OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS:
ANALYSIS AND APPROACHES

STANDARD LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1 From patterns to generalizations: sequences and series

Skills check

1 **a** $x = -3$ **b** $a = 2$ **c** $x = 4$

2 **a** $\frac{5}{12}$ **b** $\frac{53}{48}$ **c** $-\frac{6}{5}$

3 **a** 128 **b** 9 **c** −81

Exercise 1A

1 **a** $-20, -23, -26$ **b** $49, 64, 81$

c $30, 36, 42$

d $\frac{125}{2}, \frac{125}{4}, \frac{125}{8}$ or 62.5, 31.25, 15.625

e $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ **f** $\frac{5}{243}, \frac{6}{729}, \frac{7}{2187}$

2 **a** $u_n = 10 \times 5^{n-1}$, geometric

b $u_n = -6n + 47$, arithmetic

c $u_n = (-1)^{n+1}\frac{1}{3^n}$, geometric

d $u_n = u_{n-1} + u_{n-2}$, neither

e $u_n = \frac{2n-1}{2n}$, neither

f $u_n = -4 \times 3^n$, geometric

3 **a** $100, 200, 300, \ldots, u_n = 100n$, arithmetic

b $6, 3, \frac{3}{2}, \ldots, u_n = 6\left(\frac{1}{2}\right)^{n-1}$, geometric

c $70, 77, 84.7, \ldots, u_n = 70(1.1)^{n-1}$, geometric

Exercise 1B

1 **a** 1, -4, 16, -64, 256

b $3, -\frac{2}{3}, 3, -\frac{2}{3}, 3$

c -1, 2, 8, 128, 32768

d m, $3m + 5$, $9m + 20$, $27m + 65$, $81m + 200$

2 **a** $u_n = u_{n-1} - 2, u_1 = -2$

b $u_n = 4u_{n-1}, u_1 = 1$

c $u_n = \frac{u_{n-1}}{10}, u_1 = 52$

d $u_n = u_{n-1} + 5, u_1 = 14$

e $u_n = u_{n-2} \times u_{n-1}, u_1 = 2, u_2 = 3$

f $u_n = n(u_{n-1}), u_1 = 1$

Exercise 1C

1 **a** $\sum_{n=1}^{4}(-1)^n(n+1) = -2 + 3 - 4 + 5 = 2$

b $\sum_{n=2}^{6} 4n - 3 = 5 + 9 + 13 + 17 + 21 = 65$

c $\sum_{n=1}^{3} n(n+1) = 2 + 6 + 12 = 20$

d $\sum_{n=3}^{5}\frac{(-1)^{n+1}}{n-2} = 1 + \frac{-1}{2} + \frac{1}{3} = \frac{5}{6}$

2 **a** $\sum_{n=1}^{\infty} 4^n$ **b** $\sum_{n=3}^{5}\frac{n}{n+1}$ **c** $\sum_{n=1}^{100}(-1)^n\frac{1}{n}$

d $\sum_{n=1}^{8} -2$ **e** $\sum_{n=2}^{\infty} n^2 + 1$

f $\sum_{n=7}^{11} n^2 m^{n-1}$ or $\sum_{n=6}^{10}(n+1)^2 m^n$

Exercise 1D

1 $u_n = u_1 + (n-1)d$

$u_9 = 5 + (9-1)(8)$

$u_9 = 5 + (8)(8)$

$u_9 = 5 + 64$

$u_9 = 69$

2 $u_n = u_1 + (n-1)d$

$u_{11} = 40 + (11-1)(-8)$

$u_{11} = 40 + (10)(-8)$

$u_{11} = 40 - 80$

$u_{11} = -40$

3 $u_n = u_1 + (n-1)d$

$u_7 = 5.05 + (7-1)(0.32)$

$u_7 = 5.05 + (6)(0.32)$

$u_7 = 5.05 + 1.92$

$u_7 = 6.97$

4 $u_n = u_1 + (n-1)d$

$u_6 = \frac{1}{2} + (6-1)\left(\frac{1}{3}\right)$

$u_6 = \frac{1}{2} + (5)\left(\frac{1}{3}\right)$

$u_6 = \frac{1}{2} + \frac{5}{3}$

$u_6 = \frac{13}{6}$

5 $u_n = u_1 + (n-1)d$

$u_9 = x + 2 + (9-1)(3)$

$u_9 = x + 2 + (8)(3)$

$u_9 = x + 2 + 24$

$u_9 = x + 26$

6 $u_n = u_1 + (n-1)d$

$u_{12} = 3a + (12-1)(3a)$

$u_{12} = 3a + (11)(3a)$

$u_{12} = 3a + 33a$

$u_{12} = 36a$

Exercise 1E

1 $u_n = u_1 + (n-1)d$

$65 = u_1 + (21-1)(-2)$

$65 = u_1 + (20)(-2)$

$65 = u_1 - 40$

$u_1 = 105$

2 $u_n = u_1 + (n-1)d$

$u_5 \to u_1$

$u_{15} \to u_{11}$

$u_{19} \to u_{15}$

$u_{11} = u_1 + (11-1)d$

$-52.3 = -3.7 + (10)d$

$-48.6 = 10d$

$d = -4.86$

$u_{15} = u_1 + (15-1)(-4.86)$

$u_{15} = -3.7 + (14)(-4.86)$

$u_{15} = -3.7 + (14)(-4.86)$

$u_{15} = -71.74$

3 $u_n = u_1 + (n-1)d$

$2 = 11 + (n-1)(-3)$

$-9 = -3n + 3$

$-12 = -3n$

$n = 4$

4 $u_n = u_1 + (n-1)d$

$u_3 \to u_1$

$u_6 \to u_4$

$u_{14} \to u_{12}$

$u_4 = u_1 + (4-1)d$

$184 = 4 + (3)d$

$180 = 3d$

$d = 60$

$u_{12} = 4 + (12-1)(60)$

$u_{12} = 4 + (11)(60)$

$u_{12} = 4 + 660$

$u_{12} = 664$

So 14th term of the given series is 664

5 $-36 = 6 + (n-1)(-7)$

$-42 = -7n + 7$

$-49 = -7n$

$n = 7$

It is the 7th term.

6 $u_{12} = 30 + (12-1)(2)$

$u_{12} = 30 + (11)(2)$

$u_{12} = 30 + 22$

$u_{12} = 52$ seats

7 Let $u_1 = 2010$ and $d = 4$.

$2050 = 2010 + (n-1)4$

$40 = 4n - 4$

$44 = 4n$

$n = 11$

Since they are held in 2050 (since n is a natural number), the next time they will be held is 2054.

8 $82 = 40 + (n-1)(6)$

$42 = 6n - 6$

$48 = 6n$

$n = 8$

In 7 weeks

Exercise 1F

1 a $u_6 = 9(3)^{6-1}$

$u_6 = 9(3)^5$

$u_6 = 2187$

b Not geometric

c $u_7 = 6(0.75)^{7-1}$

$u_7 = 6(0.75)^6$

$u_7 = 1.0678\ldots$

$u_7 \approx 1.068$

d $u_8 = -4(-1.5)^{8-1}$

$u_8 = -4(-1.5)^7$

$u_8 = 68.34375$

$u_8 \approx 68.3$

e $u_{13} = 500\left(\frac{1}{5}\right)^{13-1}$

$u_{13} = 500\left(\frac{1}{5}\right)^{12}$

$u_{13} = \frac{500}{244140625} = \frac{4}{1953125}$

f Not geometric

g $u_{12} = 3(m)^{12-1}$

$u_{12} = 3(m)^{11}$

$u_{12} = 3m^{11}$

2 $1, 2, 4\ldots u_{30} = ?$

$u_{30} = 1(2)^{30-1}$

$u_{30} = 2^{29}$

$u_{30} = 536870912$ cents or \$5 368 709.12

3 Use an r value that is a factor of 64. For example, $r = 2$:

$u_5 = \frac{u_6}{r} = \frac{64}{2} = 32$

$u_4 = \frac{u_5}{r} = \frac{32}{2} = 16$

$u_3 = \frac{u_4}{r} = \frac{16}{2} = 8$

$u_2 = \frac{u_3}{r} = \frac{8}{2} = 4$

$u_1 = \frac{u_2}{r} = \frac{4}{2} = 2$

$\therefore$ One possible sequence is $2, 4, 8, 16, 32, \ldots$

Exercise 1G

1 $u_5 \to u_1 \quad u_{10} \to u_6 \quad u_{15} \to u_{11}$

$303.75 = 40r^{6-1}$

$r^5 = 7.59375$

$r = 1.5$

$u_{11} = 40 \times 1.5^{11-1}$

$u_{11} = 40 \times 1.5^{10}$

$u_{11} = 2306.60156\ldots$

$u_{11} \approx 2307$

2 $u_6 \to u_1 \quad u_{20} \to u_{15}$

$u_{15} = -1280\left(-\frac{4}{5}\right)^{15-1}$

$u_{15} = -1280\left(-\frac{4}{5}\right)^{14}$

$u_{15} = -1280\left(\frac{268\,435\,456}{6103\,515\,625}\right)$

$u_{15} = -56.295.. \approx -56.3$

3 $u_n = u_1 r^{n-1}$

$1 = 16r^{3-1}$

$r^2 = \frac{1}{16}$

$r = \pm\frac{1}{4}$

$r = \frac{u_2}{u_1}$

$\frac{1}{4} = \frac{x+2}{16}$

$4 = x + 2$

$x = 2$

or $\frac{-1}{4} = \frac{x+2}{16}$

$-4 = x + 2$

$x = -6$

4 $1536 = 6(2)^{n-1}$

$256 = 2^{n-1}$

$2^8 = 2^{n-1}$

$n - 1 = 8$

$n = 9$

5 $32 = 2(r)^{5-1}$

$16 = r^4$

$2^4 = r^4$

$r = 2$

6 $u_{10} = 232(1.03)^{10-1}$

$u_{10} = 232(1.03)^9$

$u_{10} = 302.70737\ldots$

$u_{10} = 303$ students

7 **a** $u_{30} = 1(2)^{30-1}$

$u_{30} = 2^{29}$

$u_{30} = 536870912$ grains

b $512 = 1(2)^{n-1}$

$2^9 = 2^{n-1}$

$n - 1 = 9$

$n = 10$th square

8 $128 = 8(r)^{5-1}$

$16 = r^4$

$r = 2$

8, 16, 32, 64, 128

Exercise 1H

1 Geometric because you are multiplying each previous height by $\frac{1}{2}$.

$u_{10} = 1\left(\frac{1}{2}\right)^{10-1}$

$u_{10} = 1\left(\frac{1}{2}\right)^{10-1}$

$u_{10} = \left(\frac{1}{2}\right)^9$

$u_{10} = \frac{1}{512}$ meters

2 Arithmetic because you are adding more money to your account every month.

$6500 = 2000 + (36 - 1)(x + 5)$

$4500 = (35)(x + 5)$

$128.57 = (x + 5)$

$x = \$123.57$

3 Arithmetic because you are adding from year to year.

$2017 = 1962 + (n - 1)(12)$

$55 = 12n - 12$

$67 = 12n$

$n = 5.58\overline{3}$

Finland did not gain independence in the year of the tiger.

4 Geometric because a rate implies you are multiplying.

$324 = 6(r)^{21-1}$

$54 = r^{20}$

$r = \sqrt[20]{54}$

$r = 1.220730\ldots$

$r \approx 122\%$

Exercise 1I

1 **a** $d = u_n - u_{n-1} = \frac{8}{15} - \frac{1}{5} = \frac{1}{3}$

$S_7 = \frac{7}{2}\left[2\left(\frac{1}{5}\right) + (7 - 1)\left(\frac{1}{3}\right)\right]$

$S_7 = \frac{7}{2}\left[\frac{2}{5} + (6)\left(\frac{1}{3}\right)\right]$

$S_7 = \frac{7}{2}\left[\frac{2}{5} + 2\right]$

$S_7 = \frac{7}{2}\left[\frac{2}{5} + \frac{10}{5}\right]$

$S_7 = \frac{7}{2}\left[\frac{12}{5}\right]$

$S_7 = \frac{42}{5}$

b $u_1 = \frac{1}{2}(-3)^1 = -\frac{3}{2}$

$u_2 = \frac{1}{2}(-3)^2 = \frac{1}{2}(9) = \frac{9}{2}$

$r = \frac{u_n}{u_{n-1}} = \frac{\frac{9}{2}}{-\frac{3}{2}} = \frac{9}{2} \times \left(-\frac{2}{3}\right) = -3$

$S_8 = -\frac{3}{2}\left(\frac{1 - (-3)^8}{1 - (-3)}\right)$

$S_8 = -\frac{3}{2}\left(\frac{1 - 6561}{1 + 3}\right)$

$S_8 = -\frac{3}{2}\left(\frac{-6560}{4}\right)$

$S_8 = 2460$

c $r = \frac{u_n}{u_{n-1}} = \frac{0.05}{0.1} = 0.5$

$S_8 = 0.1\left(\frac{1 - (0.5)^8}{1 - 0.5}\right)$

$S_8 = 0.19921875$

$S_8 \approx 0.199$

d $d = u_n - u_{n-1} = 12 - 6 = 6$

$288 = 6 + (n-1)(6)$

$282 = 6(n-1)$

$47 = n - 1$

$n = 48$

$S_{48} = \frac{48}{2}[2(6) + (48-1)(6)]$

$S_{48} = 24[12 + (47)(6)]$

$S_{48} = 24[294]$

$S_{48} = 7056$

e $u_1 = 4$, $u_2 = 8, d = 8 - 4 = 4$

As 1000 is a multiple of 4, the largest multiple of 4 less than 999 would be 996.

$996 = 4 + (n-1)(4)$

$992 = 4(n-1)$

$248 = n - 1$

$n = 249$

$S_{249} = \frac{249}{2}[2(4) + (249-1)(4)]$

$S_{249} = \frac{249}{2}[8 + 992]$

$S_{249} = 124500$

f $u_1 = (-1)^0 (2)^1 = 2$

$u_2 = (-1)^1 (2)^2 = -4$

$r = \frac{u_n}{u_{n-1}} = \frac{-4}{2} = -2$

$S_6 = 2\left(\frac{1-(-2)^6}{1-(-2)}\right)$

$S_6 = 2\left(\frac{1-64}{3}\right)$

$S_6 = 2(-21)$

$S_6 = -42$

2 $d = u_n - u_{n-1} = 26 - 22 = 4$

$S_{30} = \frac{30}{2}[2(22) + (30-1)(4)]$

$S_{30} = 15[44 + (29)(4)]$

$S_{30} = 15[44 + 116]$

$S_{30} = 15[160]$

$S_{30} = 2400$ seats

3 The series is $1 + 2 + 4 + \ldots$

$S_6 = 1\left(\frac{1-2^6}{1-2}\right)$

$S_6 = \left(\frac{1-64}{1-2}\right)$

$S_6 = \left(\frac{-63}{-1}\right)$

$S_6 = 63$ family members

4 The series is $1 + 2 + 3 + \ldots$

$d = u_n - u_{n-1} = 2 - 1 = 1$

$S_{12} = \frac{12}{2}[2(1) + (12-1)(1)]$

$S_{12} = 6[2 + (11)(1)]$

$S_{12} = 6[13]$

$S_{12} = 78$

But since there are two 12-hour cycles in a 24-hour day:

$S_{24} = 78 \times 2 = 156$ chimes

5 The series is $5 + 9 + 13 + \ldots$

$d = u_n - u_{n-1} = 9 - 5 = 4$

$S_{48} = \frac{48}{2}[2(5) + (48-1)(4)]$

$S_{48} = 24[10 + (47)(4)]$

$S_{48} = 24[10 + 188]$

$S_{48} = 24[198]$

$S_{48} = 4752$ line segments

Exercise 1J

1 a Not converging as $r = 1.5$.

b $r = \frac{u_n}{u_{n-1}} = \frac{\frac{9}{32}}{\frac{-3}{8}} = \frac{9}{32} \times \frac{8}{-3} = -\frac{3}{4}$

$S_\infty = \frac{u_1}{1-r} = \frac{-\frac{3}{8}}{1 - \frac{-3}{4}} = \frac{-\frac{3}{8}}{\frac{7}{4}} = \frac{-3}{8} \times \frac{4}{7} = \frac{-3}{14}$

c $r = \frac{u_n}{u_{n-1}} = \frac{\frac{-5}{4}}{\frac{-5}{2}} = \frac{-5}{4} \times \frac{2}{-5} = \frac{1}{2}$

$S_\infty = \frac{u_1}{1-r} = \frac{-\frac{5}{2}}{1 - \frac{1}{2}} = \frac{-\frac{5}{2}}{\frac{1}{2}} = \frac{-5}{2} \times \frac{2}{1} = -5$

d Not converging as $r = -2$.

e

$$r = \frac{u_n}{u_{n-1}} = \frac{9x-9}{27x-27} = \frac{9(x-1)}{27(x-1)} = \frac{9}{27} = \frac{1}{3}$$

$$S_\infty = \frac{27x-27}{1-\frac{1}{3}} = \frac{27x-27}{\frac{2}{3}}$$

$$= (27x-27)\left(\frac{3}{2}\right) = \frac{81}{2}x - \frac{81}{2}$$

f Not converging as $r = 2$.

g $$r = \frac{u_n}{u_{n-1}} = \frac{\frac{1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{1} = \frac{1}{2}$$

$$S_\infty = \frac{u_1}{1-r} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{2}} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} = \frac{1}{\sqrt{2}} \times \frac{2}{1} = \frac{2}{\sqrt{2}}$$

2 Any infinite geometric series where $-1 < r < 1$, $r \neq 0$

3 Any infinite geometric series where $r < -1$, or $r > 1$

4 $$S_\infty = \frac{u_1}{1-r} = \frac{12}{1-\frac{3}{5}} = \frac{12}{\frac{2}{5}} = 30$$

Total distance $= 2S_\infty - u_1 = 2(30) - 12$

$= 48$ ft.

5 $$S_\infty = \frac{u_1}{1-r} = \frac{426}{1-0.999} = \frac{426}{0.001} = 426000$$

$426000 \times 42 = 17892000$ gallons

Exercise 1K

1 a $$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

$$790 = \frac{20}{2}\left[2(-8) + (20-1)d\right]$$

$$790 = 10\left[-16 + (19)d\right]$$

$$79 = 19d - 16$$

$$95 = 19d$$

$$d = 5$$

b i $$u_n = u_1 + (n-1)d$$

$$u_{28} = -8 + (28-1)(5)$$

$$u_{28} = -8 + (27)(5)$$

$$u_{28} = -8 + 135$$

$$u_{28} = 127$$

b ii $$S_n = \frac{n}{2}\left[u_1 + u_n\right]$$

$$S_{28} = \frac{28}{2}\left[-8 + 127\right]$$

$$S_{28} = 14(119)$$

$$S_{28} = 1666$$

c $$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

$$\frac{n}{2}\left[2(-8) + (n-1)(5)\right] > 2000$$

$$n\left[-16 + (5n-5)\right] > 4000$$

$$n(5n-21) > 4000$$

$$5n^2 - 21n - 4000 > 0$$

By GDC, $n \approx -26.3$ or 30.5

Since $n > 0$, $n = 31$

2 $$1900 = \frac{40}{2}\left[2u_1 + (40-1)d\right]$$

$$1900 = 20\left[2u_1 + (39)d\right]$$

$$95 = 2u_1 + 39d$$

$$106 = u_1 + (40-1)d$$

$$106 = u_1 + 39d$$

$$39d = 106 - u_1$$

$$95 = 2u_1 + 106 - u_1$$

$$u_1 = -11$$

$$39d = 106 - -11$$

$$39d = 117$$

$$d = 3$$

3 $$S_\infty = \frac{u_1}{1-r}$$

$$20 = \frac{u_1}{1-0.2}$$

$$20 = \frac{u_1}{0.8}$$

$$u_1 = 16$$

4 $$3u_1 = \frac{u_1}{1-r}$$

$$1 - r = \frac{u_1}{3u_1}$$

$$1 - r = \frac{1}{3}$$

$$r = 1 - \frac{1}{3}$$

$$r = \frac{2}{3}$$

5 Choose any r value $-1 < r < 1, r \neq 0$.

Example:

Let $r = \frac{1}{2}$

$8 = \frac{u_1}{1-\frac{1}{2}}$

$8 = \frac{u_1}{\frac{1}{2}}$

$u_1 = 4$

$\therefore$ The series is $4 + 2 + 1 + \ldots$

6 $u_4 = u_1 r^{4-1}$

$8u_1 = u_1 r^3$

$r^3 = 8$

$r = 2$

$S_n = u_1\left(\frac{1-r^n}{1-r}\right)$

$2557.5 = u_1\left(\frac{1-2^{10}}{1-2}\right)$

$2557.5 = u_1\left(\frac{-1023}{-1}\right)$

$2557.5 = 1023u_1$

$u_1 = 2.5$

$u_{10} = 2.5(2)^{10-1}$

$u_{10} = 2.5(2)^9$

$u_{10} = 2.5(512)$

$u_{10} = 1280$

7 $2375 = 5\left(\frac{1-5^n}{1-5}\right)$

$475 = \left(\frac{1-5^n}{-4}\right)$

$-1900 = 1 - 5^n$

$5^n = 1901$

By GDC, $n = 4.69116\ldots$

A minimum of 5 rounds are required.

8 **a** $S_2 = u_1\left(\frac{1-r^2}{1-r}\right)$

$15 = u_1\left(\frac{1-r^2}{1-r}\right)$

$27 = \frac{u_1}{1-r}$

$u_1 = 27(1-r)$

$15 = 27(1-r)\left(\frac{1-r^2}{1-r}\right)$

$\frac{15}{27} = 1 - r^2$

$r^2 = 1 - \frac{15}{27}$

$r^2 = \frac{12}{27}$

$r = \pm\sqrt{\frac{12}{27}} = \pm\frac{2\sqrt{3}}{3\sqrt{3}} = \pm\frac{2}{3}$

Since the geometric series has only positive terms,

$r = \frac{2}{3}$

b $u_1 = 27\left(1-\frac{2}{3}\right)$

$u_1 = 27\left(\frac{1}{3}\right)$

$u_1 = 9$

9 **a** $r = \frac{6}{m-1}$

$r = \frac{m+8}{6}$

b **i** $\frac{6}{m-1} = \frac{m+8}{6}$

$36 = m^2 + 7m - 8$

$0 = m^2 + 7m - 44$

$0 = (m+11)(m-4)$

$m = -11 \quad m = 4$

b **ii** $r = \frac{-11+8}{6} = -\frac{3}{6} = -\frac{1}{2}$

$r = \frac{4+8}{6} = \frac{12}{6} = 2$

c **i** Since the sum of an infinite series can only be found when $-1 < r < 1, r \neq 0$, $r = -\frac{1}{2}$.

c **ii** $u_1 = m - 1 = -11 - 1 = -12$

$S_\infty = \frac{-12}{1+\frac{1}{2}}$

$S_\infty = \frac{-12}{\frac{3}{2}}$

$S_\infty = -8$

Exercise 1L

1 **a** $A = P(1+nr)$

$A = 1500(1+10(0.06))$

$A = 1500(1.6)$

$A = 2400$

$I = 2400 - 1500 = \$900$

b $A = P(1+nr)$

$A = 32000\left(1+32\left(\frac{0.0125}{4}\right)\right)$

$A = 32000(1.1)$

$A = 35200$

$I = 35200 - 32000 = 3200$ GBP

c $A = P(1+r)^n$

$A = 14168000\left(1+\frac{0.02}{12}\right)^{3\times12}$

$A = 14168000\left(1+\frac{0.02}{12}\right)^{36}$

$A = 14168000(1.0617835\ldots)$

$A = 15043348.839948\ldots$

$I = 15043348.84 - 14168000$

$I \approx 875348.84$ Yen

d $A = P(1+r)^n$

$A = 300000\left(1+\frac{0.04}{365}\right)^{2\times365}$

$A = 300000\left(1+\frac{0.04}{365}\right)^{730}$

$A = 324984.69581\ldots$

$I = 324984.70 - 300000$

$I \approx 24984.70$ Mexican Pesos

e $A = P(1+r)^n$

$A = 250000\left(1+\frac{0.0225}{12}\right)^{12\times25}$

$A = 250000\left(1+\frac{0.0225}{12}\right)^{300}$

$A = 438532.634627\ldots$

$I = 438532.63 - 250000$

$I \approx 188532.63$ Swiss Francs

2 **i** $A = P(1+nr)$

$2480000 = 2323000(1+(52\times2)r)$

$1.067585\ldots = 1+104r$

$104r = 0.067585\ldots$

$r = 0.000649855\ldots$

Fernando will pay an annual simple interest rate of 0.065%.

ii $\frac{2480000}{104} = 23846.15$

≈ 23846 Columbia Pesos.

3 $A = P(1+r)^n$

$A = 90000\left(1+\frac{0.0225}{12}\right)^{12\times5}$

$A = 90000(1.001875)^{60}$

$A = 100705.8944966\ldots$

$A \approx \$100705.89$

4 $A = P(1+r)^n$

$32546 = P\left(1+\frac{0.042}{4}\right)^{4\times5}$

$32546 = P(1.0105)^{20}$

$32546 = 1.232328\ldots P$

$P = 26410.17651144\ldots$

$P \approx \$26410.18$

5 $A = P(1+r)^n$

$10000 = 5000\left(1+\frac{0.0325}{12}\right)^{12\times n}$

$2 = (1.0027083\ldots)^{12n}$

Using the GDC, $n = 21.3567\ldots$

$n \approx 21.4$ years

6 $A = P(1+r)^n$

$50000 = P\left(1+\frac{0.055}{12}\right)^{12\times(18-5)}$

$50000 = P(1.00458333\ldots)^{156}$

$50000 = P(2.04085012\ldots)$

$P = 24499.594316\ldots$

$P \approx 24500$ Brazilian Reals

7 Oliver:

$A = P(1+r)^n$

$A = 400\left(1+\frac{0.0125}{12}\right)^{12\times5}$

$A = 425.783932\ldots$

$A \approx 425.78$ GBP

Harry:

$A = P(1+r)^n$

$A = 400(1+0.0175)^5$

$A = 436.2466257\ldots$

$A \approx 436.25$ GBP

Harry earned more than Oliver.

8 Savings account:

$A = P(1+nr)$

$A = 20000(1+n(0.012))$

$A = 20000(1+0.012n)$

GIC:

$A = P(1+r)^n$

$A = 20000\left(1+\frac{0.035}{12}\right)^{12\times 2}$

$A = 20000(1.00291666\ldots)^{24}$

$A = 21447.978280670\ldots$

$21447.978280670\ldots = 20000(1+0.012n)$

$1.0723989140\ldots = 1+0.012n$

$0.0723989140\ldots = 0.012n$

$n = 6.03324\ldots$

$n \approx 6.03$ years

Exercise 1M

1 a $r = \frac{11000}{12500} = 0.88$

$C = 12500 \times 0.88^t$, where C represents the white blood cell count and t is the time every 12 hours.

b 3 days = 72 hours = 6 12-hour periods

$C = 12500 \times 0.88^6 = 5805.0510\ldots$

$C \approx 5805$ cells/mcL

c The limitation of the general formula is that white blood cell count does not continue to decrease infinitely. Once the antibiotics killed the infection, the patient's white blood cell count would return to normal.

2 a This is an arithmetic sequence since the rate decreases by -0.2% each month.

b $U = 7.9 - 0.2(t-1)$, where U represents the unemployment rate and t is the month starting with January.

c $U = 7.9 - 0.2(12-1)$

$U = 7.9 - 0.2(11)$

$U = 5.7\%$

d It is not realistic. There will always be people you are not capable of working, or are switching jobs, or looking for jobs.

3 a This means that it takes 1.23 years for the substance to decrease to half of the original mass.

b $A = A_0\left(\frac{1}{2}\right)^{\frac{t-1}{h}}$, where A is the amount remaining after t years, A_0 is the original mass, and h is the half-life.

d $A = 52\left(\frac{1}{2}\right)^{\frac{7.2-1}{1.23}}$

$A = 52\left(\frac{1}{2}\right)^{5.0406504\ldots}$

$A = 1.57985\ldots$

$A \approx 1.58$ g

Exercise 1N

1 $x^4 + 20x^3 + 150x^2 + 500x + 625$

2 $-b^5 + 10b^4 - 40b^3 + 80b^2 - 80b + 32$

3 $64x^6 - 192x^5 + 240x^4 - 160x^3$

$+60x^2 - 12x + 1$

4 $256x^4 + 256x^3y + 96x^2y^2 + 16xy^3 + y^4$

5 $x^3 - 9x^2y + 27xy^2 - 27y^3$

6 $243x^5 + 1620x^4y + 4320x^3y^2 + 5760x^2y^3$

$+3840xy^4 + 1024y^5$

Exercise 1O

1 a ${}_{11}C_4(3x)^7(-5)^4$

$= (330)(2187x^7)(625)$

$= 451068750x^7$

b ${}_{10}C_8(x)^2(6y)^8$

$= (45)(x^2)(1679616y^8)$

$= 75582720x^2y^8$

c Since $n = 6$, there will be 7 terms in the expansion. Hence the middle term is the 4th term.

${}_6C_3(2)^3(-3y)^3$

$=20(8)(-27y^3)$

$= -4320y^3$

d The constant term will contain x^0, hence

${}_9C_9(x^4)^0(-3)^9$

$= (-3)^9$

$= -19683$

e ${}_7C_7\left(-2x^2\right)^0\left(\frac{-3}{x}\right)^7$

$= \left(\frac{-3}{x}\right)^7$

$= -\frac{2187}{x^7}$

2 **a** 1, 4, 6, 4, 1

b ${}_4C_3\left(3x\right)^1\left(-2\right)^3$

$= 4\left(3x\right)\left(-8\right)$

$= -96$

3 **a** ${}_8C_5\left(x\right)^3\left(-3\right)^5$

$= (56)\left(x^3\right)\left(-243\right)$

$= -13608x^3$

b $\left(-2x\right)\ {}_8C_4\, x^4\left(-3\right)^4$

$= -11340x^5$

4 $\left(x^3\right)^{11-r}\left(\frac{1}{x}\right)^r = x^9$

$\left(x^{33-3r}\right)\left(\frac{1}{x^r}\right) = x^9$

$\frac{x^{33-3r}}{x^r} = x^9$

$x^{33-3r-r} = x^9$

$x^{33-4r} = x^9$

$33 - 4r = 9$

$4r = 24$

$r = 6$

${}_{11}C_6\left(x^3\right)^5\left(\frac{-3}{x}\right)^6$

$462\left(x^{15}\right)\left(\frac{729}{x^6}\right)$

$336798x^9$

5 The constant term will contain x^0, hence

$x^7\left(x^4\right)^{7-r}\left(\frac{1}{x^3}\right)^r = x^0$

$x^7\left(x^{28-4r}\right)\left(\frac{1}{x^{3r}}\right) = x^0$

$x^{28-4r+7-3r} = x^0$

$x^{35-7r} = x^0$

$35 - 7r = 0$

$7r = 35$

$r = 5$

${}_7C_5\ x^7\left(\frac{x^4}{2}\right)^{7-5}\left(\frac{k}{x^3}\right)^5 = 168$

$21x^7\left(\frac{x^4}{2}\right)^2\left(\frac{k}{x^3}\right)^5 = 168$

$21x^7\left(\frac{x^8}{2^2}\right)\left(\frac{k^5}{x^{15}}\right) = 168$

$\frac{21k^5x^{15}}{4x^{15}} = 168$

$\frac{21k^5}{4} = 168$

$k^5 = 32$

$k = 2$

6 ${}_8C_3\left(2x\right)^5\left(-k\right)^3 = -387072x^5$

$(56)\left(32x^5\right)\left(-k\right)^3 = -387072x^5$

$(56)(32)\left(-k\right)^3 = -387072$

$1792\left(-k\right)^3 = -387072$

$\left(-k\right)^3 = -216$

$-k = \sqrt[3]{-216}$

$-k = -6$

$k = 6$

7 ${}_6C_3\left(a\right)^3\left(-b^2\right)^3$

$20a^3\left(-b^6\right)$

$-20a^3b^6$

The coefficient is -20.

8 $(2.52)^3 = (2 + 0.52)^3$

$(2.52)^3 = \binom{3}{0}2^3(0.52)^0 + \binom{3}{1}2^2(0.52)^1$

$+ \binom{3}{2}2^1(0.52)^2 + \binom{3}{3}2^0(0.52)^3$

$(2.52)^3 = 8 + (3)(4)(0.52) + (3)(2)(0.52)^2$

$+ (0.52)^3$

$(2.52)^3 = 8 + 12(0.52) + 6(0.52)^2 + (0.52)^3$

$(2.52)^3 = 8 + 12(0.52) + 6(0.2704)$

$+ 0.140608$

$(2.52)^3 = 8 + 6.24 + 1.6224 + 0.140608$

$(2.52)^3 = 16.003008$

$(2.52)^3 \approx 16.003$

9 a $x^5 - 25x^4 + 250x^3 - 1250x^2 + 3125x$
-3125

b Hence, the term containing x^4 will be $(2x)(250x^3) = 525x^4$

10 a $(3-2x)^4$

$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

$(-2x+3)^4$

$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

b No, when the exponent is odd, the expansions will not be the same.

11 Since there are four terms, n must be 3.

Let k be the coefficient of x and m be the coefficient of y.

Using the first term,

$\binom{3}{1}(kx)^3(my)^0 = 27x^3$

$k^3x^3 = 27x^3$

$k^3 = 27$

$k = \sqrt[3]{27}$

$k = 3$

Using the last term,

$\binom{3}{3}(3x)^0(my)^3 = -64y^3$

$(my)^3 = -64y^3$

$m^3y^3 = -64y^3$

$m^3 = -64$

$m = \sqrt[3]{-64}$

$m = -4$

$\therefore (a+b)^n = (3x-4y)^3$

12 $2^n = (1+1)^n$

$$2^n = \binom{n}{0}(1)^n(1)^0 + \binom{n}{1}(1)^{n-1}(1)^1 + \binom{n}{2}(1)^{n-2}(1)^2 + \ldots \binom{n}{n-1}(1)^1(1)^{n-1} + \binom{n}{n}(1)^0(1)^n$$

Since $1^x = 1$ for any $x \in R$,

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n-1} + \binom{n}{n}$$

Exercise 1P

1

LHS	RHS
$-2(a-4)+3(2a+6)-6(a-5)$ $-2a+8+6a+18-6a+30$ $-2a+56$ $-2(a-28)$ LHS $\equiv$ RHS	$-2(a-28)$

2

LHS	RHS
$(x-3)^2+5$ $x^2-6x+9+5$ $x^2-6x+14$ LHS $\equiv$ RHS	$x^2-6x+14$

3

LHS	RHS
$\frac{1}{m}$	$\frac{1}{m+1}+\frac{1}{m^2+m}$ $\frac{1}{m+1}+\frac{1}{m(m+1)}$ $\frac{m}{m(m+1)}+\frac{1}{m(m+1)}$ $\frac{m+1}{m(m+1)}$ $\frac{1}{m}$ RHS $\equiv$ LHS

4 a

LHS	RHS
$\frac{x-2}{x}\div\frac{3x-6}{x^2+x}$ $\frac{x-2}{x}\times\frac{x^2+x}{3x-6}$ $\frac{x-2}{x}\times\frac{x(x+1)}{3(x-2)}$ $\frac{x+1}{3}$ LHS $\equiv$ RHS	$\frac{x+1}{3}$

b $x \neq -1,0,2$

Chapter review

1 i a This sequence is not arithmetic since $18-6 \neq 6-3$. This sequence is not geometric since $\frac{18}{6} \neq \frac{6}{3}$.

ii a This sequence is arithmetic since $-12--14=-14--16=2$.

b $u_n = u_1 + (n-1)d$

$u_n = -16 + 2(n-1)$

$u_n = -16 + 2n - 2$

$u_n = 2n - 18$

c $u_{10} = 2(10) - 18 = 2$

d $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_8 = \frac{8}{2}(2(-16) + (8-1)(2))$

$S_8 = 4(-32 + (7)(2))$

$S_8 = 4(-32 + 14)$

$S_8 = 4(-18)$

$S_8 = -72$

iii a This sequence is geometric since $\frac{500}{1000} = \frac{1000}{2000} = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$

$u_n = 2000 \times \left(\frac{1}{2}\right)^{n-1}$

c $u_9 = 2000 \times \left(\frac{1}{2}\right)^{9-1}$

$u_9 = 2000 \times \left(\frac{1}{2}\right)^{8}$

$u_9 = 7.81$

d $S_n = u_n\left(\frac{1-r^n}{1-r}\right)$

$S_7 = 2000\left(\frac{1-\left(\frac{1}{2}\right)^7}{1-\frac{1}{2}}\right)$

$S_7 = 3968.75$

iv a The first few terms of this sequence is $3, 6, 12, \ldots$

This is a geometric sequence since $\frac{12}{6} = \frac{6}{3} = 2$.

b $u_n = u_1 r^{n-1}$

$u_n = 3 \times (2)^{n-1}$

c $u_5 = 3 \times (2)^{5-1}$

$u_5 = 3 \times (2)^4$

$u_5 = 3 \times 16$

$u_5 = 48$

d $S_{10} = 3\left(\frac{1-2^{10}}{1-2}\right)$

$S_{10} = 3\left(\frac{1-2^{10}}{-1}\right)$

$S_{10} = 3069$

v a The first few terms of the sequence are 105, 110, 115…

This sequence is arithmetic since $115-110=110-105=5$.

b $u_n = 105 + (n-1)5$

$u_n = 105 + 5n - 5$

$u_n = 5n + 100$

c $u_7 = 5(7) + 100$

$u_7 = 35 + 100$

$u_7 = 135$

d $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_9 = \frac{9}{2}(2(105) + (9-1)(5))$

$S_9 = 4.5(210 + (8)(5))$

$S_9 = 4.5(210 + 40)$

$S_9 = 4.5(250)$

$S_9 = 1125$

2 Renaming the terms:

$u_6 = u_1 = -5$

$u_9 = u_4 = -20$

This means we now want to find $S_{20} = S_{15}$.

We first need to find d:

$u_4 = u_1 + (n-1)d$

$-20 = -5 + (4-1)d$

$-15 = 3d$

$d = -5$

$S_{15} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$

$S_{16} = \frac{15}{2}\left(2(-5) + (15-1)(-5)\right)$

$S_{16} = 7.5\left(-10 + (14)(-5)\right)$

$S_{16} = 7.5(-10 - 70)$

$S_{16} = 7.5(-80)$

$S_{16} = -600$

3 $u_2 = -2(-4) + 3 = 8 + 3 = 11$

$u_3 = -2(11) + 3 = -22 + 3 = -19$

$u_4 = -2(-19) + 3 = 38 + 3 = 41$

$u_5 = -2(41) + 3 = -82 + 3 = -79$

The first five terms are $-4, 11, -19, 41, -79$.

4 We first need to find r:

$r = \frac{-0.1}{0.5} = -0.2$

$0.416 = 0.5\left(\frac{1-(-0.2)^{n-1}}{1--0.2}\right)$

$-0.0016 = -(-0.2)^{n-1}$

$0.0016 = (-0.2)^{n-1}$

By GDC, $n = 5$

5 Renaming the terms:

$u_3 = u_1 = 4.5$

$u_7 = u_5 = 22.78125$

This means we now want to find $u_1 = u_{-1}$

Finding r:

$u_n = u_1 r^{n-1}$

$u_5 = u_1 r^{5-1}$

$22.78125 = 4.5 \times r^4$

$5.0625 = r^4$

$\sqrt[4]{r^4} = \sqrt[4]{5.0625}$

$r = 1.5$

$u_{-1} = u_1 r^{-1-1}$

$u_{-1} = 4.5 \times 1.5^{-2}$

$u_{-1} = 2$

6 **a** An infinite sum can only be found for a converging geometric sequence.

$S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{\frac{1}{4}}{1 - -\frac{1}{2}}$

$S_\infty = \frac{\frac{1}{4}}{1 + \frac{1}{2}}$

$S_\infty = \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{1}{6}$

b $b = \frac{0.12}{0.06} = 2 > 1$ so the series is not converging

7 $u_n = u_1 + (n-1)d$

$61 = 4 + (n-1)(3)$

$61 = 4 + 3n - 3$

$60 = 3n$

$n = 20$

8 $u_4 = 8u_1$

$u_n = u_1 r^{n-1}$

$u_4 = u_1 r^{4-1}$

$8u_1 = u_1 r^3$

$8 = r^3$

$r = 2$

$765 = u_1\left(\frac{1-2^{9-1}}{1-2}\right)$

$765 = u_1\left(\frac{1-2^8}{-1}\right)$

$765 = u_1\left(\frac{1-2^8}{-1}\right)$

$765 = u_1\left(\frac{1-256}{-1}\right)$

$765 = u_1\left(\frac{-255}{-1}\right)$

$765 = 255u_1$

$u_1 = 3$

$u_9 = u_1 r^{9-1}$

$u_9 = 3 \times 2^8$

$u_9 = 768$

9 $\dfrac{6}{x-3}=\dfrac{x+2}{6}$

$36=x^2-x-6$

$0=x^2-x-42$

$0=(x-7)(x+6)$

$x=7$ or $x=-6$

10 a $55, 51.15, 47.5695, 44.239635\ldots$

b It is a geometric sequence because $\dfrac{47.5695}{51.15}=\dfrac{51.15}{55}=0.93$.

c $u_{11}=u_1r^{11-1}$

$u_{11}=55\times 0.93^{10}$

$u_{11}=26.61902\ldots$

$u_{11}\approx 26.6$ litres left in the tank

d $u_{16}=u_1r^{16-1}$

$u_{16}=55\times 0.93^{15}$

$u_{11}=18.5185474\ldots$

$u_{11}\approx 18.5$ litres

$55-18.5=36.5$ litres drained from the tank

e $S_\infty=\dfrac{u_1}{1-r}$

$S_\infty=\dfrac{55}{1-0.93}$

$S_\infty=\dfrac{55}{0.07}$

$S_\infty=785.714285\ldots$

$S_\infty\approx 785$ minutes or 13 hours and 5 minutes

11 a $u_n=u_1+(n-1)d$

$u_n=45+(n-1)(4)$

$u_n=45+4n-4$

$u_n=4n+41$

b $1\text{kg}=1000\text{g}=10$ weights

$u_{10}=4(10)+41$

$u_{10}=40+41$

$u_{10}=81$ cm

c Eventually the spring will hit the ground or the surface it is sitting on, so the length will become constant. Also, the spring could break from too much weight.

d $u_n=4n+41$

$101=4n+41$

$60=4n$

$n=15$

$15\text{x}100\text{g}=1500\text{g}$ or 1.5kg

12 $22960=20987\left(1+\dfrac{r}{12}\right)^{24}$

$1.09401\ldots=\left(1+\dfrac{r}{12}\right)^{24}$

$1+\dfrac{r}{12}=\sqrt[24]{1.09401\ldots}$

$1+\dfrac{r}{12}=1.003750\ldots$

$\dfrac{r}{12}=0.003750\ldots$

$r=0.0450009$

$r\approx 4.5\%$

13 a $1, 13, 78, 286, 715, 1287, 1716,$
$1716, 1287, 715, 286, 78, 13, 1$

Each row in Pascal's triangle is symmetric.

b You can add each set of consecutive in the 14th row to find the 15th row.

$1, (1+13), (13+78), (78+286),$
$(286+715), (715+1287),$
$(1287+1716), (1716+1716),$
$(1716+1287), (1287+715),$
$(715+286), (286+78), (78+13),$
$(13+1), 1$

$1, 14, 91, 364, 1001, 2002, 3003, 3432,$
$3003, 2002, 1001, 364, 91, 14, 1$

14 $\binom{6}{0}(3x)^6(-y)^0+\binom{6}{1}(3x)^5(-y)^1$

$+\binom{6}{2}(3x)^4(-y)^2+\binom{6}{3}(3x)^3(-y)^3$

$+\binom{6}{4}(3x)^2(-y)^4+\binom{6}{5}(3x)^1(-y)^5$

$+\binom{6}{6}(3x)^0(-y)^6$

$=729x^6+6(243x^5)(-y)+15(81x^4)(y^2)$

$+20(27x^3)(-y^3)+15(9x^2)(y^4)$

$+6(3x)(-y^5)+y^6$

$=729x^6-1458x^5y+1215x^4y^2-540x^3y^3$
$+135x^2y^4-18xy^5+y^6$

15 $\binom{8}{3}\left(\frac{3}{x^2}\right)^5\left(-4x^4\right)^3$

$56\left(\frac{243}{x^{10}}\right)\left(-64x^{12}\right)$

$56\left(\frac{243}{x^{10}}\right)\left(-64x^{12}\right)$

$-870912x^2$

The coefficient is -870 912.

