.

9 **Conceptual:** Using symmetry, state the different relationships between angles in different quadrants for each of the trigonometric values.

Answer (this is the conceptual understanding): The symmetry and periodic nature of trigonometric functions implies that different angles may lead to the same ratios.

$\cos\theta = \cos(-\theta)$, $\sin\theta = \sin(180-\theta)$, $\tan\theta = \tan(180+\theta)$

10 Adding or subtracting multiples of 360°.

11 Adding or subtracting multiples of 360°.

12 **(This is the conceptual understanding):** Special right triangles with angles of 30/60 or 45/45 degrees build understandings of the relationships in the unit circle.

13 Students may notice that $\cos^2\theta + \sin^2\theta = 1$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$

14 The trigonometric values are the same.

Developing inquiry skills

In the graph on the opening page, what do you think will happen after one second? Can you describe the next few seconds? What do you think could have happened before zero seconds?

Answer: After one second the curve will be repeated and it will continue in the pattern until the music changes or stops. The time before zero may have a similar cyclic pattern if the music was playing.

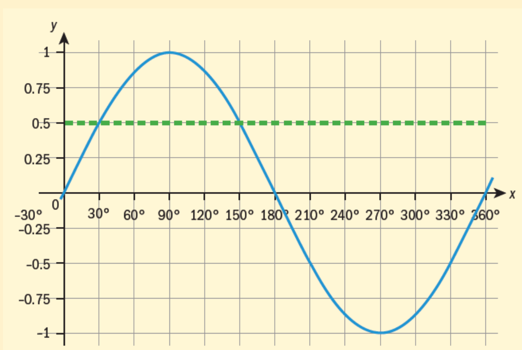
12.3 Trigonometric identities and equations

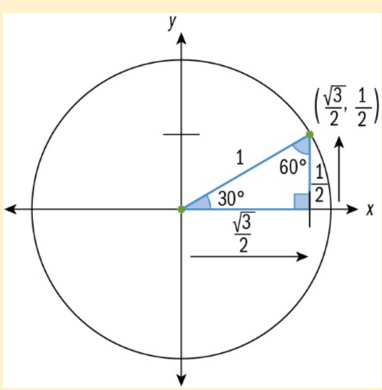
Investigation 4

Conceptual understanding:

The symmetry and periodic nature of trigonometric functions implies that different angles may lead to the same ratios and with the appropriate use of trigonometric identities algebraic manipulation the solution of a trigonometric equation can be found. The use of reference angles, appropriate trigonometric identities and algebraic manipulation allows to find the solution of a trigonometric equation.

1&2 30° or $\frac{\pi}{6}$ radians

3 

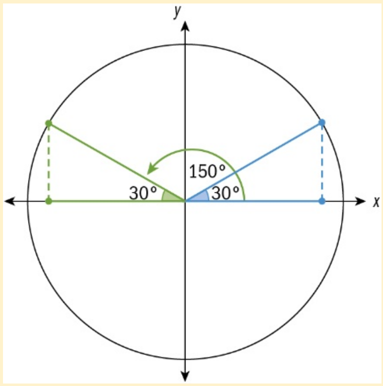
4 

5&6 All of the angles have an x value of $\frac{1}{2}$

7&8 The calculator gives a value of $30°$ or $\frac{\pi}{6}$ radians.

9 Find θ and then add or subtract as many rotations of $360°$ as you want.

10-12



All of the angles have an x value of $\frac{1}{2}$

13 & 14 The calculator gives a value of $30°$ or $\frac{\pi}{6}$ radians.

15 Factual: How can you use these patterns to solve equations?

Answer: After finding the first angle, the primary value, we are able to find a related angle in the unit circle, the secondary value and the subsequent values for the angle by taking these values and adding and subtracting multiples of 360 degrees for sine or cosine and 180 degrees for tangents.

16 Conceptual: How is trigonometry used to find all possible angles for unknown values? How is trigonometry used to find unknown values?

Answer: All possible values would follow the above method but to denote all possible solutions we could write the primary and secondary values $\pm 360n$ or $\pm 180n$ in the case of tangents.

17 Find θ and then add or subtract as many rotations of $360°$ as you want and include the primary values.

18 Discuss as a class.

Developing inquiry skills

Notice that the function can have more than one intersection when you draw horizontal lines on the opening graph. What would those lines tell you?

Answer: The time at which the amplitude is the same.

Where would there be only one point of intersection?

Answer: At the minimum/maximum amplitudes.

Is there anywhere that would be more than 2?

Answer: Yes, for amplitude 0.

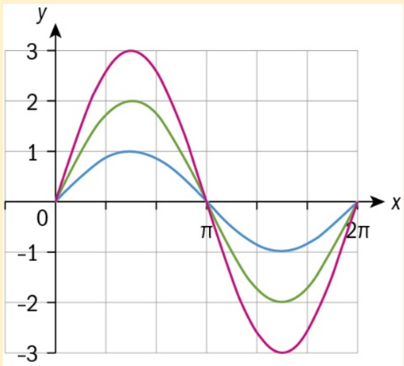
12.4 Trigonometric functions

Investigation 5

Conceptual understanding:

Transformations of the parent trigonometric function displayed by the parameters may identify the zeros, amplitude and intercepts to help sketch trigonometric functions.

1 a-c

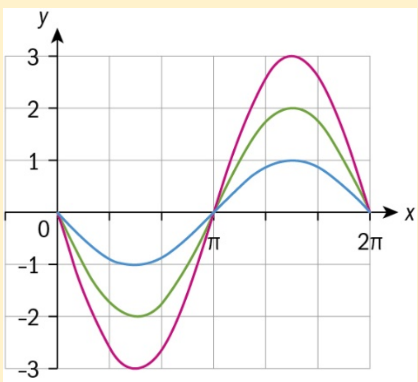


d sin x amplitude 1, 2sin x amplitude 2, 3 sin x amplitude 3.

e The coefficient of sin x is the amplitude

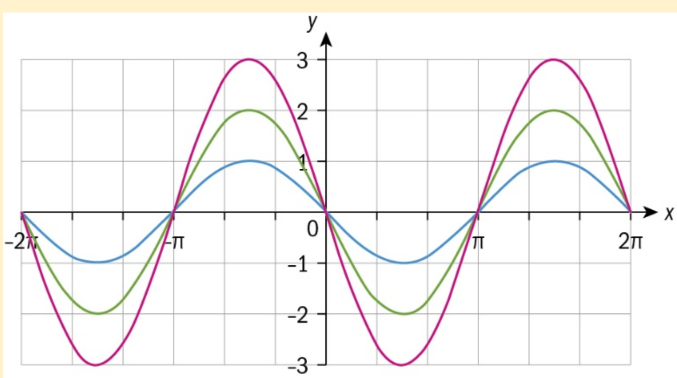
f No. The amplitude is a scalar value.

2 a-c



d The negative sign reflects the original function in the x axis

3 a-c



d A reflection in the y axis.

4 a-d 

5 The number of "waves" in the domain $0 \le x \le 2\pi$ is the coefficient of x.

6 The period of each function is 2π divided by the coefficient of x.

7 **Factual:** What do the parameters represent in $f(x) = a\sin(bx)$?

Answer: a is the amplitude and the period is $\frac{2\pi}{b}$

8 **Conceptual:** How does changing the parameters affect the graphs of trigonometric functions?

Answer:

$y = a\sin x$ is a vertical stretch of a.

$y = \sin bx$ is a horizontal stretch of $\frac{1}{b}$

$y = -\sin x$ or $y = -\cos x$ is a reflection in the x-axis

$y = \sin(-x)$ or $y = \cos(-x)$ is a reflection in the y-axis

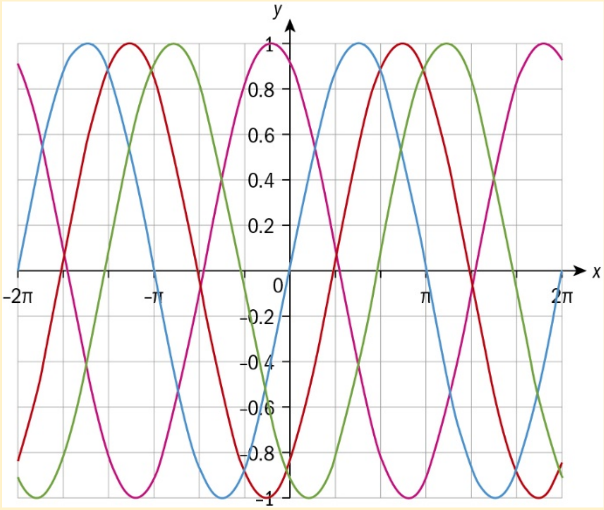
9 **Conceptual:** How can you transform trigonometric graphs?

Answer: Trigonometric graphs follow the same rules of transformations as all other functions.

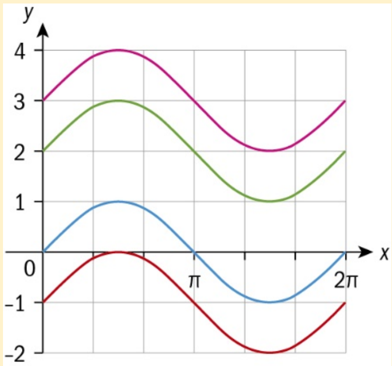
Investigation 6

Conceptual understanding:

Transformations of the parent trigonometric function displayed by the parameters may identify the zeros, amplitude and intercepts to help sketch trigonometric functions.

1 a-d 

e A horizontal translation of c units.



2 a-d

e A vertical translation

3 **Factual:** What do the parameters represent in $f(x) = \sin(x - c) + d$?