16 The sixth term will have be in the form of

$\binom{n}{5}\left(\frac{2}{x^2}\right)^{n-5}\left(-5x^5\right)^5.$

Ignoring any coefficients:

$\left(\frac{1}{x^{2n-10}}\right)\left(x^{25}\right) = x^{25}$

$\left(\frac{1}{x^{2n-10}}\right) = 1$

$x^{-2n+10} = x^0$

$-2n + 10 = 0$

$-2n = -10$

$n = 5$

17 a $\binom{9}{4}(x)^5(-3)^4$

$126\left(x^5\right)(81)$

$10206x^5$

b $-2x\left(10206x^5\right)$

$-20412x^6$

18 To determine which term is the constant, ignore the coefficients for now:

$\left(\frac{x^3}{1}\right)^{12-a}\left(\frac{1}{x}\right)^a = x^0$

$\frac{x^{36-3a}}{x^a} = x^0$

$x^{36-3a-a} = x^0$

$36 - 4a = 0$

$4a = 36$

$a = 9$

$\binom{12}{3}\left(\frac{x^3}{3}\right)^3\left(\frac{k}{x}\right)^9$

$220\left(\frac{x^9}{27}\right)\left(\frac{k^9}{x^9}\right) = \frac{112640}{27}$

$\frac{220k^9}{27} = \frac{112640}{27}$

$220k^9 = 112640$

$k^9 = 512$

$\sqrt[9]{k^9} = \sqrt[9]{512}$

$k = 2$

19 $(2x-1)(x-3) - 3(x-4)^2$

$= 2x^2 - 7x + 3 - 3\left(x^2 - 8x + 16\right)$

$= 2x^2 - 7x + 3 - 3x^2 + 24x - 48$

$= -x^2 + 17x - 45$

20 a $\frac{x^2 - x - 6}{x+4} \times \frac{x^2 - 16}{x^2 + 2}$

$= \frac{(x-3)(x+2)}{x+4} \times \frac{(x-4)(x+4)}{x(x+2)}$

$= (x-3) \times \frac{(x-4)}{x}$

$= \frac{x^2 - 7x + 12}{x}$ QED

b $x \neq -4, -2, 0$

21 a $\left(1 - \frac{x}{4}\right)^5$

$= \binom{5}{0}1^5 + \binom{5}{1}1^4\left(-\frac{x}{4}\right)$

$+ \binom{5}{2}1^3\left(-\frac{x}{4}\right)^2 + \binom{5}{3}1^2\left(-\frac{x}{4}\right)^3$

$+ \binom{5}{4}1^1\left(-\frac{x}{4}\right)^4 + \binom{5}{5}\left(-\frac{x}{4}\right)^5$ M1A1

$= 1 - \frac{5x}{4} + \frac{5x^2}{8} - \frac{5x^3}{32}$

$+ \frac{5x^4}{256} - \frac{x^5}{1024}$ A1

b Substituting $x = 0.1$ M1

$0.975^5 \approx 1 - \frac{5 \times 0.1}{4} + \frac{5 \times 0.1^2}{8}$

$= 1 - \frac{1}{8} + \frac{5}{800}$ A1

$= \frac{800}{800} - \frac{100}{800} + \frac{5}{800}$

$= \frac{705}{800}\left(= \frac{141}{160}\right)$ A1

22 a Using $u_n = a + (n-1)d$ M1

$143 = a + 14d$

$183 = a + 30d$ A1

Solving simultaneously A1

$a = 108$ A1

$d = \frac{5}{2}$ A1

b 100th term is $a+99d$ M1

$=108+99\times\frac{5}{2}$

$=355.5$ A1

23 a Money in the account would be

$3000\times1.015^{10}\ (=\$3482)$ M1A1

Therefore interest gained is

$3000\times1.015^{10}-3000=\$482$ A1

b Total amount is

3000×1.015^{11}

$+(1200\times1.015+1200\times1.015^{2}+\ldots$

$+1200\times1.015^{10})$ M1

$=3000\times1.015^{11}$

$+(1200\times1.015)\left(\frac{1.015^{10}-1}{1.015-1}\right)$

M1A1

$=\$16570$ A1

24 a 5500×1.0275^{4} M1A1

$=\$6130.42$ A1

b Consider $5500\times1.0275^{n-1}=12000$

M1A1

$1.0275^{n-1}=\frac{12000}{5500}$

Using GDC: M1

$n-1=28.76$

$n=29.76$ A1

So Brad must wait 30 years A1

25 Require ($3\times$ coefficient of term in x^5)

$+\left(1\times\text{coefficient of term in } x^4\right)$

$3\times\binom{8}{5}4^3(-2x)^5+1\times\binom{8}{4}4^4(-2x)^4$ M1A1A1

$=3\times(-114688)+1\times286720$

$=-57344$ A1

26 $\binom{n}{2}\left(1^{n-2}\right)(3x)^2=495x^2$ M1A1

$\frac{9n(n-1)}{2}=495$

$n(n-1)=110$

$n^2-n-110=0$ A1

$(n-11)(n+10)=0$ M1A1

So $n=11$ since $n>0$ A1

27 Require $\binom{8}{2}\left(x^3\right)^2\left(-\frac{2}{x}\right)^6$ M1A1

$=28\times(-2)^6$ A1

$=28\times64$

$=1792$ A1

28 a $\left(\frac{1}{2x}-x\right)^4=\binom{4}{0}\left(\frac{1}{2x}\right)^4(-x)^0$

$+\binom{4}{1}\left(\frac{1}{2x}\right)^3(-x)^1+\binom{4}{2}\left(\frac{1}{2x}\right)^2(-x)^2$

$+\binom{4}{3}\left(\frac{1}{2x}\right)^1(-x)^3+\binom{4}{4}\left(\frac{1}{2x}\right)^0(-x)^1$

M1A1

$=x^4+\frac{1}{16x^4}-2x^2-\frac{1}{2x^2}+\frac{3}{2}$ A1

b $(3-x)^3=27-27x+9x^2-x^3$ M1A1

$(3-x)^3\left(\frac{1}{2x}-x\right)^4$

$=\left(27-27x+9x^2-x^3\right)\left(x^4+\frac{1}{16x^4}-2x^2-\frac{1}{2x^2}+\frac{3}{2}\right)$

Therefore required term is $\left(27\times\frac{3}{2}\right)-\frac{9}{2}$

A1

$=36$ A1

29 $120=\frac{a}{1-0.2}$ M1A1

$120=\frac{5a}{4}$

$a=96$ A1

The 6th term is therefore

$96\times0.2^5=0.03072$ M1A1

30 $ar=180$ and $ar^5=\frac{20}{9}$ M1A1

Solving simultaneously M1

$\frac{ar^5}{ar}=\frac{20}{9\times180}=\frac{1}{81}=r^4$

Therefore $r=\frac{1}{3}$ A1

So $a=\frac{180}{r}=\frac{180}{\frac{1}{3}}=540$ A1

$S_\infty=\frac{a}{1-r}=\frac{540}{1-\frac{1}{3}}=\frac{3\times540}{2}=810$ M1A1

31 First part is geometric sum, $a=1$,

$r=1.6$, $n=16$ M1

Second part is arithmetic sum, $a=0$,

$d=-12$, $n=16$ M1

Third part is $16\times1=16$ A1

Geometric sum:

$S_{16}=\frac{1.6^{16}-1}{1.6-1}=3072.791$ A1

Arithmetic sum:

$S_{16}=\frac{16}{2}(2\times0+15\times(-12))=-1440$

A1

So $\sum_{n=0}^{n=15}\left(1.6^n - 12n + 1\right)$

$= 3072.791 - 1440 + 16 = 1648.8$ A1

32 Required distance

$= 20 + \left(2 \times \frac{5}{6} \times 20\right) + \left(2 \times \frac{5}{6} \times \frac{5}{6} \times 20\right) + \text{L}$

M1M1A1

$= 20 + \frac{\frac{100}{3}}{1 - \frac{5}{6}}$ A1

$= 20 + \frac{\frac{100}{3}}{\frac{1}{6}}$

$= 20 + 200$ A1

$= 220$ m

33 $\binom{n-1}{k} + \binom{n-1}{k-1}$

$= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$ M1A1A1

$= \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!}$ A1

$= \frac{n(n-1)! - k(n-1)! + k(n-1)!}{k!(n-k)!}$ A1

$= \frac{n(n-1)!}{k!(n-k)!}$ A1

$\left(= \frac{n!}{k!(n-k)!}\right) = \binom{n}{k}$

34 Consider multiples of 7:

504 is the first multiple and 1400 is the final multiple

$1400 = 504 + 7(n-1)$ M1

$\Rightarrow n = 129$ A1

So the sum of the multiples of 7 is

$S_{129} = \frac{129}{2}\left(2 \times 504 + 7 \times (129-1)\right)$

$= 122808$ M1A1

Sum of the integers from 500 to 1400 (inclusive) is

$S_{901} = \frac{901}{2}\left(2 \times 500 + 1 \times (901-1)\right)$

$= 855950$ M1A1

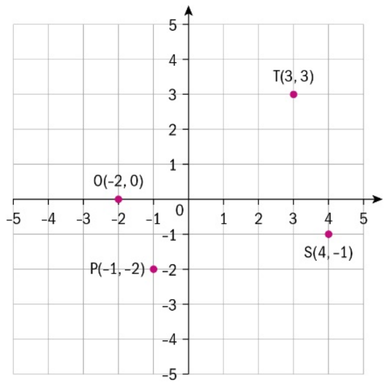
Therefore require

$855950 - 122808 = 733142$ A1

2 Representing relationships: introducing functions

Skills check

1



2 A (3, 0), B (-2, 4), E (-1, 1), R (2, -1)

3 **a** $4(2) - 3(-3) = 17$

b $(2)^2 - (-3)^2 = -5$

c $2 + -3 + -\frac{1}{2} = -\frac{3}{2}$

d $-6\left(-\frac{1}{2}\right)^2 = -\frac{3}{2}$

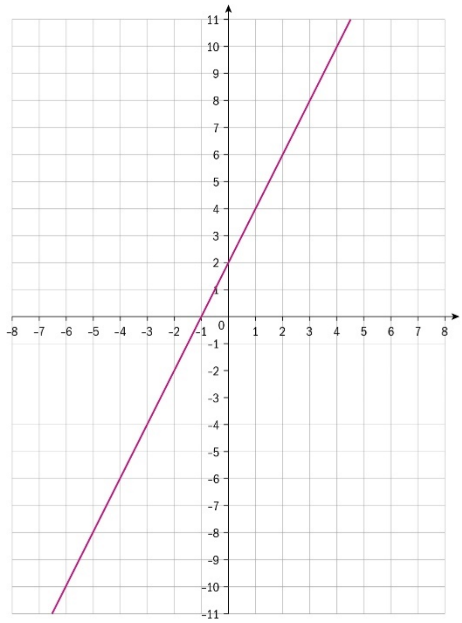
4 **a** $8x = 16 \rightarrow x = 2$

b $4x = 2 \rightarrow x = \frac{1}{2}$

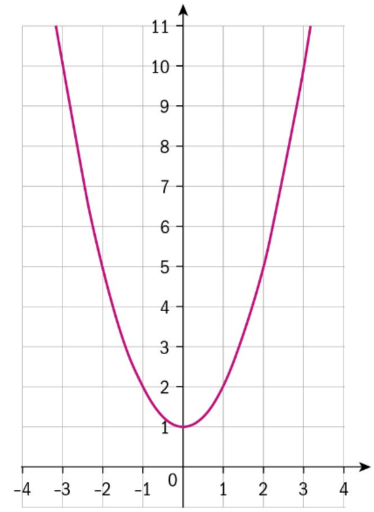
c $x - 10 = 3 \rightarrow x = 13$

d $12x = -12 \rightarrow x = -1$

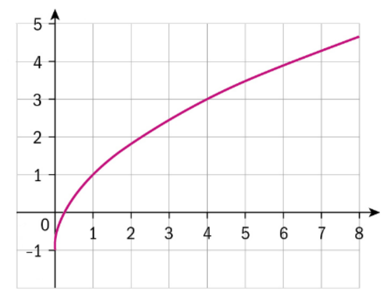
5 **a**



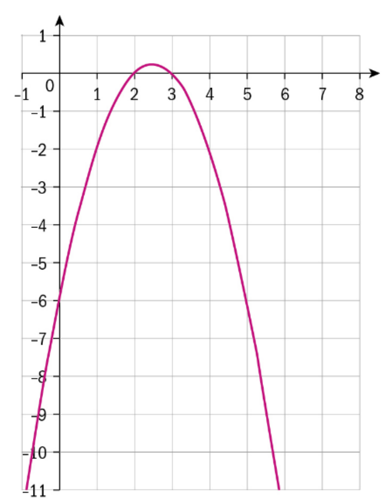
b



c



d



Exercise 2A

1 **a** If the marbles are identical of mass a then the function takes x to ax. This function satisfies the vertical test line.

b This is a function because for s sides of a polygon the sum of the interior angles of the polygon is $(s-2)\cdot 180^0$ which satisfies the vertical test line.

c This is not a function. If a ticket for an adult is 10 pounds and a ticket for a student is 5 pounds then if we have one adult and 2 students the total cost is 20 pounds, for 3 students the total cost is 25 pounds, therefore we will have more than one y-value for the same $x = 1$ coordinate (adult movie tickets purchased).

d This is a function. Each x-coordinate has a unique y-coordinate, $y = x + 1$.

e This is not a function. It does not satisfy the vertical line test for example at $x = 0$.

f This is a function. We see that every x-coordinate has a unique y-coordinate.

g This is a function. Every x-coordinate has a unique y-coordinate.

h This is a function. This is seen when drawing the graph of the function $y = -2x + 6$ and applying the vertical line test.

i This is not a function. We see that for $x = 3$ there are infinitely many y-coordinate values.

j This is a function. This is seen when drawing the graph of the function $y = x^2$ and applying the vertical line test.

k This is a function. Apply the vertical line test.

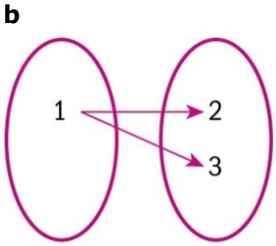
l This is a function. Apply the vertical line test.

m This is not a function. This is showed at $x = -2$ which has 2 distinct y-coordinate values.

2 **a**

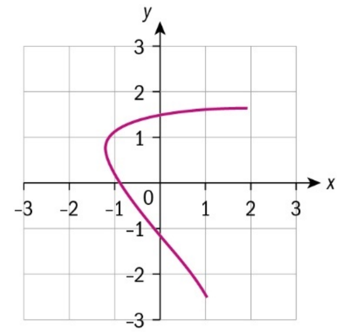
X	2	2	3
Y	0	1	4

b



c $y = \begin{cases} \sqrt{x}, x \geq 0 \\ x, x \geq 0 \end{cases}$

d



3 In a function for every x-coordinate there exists a unique y-coordinate which satisfies the definition of a relation. On the other hand, the examples from the previous questions are all relations but none of them are functions.

"All functions are relations, but not all relations are functions."

Exercise 2B

1 **a** $g(-4) = -(-4)^2 + 2 = -16 + 2 = -14$

b $f(-9) = 5\cdot(-9) - 1 = -45 - 1 = -46$

c $C(100) = 20\cdot 100 + 250$

$= 2000 + 250 = 2250$

d $h(5) = -4$

e $f(2) = 3$

f $f(-3) = 5$

g $f(-1) = 1$

2 **a** $f(-3) = -3(-3)^2 - 1$

$= -3(9) - 1 = -27 - 1 = -28$

b $g(15) = -4(15) + 7 = -60 + 7 = -53$

c $f(1) + g(-1) = -3(1)^2 - 1 + -4(-1) + 7$

$= -3 - 1 + 4 + 7 = 7$

d 6

e $f(x-2) = -3(x-2)^2 - 1$

$= -3(x^2 - 4x + 4) - 1$

$= -3x^2 + 12x - 12 - 1$

$= -3x^2 + 12x - 13$

f $g(n) = -4n + 7$

g $f(1) \times h(1) = \left(-3(1)^2 - 1\right)(6)$

$= (-4)(6) = -24$

h $f(x+1) = -3(x+1)^2 - 1$

$= -3(x^2 + 2x + 1) - 1$

$= -3x^2 - 6x - 3 - 1$

$= -3x^2 - 6x - 4$

$g(x-2) = -4(x-2) + 7$

$= -4x + 8 + 7$

$= -4x + 15$

$f(x+1) \times g(x-2)$

$= (-3x^2 - 6x - 4)(-4x + 15)$

$= 12x^3 - 45x^2 + 24x^2 - 90x + 16x - 60$

$= 12x^3 - 21x^2 - 74x - 60$

3 a Yes, it is a function. Every value of t will yield only one value of d.

b $d(t) = -75t + 275$

c $d(0) = -75(0) + 275 = 275\,\text{km}$

d $0 < t < n$, where n is the amount of time it takes to drive to Perth.

4 a Yes, it is a function. Every temperature in Celsius will only yield one temperature in Fahrenheit.

b $F(17)$ is asking what temperature in ºF is equivalent to 17ºC.

$F(17) = \frac{9}{5}(17) + 32 = 62.6\,^{o}\text{F}$.

c $F(C) = 100$ is asking what temperature in ºC is equivalent to 100ºF.

$\frac{9}{5}C + 32 = 100$

$\frac{9}{5}C = 68$

$C = 37.\overline{7} \approx 37.8\,^{0}\text{C}$

d $F(0) = \frac{9}{5}(0) + 32 = 32\,^{o}\text{F}$

e $F(100) = \frac{9}{5}(100) + 32 = 212\,^{o}\text{F}$

f $F(38.75) = \frac{9}{5}(38.75) + 32 = 101.75\,^{o}\text{F}$

g $\frac{9}{5}C + 32 = 350$

$\frac{9}{5}C = 318$

$C = 176.\overline{6} \approx 177\,^{o}\text{C}$

5 a $C(g) = 10g + 25$

b $g < 0$

c $C(14) = 10(14) + 25 = 165$

d $C(g) = 100$,

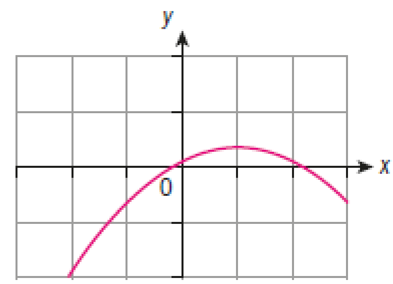
$10g + 25 = 100$

$10g = 75$

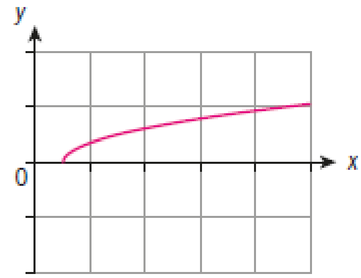
$g = 7.5\,\text{gigs}$

Exercise 2C

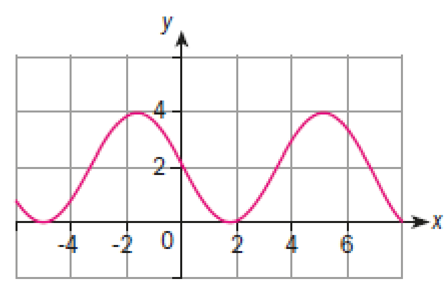
1 a



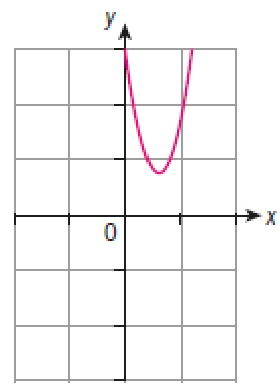
b



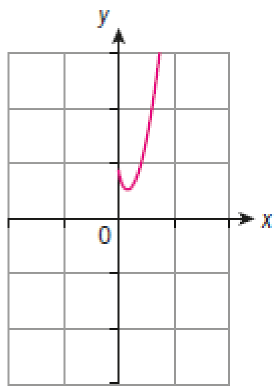
c



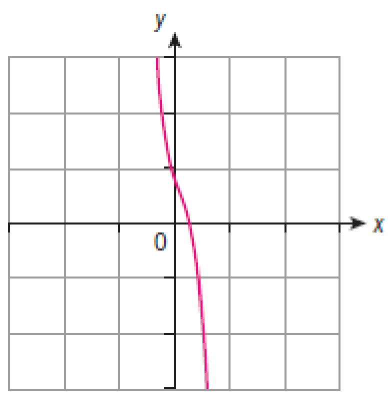
2 a



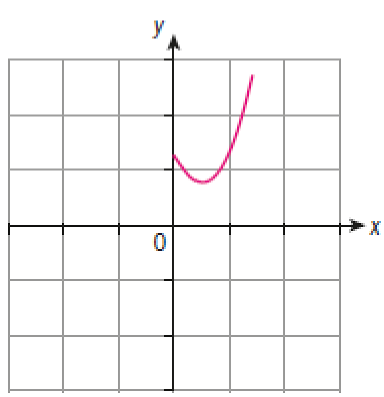
b



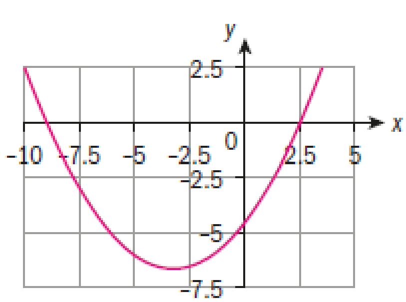
c



d

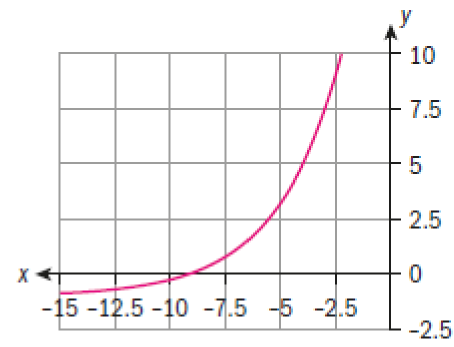


3 a



x-intercepts: (-9,0), (2.5,0)
y-intercepts: (0, -4.5)
minimum: (-3.25, -6.61)

b



x-intercept: (-9.17,0),
y-intercept: (0, 23)

Exercise 2D

1 a Not a function

b Domain: $\{-5,-2,3\}$

Range: $\{4,6,14\}$

c Domain: $\{-12,-8,-5\}$

Range: $\{-8,7\}$

d Not a function

e Domain: $x \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

Range: $y \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

f Domain: $x \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

Range: $\{4\}$

g Not a function

h Domain: $x \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

Range: $3 \le y \le 5, [3,5]$

i Domain: $x \ge 0, [0,\infty[$ or $[0,\infty)$

Range: $y \le 0,]-\infty,0]$ or $(-\infty,0]$

j Domain: $x \ge 2.5, [2.5,\infty[$ or $[2.5,\infty)$

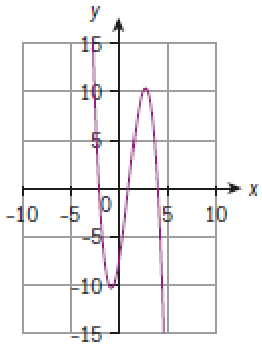
Range: $y \ge 0, [0,\infty[$ or $[0,\infty)$

k Domain: $x \le -1$ or $x \ge 1,]-\infty,-1] \cup [1,\infty[$ or $(-\infty,-1] \cup [1,\infty)$

Range: $y \ge 3, [3,\infty[$ or $[3,\infty)$

L It is not a function. For $x = 0$, $f(0) = 0$ and $f(0) = 4$.

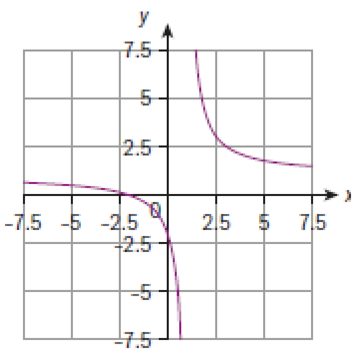
2 a



Domain: $x \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

Range: $y \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

b



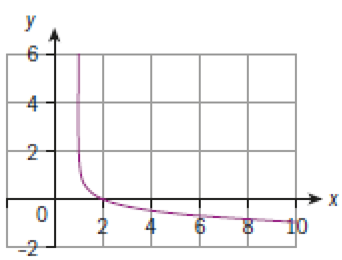
Domain: $x \in \mathrm{R}$, $x \neq 1$, $]-\infty,1[\cup]1,\infty[$

or $(-\infty,1)\cup(1,\infty)$

Range: $y \in \mathrm{R}$, $y \neq 1$, $]-\infty,1[\cup]1,\infty[$

or $(-\infty,1)\cup(1,\infty)$

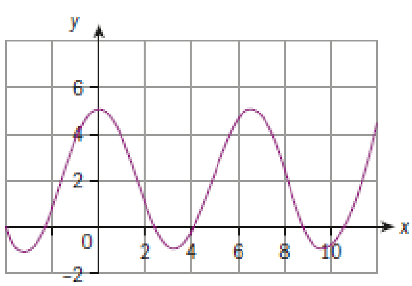
c



Domain: $x > 1$, $]1,\infty[$ or $(1,\infty)$

Range: $y \in$ ¡ , $]-\infty,\infty[$ or $(-\infty,\infty)$

d

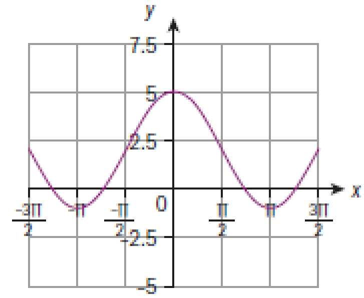


Domain: $x \in \mathrm{R}$, $]-\infty,\infty[$ or $(-\infty,\infty)$

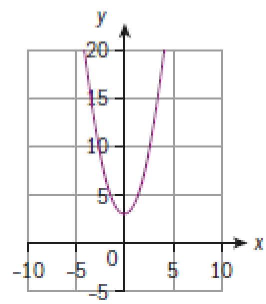
Range: $-1 \leq y \leq 5$, $[-1,5]$

3 Answer will vary.

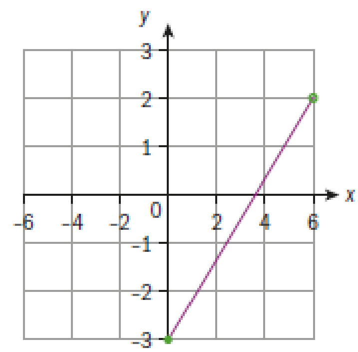
a



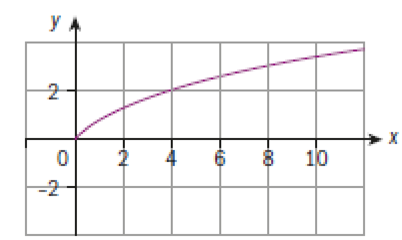
b



c



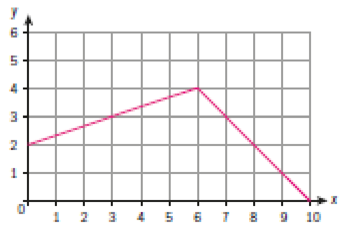
d



4 a i $f(6) = \frac{6}{3} + 2 = 4$

ii $f(8) = -8 + 10 = 2$

b



c $0 \leq x \leq 10$; $0 \leq y \leq 4$

5 $f(x) = \begin{cases} -x, & -3 \leq x \leq -1 \\ 2x+3, & -1 < x \leq 2 \\ 7, & 2 < x \leq 6 \end{cases}$

It is also possible to include -1 in the second interval rather than the first and 2 in the third interval rather than the second.

Exercise 2E

1 a $C(n) = 40 + 21n$, where C is the cost and n is the number of hours.

b Domain: $n \geq 0, [0, \infty[$ or $[0, \infty)$

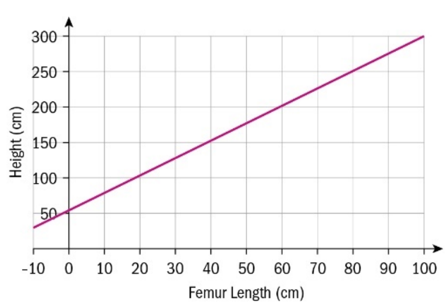
Range: $C(n) \geq 40, [40, \infty[$ or $(40, \infty)$

c $C(4) = 40 + 21(4)$

$C(4) = 40 + 84$

$C(4) = \$124$

2 a



b Domain e.g: $10 \leq f \leq 80$

Range e.g: $75.8 \leq h \leq 228$

c $h(51) = 2.47(51) + 54.10$

$h(51) = 180.07$

$h(51) \approx 180$ cm

d $161 = 2.47f + 54.10$

$2.47f = 106.9$

$f = 43.27935\ldots$

$f \approx 43.3$ cm

3 a $I(t) = 10000\left(1 + \frac{0.025}{12}\right)^{12t}$

b The equation satisfies the vertical line test.

c Domain: $t \in$ ¡ $, t \geq 0$

Range: $[10000, \infty)$.

d $20000 = 10000\left(1 + \frac{0.025}{12}\right)^{12t}$

$2 = \left(1 + \frac{0.025}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \frac{0.025}{12}\right)$

$12t = \dfrac{\ln 2}{\ln\left(1 + \frac{0.025}{12}\right)}$

$12t = 333.0571$

$t = 27.755\ldots$

Javier needs 27 years and 10 months to double his money.

Exercise 2F

1 a i $f(g(x)) = -(4x-2)^2 + 5(4x-2)$

$f(g(x)) = -(16x^2 - 16x + 4)$
$+ 5(4x-2)$

$f(g(x)) = -16x^2 + 16x - 4$
$+ 20x - 10$

$f(g(x)) = -16x^2 + 36x - 14$

ii $f(f(x)) = -(-x^2 + 5x)^2 + 5(-x^2 + 5x)$

$f(f(x)) = -(x^4 - 10x^3 + 25x^2)$
$+ 5(-x^2 + 5x)$

$f(f(x)) = -x^4 + 10x^3 - 25x^2$
$- 5x^2 + 25x$

$f(f(x)) = -x^4 + 10x^3 - 30x^2 + 25x$

iii $f(h(x)) = -(\sqrt{x} + 1)^2 + 5(\sqrt{x} + 1)$

$f(h(x)) = -(x + 2\sqrt{x} + 1) + 5(\sqrt{x} + 1)$

$f(h(x)) = -x - 2\sqrt{x} - 1 + 5\sqrt{x} + 5$

$f(h(x)) = -x + 3\sqrt{x} + 4$

iv $g \circ h(x) = 4(\sqrt{x} + 1) - 2$

$g \circ h(x) = 4\sqrt{x} + 4 - 2$

$g \circ h(x) = 4\sqrt{x} + 2$

v $f(-1) = -(-1)^2 + 5(-1) = -1 - 5 = -6$

$f \circ f(-1) = f(-6)$

$= -(-6)^2 + 5(-6)$

$= -36 - 30 = -66$

$f \circ f \circ f(-1) = f(-66)$

$= -(-66)^2 + 5(-66)$

$= -4356 - 330$

$= 4686$

vi $g(h(9)) = 4\left(\sqrt{9} + 1\right) - 2$

$g(h(9)) = 4(3 + 1) - 2$

$g(h(9)) = 4(4) - 2$

$g(h(9)) = 16 - 2$

$g(h(9)) = 14$

vii. $g \circ f(2) = 4\left(-(2)^2 + 5(2)\right) - 2$

$g \circ f(2) = 4(-4 + 10) - 2$

$g \circ f(2) = 4(6) - 2$

$g \circ f(2) = 24 - 2$

$g \circ f(2) = 22$

$f \circ g(2) = -(4(2) - 2)^2 + 5(4(2) - 2)$

$f \circ g(2) = -(8 - 2)^2 + 5(8 - 2)$

$f \circ g(2) = -(6)^2 + 5(6)$

$f \circ g(2) = -36 + 30$

$f \circ g(2) = -6$

$g \circ f(2) + f \circ g(2) = 22 - 6 = 16$

b **i** $x \in \mathbb{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

ii $x \in \mathbb{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

iii $x \geq 0$ or $[0, \infty[$ or $[0, \infty)$

iv $x \geq 0$ or $[0, \infty[$ or $[0, \infty)$

2 Answers will vary

3 **a** $f(g(x)) = -2(4x - 1) + 5$

$f(g(x)) = -8x + 2 + 5$

$f(g(x)) = -8x + 7$

b $f(g(x)) = -8x + 7 = 12$

$-8x = 5$

$x = -\frac{5}{8}$

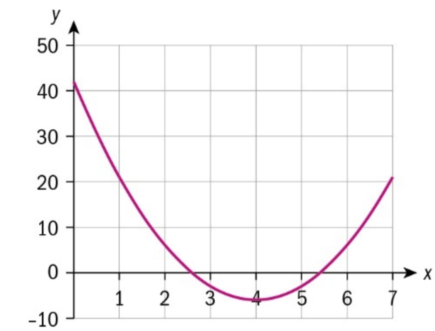
4 **a** $f(g(x)) = 3(-x + 4)^2 - 6$

$f(g(x)) = 3(x^2 - 8x + 16) - 6$

$f(g(x)) = 3x^2 - 24x + 48 - 6$

$f(g(x)) = 3x^2 - 24x + 42$

b



c Domain: $x \in \mathbb{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

Range: $y \geq -6$ or $[-6, \infty[$ or $[-6, \infty)$

5 **a** $f(x) = x + 25$

b $g(x) = 1.06x$

c $f(g(x)) = 1.06x + 25$; this represents only paying tax on the price of the fridge.

$g(f(x)) = 1.06(x + 25)$; this represents paying tax on both the price of the fridge and the delivery fee.

d $f(g(x))$

Exercise 2G

1 **a** **i** $g(-2) = 3$

$f(g(-2)) = f(3) = 2$

ii $f(5) = 6$

$g(f(5)) = g(6) = -3$

iii $g(3) = 11$

$g(g(3)) = g(11) = 0$

b For $f(x)$,

domain: $\{2, 3, 5, 10\}$

range: {-4, 1, 2, 6}

For $g(x)$:

Domain: {−2, 3, 6, 11}

Range: {−3, 0, 3, 11}

2 a $g(3)=7$

$f \circ g(3)=f(7)=-2$

b $f(-1)=9$

$g \circ f(-1)=g(9)=-4$

c $f(9)=-1$

$f \circ f(9)=f(-1)=9$

3 a $f(0)=0$

$g(f(0))=g(0)=-4$

b $f(1)=1$

$g(f(1))=g(1)=-3$

c $g(-2)=0$

$f(g(-2))=f(0)=0$

d $g(-0)=-4$

$f(g(0))=f(-4)=4$

4 Answers will vary; $g(x)$ contain a point with an x-coordinate of -1. ; $f(x)$ must have a point with a y-coordinate of 2.

Exercise 2H

1 a $f(n)=n-100$

$g(n)=2.20n$

b $f(n)$ represents that you receive commission on every new person who signs up after the first 100 people.

$g(n)$ represents that you receive 2.20 GBP for each person (after the first 100) who sign up.

c i $f(224)=224-100=124$

ii $g(124)=2.20(124)=272.80\text{ GBP}$

d i $S(276)=2.20(276-100)$

$=2.20(176)=387.20\text{ GBP}$

ii $114.40=2.20(x-100)$

$52=x-100$

$x=152$ people

2 a $f(x)=x-25$; this could represent \$25 off the price of the TV

$g(x)=1.10x$; this could represent a tax of 10%.

b i You paid $699.99-25=\$674.99$.

ii After tax, the TV cost

$1.10\times 674.99=742.489$

$\approx \$742.49$

c i $P(x)=1.10(x-25)+49.99$

ii $P(525.99)$

$=1.10(525.99-25)+49.99$

$=601.079\approx \$601.08$

3 Answer will vary.

Exercise 2I

1 a $f(g(x))=-2\left(\frac{1}{2}x+2\right)+2$

$f(g(x))=-x-4+2$

$f(g(x))=-x-2$

Since $f(g(x))\neq x$, these are not inverses.

b $f(g(x))=2\left(\sqrt[3]{\frac{x-3}{2}}\right)^3+3$

$f(g(x))=2\left(\frac{x-3}{2}\right)+3$

$f(g(x))=x-3+3$

$f(g(x))=x$

$g(f(x))=\sqrt[3]{\frac{(2x^3+3)-3}{2}}$

$g(f(x))=\sqrt[3]{\frac{2x^3}{2}}$

$g(f(x))=\sqrt[3]{x^3}$

$g(f(x))=x$

Since $f(g(x))=g(f(x))=x$, these are inverses.

c $f(g(x))=\sqrt{3\left(\frac{x^2}{3}+\frac{2}{3}\right)-2}$

$f(g(x))=\sqrt{x^2+2-2}$

$f(g(x))=\sqrt{x^2}$

$f(g(x))=x$

$g(f(x))=\frac{\left(\sqrt{3x-2}\right)^2}{3}+\frac{2}{3}$

$$g(f(x)) = \frac{3x-2}{3} + \frac{2}{3}$$

$$g(f(x)) = \frac{3x-2+2}{3}$$

$$g(f(x)) = \frac{3x}{3}$$

$$g(f(x)) = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

d $$g(h(x)) = -\frac{3}{4}\left(-\frac{4x-20}{3}\right) + 5$$

$$g(h(x)) = \frac{12x-60}{12} + 5$$

$$g(h(x)) = x - 5 + 5$$

$$g(h(x)) = x$$

$$h(g(x)) = -\frac{4\left(-\frac{3}{4}x+5\right)-20}{3}$$

$$h(g(x)) = -\frac{-3x+20-20}{3}$$

$$h(g(x)) = \frac{3x}{3}$$

$$h(g(x)) = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

2 x-intercept: $0 = -4x + 2$

$$x = \frac{1}{2}$$

$$\therefore \left(\frac{1}{2}, 0\right)$$

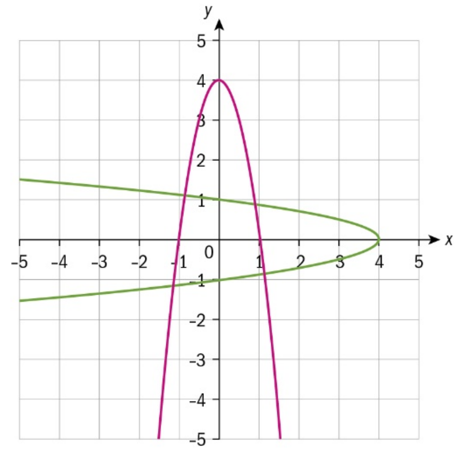
y-intercept: $y = -4(0) + 2$

$$y = 2$$

$$\therefore (0, 2)$$

Since you only need two points to graph a line, you can switch the coordinates to find two point that the inverse passes through: $\left(0, \frac{1}{2}\right)$ and $(2, 0)$.

3 a i & ii



iii $f(x) = -4x^2 + 4$

$$y = -4x^2 + 4$$

$$x = -4y^2 + 4$$

$$4y^2 = -x + 4$$

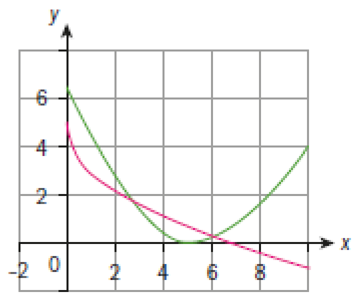
$$y^2 = -\frac{x}{4} + 1$$

$$\sqrt{y^2} = \pm\sqrt{-\frac{x}{4} + 1}$$

$$y = \pm\sqrt{-\frac{x}{4} + 1}$$

$$f^{-1}(x) = \pm\sqrt{-\frac{x}{4} + 1}$$

b i & ii



iii $g(x) = -2\sqrt{x} + 5$

$$y = -2\sqrt{x} + 5$$

$$x = -2\sqrt{y} + 5$$

$$2\sqrt{y} = -x + 5$$

$$\sqrt{y} = \frac{-x+5}{2}$$

$$\left(\sqrt{y}\right)^2 = \left(\frac{-x+5}{2}\right)^2$$

$$g^{-1}(x) = \left(\frac{-x+5}{2}\right)^2, x \geq 0$$

Note 1: The domain restriction is needed since the original function $g(x) = -2\sqrt{x} + 5$ would have the same restriction.

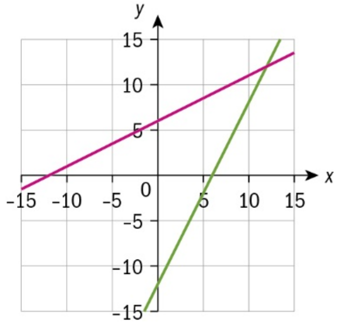
Note 2: The inverse, $g^{-1}(x)$, can be simplified further if desired:

$$y = \frac{(-x+5)^2}{4}$$

$$y = \frac{x^2 - 10x + 25}{4}$$

$$y = \frac{1}{4}x^2 - \frac{5}{2}x + \frac{25}{4}, x \geq 0$$

c **i & ii**



iii $g : x \to \frac{1}{2}x + 6$

$$y = \frac{1}{2}x + 6$$

$$x = \frac{1}{2}y + 6$$

$$\frac{1}{2}y = x - 6$$

$$y = 2x - 12$$

$$g^{-1}(x) = 2x - 12$$

4 **a** **i** Domain: $x \geq 2.875$ or $[2.875, \infty[$ or $[2.875, \infty)$.

Range: $y \geq 1.25$ or $[1.25, \infty[$ or $[1.25, \infty)$

ii Domain: $x \in \mathrm{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

Range: $y \in \mathrm{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

iii Domain: $x > 3$ or $]3, \infty[$ or $(3, \infty)$

Range: $y \in \mathrm{R}$ or $]-\infty, \infty[$ or $(-\infty, \infty)$

iv Domain: $x \leq 1$ or $[-\infty, 1]$ or $(-\infty, 1]$

Range: $y \leq 2$ or $[-\infty, 2]$ or $(-\infty, 2]$

b The domain of the function becomes the range of its inverse, and the range of the function becomes the domain of its inverse.

5 Answers will vary. In order for the function to be a one-to-one function, the inverse must be a function.

6 **a** $f(x) = 2x - 5 = 11$

$$2x - 5 = 11$$

$$2x = 16$$

$$x = 8$$

b $f(x) = 2x - 5$

$$y = 2x - 5$$

$$x = 2y - 5$$

$$2y = x + 5$$

$$y = \frac{x+5}{2}$$

$$f^{-1}(x) = \frac{x+5}{2}$$

c $f^{-1}(11) = \frac{11+5}{2}$

$$f^{-1}(11) = \frac{16}{2}$$

$$f^{-1}(11) = 8$$

d $f(x) = 11$ gives the same answer as $f^{-1}(11)$.

e $f(x) = y \Leftrightarrow x = f^{-1}(y)$

7 $f(x) = -2x - 1$

$$y = -2x - 1$$

$$x = -2y - 1$$

$$2y = -x - 1$$

$$y = \frac{-x-1}{2}$$

$$f^{-1}(x) = \frac{-x-1}{2}$$

$$g \circ f^{-1}(x) = -3\left(\frac{-x-1}{2}\right)^2$$

$$g \circ f^{-1}(x) = -3\frac{(-x-1)^2}{4}$$

$$g \circ f^{-1}(x) = -3\frac{(x^2+2x+1)}{4}$$

$$g \circ f^{-1}(x) = \frac{-3x^2-6x-3}{4}$$

$$g\circ f^{-1}(-1)=\frac{-3(-1)^2-6(-1)-3}{4}$$

$$g\circ f^{-1}(-1)=\frac{-3(1)-6(-1)-3}{4}$$

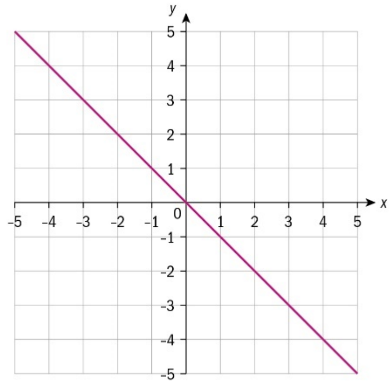
$$g\circ f^{-1}(-1)=\frac{-3+6-3}{4}$$

$$g\circ f^{-1}(-1)=\frac{0}{4}$$

$$g\circ f^{-1}(-1)=0$$

Exercise 2J

1



2 **a** $f(f(x))=x$

$3-(3-x)=x$

$3-3+x=x$

$x=x$

b $f(f(x))=x$

$-2-(-2-x)=x$

$-2+2+x=x$

$x=x$

c $f(f(x))=x$

$\frac{1}{2}-\left(\frac{1}{2}-x\right)=x$

$\frac{1}{2}-\frac{1}{2}-x=x$

$x=x$

d $f(x)=n-x, n\in\mathbb{R}$ is a self-inverse.

3 $f(f(x))=x$

$$\frac{-\left(\frac{-x-2}{5x+1}\right)-2}{5\left(\frac{-x-2}{5x+1}\right)+1}=x$$

$$\frac{\frac{(x+2)}{(5x+1)}-\frac{2(5x+1)}{(5x+1)}}{\frac{(-5x-10)}{(5x+1)}+\frac{(5x+1)}{(5x+1)}}=x$$

$$\frac{\frac{x+2-10x-2}{5x+1}}{\frac{-9}{5x+1}}=x$$

$$\frac{-9x}{-9}=x$$

$x=x$

4 $f(f(x))=x$

$$\frac{2\left(\frac{2x-4}{x+m}\right)-4}{\left(\frac{2x-4}{x+m}\right)+m}=x$$

$$\frac{\left(\frac{4x-8}{x+m}\right)-4\left(\frac{x+m}{x+m}\right)}{\left(\frac{2x-4}{x+m}\right)+m\left(\frac{x+m}{x+m}\right)}=x$$

$$\frac{\frac{4x-8-4x-4m}{x+m}}{\frac{2x-4+xm+m^2}{x+m}}=x$$

$$\frac{-4m-8}{2x-4+xm+m^2}=x$$

$-4m-8=x\left(2x-4+xm+m^2\right)$

$-4m-8=2x^2-4x+x^2m+m^2x$

$-4m-8=(2+m)x^2+(m^2-4)x$

Since there is no x^2 on the left-hand side:

$2+m=0$

$m=-2$

No solution because for m = -2 we get the constant function $y=2$, which has no inverse.

Chapter review

1 **a** Yes **b** No **c** Yes **d** No
e Yes **f** Yes **g** Yes **h** No
i Yes **j** Yes **k** Yes **l** No
m No

2 **a** Domain: {-5, -1, 0, 1, 4, 9}
Range: {-8, -1, 0, 1, 6, 9}

b Domain: {0, 2, 3, 4}
Range: {1, 2}

c Domain: {-8, -5, 0, 1}
Range: {-2, 2, 3}

d Domain: $x\in\mathbb{R}$ or $(-\infty,\infty)$ or $]-\infty,\infty[$

Range: $y \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

e Domain: $-3 \le x \le 3$ or $[-3, 3]$

Range: $-3 \le x \le -1$ or $[-3, -1]$

f Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $x \ge -12.25$ or $[12.25, \infty)$ or $[12.25, \infty[$

g Domain: $x \ge 0$ or $[0, \infty)$ or $[0, \infty[$

Range: $y \le 1$ or $(-\infty, 1]$ or $]-\infty, 1]$

h Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $x \ge 5$ or $[5, \infty)$ or $[5, \infty[$

3 a $f(3) = 3^2 - 6$

$f(3) = 9 - 6$

$f(3) = 3$

b $f(-2) = (-2)^2 - 6$

$f(-2) = 4 - 6$

$f(-2) = -2$

c $g(-6) = -2(-6)$

$g(-6) = 12$

d $f(1) + h(2) = (1)^2 - 6 - 4$

$f(1) + h(2) = 1 - 6 - 4$

$f(1) + h(2) = -9$

e $2f(0) - 2g(-1) = 2(0^2 - 6) - 2(-2(-1))$

$2f(0) - 2g(-1) = -12 - 2(2)$

$2f(0) - 2g(-1) = -12 - 4$

$2f(0) - 2g(-1) = -16$

f $h(0) \times f(-1) = -4((-1)^2 - 6)$

$h(0) \times f(-1) = -4(1 - 6)$

$h(0) \times f(-1) = -4(-5)$

$h(0) \times f(-1) = 20$

g $g(x) = -2x$

$y = -2x$

$x = -2y$

$y = \dfrac{x}{-2}$

$g^{-1}(x) = \dfrac{x}{-2}$

$g^{-1}(-3) = \dfrac{-3}{-2}$

$g^{-1}(-3) = \dfrac{3}{2}$

h $f(g(x)) = (-2x)^2 - 6$

$f(g(x)) = 4x^2 - 6$

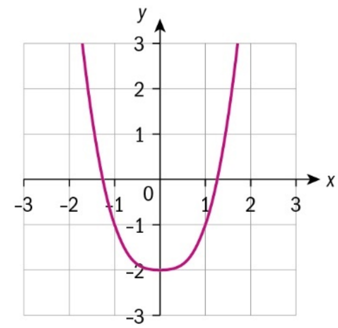
i $f \circ g^{-1}(x) = \left(\dfrac{x}{-2}\right)^2 - 6$

$f \circ g^{-1}(x) = \dfrac{x^2}{4} - 6$

$f \circ g^{-1}(x) = \dfrac{x^2}{4} - \dfrac{24}{4}$

$f \circ g^{-1}(x) = \dfrac{x^2 - 24}{4}$

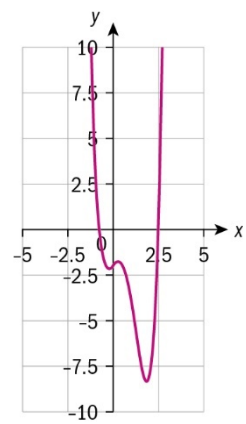
4 a



Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $y \ge -2$ or $[-2, \infty)$ or $[-2, \infty[$

b



Domain: $x \in \mathbb{R}$ or $(-\infty, \infty)$ or $]-\infty, \infty[$

Range: $y \ge -8.38$ or $[-8.38, \infty)$ or $[-8.38, \infty[$

5 a $f(g(x)) = -4\left(-\dfrac{x-2}{4}\right) + 2$

$$f(g(x)) = x - 2 + 2$$

$$f(g(x)) = x$$

$$g(f(x)) = -\frac{(-4x+2)-2}{4}$$

$$g(f(x)) = \frac{4x-2+2}{4}$$

$$g(f(x)) = \frac{4x}{4}$$

$$g(f(x)) = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

b $$f(g(x)) = \frac{1}{2}\left(-\frac{x-2}{4}\right) - 4$$

$$f(g(x)) = \frac{-x+2}{8} - 4$$

$$f(g(x)) = \frac{-x+2}{8} - \frac{32}{8}$$

$$f(g(x)) = \frac{-x-30}{8}$$

Since $f(g(x)) \neq x$, these are not inverses.

c $$f(g(x)) = \frac{1}{2}\left(2x + \frac{1}{4}\right)^2 + 4$$

$$f(g(x)) = \frac{1}{2}\left(4x^2 + x + \frac{1}{16}\right) + 4$$

$$f(g(x)) = 2x^2 + \frac{1}{2}x + \frac{1}{32} + 4$$

Since $f(g(x)) \neq x$, these are not inverses.

d $$f(g(x)) = \frac{2\left(\frac{3+x}{3x-2}\right) + 3}{3\left(\frac{3+x}{3x-2}\right) - 1}$$

$$f(g(x)) = \frac{\frac{6+2x}{3x-2} + 3}{\frac{9+3x}{3x-2} - 1}$$

$$f(g(x)) = \frac{\frac{6+2x}{3x-2} + \frac{9x-6}{3x-2}}{\frac{9+3x}{3x-2} - \frac{3x-2}{3x-2}}$$

$$f(g(x)) = \frac{\frac{11x}{3x-2}}{\frac{11}{3x-2}}$$

$$f(g(x)) = \frac{11x}{11}$$

$$f(g(x)) = x$$

$$g(f(x)) = \frac{3 + \left(\frac{2x+3}{3x-1}\right)}{3\left(\frac{2x+3}{3x-1}\right) - 2}$$

$$g(f(x)) = \frac{3 + \left(\frac{2x+3}{3x-1}\right)}{\left(\frac{6x+9}{3x-1}\right) - 2}$$

$$g(f(x)) = \frac{\frac{9x-3}{3x-1} + \frac{2x+3}{3x-1}}{\frac{6x+9}{3x-1} - \frac{6x-2}{3x-1}}$$

$$g(f(x)) = \frac{\frac{11x}{3x-1}}{\frac{11}{3x-1}}$$

$$g(f(x)) = \frac{11x}{11}$$

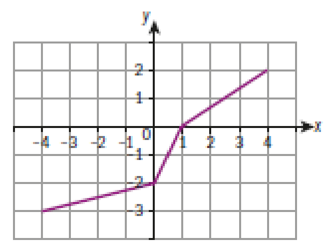
$$g(f(x)) = x$$

Since $f(g(x)) = g(f(x)) = x$, these are inverses.

6 a i -4 **ii** 4

b $-4 \le x \le 4$ or $[-4, 4]$

c



7 $$(f \circ g)(x) = (g(x) + 2)^3$$

$$-8x^6 = (g(x)+2)^3$$

$$\sqrt[3]{-8x^6} = \sqrt[3]{(g(x)+2)^3}$$

$$-2x^2 = g(x) + 2$$

$$g(x) = -2x^2 - 2$$

8 $$(f \circ h^{-1})(-2) = f(h^{-1}(-2))$$

Since $h(16) = -2, h^{-1}(-2) = 16$

$$f(h^{-1}(-2)) = f(16) = 2\sqrt{16} + 16^2$$

$$f(h^{-1}(-2)) = 2(4) + 256$$

$$f(h^{-1}(-2)) = 8 + 256$$

$f\left(h\left(^{-1}(-2)\right)\right) = \; 264$

9 For a function to be a self-inverse, we must show that $f\left(f\left(x\right)\right) = x$:

$$f\left(f\left(x\right)\right) = -\frac{3}{\left(-\frac{3}{x}\right)}$$

$$f\left(f\left(x\right)\right) = -3 \times \frac{-x}{3}$$

$$f\left(f\left(x\right)\right) = x$$

10 a $-24 \le f\left(x\right) \le 26$ A1A1

b $f\left(x\right) = \{-4, -2, 0, 2, 4, 6\}$ A1A1

c $0 \le f\left(x\right) \le 100$ A1A1

d $125 \le f\left(x\right) \le 250$ A1A1

11 a $f\left(-2\right) = \left(4 \times -2\right) - 2 = -8 - 2 = -10$

M1A1

b $g\left(-2\right) = \left(-2\right)^2 - 8\left(-2\right) + 15$

$= 4 + 16 + 15 = 35$ M1A1

c $y = 4x - 2$

$x = 4y - 2$

$y = \frac{x+2}{4}$ M1

$f^{-1}\left(x\right) = \frac{x+2}{4}$ A1

d $x^2 - 8x + 15 = 35$ M1

$x^2 - 8x - 20 = 0$

$\left(x - 10\right)\left(x + 2\right) = 0$ M1

$x = 10$ or $x = -2$ A1A1

12 a $f\left(x\right) = 128\left(\frac{3}{2}\right) - 15 = 177$ M1A1

b $f\left(-3\right) = 128\left(-3\right) - 15 = -399$ M1A1

$f\left(15\right) = 128\left(15\right) - 15 = 1905$ A1

Range is $-399 < f\left(x\right) < 1905$ A1

c Solving $128a - 15 = 1162.6$ M1

$a = 9.2$ A1

13 a Domain is $-3 \le x \le 3$ A1A1

Range is $-1 \le f\left(x\right) \le 1$ A1A1

b Domain is $-1.5 \le x \le 5$ A1A1

Range is $-5 \le f\left(x\right) \le 4$ A1A1

c Domain is $0 \le x \le 24$ A1A1

Range is $0 \le f\left(x\right) \le 12$ A1A1

d Domain is $-3 \le x \le 3$ A1A1

Range is $0 \le f\left(x\right) \le 9$ A1A1

14 a Solving $3x - 10 = 5$ and $3x - 10 = 50$

M1A1A1

Domain is $5 < x < 20$ A1

b $ff\left(10\right) = f\left(f\left(10\right)\right)$ M1

$= f\left(20\right)$ A1

$= 50$ A1

c $f^{-1}\left(x\right) = \frac{x+10}{3}$ M1A1

d Range is $5 < f^{-1}\left(x\right) < 20$ A1A1

15 a NOT a function, since, eg. the value of $x = 5$ is related to more than one co-ordinate on the y-axis A1R1

b This is a function. Each value of x is related to only one value for y

A1R1

c This is a function. Each value of x is related to only one value for y

A1R1

d This is a function. Each value of x is related to only one value for y

A1R1

16 a $y = \frac{k}{x-1} + 1$

$x = \frac{k}{y-1} + 1$

$x\left(y - 1\right) = k + y - 1$ M1

$xy - x = k + y - 1$ A1

$xy - y = k + x - 1$

$y\left(x - 1\right) = k + x - 1$

$y = \frac{k + x - 1}{x - 1}$

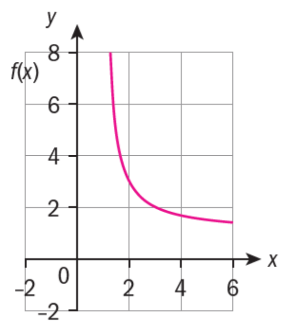
$y = \frac{k}{x-1} + 1$ A1

$f^{-1}\left(x\right) = \frac{k}{x-1} + 1$

So f is self-inverse

b Range is $f\left(x\right) > 1$, $f\left(x\right) \in \mathbb{R}$ A1A1

c



A1A1

17 a Range is $f(x) \neq -\frac{2}{3}$, $(f(x) \in \mathbb{R})$

A1

b $y = \frac{1-2x}{3x+6}$

$x = \frac{1-2y}{3y+6}$

$x(3y+6) = 1-2y$ M1A1

$3xy + 6x = 1-2y$

$2y + 3xy = 1-6x$

$y(2+3x) = 1-6x$

$y = \frac{1-6x}{2+3x}$

$f^{-1}(x) = \frac{1-6x}{2+3x}$ A1

c Domain is, $x \neq -\frac{2}{3}$, $(x \in \mathbb{R})$ A1

Range is $f(x) \neq -2$, $(f(x) \in \mathbb{R})$

A1

18 a $x^2 = 2x - 1$ M1

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$ A1

$x = 1$ A1

b $fg(x) = (2x-1)^2$ A1

$gf(x) = 2x^2 - 1$ A1

$(2x-1)^2 = 2x^2 - 1$ M1

$4x^2 - 4x + 1 = 2x^2 - 1$

$2x^2 - 4x + 2 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$ M1

$x = 1$ A1

19 a $C = 430 + 14.5p$ M1A1

b $f(p)$ is a function since it passes the vertical line test R1

c $C = 430 + 14.5p$

$\frac{C - 430}{14.5} = p$ M1

$f^{-1}(p) = \frac{p - 430}{14.5}$ A1

d $f^{-1}(1000) = \frac{1000 - 430}{14.5} = 39.3$

M1

She can therefore invite a maximum of 39 people A1

e $C = 430 + 14.5 \times 16 = \$662$ M1

$\frac{662}{16} = 41.375$ A1

Katie will therefore need to charge a minimum of $\$41.38$ per head A1

20 a $h(x) \geq 2$, $(h(x) \in \mathbb{R})$ A1

b $y = \frac{x}{3} + 2$

$x = \frac{y}{3} + 2$

$3x = y + 6$ M1A1

$y = 3x - 6$

$h^{-1}(x) = 3x - 6$ A1

c $hh(x) = \frac{\frac{x}{3}+2}{3} + 2$ M1A1

$= \frac{x}{9} + \frac{2}{3} + 2$

$= \frac{x}{9} + \frac{8}{3}$ A1

d $\frac{x}{3} + 2 = 3x - 6$ M1

$\frac{8x}{3} = 8$

$x = 3$ A1

e Because $h(x)$ and $h^{-1}(x)$ both intersect on the line $y = x$ R1

21 $x^2 + 4x - 11 = (x+2)^2 - 15$ M1A1

Therefore $h(x) = x + 2$ A1

$g(x) = x^2$ A1

$f(x) = x - 15$ A1

22 a $(f(x) \in \mathbb{R})$, $f(x) \geq -4$ A1

b $(g(x) \in \mathbb{R})$, $g(x) \neq 0$ A1

c $(h(x) \in \mathbb{R})$, $h(x) > 0$ A1

d $gf(x) = \frac{1}{(x^2-4)+1}$ M1

$= \frac{1}{x^2-3}$ A1

e $\frac{1}{x^2-3} = 9$ M1

$x^2 - 3 = \frac{1}{9}$

$x^2 = \dfrac{28}{9}$

$x = \pm\dfrac{2\sqrt{7}}{3}$ A1

f $gh(x) = \dfrac{1}{2^x + 1}$ M1A1

$\dfrac{1}{2^x + 1} > \dfrac{1}{17}$ M1

$2^x + 1 < 17$

$2^x < 16$ A1

$x < 4$ A1

23 a $-8 \le p(x) \le 8$ A1A1

b $p^{-1}(x) = \sqrt[3]{x}$, $-8 \le x \le 8$, $(x \in \mathbb{R})$

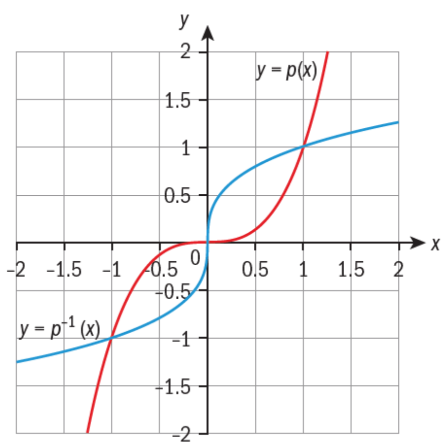
A1A1

c Using GDC, or ex., solving $x^3 = x$

M1

$x = -1$, $x = 0$, $x = 1$ A1

d



A1A1

24 a $y = \dfrac{3x + 5}{4x - 3}$

$x = \dfrac{3y + 5}{4y - 3}$

$x(4y - 3) = 3y + 5$ M1A1

$4xy - 3x = 3y + 5$

$4xy - 3y = 3x + 5$

$y(4x - 3) = 3x + 5$

$y = \dfrac{3x + 5}{4x - 3}$ A1

So $r(x)$ is self-inverse

b $rrrrrr(5) = rrrr(5) = rr(5) = 5$ M1A1

25 a $x^2 - 6x + 13 = (x - 3)^2 + 4$ M1A1

Therefore $k = 3$ A1

b $y = (x - 3)^2 + 4$

$x = (y - 3)^2 + 4$

$(y - 3)^2 = x - 4$ M1A1

$y - 3 = \sqrt{x - 4}$

$y = 3 + \sqrt{x - 4}$

$f^{-1}(x) = 3 + \sqrt{x - 4}$ A1

c The domain of $f^{-1}(x)$ is $x \ge 4$,

$(x \in \mathbb{R})$ A1

The range of $f^{-1}(x)$ is $f(x) \ge 3$,

$(f(x) \in \mathbb{R})$ A1

26 $(x - 3)^2 = x^2 - 6x + 9$ M1

$2(x - 3)^2 = 2x^2 - 12x + 18$ M1A1

$2(x - 3)^2 + 12x = 2x^2 + 18$

Therefore $g(x) = 2x^2 + 12x$ A1

3 Modelling relationships: linear and quadratic functions

Skills Check

1 a $x=-3$ **b** $t=\pm\sqrt{7}$ **c** $a=-\frac{9}{2}$

2 a $3m(m-5)$ **b** $(x+6)(x-6)$ **c** $(n+1)(n+7)$ **d** $(x+1)(4x-3)$

e $9x(x+2)$ **f** $(a+1)(2a-5)$ **g** $(3x+2)(4x-1)$ **h** $(4a+7b)(4a-7b)$

Exercise 3A

1 a Using the points (-1,0) and (1,-1) on the graph, $m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-1)-0}{1-(-1)}=\frac{-1}{2}$

b Using the points (−5,0) and (0,2) on the graph, $m=\frac{y_2-y_1}{x_2-x_1}=\frac{2-0}{0-(-5)}=\frac{2}{5}$

2 a $m=\frac{y_2-y_1}{x_2-x_1}=\frac{11-8}{8-4}=\frac{3}{4}$

b $m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-4)-2}{4-(-2)}=\frac{-6}{6}=-1$

c $m=\frac{y_2-y_1}{x_2-x_1}=\frac{8-1}{7-(-7)}=\frac{7}{14}=\frac{1}{2}$

3 As the line joining the scatter plot (drawn up with t on the x-axis and h on the y-axis) is linear, the gradient can be found by using any two points in the scatterplot: $m=\frac{h_2-h_1}{t_2-t_1}=\frac{(4.15)-4.3}{30-20}=\frac{-0.15}{10}=-0.015$. This is the rate of change of the height of the candle, i.e. how fast it is burning down in cm/s.

4 a You can use the Pythagorean theorem to find the coordinate of B: as the elevation of B above A is 70m and the direct distance is 350m,

$x_B=\sqrt{350^2-70^2}=\sqrt{122500-4900}$.

$=\sqrt{117600}\approx 342.93$

Coordinates of B are $(342.93,100)$.

b $m=\frac{y_2-y_1}{x_2-x_1}\approx\frac{100-30}{342.93}$

$=\frac{70}{342.93}\approx 0.20$

c As the gradient is given by $\frac{\text{rise}}{\text{run}}$ itself,

$\text{grade}=\text{gradient}\times 100\%\approx 20\%$.

Exercise 3B

1 a They are not parallel, as their gradients are not the same, and not perpendicular, as both gradients are positive.

b They are parallel, as $m_1=-4=m_2$.

2 As $m*\frac{4}{3}=-1, m=\frac{-3}{4}$. Therefore

$\frac{-3}{4}=\frac{5-2}{x-3}=\frac{3}{x-3}$, which is rearranged to

$x-3=\frac{3*4}{-3}=-4$, yielding $x=-1$.

3 a For the first segment, the gradient is given as $m_1=\frac{320-0}{40-0}=\frac{320}{40}=8$. The gradient of the second segment $m_2=\frac{560-320}{60-40}=\frac{240}{20}=12$.

b This shows that Liam earns 8 dollars per hour regular wage (for the first 40 hours) and 12 dollars per hour worked overtime.

Exercise 3C

1 a The gradient is 3, y-intercept is −7.

b The gradient is $-\frac{2}{3}$, y-intercept is 4.

c This could be written as $y=0x-2$; thus, the gradient is 0 and the y-intercept is −2.

2 $y=\frac{1}{5}x+1$ as the gradient is $\frac{1}{5}$ and the y-intercept is 1

3 a The gradient is equal to the gradient of $y=4x-3$, which is 4, and the y-intercept is -1. Thus $y=4x-1$.

b $m=\frac{12}{4}=3$ and thus $3(1)+a=10$.

Therefore $a=10-3=7$.

Thus $y=3x+7$

4 a The x-coordinate remains constant so the equation is $x=8$.

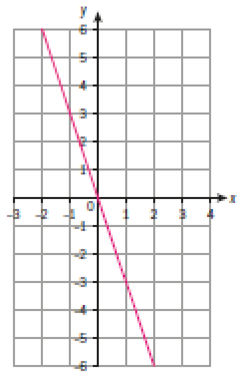
b The y-coordinate remains constant so the equation is $y=-10$

c As horizontal lines are perpendicular to vertical lines, the line is vertical and the equation is $x=9$.

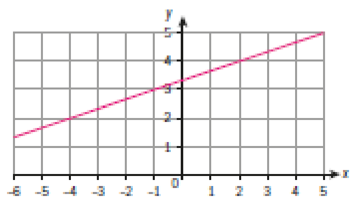
d The lines intersect at the point where $x=-2$ and $y=7$.

Exercise 3D

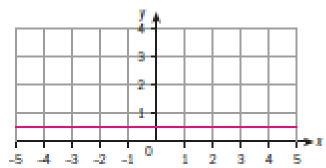
1 **a** The line goes through $(0,0)$ and through $(1,-3)$.



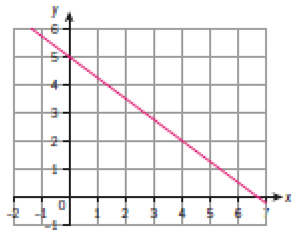
b The point $(-4,2)$ is on the line and so is $(-4+3,2+1)=(-1,3)$.



c The line is horizontal at $y=\frac{1}{2}$



d The line goes through $(0,5)$ and through $(4,2)$.



2 $y-6=-3(x-2)$

3 **a** $m=\frac{6}{-2}=-3$

b $y+4=-3(x+3)$ and $y-2=-3(x+5)$ corresponding to the two points given.

c $y+4=-3(x+3)$

$\leftrightarrow y=-3(x+3)-4=-3x-9-4$

$=-3x-13$

$y-2=-3(x+5)$

$\leftrightarrow y=-3(x+5)+2=-3x-15+2$

$=-3x-13$

Exercise 3E

1 **a** $y=\frac{1}{6}x-3$

$-y+\frac{1}{6}x-3=0$

$-6y+x-18=0$

b $y=-\frac{2}{3}x+4$

$-y-\frac{2}{3}x+4=0$

$3y+2x-12=0$

c $y-2=-(x+3)$

$y-2=-x-3$

$y+x-2=-3$

$y+x+1=0$

2 **a** $3x+y-5=0$

$y-5=-3x$

$y=-3x+5$

b $2x-4y+8=0$

$\frac{1}{2}x-y+2=0$

$y=\frac{1}{2}x+2$

c $5x+2y+7=0$

$\frac{5}{2}x+y+\frac{7}{2}=0$

$y=-\frac{5}{2}x-\frac{7}{2}$

3 **a** x-intercept:

$x+2\times 0+6=0$

$x+6=0$

$x=-6$

The x-intercept is $(-6,0)$.

y-intercept:

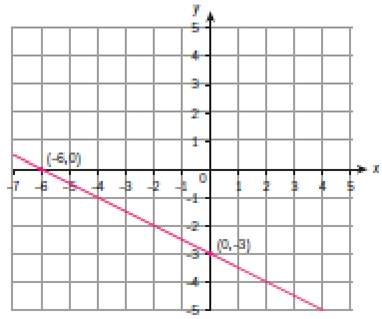
$0+2y+6=0$

$2y+6=0$

$2y=-6$

$y=-3$

The y-intercept is $(0,-3)$.



b x-intercept:

$2x - 6 * 0 + 8 = 0$

$2x + 8 = 0$

$2x = -8$

$x = -4$

The x-intercept is $(-4, 0)$.

y-intercept:

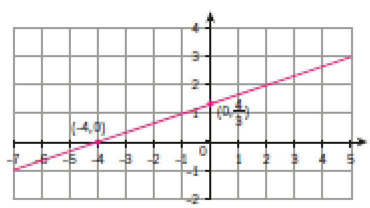
$2 \times 0 - 6y + 8 = 0$

$-6y + 8 = 0$

$6y = 8$

$y = \frac{4}{3}$

The y-intercept is $\left(0, \frac{4}{3}\right)$.



Exercise 3F

1 **a** $(-2, -5)$ **b** $(0.75, 2.5)$
c $(-3.58, -8.19)$ **d** $(1.18, 1.12)$

2 **a** 0.9 **b** -5.05

3 \$1666.67

Exercise 3G

1 **a** $f(3) = -3 + 5 = 2$

b $g(0) = 2 * 0 + 3 = 3$

c $h(6) - g(1) = \left(\frac{1}{3} * 6 - 4\right) - (2 * 1 + 3)$

$= (-2) - 5 = -7$

d $f(2) + g(-1) = (-2 + 5) + (2 * (-1) + 3)$

$= 3 + 1 = 4$

e $(f \circ g)(4) = -g(4) + 5 = -11 + 5 = -6$

f $(h \circ f)(-7) = \frac{1}{3} f(-7) - 4$

$= \frac{1}{3} * 12 - 4 = 4 - 4 = 0$

g $(f \circ g)(x) = -g(x) + 5$

$= -(2x + 3) + 5 = -2x + 2$

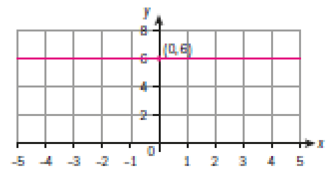
h $(h \circ f)(x) = \frac{1}{3} f(x) - 4 = \frac{1}{3}(-x + 5) - 4$

$= -\frac{1}{3}x + \frac{5}{3} - 4 = -\frac{1}{3}x - \frac{7}{3}$

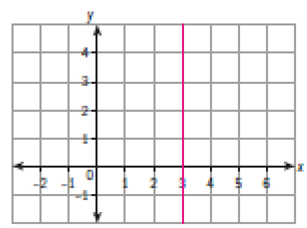
2 **a** As any real number can be inserted for x and any real number can be obtained as $3x + 8$ for an x, both domain and range are all real numbers.

b Just as above, domain and range are all real numbers.

3 **a** The line $y = 6$ has range $\{6\}$ as only 6 can be obtained for y.



b No vertical line is a function as the y corresponding to the x-coordinate of the x-intercept is not unique (in fact, any y corresponds to it).



4 **a** $x = \frac{1}{2}y + 4$

$2x = y + 8$

$f^{-1}(x) = 2x - 8$

b $x = -3y + 9$

$x - 9 = -3y$

$f^{-1}(x) = -\frac{1}{3}x + 3$

Exercise 3H

1 **a** $x = 4y - 5$

$4y = x + 5$

$y = \frac{1}{4}x + \frac{5}{4}$

$f^{-1}(x) = \frac{1}{4}x + \frac{5}{4}$

b $x = -\frac{1}{6}y + 3$

$-\frac{1}{6}y = x - 3$

$y = -6x + 18$

$f^{-1}(x) = -6x + 18$

c $x = 0.25y + 1.75$

$4x = y + 7$

$y = 4x - 7$

$f^{-1}(x) = 4x - 7$

2 The graph of the inverse function is obtained by mirroring the graph of f at the line $y = x$.

3 a $f(55) = 10 * 55 + 65 = 615$

b $x = 10y + 65$

$10y = x - 65$

$y = 0.1x - 6.5$

$f^{-1}(x) = 0.1x - 6.5$

x here represents the money available in CAD and $f^{-1}(x)$ is the number of t-shirts one can buy with x dollars.

c $y = 0.1 * 5065 - 6.5 = 506.5 - 6.5 = 500$

Exercise 3I

1 a The gradient can be computed from any two points on the line; in this case, a force F of 160 Newtons leads to an extension d of 5 centimetres, while no force (i. e. a force of 0 Newtons) leads to no extension (0 centimetres).

Therefore the y-intercept is $(0,0)$ and the gradient is $\frac{5-0}{160-0} = \frac{1}{32}$. This gives the model $d = \frac{1}{32}F$.

b $d = \frac{1}{32} * 370 = 11.5625$ cm.

2 a The gradient is given by $\frac{680-600}{2000-1500} = \frac{80}{500} = 0.16$. As $(1500, 600)$ is on the graph, a point-gradient form of the equation of the line is $y - 600 = 0.16(x - 1500)$. We find the gradient-intercept form:

$y = 0.16(x - 1500) + 600 = 0.16x - 240 + 600 = 0.16x + 3$

.

b The y-intercept represents Frank's basic weekly salary of £360. The gradient shows that Frank's commission is 16% of his sales.

c $y = 0.16 * 900 + 360 = 504$ pounds.

3 a Let y be the total cost in dollars and x the number of months of membership.

For Plan A: $y = 9.99x + 79.99$

For Plan B: $y = 20x$

b We would like to know after how many months the amount paid under each plan is the same (From then onwards, Plan A will be more cost-effective). We therefore solve:

$9.99x + 79.99 = 20x$

$79.99 = 10.01x$

$x = \frac{79.99}{10.01} \approx 7.99$.

Therefore, Plan A is more cost-effective from 8 months onwards.

4 a In the first 40 hours, his pay in pounds is given by $p = \frac{320}{40}h = 8h$. From then on, his pay is given by

$p - 320 = \frac{560-320}{60-40}(h - 40) = 12(h - 40)$

. In gradient-intercept form, this is $p = 12h - 160$.

$$p(h) = \begin{cases} 8h, & 0 \le h \le 40 \\ 12h - 360, & 40 < h \le 60 \end{cases}$$

b i $p = 8 \times 22 = 176$ pounds

ii $p = 12 \times 47 - 160 = 404$ pounds.

5 a $q = -6.5 \times 200 + 3000 = 1700$

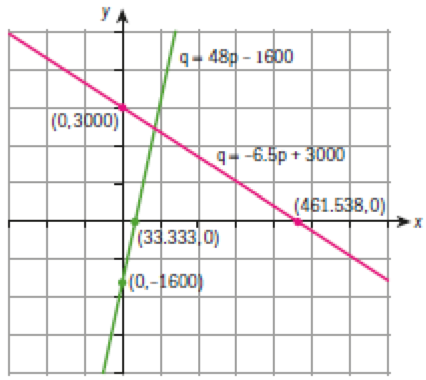
b They will drop by $6.5 \times 20 = 130$ printers a month.

c $2000 = 48p - 1600$

$48p = 3600$

$p = \frac{3600}{48} = 75$ Euro.

d



Use the "solve" function of the GDC.

e Solving $-6.5p + 3000 = 48p - 1600$

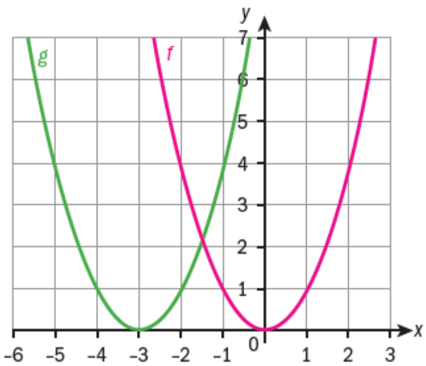
$\Rightarrow p = 84.40..$

So p = €84.40

Then $q = 48 \times 84.40.. - 1600 = 2451$ printers

Exercise 3J

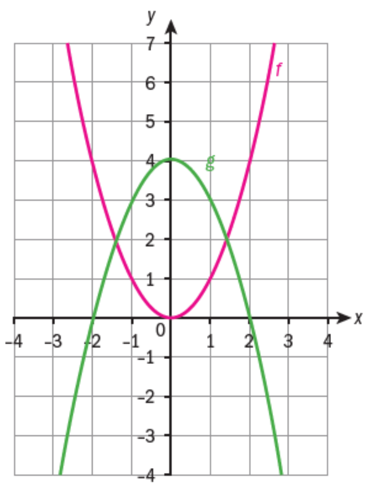
1 a



Axis: x = -3, vertex: (-3, 0)

The graph is translated to the left by 3 units.

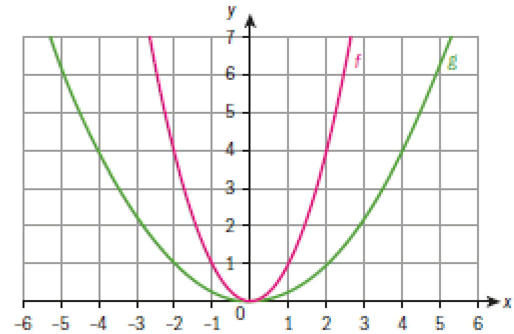
b



Axis: x = 0, vertex: (0, 4)

The graph is reflected about the x -axis and shifted upwards by 4 units.

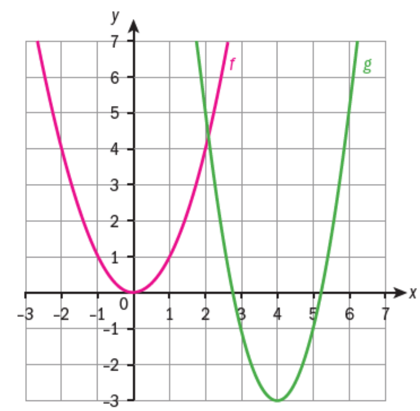
c



Axis: x = 0, vertex: (0, 0)

The graph is compressed vertically with scale factor $\frac{1}{4}$.

d



Axis: x = 4, vertex: (4, -3)

The graph is first translated to the right by 4 units, then stretched vertically with scale factor 2 and finally translated downwards by 3 units.

2 a It is compressed vertically by a scale factor of $\frac{1}{4}$. Thus, the function is given by $g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^2$.

b It is stretched vertically by a scale factor of 2 and reflected along the x -axis. Thus, the function is given by $g(x) = -2f(x) = -2x^2$.

c It is translated to the right by 3 and upwards by 2 units. Thus, the function is given by

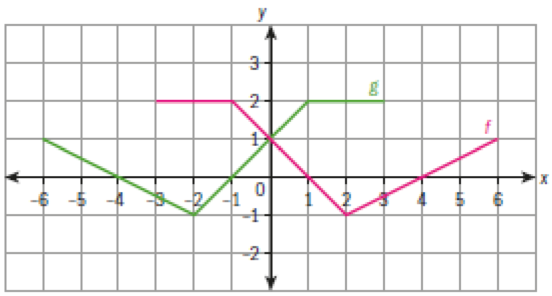
$g(x) = f(x-3) + 2 = (x-3)^2 + 2$.

d It is stretched vertically by a scale factor of 1.5, translated to the left by 3 and downwards by 5 units. Thus, the function is given by

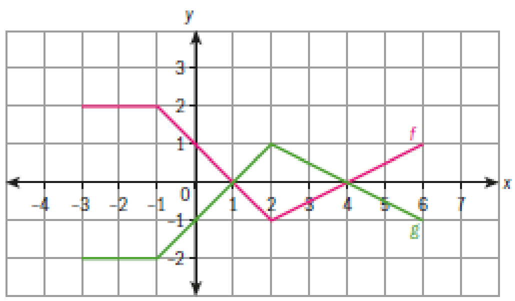
$g(x) = 1.5f(x+3) - 5 = 1.5(x+3)^2 - 5$.

Exercise 3K

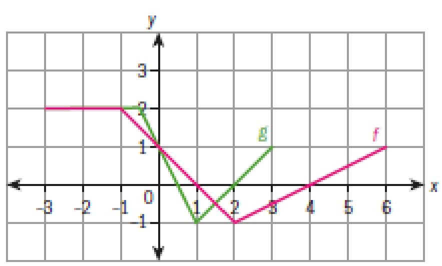
1 a The graph is reflected about the y-axis.



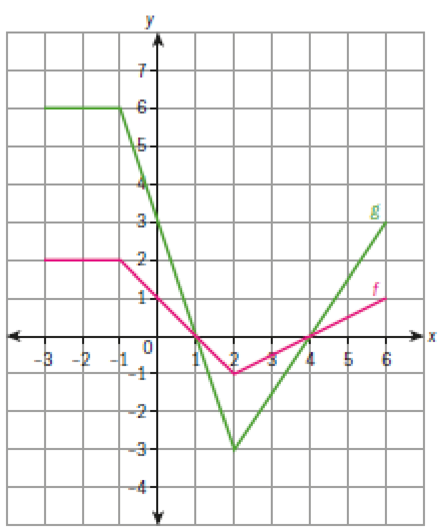
b The graph is reflected about the x-axis.



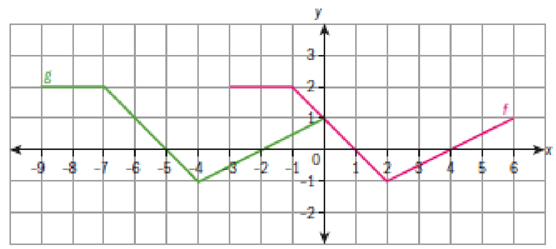
c The graph is compressed horizontally with scale factor $\frac{1}{2}$.



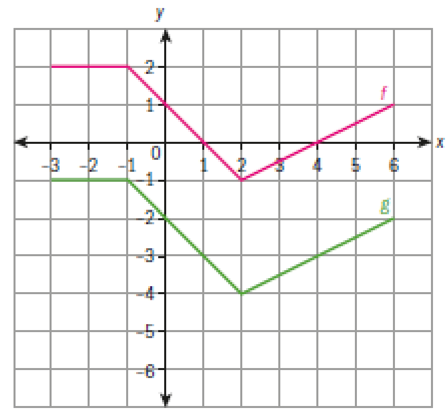
d The graph is stretched vertically with scale factor 3.



e The graph is translated to the left by 6 units.



f The graph is translated downwards by 3 units.



2 a The graph of r is stretched by a scale factor of 2. Thus $r(x) = 2f(x)$.

The graph of s is translated to the right by 3 units and reflected about the x-axis. Thus $s(x) = -f(x-3)$.

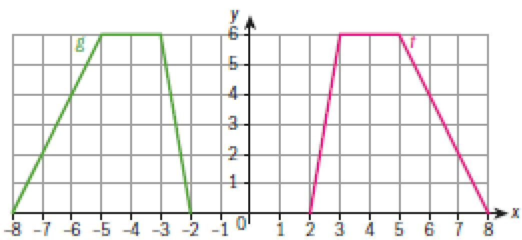
b The graph of r is reflected about the y-axis. Thus $r(x) = f(-x)$.

The graph of s is stretched horizontally by a scale factor of 2 and translated downwards by 4 units. Thus

$$s(x) = f\left(\frac{1}{2}x\right) - 4$$

3 a $0 \le y \le 6$

b It is reflected about the y-axis.



c $2 \le -x \le 8$, which is equivalent to $-8 \le x \le -2$.

d The range of g is the same as the range of f. $0 \le y - c \le 6$ is equivalent to $c \le y \le 6 + c$, so $c = -4$. Thus $h(x) = g(x) - 4$.

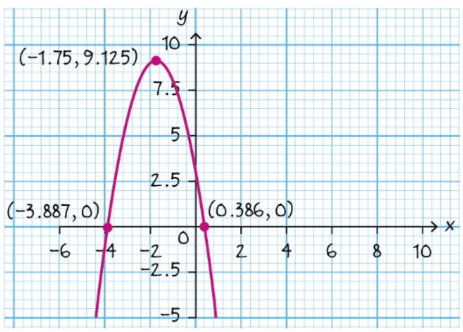
e $h(x) = g(x) - 4 = f(-x) - 4$

Exercise 3L

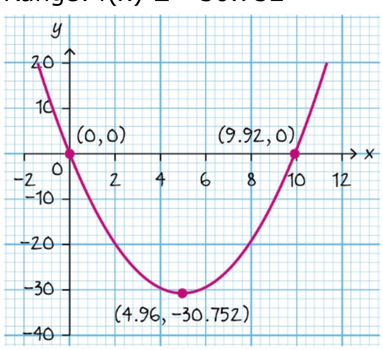
1 x-intercepts: (−2.81, 0), (0.475, 0);
y-intercept: (0, −4);
vertex: (−1.17, −8.08)

2 x-intercepts: none;
y-intercept: (0, −3);
vertex: (0.726, −0.785)

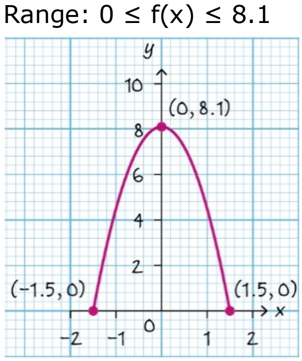
3 Domain: $x \in$ ¡
Range: f(x) ≤ 9.125



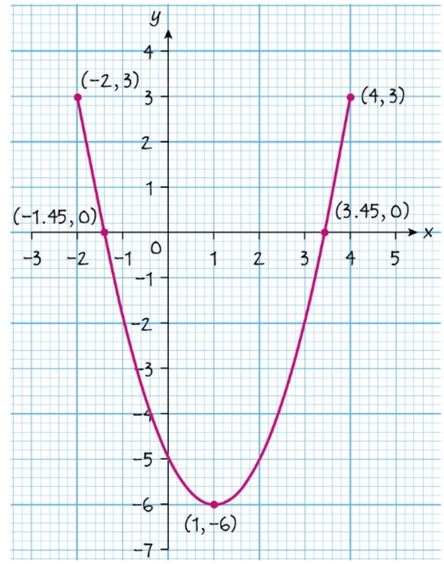
4 Domain: $x \in$ ¡
Range: f(x) ≥ −30.752



5 Range: 0 ≤ f(x) ≤ 8.1



6 Range: $-6 \le f(x) \le 3$



Exercise 3M

1 **a** $x = 3$ is the axis of symmetry and $(3,4)$ the coordinates of the vertex.

b $x = 1$ is the axis of symmetry and $(1,-5)$ the coordinates of the vertex.

c $x = -3$ is the axis of symmetry and $(-3,2)$ the coordinates of the vertex.

d $x = -6$ is the axis of symmetry and $(-6,-5)$ the coordinates of the vertex.