Answer: A horizontal translation of c and a vertical translation of d.

4,5 **Conceptual:** How does changing the parameters affect the graphs of trigonometric

Answer: $y = \sin(x - c)$ *or* $y = \cos(x - c)$ translates the function c units horizontally

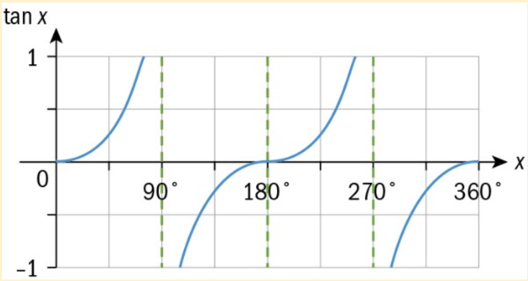
$y = d + \sin x$ *or* $y = d + \cos x$ translates the function d units vertically

Investigation 7

1 Completed table:

x°	*0*	*20*	*40*	*60*	*80*	*90*	*100*	*120*	*140*	*160*	*180*
tanx	*0*	*0.364*	*0.839*	*1.73*	*5.67*	*undefined*	*-5.67*	–1.73	–0.839	–0.364	*0*

x°	*200*	*220*	*240*	*260*	*270*	*280*	*300*	*320*	*340*	*360*
tanx	*0.364*	*0.839*	*1.73*	*5.67*	*undefined*	–5.67	–1.73	–0.839	–0.364	*0*



2

3 0, 180, 360.

4 There are asymptotes.

5-6 Yes, because there is a regular, repeating pattern

7 A tangent function is unlike a sine or cosine function in that it has asymptotes.

8 The tangent function is not like the sine and cosine functions in that they seem more like waves, but it is a periodic function because of the repeated pattern.

Investigation 8

Conceptual understanding:

Trigonometric periodic functions and their transformations may be used to solve equations and also applied to real life situations, such as tides, positions on a Ferris wheel etc.

1 **Factual:** What real-life situations can be modelled with graphs of trigonometric functions?

Answer: Situations involving repeated behaviour such as Ferris wheels, tides, lunar phases, biorhythms etc.

2 **Conceptual:** How can you solve real-life problems using the graphs of trigonometric functions?

Answer: Modelling situations with repeated patterns as a graph. Making a scatter plot and searching for patterns often reveals periodic behaviour.

3 The distance from the centre of the wheel remains constant, but the height above the ground changes.

4 2 metres

5 137 metres

6 15 mins

7 7.5 mins, 69.5 metres

8 22.5 mins, 69.5 metres

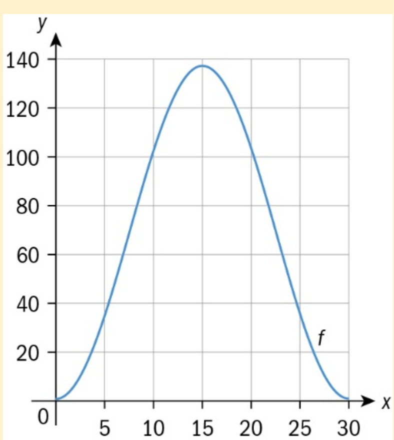
9 Completed table:

Position	*P0*	*P1*	*P2*	*P3*	*P4*	*P5*	*P6*	*P7*	*P8*	*P9*	*P10*	*P11*
Time (mins)	*0*	*2.5*	*5*	*7.5*	*10*	*12.5*	*15*	*17.5*	*20*	*22.5*	*25*	*27.5*
Height (metres)	*2*	*9.04*	*35.75*	*69.5*	*103.25*	*128*	*137*	*128*	*103.25*	*69.5*	*35.75*	*9.04*

10 **Conceptual:** How can you model periodic behaviour?

Answer (this is the conceptual understanding): Trigonometric periodic functions and their transformations may be used to solve equations and also applied to real life situations, such as tides, positions on a Ferris wheel etc.

11 $y = 68.1\sin(0.209x - 1.57) + 69.1$



12 Amplitude 67.5. Period 30. Vertical shift 69.5

13 $h(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$

14 Students checking their model.

15 **Factual:** What kinds of information can be modelled using periodic graphs?

Answer: Discuss ideas as a class.

16 124 metres.

17 **Factual:** What equations can be solved with trigonometric functions?

Answer: Sin, cosine or tangent equations.

Developing inquiry skills

Can you find the equation of the function from the opening problem?

Would you use sine or cosine?

What would be the difference if you used sine and your friend used cosine?

Answer: The graphs would both be suitable but the cosine graph would be the sine function translated ninety degrees horizontally to left.

The Sound of Mathematics

Approaches to Learning: Research, Critical Thinking, Using Technology

Exploration Criteria: Mathematical Communication (B); Personal Engagement (C); Use of Mathematics (E)

IB Topic: Trigonometric Functions

Introduction

When choosing an exploration topic, students are often encouraged to look for links between one of their interests and mathematics. A topic that is frequently considered by students, and not always handled well, is music and mathematics. Students' explorations tend to be based mainly on research pieces without any real Personal Engagement beyond personal interest, which is not enough to reach the higher levels of Criterion C.

In this task, students are encouraged to start by considering the usefulness of a mind map in producing ideas for ideas to explore within the field of music, and then to consider one of their ideas to brainstorm further.

One of the aims of this task is to help students to appreciate that sound consists simply of travelling waves. Students will consider the relationship between pitch, frequency and period. However, a more powerful aspect of the task is also the opportunity for students to consider where what they know could lead beyond pure research. Students are asked to think of ideas of how they might explore the way trigonometric functions are related to different sounds and musical notes in various different situations. Students also consider what technology it may be appropriate to use.

Brainstorm

This is an opportunity to discuss mind mapping if this has not already been done in the course. Students will probably be familiar with mind maps from other subjects too.

Split students into groups of three or four and provide them with large pieces of paper or a whiteboard, giving them time to brainstorm and construct mind maps. The whole class can then share their mind maps and discuss.

Research

When a sound is made, say on a musical instrument, it produces **vibrations** in the air. These vibrations vary depending on the sound. Particles vibrating because of the sound cause other nearby particles to vibrate, and so on. These vibrating particles make up a sound wave. Sound waves travel at different speeds and with variations in pressure through air, liquids and solids. (Sound waves are sometimes called pressure waves). A sound wave can be modelled using a **sine curve**.

The **frequency** determines the **pitch** of a sound (the greater the frequency, the higher the pitch). Frequency is measured in **Hertz** (Hz) and is the number of full **periods** of the soundwave per second. Waves with different Hertz values each have distinct sounds.

The **amplitude** determines the **loudness** (the greater the amplitude, the higher the volume).

Encourage students to look up, discuss or research further any of the words in bold that are unfamiliar to them.

This is a good opportunity for individual research as well as class discussion.

Students now have an opportunity to summarize and use what has been learnt in the chapter.

The **period** goes from one peak to the next (or from any point to the next matching point).

The **amplitude** is the height from the centre line to the peak (or to the trough). Or you can measure the height from highest to lowest points and divide that by 2.

Frequency is how often something happens per unit of time.

For a sine wave with the basic form $y = a\sin(bt)$:

a is the amplitude

b is connected to the period and frequency of the function

y is the output value of the trigonometric function.

$\text{Period} = \frac{2\pi}{b}$ and also $\text{Period} = \frac{1}{\text{Frequency}}$

A sound with frequency of 440 Hz, for example, means there are 440 periods per second (i.e. 1 period per $\frac{1}{440}$ second).

In this case, y is the pressure.

For a sound of 440 Hz:

period = $2\pi/b$

$1/440 = 2\pi/b$

$b = 880\pi$

So, a sound with frequency 440 Hz is modelled by $y = a\sin(880\pi t)$.

For **extension**, students could also investigate the sine wave equations for other musical notes.

Technology

Here are some examples of programmes that can be used to consider sound waves:

Audacity “Audacity” is a free programme that can be downloaded and is designed for sound analysis and editing. Using this programme, it is possible to record or generate a sound and view a graphical representation of its sound wave.

Note: the measurement setting at the bottom needs to be set to “Length” and the units to hh:mm:ss milliseconds. The Record button will record sounds and the Generate tab will produce sounds from the computer to analyse. Students will need to zoom in on the display. The mouse can be used to highlight repetitions of the period and the amount of time taken to complete these periods. This divided by the number of periods will give the actual period.

CBL (Calculator-Based Laboratory system) with connected microphone linked to a TI graphing calculator: This can collect sound data and store it in a graphing calculator. The calculator can then graph the sound as a function of time.

Wolfram Alpha: This can be used to play notes with different frequencies.

(There are many other similar free and paid programmes and packages available. You could check which programmes are available in your school's music department, if there is one).

Design an Investigation: Here are some possible ideas. This is by no means exhaustive:

Students could use the Generate function in Audacity to create tones with different frequencies, and graph and compare the corresponding sine curves. They can also generate tones with the same frequency, but different amplitudes.

Students could record a steady tone produced by their own voices. They should try to make sure they keep the pitch steady and record the sound for long enough (say 15 seconds). They could then use Audacity or the CBL software to determine whether the note is pitch perfect.

Students could play notes on an instrument and measure how long it takes for the sound wave to complete, for example, ten periods. From this information they could determine the period and the frequency of this sound and compare the curves produced for different notes—perhaps looking at the relationship between the different notes in a full scale.

Students could try to generate different notes by adjusting the level of water in a bottle and blowing across the top of it. By altering the amount of water in the bottle, can they produce a note of a particular frequency? Can they use this to produce a musical instrument?

Bring in different musical instruments and play the same pure note on each. Observe the differences between the sine functions when the instrument and volume of the note changes.