2 **a** The y-intercept is given by $(0,5)$, the axis of symmetry is at $x = -\frac{-8}{2} = 4$ and the vertex is at
$(4, f(4)) = (4, 16 - 32 + 5) = (4, -11)$.

b The y-intercept is given by $(0,2)$, the axis of symmetry is at $x = -\frac{-6}{6} = 1$ and the vertex is at
$(1, f(1)) = (1, 3 - 6 + 2) = (1, -1)$.

c The y-intercept is given by $(0,-11)$, the axis of symmetry is at
$x = -\frac{-8}{-4} = -2$ and the vertex is at
$(-2, f(-2)) = (-2, -8 + 16 - 11) = (-2, -3)$.

d The y-intercept is given by $(0,3)$, the axis of symmetry is at $x = -\frac{6}{4} = -\frac{3}{2}$
and the vertex is at
$\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right) = \left(-\frac{3}{2}, \frac{9}{2} - 9 + 3\right) = \left(-\frac{3}{2}, -\frac{3}{2}\right)$
.

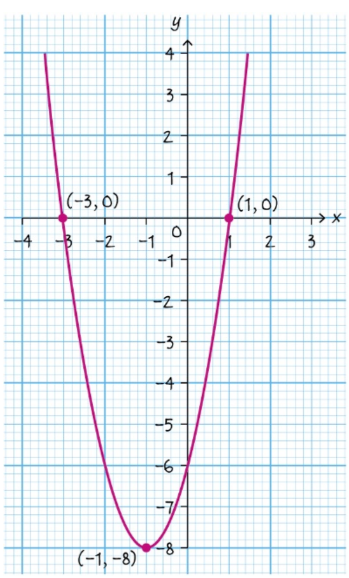
3 **a** The x -intercepts are at $(2,0)$ and $(4,0)$. The axis of symmetry lies at $x=\frac{2+4}{2}=\frac{6}{2}=3$. The vertex is at $(3,f(3))=(3,1*(-1))=(3,-1)$.

b The x -intercepts are at $(-3,0)$ and $(1,0)$. The axis of symmetry lies at $x=\frac{-3+1}{2}=\frac{-2}{2}=-1$. The vertex is at $(-1,f(-1))=(-1,4*2*(-2))=(-1,-16)$.

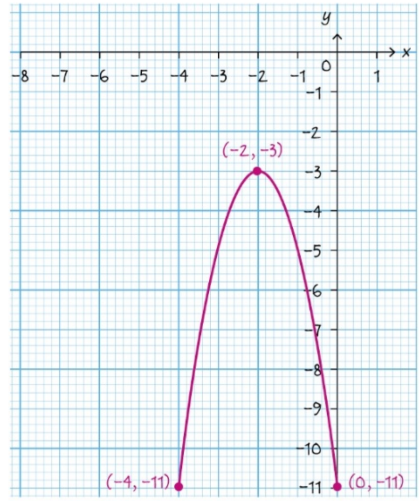
c The x -intercepts are at $(-5,0)$ and $(3,0)$. The axis of symmetry lies at $x=\frac{-5+3}{2}=\frac{-2}{2}=-1$. The vertex is at $(-1,f(-1))=(-1,-(4*(-4))=(-1,16)$.

d The x -intercepts are at $(-3,0)$ and $(-2,0)$. The axis of symmetry lies at $x=\frac{-3-2}{2}=\frac{-5}{2}$. The vertex is at $\left(\frac{-5}{2},f\left(\frac{-5}{2}\right)\right)=\left(\frac{-5}{2},2*\frac{1}{2}*\left(-\frac{1}{2}\right)\right)$
$=\left(\frac{-5}{2},-\frac{1}{2}\right)$

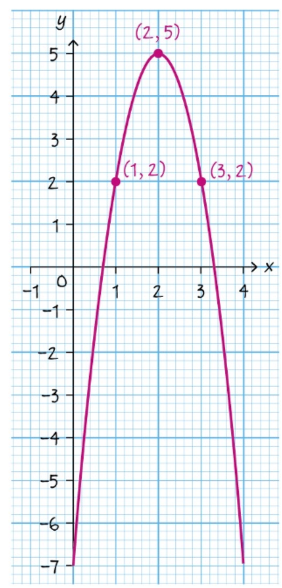
4 **a**



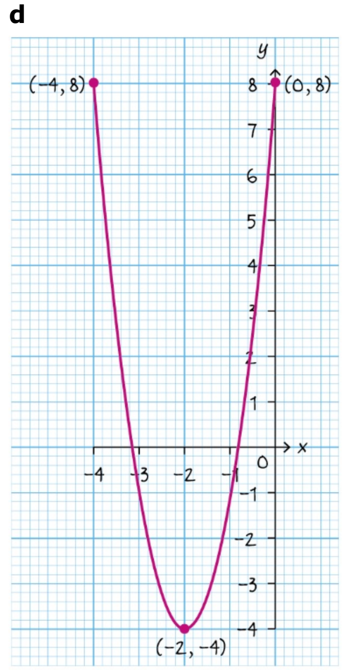
b



c



d



Exercise 3N

1 a $f(x)=(x-2)(x+9)$. The x-intercepts are $(2,0)$ and $(-9,0)$ (from the intercept form), and the y-intercept is $(0,-18)$ (from the standard form).

b $f(x)=(3x-5)(x-2)=3(x-2)\left(x-\frac{5}{3}\right)$.

The x-intercepts are $(2,0)$ and $\left(\frac{5}{3},0\right)$ (from the intercept form), and the y-intercept is $(0,10)$ (from the standard form).

c $f(x)=\frac{1}{2}\left(x^2+6x+8\right)=\frac{1}{2}(x+2)(x+4)$.

The x-intercepts are $-(2,0)$ and $(-4,0)$ (from the intercept form), and the y-intercept is $(0,4)$ (from the standard form).

d $f(x)=-(x-4)(4x-2)$

$=-4(x-4)\left(x-\frac{1}{2}\right)$

The x-intercepts are $(4,0)$ and $\left(\frac{1}{2},0\right)$ (from the intercept form), and the y-intercept is $(0,-8)$ (from the standard form).

2 a $f(x)=4x^2+16x-20$. The x-intercepts are $(1,0)$ and $(-5,0)$ (from the intercept form), and the y-intercept is $(0,-20)$ (from the standard form).

b $f(x)=-2x^2-16x-14$. The x-intercepts are $(-7,0)$ and $(-1,0)$ (from the intercept form), and the y-intercept is $(0,-14)$ (from the standard form).

3 a $f(x)=-3x^2-6x-9$. The vertex is at $(-1,-6)$ (from the vertex form) and the y-intercept is $(0,-9)$.

b $f(x)=\frac{1}{2}x^2-4x+11$. The vertex is at $(4,3)$ (from the vertex form) and the y-intercept is $(0,11)$.

4 a $f(x)=(x-4)(x+2)$. Thus

i $a=1$ **ii** $p=4$ **iii** $q=-2$

b i The x-intercepts are at $(4,0)$ and $(-2,0)$

ii The y-intercept is at $(0,-8)$

c The axis of symmetry is at $x=\frac{4-2}{2}=1$ Thus the vertex is at $(1,f(1))=(1,(-3)\times 3)=(1,-9)$.

5 a i The vertex is at $(3,-2)$.

ii The axis of symmetry is at $x=3$.

b $f(x)=x^2-6x+7$

c B is the y-intercept of the graph, and its coordinates are $\left(0,(-3)^2-2\right)=(0,7)$.

d By symmetry, $p=6$ as $6-3=3-0$.

6 a $h(x)=(x-2)^2-2(x-2)-3$

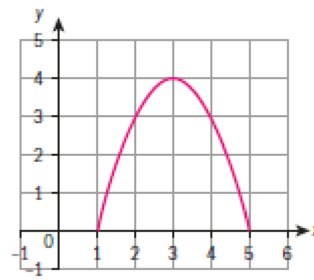
$=x^2-6x+5$

b The axis of symmetry lies at $x=-\frac{6}{2}=3$

c The vertex is at $(3,h(3))=(3,9-18+5)=(3,-4)$.

d $h(x)=(x-5)(x-1)$

e The graph is the same as that of $h(x)$, but reflected about the x-axis.



Exercise 3O

1 a The vertex is at $(2,-16)$ and the y-intercept is at $(0,-12)$. Thus

$f(x)=a(x-2)^2-16$, and

$-12=a(-2)^2-16=4a-16$. Thus $a=1$

. In standard form,

$f(x)=(x-2)^2-16=x^2-4x+4-16$

$=x^2-4x-12$

b $f(x)=a(x-1)(x+3)$ from the x-intercepts. $3=a*(-1)*3=-3a$. Thus $a=-1$. In standard form, $f(x)=-(x-1)(x+3)=-x^2-2x+3$.

c $f(x)=a(x-5)(x-1)$ from the x-intercepts. $-12=a*(-1)*3=-3a$. Thus $a=4$. In standard form, $f(x)=4(x-5)(x-1)=4x^2-24x+20$.

d The vertex is at $(2,-6)$. Thus $f(x)=a(x-2)^2-6$, and $6=a(2)^2-6=4a-6$. Thus $a=3$. In standard form, $f(x)=3(x-2)^2-6=3x^2-12x+6$.

e $f(x)=a(x-2)(x+5)$ from the x-intercepts. $3=a*(-1)*6=-6a$. Thus $a=-\frac{1}{2}$. In standard form, $f(x)=-\frac{1}{2}(x-2)(x+5)=-\frac{1}{2}x^2-\frac{3}{2}x+5$

f The vertex is at $(-10,60)$. Thus $f(x)=a(x+10)^2+60$, and $45=a(-5)^2+60=25a+60$. Thus $a=-\frac{3}{5}$. In standard form, $f(x)=-\frac{3}{5}(x+10)^2+60=-\frac{3}{5}x^2-12x$.

2 a In intercept form, $f(x)=a(x-3)(x+1)$ Therefore, the axis of symmetry is at $x=\frac{3-1}{2}=1$.

b The vertex is at $(1,4)$ as $x=1$ is the axis of symmetry and 4 the maximum value.

c Since the vertex is at $(1,4)$, $h=1$ and $k=4$. So $f(x)=a(x-1)^2+4$. As we also know that $f(3)=0$, $4a+4=0$ and thus $a=-1$. So $f(x)=-(x-1)^2+4$

d $g(x)=f(x-4)-5$

$=-(x-5)^2-1$

$=-\left(x^2-10x+25\right)-1$

$=-x^2+10x-26$

3 a The vertex is at $(4,80)$. The model rocket is predicted to reach a maximum of 80 m, 4 s after it is launched.

b In intercept form, $h(t)=at(t-8)$. Inserting the coordinates of the vertex, we obtain $80=a\times4\times(-4)=-16a$.

Thus $a=-5$. Overall, $h(t)=-5t(t-8)$ $0\le t\le 8$

c $h(2.4)=-5\times2.4\times(-5.6)=67.2$.

Therefore, the rocket is predicted to be 67.20 metres high.

Exercise 3P

1 a $x^2-4x+3=(x-3)(x-1)$. Thus $x=1$ or $x=3$.

b $x^2-x-20=(x-5)(x+4)$. Thus $x=5$ or $x=-4$.

c $x^2-8x+12=(x-6)(x-2)$. Thus $x=2$ or $x=6$.

d $x^2-121=(x-11)(x+11)$. Thus $x=11$ or $x=-11$.

e $x^2+x-42=(x-6)(x+7)$. Thus $x=6$ or $x=-7$.

f $x^2-8x+16=(x-4)^2$. Thus $x=4$.

2 a $2x^2+x-3=(2x+3)\ (x-1)$. Thus $x=1$ or $x=-\frac{3}{2}$.

b $3x^2+5x-12=(3x-4)\ (x+3)$. Thus $x=\frac{4}{3}$ or $x=-3$.

c $4x^2+11x+6=(x+2)\ (4x+3)$. Thus $x=-2$ or $x=-\frac{3}{4}$.

d $9x^2-49=\left(x-\frac{7}{3}\right)\ \left(x+\frac{7}{3}\right)$. Thus $x=\frac{7}{3}$ or $x=-\frac{7}{3}$.

e $4x^2-16x+7=(2x-7)(2x-1)$. Thus $x=\frac{7}{2}$ or $x=\frac{1}{2}$.

f $12x^2+11x-5=(3x-1)(4x+5)$. Thus $x=\frac{1}{3}$ or $x=-\frac{5}{4}$.

Exercise 3Q

1 **a** $(x^2 - x - 20) - (2x + 8) = x^2 - 3x - 28$

$= (x - 7)(x + 4)$

Thus $x = 7$ or $x = -4$.

b $(2x^2 - 3x - 8) - (-x^2 + 2x)$

$= 3x^2 - 5x - 8 = (3x - 8)(x + 1)$

Thus $x = \frac{8}{3}$ or $x = -1$.

c $(4x^2 + 20) - (3x^2 + 10x - 4)$

$= x^2 - 10x + 24 = (x - 6)(x - 4)$

Thus $x = 4$ or $x = 6$.

d $(3x^2 + 15x) + (x + 5)$

$= 3x^2 + 16x + 5 = (3x + 1)(x + 5)$

Thus $x = -\frac{1}{3}$ or $x = -5$.

e $3(x + 2)(x - 2) - (5x)$

$= 3x^2 - 5x - 12 = (3x + 4)(x - 3)$

Thus $x = -\frac{4}{3}$ or $x = 3$.

f For $x \neq 0$, $x + 8 = \frac{-15}{x}$ if and only if

$x^2 + 8x = -15$.

$x^2 + 8x + 15 = (x + 3)(x + 5)$ and thus

$x = -3$ or $x = -5$.

2 **a** $(f \circ g)(x) = (2x + 1)^2 - 2$

$= 4x^2 + 4x + 1 - 2 = 4x^2 + 4x - 1$

b $(4x^2 + 4x - 1) - (x^2 + 5x + 3)$

$= 3x^2 - x - 4 = (3x - 4)(x + 1)$

Thus $x = \frac{4}{3}$ or $x = -1$.

Exercise 3R

1 $x^2 - 8x + 16 = (x - 4)^2 = 10$. Thus

$x = \pm\sqrt{10} + 4$.

2 $x^2 + 20x + 100 = (x + 10)^2 = 15$. Thus

$x = \pm\sqrt{15} - 10$.

3 $x^2 + 12x + 36 = (x + 6)^2 = 12$. Thus

$x = \pm\sqrt{12} - 6$.

4 $x^2 - 10x + 25 = (x - 5)^2 = 27$. Thus

$x = \pm\sqrt{27} + 5$.

5 $4x^2 + 3x + 2 = -x + 5$

$4x^2 + 4x = 3$

$4(x^2 + x) = 3$

$4\left(x + \frac{1}{2}\right)^2 = 4$

$\left(x + \frac{1}{2}\right)^2 = 1$

$x = -\frac{1}{2} \pm 1$

$f\left(-\frac{3}{2}\right) = \frac{3}{2} + 5 = 6.5$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + 5 = 4.5$

6 (1.18, 7.35), (−1.96, 1.07)

7 (1, 5)

8 (2.72, 7.64), (0.613, −0.0872)

9 $x = -0.802, 1.80$

10 $x = -2.91, 0.915$

Exercise 3S

1 $\left(\frac{b}{2}\right)^2 = 6^2 = 36$. Therefore consider

$x^2 + 12x + 36 = 2 + 36 = 38$. This factorises to $(x + 6)^2 = 38$, giving $x = -6 \pm \sqrt{38}$

2 $\left(\frac{b}{2}\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$. Therefore consider

$x^2 - 3x + \frac{9}{4} = 2 + \frac{9}{4} = \frac{17}{4}$. This factorises to

$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4}$, giving

$x = \frac{\pm\sqrt{17}}{\sqrt{4}} + \frac{3}{2} = \frac{3 \pm \sqrt{17}}{2}$

3 $x^2 - 6x + 4 = 0$ is equivalent to

$x^2 - 6x = -4$. $\left(\frac{b}{2}\right)^2 = (-3)^2 = 9$. Therefore consider $x^2 - 6x + 9 = -4 + 9 = 5$. This factorises to $(x - 3)^2 = 5$, giving $x = 3 \pm \sqrt{5}$

4 $x^2 - 12x + 4 = 0$ is equivalent to

$x^2 - 12x = -4$. $\left(\frac{b}{2}\right)^2 = (-6)^2 = 36$.

Therefore consider

$x^2 - 12x + 36 = -4 + 36 = 32$. This

factorises to $(x-6)^2 = 32$, giving

$x = 6 \pm \sqrt{32} = 6 \pm 4\sqrt{2}$

5 $x^2 + 5x - 4 = 0$ is equivalent to $x^2 + 5x = 4$

$\left(\frac{b}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$. Therefore consider

$x^2 + 5x + \frac{25}{4} = 4 + \frac{25}{4} = \frac{41}{4}$. This factorises

to $\left(x + \frac{5}{2}\right)^2 = \frac{41}{4}$, giving

$x = \frac{\pm\sqrt{41}}{\sqrt{4}} - \frac{5}{2} = \frac{-5 \pm \sqrt{41}}{2}$

6 $x^2 + x - 11 = 0$ is equivalent to $x^2 + x = 11$

$\left(\frac{b}{2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore consider

$x^2 + x + \frac{1}{4} = 11 + \frac{1}{4} = \frac{45}{4}$. This factorises to

$\left(x + \frac{1}{2}\right)^2 = \frac{45}{4}$, giving

$x = \frac{\pm\sqrt{45}}{\sqrt{4}} - \frac{1}{2} = \frac{-1 \pm \sqrt{45}}{2} = \frac{-1 \pm 3\sqrt{5}}{2}$

Exercise 3T

1 $2x^2 + 16x = 10$ is equivalent to

$x^2 + 8x = 5$. $\left(\frac{b}{2}\right)^2 = 4^2 = 16$. Therefore

consider $x^2 + 8x + 16 = 5 + 16 = 21$. This

factorises to $(x+4)^2 = 21$, giving

$x = -4 \pm \sqrt{21}$.

2 $5x^2 - 30x = 10$ is equivalent to

$x^2 - 6x = 2$. $\left(\frac{b}{2}\right)^2 = (-3)^2 = 9$. Therefore

consider $x^2 - 6x + 9 = 2 + 9 = 11$. This

factorises to $(x-3)^2 = 11$, giving

$x = 3 \pm \sqrt{11}$.

3 $6x^2 - 12x - 3 = 0$ is equivalent to

$x^2 - 2x = \frac{1}{2}$. $\left(\frac{b}{2}\right)^2 = (-1)^2 = 1$. Therefore

consider $x^2 - 2x + 1 = \frac{1}{2} + 1 = \frac{3}{2}$. This

factorises to $(x-1)^2 = \frac{3}{2}$, giving

$x = 1 \pm \sqrt{\frac{3}{2}}$.

4 $6x(x+8) = 12$ is equivalent to

$x(x+8) = x^2 + 8x = 2$. $\left(\frac{b}{2}\right)^2 = 4^2 = 16$.

Therefore consider

$x^2 + 8x + 16 = 2 + 16 = 18$. This factorises

to $(x+4)^2 = 18$, giving

$x = -4 \pm \sqrt{18} = -4 \pm 3\sqrt{2}$.

5 $2x^2 + x - 6 = 0$ is equivalent to $x^2 + \frac{1}{2}x = 3$

. $\left(\frac{b}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$. Therefore consider

$x^2 + \frac{1}{2}x + \frac{1}{16} = 3 + \frac{1}{16} = \frac{49}{16}$. This factorises

to $\left(x + \frac{1}{4}\right)^2 = \frac{49}{16}$, giving

$x = \frac{\pm\sqrt{49}}{\sqrt{16}} - \frac{1}{4} = \frac{-1 \pm \sqrt{49}}{4} = \frac{-1 \pm 7}{4}$. This

means that x is either $\frac{3}{2}$ or -2.

6 $2x(x+8) + 12 = 0$ is equivalent to

$x(x+8) = x^2 + 8x = -6$. $\left(\frac{b}{2}\right)^2 = 4^2 = 16$.

Therefore consider

$x^2 + 8x + 16 = -6 + 16 = 10$. This factorises

to $(x+4)^2 = 10$, giving $x = -4 \pm \sqrt{10}$.

7 **a** Revenue is equal to cost when $R(x) = C(x)$, i. e. when

$35x - 0.25x^2 = 300 + 15x$.

b This is equivalent to $-0.25x^2 + 20x = 300$, which is in turn equivalent to $x^2 - 80x = -1200$.

$\left(\frac{b}{2}\right)^2 = (-40)^2 = 1600$. Therefore

consider

$x^2 - 80x + 1600 = -1200 + 1600 = 400$.

This factorises to $(x-40)^2 = 400$,

giving $x = 40 \pm \sqrt{400} = 40 \pm 20 = 20, 60$.

c The break-even points lie at $x = 20$ and $x = 60$.

d We will want to find where the maximum of the equation $P(x) = R(x) - C(x)$ lies. This will just be the coordinates of the vertex, since the leading coefficient is negative.

$P(x) = R(x) - C(x)$

$= -0.25x^2 + 20x - 300$

In vertex form, this is $P(x) = -0.25(x-40)^2 + 100$. Thus, the maximal profit is reached at 40 subscribers.

e As seen from the vertex form above, the vertex has coordinates (40,100) and therefore the maximal profit is equal to 100 thousand Euros.

Exercise 3U

1 a $x = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \frac{\sqrt{24}}{2} = -2 \pm \sqrt{6}$

b $x = \frac{8 \pm \sqrt{64-60}}{6} = \frac{8 \pm 2}{6}$; that is, $x = 1$ or $x = \frac{5}{3}$.

c $x = \frac{5 \pm \sqrt{25+16}}{4} = \frac{5 \pm \sqrt{41}}{4}$

2 a $x^2 + 3x - 9 = 0$. Thus $x = \frac{-3 \pm \sqrt{9+36}}{2} = -\frac{-3 \pm \sqrt{45}}{2}$ $= \frac{-3 \pm 3\sqrt{5}}{2}$

b $3x^2 - 4x - 2 = 0$. Thus $x = \frac{4 \pm \sqrt{16+24}}{6} = -\frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$

c $-x^2 + 2x + 2 = 0$. Thus $x = \frac{-2 \pm \sqrt{4+8}}{-2} = 1 \pm \sqrt{3}$

d $3x^2 + 4x + 10 = 0$. Thus $x = \frac{-4 \pm \sqrt{16-120}}{6} = \frac{-4 \pm \sqrt{-104}}{6}$. As -104 has no real square root, the equation has no real solution.

e $-2x^2 + 10x - 9 = 0$. Thus $x = \frac{-10 \pm \sqrt{100-72}}{-4} = \frac{5 \pm \sqrt{7}}{2}$

f $2x^2 - 9x + 9 = 0$. Thus $x = \frac{9 \pm \sqrt{81-72}}{4} = -\frac{9 \pm 3}{4}$; that is, $x = 3$ or $x = \frac{3}{2}$.

g $(x+3)(x+1) = 2x(x-1)$. This is equivalent to $x^2 + 4x + 3 = 2x^2 - 2x$, which simplifies to $x^2 - 6x - 3 = 0$. Thus $x = \frac{6 \pm \sqrt{36+12}}{2} = 3 \pm \sqrt{12} = 3 \pm 2\sqrt{3}$.

3 a $x = \frac{-5 \pm \sqrt{25+144}}{12} = \frac{-5 \pm \sqrt{169}}{12}$ $= \frac{-5 \pm 13}{12}$; that is, $x = \frac{2}{3}$ or $x = -\frac{3}{2}$.

b $x = \frac{4 \pm \sqrt{16-8}}{4} = -\frac{4 \pm \sqrt{8}}{4} = \frac{2 \pm \sqrt{2}}{2}$

c $x = \frac{-2 \pm \sqrt{4+16}}{-2} = 1 \pm \sqrt{5}$

4 a $c = -2$

b $2x^2 - 4x - 2 = 2(x^2 - 2x - 1)$ $= 2(x-1)^2 - 4$. Therefore the vertex is at $(1,-4)$.

c Using the quadratic formula: $x = \frac{4 \pm \sqrt{16+16}}{4} = 1 \pm \sqrt{2}$. Therefore $r = 1$ and $s = 2$.

Exercise 3V

1 a $\Delta = (-5)^2 - 4 \times 1 \times 9 = 25 - 36 = -11$. Therefore the equation has no real roots.

b $\Delta = 7^2 - 4 \times 6 \times (-3) = 49 + 72 = 121$. Therefore the equation has two distinct real roots.

c $\Delta = (-4)^2 - 4 \times 1 \times 15 = 16 - 60 = -44$. Therefore the equation has no real roots.

d $\Delta = 4^2 - 4 \times 3 \times (-8) = 16 + 96 = 112$. Therefore the equation has two distinct real roots.

e $\Delta = (-4)^2 - 4 \times 1 \times 4 = 16 - 16 = 0$. Therefore the equation has two equal real roots.

f $\Delta = (-1)^2 - 4 \times 5 \times 10 = 1 - 200 = -199$. Therefore the equation has no real roots.

2 a $\Delta = 3^2 - 4k = 9 - 4k$. This is positive whenever $k < \frac{9}{4}$.

b $\Delta = 20^2 - 20k = 400 - 20k$. This is positive whenever $k < 20$.

3 a $\Delta = 5^2 - 4p = 25 - 4p$. This is 0 if and only if $p = \frac{25}{4}$.

b $\Delta = (-12)^2 - 12p = 144 - 12p$. This is 0 if and only if $p = 12$.

c $\Delta = (-2p)^2 - 32 = 4p^2 - 32$. This is 0 if and only if $p^2 = 8$, which holds for $p = \pm\sqrt{8} = \pm 2\sqrt{2}$.

d $\Delta = (-3p)^2 + 8p = 9p^2 + 8p = p(9p + 8)$. This is 0 if and only if $p = 0$ or $p = -\dfrac{8}{9}$.

4 a $\Delta = (-2)^2 - 4m = 4 - 4m$. This is negative if and only if $m > 1$.

b $\Delta = (-6)^2 - 12m = 36 - 12m$. This is negative if and only if $m > 3$.

c $\Delta = 5^2 - 4(m-2) = 33 - 4m$. This is negative if and only if $m > \dfrac{33}{4}$.

Exercise 3W

1 a We need to find the x-intercepts. By the quadratic formula, $x = \dfrac{-5 \pm \sqrt{25+24}}{6} = \dfrac{-5 \pm 7}{6}$. Since the coefficient of x^2 is positive, the parabola will be concave up. Thus the inequality is satisfied whenever $x \le -2$ or $x \ge \dfrac{1}{3}$

b $x^2 \le 5$ if and only if $-\sqrt{5} \le x \le \sqrt{5}$.

c This is equivalent to $x^2 + 4x - 6 < 0$. By the quadratic formula, $x = \dfrac{-4 \pm \sqrt{16+24}}{2} = -2 \pm \sqrt{10}$. As the parabola is concave up, the inequality is satisfied whenever $-2 - \sqrt{10} < x < -2 + \sqrt{10}$.

2 a $x \le -0.245$ or $x \ge 12.2$

b $-\dfrac{2}{3} \le x \le 3$

c $-0.890 \le x \le 1.26$

3 a $\Delta = k^2 - 16$. This is positive whenever $k > 4$ or $k < -4$.

b $\Delta = 4k^2 - 12$. This is positive whenever $k > \sqrt{3}$ or $k < -\sqrt{3}$.

4 $\Delta = 36m^2 - 4m = m(36m - 4)$. The zeroes of this equation are at $m = 0$ and $m = \dfrac{1}{9}$. As the parabola described by Δ is concave up, this is negative if and only if $0 < m < \dfrac{1}{9}$

5 $\Delta = 36k^2 - 4k(k+2) = 32k^2 - 8k$ $= 8k(4k-1)$. The zeroes of this equation are at $k = 0$ and $k = \dfrac{1}{4}$. As the parabola described by Δ is concave up, this is positive if and only if $k < 0$ or $k > \dfrac{1}{4}$.

6 a $\Delta = p^2 - 48$

b As the graph has no x-intercepts, $p^2 - 48 < 0$. This means that $-\sqrt{48} < p < \sqrt{48}$, which can be simplified as $-4\sqrt{3} < p < 4\sqrt{3}$.

c As $6^2 = 36 < 48 < 49 = 7^2$, $m = 6$.

d $3x^2 + 6x + 4 = 3\left(x^2 + 2x + \dfrac{4}{3}\right)$

$= 3\left((x+1)^2 + \dfrac{1}{3}\right) = 3(x+1)^2 + 1$

Thus $a = 3$, $h = -1$ and $k = 1$

Exercise 3X

1 $24 = \dfrac{1}{2}h(2h+4)$

$48 = 2h^2 + 4h$

$2h^2 + 4h - 48 = 0$

$h^2 + 2h - 24 = 0$

$(h+6)(h-4) = 0$

$h = 4, -6$

h must be positive

So $h = 4$ m

$b = 2h + 4 = 12$ m

2 a $h(3) = 2 + 20(3) - 4.9(3^2) = 17.9$ m

b $2 + 20t - 4.9t^2 = 6$

$4.9t^2 - 20t + 4 = 0$

$t = \dfrac{20 \pm \sqrt{400 - 78.4}}{9.8}$

$t = 0.211$ seconds, 3.87 seconds

c Maximum height when: $t = -\dfrac{b}{2a} = \dfrac{20}{9.8}$

$h = 2 + 20\left(\dfrac{20}{9.8}\right) - 4.9\left(\dfrac{20}{9.8}\right)^2$

$h = 22.4$ metres

3 a Fare $= 5.50 - 0.05x$

b Number of riders $= 800 + 10x$

c Revenue $= (5.50 - 0.05x)(800 + 10x)$

$= 4400 - 40x + 55x - 0.5x^2$
$= 4400 + 15x - 0.5x^2$

d $4400 + 15x - 0.5x^2 = 4500$

$0 = 0.5x^2 - 15x + 100$
$x = 10, 20$

10 or 20 decreases

e $4400 + 15x - 0.5x^2 > 0$

Using GDC: $x < 110$

4 a $y = -(x-2)^2 + 4 = -x(x-4)$

or $y = -x^2 + 4x$

b If the center of the object is aligned with the center of the archway, it spans form $x = 0.5$ to $x = 3.5$. Evaluating the function at $x = 0.5$ and $x = 3.5$ gives 1.75. Since 1.6 < 1.75, the object will fit through the archway.

5 a $A(x) = x(155 - x) = 155x - x^2$

b Maximum area occurs at:

$x = \frac{-b}{2a} = \frac{155}{2} = 77.5$

$w = \frac{310 - 2(77.5)}{2} = 77.5$

Dimension: 77.5 metres by 77.5 metres

c No; The touchline would not be longer than the goal line and 77.5 metres is less that the minimum of 90 metres for the touchline.

d $90 \le x \le 120$ (If the goal line restrictions are also taken into consideration the answer is $90 \le x \le 110$.

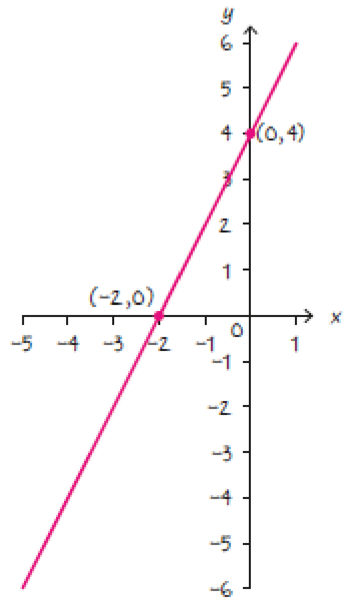
e Maximum occurs when $x = 90$

$w = \frac{310 - 2(90)}{2} = \frac{310 - 180}{2} = 65$

Area $= 90 \times 65 = 5850 \text{ m}^2$

Chapter review

1 a



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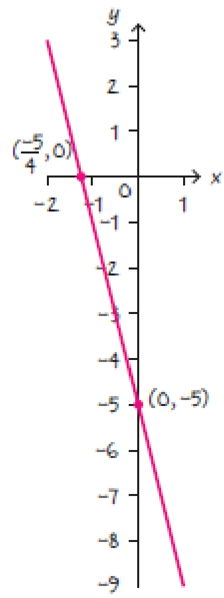
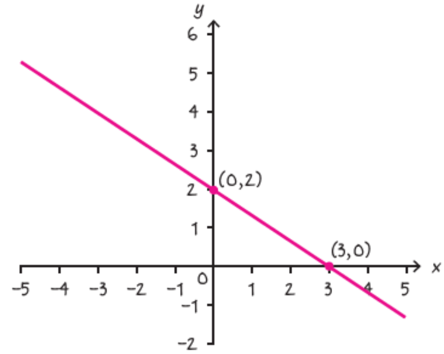


c



2 a $m = \frac{2 - -1}{-4 - 8} = \frac{3}{-12} = -\frac{1}{4}$

$y - 2 = -\frac{1}{4}(x + 4)$

$y - 2 = -\frac{1}{4}x - 1 \Rightarrow y = -\frac{1}{4}x + 1$

b $y = \frac{1}{2}x - 5$

c $m = \frac{-1}{\left(-\frac{2}{3}\right)} = \frac{3}{2}$

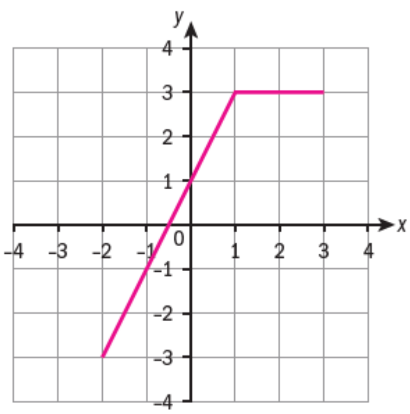
$y - 4 = \frac{3}{2}(x - 2)$

$y - 4 = \frac{3}{2}x - 3 \Rightarrow y = \frac{3}{2}x + 1$

d $y = -4$

3 a $f(1) = 3, f(2) = 3$

b



4 a Vertical stretch with scale factor 2, horizontal translation right 3

b Vertical dilation with scale factor $\frac{1}{2}$, vertical translation up 5

c Reflection in the x-axis, horizontal translation left 2, vertical translation down 1

d Horizontal dilation with scale factor $\frac{1}{3}$

e Reflection in the y-axis, vertical translation up 6

5 a x – intercepts: $2(x - 3)(x + 7) = 0$

$\Rightarrow x = 3, -7 \therefore (3, 0), (-7, 0)$

Axis of symmetry occurs at midpoint of x-intercepts

$x = \frac{3 + -7}{2} \Rightarrow x = -2$

b Found from the function

Axis of symmetry: $x = 4$, Vertex: $(4, 2)$

c Axis of symmetry: $\frac{-b}{2a} = \frac{4}{-2} = -2$

$x = -2$

y-intercept found from the function: $(0, 6)$

6 a $3x^2 + 18x + 20 = 3(x^2 + 6x) + 20$

$= 3((x + 3)^2 - 9) + 20$

$= 3(x + 3)^2 - 27 + 20$

$= 3(x + 3)^2 - 7$

i $a = 3$ **ii** $h = -3$ **iii** $k = -7$

b $(-3, -7)$

c $(-3 + 5, -7 - 3) = (2, -10)$

7 a $(x - 3)^2 = 64$

$x - 3 = \pm 8$

$x = -5, 11$

b $(x + 2)^2 = 7$

$x + 2 = \pm\sqrt{7}$

$x = -2 - \sqrt{7}, -2 + \sqrt{7}$

c $x^2 + 14x + 49 = 0$

$(x + 7)^2 = 0 \Rightarrow x = -7$

d $x^2 + x - 12 = 0$

$(x + 4)(x - 3) = 0 \Rightarrow x = -4, 3$

e $3x^2 + 4x - 7 = 0$

$(3x + 7)(x - 1) = 0 \Rightarrow x = 1, -\frac{7}{3}$

8 Equal real root: $b^2 - 4ac = 0$

$9k^2 - 16 = 0 \Rightarrow k^2 = \frac{16}{9} \Rightarrow k = -\frac{4}{3}, \frac{4}{3}$

9 From the x-intercepts:

$f(x) = a(x + 4)(x - 2) = ax^2 + 2ax - 8a$

From the y-intercept:

$-8a = -16 \Rightarrow a = 2$

$f(x) = 2x^2 + 4x - 16$

10 Using GDC solver

a -0.679, 3.68 **b** -4.92, 1.42

11 a $t = 0$, $h = 18$ m

b Maximum height occurs when:

$x = \frac{-b}{2a} = \frac{13}{9.8}$

$h = 18 + 13\left(\frac{13}{9.8}\right) - 4.9\left(\frac{13}{9.8}\right)^2$

$h = 26.6$ m

c $18 + 13t - 4.9t^2 = 0$

$t = -1.00, 3.66$ as $t > 0$

Time taken = 3.66 seconds

d $0 \le t \le 3.66$

e $18 + 13t - 4.9t^2 = 23$

$-4.9t^2 + 13t - 5 = 0$

$t = 0.4667.., 2.1863..$

2.1863.. − 0.4667.. = 1.72 seconds

12 a $A(-4,0)$; $B(0,7)$; $C(4,0)$

b $m = \dfrac{7-0}{0-4} = -\dfrac{7}{4}$

$y - 0 = -\dfrac{7}{4}(x-4) \Rightarrow y = -\dfrac{7}{4}x + 7$

c $2p$ cm by $-1.75p + 7$ cm

d Area $= 2p(-1.75p + 7) = -3.5p^2 + 14p$

e Maximum area occurs when

$p = \dfrac{-b}{2a} = \dfrac{14}{7} = 2$

So dimensions are 4 cm by 3.5 cm

f Area = $4 \times 3.5 =$ 14 cm^2

13 a $-7x - 12y + 168 = 0$

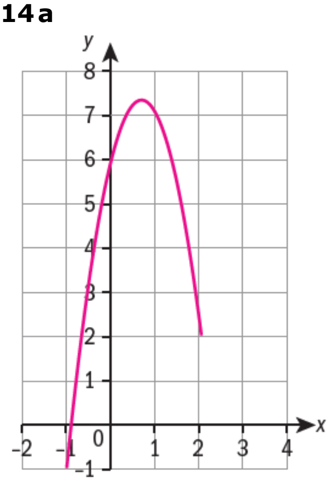
$12y = -7x + 168$ M1

$y = -\dfrac{7x}{12} + 14$ A1

b $A(24,0)$ and $B(0,14)$ A1A1

c Area $= \dfrac{1}{2} \times 24 \times 14 = 168$ units^2 M1A1

14 a



M1A1

b $(0, 5.9)$ and $(-0.885, 0)$ A1A1

c $-1.1 \le f(x) \le 7.35$ A1A1

15 a $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ M1

$x = \dfrac{6 \pm \sqrt{208}}{2}$ A1

$x = 3 \pm \sqrt{52}$ A1

$x = 3 \pm 2\sqrt{13}$ A1

b Using GDC

$3 - 2\sqrt{13} \le x \le 3 + 2\sqrt{13}$ M1A1

16 a $3(x-1)^2 - 18 = 3(x^2 - 2x + 1) - 18$

M1

$= 3x^2 - 6x - 15$ A1

b $(1, -18)$ A1

c $x = 1$ A1

d $f(x) \in$ ¡ , $f(x) \ge -18$ A1A1

e $g(x) = 3((x-2)-1)^2 - 18 - 1$ M1A1

$= 3(x-3)^2 - 19$

$= 3(x^2 - 6x + 9) - 19$

$= 3x^2 - 18x + 8$ A1

17 a $8x^2 + 6x - 5 = 0$

$(4x+5)(2x-1) = 0$ M1A1

$4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$ A1

$2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$ A1

b $8x^2 + 6x - 5 - k = 0$

No real solutions

$\Rightarrow b^2 - 4ac < 0$ M1

$36 - 4 \times 8 \times (-5 - k) < 0$ A1

$36 + 32(5 + k) < 0$

$5 + k < -\dfrac{36}{32} \Rightarrow k < -\dfrac{36}{32} - 5$

$k < -\dfrac{9}{8} - \dfrac{40}{8} \Rightarrow k < -\dfrac{49}{8}$ A1

18 a $x^2 - 10x + 27$

$= (x-5)^2 - 25 + 27$ M1A1

$= (x-5)^2 + 2$ A1

b Coordinates of the vertex is $(5, 2)$

A1

c Equation of symmetry is $x = 5$ A1

19 a At $(10,0)$, $0=10^2+10b+c$, so

$10b+c=-100$ M1A1

Line of symmetry is $x=-\frac{b}{2}$, so $b=-5$

A1

Solving simultaneously gives
$-50+c=-100$

So $c=-50$ A1

Therefore the equation is
$y=x^2-5x-50$

b Setting $x=0$ gives the y-intercept of
$(0,-50)$ A1

Setting y = 0 and solving gives the x-intercept of $(-5,0)$ A1

20 a $f(x)=2\left[x^2-2x-4\right]$ M1

$=2\left[(x-1)^2-1-4\right]$ A1

$=2\left[(x-1)^2-5\right]$

$=2(x-1)^2-10$ A1

b A horizontal translation right 1 unit
A1

A vertical stretch with scale factor 2
A1

A vertical translation down 10 units
A1

21 a Two real roots $\Rightarrow b^2-4ac=0$ M1

$36-4(2k)(k)=0$ A1

$36-8k^2=0$

$k^2=\frac{36}{8}=\frac{9}{2}\Rightarrow\ k=\pm\frac{\sqrt{3}}{2}$ A1A1

b Equation of line of symmetry is
$x=-\frac{b}{2a}=-\frac{6}{4k}=-\frac{3}{2k}$ M1A1

Therefore $\frac{3}{2k}=1\ \Rightarrow k=\frac{3}{2}$ A1

c $k=2\Rightarrow 4x^2+6x+2=0$

$2x^2+3x+1=0$

$(2x+1)(x+1)=0$ M1

$x=-\frac{1}{2}$ or $x=-1$ A1A1

22 a $A'(-6,10), B'(0,-16), C'(1,9)$

and $D'(7,-10)$ A4

b $A(12,13), B(0,-13), C(-2,12)$

and $D(-14,-7)$ A4

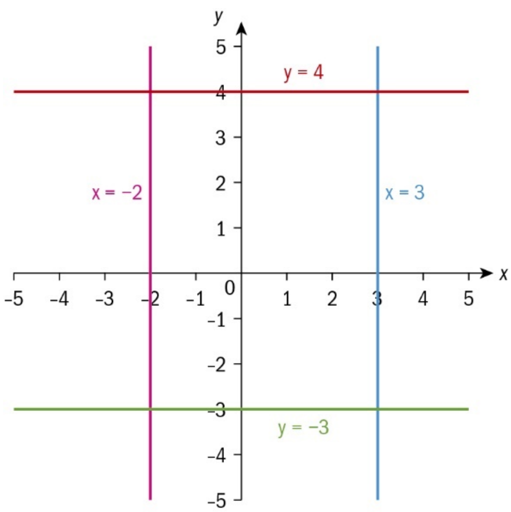
4 Equivalent representations: rational functions

Skills check

1 **a** $x = -5$ **b** $x = 6$

c $2x = 5 \Rightarrow x = \frac{5}{2}$

2



Exercise 4A

1 **a** $\frac{1}{3}$ **b** $\frac{1}{5}$ **c** $-\frac{1}{2}$

d $-\frac{1}{1} = -1$ **e** $\frac{5}{3}$ **f** $\frac{7}{22}$

g $-\frac{9}{8}$ **h** $\frac{1}{2\frac{3}{4}} = \frac{1}{\frac{2\cdot4+3}{4}} = \frac{4}{11}$

2 **a** $1.5 = \frac{3}{2} \Rightarrow \frac{1}{1.5} = \frac{2}{3}$ **b** $\frac{1}{x}$

c $\frac{1}{2x}$ **d** $\frac{1}{4y}$ **e** $\frac{4}{3x}$

f $\frac{t}{d}$ **g** $\frac{4d}{3}$ **h**

$\frac{x-3}{x+2}$

3 **a** $4\cdot\frac{1}{4} = \frac{4}{4} = 1$

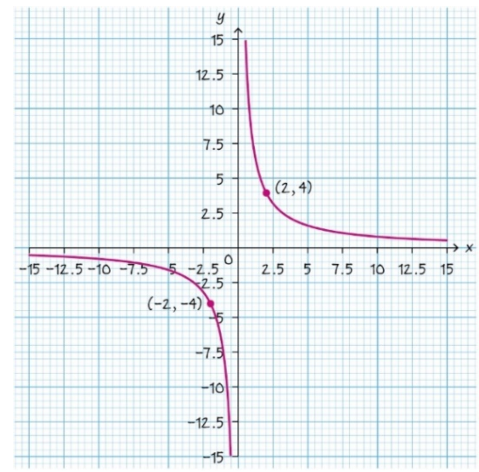
b $\frac{7}{11}\cdot\frac{11}{7} = \frac{7\cdot11}{7\cdot11} = \frac{77}{77} = 1$

c $\frac{2}{x}\cdot\frac{x}{2} = \frac{2x}{2x} = 1$

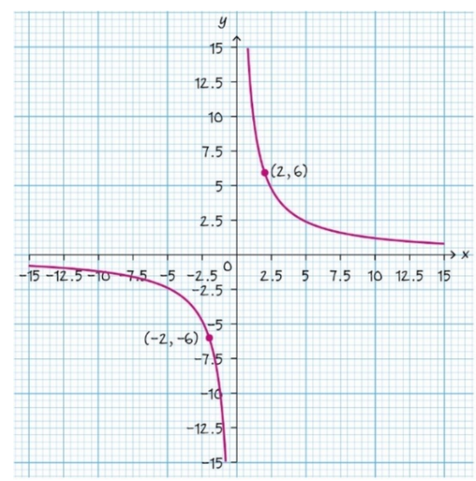
d $\frac{x-1}{x-2}\cdot\frac{x-2}{x-1} = \frac{(x-1)(x-2)}{(x-1)(x-2)} = 1$

Exercise 4B

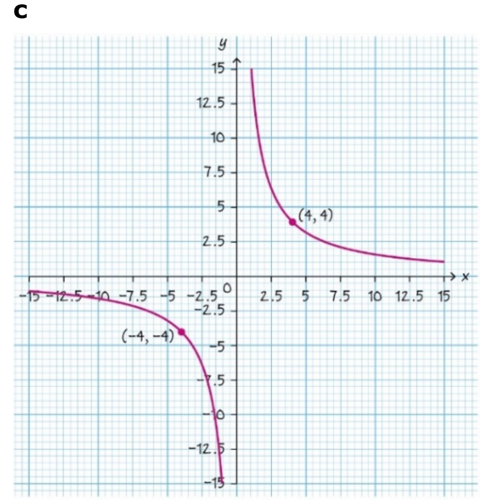
1 **a**



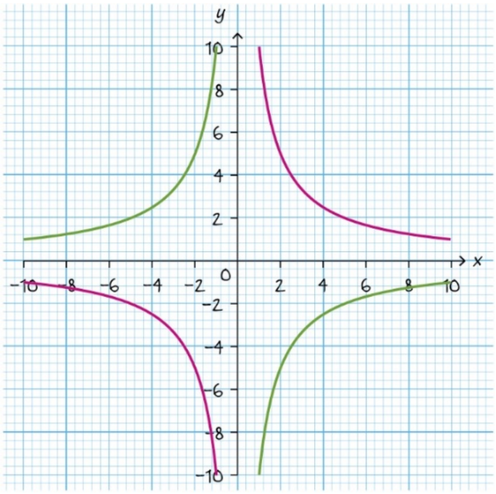
b



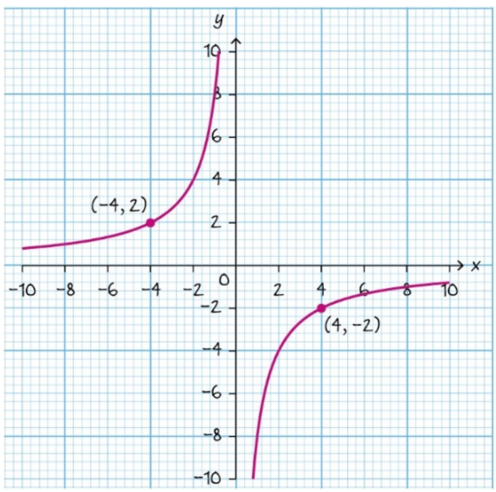
c



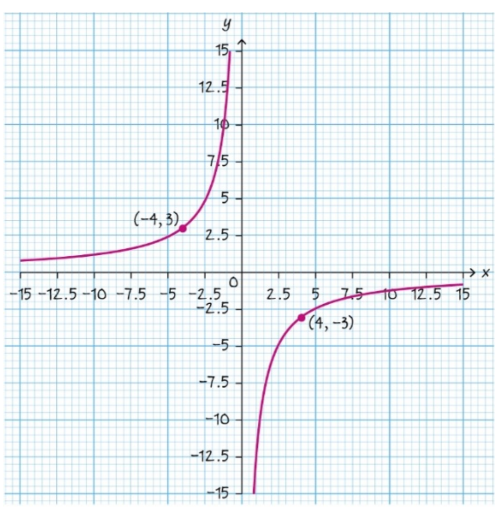
2



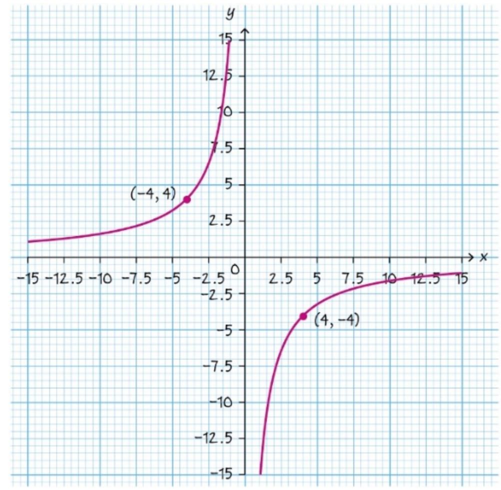
3 a



b



c

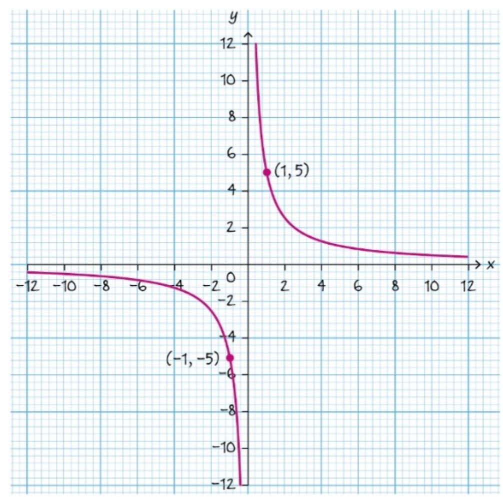


d The curves are in the opposite quadrants. The negative reflects the function in the x-axis.

4 x = 0, y = 0

Domain: $x \in$ ¡ $, x \neq 0$

Range: $y \in$ ¡ $, y \neq 0$



Exercise 4C

1 a $x = 2 \Rightarrow y = \frac{2}{x} = \frac{2}{2} = 1$

b $y = 4$

$$y = \frac{2}{x}$$

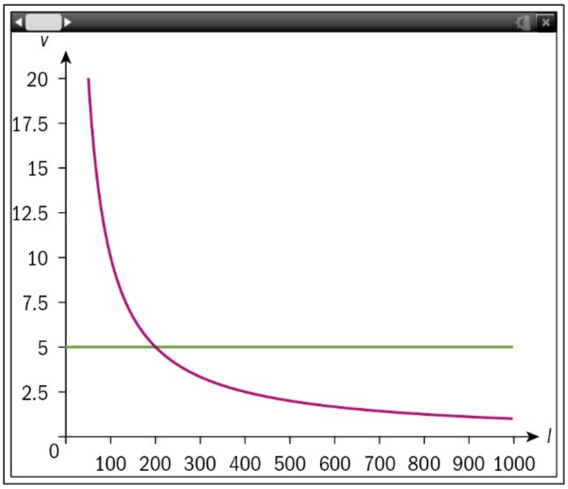
$$\frac{2}{x} = 4$$

$$x = \frac{2}{4}$$

$$x = 0.5$$

Chamse spends 30 seconds brushing her teeth.

2 **a** and **c**



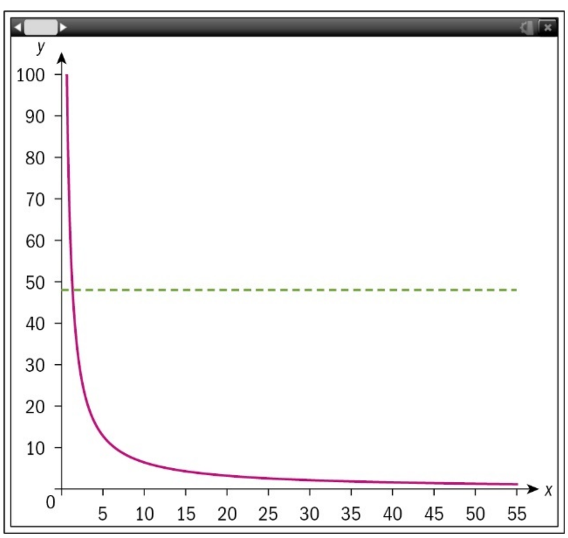
b $l = 10 \Rightarrow v = \dfrac{1000}{10} = 100$ Hz

c A string 5 cm long has vibrations of frequency 200 Hz.

3 **a** $y = \dfrac{64}{16} = 4$ videos of length 16 minutes

b $y = \dfrac{64}{x}$ is the equation that models the number of videos of x minutes.

c and **d**



1.33 minutes

Exercise 4D

1 **a** $y = \dfrac{1}{x+1}$

The vertical asymptote is at $x = -1$ and the horizontal asymptote at $y = 0$.

The domain is $x \in$ ¡ $, x + 1 \neq 0 \Leftrightarrow x \neq -1$.

The range is $y \in$ ¡ , ¡ $- \{0\}$.

b $y = \dfrac{1}{x-5}$

The vertical asymptote is at $x = -(-5) = 5$ and the horizontal asymptote at $y = 0$.

The domain is $x \in$ ¡ $, x - 5 \neq 0 \Leftrightarrow x \neq 5$.

The range is $y \in$ ¡ , ¡ $- \{0\}$.

c $y = \dfrac{-1}{x-4}$

The vertical asymptote is at $x - 4 = 0 \Leftrightarrow x = 4$ and the horizontal asymptote at $y = 0$.

The domain is $x \in$ ¡ $, x - 4 \neq 0 \Leftrightarrow x \neq 4$.

The range is $y \in$ ¡ , ¡ $- \{0\}$.

d $y = \dfrac{5}{x+5}$

The vertical asymptote is at $x + 5 = 0 \Leftrightarrow x = -5$ and the horizontal asymptote at $y = 0$.

The domain is $x \in$ ¡ $, x + 5 \neq 0 \Leftrightarrow x \neq -5$.

The range is $y \in$ ¡ , ¡ $- \{0\}$.

e $y = \dfrac{12}{x+1} + 2$

The vertical asymptote is at $x + 1 = 0 \Leftrightarrow x = -1$ and the horizontal asymptote at $y = 2$.

The domain is $x \in$ ¡ $, x + 1 \neq 0 \Leftrightarrow x \neq -1$.

The range is $y \in$ ¡ , ¡ $- \{2\}$.

f $y = \dfrac{12}{x+1} - 2$

The vertical asymptote is at $x + 1 = 0 \Leftrightarrow x = -1$ and the horizontal asymptote at $y = -2$.

The domain is $x \in$ ¡ $, x + 1 \neq 0 \Leftrightarrow x \neq -1$.

The range is $y \in$ ¡ , ¡ $- \{-2\}$.

g $y = \dfrac{4}{x-3} + 2$

The vertical asymptote is at $x - 3 = 0 \Leftrightarrow x = 3$ and the horizontal asymptote at $y = 2$.

The domain is $x \in$ ¡ $, x - 3 \neq 0 \Leftrightarrow x \neq 3$.

The range is $y \in$ ¡ , ¡ $- \{2\}$.

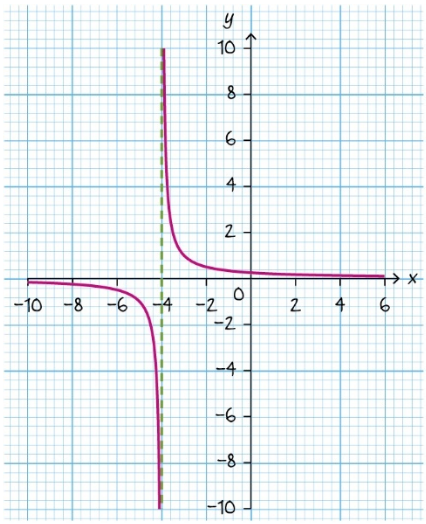
h $y = \dfrac{-4}{x-4} - 4$

The vertical asymptote is at $x - 4 = 0 \Leftrightarrow x = 4$ and the horizontal asymptote at $y = -4$.

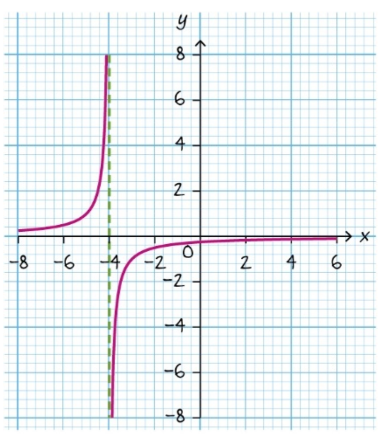
The domain is $x \in \mathbb{R}, x - 4 \neq 0 \Leftrightarrow x \neq 4$.

The range is $y \in \mathbb{R}, \mathbb{R} - \{-4\}$.

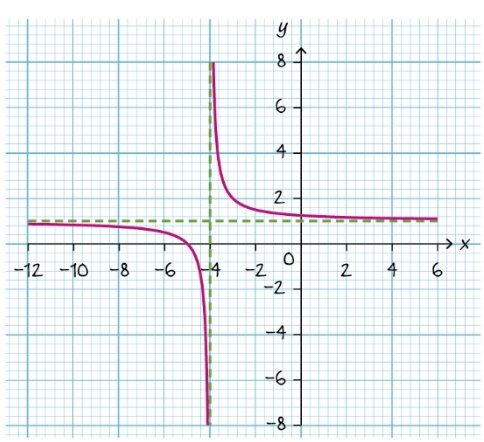
2 **a** $x \in \mathbb{R}, x \neq -4 \quad y \in \mathbb{R}, y \neq 0$



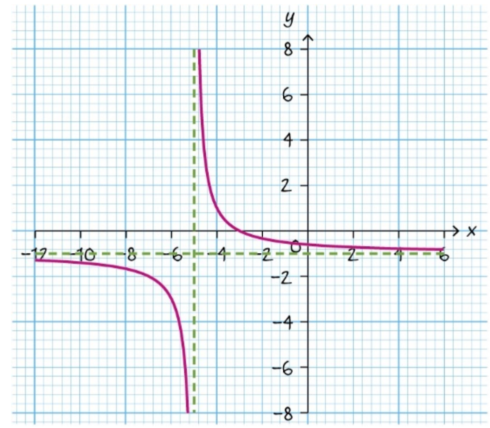
b $x \in \mathbb{R}, x \neq -4 \quad y \in \mathbb{R}, y \neq 0$



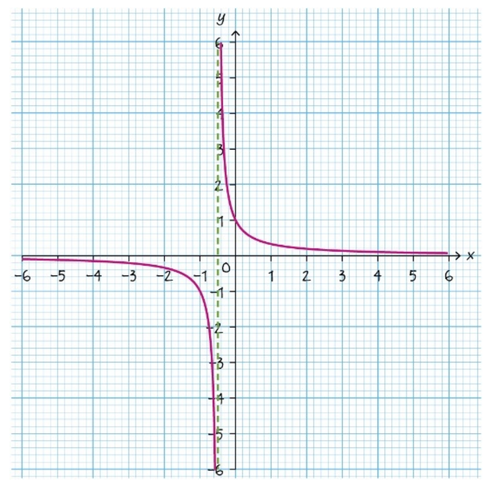
c $x \in \mathbb{R}, x \neq -4 \quad y \in \mathbb{R}, y \neq 1$



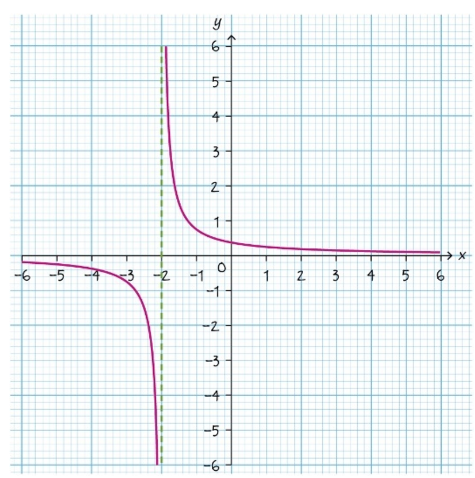
d $x \in \mathbb{R}, x \neq -5 \quad y \in \mathbb{R}, y \neq 1$



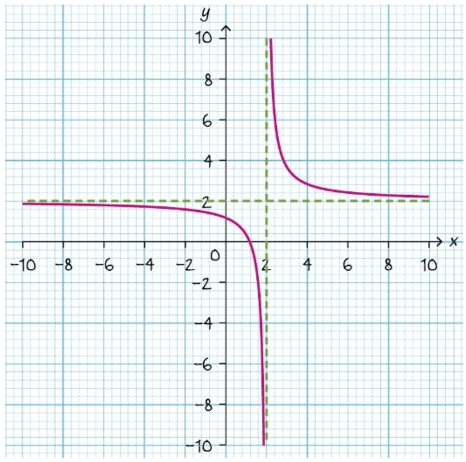
e $x \in \mathbb{R}, x \neq -0.5 \quad y \in \mathbb{R}, y \neq 0$



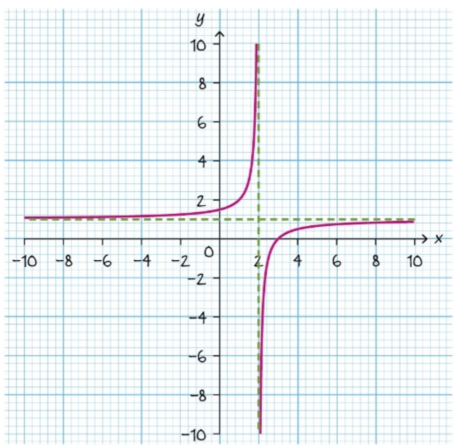
f $x \in \mathbb{R}, x \neq -2 \quad y \in \mathbb{R}, y \neq 0$



g $x \in \mathbb{R}, x \neq 2 \qquad y \in \mathbb{R}, y \neq 2$



h $x \in \mathbb{R}, x \neq 2 \qquad y \in \mathbb{R}, y \neq 1$



3 **a** **2:** Translation of 2 units right

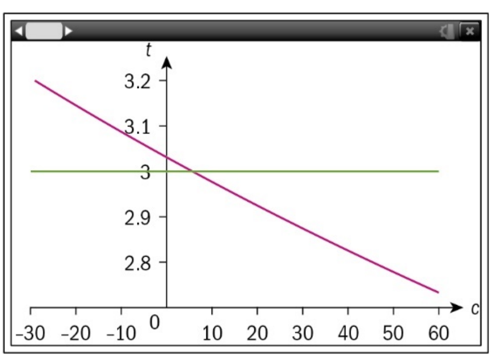
b **5:** Reflection in $y = 0$ and a translation of 2 units right

c **1:** Translation of 2 units right and 2 units up

d **4:** Translation of 2 units right and 2 units down

e **3:** Translation of 2 units right and vertical stretch by a factor of 3

4 **a**



b 5.56

c $t = 6$

$$6 = \frac{1000}{0.6c + 330}$$

$$6(0.6c + 330) = 1000$$

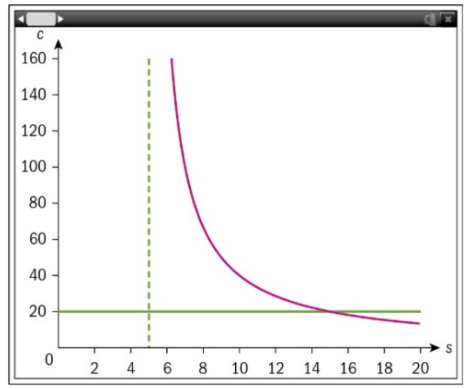
$$3.6c + 1980 = 1000$$

$$c = \frac{1000 - 1980}{3.6}$$

$$c = -272.22°$$

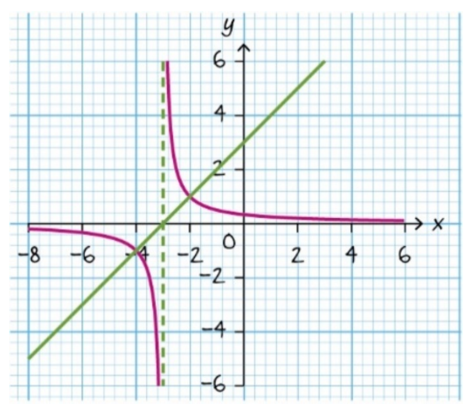
5 **a** $c = \dfrac{200}{s - 5}$

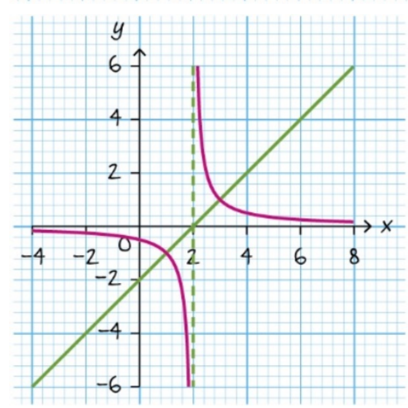
The vertical asymptote is at $s - 5 = 0 \Leftrightarrow s = 5$ and the horizontal asymptote at $c = 0$.



b 15 sessions.

6





The linear function is a line of symmetry for the rational function. The linear function crosses the x-axis at the same place as the vertical asymptote.

Exercise 4E

1 a $y = \dfrac{x+1}{x-1} \Rightarrow a = 1, b = 1, c = 1, d = -1$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{(-1)}{1} = 1$ and the horizontal

asymptote at $y = \dfrac{a}{c} = \dfrac{1}{1} = 1$

Domain $x \in \mathbb{R}, x \neq 1$.

Range $y \in \mathbb{R}, y \neq 1$.

b $y = \dfrac{2x+3}{x+1} \Rightarrow a = 2, b = 3, c = 1, d = 1$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{1}{1} = -1$ and the horizontal

asymptote at $y = \dfrac{a}{c} = \dfrac{2}{1} = 2$.

Domain $x \in \mathbb{R}, x \neq -1$.

Range $y \in \mathbb{R}, y \neq 2$.

c $y = \dfrac{6x-1}{2x+4} \Rightarrow a = 6, b = -1, c = 2, d = 4$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{4}{2} = -2$ and the horizontal

asymptote at $y = \dfrac{a}{c} = \dfrac{6}{2} = 3$.

Domain $x \in \mathbb{R}, x \neq -2$.

Range $y \in \mathbb{R}, y \neq 3$.

d $y = \dfrac{2-3x}{5-4x} \Rightarrow a = -3, b = 2, c = -4, d = 5$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{5}{(-4)} = 1.25$ and the

horizontal asymptote at

$y = \dfrac{a}{c} = \dfrac{-3}{-4} = 0.75$.

Domain $x \in \mathbb{R}, x \neq 1.25$.

Range $y \in \mathbb{R}, y \neq 0.75$.

e $y = \dfrac{9x-2}{6-3x} \Rightarrow a = 9, b = -2, c = -3, d = 6$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{6}{(-3)} = 2$ and the horizontal

asymptote at $y = \dfrac{a}{c} = \dfrac{9}{(-3)} = -3$.

Domain $x \in \mathbb{R}, x \neq 2$.

Range $y \in \mathbb{R}, y \neq -3$.

2 i B

$a = 1, b = -3, c = 1, d = 2$

Vertical asymptote: $x = -\dfrac{d}{c} = -2$

Horizontal asymptote: $y = \dfrac{a}{c} = 1$

ii A

$a = 0, b = 4, c = 1, d = 0$

Vertical asymptote: $x = -\dfrac{d}{c} = 0$

Horizontal asymptote: $y = \dfrac{a}{c} = 0$

iii D

$a = -2, b = 3, c = 1, d = 2$

Vertical asymptote: $x = -\dfrac{d}{c} = -2$

Horizontal asymptote: $y = \dfrac{a}{c} = -2$

iv C

$a = 2, b = -3, c = 1, d = 2$

Vertical asymptote: $x = -\dfrac{d}{c} = -2$

Horizontal asymptote: $y = \dfrac{a}{c} = 2$

3 $y = \dfrac{x-p}{x-q} \Rightarrow a = 1, b = -p, c = 1, d = -q$

The vertical asymptote is at

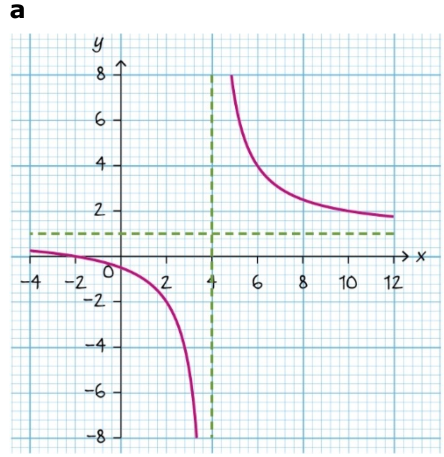
$x = -\dfrac{d}{c} = -\dfrac{(-q)}{1} = q$ and the horizontal

asymptote at $y = \dfrac{a}{c} = \dfrac{1}{1} = 1$.

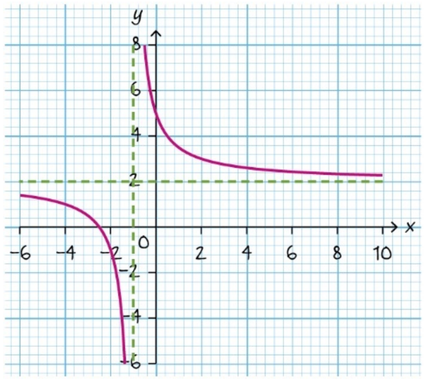
Domain $x \in \mathbb{R}, x \neq q$.

Range $y \in \mathbb{R}, y \neq 1$.

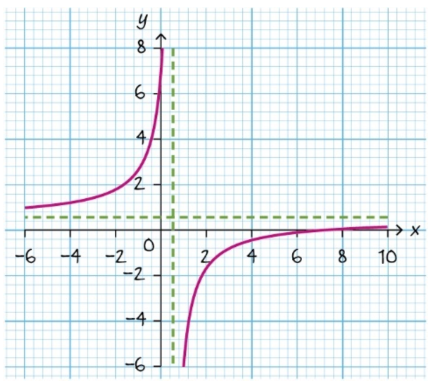
4 a



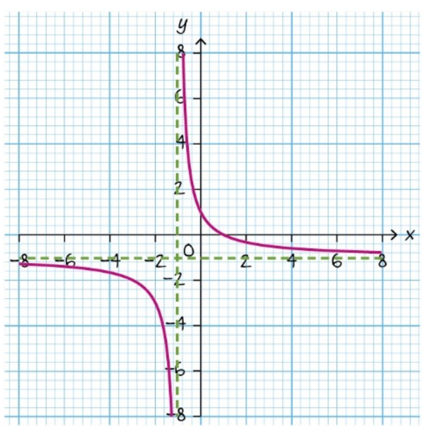
b



c



d



5 **a** $\dfrac{5}{2x}+\dfrac{x+7}{x+4}=2$

$\dfrac{5(x+4)+2x(x+7)}{2x(x+4)}=2$

$5x+20+2x^2+14x=4x(x+4)$

$2x^2+19x+20=4x^2+16x$

$2x^2-3x-20=0$

$2x^2-8x+5x-20=0$

$2x(x-4)+5(x-4)=0$

$(x-4)(2x+5)=0$

So $x=4$ and $x=\dfrac{-5}{2}$.

b $\dfrac{2x-3}{x+1}=\dfrac{x+6}{x-2}$

$(2x-3)(x-2)=(x+1)(x+6)$

$2x^2-3x-4x+6=x^2+6x+x+6$

$x^2-14x=0$

$x(x-14)=0$

So $x=0$ and $x=14$.

c $7-\dfrac{5}{x-2}=\dfrac{10}{x+2}$

$\dfrac{7(x-2)-5}{x-2}=\dfrac{10}{x+2}$

$\dfrac{7x-19}{x-2}=\dfrac{10}{x+2}$

$(7x-19)(x+2)=10(x-2)$

$7x^2+14x-19x-38=10x-20$

$7x^2-15x-18=0$

$(x-3)(7x+6)=0$

So $x=3$ and $x=-\dfrac{6}{7}$.

d $\dfrac{x+5}{x+8}=1+\dfrac{6}{x+1}$

$\dfrac{x+5}{x+8}==\dfrac{x+1+6}{x+1}$

$\dfrac{x+5}{x+8}=\dfrac{x+7}{x+1}$

$(x+5)(x+1)=(x+8)(x+7)$

$x^2+6x+5=x^2+15x+56$

$9x+51=0$

$x=-\dfrac{51}{9}=-\dfrac{17}{3}$

6 $x=3$ is the extraneous solution.
Therefore the solution to Will's equation is $x=2$.

7 **a** $f(x)=\dfrac{x+3}{x-2}$

$x=\dfrac{y+3}{y-2}$

$x(y-2)=y+3$

$xy-2x=y+3$

$xy-y=2x+3$

$y(x-1)=2x+3$

$y=\dfrac{2x+3}{x-1}$

$f^{-1}(x)=\dfrac{2x+3}{x-1}$

b $f(x) = \dfrac{7-2x}{x}$

$x = \dfrac{7-2y}{y}$

$xy = 7-2y$

$y(x+2) = 7$

$y = \dfrac{7}{x+2}$

$f^{-1}(x) = \dfrac{7}{x+2}$

c $f(x) = \dfrac{1+7x}{9-x}$

$x = \dfrac{1+7y}{9-y}$

$x(9-y) = 1+7y$

$9x - xy = 1+7y$

$y(7+x) = 9x-1$

$y = \dfrac{9x-1}{7+x}$

$f^{-1}(x) = \dfrac{9x-1}{x+7}$

d $f(x) = \dfrac{5-11x}{x+6}$

$x = \dfrac{5-11y}{y+6}$

$x(y+6) = 5-11y$

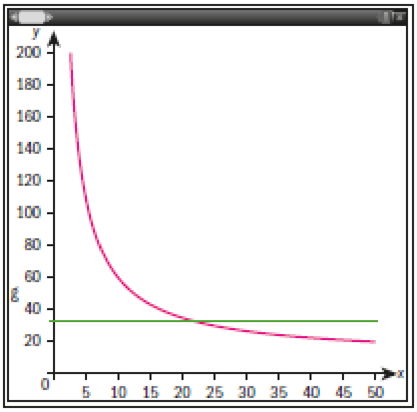
$xy+6x = 5-11y$

$y(x+11) = 5-6x$

$y = \dfrac{5-6x}{x+11}$

$f^{-1}(x) = \dfrac{5-6x}{x+11}$

8 **a** and **c**



b 20

c $M(s) = \dfrac{10s+500}{s} = 20$

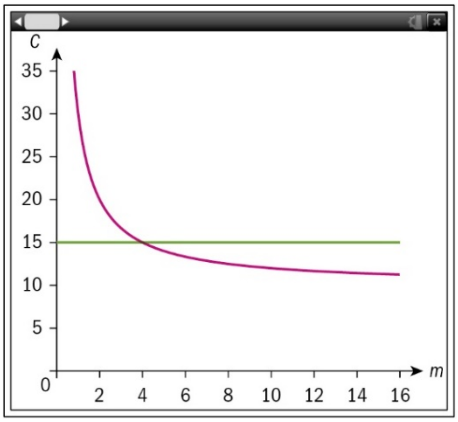
$10s + 500 = 20s$

$500 = 10s$

$s = 50$

9 **a** $C(m) = \dfrac{20+10m}{m}$ as 20 is the initial cost and them for every month there is another 10 AUD cost.

b



c 4 months

d The price will get closer to the horizontal asymptote $y = 10$.

10 a $f(x) = m + \dfrac{6}{x-n}$

$= \dfrac{m(x-n)+6}{x-n}$

$= \dfrac{mx - mn + 6}{x-n}$

$a = m$

$b = 6 - mn$

$c = 1$

$d = -n$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{(-n)}{1} = n = 5.$

Hence $n = 5$.

b $f(7) = 7$

$f(7) = m + \dfrac{6}{7-5} = m + \dfrac{6}{2}$

$f(7) = m + 3 = 7$

$m = 4$

c The vertical asymptote is at

$x = \dfrac{a}{c} = \dfrac{4}{1} = 4.$

11 a $y = \dfrac{4}{x-2} + 3 = \dfrac{4+3(x-2)}{x-2} = \dfrac{3x-2}{x-2}$

$a = 3$

$b = -2$

$c = 1$

$d = -2$

The horizontal asymptote is at

$y = \frac{a}{c} = \frac{3}{1} = 3.$

b The vertical asymptote is at

$x = -\frac{d}{c} = -\frac{(-2)}{1} = 2.$

c The x-intercept is when $y = 0$.

$\frac{3x-2}{x-2} = 0$

$3x = 2$ The point is

$x = \frac{2}{3}$

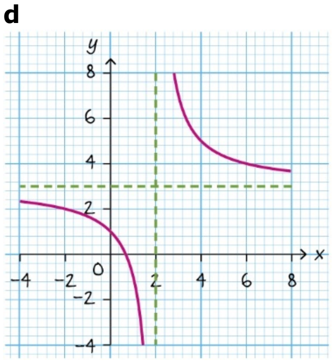
$(\frac{2}{3}, 0) = (0.667, 0).$

The y-intercept is when $x = 0$.

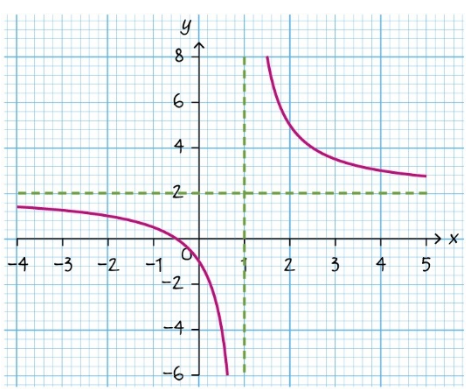
$\frac{3 \cdot 0 - 2}{0 - 2} = \frac{-2}{-2} = 1 = y$

The point is $(0, 1)$.

d



12 a



b

$f(x) = \frac{2x+1}{x-1} \Rightarrow a = 2, b = 1, c = 1, d = -1$

The horizontal asymptote is at

$y = \frac{a}{c} = \frac{2}{1} = 2.$

The vertical asymptote is at

$x = -\frac{d}{c} = -\frac{(-1)}{1} = 1.$

c $f(x) = 0$

$\frac{2x+1}{x-1} = 0$

$2x + 1 = 0$

$x = -\frac{1}{2}$

The x-intercept of f is at point

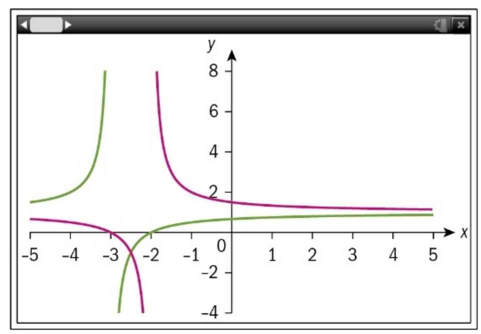
$(-\frac{1}{2}, 0) = (-0.5, 0).$

13 a $g \circ f(x) = g(f(x))$

$= g(\frac{x+2}{x+3}) = \frac{1}{\frac{x+2}{x+3}}$

$= \frac{x+3}{x+2}$

b



$x = -2.5$

Chapter review

1 a $y = \frac{2}{x} \Rightarrow a = 0, b = 2, c = 1, d = 0$

The horizontal asymptote is at

$y = \frac{a}{c} = \frac{0}{1} = 0.$

The vertical asymptote is at

$x = -\frac{d}{c} = -\frac{0}{1} = 0.$

Domain: $x \in \mathbb{R}, x \neq 0$

Range: $y \in \mathbb{R}, y \neq 0$

b $y = \frac{1}{x+8} \Rightarrow a = 0, b = 1, c = 1, d = 8$

The horizontal asymptote is at

$y = \frac{a}{c} = \frac{0}{1} = 0.$

The vertical asymptote is at

$x = -\frac{d}{c} = -\frac{8}{1} = -8.$

Domain: $x \in \mathbb{R}, x \neq -8$

Range: $y \in \mathbb{R}, y \neq 0$

c

$y = \dfrac{x}{2x - 10} \Rightarrow a = 1, b = 0, c = 2, d = -10$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{1}{2}.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{-10}{2} = 5.$

Domain: $x \in \mathbb{R}, x \neq 5$

Range: $y \in \mathbb{R}, y \neq \dfrac{1}{2}$

d $y = \dfrac{3}{x-2} + 3 = \dfrac{3 + 3(x-2)}{x-2} = \dfrac{3x-3}{x-2}$

$\Rightarrow a = 3, b = -3, c = 1, d = -2$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{3}{1} = 3.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{-2}{1} = 2.$

Domain: $x \in \mathbb{R}, x \neq 2$

Range: $y \in \mathbb{R}, y \neq 3$

e $y = \dfrac{2x}{x-9} \Rightarrow a = 2, b = 0, c = 1, d = -9$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{2}{1} = 2.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{-9}{1} = 9.$

Domain: $x \in \mathbb{R}, x \neq 9$

Range: $y \in \mathbb{R}, y \neq 2$

f $y = \dfrac{8x-5}{2x+4} \Rightarrow a = 8, b = -5, c = 2, d = 4$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{8}{2} = 4.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{4}{2} = -2.$

Domain: $x \in \mathbb{R}, x \neq -2$

Range: $y \in \mathbb{R}, y \neq 4$

g $y = \dfrac{1-x}{x+4} \Rightarrow a = -1, b = 1, c = 1, d = 4$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{-1}{1} = -1.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{4}{1} = -4.$

Domain: $x \in \mathbb{R}, x \neq -4$

Range: $y \in \mathbb{R}, y \neq -1$

h $y = \dfrac{2x-1}{2x+6} - 4 = \dfrac{2x - 1 - 4(2x+6)}{2x+6}$

$= \dfrac{2x - 1 - 8x - 24}{2x+6} = \dfrac{-6x - 25}{2x+6}$

$\Rightarrow a = -6, b = -25, c = 2, d = 6$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{-6}{2} = -3.$

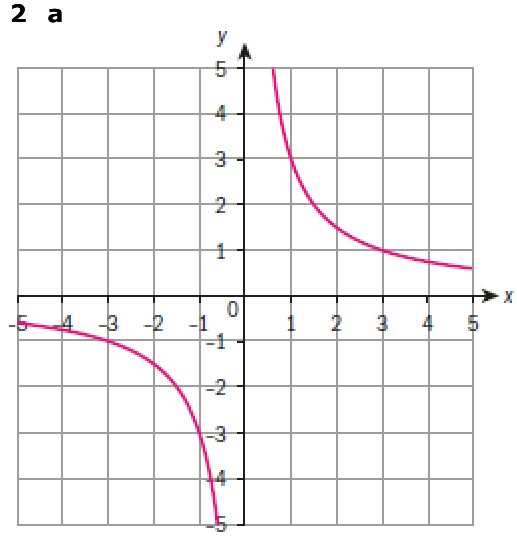
The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{6}{2} = -3.$

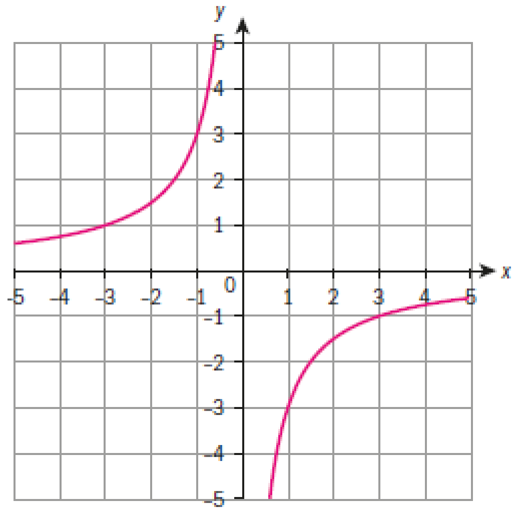
Domain: $x \in \mathbb{R}, x \neq -3$

Range: $y \in \mathbb{R}, y \neq -3$

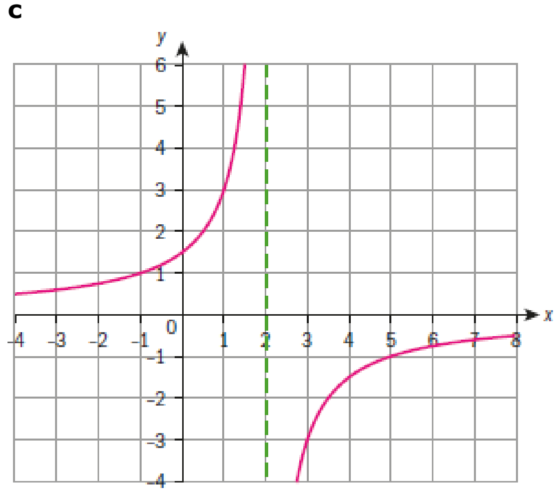
2 a



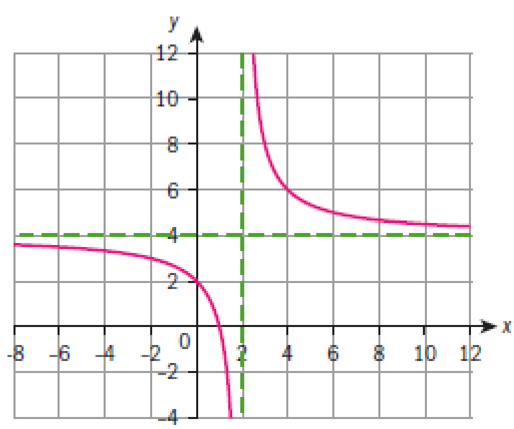
b



c



d



3 a $f(x) = \dfrac{1}{x-1} + 2 = \dfrac{1+2(x-1)}{x-1} = \dfrac{2x-1}{x-1}$

$\Rightarrow a = 2, b = -1, c = 1, d = -1$

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{2}{1} = 2.$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{-1}{1} = 1.$

b The x-intercept is $(\frac{1}{2}, 0) = (0.5, 0)$ as:

$f(x) = 0$

$\dfrac{2x-1}{x-1} = 0$

$2x - 1 = 0$

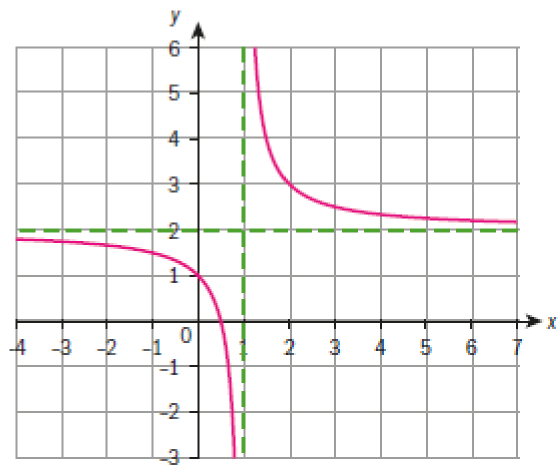
$x = \dfrac{1}{2}$

The y-intercept is $(0, 1)$ as:

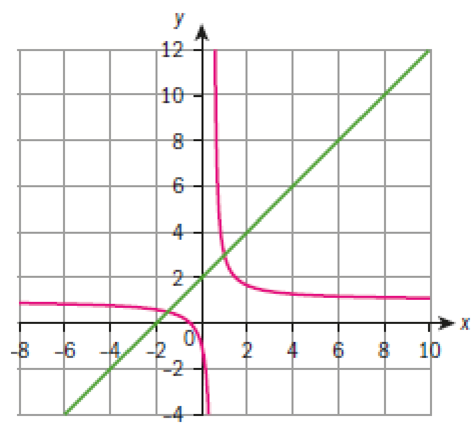
$x = 0$

$f(0) = \dfrac{2 \cdot 0 - 1}{0 - 1} = 1$

c



4



$x = -1.5,\ 1$

5 a 1.29, 2.71 **b** 2.71 **c** 1.27

6 a $f(x) = 0$

$\dfrac{2x-8}{1-x} = 0$

$2x - 8 = 0$

$x = \dfrac{8}{2} = 4$

The x-intercept is therefore $(4,0)$.

b $f(x)=\dfrac{2x-8}{1-x}$

$\Rightarrow a=2, b=-8, c=-1, d=1$

The vertical asymptote is at

$x=-\dfrac{d}{c}=-\dfrac{1}{-1}=1.$

c The horizontal asymptote is at

$y=\dfrac{a}{c}=\dfrac{2}{-1}=-2.$

7 a $f(x)=\dfrac{ax+b}{x-d}$

The vertical asymptote is at

$x=-\dfrac{-d}{1}=d.$

The horizontal asymptote is at

$y=\dfrac{a}{1}=a.$

Hence $3=d$ and $2=a$.

b $f(1)=\dfrac{a+b}{1-d}=\dfrac{2+b}{1-3}=-4$

$f(1)=\dfrac{2+b}{-2}=-4$

$2+b=8$

$b=6.$

8 a $f(x)=\dfrac{5}{x-m}+n=\dfrac{5+n(x-m)}{x-m}$

$=\dfrac{nx-mn+5}{x-m}$

$a=n, b=-mn+5, c=1, d=-m$

$4=-\dfrac{d}{c}=-\dfrac{-m}{1}=m$

b $f(0)=7$

$f(0)=\dfrac{n\cdot 0-4n+5}{0-4}=\dfrac{-4n+5}{-4}=7$

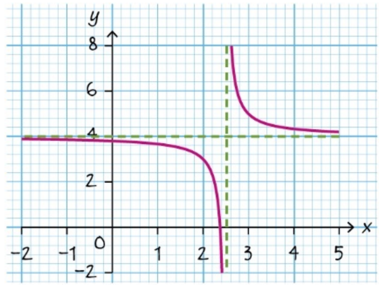
$4n-5=28$

$4n=33$

$n=\dfrac{33}{4}$

c $y=\dfrac{\frac{33}{4}}{1}=\dfrac{33}{4}$

9 a The x-intercept is $\left(-\dfrac{1}{2},0\right)$



b $x = 2.5$, $y = 4$

c 2.375 **d** 3.8

10 a $x=\dfrac{2y+1}{y-1}$

$x(y-1)=2y+1$

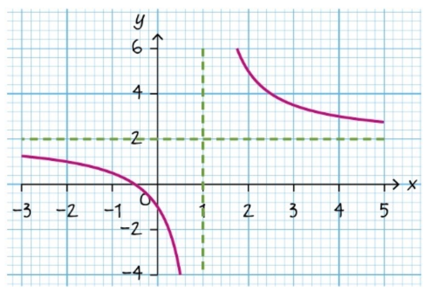
$xy-x=2y+1$

$y(x-2)=x+1$

$y=\dfrac{x+1}{x-2}$

$f^{-1}(x)=\dfrac{x+1}{x-2}$

b



c $a=2, b=1, c=1, d=-1$

The vertical asymptote is at

$x=-\dfrac{d}{c}=-\dfrac{-1}{1}=1.$

The horizontal asymptote is at

$y=\dfrac{a}{c}=\dfrac{2}{1}=2.$

d $f(x)=0$

$\dfrac{2x+1}{x-1}=0$

$2x+1=0$

$x=-\dfrac{1}{2}$

The x-intercept is $(-\dfrac{1}{2},0)$.

e $f(x) = f^{-1}(x)$

$$\frac{2x+1}{x-1} = \frac{x+1}{x-2}$$

$$(2x+1)(x-2) = (x-1)(x+1)$$

$$2x^2 - 3x - 2 = x^2 - 1$$

$$x^2 - 3x - 1 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9+4}}{2}$$

$$= \frac{3 \pm \sqrt{13}}{2} = -0.303, 3.30$$

11 a $f(x) = \dfrac{1}{x-2}$

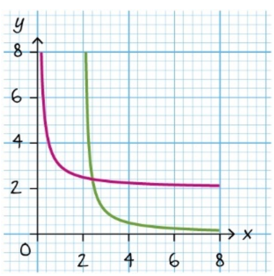
$$x = \frac{1}{y-2}$$

$$xy - 2x = 1$$

$$y = \frac{1+2x}{x}$$

$$f^{-1}(x) = \frac{1+2x}{x} = \frac{1}{x} + 2$$

b



c $\dfrac{1}{x-2} = \dfrac{1+2x}{x}$

$$x = (1+2x)(x-2)$$

$$x = x + 2x^2 - 2 - 4x$$

$$2x^2 - 4x - 2 = 0$$

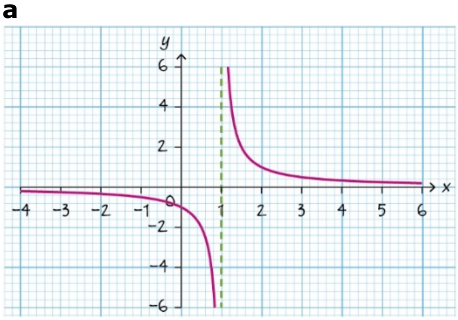
$$x^2 - 2x - 1 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{8}}{2}$$

$$x > 2$$

Hence the solution is $x = 2.41$.