Students could use a programme like Wolfram Alpha to find what frequencies are and are not audible to the human ear. They could design a hearing test, for example, by comparing the different pitches people can hear. This study could, for instance, be related to age.

Students could consider more complicated sounds. They could sum together several sine waves and see what sounds are produced. This could lead to research into the meaning of consonance and dissonance. Students could try to make and record sounds that are not periodic.

The intention is not necessarily that students do the full exploration here (this may be too time-consuming). This of course would be good. Students may even want to use the ideas as their actual exploration. The idea is that students have engaged in the planning of the exploration and this should be useful for the actual IA.

Extension

Students could consider how they might collect readings such as tide times, sunrise or sunset times, the height of a spring or pendulum, temperature over the course of a day/year, etc.

Having found the data and plotted t, students could then try to find the model that best fits the data. This may be in the form $y = a \sin(bt)$, although some more complicated data may require looking for a function of the form $y = a \sin(bt + c) + d$, with different phase shifts and starting points to consider too.

13 Modelling change: more calculus

Essential understandings

Calculus describes rates of change between two variables. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximising or minimising a quantity, such as cost or profit
- Derivatives and integrals describe real-world kinematics problems in two and three dimensional space by examining displacement, velocity and acceleration.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
The cosine function describes the gradient of the tangent lines to the graph of the sine function for each value in the domain of the sine function.	Investigation 1
The derivative of a function at a certain point represents the instantaneous rate of change of the function at that point.	Investigation 2
Differentiation rules provide the basis for finding antiderivatives (or indefinite integrals) of many functions.	Investigation 3
Integration by substitution represents the reverse process of the chain rule for differentiation.	Investigation 4
Displacement represents the distance between the initial position and final position of an object moving along a line and total distance represents the total length of the path the object travels.	Investigation 5

Syllabus sections covered in this chapter:

- SL5.6
- SL5.8
- SL5.9
- SL5.10

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
N/A	Page 549: Example 4 Page 553: Example 6 Page 570: Example 15	Page 533: Example 5&6 Page 554: Example 7 Page 556: Example 9 Page 565: Example 12 Page 565: Example 13 Page 569: Example 14	Pages 551, 558, 566, 576

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 576	Page 577	N/A

13.1 Derivatives with sine and cosine

Investigation 1

Conceptual understanding:

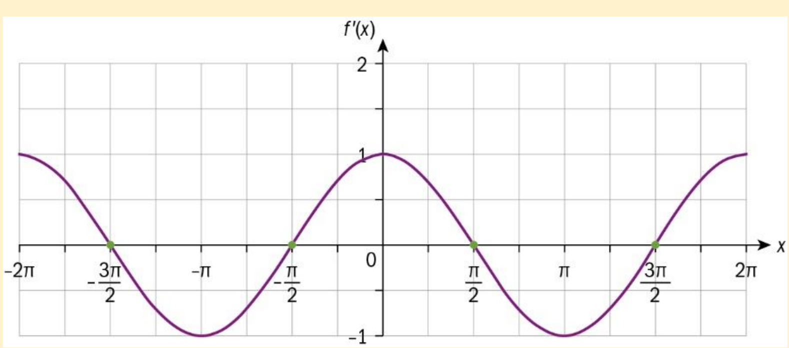
The cosine function describes the gradient of the tangent lines to the graph of the sine function for each value in the domain of the sine function.

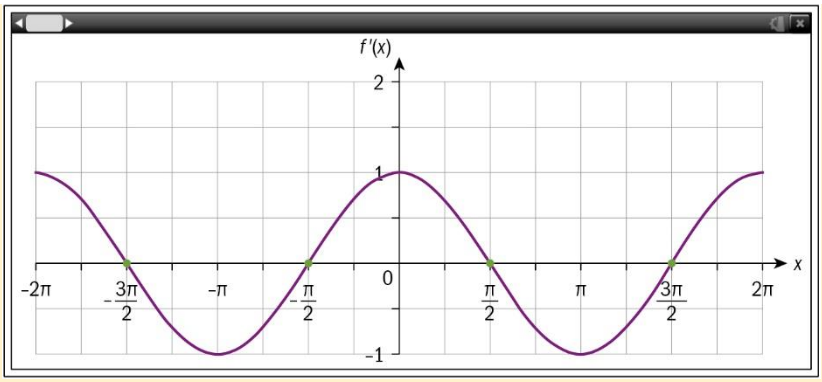
1 $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

2 $f'(x) > 0: \left(-2\pi, -\frac{3\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$

$f'(x) < 0: \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

3



4 

5 Students should see that $\frac{d}{dx}(\sin x) = \cos x$ for the following reasons:

Because the graphs of $y = \frac{d}{dx}(\sin x)$ and $y = \cos x$ are the same.

Because the table of values which the GDC gives for $y = \frac{d}{dx}(\sin x)$ are the same points as those in a table of values for $y = \cos x$.

6 **Factual:** What is the derivative of $f(x) = \sin x$?

Answer: The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

7 **Conceptual:** What does the cosine function tell you about the graph of the sine function for any given value in the domain of the sine function?

Answer (this is the conceptual understanding): The cosine function describes the gradient of the tangent lines to the graph of the sine function for each value in the domain of the sine function.

8 **Factual:** What is the derivative of $f(x) = \cos x$?

Answer: The derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$.

Developing inquiry skills

In the opening scenario, you were given that the number of gallons of water in storage tank A at time t hours is modelled by $V(t) = -500 sin(0.3t) + 150t + 1450$. You should now have the knowledge to find the function that gives the rate at which the amount of water in the tank A is changing at time t hours and to find that rate of change at t=15 hours.

Answer: V′(t) = -150sin(0.3t) + 150, V′(15) = 182 gallons/hour (3 s.f.)

13.2 Applications of derivatives

Investigation 2

Conceptual understanding:

The derivative of a function at a certain point represents the instantaneous rate of change of the function at that point.

1 **a** \$34 250

b \$34 349.90

c \$55 250

d \$55 289.90

2 **a** \$99.90

b \$39.90

3 **Conceptual:** Why is the cost of producing one additional item not constant?

Answer: The cost function is quadratic and so there is not a constant rate of change between cost, C, and the number of bicycles, x.

4 **a** \$100

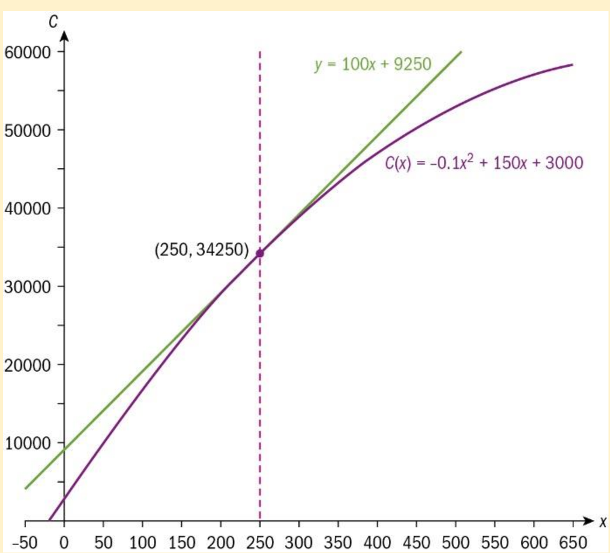
b \$40

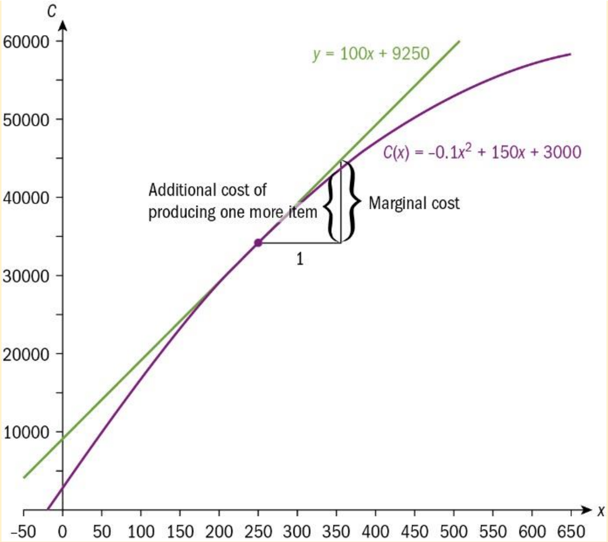
The derivative is the rate at which the cost function is changing as the number of bicycles changes.

5 The values are approximately equal.

6 **Conceptual:** Why can the marginal cost function – the derivative of the cost function – be used to approximate the cost incurred for producing one additional item?

Answer: The tangent line to a graph of a function tells you the rate at which cost-per-item is changing at the point of tangency. The diagrams below illustrate this fact, and you may want to show these to students.





7 **Conceptual:** What represents the instantaneous rate of change of a function at a given point?

Answer (this is the conceptual understanding): The derivative of a function at a certain point represents the instantaneous rate of change of the function at that point.

13.3 Integration with sine, cosine and substitution

Investigation 3

Conceptual understanding:

Differentiation rules provide the basis for finding antiderivatives (or indefinite integrals) of many functions.

1 $\frac{d}{dx}\left(e^{-2x}\right) = -2e^{-2x} \Rightarrow \int -2e^{-2x}dx = e^{-2x} + C$.

$\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$.

$\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$.

2 **Factual:** What is $\int \sin x\,dx$?

Answer: $\int \sin x\,dx = -\cos x + C$

3 **Factual:** What is $\int \cos x\,dx$?

Answer: $\int \cos x\,dx = \sin x + C$

4 **Conceptual:** What can you use as basis for determining the antiderivatives of many functions?