12 a



b $g(x) = \dfrac{1}{x-3} + 3$

c $g(x) = 0$

$$\frac{1}{x-3} + 3 = 0$$

$$\frac{1}{x-3} = -3$$

$$x - 3 = -\frac{1}{3}$$

$$x = 3 - \frac{1}{3} = \frac{8}{3}$$

The x-intercept is $(2.67, 0)$.

$$x = 0$$

$$g(0) = -\frac{1}{3} + 3 = \frac{8}{3}$$

The y-intercept is $(0, 2.67)$.

d $g(x) = \dfrac{1}{x-3} + 3 = \dfrac{1+3(x-3)}{x-3}$

$$= \frac{1+3x-9}{x-3} = \frac{3x-8}{x-3}$$

$$\Rightarrow a = 3, b = -8, c = 1, d = -3$$

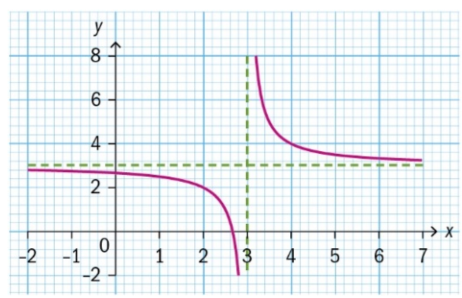
The vertical asymptote is at

$$x = -\frac{d}{c} = -\frac{-3}{1} = 3.$$

The horizontal asymptote is at

$$y = \frac{a}{c} = \frac{3}{1} = 3.$$

e



13 a $f(x) = 2x + 3$

$$x = 2y + 3$$

$$2y = x - 3$$

$$y = \frac{x-3}{2}$$

$$f^{-1}(x) = \frac{x-3}{2}$$

b $g \circ f^{-1}(x) = g\left(\dfrac{x-3}{2}\right) = \dfrac{5}{4\dfrac{x-3}{2}}$

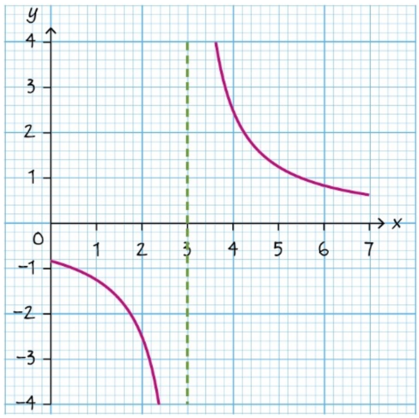
$$= \frac{5}{2(x-3)} = \frac{5}{2x-6}$$

c $x = 0 \Rightarrow h(0) = \dfrac{5}{2 \cdot 0 - 6} = -\dfrac{5}{6}$

The y-intercept of h is

$(0, -\frac{5}{6}) = (0, -0.833)$.

d



e $h(x) = \dfrac{5}{2x-6}$

$x = \dfrac{5}{2y-6}$

$x(2y-6) = 5$

$2xy - 6x = 5$

$y = \dfrac{5+6x}{2x}$

$h^{-1}(x) = \dfrac{5+6x}{2x}$

The x-intercept of h^{-1} is given by

$h^{-1}(x) = 0$

$\dfrac{5+6x}{2x} = 0$

$5 + 6x = 0$

$x = -\dfrac{5}{6}$

The point is therefore

$(-\frac{5}{6}, 0) = (-0.833, 0)$.

f $a = 6, b = 5, c = 2, d = 0$

The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{0}{2} = 0$.

14 $f(x) = 2 + \dfrac{10}{x-4} = \dfrac{2(x-4)+10}{x-4} = \dfrac{2x+2}{x-4}$

$a = 2, b = 2, c = 1, d = -4$

a The vertical asymptote is at

$x = -\dfrac{d}{c} = -\dfrac{-4}{1} = 4$.

The horizontal asymptote is at

$y = \dfrac{a}{c} = \dfrac{2}{1} = 2$.

b The domain is $x \in \mathbb{R}, x - 4 \neq 0 \Leftrightarrow x \neq 4$.

The range is $y \in \mathbb{R}, \mathbb{R} - \{2\}$.

c The x-intercept:

$f(x) = 0$

$\dfrac{2x+2}{x-4} = 0$

$2x + 2 = 0$

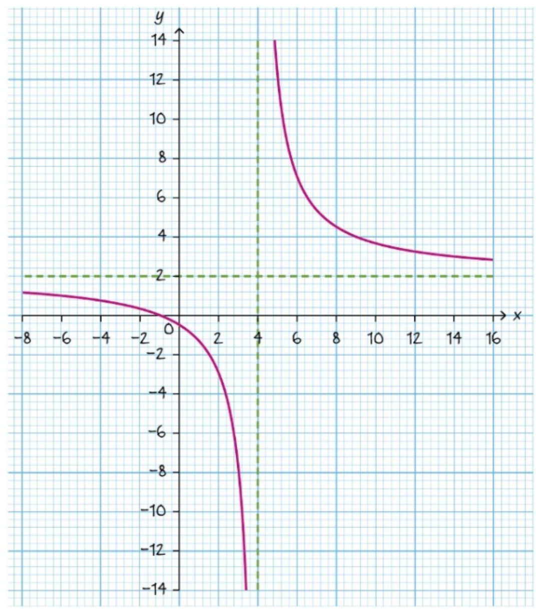
$x = -1$

The point $(-1, 0)$.

The y-intercept: $f(0) = \dfrac{2}{-4} = -0.5$

The point $(0, -0.5)$.

d



e Horizontal shift of 4 units right and a vertical shift of 2 units up.

15 a $x \in \mathbb{R}, x \neq -2$ A1

b $f(x) \in \mathbb{R}, f(x) \neq \dfrac{3}{2}$ A1

c When $x = 0$, $f(x) = -\dfrac{20}{4} = -5$.

So one coordinate is $(0. -5)$ A1

When $y = 0$, $x = \dfrac{20}{3}$

So the other coordinate is $\left(\dfrac{20}{3}, 0\right)$

A1

16 a Domain is $x \in \mathbb{R}, x \neq -2$

Range is $f(x) \in \mathbb{R}, f(x) \neq 0$ A1A1

b Domain is $x \in \mathbb{R}, x \neq -2$

Range is $f(x) \in \mathbb{R}, f(x) \neq 4$ A1A1

c Domain is $x \in \mathbb{R}, x \neq 0$

Range is $f(x) \in \mathbb{R}, f(x) \neq 4$ A1A1

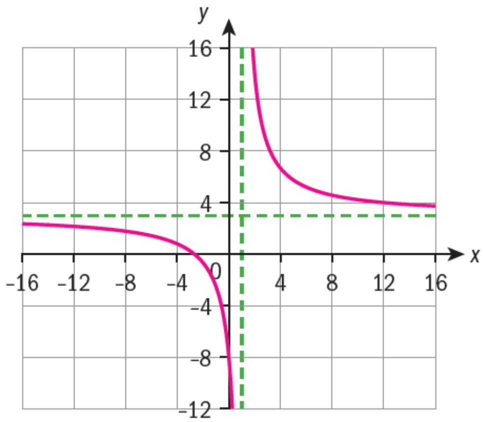
d Domain is $x \in \mathbb{R}\ ,\ x \neq 0$

Range is $f(x) \in \mathbb{R}\ ,\ f(x) \neq 0$ A1A1

17 a $x = 1$ A1

b $y = 3$ A1

c



A3

18 a $y = 10$ A1

b $x = 2$ A1

c $f(x) = 10 + \dfrac{3}{2-x} = \dfrac{10(2-x)+3}{2-x}$

M1A1

$= \dfrac{-10x+23}{-x+2}$ A1

19 a Vertical asymptote occurs when

$c + 8x = 0$ M1

$c + 8\left(-\tfrac{3}{4}\right) = 0$

$c = 6$ A1

b $y = \dfrac{a+bx}{6+8x}$

Substituting the first coordinate:

M1

$\dfrac{2}{5} = \dfrac{a+\frac{1}{2}b}{10}$

$4 = a + \tfrac{1}{2}b$

$8 = 2a + b\ \ (1)$ A1

Substituting the second coordinate:

$-\dfrac{3}{38} = \dfrac{a+4b}{38}$

$-3 = a + 4b\,(2)$ A1

Solving **(1)** and **(2)** simultaneously:

$a = 5$ A1

$b = -2$ A1

20 a 6 A1

b $P = \dfrac{18(1+0.82\times 12)}{3+(0.034\times 12)} \approx 57$ M1A1

c Solving $100 = \dfrac{18(1+0.82t)}{3+0.034t}$ M1

$300 + 3.4t = 18(1+0.82t)$

$300 + 3.4t = 18 + 14.76t$

$282 = 11.36t$

$t = \dfrac{282}{11.36} = 24.8$ months A1

d A horizontal asymptote exists at

$P = \dfrac{18\times 0.82}{0.034} = 434.12$ M1A1

Therefore for $t \geq 0\ ,\ P < 435$ R1

21 a $f(x) = \dfrac{17-10x}{2x-1} = \dfrac{12+5-10x}{2x-1}$ M1A1

$= \dfrac{12+5(1-2x)}{2x-1}$ A1

$= \dfrac{12-5(2x-1)}{2x-1}$

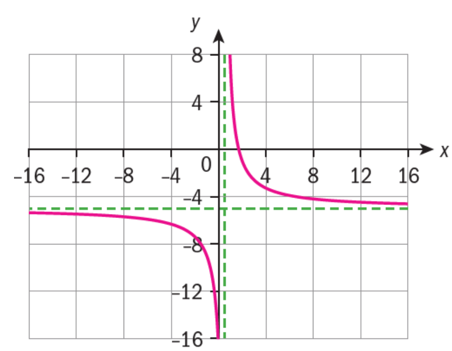
$= \dfrac{12}{2x-1} - \dfrac{5(2x-1)}{(2x-1)}$

$= \dfrac{12}{2x-1} - 5$ A1

b $x = \dfrac{1}{2}$ A1

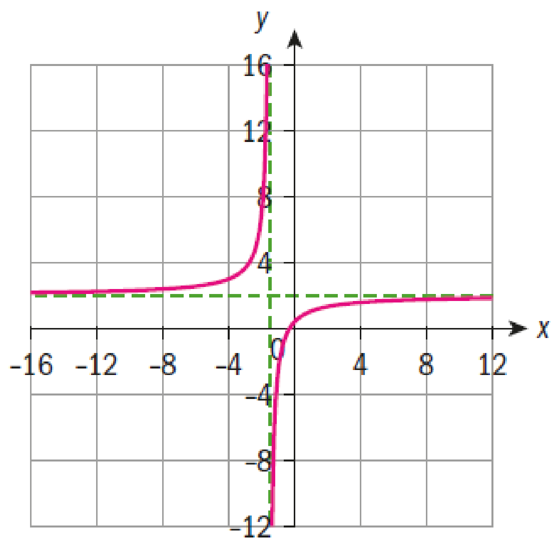
c $y = -5$ A1

d



A3

22



A2

Asymptotes are $x = -\frac{3}{2}$ and $y = 2$

A1A1

Intersections with axes are at $\left(0, \frac{1}{3}\right)$ and $\left(-\frac{1}{4}, 0\right)$

A1A1

5 Measuring change: differentiation

Skills check

1 **a** $\dfrac{-3-0}{-4-0}=\dfrac{3}{4}$

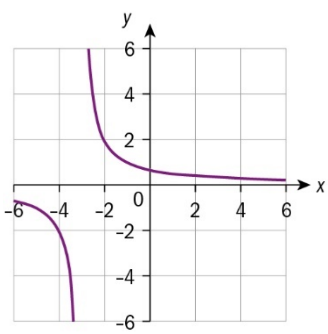
b $\dfrac{-1-2}{4-\left(-\frac{3}{4}\right)}=-\dfrac{12}{19}$

2 **a** $7\sqrt{x}=7x^{\frac{1}{2}}$

b $\dfrac{1}{x^2}=x^{-2}$

c $\dfrac{8}{5\sqrt{x^3}}=\dfrac{8}{5}x^{-\frac{3}{2}}$

3



4 Since $|\frac{1}{2}|<1$,

$$\sum_{n=0}^{\infty}5\left(\frac{1}{2}\right)^n=5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{5(1)}{1-\frac{1}{2}}=10$$

Exercise 5A

1 $\lim\limits_{x\to3^-}\left(x^2+1\right)=\lim\limits_{x\to3^+}\left(x^2+1\right)=10$

2 $\lim\limits_{x\to1^-}(5-2x)=\lim\limits_{x\to1^+}(5-2x)=3$

3 $\lim\limits_{x\to0^-}\left(\dfrac{2x^2-x}{x}\right)=\lim\limits_{x\to0^+}\left(\dfrac{2x^2-x}{x}\right)=-1$

4 $\lim\limits_{x\to1^-}\left(\dfrac{x^2-x}{x-1}\right)=\lim\limits_{x\to1^+}\left(\dfrac{x^2-x}{x-1}\right)=1$

Exercise 5B

1 Vertical asymptote at $x=\dfrac{1}{6}$

since $\lim\limits_{x\to\frac{1}{6}^-}f(x)=-\infty$ and $\lim\limits_{x\to\frac{1}{6}^+}f(x)=\infty$

Horizontal asymptote at $y=\dfrac{1}{2}$

since $\lim\limits_{x\to\pm\infty}f(x)=\dfrac{1}{2}$

2 Vertical asymptotes at $x=\pm\sqrt{3}$

since $\lim\limits_{x\to-\sqrt{3}^-}=-\infty$ and $\lim\limits_{x\to-\sqrt{3}^+}=\infty$

and $\lim\limits_{x\to\sqrt{3}^-}=\infty$ and $\lim\limits_{x\to\sqrt{3}^+}=-\infty$

Horizontal asymptote at $y=-1$

since $\lim\limits_{x\to\pm\infty}g(x)=-1$

3 Vertical asymptote at $x=1$

since $\lim\limits_{x\to1^-}f(x)=\infty$ and $\lim\limits_{x\to1^+}f(x)=-\infty$

Horizontal asymptote at $y=-1$

since $\lim\limits_{x\to\pm\infty}h(x)=-1$

4 Vertical asymptotes at $x=\pm\sqrt{2}$

since $\lim\limits_{x\to-\sqrt{2}^-}=\infty$ and $\lim\limits_{x\to-\sqrt{2}^+}=-\infty$

and $\lim\limits_{x\to\sqrt{2}^-}=\infty$ and $\lim\limits_{x\to\sqrt{2}^+}=-\infty$

Horizontal asymptote at $y=0$

since $\lim\limits_{x\to\pm\infty}\left(-\dfrac{5x}{x^2-2}\right)=0$

Exercise 5C

1 $f'(x)=7x^{7-1}=7x^6$

2 $f'(x)=18x^{18-1}=18x^{17}$

3 $f'(x)=-\dfrac{1}{2}x^{-\frac{1}{2}-1}=-\dfrac{1}{2}x^{-\frac{3}{2}}$

4 $f(x)=\sqrt[5]{x}=x^{\frac{1}{5}}\Rightarrow f'(x)=\dfrac{1}{5}x^{\frac{1}{5}-1}=\dfrac{1}{5}x^{-\frac{4}{5}}$

5 $f(x)=\dfrac{1}{\sqrt{x}}=x^{-\frac{1}{2}}\Rightarrow f'(x)=-\dfrac{1}{2}x^{-\frac{1}{2}-1}=-\dfrac{1}{2}x^{-\frac{3}{2}}$

6 $f(x)=\sqrt[4]{x^3}=x^{\frac{3}{4}}\Rightarrow f'(x)=\dfrac{3}{4}x^{\frac{3}{4}-1}=\dfrac{3}{4}x^{-\frac{1}{4}}$

Exercise 5D

1 **a** $\dfrac{dy}{dx}=4x^3-x$

b $f(x)=5x(x^2-1)=5x^3-5x$

$\therefore f'(x)=15x^2-5$

c $f'(x)=24x^3-6x$ **d** $\dfrac{ds}{dt}=4t+3$

e $\dfrac{dv}{dt}=-9.8$ **f** $\dfrac{dc}{dx}=24$

2 a $f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}} \quad \therefore f'(x) = 3x^{-\frac{1}{2}}$

b $f(x) = 5\sqrt[5]{x^3} = 5x^{\frac{3}{5}} \quad \therefore f'(x) = 3x^{-\frac{2}{5}}$

c $f(x) = \dfrac{2}{x} - 3\sqrt{x} = 2x^{-1} - 3x^{\frac{1}{2}}$

$\therefore f'(x) = -2x^{-2} - \dfrac{3}{2}x^{-\frac{1}{2}}$

3 a $f(x) = \dfrac{3}{2x^2} = \dfrac{3}{2}x^{-2} \quad \therefore f'(x) = -3x^{-3}$

b $f(x) = \dfrac{3}{(2x)^2} = \dfrac{3}{4x^2} = \dfrac{3}{4}x^{-2}$

$\therefore f'(x) = -\dfrac{3}{2}x^{-3}$

c $f'(x) = 12\pi x^2$

d $f(x) = (x+1)^2 = x^2 + 2x + 1$

$\therefore f'(x) = 2x + 2$

e $f(x) = \dfrac{x^3 + x - 3}{x} = x^2 + 1 - \dfrac{3}{x}$

$= x^2 + 1 - 3x^{-1} \therefore f'(x) = 2x + 3x^{-2}$

f $f(x) = (2x-1)(x^2+3)$

$= 2x^3 - x^2 + 6x - 3$

$\therefore f'(x) = 6x^2 - 2x + 6$

4 a $y = 1 + x\sqrt{x} = 1 + x^{\frac{3}{2}} \quad \therefore \dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}}$

b $y = \dfrac{7}{x^2} - \dfrac{1}{\sqrt{x}} = 7x^{-2} - x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -14x^{-3} + \dfrac{1}{2}x^{-\frac{3}{2}}$

c $y = \sqrt[3]{x} + \sqrt[4]{x} = x^{\frac{1}{3}} + x^{\frac{1}{4}}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}} + \dfrac{1}{4}x^{-\frac{3}{4}}$

Exercise 5E

1 a $\dfrac{dy}{dx} = 2x - 4$ so the gradient at

$x = -1$ is $2(-1) - 4 = -6$

b $y = \dfrac{2x^5 - 5}{x} = 2x^4 - \dfrac{5}{x} = 2x^4 - 5x^{-1}$

$\therefore \dfrac{dy}{dx} = 8x^3 + 5x^{-2}$

so the gradient at $(1, -3)$ is

$8(1)^3 + 5(1)^{-2} = 13$

c $f(x) = \sqrt[4]{x} + \dfrac{8}{\sqrt{x}} = x^{\frac{1}{4}} + 8x^{-\frac{1}{2}}$

$\therefore f'(x) = \dfrac{1}{4}x^{-\frac{3}{4}} - 4x^{-\frac{3}{2}}$

so $f'(1) = \dfrac{1}{4}(1)^{-\frac{3}{4}} - 4(1)^{-\frac{3}{2}} = -\dfrac{15}{4}$

2 $f'(x) = 2x^2 - 9x - 3 \quad \therefore 2 = 2x^2 - 9x - 3$

$\Rightarrow 2x^2 - 9x - 5 = (2x+1)(x-5) = 0$

so $x = -\dfrac{1}{2}$ or $x = 5$

when $x = -\dfrac{1}{2}$, $y = f\left(-\dfrac{1}{2}\right) = \dfrac{199}{24}$

when $x = 5$, $y = f(5) = -\dfrac{217}{6}$

so $\left(-\dfrac{1}{2}, \dfrac{199}{24}\right)$ and $\left(5, -\dfrac{217}{6}\right)$

Exercise 5F

1 $y = \dfrac{1 - 2x}{x^2} = \dfrac{1}{x^2} - \dfrac{2}{x} = x^{-2} - 2x^{-1}$

$\therefore \dfrac{dy}{dx} = -2x^{-3} + 2x^{-2}$

Therefore, the gradient at $\left(2, -\dfrac{3}{4}\right)$

is $-2(2)^{-3} + 2(2)^{-2} = \dfrac{1}{4}$

So the gradient of the normal at this point is -4

$\therefore y - \left(-\dfrac{3}{4}\right) = -4(x-2) \Rightarrow y = -4x + \dfrac{29}{4}$

2 $\dfrac{dy}{dx} = -3x^2 + 4x$

So the gradient at $x = -1$ is -7

$\dfrac{dy}{dx} = -7 \Rightarrow 3x^2 - 4x - 7$

$= (3x - 7)(x + 1) = 0$

$\therefore x = \dfrac{7}{3}$ or $x = -1$ (i.e. the tangent itself)

$y\left(\dfrac{7}{3}\right) = -\dfrac{22}{27}$

$\therefore y + \dfrac{22}{27} = -7\left(x - \dfrac{7}{3}\right) \Rightarrow y = -7x + \dfrac{419}{27}$

3 $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$

$\dfrac{dy}{dx} = -3 \Rightarrow 1 - \dfrac{1}{x^2} = -3 \Rightarrow x = \pm\dfrac{1}{2}$

$y\left(\pm\dfrac{1}{2}\right) = \pm\dfrac{5}{2}$

Gradient of normal is $\dfrac{1}{3}$

$\therefore y - \left(\pm\dfrac{5}{2}\right) = \dfrac{1}{3}\left(x - \left(\pm\dfrac{1}{2}\right)\right)$

$\Rightarrow y = \dfrac{1}{3}x \pm \dfrac{5}{2} \mp \dfrac{1}{6}$

$\therefore y = \dfrac{1}{3}x + \dfrac{7}{3}$ and $y = \dfrac{1}{3}x - \dfrac{7}{3}$

4 $f'(x) = 6x - 2k$

$f'(1) = 6 - 2k = 10 \Rightarrow k = -2$

5 $f'(x) = 3x^2 - 2x - 2 = 0$

$\Rightarrow x^2 - \dfrac{2}{3}x - \dfrac{2}{3} = \left(x - \dfrac{1}{3}\right)^2 = \dfrac{7}{9}$

$\therefore x = \dfrac{1 \pm \sqrt{7}}{3}$

Coordinates are

$\left(\dfrac{1+\sqrt{7}}{3}, \dfrac{7-14\sqrt{7}}{27}\right), \left(\dfrac{1-\sqrt{7}}{3}, \dfrac{7+14\sqrt{7}}{27}\right)$

6 $g(x) = \dfrac{1}{x^n} = x^{-n} \quad \therefore g'(x) = -nx^{-n-1}$

$\Rightarrow xg'(x) + ng(x) = x(-nx^{-n-1}) + nx^{-n}$

$= -nx^{-n} + nx^{-n} = 0$

7 **a** $f'(x) = 15ax^2 - 4bx + 4c$

b $f'(x) \geq 0 \Rightarrow 15ax^2 - 4bx + 4c \geq 0$

$\Rightarrow x^2 - \dfrac{4b}{15a}x + \dfrac{4c}{15a} \geq 0$

$\Rightarrow \left(x - \dfrac{2b}{15a}\right)^2 - \dfrac{4b^2}{225a^2} + \dfrac{4c}{15a} \geq 0$

The LHS is valid for all real x and attains its minimum at $x = \dfrac{2b}{15a}$ so

$-\dfrac{4b^2}{225a^2} + \dfrac{4c}{15a} \geq 0 \Rightarrow b^2 \leq 15ac$

8 $f(x) = -20x^{-1} + 1 \qquad$ for $x > 0$

$\therefore f'(x) = 20x^{-2}, g'(x) = 5$ for $x \in \mathbb{R}$

$f'(x) = g'(x)$ when $20x^{-2} = 5$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

But $x > 0$ so $x = 2$ only

Exercise 5G

1 **a** $y = u^5$ where $u = 2x + 3$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (5u^4)(2) = 10u^4$

$= 10(2x+3)^4$

b $y = \sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$

$y = u^{\frac{1}{2}}$ where $u = 1 - 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(\dfrac{1}{2}u^{-\frac{1}{2}}\right)(-2)$

$= -u^{-\frac{1}{2}} = -(1-2x)^{-\frac{1}{2}}$

c $y = -\dfrac{3}{\sqrt{2x^2-1}} = -3u^{-\frac{1}{2}}$

where $u = 2x^2 - 1$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(\dfrac{3}{2}u^{-\frac{3}{2}}\right)(4x)$

$= 6x(2x^2-1)^{-\frac{3}{2}}$

d $y = 2\left(x^2 - \dfrac{2}{x}\right)^3 = 2u^3$

where $u = x^2 - \dfrac{2}{x}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (6u^2)\left(2x + \dfrac{2}{x^2}\right)$

$= 12\left(x^2 - \dfrac{2}{x}\right)^2\left(x + \dfrac{1}{x^2}\right)$

2 At $x = 0$, $y = 6$ so tangent passes through $(0,6)$

$y = 6(1-2x)^{\frac{1}{3}} = 6u^{\frac{1}{3}}$ where $u = 1 - 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(2u^{-\frac{2}{3}}\right)(-2) = -4u^{-\frac{2}{3}}$

$= -4(1-2x)^{-\frac{2}{3}}$

so the gradient at $x = 0$ is -4

$\therefore y - 6 = -4(x - 0) \Rightarrow y = -4x + 6$

3 When $x = 1$, $y = 1$ so $a = \sqrt{1+b}$

$y = a(1+bx)^{-\frac{1}{2}} = au^{-\frac{1}{2}}$ where $u = 1 + bx$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{a}{2}u^{-\frac{3}{2}}\right)(b) = -\frac{ab}{2}u^{-\frac{3}{2}}$

$= -\frac{ab}{2}(1+bx)^{-\frac{3}{2}}$

At $(1,1)$, $\frac{dy}{dx} = -\frac{3}{8}$

$\Rightarrow -\frac{ab}{2}(1+b)^{-\frac{3}{2}} = -\frac{b}{2a^2} = -\frac{3}{8}$

so $b = \frac{3a^2}{4}$

$\Rightarrow a = \sqrt{1+\frac{3a^2}{4}} \Rightarrow a^2 = 1 + \frac{3a^2}{4}$

$\Rightarrow a = 2 \quad (a > 0) \therefore b = a^2 - 1 = 3$

4 $y = \frac{4}{(3-x)^3} = 4u^{-3}$ where $u = 3 - x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-12u^{-4})(-1) = 12u^{-4}$

$= 12(3-x)^{-4}$

so at $x = 1$, $\frac{dy}{dx} = \frac{3}{4}$ and therefore

the normal has gradient $-\frac{4}{3}$

$\therefore y - \frac{1}{2} = -\frac{4}{3}(x-1) \Rightarrow y = -\frac{4}{3}x + \frac{11}{6}$

5 $\frac{dy}{dx} = -9x^2 + 2$

Curve horizontal when $\frac{dy}{dx} = 0$

So $x = \pm\sqrt{\frac{2}{9}} = \pm\frac{\sqrt{2}}{3}$

Exercise 5H

1 a $y = x^2(2x-1) = uv$ where $u = x^2$

and $v = 2x - 1$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2x)(2x-1) + x^2(2)$

$= 6x^2 - 2x = 2x(3x-1)$

b $y = (2x-3)(x+3)^3 = uv$ where

$u = 2x - 3$ and $v = (x+3)^3$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2)(x+3)^3 + (2x-3)\left(3(x+3)^2\right)$

$= (x+3)^2(8x-3)$

c $y = x\sqrt{2-3x} = uv$ where $u = x$

and $v = \sqrt{2-3x}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (1)\sqrt{2-3x} + x\left(-\frac{3}{2\sqrt{2-3x}}\right)$

$= \frac{4-9x}{2\sqrt{2-3x}}$

d $y = (2x+1)(x^2-x+1)^2 = uv$ where

$u = 2x + 1$ and $v = (x^2-x+1)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2)(x^2-x+1)^2$

$\quad + 2(2x+1)(2x-1)(x^2-x+1)$

$= 2x(5x-1)(x^2-x+1)$

e $y = (2-3x)\sqrt{x+2} = uv$

where $u = 2 - 3x$ and $v = \sqrt{x+2}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (-3)\sqrt{x+2} + (2-3x)\left(\frac{1}{2\sqrt{x+2}}\right)$

$= \frac{-10-9x}{2\sqrt{x+2}}$

2 a $y = \sqrt{x+1}(3-x)^2 = uv$ where

$u = \sqrt{x+1}$ and $v = (3-x)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \left(\frac{1}{2\sqrt{x+1}}\right)(3-x)^2 + \sqrt{x+1}(-2(3-x))$

$\therefore \frac{dy}{dx} = \frac{(x-3)^2 + 4(x-3)(x+1)}{2\sqrt{x+1}}$

$= \frac{(x-3)(x-3+4(x+1))}{2\sqrt{x+1}}$

$= \frac{(x-3)(5x+1)}{2\sqrt{x+1}}$

b Using the result from part a,

$x - 3 = 0 \Rightarrow x = 3$

or $5x + 1 = 0 \Rightarrow x = -\frac{1}{5}$

3 $y = x(1-2x)^{-1} = uv$ where $u = x$

and $v = (1-2x)^{-1}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (1)(1-2x)^{-1} + x\left(2(1-2x)^{-2}\right) = \frac{1}{(1-2x)^2}$

so the gradient at (0,0) is 1 and the normal therefore has gradient -1

$\therefore y = -x$

Exercise 5I

1 a $y = \frac{1+3x}{5-x} = \frac{u}{v}$ where $u = 1+3x$

and $v = 5-x$, $u' = 3, v' = -1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(5-x)(3)-(1+3x)(-1)}{(5-x)^2} = \frac{16}{(5-x)^2}$

b $y = \frac{\sqrt{x}}{2-x} = \frac{u}{v}$ where $u = \sqrt{x}$

and $v = 2-x$

$u' = \frac{1}{2\sqrt{x}}, v' = -1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(2-x)\left(\frac{1}{2\sqrt{x}}\right)-\left(\sqrt{x}\right)(-1)}{(2-x)^2}$

$= \frac{2+x}{2(2-x)^2\sqrt{x}}$

c $y = \frac{1+2x}{\sqrt{1-x^2}} = \frac{u}{v}$ where $u = 1+2x$

and $v = \sqrt{1-x^2}$

$u' = 2, v' = -\frac{x}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{\sqrt{1-x^2}\,(2)-(1+2x)\left(-\frac{x}{\sqrt{1-x^2}}\right)}{1-x^2}$

$= \frac{x+2}{(1-x^2)^{\frac{3}{2}}}$

d $y = \frac{1+3x}{x^2+1} = \frac{u}{v}$ where $u = 1+3x$

and $v = x^2+1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x^2+1)(3)-(1+3x)(2x)}{(x^2+1)^2}$

$= \frac{3-2x-3x^2}{(x^2+1)^2}$

2 $f'(x) = \frac{-3x^2+4x+3}{(x+1)^2}$, $f'(0) = 3$

so normal at this point has gradient $-\frac{1}{3}$ and passes through $(0,-2)$

$\therefore y = -\frac{1}{3}x - 2$

3 $f(x) = \frac{x^3+x^2+x+1}{x} = \frac{u}{v}$

where $u = x^3+x^2+x+1$ and $v = x$

$f'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{x(3x^2+2x+1)-(x^3+x^2+x+1)(1)}{x^2}$

$= \frac{2x^3+x^2-1}{x^2}$

$f'(x) = 1 \Rightarrow 2x^3+x^2-1 = x^2 \Rightarrow x^3 = \frac{1}{2}$

$\therefore x = \frac{1}{\sqrt[3]{2}}$

Exercise 5J

1 a

$y = (x-1)(x+3)^2 = uv$ where

$u = x-1$ and $v = (x+3)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (x+3)^2 + 2(x-1)(x+3)$

$= (x+3)(x+3+2(x-1))$

$= (x+3)(3x+1)$

b Most easily done using the product (and chain) rule:

$y = (x+1)\sqrt{1-2x} = uv$

where $u = x+1$ and $v = \sqrt{1-2x}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \sqrt{1-2x} - \frac{x+1}{\sqrt{1-2x}} = -\frac{3x}{\sqrt{1-2x}}$

c Most easily done using the quotient rule:

$y = \frac{x+1}{x-1} = \frac{u}{v}$ where $u = x+1$

and $v = x-1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$

d Most easily done by chain rule (quotient rule also valid)

$y = 2\left(x^4 - 2x + 1\right)^{-1} = 2u^{-1}$

where $u = x^4 - 2x + 1$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{2}{u^2}\right)\left(4x^3 - 2\right)$

$= \frac{4\left(1-2x^3\right)}{\left(x^4 - 2x + 1\right)^2}$

2 $f(x) = \frac{1+\sqrt{x}}{x-1} = \frac{u}{v}$ where $u = 1 + \sqrt{x}$

and $v = x - 1$

$f'(x) = \frac{vu' - uv'}{v^2} = \frac{\frac{x-1}{2\sqrt{x}} - \left(1+\sqrt{x}\right)}{(x-1)^2}$

$= -\frac{x + 2\sqrt{x} + 1}{2\sqrt{x}(x-1)^2} = -\frac{\left(\sqrt{x}+1\right)^2}{2\sqrt{x}(x-1)^2}$

$f'(9) = -\frac{(3+1)^2}{2(3)(9-1)^2} = -\frac{1}{24}$

Tangent: $y - \frac{1}{2} = -\frac{1}{24}(x-9)$

$\Rightarrow y = \frac{7}{8} - \frac{x}{24}$

Normal: $y - \frac{1}{2} = 24(x-9)$

$\Rightarrow y = 24x - \frac{431}{2}$

Exercise 5K

1 a i $x > 0$

ii Nowhere

b i $x \in (-\infty, -1) \cup (-1, 0)$ □

ii $x \in (0,1) \cup (1, \infty)$

c i $x \in (-\infty, -0.215) \cup (1.55, \infty)$

ii $x \in (-0.215, 1.55)$

d i $x \in (-\infty, -1) \cup (1, \infty)$

ii $x \in (-1, 1)$

2 a $f'(x) = -3x^2$

Increasing: nowhere

Decreasing: $\forall x \in$ ¡

b $f'(x) = 4x$

Increasing: $x > 0$

Decreasing: $x < 0$

c $f'(x) = -\frac{1}{2\sqrt{x-1}}$

Increasing: nowhere

(note the function is only valid here for $x > 1$)

Decreasing: $x \in (1, \infty)$

d $f'(x) = \frac{1}{2\sqrt{x}} - 2$

Increasing: $x \in \left(0, \frac{1}{16}\right)$

(note the function is only valid here for $x > 0$)

Decreasing: $x \in \left(\frac{1}{16}, \infty\right)$

Exercise 5L

1 a $f'(x) = 2x$, $f'(x) = 0 \Rightarrow x = 0$

$f(x)$ decreasing for $x < 0$ and increasing for $x > 0$ so this is a local minimum

$\therefore (0, -2)$ is a local minimum point

Graphically, this is a positive parabola so the turning point must be a local minimum (students should draw this)

b $f'(x) = 1 - \frac{1}{\sqrt{x}}$, $f'(x) = 0 \Rightarrow x = 1$

$f(x)$ decreasing for $0 < x < 1$ and increasing for $x > 1$ so this is a local minimum

$\therefore (1, -1)$ is a local minimum point

Graphically, the graph is continuous, begins at $(0,0)$ and $\lim_{x\to\infty} f(x) = \infty$ so the turning point at $(1, -1)$ must be a local minimum point (and in fact this case a global minimum).

(Students should draw this.)

c $f'(x) = 3x^2 - 12x = 3x(x-4)$

$f'(x) = 0 \Rightarrow x = 0$ or $x = 4$
Consider the point $(0, 2)$,
$f(x)$ increasing for $x < 0$ and decreasing for $0 < x < 4$ so this is a local maximum
Consider the point $(4, -30)$
$f(x)$ decreasing for $0 < x < 4$ and increasing for $x > 4$ so this is a local minimum
$\therefore (0, 2)$ is a local maximum
and $(4, -30)$ is local minimum
Graphically, this is a positive cubic, so the first turning point is a maximum and the second point a minimum
(students should draw this)

2 $f'(x) = 3ax^2 + 4x = x(3ax + 4)$

$f'(x) = 0$ when $x = 0$ or $x = -\dfrac{4}{3a}$

It is given that the turning point, away from $x = 0$, occurs at $x = 1$

$\therefore 1 = -\dfrac{4}{3a} \Rightarrow a = -\dfrac{4}{3}$

3 $p(0) = d = 1$ so $d = 1$

$p(-1) = -a + b - c + 1 = -3$

so $a - b + c = 4$

$p'(x) = 3ax^2 + 2bx + c = x(3ax + 2b) + c$

$p'(0) = 3 \Rightarrow c = 3 \Rightarrow a - b = 1$

$p'(-1) = 0 \Rightarrow -(-3a + 2b) + 3 = 0$

$\Rightarrow a = \dfrac{2b}{3} - 1$

$\therefore \dfrac{2b}{3} - 1 - b = 1 \Rightarrow b = -6 \Rightarrow a = -5$

so $a = -5,\ b = -6,\ c = 3,\ d = 1$

4 $\dfrac{dy}{dx} = 3x^2 + 2ax = x(3x + 2a) = 0$

$\therefore \dfrac{dy}{dx} = 0$ when $x = 0$ or $x = -\dfrac{2a}{3}$

$\therefore -\dfrac{2a}{3} = 4 \Rightarrow a = -6$

$y(4) = 64 - 6(16) + b = b - 32 = -11$

$\Rightarrow b = 21$

so the local maximum is at $(0, 21)$

Exercise 5M

1 $f'(x) = 5x^{\frac{3}{2}} \quad \therefore f''(x) = \dfrac{15}{2}x^{\frac{1}{2}}$

2 $f'(x) = \dfrac{x^3}{3} - 4x + 5$

$\therefore f''(x) = x^2 - 4$

$f''(x) = x^2 - 4 = 0 \Rightarrow x = \pm 2$

3 $f'(x) = -2(5 - 4x)^{-\frac{1}{2}}$

$\therefore f''(x) = -4(5 - 4x)^{-\frac{3}{2}}$

4 $\dfrac{d^2y}{dx^2} = 2a^2$

(terms of order x and constants disapper upon differentiating twice)

$\therefore 2a^2 = 8 \Rightarrow a = \pm 2$

Exercise 5N

1 a $\dfrac{dy}{dx} = 3x^2 - 1$

$\dfrac{d^2y}{dx^2} = 6x \Rightarrow 0$ at $x = 0$

Coordinates of point of inflexion are $(0, 0)$

b $\dfrac{d^2y}{dx^2} = 6x > 0 \Rightarrow x > 0$

Function concave up on $]0, \infty[$

c $\dfrac{d^2y}{dx^2} = 6x < 0 \Rightarrow x < 0$

Function concave down on $]-\infty, 0[$

2 a $\dfrac{dy}{dx} = 4x^3 - 3$

$\dfrac{d^2y}{dx^2} = 12x^2 > 0$

There are no points of inflexion

b $\dfrac{d^2y}{dx^2} = 12x^2 > 0$

Functions is concave up throughout its domain

c Function is never concave down

3 a $\dfrac{dy}{dx} = 3x^2 - 12x - 12$

$\dfrac{d^2y}{dx^2} = 6x - 12 = 0$ at $x = 2$

Coordinates of point of inflexion are $(2, -38)$

b $\dfrac{d^2y}{dx^2} = 6x - 12 > 0 \Rightarrow x > 2$

Function is concave up on $]2, \infty[$

c $\dfrac{d^2y}{dx^2} = 6x - 12 < 0 \Rightarrow x < 2$

Function is concave down on $]-\infty, 2[$

4 a $\frac{dy}{dx} = 3x^2 + 2x$

$\frac{d^2y}{dx^2} = 6x + 2 = 0$ at $x = -\frac{1}{3}$

Coordinates of point of inflexion are $\left(-\frac{1}{3}, -\frac{25}{27}\right)$

b $\frac{d^2y}{dx^2} = 6x + 2 > 0 \Rightarrow x > -\frac{1}{3}$

Function is concave up on $]-\frac{1}{3}, \infty[$

c $\frac{d^2y}{dx^2} = 6x + 2 < 0 \Rightarrow x < -\frac{1}{3}$

Function is concave up on $]-\infty, -\frac{1}{3}[$

5 a $\frac{dy}{dx} = 12x^2 - 4x^3$

$\frac{d^2y}{dx^2} = 24x - 12x^2 = 0$ at $x = 0, 2$

Coordinates of point of inflexion are $(0, 0), (2, 16)$

b $\frac{d^2y}{dx^2} = 24x - 12x^2 > 0 \Rightarrow 0 < x < 2$

Function is concave up for $0 < x < 2$

c $\frac{d^2y}{dx^2} = 24x - 12x^2 < 0 \Rightarrow 0 > x, x > 2$

Function is concave down for $x < 0$; $x > 2$

6 a $\frac{dy}{dx} = 3x^2 - 6x + 3$

$\frac{d^2y}{dx^2} = 6x - 6 = 0$ at $x = 1$

Coordinates of point of inflexion are $(1, 0)$

b $\frac{d^2y}{dx^2} = 6x - 6 > 0 \Rightarrow x > 1$

Function is concave up on $]1, \infty[$

c $\frac{d^2y}{dx^2} = 6x - 6 < 0 \Rightarrow x < 1$

Function is concave down on $]-\infty, 1[$

7 a $\frac{dy}{dx} = 8x^3 + 3x^2$

$\frac{d^2y}{dx^2} = 24x^2 + 6x = 0$ at $x = 0, -0.25$

Coordinates of point of inflexion are $(-0.25, 0.992), (0, 1)$

b $\frac{d^2y}{dx^2} = 24x^2 + 6x > 0 \Rightarrow x > 0, x < -0.25$

Function is concave up for $x > 0$; $x < -0.25$

c $\frac{d^2y}{dx^2} = 24x^2 + 6x < 0 \Rightarrow -0.25 < x < 0$

Function is concave down for $-0.25 < x < 0$

8 a $\frac{dy}{dx} = 4x^3 - 12x^2 + 16$

$\frac{d^2y}{dx^2} = 12x^2 - 24x = 0$ at $x = 0, 2$

Coordinates of point of inflexion are $(0, -16)$, $(2, 0)$

b $\frac{d^2y}{dx^2} = 12x^2 - 24 > 0 \Rightarrow x < 0, x > 2$

Function is concave up when $x < 0$, $x > 2$

c $\frac{d^2y}{dx^2} = 12x^2 - 24 < 0 \Rightarrow 0 < x < 2$

Function is concave down when $0 < x < 2$

9 a $f'(x) = 3x^2 + 4x$

$3x^2 + 4x = 0 \Rightarrow x = 0, -\frac{4}{3}$

$f''(x) = 6x + 4$

$f''(0) = 4$

$f''\left(-\frac{4}{3}\right) = -4$

$f''(x) = 0 \Rightarrow x = -\frac{2}{3}$

Non-horizontal inflexion at $\left(-\frac{2}{3}, \frac{43}{27}\right)$

b $f'(x) = 3(x-1)^2$

$3(x-1)^2 = 0 \Rightarrow x = 1$

$f''(x) = 6(x-1) = 0$ at $x = 1$

$f''(1.1) = 0.6 > 0$

$f''(0.9) = -0.6 < 0$

Second derivative = 0 at $x = 1$, there is a change in concavity at $x = 1$.

Therefore there is a horizontal inflexion at (1, 0)

c $f'(x) = -12x^3 - 24x^2$

$-12x^3 - 24x^2 = 0 \Rightarrow x = 0, -2$

$f''(x) = -36x^2 - 48x = 0 \text{ at } x = 0, -\frac{4}{3}$

$f''(0.1) = -\frac{129}{25}$

$f''(-0.1) = \frac{111}{25}$

First and second derivatives = 0 at $x = 0$, and there is a change in concavity at $x = 0$.

Therefore there is a horizontal inflexion at (0, 2)

$f''(-2) = -48$

Second derivative = 0 at $x = -\frac{4}{3}$, and there is a change in concavity at $x = -\frac{4}{3}$

Therefore there is a non-horizontal point of inflexion at $\left(-\frac{4}{3}, \frac{310}{27}\right)$

d $f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$

First derivative has no roots, therefore there are no points of inflexion.

10 a i $f'(x) = 3x^2 - 6x - 6$

$3x^2 - 6x - 6 = 0 \Rightarrow x = -0.732, 2.73$

$f''(x) = 6x - 6$

$f''(-0.732) = -10.392$

$f''(2.73) = 10.38$

Therefore local max at (−0.732, 3.39) and local min at (2.73, −17.4)

ii $f''(x) = 6x - 6 = 0 \Rightarrow x = 1$

Non-horizontal inflexion at (1, −7)

iii Increasing: $x < -0.732$ or $x > 2.73$ decreasing for $-0.732 < x < 2.73$

iv Concave downward $x < 1$ and concave upward for $x > 1$

b i $f'(x) = 2(x - 1) \Rightarrow x = 1$

$f''(x) = 2$

$f(1) = 2$

Therefore local min at (1, 0)

ii $f''(x) = 2$ therefore there are no inflexion points

iii Increasing for $x > 1$

decreasing for $x < 1$

iv Concave upward for $x \in \mathbb{R}$

c i $f'(x) = 12x^3 + 12x^2 = 0 \Rightarrow x = -1, 0$

$f''(x) = 36x^2 + 24x$

$f''(-1) = 60$

$f''(0) = 0$

Therefore local min at (−1, −3)

ii $f''(x) = 36x^2 + 24x = 0 \Rightarrow x = -\frac{2}{3}, 0$

Horizontal inflexion point at (0, −2)

Non-horizontal inflexion point at $\left(-\frac{2}{3}, -\frac{70}{27}\right)$

iii Increasing $x > -1$

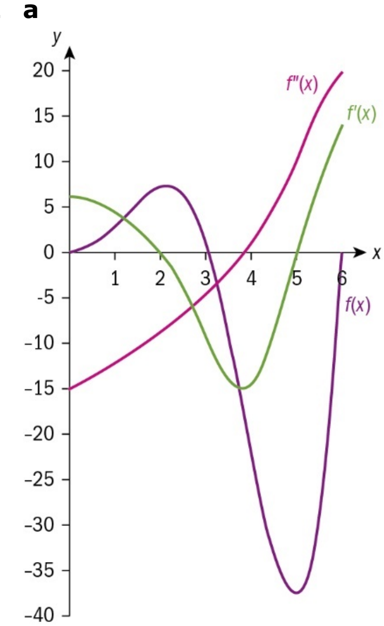
Decreasing $x < -1$

iv Concave upward $x < -\frac{2}{3}$ or $x > 0$

Concave downward $-\frac{2}{3} < x < 0$

Exercise 50

1 a



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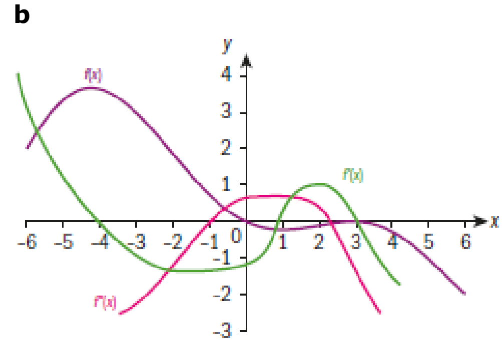

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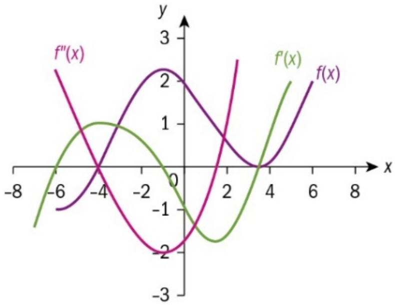
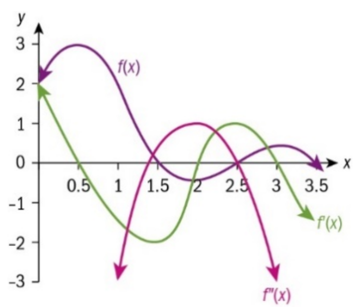
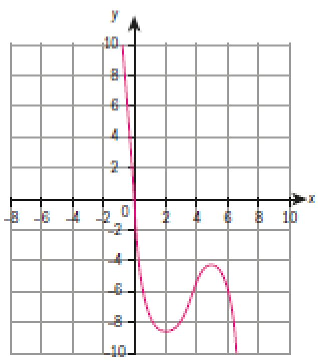


d

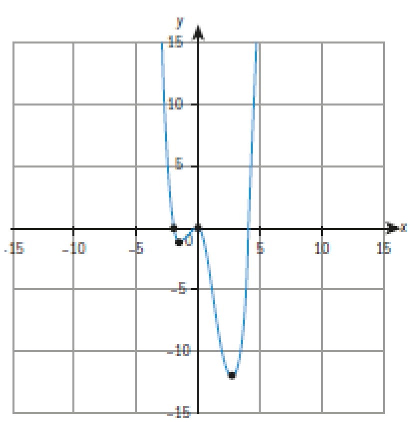


Exercise 5P

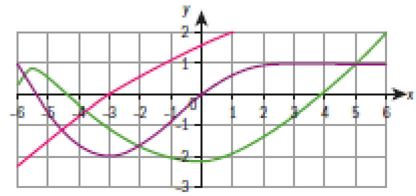
1 a



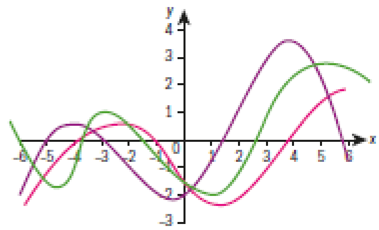
b



2 a



b



Exercise 5Q

1 a $L = \dfrac{100}{x}$

b $P = 2x + \dfrac{100}{x}$ assuming clarification to the question is made, as written in the comments

c $P'(x) = 2 - \dfrac{100}{x^2}$

$P'(x) = 0 \Rightarrow x = \sqrt{50} = 5\sqrt{2} \quad (x > 0)$

This must be a minimum because

$\lim_{x \to 0^+} P(x) = \infty$ and $\lim_{x \to \infty} P(x) = \infty$

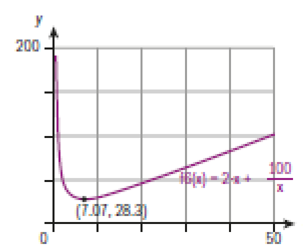
and this is the only turning point (and the function is continuous)

So $x = 5\sqrt{2}$ $\therefore P\left(5\sqrt{2}\right) = 2\left(5\sqrt{2}\right) + \dfrac{100}{5\sqrt{2}}$

$= 10\sqrt{2} + 10\sqrt{2}$

$= 20\sqrt{2}$ (measured in metres)

d



2 $\dfrac{dy}{dx} = 600 + 30x - 3x^2$

$\dfrac{dy}{dx} = 0 \Rightarrow x^2 - 10x - 200$

$= (x - 20)(x + 10) = 0$

$\therefore x = 20$ since $x > 0$

$y(20) = 600(20) + 15(20)^2 - (20)^3 = 10000$

Therefore, maximum profit is \$10 000

3 **a** $h = \dfrac{216}{s^2}$

b $A = s^2 + \dfrac{864}{s}$

c $\dfrac{dA}{ds} = 2s - \dfrac{864}{s^2} = 0$

$\Rightarrow s^3 = 432$ so $s = \sqrt[3]{432}$

4 The length of each side of the square is s

Therefore the total length of wire used for the rectangle is $150 - 4s$

Since the length is twice the length of the width,

the length of the rectangle is $50 - \dfrac{4s}{3}$

and the width of the rectangle is $25 - \dfrac{2s}{3}$

so the total area enclosed by the square and rectangle is

$$A = s^2 + \left(50 - \frac{4s}{3}\right)\left(25 - \frac{2s}{3}\right)$$

$$= s^2 + 2\left(25 - \frac{2s}{3}\right)^2$$

$$= s^2 + 2\left(625 - \frac{100s}{3} + \frac{4s^2}{9}\right)$$

$$= \frac{17s^2}{9} - \frac{200s}{3} + 1250$$

$$\frac{dA}{ds} = \frac{34}{9}s - \frac{200}{3} = 0 \Rightarrow s = \frac{300}{17}$$

5 Let the length of the shorter side of the base be l, so the longer side measures $2l$

Therefore the area of the base is $2l^2$.

Let the height be h

$V = 2l^2h = 10 \Rightarrow h = \dfrac{5}{l^2}$

So the total cost is

$$C = 2l^2(10) + 2(2l)(h)(6) + 2(l)(h)(6)$$

$$= 20l^2 + 36lh = 20l^2 + \frac{180}{l}$$

$$\frac{dC}{dl} = 40l - \frac{180}{l^2} = 0 \Rightarrow l = \left(\frac{9}{2}\right)^{\frac{1}{3}}$$

$$\therefore C_{min} = 20\left(\frac{9}{2}\right)^{\frac{2}{3}} + 180\left(\frac{2}{9}\right)^{\frac{1}{3}}$$

$= 164$ (to nearest dollar)

Exercise 5R

1 **a** $v(t) = 3t^2 - 3,\ a(t) = 6t$

b $s(0) = 1,\ v(0) = -3,\ a(0) = 0$

At this instant, the particle is 1 metre from the origin in the positive direction, travelling towards the origin at 3m/s, and is not accelerating

c The particle is moving away from the origin at 9m/s and is accelerating away from the origin at 12m/s^2

d The change of sign of $v(t)$ occurs at $t = 1$ $(t > 0)$

e $t > 1$

f $s(0) = 1,\ s(1) = -1$

so travels 2m in this period

$s(1) = -1,\ s(3) = 19$

so 20m travelled in this period

$\therefore$ Altogether distance travelled is 22m

2 **a** 1m

b $s'(t) = 12 - 3t^2 = 0 \Rightarrow t = 2$

$s(2) = 17$ and this is clearly a maximum so 17m

c $v(t) = 12 - 3t^2$

$v(0) = 12,\ v(1) = 9,\ v(3) = -15$

d $16 + 17 = 33$ so 33m

3 $s'(t) = 15 - 10t \Rightarrow t = \dfrac{3}{2}$

clearly attains maximum here as the function is a negative parabola

$$s_{max} = s\left(\frac{3}{2}\right) = 15\left(\frac{3}{2}\right) - 5\left(\frac{3}{2}\right)^2 = \frac{45}{4}$$

4 **a** $s(0) = 10$ so 10m

b $s(t) = 0 \Rightarrow t^2 - 5t - 10 = 0$

$\therefore t = \dfrac{5 + \sqrt{65}}{2} \approx 6.53$ $(t > 0)$

c $v(t) = s'(t) = 5 - 2t$

$v\left(\dfrac{5 + \sqrt{65}}{2}\right) = -8.06\,\text{ms}^{-1}$

$a(t) = v'(t) = -2\,\text{ms}^{-2}$

As both the velocity and acceleration are negative, the diver is speeding up as he/she hit the water.

5 $h_0 = 0,\ v_0 = 50 \quad \therefore h(t) = 50t - 4.9t^2$

$h'(t) = 50 - 9.8t = 0 \Rightarrow t = \dfrac{50}{9.8}$

(clearly maximum here)

$h_{\max} = h\left(\dfrac{50}{9.8}\right) = 127.551$

so maximum height is 127.6m to 1d.p.

$t_{ground} = \dfrac{50}{4.9} = 10.2041$

so hits ground after 10.2s to 1d.p.

6 **a** $t = 0,\ t = 3,\ t = 6,\ t = 11$

b **i** Eastward is positive $\rightarrow 0 < t < 3;\ 6 < t < 11$

ii Westward is negative $\rightarrow 3 < t < 6$

c **i** $t = 1.5$ **ii** $t = 4.5$

d $t = 1.5$ and $t = 4.5$

e Speeding up: $t \in (0, 1.5)$;

$t \in (3, 4.5); t \in (6, 9)$

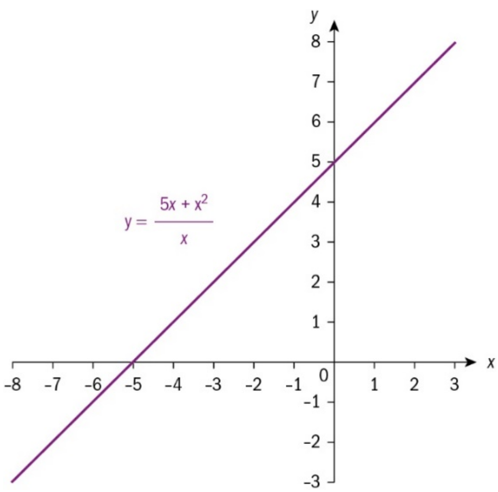
Slowing down: $t \in (1.5, 3)$;

$t \in (4.5, 6); t \in (9, 11)$

Chapter Review

1 **a** $y = 2$ **b** $a = 2$

2 **a**



b $\lim\limits_{x\to 0^-} \dfrac{5x + x^2}{x} = \lim\limits_{x\to 0^+} \dfrac{5x + x^2}{x} = 5$

3 **a** $y = 6,\ x = \pm 3$ **b** $y = 0,\ x = -3$

4 **a** Using the product rule,

$$\frac{dy}{dx} = -10(1-2x)^4(3x-2)^6 + 18(1-2x)^5(3x-2)^5$$

$$= (1-2x)^4(3x-2)^5\left(-10(3x-2) + 18(1-2x)\right)$$

$$= 2(1-2x)^4(3x-2)^5(19-33x)$$

b $y = \dfrac{x-3}{x(x-3)} = \dfrac{1}{x}$ so $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

c $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{4}{3}x^{-\frac{2}{3}}$

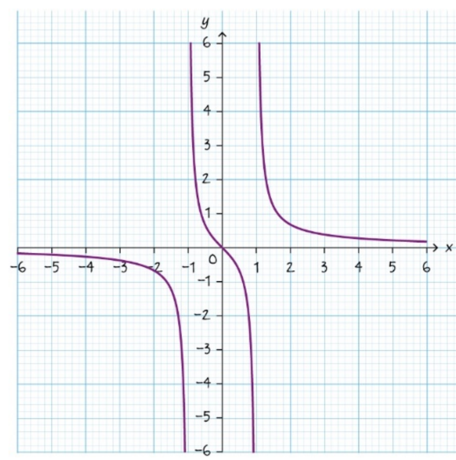
5 **a** $y = 0,\ x = \pm 1$

b Using the quotient rule,

$$\frac{dy}{dx} = \frac{(x^2-1) - x(2x)}{(x^2-1)^2}$$

$$= -\frac{x^2+1}{(x^2-1)^2} < 0 \quad \text{for all } x \in ¡$$

c



6 $\dfrac{dy}{dx} = 3x^2 - 6x - 9 = 3(x-3)(x+1)$

$\therefore \dfrac{dy}{dx} = 0 \Rightarrow x = -1$ or $x = 3$

$y(-1) = -1 - 3 + 9 + 2 = 7$

$y(3) = 27 - 27 - 27 + 2 = -25$

So $y = -25$ and $y = 7$

7 Using the quotient rule,

$$\frac{dy}{dx} = \frac{2(x-1) - 2x}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

$\therefore$ at $(2, 4)$ the gradient is -2

So the tangent at this point is

$y-4=-2(x-2) \Rightarrow y=-2x+8$

The gradient at $(3,3)$ is $-\frac{1}{2}$ so the gradient of the normal at this point is 2

Therefore the normal at this point is

$y-3=-\frac{1}{2}(x-3) \Rightarrow y=-\frac{x}{2}+\frac{9}{2}$

$\therefore -\frac{x}{2}+\frac{9}{2}=-2x+8 \Rightarrow x=\frac{7}{3}$

$\Rightarrow y=-2\left(\frac{7}{3}\right)+8=\frac{10}{3} \therefore P\left(\frac{7}{3},\frac{10}{3}\right)$

8 $f'(x)=6x^2-3$

$f'(1)=3$

So the normal to the curve at this point has gradient $-\frac{1}{3}$

$\therefore y-0=-\frac{1}{3}(x-1)=-\frac{x}{3}+\frac{1}{3}$

9 $\frac{\mathrm{d}y}{\mathrm{d}x}=-3x^2+4x$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-4 \Rightarrow 3x^2-4x-4=0$

$\Rightarrow (3x+2)(x-2)=0$

$\therefore x=-\frac{2}{3}$ or $x=2$

$y\left(-\frac{2}{3}\right)=\frac{8}{27}+2\left(\frac{4}{9}\right)+1=\frac{59}{27}$

$y(2)=-8+8+1=1$

$\therefore \left(-\frac{2}{3},\frac{59}{27}\right)$ and $(2,1)$

10 Using the quotient rule,

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x(x+1)-x^2}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}=\frac{x(x+2)}{(x+1)^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x=0$ or $x=-2$

Students may either use first derivative or second derivative test here

e.g. second derivative test:

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2(x+1)^3-2(x^2+2x)(x+1)}{(x+1)^4}$

$=\frac{2}{(x+1)^3}$

The second derivative is negative at $x=-2$ and positive at $x=0$

$\therefore (-2,-4)$ is a local maximum and $(0,0)$ is a local minimum

11 a $f(x)=0 \Rightarrow x=-1$ **b** $y=0,\ x=0$

c $f(x)=9\left(\frac{1}{x}+\frac{1}{x^2}\right)$

$\Rightarrow f'(x)=9\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)$

$\therefore f'(x)=0 \Rightarrow x=-2$

$y=f(-2)=-\frac{9}{4}$

$f''(x)=9\left(\frac{2}{x^3}+\frac{6}{x^4}\right)$

$f''(-2)=9\left(-\frac{1}{4}+\frac{3}{8}\right)=\frac{9}{8}>0$

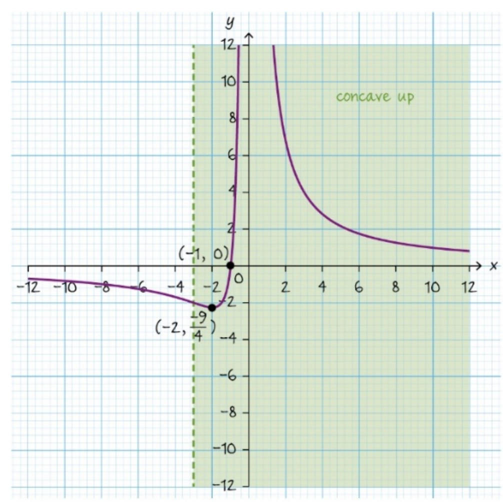
so $\left(-2,-\frac{9}{4}\right)$ is a local minimum

d For $f(x)$ to be concave up $f''(x)>0$

$9\left(\frac{2}{x^3}+\frac{6}{x^4}\right)>0 \Rightarrow 2x+6>0$

$x>-3$

e



12 a $f(x)=0 \Rightarrow \sqrt{x}\left(\sqrt{x}-b\right)=0$

$\Rightarrow x=0$ or $x=b^2$

b $f'(x)=1-\frac{b}{2\sqrt{x}}$

i $f'(x)>0$ when $x>\frac{b^2}{4}$

ii $f'(x)<0$ when $0<x<\frac{b^2}{4}$

c $f''(x)=\frac{b}{4x^{\frac{3}{2}}}$

i $f''(x)>0$ when $b>0$

ii $f''(x)<0$ when $b<0$

13 a $v(t)=49-4.9t$

b $v(t)=0 \Rightarrow t=10$

$h(10) = 49(10) - 2.45(10)^2$
$= 490 - 245 = 245$
So 245m

14a $v(0) = -2$

b $v(t) = 0 \Rightarrow (1+t)^2 = 4t + 9$

$t^2 - 2t - 8 = (t-4)(t+2) = 0$

So $t = 4$

c $a(t) = 1 - \dfrac{2}{\sqrt{4t+9}}$

$a(4) = 1 - \dfrac{2}{\sqrt{25}} = \dfrac{3}{5}$

d Always speeding up since acceleration is always positive

15a $f'(x) = 4x^3 - 6x^2 - 2x + 3$ A1

b $g'(x) = \dfrac{-4(x^2+1) - (-4x)\cdot 2x}{(x^2+1)^2}$ M1A1

$g'(x) = \dfrac{4x^2 - 4}{(x^2+1)^2}$ A1

c $h'(x) = 1\cdot(x-7) + (x+2)\cdot 1$ M1

$h'(x) = 2x - 5$ A1

d $i'(x) = 3\cdot 2\cdot(2x+3)^2$ M1

$i'(x) = 6(2x+3)^2$ A1

16a Graph 1 A1
as the gradient of the tangent at any point is non-positive and therefore different from 1. R1

b Graph 2 A1
as y increases as x increases R1

c Graph 3 A1
as the other two functions are not defined at infinity R1

d Graph 1 A1
as the function is decreasing. R1

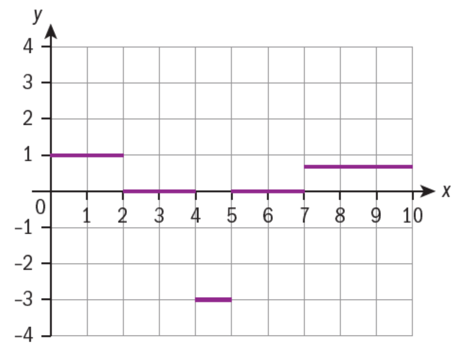
17a i $0 \le t \le 2$, $4.6 \le t \le 5$ A1A1
and $8.5 \le t \le 10$ A1
ii $2 \le t \le 4$ and $5 \le t \le 7$ A1A1
iii $4.6 \le t \le 8.5$ A1

b $f(t) = 2t$, $g(t) = 2$

$h(t) = -3t + 14$, $i(t) = -1$

$f(t) = \dfrac{1}{3}(2t - 17)$ A4

c Up to two correct branches correct A1; all branches correct A2; all branches correct and labels and scale also correct A3



18a Letting x represent the number of \$10 increases above \$320. Then rental income is

$R(x) = (320 + 10x)(200 - 5x)$ A1

$R'(x) = 400 - 100x = 0$ M1

$x = 4$ A1
Which corresponds to \$360 rent R1

b **i** $200 - 5 \times 4 = 180$ A1
ii $360 \times 180 = \$64800$ M1A1

19 a $h(4) = 370$

and $h(5) = 438$ (3 s.f.) A1A1

b $v(t) = h'(t) = 112 - 9.8t$ M1A1

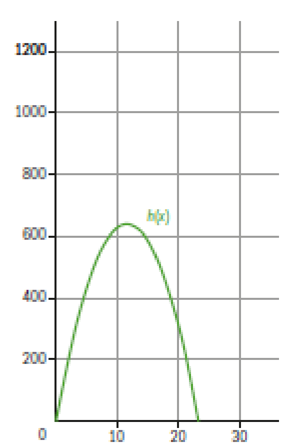
c $v(t) = 0 \Rightarrow 112 - 9.8t = 0$ M1

$t = 11.4$ (3 s.f.) A1

d Double x-coordinate of maximum or determine zero M1

22.8 (3 s.f.) A1

e



Shape A1
Domain $0 \le x \le 22.9$ (3 sf) A1
Maximum 640 (3 sf) A1

f $v(22.9) = -112\,\text{ms}^{-1}$ M1A1

g $a(t) = v'(t) = -9.8$ M1A1

which is constant A1AG

20 a i $\left(\dfrac{f}{g}\right)'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2}$

M1

$= \dfrac{10 \times 4 - 9 \times \left(-\dfrac{4}{3}\right)}{4^2}$ A1

$= \dfrac{52}{16}\left(= \dfrac{13}{4} = 3.25\right)$ A1

ii $(g \circ f)'(1) = g'(f(1))f'(1)$ M1

$= -\dfrac{4}{3} \times 4 = -\dfrac{16}{3}$ A1

b i False A1

as derivative changes sign. R1

b False A1
as the derivatives at these points are not negative reciprocals.
R1

21 a $\dfrac{N(3) - N(1)}{3 - 1} = 1410$ M1A1

$\dfrac{N(5) - N(4)}{5 - 4} = 2220$ A1

the first period the number of cases is increasing in average 1410 per day; in the second period it increases in average 2220 per day.

b $\dfrac{\mathrm{d}N}{\mathrm{d}t} = 900t - 90t^2$ M1A1

c After 10 days (reaches 15 000 cases)
M1A1

d $\dfrac{\mathrm{d}^2N}{\mathrm{d}t^2} = 900 - 180t$ M1A1

which gives the variation of the rate at which the spread of the disease spreads. R1

22 a $x = \dfrac{y + 2}{y - 1}$ M1

$x(y - 1) = y + 2$ M1

$xy - y = x + 2$ A1

$g^{-1}(x) = \dfrac{x + 2}{x - 1} = g(x)$ A1AG

b $(h \circ g^{-1})'(x) = h'(g^{-1}(x))(g^{-1})'(x)$

M1A1

$= 2 \cdot \dfrac{x + 2}{x - 1} \cdot \left(-\dfrac{3}{(x - 1)^2}\right) = \left(-\dfrac{6(x + 2)}{(x - 1)^3}\right)$

A1

$(h' \circ g^{-1})(x) = h'(g^{-1}(x))$ M1

$= 2 \cdot \dfrac{x + 2}{x - 1}$ A1

$(h \circ g^{-1})'(x) \neq (h' \circ g^{-1})(x)$ AG

6 Representing data: statistics for univariate data

Skills check

1 **a** Mean $= \dfrac{2+3+4+5+6}{5} = 4$

b Mean $= \dfrac{13+9+7+12+15+19+2}{7} = 11$

2 **a** The number that occurs most often is 5

b The numbers that occur most often are 1 and 7. The data is bimodal

3 **a** The median is the middle number, 6

b Arrange the data in order of size. $2,3,5,7,8,9$.

The median is in between 5 and 7.

$\frac{1}{2}(5+7) = 6$

Exercise 6A

1 **a** Discrete

b Continuous

c Continuous

d Discrete

2 **a** Stratified sampling

b Systematic sampling

c Simple random sampling

d Quota sampling

3 **a** Stratified sampling

b Stratified sampling

c Systematic sampling

d Simple random sampling

e Quota sampling

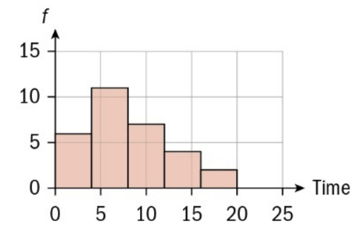
Exercise 6B

1 **a** Continuous

b The frequency table is given here (note difference between this one and the one in the solutions provided)

Time (t)	f
$0 \le t \le 4$	6
$4 < t \le 8$	11
$8 < t \le 12$	7
$12 < t \le 16$	4
$16 < t \le 20$	2

c The histogram is given here (note difference between this one and the one in the solutions provided)



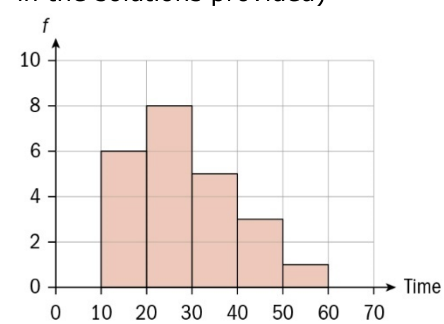
d The data is right or positively skewed

2 **a** Continuous

b The frequency table is given here (note difference between this one and the one in the solutions provided)

Time (t)	f
$10 \le t \le 20$	6
$20 < t \le 30$	8
$30 < t \le 40$	5
$40 < t \le 50$	3
$50 < t \le 60$	1

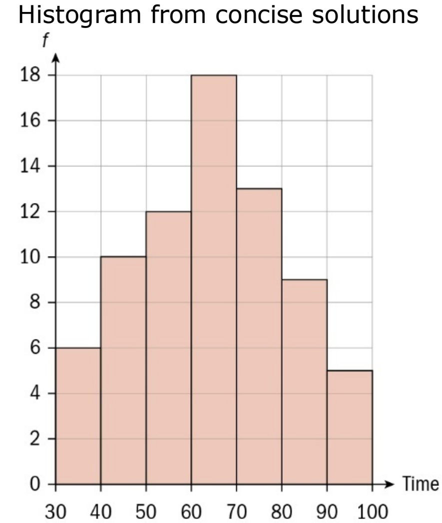
c The histogram is given here (note difference between this one and the one in the solutions provided)



d The data is right or positively skewed

3 **a** Continuous

b Histogram from concise solutions



c The data is neither right nor left skewed, it has normal distribution

4 **a** Frequency table from concise solutions

b The data is left or negatively skewed

5 **a** The frequency table is given here (note difference between this one and the one in the solutions provided)

Hours	Days
$0 < h \le 1$	1
$1 < h \le 2$	2
$2 < h \le 3$	3
$3 < h \le 4$	4
$4 < h \le 5$	6
$5 < h \le 6$	8
$6 < h \le 7$	6

b The data is left or negatively skewed

Exercise 6C

1 **a** The number that occurs most often is 8

b The number that occurs most often is 4

c The number that occurs most often is 13

d Each number occurs only once, so there is no mode

e The numbers that occur most often are 2 and 4. The data is bimodal

2 **a** The shoe size with the highest frequency is 10

b The modal mark range is $60 < y \le 80$

3 **a** **i** The mode is 3

ii The modal range is $30 < x \le 35$

b **i** Discrete data, since the scale on the x-axis is given as discrete values.

ii Continuous data, since there is a continuous scale of values on the x-axis.

Exercise 6D

1 **a** The mean is 3.65

b The mean is 12.8056

c The mean is 3.35

2 **a**

x	f	Mid value (m)	fm
$0 \le x \le 10$	18	5	90
$10 < x \le 20$	14	15	210
$20 < x \le 30$	12	25	300
$30 < x \le 40$	9	35	315
$40 < x \le 50$	7	45	315
	$\Sigma f = 60$		$\Sigma fm = 1230$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{1230}{60} = 20.5$$

b

x	f	Mid value (m)	fm
$0 \le x \le 12$	4	6	24
$12 < x \le 24$	0	18	0
$24 < x \le 36$	8	30	240
$36 < x \le 48$	15	42	630
$48 < x \le 60$	13	54	702
$60 < x \le 72$	7	66	462
	$\Sigma f = 47$		$\Sigma fm = 2058$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{2058}{47} = 43.7872 \approx 43.8$$

c

x	f	Mid value (m)	fm
$1 \le x \le 1.5$	4	1.25	5
$1.5 < x \le 2$	6	1.75	10.5
$2 < x \le 2.5$	7	2.25	15.75
$2.5 < x \le 3$	7	2.75	19.25
$3 < x \le 3.5$	5	3.25	16.25
	$\Sigma f = 29$		$\Sigma fm = 66.75$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{66.75}{29} = 2.30172 \approx 2.30$$

3 Mean

$$= \frac{0 \times 6 + 1 \times 5 + 2 \times 4 + 3 \times 7 + 4 \times 10 + 5 \times 4}{6 + 5 + 4 + 7 + 10 + 4}$$

$$= \frac{94}{36} = 2.6111 \approx 2.6 \text{ cups of coffee}$$

4 **a** Phil had

$2 + 4 + 4 + 6 + 10 + 15 + 4 + 5$

$= 50$ tomato plants

b The modal number of tomatoes per plant was 8

c Mean =

$$\frac{3 \times 2 + 4 \times 4 + 5 \times 4 + 6 \times 6 + 7 \times 10 + 8 \times 15 + 9 \times 4 + 10 \times 5}{50}$$

$$= \frac{177}{25} = 7.08$$

5 The mean number of fish caught per day was

$\frac{0\times1+1\times5+2\times4+3\times2+4\times3+5\times5+6\times3+7\times2+8\times3+9\times1+10\times2}{1+5+4+2+3+5+3+2+3+1+2}$

$=\frac{141}{31}\approx 4.55$

6 The mean amount received per day is

$\frac{5\times6+15\times14+25\times15+35\times8+45\times2}{6+14+15+8+2}$

$=\frac{197}{9}=\$21.89$

7 **a** There are $7+12+10+9+7+6+6+3$

$=60$ families represented

b The data is right or positively skewed

c The mode of the data is 2 children per family

d The mean number of children is

$\frac{7\times1+12\times2+10\times3+9\times4+7\times5+6\times6+6\times7+3\times8}{60}$

$=\frac{39}{10}=3.9$

8 **a** There are $40+60+80+30+10$

$=220$ people in the village

b The modal class is $40<a\le60$

c The mean age of the villagers is

$\frac{40\times10+60\times30+80\times50+30\times70+10\times90}{220}$

$=\frac{460}{11}\approx 41.8$

9 In the set of numbers, each appears only once, so therefore for 2 to be the mode, $a=2$. Given that the mean is 5, we have

$5=\frac{1+2+2+4+5+6+b+8+10}{9}$,

$5\times9=38+b$, $45=38+b$, $b=45-38$, $b=7$

10 Given that the mean of the numbers is 23, we have to find x

$23=\frac{8+x+17+(2x+3)+45}{5}$

$23\times5=73+3x$

$115=73+3x$

$115-73=3x$

$42=3x$

$x=\frac{42}{3}$,

$x=14$

11 Starting with 2 and 3, we know that as 4 is the mode, it must occur at least twice, start by assuming that there are two 4s then x and y are the remaining two numbers, the mean is 4, so

$4=\frac{2+3+4+4+x+y}{6}$

$6\times4=13+x+y$

$24-13=x+y$

$11=x+y$

As x and y are positive integers less than 8, the only possible solution is if $x=5$ and $y=6$ (or $y=5$ and $x=6$), so the numbers are $2,3,4,4,5,6$

12 The mean mass of the students is

$\frac{52\times8+44\times12}{20}=\frac{236}{5}=47.2$ kg

Exercise 6E

1 **a** The median is the middle number, 18

b The middle number lies between 18 and 19, $\frac{18+19}{2}=\frac{37}{2}=18.5$

c Arranging the numbers in size order $1,2,4,5,5$, the middle number is 4

d The numbers are already in size order (reversed), so the middle number is the median, the middle number lies between 3 and 4, $\frac{3+4}{2}=\frac{7}{2}=3.5$

e $2,4,5,5,6,7,7,8,8,10$. The middle number is between 6 and 7,

$\frac{6+7}{2}=\frac{13}{2}=6.5$

f $2,3,5,5,6,8$. The median is 5

2 Total number of days

$=2+4+3+7+11+18+6+2=53$

Median $=\left(\frac{n+1}{2}\right)^{th}=\left(\frac{53+1}{2}\right)^{th}=27^{th}=9$

3 **a** Mean

$=\frac{2000000\times1+1000\times10+600\times14+200\times25}{1+10+14+25}$

$=\frac{2023400}{50}=\$40468$

b The mode is the most common number, 200

c The median number is the number in the $\left(\frac{50+1}{2}\right)^{th}=25.5^{th}$ position, that is the number between 200 and 600,

$\frac{200+600}{2}=400$

Exercise 6F

1 **a** The median is the middle number, 8

b $Q_1 = \left(\frac{n+1}{4}\right)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd} = 7$

c $Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} = \left(\frac{3(11+1)}{4}\right)^{th}$

$= 9^{th} = 12$

d $\text{IQR} = Q_3 - Q_1 = 12 - 7 = 5$

e Range = largest – smallest

= 15 – 3 = 12

2 In ascending order, the numbers are

$2, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15$

a Median

$= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{12+1}{2}\right)^{th} = 6.5^{th} = 6$

b Q_1 is the median of the lower half of the numbers, 5

c Q_3 is the median of the upper half of the numbers, it lies between 8 and 9, 8.5

d $\text{IQR} = Q_3 - Q_1 = 8.5 - 5 = 3.5$

e $\text{Range} = \text{largest} - \text{smallest} = 15 - 2 = 13$

3 Sorting the number of sit-ups into ascending order,

$2, 10, 10, 12, 14, 16, 16, 20, 25, 25,$

$28, 30, 37, 40, 45, 50$

a Median

$= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{16+1}{2}\right)^{th} = 8.5^{th}$

$= \frac{20+25}{2} = 22.5$

b Q_1 is the median of the lower half of the numbers, $\frac{12+14}{2} = 13$

Q_3 is the median of the upper half of the numbers, $\frac{30+37}{2} = 33.5$

$\text{IQR} = Q_3 - Q_1 = 33.5 - 13 = 20.5$

c On 8 of the 16 days, Lincy did more than 22.5 sit-ups.

d The 'middle half' of the number of sit-ups Lincy did was between 13 and 33.5.

e On 4 of the 16 days, Lincy did more than 33.5 sit-ups.