Answer (this is the conceptual understanding): Differentiation rules provide the basis for finding antiderivatives (or indefinite integrals) of many functions.

Investigation 4

Conceptual understanding:

Integration by substitution represents the reverse process of the chain rule for differentiation.

1 **a** $\frac{du}{dx} = 2x - 4$

b $y = \frac{1}{3}u^3$; $\frac{dy}{du} = u^2$

c $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = u^2(2x-4) = (x^2-4x)^2(2x-4)$. This is the chain rule for differentiation.

2 $\int (x^2-4x)^2(2x-4)\,dx = \frac{1}{3}(x^2-4x)^3 + C$

3 $\frac{du}{dx} = 2x - 4$

$du = (2x-4)dx$

$\int (x^2-4x)^2(2x-4)\,dx = \int u^2 du$

$\int u^2 du = \frac{1}{3}u^3 + C$

$\int (x^2-4x)^2(2x-4)\,dx = \frac{1}{3}(x^2-4x)^3 + C$

4 **Conceptual:** The integration by substitution method is the reverse process of which method of differentiation?

Answer (this is the conceptual understanding): Integration by substitution represents the reverse process of the chain rule for differentiation.

5 **Conceptual:** When may the substitution method of integration apply?

Answer: Integration by substitution may apply to the product of a composite function with the derivative of the 'inner' function of the composite factor.

13.4 Kinematics and accumulating change

Investigation 5

Conceptual understanding:

Displacement represents the distance between the initial position and final position of an object moving along a line and total distance represents the total length of the path the object travels.

1 $v'(t) = 6t^2 - 42t + 60$

2 **a** $p = s(7) = 80$

b $6t^2 - 42t + 60 = 0 \Rightarrow m = 2$ and $n = 5$

c $q = s(5) = 28$

3 **a** $s(7)-s(0)=80-3=77\,\text{m}$; At 7 seconds, the object is 77 m to the right of its position at 0 seconds.

b $\int_0^7 v(t)\,dt=77$; The antiderivative of velocity function is the displacement (or position) function. Therefore, $\int_0^7 v(t)\,dt=s(t)\big|_0^7=s(7)-s(0)$, which is the displacement of the object after 7 seconds.

4 **Factual:** What does $\int_a^b v(t)\,dt$ represent?

Answer: $\int_a^b v(t)\,dt$ represents displacement from time a to time b.

5 Total distance travelled is $52+27+52=131$ m.

6 **Conceptual:** Is displacement the same of the total distance travelled? Why or why not?

Answer (this is the conceptual understanding): No. Displacement represents the distance between the initial position and final position of an object moving along a line and total distance represents the total length of the path the object travels.

7 $\int_0^7 |v(t)|\,dt=131$, which is the same as the distance found in question 5.

8 **Factual:** What does $\int_a^b |v(t)|\,dt$ represent?

Answer: $\int_a^b |v(t)|\,dt$ represents the distance travelled from time a to time b.

Reflect: When will the total distance travelled be equal to the displacement for a given time interval?

Answer: The total distance traveled will be equal to the displacement on any interval where $v(t)\geq 0$.

Reflect: What does the definite integral of the rate of change of a quantity represent?

Answer (this is the conceptual understanding): The definite integral of the rate of change of a quantity over a given interval represents the net change of that quantity over the interval.

Developing inquiry skills

You have seen how the definite integral of a rate of change over a given time is equal to the net change in a quantity over that time period. Return to the opening scenario for this chapter and use this knowledge to find the amount of water in storage tank B at t=12 hours.

Answer: Amount of water in tank B at time 12 hours $=1800+\int_0^{12}\left(E(t)-R(t)\right)dt\approx 1540$ gallons

How could you find the minimum and maximum amounts of water in tank B during the time $0\leq t\leq 24$ hours?

Answer: Find the values for which E′(t) – R′(t) = 0.

Be the Particle

Approaches to Learning/Learner Profile: Collaboration, Communication

Exploration Criteria: Personal Engagement (C), Use of Mathematics (E)

IB Topic: Calculus, Kinematics—Motion in a Straight Line

Introduction

In this task, students get the opportunity to try to “act out” the motion of a particle moving in a straight line given a velocity function from which they have found a displacement function and acceleration function. If the equipment (motion detector or filming equipment) is available, they can then test how accurate they are. Otherwise, it is possible just to generate discussions based around the questions posed.

The use of a motion detector or filming and then modelling on Logger Pro would constitute good Personal Engagement (Criterion C) in an exploration.

Motion of a Particle in a Straight Line

Put the students into groups of three or four.

If possible, ensure that each group includes a mix of physics students and non-physics students, as it is often the case that physics students have encountered this topic before.

When $t = 0$, $v(0) = 10$ units/sec.

The particle is stationary when the velocity is zero.

$(t - 2)(t - 5) = 0$, so the particle is stationary at $t = 2$ and $t = 5$.

At $t = 6$ the particle is moving at 4 units/sec.

$s(t) < 0$ means that the particle is 5 units to the left of the origin.

$s(1)$ is the position of the particle after 1 second.

The displacement function $s(t)$ is found by integrating the velocity function and using the initial condition given to find the “$+ c$”:

$$s(t) = \frac{t^3}{3} - \frac{7t^2}{2} + 10t - 5, 0 \le t \le 6$$

At $t = 6$ the particle is 1 unit to the right of the origin. ($s(6) = 1$)

The particle's displacement at each second of the journey is:

t (secs)	0	1	2	3	4	5	6
$s(t)$ (units)	-5	$1\frac{5}{6}$	$3\frac{2}{3}$	$2\frac{1}{2}$	$\frac{1}{3}$	$-\frac{5}{6}$	1

Acceleration is the derivative of the velocity function. $a(t) = 2t - 7$.

Initial acceleration is -7 units/s^2.

The particle changes direction at $t = 2$ and $t = 5$.

Total distance travelled is the area under the velocity/time graph = 15 units.

Make sure that students produce the graphs correctly.

Remind students to include axis labels and a title.

Students can use their GDC to draw the graphs.

For **extension**, you could discuss, for example, the shape of each graph, the axis intercepts, the gradient at different parts on each graph, any maximum and minimum points, when each graph is increasing/decreasing, etc.

Be the Particle for 6 Seconds

Give students the opportunity to discuss how they will walk the path of the particle.

You could ask, for instance:

What do you need?

Who will do which job?

Before starting this activity, prepare by marking a scale from −5 to 5, where 0 is the origin, on the classroom board or wall or corridor, etc.

If students have access to this equipment, to check the accuracy of the attempt students could compare the motion with:

a motion detector, or

a graphing programme such as Logger Pro.

You might want to discuss this equipment with the students:

A motion detector measures the time it takes for a high frequency sound pulse to travel from the detector to an object and back. This will determine the position of the object at different times.

The data can be fed into Logger Pro, which can then use the change in position to calculate the object's velocity and acceleration (which can be displayed either as a table or a graph).

If students do not have access to a motion detector, you could film some of the students' attempts from the side and analyse which one they think is the best, using Logger Pro and inserting the movie they want to analyse. They can then set a scale and track the moving object (in this case picking a point on the "particle's" body that is easily visible in every shot). This will produce a graph of time (t) against displacement (s), from which the velocity and acceleration graphs can be produced.

The comparison of the two cubic graphs will depend on the accuracy of the attempt.

To check how similar their cubic graph is to the cubic produced above, students can overlay a plot of the required displacement graph to look for similarities.

To improve the model found, they can try again.

Extension

Make sure that students produce the graphs correctly.

Remind students to include axis labels and a title.

Students can use their GDC to draw the graphs.

14 Valid comparisons and informed decisions: probability distribution

Essential understandings

Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Probability provides important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed.

Content-specific Conceptual understandings

This chapter leads to the following content-specific Conceptual understandings listed in the subject guide:

- Approximation in data can approach the truth but may not always achieve it.

We have taken these suggested content-specific Conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own Conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these Conceptual understandings:

Conceptual understandings	Investigation
Discrete and Continuous variables vary in applicable situations that depend on countable possible values or values that can take any value in a range.	Investigation 1
The probability distribution of a random variable represents all the values the variable can take, and sums to 1.	Investigation 2
The mean of a frequency distribution (for multiple repeated events), or the mean of a probability distribution (for a single event) both lead to finding the expected value of a random variable.	Investigation 3
For large numbers of trials the binomial theorem may be more useful and efficient than using a tree diagram.	Investigation 5
The parameters of a binomial distribution function determine the values and distribution of the resulting probabilities.	Investigation 6
The expected value of the binomial random variable can be evaluated by multiplying the number of trials by the probability of success; an application of the formula for the expected number of occurrences.	Investigation 7
The distribution of outcomes of many real- life events involving continuous data can be approximated by the normal curve.	Investigation 8
For a perfect normal distribution the curve is bell-shaped, the data is symmetrical about the mean (μ), the mean, mode and median are the same.	Investigation 9
The parameters of a normal distribution determine the location and spread of a normal curve.	Investigation 10
The area underneath normal cumulative distribution function facilitates predictions of probabilities for a given interval of the random variable.	Investigation 11
More than half the data of a normal distribution falls within one standard deviation of the mean, while the majority of data falls with three standard deviations of the mean.	Investigation 12

Syllabus sections covered in this chapter:

- SL4.3*
- SL4.7*
- SL4.8*
- SL4.9*
- SL4.12

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 583: Measures of central tendency, Solving linear equations	Page 596: Example 4 Page 597: Example 6 Page 615: Example 14 Page 617: Example 16	Page 590: Example 3 Page 596: Example 4 Page 598: Example 7 Page 610: Example 11 Page 613: Example 13 Page 617: Example 14	Pages 590, 602, 619

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 620	Page 622	Page 622

14.1 Random Variables

Investigation 1

Conceptual understanding:
Discrete and Continuous variables vary in applicable situations that depend on countable possible values or values that can take any value in a range.

1 **i** 0, 1, 2, 3

ii e.g. 0, 1, 2, 3 etc

iii e.g. 176cm, 167.54cm, 170.1654cm, 5ft9in

iv Eg. 9.89s, 12.6s, 71.2s, 10.14396s

2 **i** e.g. 4, 2.5, -3 etc

ii e.g. 0.5, -2 etc

iii e.g. negative heights; some unreasonably large or small heights

iv e.g. negative times; some unreasonably quick times

3 **a** The outcomes are specific (discrete).

b The outcomes can take any value within a range (continuous) – only limited by the accuracy of the measuring equipment.

c Theoretically there is no upper bound to (ii), whereas (i) can only take these values.

d (i) takes specific values and (iii) takes values within a range.

4 **a** Discrete **b** Continuous **c** Discrete **d** Continuous

5 **a** We would need to ask. People may lie about their age.

b The accuracy of the measuring instrument. Banana may be curved.

c What if Cat is not entirely white?

d The accuracy of the measuring instrument. Donut may not be perfectly circular.

6 **Factual:** What are specific examples of continuous and discrete random variables?

Answer: Students should list some examples of each.

7 **Conceptual:** What are the distinguishing features of continuous and discrete random variables?

Answer (this is the conceptual understanding): Discrete and continuous variables vary in applicable situations that depend on countable possible values or values that can take any value in a range.

TOK
Is it possible to reduce all human behaviour to a set of statistical data?

Answer: A class discussion might go like this:

Social scientists use statistics to study human behaviour, for example, population, income, birth rates. Can you think of some more examples of this?

The United Nations and World Health Organisation collect data and use it to help plan aid and development programs.

Can you think of some aspects of human behaviour from other areas of knowledge that cannot be measured?

Investigation 2

Conceptual understanding:
The probability distribution of a random variable represents all the values the variable can take, and sums to 1.

1 0, 1, 2, 3, 4, 5

2 *M* can take values 1, 2, 4

3 **Factual:** Which of the following methods, which you learned in Chapter 8 (Probability), would be best to use in order to calculate these probabilities?

Answer: Sample space diagram

4 Students should produce a table similar to the following.

		Score on dice 1											
		1		2		3		4		5		6	
Score on dice 2	**1**	0	4	1	2	2	2	3	1	4	1	5	1
	2	1	2	0	4	1	2	2	2	3	1	4	1
	3	2	2	1	2	0	4	1	2	2	2	3	1
	4	3	1	2	2	1	2	0	4	1	2	2	2
	5	4	1	3	1	2	2	1	2	0	4	1	2
	6	5	1	4	1	3	1	2	2	1	2	0	4
		diff.	M	diff.	M	diff.	M	diff.	M	diff.	M	diff.	M

$P(M=1)=\frac{12}{36}$, $P(M=2)=\frac{18}{36}$, $P(M=3)=0$, $P(M=4)=\frac{6}{36}$

5 Completed table:

m	1	2	4
P(*M* = *m*)	$\frac{12}{36}$	$\frac{18}{36}$	$\frac{6}{36}$

6 Probabilities add up to 1.

7 **Conceptual:** What does the probability distribution of a discrete random variable add up to? Why is this the case?

Answer (this is the conceptual understanding): The probability distribution of a random variable represents all the values the variable can take, and sums to 1.

TOK

What does it mean to say that mathematics can be regarded as a formal game lacking in essential meaning?

Answer: This can lead to a debate about what is a game and what is mathematics.

What is mathematics?

"I think many physicists, including myself, agree that there should be some complete description of the universe and the laws of nature. Implicit in that assumption is the universe is intrinsically mathematical." – Simeon Hellerman

What is Mathematics? “Very simple: Mathematics is the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter. Since the 17th century, mathematics has been an indispensable adjunct to the physical sciences and technology, and in more recent times it has assumed a similar role in the quantitative aspects of the life sciences”. - Jorge Morales Pedraza.

Investigation 3

Conceptual understanding:
The mean of a frequency distribution (for multiple repeated events), or the mean of a probability distribution (for a single event) both lead to finding the expected value of a random variable.

1

m	1	2	4
$P(M = m)$	$\frac{12}{36}$	$\frac{18}{36}$	$\frac{6}{36}$

2

m	1	2	4
Expected frequency	12	18	6

3 $(1\times12)+(2\times18)+(4\times6)=72$

mean = $\frac{72}{36}=2$

4 **Factual:** What does the mean of this frequency distribution tell you?

Answer: If we conducted the experiment 36 times we would expect, on average, to move 2 squares each go (sometimes we would move more, and sometimes less).

5

m	1	2	4
Expected frequency	$\frac{100}{3}$	50	$\frac{50}{3}$

6

$$\left(1\times\frac{100}{3}\right)+(2\times50)+\left(4\times\frac{50}{3}\right)=2$$

7 **Factual:** What do you notice from your answers to questions 3 and 6?

Answer: The mean value of M is the same, no matter how many times you repeat the experiment.

8 **Factual:** What would the mean value of M be if the experiment were repeated 10 times? Or 1000 times? Or just once?

Answer: In every case, the mean would be 2.

9 **Conceptual:** What does the mean of a frequency distribution (for multiple events) or the mean of a probability distribution (for just one event) tell you about the random variable?

Answer (this is the conceptual understanding): The mean of a frequency distribution (for multiple repeated events), or the mean of a probability distribution (for a single event) both lead to finding the expected value of a random variable.

TOK

Do you think that people from very different backgrounds are able to follow mathematical arguments, as they possess deductive ability?

Answer: You might want to use this TOK session to begin a discussion about whether all people possess the same reasoning ability at the same level. Some might argue that our ability to reason distinguishes us from the rest of the animal kingdom. You might want to ask where students have used reason today. As a counterclaim you could point to people who have a diminished capacity for reasoning, such as the mentally disabled, and ask if they are not human. An attention-grabbing debate is sure to follow!

Developing inquiry skills

In the opening problem, you were faced with a number of games. Consider the first one.

If you played this game a few times, do you think you would end up with more money than you started with?

Answer: Students should make a guess based on intuition for the problem

How could we tell?

Answer: Calculate the expected value

Define a random variable for the winnings from playing the game once.

Answer: Let W = amount of winnings (note that this is not the score on the dice)

Draw a probability distribution for your random variable.

Answer:

w	20	0	-10
P(*W* = *w*)	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

Calculated the expected winnings from the game.

Answer:

$$E(W) = \left(20 \times \frac{1}{6}\right) + \left(0 \times \frac{2}{6}\right) + \left(-10 \times \frac{3}{6}\right) = -\frac{5}{3}$$

Expected winnings are therefore -\$1.67 (there is an expected loss of\$1.67 each time the game is played)

How much would you expect to win or lose if you played the game 10 times? 100 times?

Answer: For 10 times expected loss is \$16.67 and for 100 times it is \$166.67

Is the game 'fair'?

Answer: No, because you expect to lose money.

How could you define a fair game?

A game where the expected winnings are 0. (E(W)=0)

What adjustment could you make to the prizes that would ensure that this is a fair game?

Answer: We want E(W) = 0

So, for example, you could increase the winnings when rolling a 6 to £x, where x is such that E(W) = 0.

$$E(W) = \left(x \times \frac{1}{6}\right) + \left(0 \times \frac{2}{6}\right) + \left(-10 \times \frac{3}{6}\right) = 0$$

$$\frac{x}{6} - 5 = 0$$

$$\frac{x}{6} = 5$$

$$x = 30$$

The first prize could be changed to \$40.

Other solutions and changes are possible too.

14.2 The binomial distribution

Investigation 4

1 Students should complete two tables; one for each person in the pair.

2 $P(C)=\frac{1}{5}$, $P(W)=\frac{4}{5}$

3 X = 0,1,2,3,4,5,6

4 $P(X=0)=\left(\frac{4}{5}\right)^6=\frac{4096}{15625}=0.262$

5 CWWWWW

WCWWWW

WWCWWW

WWWCWW

WWWWCW

WWWWWC

6 For each of the six possibilities listed above

$$\text{probability}=\left[\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\right]=\left[\frac{1}{5}\times\left(\frac{4}{5}\right)^5\right]=\frac{1024}{15625}$$

7 $P(X=1)=6\times\frac{1}{5}\times\left(\frac{4}{5}\right)^5=\frac{6144}{15625}$

8 There are 15 ways to guess 2 cards correctly. Students could try to list them.

For a single chain of 6 guesses where a students gets 2 correct,

$$\text{probability}=\left[\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\right]=\left[\left(\frac{1}{5}\right)^2\times\left(\frac{4}{5}\right)^4\right]=\frac{256}{15625}$$

Hence, $P(X=2)=15\times\left(\frac{1}{5}\right)^2\times\left(\frac{4}{5}\right)^4=\frac{768}{3125}$

9 $P(X=2)=15\times(P(C))^2\times(P(W))^4$

10 Completed table:

$X=x$	$P(X=x)$	
$X=0$	$1\times(P(C))^0\times(P(W))^6$	$\frac{4096}{15625}$
$X=1$	$6\times(P(C))^1\times(P(W))^5$	$\frac{6144}{15625}$
$X=2$	$15\times(P(C))^2\times(P(W))^4$	$\frac{768}{3125}$

11 The powers add up to 6 in each case, which is the number of times the game is repeated.

12 The Binomial Theorem

TOK

A model might not be a perfect fit for a real-life situation, and the results of any calculations will not necessarily give a completely accurate depiction. Does this make it any less useful?

Answer: We often use a theoretical distribution, such as the binomial theorem, to describe a random variable that occurs in a real-life situation. This process is called modelling and enables us to make calculations and possibly predict.

However, it is unusual for a model to fit a real-life situation perfectly and we have to be ready for some error, which is a normal situation in life. However, we can often give a percentage chance of an event like forecasting rain or the success of a medical procedure.

Reflect: How can you tell if a real-life situation can be modelled using the binomial distribution?

Answer: Specific conditions exist such that a real-life situation can be modelled using the binomial distribution.

Investigation 5

Conceptual understanding:

For large numbers of trials the binomial theorem may be more useful and efficient than using a tree diagram.

1 There would be 64 branches!

2 **Factual:** What would be the issue with drawing a tree diagram to represent this situation?

Answer: The tree diagram would be very large – difficult to draw and very time consuming.

3 **a** Yes. 6 trials

b 2 possible outcomes – getting a head = success, getting a tail = failure

c Yes, $p = \frac{2}{3}$

d Yes, trials are independent. One toss of the coin does not affect the next toss.

4 $P(HHTTTT) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{4}{729}$

5 Combinations of 2 heads and 4 tails listed below.

HHTTTT	THHTTT	TTHHTT	TTTHHT	TTTTHH
HTHTTT	THTHTT	TTHTHT	TTTHTH	
HTTHTT	THTTHT	TTHTTH		
HTTTHT	THTTTH			
HTTTTH				

6 Each has some combination of $\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$

7 Recall to calculate $\binom{6}{2}$ we can use the

i Calculator, or

ii the formula $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5}{2} = 15$, or

iii the 3rd entry on the 6th row of Pascal's Triangle

1 6 15 15 6 1

So there are 15 combinations of 2 heads in 6 throws.

8 P(2 heads in 6 tosses) = $\binom{6}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 = 15 \times \frac{4}{729} = \frac{20}{243} = 0.0823$ (3*sf*)

9 **Conceptual:** For a discrete random variable which is binomially distributed, how does the Binomial theorem help to calculate probabilities?

Answer (this is the conceptual understanding): For large numbers of trials the binomial theorem may be more useful and efficient than using a tree diagram.

10 **Conceptual:** Why is using a tree diagram not always efficient?

Answer: When there are large numbers of trials the tree diagram will become too large and impractical to draw as the number of branches is so great.

International-mindedness

The Galton board, also known as a quincunx or bean machine, is a device for statistical experiments named after English scientist Sir Francis Galton. It consists of an upright board with evenly spaced nails or pegs driven into its upper half, where the nails are arranged in staggered order, and a lower half divided into a number of evenly-spaced rectangular slots. In the middle of the upper edge, there is a funnel into which balls can be poured. Each time a ball hits one of the nails, it can bounce right or left with the same probability. This process gives rise to a binomial distribution of in the heights of heaps of balls in the lower slots and the shape of a normal or bell curve.

Good simulations and explanations may be found online such as https://youtu.be/6YDHBFVIvIs

Investigation 6

Conceptual understanding:

The parameters of a binomial distribution function determine the values and distribution of the resulting probabilities.

1

$r = 2$, $p = 0.5$, $n =$	$P(X = r)$
2	0.25
3	0.375
4	0.375
5	0.3125
6	0.23438
7	0.16406
8	0.10938

2	r = 2, n = 5, p =	P(X = r(= 2))
	0.1	0.0729
	0.2	0.2048
	0.3	0.3087
	0.4	0.3456
	0.5	0.3125
	0.6	0.2034
	0.7	0.1323
	0.8	0.0512
	0.9	0.0081

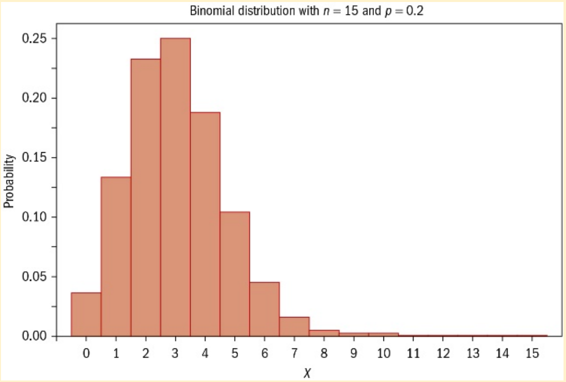
3 **Conceptual:** How do the parameters of the binomial distribution affect the resulting probabilities?

Answer (this is the conceptual understanding): The parameters of a binomial distribution function determine the values and distribution of the resulting probabilities.

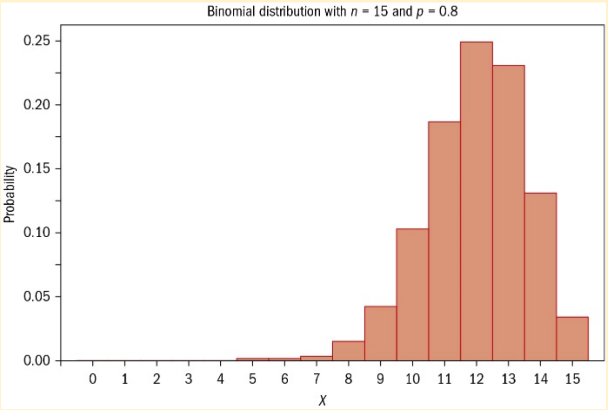
4 **Conceptual:** How do the parameters affect the graph of the binomial distribution?

Answer: Further observations, which you might want to convey to students:

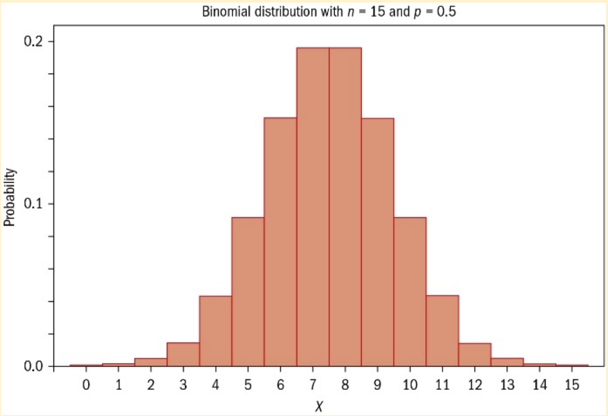
For small p and small n, the binomial distribution is what we call skewed right. That is, the bulk of the probability falls in the smaller numbers 0, 1, 2,..., and the distribution tails off to the right. For example, here is a graph of the binomial distribution when n = 15 and p = 0.2.



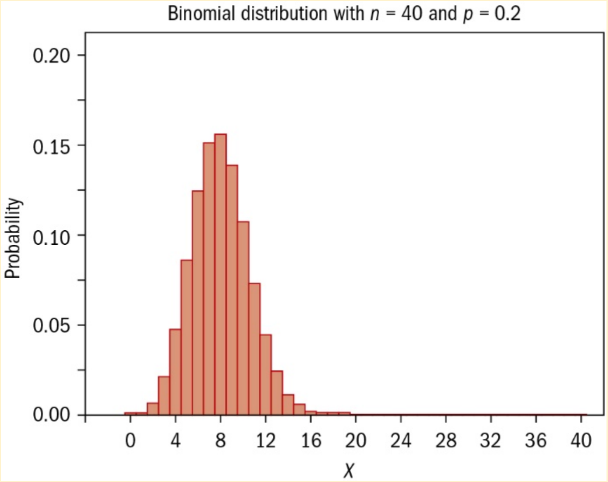
For large p and small n, the binomial distribution is what we call skewed left. That is, the bulk of the probability falls in the larger numbers n, n – 1, n – 2 ,... and the distribution tails off to the left. For example, here is a graph of the binomial distribution when n = 15 and p = 0.8.



For p = 0.5 and large and small n, the binomial distribution is what we call symmetric. That is, the distribution is without skewness. For example, here is a graph of the binomial distribution when n = 15 and p = 0.5.



For small p and large n, the binomial distribution approaches symmetry. For example, if p = 0.2 and n is small, we'd expect the binomial distribution to be skewed to the right. For large n, however, the distribution is nearly symmetric. For example, here is a graph of the binomial distribution when n = 40 and p = 0.2.



TOK

How can we trust the data collected from humans?

Answer: What happens when we ask people questions?

Repeatability – If you get the same results, does repetition prove a theory?

Controllability – humans can be unpredictable

People lie sometimes - why?

Subjects change behaviour when being observed. Research the Hawthorne effect.

What about the questions being asked? Is there a hidden agenda - who is asking?

Prejudice and leading questions - encouraging a particular answer.

You might want to use www.youtube.com/watch?v=G0ZZJXw4MTA as a stimulus.

Investigation 7

Conceptual understanding:
The expected value of the binomial random variable can be evaluated by multiplying the number of trials by the probability of success; an application of the formula for the expected number of occurrences.

1 Students may answer intuitively that the expected number of heads in 3 tosses is $\frac{2}{3} \times 3 = 2$. In this investigation, they will use the formula $E(X) = \sum_x xP(X = x)$ to find an expression for the expectation of any binomially distributed random variable.

2

$X = x$	$P(X = x)$	$x \cdot P(X = x)$
0	$\binom{3}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	*0*
1	$\binom{3}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = \frac{2}{9}$	$\frac{2}{9}$
2	$\binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{4}{9}$	$\frac{8}{9}$
3	$\binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{8}{27}$	$\frac{8}{9}$

3 $E(X) = \sum_x xP(X = x) = \frac{2}{9} + \frac{8}{9} + \frac{8}{9} = 2 = np$

4 **Conceptual:** What is the relationship between the number of trials, the probability of success and the expected value for a binomial distribution?

Answer: $E(X) = 2 = 3 \times \frac{2}{3} = np$

Developing inquiry skills

Consider now the second option from the original problem.

If you played this game a few times, do you think you would end up with more money than you started with?

Answer: Students use intuition to decide. Emphasise the large prize.

How could we tell?

Define a random variable for the winnings from playing the game once.

Answer: Let W be amount of winnings

Does this experiment fit a binomial distribution?

Answer: Yes – fixed number of trials (4, since you toss 4 coins); two possible outcomes (Heads = success, Tails = failure); constant probability of success (p=0.5); trials are independent.

What are the parameters?

Answer: n=4, p=0.5

What is the probability that you will win \$100? Your money back? \$10? \$0?

Answer: Let X represent the number of heads in 4 tosses of the coin. Then

$$P(X=0)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$$P(X=1)=4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)=\frac{4}{16}$$

$\therefore$ probability of \$0 is $\frac{1}{16}+\frac{4}{16}=\frac{5}{16}$

$$P(X=2)=6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{6}{16}$$

$\therefore$ probability of \$10 is $\frac{6}{16}$

$$P(X=3)=4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3=\frac{4}{16}$$

$\therefore$ probability of money back is $\frac{4}{16}$

$$P(X=4)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

$\therefore$ probability of \$100 is $\frac{1}{16}$

Draw a probability distribution table for your random variable representing the winnings.

Answer:

W	80	0	-10	-20
P(W=w)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{5}{16}$

Calculate the expected winnings from the game.

Answer: $E(W)=\left(80\times\frac{1}{16}\right)+\left(0\times\frac{4}{16}\right)+\left(-10\times\frac{6}{16}\right)+\left(-20\times\frac{5}{16}\right)=\frac{80}{16}+0-\frac{60}{16}-\frac{100}{16}=-\frac{80}{16}=-5$

This is a loss of \$5

How much would you expect to win or lose if you played the game 10 times? 100 times?

Answer: For 10 times, expected loss is \$50; for 100 times, expected loss is \$500.

Is the game 'fair'?

Answer: No, because $E(W)\neq 0$

What adjustment could you make to the prizes that would ensure that this is a fair game?

Answer: We want E(W) = 0

There are a number of ways to do this. For example, increase the prize for 4 heads to \$x. Then

$$E(W) = \left(x \times \frac{1}{16}\right) + \left(0 \times \frac{4}{16}\right) + \left(-10 \times \frac{6}{16}\right) + \left(-20 \times \frac{5}{16}\right) = 0$$

$$\frac{x}{16} - \frac{60}{16} - \frac{100}{16} = 0$$

$$\frac{x}{16} - 10 = 0$$

$$\frac{x}{16} = 10$$

$$x = 160$$

The first prize could be changed to \$180.

Other solutions and changes are possible

14.3 The normal distribution

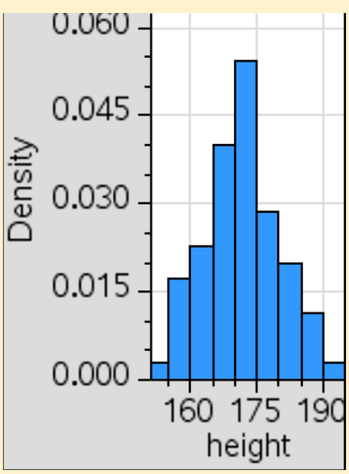
Investigation 8

Conceptual understanding:
The distribution of outcomes of many real- life events involving continuous data can be approximated by the normal curve.

1 **Factual:** Using words, describe how you would expect the heights of the 70 students to be distributed.

Answer: Students should identify that the heights of most students lie around the mean, with a minority of very short and very tall students. They should begin to get to grips with the features of a normal distribution, even though the term has not yet been introduced.

2 Histogram could be hand-drawn, or constructed using a computer or calculator package. Any group size is fine; the key idea is that students visually see the shape of the distribution.



3 **Factual:** Where is the peak of the histogram? What does the peak of the histogram represent?

Answer: The peak of the histogram is 170 to 175cm (If students have grouped the data differently, their answers may vary slightly. However, regardless of the grouping, the peak will be in the middle). This peak represents the heights which occur with the greatest frequency.

4 Yes, histogram is roughly symmetrical about the peak of the data.

5 Students should draw a smooth, bell-shaped curve.

6 Bell-shaped and roughly symmetrical.

7 Examples might include:

size of things produced by machines

errors in measurements

blood pressure

marks on a test

8 Note: This is a good opportunity to discuss Internal Assessments as students will collect data and will need to consider reliability and collection methods.

9 **Conceptual:** How can you approximate the distribution of many real-life events that involve continuous data?

Answer (this is the conceptual understanding): The distribution of outcomes of many real-life events involving continuous data can be approximated by the normal curve.

Investigation 9

Conceptual understanding:
For a perfect normal distribution the curve is bell-shaped, the data is symmetrical about the mean (μ), the mean, mode and median are the same.

1 For the given data listing students' heights:

Mean	171.1
Mode	170
Median	171

2 **Factual:** What do you notice about the values of the mean, median, and mode?

Answer: They are all roughly the same.

3 **Conceptual:** Can you explain why your observation from question 2 is the case? Will this always be the case for a normally distributed data set?

Answer: Because the data is symmetrical, the mean, median and mode should always be in the middle

4 **Conceptual:** What are the characteristics of a perfect normal distribution?

Answer (this is the conceptual understanding): For a perfect normal distribution the curve is bell-shaped, the data is symmetrical about the mean (μ), the mean, mode and median are the same.

Investigation 10

Conceptual understanding:

The parameters of a normal distribution determine the location and spread of a normal curve.

Go to https://www.desmos.com/calculator/jxzs8fz9qr

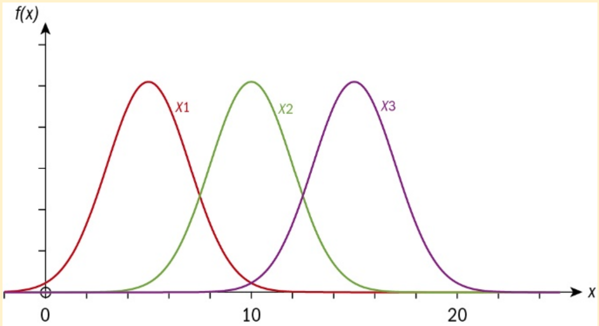
The first curve you will see when you load the page is a normal distribution curve with a mean (μ = b) of 0 and a standard deviation (σ = a) of 1.

This is called the standard normal distribution curve. The Standard Normal Distribution is the Normal Distribution where $\mu = 0$ and $\sigma = 1$ and we write $Z \sim N(0,1)$

1 Along the line $x = 0$

2 It moves to $x = b$

3

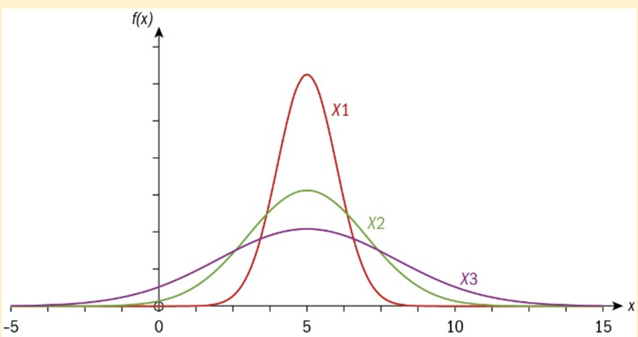


the standard deviation describes the spread of the distribution. The higher the standard deviation the wider the normal curve will be.

Change the value of a (the standard deviation) and keep the value of the mean the same.

4 The mean stays the same but the graph spreads out (whilst retaining the same area under the graph)

5



6 **Conceptual:** How do the parameters of a normal distribution affect the shape and position of the curve?

Answer (this is the conceptual understanding): The parameters of a normal distribution determine the location and spread of a normal curve.

Investigation 11

Conceptual understanding:
The area underneath normal cumulative distribution function facilitates predictions of probabilities for a given interval of the random variable.

1 **a** $\frac{51}{70}$

b This will depend on the bar widths chosen for the histogram, but will likely be very similar to the above probability.

2 You would need to know the equation, $y = f(x)$, of the normal curve which fits your histogram.

The definite integral $\int_{-\infty}^{177} f(x)\,dx$ gives the area under the curve to the left of 177, which is the probability you require.

3 Counting the students less than 177 cm uses raw data, whereas the other methods use processed / grouped data.

4 Counting the students less than 177 cm, since the raw data is the most accurate.

5 The raw data may not always be available, or there may be too much of it to deal with easily.

6 1 (from this data)

7 **Conceptual:** What does your answer to question 6 tell you about the probability of a random variable taking a value in a given range?

Answer (this is the conceptual understanding): We can use the area under a normal distribution curve of a particular range to approximate a probability within that range, because probabilities always lie between 0 and 1.

Investigation 12

Conceptual understanding:

More than half the data of a normal distribution falls within one standard deviation of the mean, while the majority of data falls with three standard deviations of the mean.

1 Given that $Z \sim N(0,1)$ find

a $P(-1 < Z < 1)$, = 0.6827

b $P(-2 < Z < 2)$, = 0.9545

c $P(-3 < Z < 3)$ = 0.9973

2 The probability that Z falls within

a 1 standard deviation from the mean is 0.6827

b 2 standard deviations from the mean is 0.9545

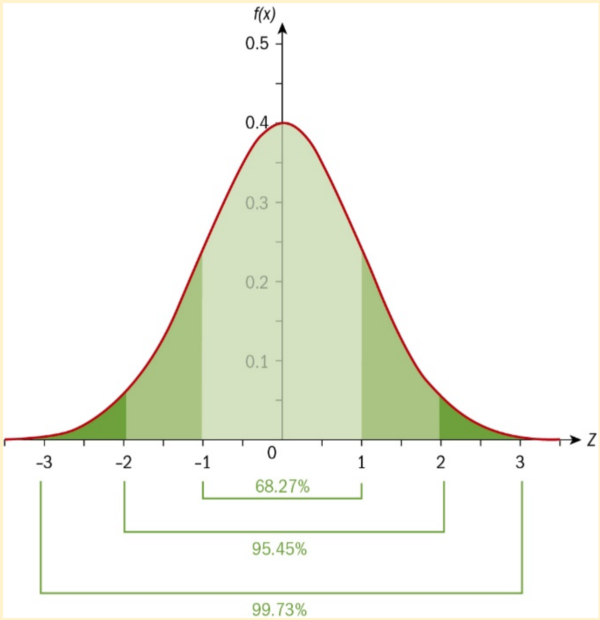
c 3 standard deviations from the mean is 0.6973.

3 **a** 68.27%

b 95.45%

c 99.73%

4



5 **Conceptual:** For a standard normal distribution, how do standard deviations from the mean represent the data?

Answer (this is the conceptual understanding): More than half the data of a normal distribution falls within one standard deviation of the mean, while the majority of data falls with three standard deviations of the mean.

6 **Conceptual:** How could we calculate how much data falls within 1.5 standard deviations of the mean?

Answer: Find $P(-1.5 < Z < 1.5)$

Investigation 13

1 Students might give intuitive answers about whether Millie's wrist size is small or large, making judgements by asking questions such as

What is my own wrist circumference?

How old is Millie?

What is the average wrist circumference for a girl of Millie's age?

etc.

2 2.25 cm below the mean.

3 Since Millie's wrist circumference is less than the mean, you might conclude that she has a small wrist. However, you need to know the standard deviation of wrist sizes in order to quantify how small.

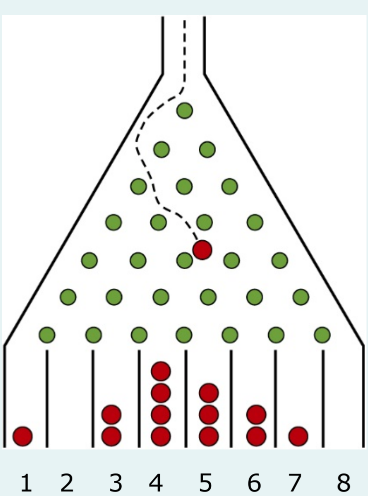
4 Millie is between 1 and 2 standard deviations below the mean, so by the results from the previous investigation, her wrist circumference is in the bottom 32% of girls her age.

5 $\frac{x-\mu}{\sigma}$ will tell you exactly how many standard deviations Millie is from the mean. In this case, the distance between Millie's wrist circumference and the mean is one and a half that of the 'typical' distance of all the scores from the mean.

Developing inquiry skills

Consider now the third option from the original problem.

PAY $30 and drop a marble into this machine



If it goes into box 1 or 8 you win $500!

If it goes into 2 or 7 you win \$80

If it goes into 3 or 6 you get your money back

If it goes into 4 or 5 you get nothing!

Is this a game that you think that if you played a few times you would end up with more than you started with?

Maybe! Discuss intuitive response

What path does the ball need to take in order to go in to box number 1?

Answer: LLLLLLL

What is the probability of this happening?

Answer: $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$

What possible paths can the ball take in order to go into box number 2?

Answer: LLLLLLR, LLLLLRL, LLLLRLL, LLLRLLL, LLRLLLL, LRLLLLL and RLLLLLL

The 'game' is called a Quincunx or Galton Board or Bean Machine. Sir Edward Galton, a British scientist, originally devised it for probability experiments.

What is the relationship between the Quincunx and the binomial distribution?

Answer: In this case there are 7 trials, constant probability of success $\left(\frac{1}{2}\right)$, 2 outcomes, independent.

What is the probability that you will win \$500? \$50? \$10? \$0?

Answer: $P(W = 470) = 2 \times \left(\frac{1}{2}\right)^7 = \frac{2}{128}$

$P(W = 20) = 2 \times 7 \times \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right) = \frac{14}{128}$

$P(W = -20) = 2 \times \binom{7}{2} \times \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^2 = \frac{42}{128}$

$P(W = -30) = 2 \times \binom{7}{3} \times \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3 = \frac{70}{128}$

Draw a probability distribution for your random variable.

Answer:

W	470	50	0	-30
P(W)	$\frac{2}{128}$	$\frac{14}{128}$	$\frac{42}{128}$	$\frac{70}{128}$

Calculate the expected winnings from the game.

Answer: $E(W) = \left(470 \times \frac{2}{128}\right) + \left(50 \times \frac{14}{128}\right) + \left(0 \times \frac{42}{128}\right) + \left(-30 \times \frac{70}{128}\right) = \frac{-460}{128} = -3.59375$

This is a loss of \$3.59

How much would you expect to win or lose if you played the game 10 times? 100 times?

Answer: For 10 times expected loss is \$35.94 and for 100 times it is \$359.37

Is the game 'fair'?

Answer: No

Interestingly if a large number of balls are dropped into the quincunx then the shape of the resulting balls approximates a normal distribution.

This can be seen in the simulation at this site: https://www.mathsisfun.com/data/quincunx.html

You can alter the number of boxes and the speed that the balls drop. It is also possible to change the probability that the ball falls to one side or the other.

What affect does this have to the distribution of the balls?

Answer: With an increased number of balls the shape of the distribution begins to resemble a normal distribution. The distribution becomes skewed.

Fair Game

Approaches to Learning/Learner Profile: Collaboration, Communication, Self-Management

Exploration Criteria: Presentation (A); Mathematical Communication (B); Personal Engagement (C); Reflection (D); Use of Mathematics (E)

IB Topic: Probability, Expected Value, Probability Distributions

Introduction

This task can be great fun for a class to get involved in, or even a whole year group. It is also possible to invite other classes from different parts of the school where this is viable, and it produces great discussions for many age groups. Sensitivity is obviously required around the topic of gambling, and clearly no actual money should change hands. This may be a time-consuming task but, given the right occasion (end or beginning of term, special mathematics day, etc.), it can be very rewarding and a lot of fun.

The aim is for students to engage in a real-life probability exercise and improve their understanding of probability, probability distributions, expected value and the concept of a fair game. However, there is a lot more to it than that. Students will need to present coherently, be organized, communicate mathematically and reflect on their outcomes. There are also aspects of psychology, advertising, ethics, etc. that may come up in various discussions.

Students should definitely be given an opportunity for Reflection (Criterion D) on the success of their game and consider what improvements they would have to make for this to be a worthwhile experience. The final "product" here is two-fold—first, it is the game itself, but more important is the write-up or video that explains the game, its success, the mathematics and any improvements.

If students have not had a chance to look at the three problems in the "Developing Inquiry Skills" section, then it may be a good idea to go back and attempt the questions at the end of each section before starting this task.

The Task

Put students into pairs, or groups of three, to design their game.

To start a discussion, you could ask:

What type of game would you play at a carnival, fairground or amusement park?

Students can use dice, spinners, balls, plastic darts, marbles or anything that involves probability and a chance to win.

When thinking about how to make a profit, you could ask:

What is a fair game?

Is a fair game likely to give you a profit?

A fair game is a game that is not biased toward any player. All players have the same chance of winning.

A fair game is not likely to give an individual a profit.

Each school context for the playing of the games and the resources available will be different. Ensure that students are clear about what is required of them.

There are three main parts to this task:

Part 1—Design and understand the probabilities involved in your game.

Part 2—Set up the game in a "class fair".

Part 3—Reflect on the success or otherwise of your game.

Part 1—Design and understand the probabilities involved in your game

Consider the amount of time that will be required for this. There will probably be an expectation that some of this will take place outside class.

Part 2—Set up the game in a "class fair"

Think of the available space and how to distribute this space between the groups.

One student from each group must remain at their game to supervise its playing, take the payments and award the prizes.

The others students may visit the other games to play.

Swap the roles regularly so that everyone gets a chance to both supervise their game and to play the other games.

Part 3—Reflect on the success or otherwise of your game

Ensure it is clear that they do not necessarily need to answer extensively all the questions listed. They are just for guidance.

Extension

Gambling is a controversial topic and will clearly need to be treated differently depending on the context of the school. The Mathematics Guide suggests questions that could be considered during this chapter, and these are adapted here for this extension. These questions could form the basis of a class debate, a written or recorded response or a blog post. The overlap with Theory of Knowledge is significant but these questions also have some mathematical foundation to them.

What does "the house always wins" mean?

This could be discussed in the context of expected values $E(X) < 0$.

This Quora post has some interesting comments regarding this:

www.quora.com/Casinos-Why-does-the-house-always-win

Could mathematics and mathematicians help increase incomes in gambling?

Again, students could use the concept of expected values. There is also mathematics that can be discussed around probabilities of particular lottery numbers coming up and comparing the probabilities in various national lotteries, as there are many misconceptions here.

How would a mathematician explain luck?

Luck has many aspects to it and mathematics is one of them—the idea that given enough opportunities then some will be lucky (and others will be unlucky), as all results are theoretically possible.

This Quora post is interesting on this:

www.quora.com/What-is-luck-Is-there-a-mathematical-explanation