4 Sorting the number of cars into ascending order,

$20, 20, 25, 30, 35, 35, 35, 35, 45, 45, 50.$

$\text{IQR} = Q_3 - Q_1.$

$Q_1 = \left(\frac{n+1}{4}\right)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd} = 25$

$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} = \left(\frac{3(11+1)}{4}\right)^{th} = 9^{th} = 45$

$\text{IQR} = Q_3 - Q_1 = 45 - 25 = 20$

5 Q_1 is the median of the lower half of the numbers, 2

Q_3 is the median of the upper half of the numbers, it lies between 4 and 5, 4.5

$\text{IQR} = Q_3 - Q_1 = 4.5 - 2 = 2.5$

6 **a** **i** Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{12+1}{2}\right)^{th} = 6.5^{th}$

$\Rightarrow \frac{9+r}{2} = 9.5$

$9.5 = \frac{9+r}{2}$

$9.5 \times 2 = 9 + r$

$19 - 9 = r$

$r = 10$

ii Q_3 is the median of the upper half of the numbers, it is between s and 13

$13 = \frac{s+13}{2}$

$21 \times 3 = s + 13$

$26 - 13 = s$

$s = 13$

b The value of t can be found as follows

$10 = \frac{5+6+7+7+9+9+10+10+13+13+13+t}{12},$

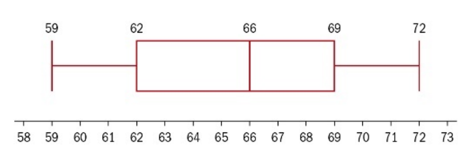
$10 \times 12 = 102 + t$

$120 - 102 = t$

$t = 18$

Exercise 6G

1



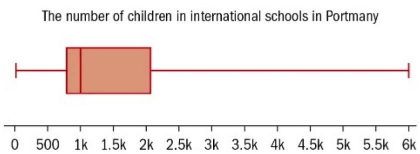
2 **a** The minimum time was 30.1

b The maximum time was 35

c The median time was 32.5

d The IQR was $33.1 - 31.9 = 1.2$

3 a



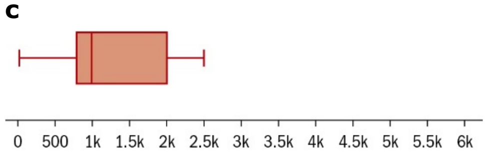
b $Q_3 + 1.5(\text{IQR})$

$= 2067.5 + 1.5 \times 1272.5$

$= 3976.25 < 6000$

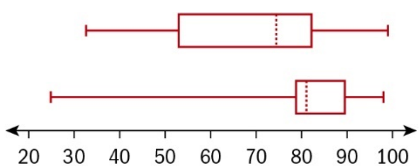
so it is an outlier

c



d The outlier was removed because it distorted the analysis

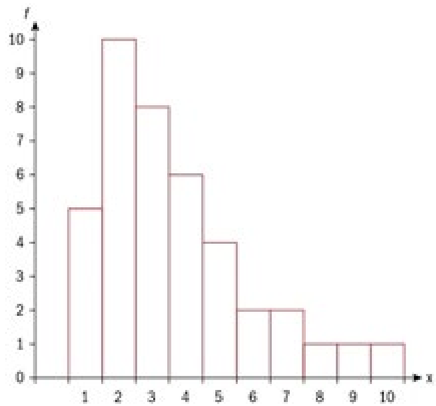
4 a



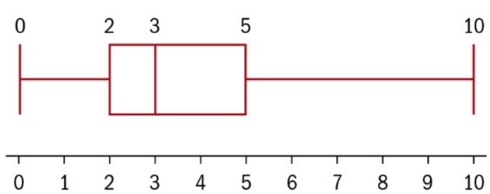
b The morning exam

c This means that there is a bigger difference between the 25% and the 75% of the scores

5 a



b



c The data is right or positively skewed

6 1A, 2C, 3B

Exercise 6H

1 a The longest time taken was 18 minutes

b The median is 11 minutes

c IQR = 13.6 – 8.2 = 5.4 minutes

d k = 15.6 minutes

2 a The median is 40 minutes

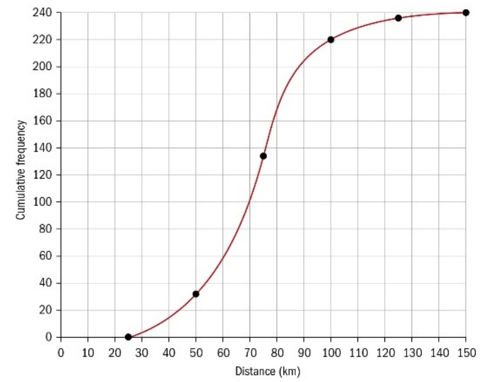
b IQR = 50 – 30 = 20

c 53

3 a

Distance	$0 \le d \le 25$	$25 < d \le 50$	$50 < d \le 75$	$75 < d \le 100$	$100 < d \le 125$	$125 < d \le 150$
CF	0	32	134	220	236	240

b



c The median is 73 km

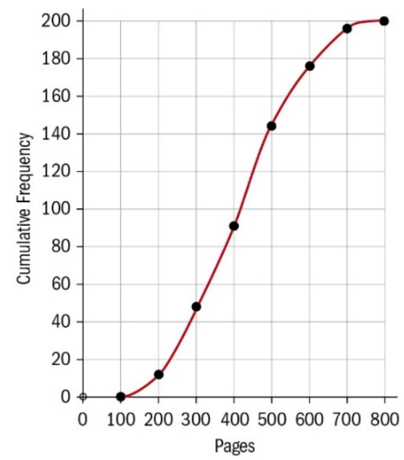
d IQR = 82 – 60 = 22 km

e 3 cars

4 a

Pages	CF
$100 \le p \le 200$	12
$200 < p \le 300$	48
$300 < p \le 400$	90
$400 < p \le 500$	143
$500 < p \le 600$	176
$600 < p \le 700$	196
$700 < p \le 800$	200

b



c The median is 420

d IQR = 510 – 300 = 210

e 80 students

5 1C, 2B, 3A

Exercise 6I

1 a $\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{4+6+7+7+5+1+2+3}{8}$

$= 4.375$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{4^2+6^2+7^2+7^2+5^2+1^2+2^2+3^2}{8} - 4.375^2$

$= 23.625 - 19.1406 \approx 4.48$

$\sigma = \sqrt{\sigma^2} \approx 2.12$

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{2+5+8+7+1+3+9+11+4+2}{10}$

$= 5.2$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{2^2+5^2+8^2+7^2+1^2+3^2+9^2+11^2+4^2+2^2}{10} - 5.2^2$

$= 37.4 - 37.04 \approx 10.4$

$\sigma = \sqrt{\sigma^2} \approx 3.22$

c $\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{-4+-2+0+3+-5}{5} = -1.6$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{(-4)^2+(-2)^2+0^2+3^2+(-5)^2}{5} - 5.2^2$

$= 10.8 - 2.56 = 8.24$

$\sigma = \sqrt{\sigma^2} \approx 2.87$

d $\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{1+2+3+4+5}{5} = 3$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2+2^2+3^2+4^2+5^2}{5} - 3^2$

$= 11 - 9 = 2$

$\sigma = \sqrt{\sigma^2} \approx 1.41$

e $\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{1+2+3+4+5+500}{6} \approx 85.8$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2+2^2+3^2+4^2+5^2+500^2}{6} - 85.833^2$

$= 41675.833 - 7367.36 \approx 34308$

$\sigma = \sqrt{\sigma^2} \approx 185.2$

2 a $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{1\times 3+2\times 8+3\times 6+4\times 6+5\times 7}{3+8+6+6+7} = 3.2$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2\times 3+2^2\times 8+3^2\times 6+4^2\times 6+5^2\times 7}{3+8+6+6+7} - 3.2^2$

$= 12 - 10.24 = 1.76$

$\sigma = \sqrt{\sigma^2} \approx 1.33$

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{1\times 5+3\times 12+5\times 16+7\times 22+9\times 27+11\times 30+13\times 18}{5+12+16+22+27+30+18}$

≈ 8.32

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2\times 5+3^2\times 12+5^2\times 16+7^2\times 22+9^2\times 27+11^2\times 30+13^2\times 18}{5+12+16+22+27+30+18} - 8.323^2$

$= 80.385 - 69.273 = 11.111$

$\sigma = \sqrt{\sigma^2} = 3.33$

c $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{5\times 18+15\times 14+25\times 13+35\times 11+45\times 6}{18+14+13+11+6}$

≈ 20.6

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{5^2\times 18+15^2\times 14+25^2\times 13+35^2\times 11+45^2\times 6}{18+14+13+11+6} - 20.645^2$

$= 602.419 - 426.223 = 176.197$

$\sigma = \sqrt{\sigma^2} \approx 13.3$

3 a $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{1\times 2+2\times 2+3\times 4+4\times 10+5\times 12+6\times 2+7\times 2+18\times 1}{2+2+4+10+12+2+2+1}$

$= 4.63$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2\times 2+2^2\times 2+3^2\times 4+4^2\times 10+5^2\times 12+6^2\times 2+7^2\times 2+18^2\times 1}{2+2+4+10+12+2+2+1} - 4.629^2$

$= 28.571 - 21.424 = 7.148$

$\sigma = \sqrt{\sigma^2} = 2.67$

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{1\times 2+2\times 2+3\times 4+4\times 10+5\times 12+6\times 2+7\times 2}{2+2+4+10+12+2+2}$

$= 4.24$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2\times 2+2^2\times 2+3^2\times 4+4^2\times 10+5^2\times 12+6^2\times 2+7^2\times 2}{2+2+4+10+12+2+2} - 4.235^2$

$= 19.882 - 17.938 = 1.945$

$\sigma = \sqrt{\sigma^2} = 1.40$

4 $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{93 + 86.2 + 80 + 64 + 60.6 + 50 + 50 + 47.3 + 46.6 + 46}{10}$

$= 62.37$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{93^2 + 86.2^2 + 80^2 + 64^2 + 60.6^2 + 50^2 + 50^2 + 47.3^2 + 46.6^2 + 46^2}{10} - 62.37^2$

$= 4177.27 - 3890.02 = 287.248$

$\sigma = \sqrt{\sigma^2} = 16.9$

5 $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{150 \times 3 + 250 \times 6 + 350 \times 11 + 450 \times 5}{3 + 6 + 11 + 5} = 322$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{150^2 \times 3 + 250^2 \times 6 + 350^2 \times 11 + 450^2 \times 5}{3 + 6 + 11 + 5} - 322^2$

$= 112100 - 103684 = 8416$

$\sigma = \sqrt{\sigma^2} = 91.7$

6 **a** $6 + 8 + 6 + 3 + 1 = 24$ months

b The modal range is 20 to 30 hours

c $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{15 \times 6 + 25 \times 8 + 35 \times 6 + 45 \times 3 + 55 \times 1}{6 + 8 + 6 + 3 + 1}$

$= 28.75$

d $\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{15^2 \times 6 + 25^2 \times 8 + 35^2 \times 6 + 45^2 \times 3 + 55^2 \times 1}{6 + 8 + 6 + 3 + 1} - 28.75^2$

$= 950 - 826.563 = 123.438$

$\sigma = \sqrt{\sigma^2} = 11.1$

Exercise 6J

1 **a** mean $= \dfrac{1 + 3 + 5 + 5 + 8}{5} = 4.4$

median = 5

mode = 5

b mean $= \dfrac{5 + 7 + 9 + 9 + 12}{5} = 8.4$

median = 9

mode = 9

c Adds 4 to mean, median and mode

2 **a** $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{7 + 9 + 3 + 0 + 1 + 8 + 6 + 4 + 10 + 5 + 5}{11} = 5.2727$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{7^2 + 9^2 + 3^2 + 0^2 + 1^2 + 8^2 + 6^2 + 4^2 + 10^2 + 5^2 + 5^2}{11} - 5.2727^2$

$= 36.909 - 27.802 = 9.11$

$\sigma = \sqrt{\sigma^2} = 3.02$

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{21 + 27 + 9 + 0 + 9 + 24 + 18 + 12 + 30 + 15 + 15}{11}$

$= 15.8182$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{21^2 + 27^2 + 9^2 + 0^2 + 9^2 + 24^2 + 18^2 + 12^2 + 30^2 + 15^2 + 15^2}{11} - 15.8182^2$

$= 332.182 - 250.215\ ;\ 82.0$

$\sigma = \sqrt{\sigma^2} = 9.054$

c The mean is multiplied by 3, and since the variance is multiplied by 9, standard deviation (which is square root of variance) is multiplied by 3.

3 mean $= 17.2 + 4 = 21.2$

median $= 17 + 4 = 21$

standard deviation $= 0.5$

4 The mean, median and standard deviation will double

5 The new variance is $9^2 x = 81x$

Chapter Review

1 **a** mode = 1

b median $= \left(\dfrac{10 + 1}{2}\right)^{th} = 5.5^{th} = \dfrac{4 + 2}{2} = 3$

c mean $= \dfrac{2 + 8 + 1 + 5 + 0 + 4 + 4 + 1 + 1 + 6}{10}$

$= \dfrac{16}{5} = 3.2$

d range $= 8 - 0 = 8$

2 $\dfrac{9 \times 420 + 3 \times 740}{12} = 500$

3 **a** mode = 3

b median $= \left(\dfrac{50 + 1}{2}\right)^{th} = 25.5^{th} = 3$

c mean $= \dfrac{0 \times 4 + 1 \times 8 + 2 \times 10 + 3 \times 20 + 4 \times 4 + 5 \times 3 + 6 \times 1}{50}$

$= \dfrac{5}{2} = 2.5$

4 The mean will increase by 4 and the standard deviation will stay the same; mean $= 21.9$, standard deviation $= 1.1$

5 **a** mean $= \dfrac{736}{23} = 32$

b mean $= \dfrac{736+24+15}{23+2} = 31$

6 **a** The mean will increase by 10 and the standard deviation will stay the same; mean $= 58$, standard deviation $= 5$

b The mean will increase by a factor of 10 and the variance will increase by a factor of 10^2; mean $= 480$,

variance $= 5^2 \times 100 = 2500$

7 **a** 40

b 60

c $50 = c - 40 \Rightarrow c = 50 + 40 = 90$

d $\text{IQR} = 24 = 74 - d \Rightarrow d = 74 - 24 = 50$

8 **a** 800 students

b 65 marks

c $\text{IQR} = 75 - 55 = 20$

d 100 students

e No, because there are 100 students who scored more than 80 marks, this is not 10%

f $k = 40$

9 1B, 2C, 3A

10 **a** $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{15+12+22+30+25+7+19+33+19+41+53+12+3+8+6+17}{16}$

$= 20.125$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{15^2+12^2+22^2+30^2+25^2+7^2+19^2+33^2+19^2+41^2+53^2+12^2+3^2+8^2+6^2+17^2}{16}$

$- 20.125^2$

$= 579.375 - 405.016 = 174.359$

$\sigma = \sqrt{\sigma^2} = 13.2$

b Write the numbers in size order;

3, 6, 7, 8, 12, 12, 15, 17, 19, 19, 22, 25, 30, 33, 41, 53

then find Q_1 as the median of the first half of the list,

$Q_1 = \left(\dfrac{8+1}{2}\right)^{th} = 4.5^{th} = \dfrac{8+12}{2} = 10$ and

find Q_2 as the median of the second half of the list,

$Q_1 = \left(8 + \dfrac{8+1}{2}\right)^{th} = 12.5^{th} = \dfrac{25+30}{2} = 27.5$

, so $\text{IQR} = 27.5 - 10 = 17.5$

11 mode $= 4$

mean

$= \dfrac{2\times3+3\times4+4\times10+5\times3+6\times2+7\times2}{3+4+10+3+2+2}$

$= 4.125$

median $= \left(\dfrac{24+1}{2}\right)^{th} = 12.5^{th} = 4$

$\sigma^2 = \dfrac{2^2\times3+3^2\times4+4^2\times10+5^2\times3+6^2\times2+7^2\times2}{3+4+10+3+2+2} - 4.125^2$

$= 18.875 - 17.0156 = 1.8593$

$\sigma = 1.36$

12 **a** mean $= \dfrac{2.5\times15+7.5\times11+12.5\times9+17.5\times12+22.5\times6}{15+11+9+12+6}$

≈ 10.9

median $= \left(\dfrac{53+1}{2}\right)^{th} = 27^{th} = 12.5$

$\sigma^2 = \dfrac{2.5^2\times15+7.5^2\times11+12.5^2\times9+17.5^2\times12+22.5^2\times6}{15+11+9+12+6} - 10.896^2$

$= 166.627 - 118.728 = 47.8996$

$\sigma = 6.92$

b Because we are using the midpoint of each range, as opposed to the actual original data, which assumes that the number of items is equally spread throughout the class interval.

13 Given that the mean number of watches is 2.5, we have to find k

$2.5 = \dfrac{0\times11+1\times7+2\times6+3\times k+4\times8+5\times10}{11+7+6+k+8+10}$,

$2.5\times(42+k) = 101+3k$

$105 + 2.5k = 101 + 3k$

$105 - 101 = (3 - 2.5)k$

$4 = 0.5k$

$k = \dfrac{4}{0.5}$

$k = 8$

14 **a** 80 bats

b 50 grams

c $\dfrac{5}{80} = 0.0625 = 6.25\%$

d $a = 10$, $c = 80 - 75 = 5$

e $b = 75 - 55 = 20$

f $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{15\times10+45\times45+75\times20+105\times5}{10+45+20+5}$

$= 52.5$

$\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{15^2\times10+45^2\times45+75^2\times20+105^2\times5}{10+45+20+5} - 52.5^2$

$= 3262.5 - 2756.25 = 506.25$

$\sigma = \sqrt{\sigma^2} = 22.5$

15 a $50 = 3 + 11 + 16 + m + 8$

$\Rightarrow m = 50 - 38 = 12$

$n = 14 + 16 = 30$

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{10 \times 3 + 15 \times 11 + 20 \times 16 + 25 \times 12 + 30 \times 8}{3 + 11 + 16 + 12 + 8}$

$= 21.1$

c $\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{10^2 \times 3 + 15^2 \times 11 + 20^2 \times 16 + 25^2 \times 12 + 30^2 \times 8}{3 + 11 + 16 + 12 + 8} - 21.1^2$

$= 477.5 - 445.21 = 32.3$

16 a Discrete

b $\bar{x} = \dfrac{\Sigma x}{n}$

$= \dfrac{1 \times 41 + 2 \times 60 + 3 \times 52 + 4 \times 32 + 5 \times 15 + 6 \times 8}{41 + 60 + 52 + 32 + 15 + 8}$

≈ 2.73

c $\sigma^2 = \dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2$

$= \dfrac{1^2 \times 41 + 2^2 \times 60 + 3^2 \times 52 + 4^2 \times 32 + 5^2 \times 15 + 6^2 \times 8}{41 + 60 + 52 + 32 + 15 + 8} - 2.731^2$

$= 9.25 - 7.4571 = 1.7929$

$\sigma = \sqrt{\sigma^2} = 1.34$

d 1 standard deviation above the mean is $2.731 + 1.339 = 4.07$, so $15 + 8 = 23$ families have more than one standard deviation above the mean mobile devices

17 a Discrete A1

b Continuous A1

c Continuous A1

d Discrete A1

18 a As the mode is 5 there must be at least another 5.

R1

So we have 1, 3, 5, 5, 6 with another number to be placed in order R1

The median will be the average of the 3rd and 4th pieces of data. R1

For this to be 4.5 the missing piece of data must be a 4.

Thus a=5, b=4 A1 A1

b $\bar{x} = \dfrac{1 + 3 + 4 + 5 + 5 + 6}{6} = \dfrac{24}{6} = 4$

M1 A1

19 a An outlier is further than 1.5 times the IQR below the lower quartile or above the upper quartile. A1

b **i** mode = 8 A1

ii median = 7 A1

iii lower quartile = 3 A1

iv upper quartile = 9 A1

c IQR = 6

$1.5 \times \text{IQR} = 9$

19 − 9 = 10 M1

19 is the (only) outlier A1

20 a $\dfrac{\sum x}{10} = 70 \Rightarrow \sum x = 700$ A1

Let the new student's mass be s.

$\dfrac{\sum x + s}{11} = 72$ M1

$700 + s = 792$ A1

So s = 92kg A1

b IQR = 10 A1

$76 + 1.5 \times IQR = 76 + 15 = 91$ M1

So new student's mass of 92 is an outlier R1

21 a 200 A1

b 35 A1

c Using mid-points 5, 15, 25… as estimates for each interval M1

i Estimate for mean is 22.25 A2

ii Estimate for standard deviation is 11.6 (3sf) A2

d Median is approximately the 100th piece of data which lies in the interval $20 < h \le 30$. A1

Will be 15 pieces of data into this interval

Estimate is $20 + \dfrac{15}{50} \times 10 = 23$ M1A1

22 a Discrete A1

b 5 A1

c **i** 4.79 (3sf) A2

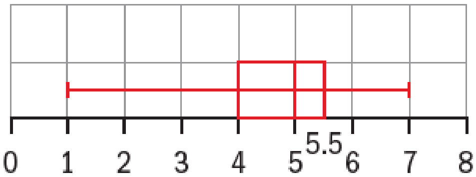
ii 1.62 (3sf) A2

d **i** 5 A1

ii 4 A1

iii 5.5 A1

e



A1 general shape

A1 median

A1 quartiles

f IQR=1.5

$1.5 \times 1.5 = 2.25$ (A1)

5.5 + 2.25 = 7.75

4 - 2.25 = 1.75 M1

So the 2 (unhappy) candidates with grade 1 are outliers A1

23 a

x	Frequency	Cumulative frequency
0	10	10
1	7	17
2	11	28
3	13	41
4	15	56
5	15	71
6	12	83
7	10	93
8	4	97
9	2	99
10	1	100

A4 for 6 correct
A3 for 4 or 5 correct
A2 for 2 or 3 correct
A1 for 1 correct

b **i** 4
ii 2
iii 6 A1A1A1

c **i** 4.05 (3sf)
ii $(2.4140...)^2 = 5.83$ (3sf) A1(M1)A1

d No. It is bimodal at x= 4 and 5. 24 A1R1

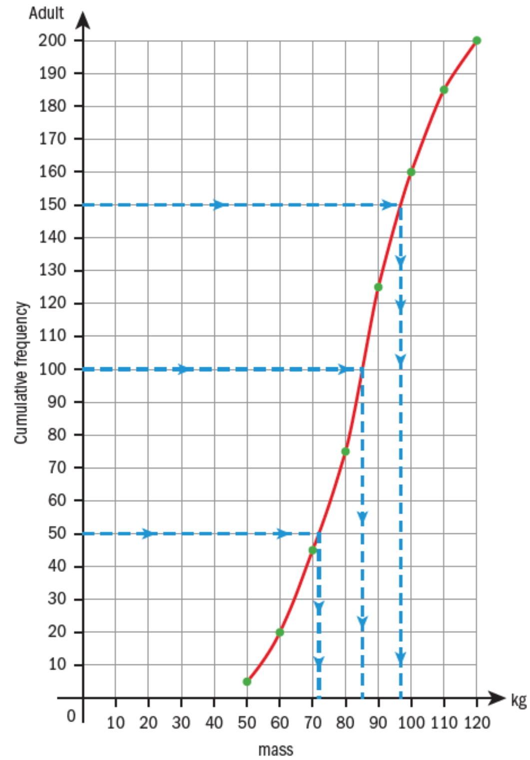
24 a $80 < w \le 90$ A1

b

mass	cumulative frequency
$40 < w \le 50$	5
$50 < w \le 60$	20
$60 < w \le 70$	45
$70 < w \le 80$	75
$80 < w \le 90$	125
$90 < w \le 100$	160
$100 < w \le 110$	185
$110 < w \le 120$	200

A2 numbers A1 labelling

c



A1A1scales A3 points and curve

d **i** 85
ii 73
iii 97 A1A1A1
M1 lines

25 a **i** 7.5
ii 6.125 A1A2
b **i** 6
ii 6.9 A1A2
c Sally's had the greater median R1
d Rob's had the greater mean R1

26 a

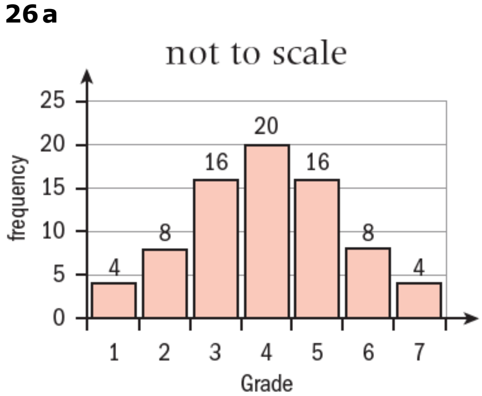


chart A1; scaleA2

b **i** 4
ii 4
iii 4 A1A1A1

c The values of the median and the mean are the same due to the symmetry of the bar chart. A1R1

7 Modelling relationships between two data sets: statistics for bivariate data

Skills check

1 **a** 1296 **b** 64 **c** 343

2 **a** 5 **b** 4 **c** 3 **d** 3

3 **a** $y = 4x - 2$ **b** $y = 1 - 2x$

Exercise 7A

1 **a** There is a strong, positive, linear correlation

b There is a weak, negative, linear correlation

c There is a strong, negative, linear correlation

d There is a weak, positive, linear correlation

e There is no correlation

2 **i** **a** Positive **b** Linear
c Strong

ii **a** Positive **b** Linear
c Moderate

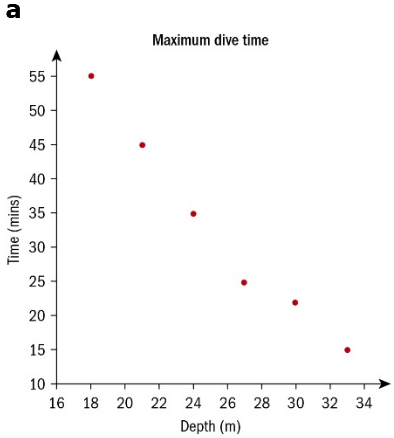
iii **a** Positive **b** Linear
c Weak

iv **a** No correlation **b** Non linear
c Zero

v **a** Negative **b** Linear
c Strong

vi **a** Negative **b** Non linear
c Strong

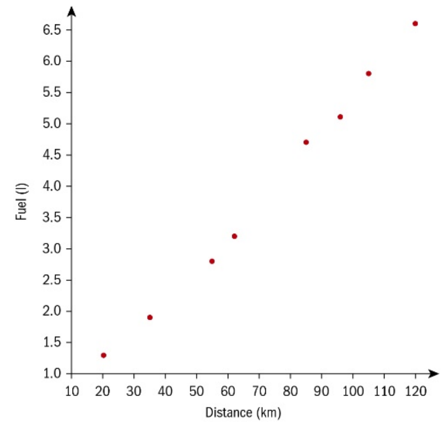
3 **a**



b There is a strong, negative, linear correlation

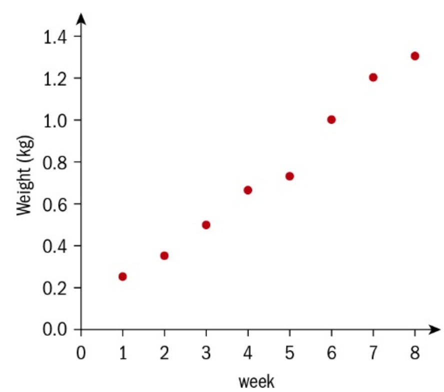
c As the maximum depth increases the time at that depth decreases

4 **a**



b There is a strong, positive, linear correlation

5 **a**



b There is a strong, positive, linear correlation

c As the kitten gets older, it gets heavier

Exercise 7B

1 **a**

x	y	x^2	y^2	xy
20	250	400	62500	5000
24	300	576	90000	7200
30	350	900	122500	10500
40	360	1600	129600	14400
50	480	2500	230400	24000
75	580	5625	336400	43500
80	750	6400	562500	60000
90	840	8100	705600	75600
100	900	10000	810000	90000
120	1000	14400	1000000	120000
$\Sigma x = 629$	$\Sigma y = 5810$	$\Sigma x^2 = 50501$	$\Sigma y^2 = 4049500$	$\Sigma xy = 450200$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 50501 - \frac{629^2}{10} = \frac{109369}{10}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 4049500 - \frac{5810^2}{10} = 673890$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 450200 - \frac{629 \times 5810}{10} = 84751$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{84751}{\sqrt{\frac{109369}{10} \times 673890}} = 0.987$$

b There is a strong positive correlation

c As the floor area increases, house price increases

2 a

x	y	x^2	y^2	xy
1	40000	1	1600000000	40000
2	36500	4	1332250000	73000
3	31000	9	961000000	93000
4	26658	16	710648954	106632
5	24250	25	588062500	121250
6	19540	36	381811600	117240
7	19100	49	364810000	133700
8	18750	64	351562500	150000
9	15430	81	238084900	138870
10	12600	100	158760000	126000
$\Sigma x =$ 55	$\Sigma y =$ 243828	$\Sigma x^2 =$ 385	$\Sigma y^2 =$ 6686990464	$\Sigma xy =$ 1099692

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 385 - \frac{55^2}{10} = \frac{165}{2}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 6686990464 - \frac{243828^2}{10}$$

$$= \frac{3708905528}{5}$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 1099692 - \frac{55 \times 243828}{10}$$

$$= -241362$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-241362}{\sqrt{\frac{165}{2} \times \frac{3708905528}{5}}} = -0.976$$

b There is a strong negative correlation

c The price of the motorbike can never fall below 0.

3 a

x	y	x^2	y^2	xy
148	34	21904	1156	5032
153	38	23409	1444	5814
165	42	27225	1764	6930
142	36	20164	1296	5112
155	42	24025	1764	6510
141	32	19881	1024	4512
171	40	29241	1600	6840
154	34	23716	1156	5236
170	40	28900	1600	6800
168	38	28224	1444	6384
Σx = 1567	Σy = 376	Σx^2 = 246689	Σy^2 = 14248	Σxy = 59170

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 246689 - \frac{1567^2}{10} = \frac{11401}{10}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 14248 - \frac{376^2}{10} = \frac{552}{5}$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 59170 - \frac{1567 \times 376}{10} = \frac{1254}{5}$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{\frac{1254}{5}}{\sqrt{\frac{11401}{10} \times \frac{552}{5}}} = 0.707$$

b There is a moderate positive correlation

4 a

x	y	x^2	y^2	xy
6	78	36	6084	468
4	80	16	6400	320
7	86	49	7396	602
5	88	25	7744	440
1	66	1	4356	66
2	70	4	4900	140
4	78	16	6084	312
6	95	36	9025	570
8	97	64	9409	776
4	76	16	5776	304
$\Sigma x = 47$	$\Sigma y = 814$	$\Sigma x^2 = 263$	$\Sigma y^2 = 67174$	$\Sigma xy = 3998$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 263 - \frac{47^2}{10} = \frac{421}{10}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 67174 - \frac{814^2}{10} = \frac{4572}{5}$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 3998 - \frac{47 \times 814}{10} = \frac{861}{5}$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{\frac{861}{5}}{\sqrt{\frac{421}{10} \times \frac{4572}{5}}} = 0.878$$

b There is a strong positive correlation

c Yes

5 a

x	y	x^2	y^2	xy
3.9	10	15.21	100	39
2.7	14	7.29	196	37.8
3.8	5	14.44	25	19
2.4	8	5.76	64	19.2
1.7	24	2.89	576	40.8
2.6	17	6.76	289	44.2
4.0	21	16	441	84
3.7	7	13.69	49	25.9
$\Sigma x = 24.8$	$\Sigma y = 106$	$\Sigma x^2 = 82.04$	$\Sigma y^2 = 1740$	$\Sigma xy = 309.9$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 82.04 - \frac{24.8^2}{8} = 5.16$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 1740 - \frac{106^2}{8} = 335.5$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 309.9 - \frac{24.8 \times 106}{8} = -18.7$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-18.7}{\sqrt{5.16 \times 335.5}} = -0.449$$

b There is a weak negative correlation

c Yes, because the correlation is only weak

6 a

x	y	x^2	y^2	xy
52	60	2704	3600	3120
60	68	3600	4624	4080
62	66	3844	4356	4092
65	69	4225	4761	4485
68	75	4624	5625	5100
76	82	5776	6724	6232
77	83	5929	6889	6391
78	84	6084	7056	6552
80	88	6400	7744	7040
84	90	7056	8100	7560
85	93	7225	8649	7905
95	92	9025	8464	8740
$\Sigma x = 882$	$\Sigma y = 950$	$\Sigma x^2 = 66492$	$\Sigma y^2 = 76592$	$\Sigma xy = 71297$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 66492 - \frac{882^2}{12} = 1665$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 76592 - \frac{950^2}{12} = 1383.67$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 71297 - \frac{882 \times 950}{12} = 1472$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{1472}{\sqrt{1665 \times 1383.67}} = 0.970$$

b There is a strong positive correlation

c More practice questions will likely increase the overall grade

Exercise 7C

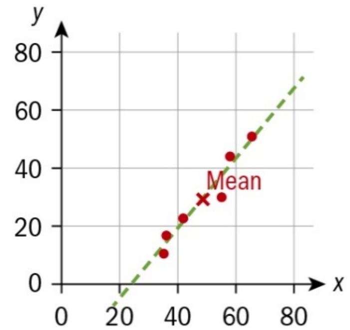
1 a mean $= \dfrac{36+55+42+35+58+65}{6}$

$= 48.5$

b mean $= \dfrac{17+30+23+11+44+51}{6}$

$= 29.333$

c



2 a mean age

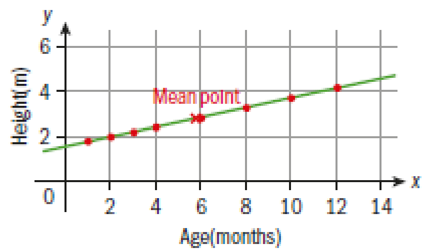
$= \dfrac{1+2+3+4+6+8+10+12}{8} = 5.75$

mean height

$= \dfrac{1.78+1.98+2.17+2.40+2.82+3.26+3.71+4.14}{8}$

$= 2.7825$

b



c $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} \times (x - x_1)$

$y - 2.7825 = \dfrac{2.7825 - 2}{5.75 - 2} \times (x - 5.75)$

$y = 0.209x + 1.583$

d $y = 0.209 \times 9 + 1.583 = 3.46$ m

e $y = 0.209 \times 120 + 1.583 = 26.6$ m

f Not reliable as it is known that giraffes only grow to 6 m

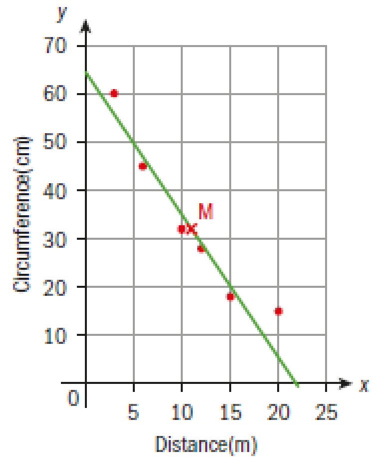
3 a mean $= \dfrac{3+6+10+12+15+20}{6}$

$= 11$ km

b mean $= \dfrac{60+45+32+28+18+15}{6}$

$= 33{,}000$ Rupees

c



d $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} \times (x - x_1)$

$y - 33 = \dfrac{33 - 50}{11 - 5} \times (x - 11)$

$y = -2.833x + 64.1667$

e $y = -2.833 \times 8 + 64.1667 = 41.5$

$\Rightarrow$ 41,500 Rupees

f $50 = -2.833x + 64.1667$

$\Rightarrow x = \dfrac{50 - 64.1667}{-2.833} = 5$ km

g $y = -2.833 \times 30 + 64.1667$

$= -20.8233$

$\Rightarrow -20{,}823$ Rupees

Not suitable to extrapolate, negative rent is not correct

Exercise 7D

1 A student who plays no sport will spend 35 hours on homework and each day spent playing sport reduces the hours of homework by 30 minutes

2 A person who has no friends who are criminals has 1 conviction and adding one extra criminal friends leads to 6 extra convictions

3 A brand new speaker is worth \$300 and as it gets older, its value decreases by \$40 per year

4 a

x	y	x^2	y^2	xy
6	157	36	24649	942
7	155	49	24025	1085
8	147	64	21609	1176
8.5	142	72.25	20164	1207
9	138	81	19044	1242
9.5	132	90.25	17424	1254
10	134	100	17956	1340
11	127	121	16129	1397
11.5	120	132.25	14400	1380
12	115	144	13225	1380
$\Sigma x = 92.5$	$\Sigma y = 1367$	$\Sigma x^2 = 889.75$	$\Sigma y^2 = 188625$	$\Sigma xy = 12403$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 889.75 - \frac{92.5^2}{10} = 34.125$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 188625 - \frac{1367^2}{10} = 1756.1$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 12403 - \frac{92.5 \times 1367}{10} = -241.75$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-241.75}{\sqrt{34.125 \times 1756.1}} = -0.988$$

b $y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{-241.75}{34.125} = -7.084 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{1367}{10} + 7.084 \times \frac{92.5}{10} = 202.227$$

so $y = 202 - 7.084x$

c $y = 202.227 - 7.084 \times 7.5 = 149 \text{ kmh}^{-1}$

d As the time taken to accelerate from 0 to 90 increases by 1 second, the top speed decreases by 7.08

5 a

x	y	x^2	y^2	xy
80	74	6400	5476	5920
73	62	5329	3844	4526
95	93	9025	8649	8835
84	75	7056	5625	6300
67	73	4489	5329	4891
88	81	7744	6561	7128
69	58	4761	3364	4002
92	90	8464	8100	8280
90	84	8100	7056	7560
$\Sigma x = 738$	$\Sigma y = 690$	$\Sigma x^2 = 61368$	$\Sigma y^2 = 54004$	$\Sigma xy = 57442$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 61368 - \frac{738^2}{9} = 852$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 54004 - \frac{690^2}{9} = 1104$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 57442 - \frac{738 \times 690}{9} = 862$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{862}{\sqrt{852 \times 1104}} = 0.889$$

b $y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{862}{852} = 1.012 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{690}{9} - 1.012 \times \frac{738}{9} = -6.30$$

so $y = 1.01x - 6.30$

c $y = 1.012 \times 75 - 6.30 = 69.6$

6 a

x	y	x^2	y^2	xy
35	13	1225	169	455
38	18	1444	324	684
42	27	1764	729	1134
45	28	2025	784	1260
47	36	2209	1296	1692
48	34	2304	1156	1632
50	40	2500	1600	2000
$\Sigma x = 305$	$\Sigma y = 196$	$\Sigma x^2 = 13471$	$\Sigma y^2 = 6058$	$\Sigma xy = 8857$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 13471 - \frac{305^2}{7} = 181.714$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 6058 - \frac{196^2}{7} = 570$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 8857 - \frac{305 \times 196}{7} = 317$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{317}{\sqrt{181.714 \times 570}} = 0.985$$

There is a strong, positive correlation

b $y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{317}{181.714} = 1.744 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{196}{7} - 1.744 \times \frac{305}{7} = -48.0321$$

so $y = 1.74x - 48.0$

c $y = 1.74 \times 40 - 48.0 = 21.6$ cm

d For every cm that the cat grows in length, it grows 1.74 cm in height

7 a $a = -8.46$ and $b = 33.0$

b $y = 32.95 - 8.46 \times 3.5 = 3.34$

$\Rightarrow$ 3 mudbugs

c

x	y	x^2	y^2	xy
0.5	30	0.25	900	15
1	28	1	784	28
1.5	14	2.25	196	21
2	18	4	324	36
2.5	10	6.25	100	25
3	7	9	49	21
4	1	16	1	4
$\Sigma x = 14.5$	$\Sigma y = 108$	$\Sigma x^2 = 38.75$	$\Sigma y^2 = 2354$	$\Sigma xy = 150$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 38.75 - \frac{14.5^2}{7} = 8.714$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 2354 - \frac{108^2}{7} = 687.714$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 150 - \frac{14.5 \times 108}{7} = -73.714$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-73.714}{\sqrt{8.714 \times 687.714}} = -0.952$$

d Strong, negative

8 a

x	y	x^2	y^2	xy
28	66	784	4356	1848
33	70	1089	4900	2310
35	85	1225	7225	2975
42	94	1764	8836	3948
40	96	1600	9216	3840
38	80	1444	6400	3040
$\Sigma x = 216$	$\Sigma y = 491$	$\Sigma x^2 = 7906$	$\Sigma y^2 = 40933$	$\Sigma xy = 17961$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 7906 - \frac{216^2}{6} = 130$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 40933 - \frac{491^2}{6} = 752.833$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 17961 - \frac{216 \times 491}{6} = 285$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{285}{\sqrt{130 \times 752.833}} = 0.911$$

b $b = \frac{S_{xy}}{S_{xx}} = \frac{285}{130} = 2.19$ and

$$a = \bar{y} - b\bar{x} = \frac{491}{6} - 2.192 \times \frac{216}{6} = 2.92$$

c If a student scores 1 mark better in the IB diploma then they will do 2.19% better in their first year at university

d $y = 2.19x + 2.92$

$= 2.19 \times 30 + 2.92 = 68.7\%$

9 a

x	y	x^2	y^2	xy
25	200	625	40000	5000
40	260	1600	67600	10400
65	350	4225	122500	22750
53	360	2809	129600	19080
46	260	2116	67600	11960
30	250	900	62500	7500
50	310	2500	96100	15500
74	600	5476	360000	44400
70	450	4900	202500	31500
$\Sigma x = 453$	$\Sigma y = 3040$	$\Sigma x^2 = 25151$	$\Sigma y^2 = 1148400$	$\Sigma xy = 168090$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 25151 - \frac{453^2}{9} = 2350$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 1148100 - \frac{3040^2}{9} = 121256$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 168090 - \frac{453 \times 3040}{9} = 15076.7$$

$y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{15076.7}{2350} = 6.416 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{3040}{9} - 6.416 \times \frac{453}{9} = 14.858$$

so $y = 6.416x + 14.858$

b **i** Each additional pizza costs \$6.42

ii When no pizzas are made, there is a cost of \$14.86

c $y = 6.416 \times 60 + 14.858 = \$399.82$

d **i** Not reliable as 5000 is not close to the domain used

ii $100 = 6.416x + 14.858$

$$x = \frac{100 - 14.858}{6.416}$$

$x = 13.27$

13 pizzas

10 a

x	y	x^2	y^2	xy
1	115	1	13225	115
2	110	4	12100	220
3	92	9	8464	276
4	89	16	7921	356
5	80	25	6400	400
8	63	64	3969	504
9	59	81	3481	531
10	54	100	2916	540
$\Sigma x = 42$	$\Sigma y = 662$	$\Sigma x^2 = 300$	$\Sigma y^2 = 58476$	$\Sigma xy = 2942$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 300 - \frac{42^2}{8} = 79.5$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 58476 - \frac{662^2}{8} = 3695.5$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 2942 - \frac{42 \times 662}{8} = -533.5$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-533.5}{\sqrt{79.5 \times 3695.5}} = -0.984$$

b $b = \frac{S_{xy}}{S_{xx}} = \frac{-533.5}{79.5} = -6.71$ and

$$a = \bar{y} - b\bar{x} = \frac{662}{8} + 6.711 \times \frac{42}{8} = 117.98$$

c $y = 117.98 - 6.711 \times 6 = 77.717$

$= ¥78000$

Exercise 7E

1 a

x	y	x^2	y^2	xy
12	45	144	2025	540
15	44	225	1936	660
18	45	324	2025	810
18	42	324	1764	756
22	40	484	1600	880
25	34	625	1156	850
30	26	900	676	780
$\Sigma x = 140$	$\Sigma y = 276$	$\Sigma x^2 = 3026$	$\Sigma y^2 = 11182$	$\Sigma xy = 5276$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 3026 - \frac{140^2}{7} = 226$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 11182 - \frac{276^2}{7} = 299.714$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 5276 - \frac{140 \times 276}{7} = -244$$

$y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{-244}{226} = -1.0797 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{276}{7} + 1.0797 \times \frac{140}{7} = 61.0226$$

so $y = 61.0 - 1.08x$,

$y = 61.0 - 1.08 \times 20 = 39.4 \Rightarrow 39$ tickets

b $x = a + by$, where

$$b = \frac{S_{xy}}{S_{yy}} = \frac{-244}{299.714} = -0.814 \text{ and}$$

$$a = \bar{x} - b\bar{y} = \frac{140}{7} + 0.814 \times \frac{276}{7} = 52.1$$

so $x = 52.1 - 0.814y$,

$y = 52.1 - 0.814 \times 35 = 23.61 \Rightarrow \$24$

2 a

x	y	x^2	y^2	xy
2	6	4	36	12
3	10	9	100	30
5	22	25	484	110
7	33	49	1089	231
8	42	64	1764	336
10	56	100	3136	560
$\Sigma x = 35$	$\Sigma y = 169$	$\Sigma x^2 = 251$	$\Sigma y^2 = 6609$	$\Sigma xy = 1279$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 6609 - \frac{169^2}{6} = 1848.83$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 1279 - \frac{35 \times 169}{6} = 293.167$$

$x = a + by$, where

$$b = \frac{S_{xy}}{S_{yy}} = \frac{293.167}{1848.83} = 0.159 \text{ and}$$

$$a = \bar{x} - b\bar{y} = \frac{35}{6} - 0.159 \times \frac{169}{6} = 1.35 \text{ so}$$

$x = 1.35 + 0.159y$

b $x = 1.35 + 0.159 \times 50 = 9.3$ mins

3

x	y	x^2	y^2	xy
90	87	8100	7569	7830
88	57	7744	3249	5061
65	52	4225	2704	3380
92	76	8464	5776	6992
50	30	2500	900	1500
67	67	4489	4489	4489
100	96	10000	9216	9600
100	74	10000	5476	7400
73	65	5329	4225	4745
90	87	8100	7569	7830
83	78	6889	6084	6474
94	89	8836	7921	8366
83	78	6889	6084	6474
$\Sigma x = 1075$	$\Sigma y = 936$	$\Sigma x^2 = 91565$	$\Sigma y^2 = 71262$	$\Sigma xy = 80096$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 71262 - \frac{936^2}{13} = 3870$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 80096 - \frac{1075 \times 936}{13} = 2696$$

$x = a + by$, where

$$b = \frac{S_{xy}}{S_{yy}} = \frac{2696}{3870} = 0.697 \text{ and}$$

$$a = \bar{x} - b\bar{y} = \frac{1075}{13} - 0.697 \times \frac{936}{13} = 32.508$$

so $x = 32.5 + 0.697y$,

$x = 32.508 + 0.697 \times 52 = 68.752$

$\Rightarrow$ 69 marks in mathematics

4 a

x	y	x^2	y^2	xy
1	180	1	32400	180
5	164	25	26896	820
9	148	81	21904	1332
12	120	144	14400	1440
14	118	196	13924	1652
19	90	361	8100	1710
21	85	441	7225	1785
24	82	576	6724	1968
30	65	900	4225	1950
34	60	1156	3600	2040
$\Sigma x = 169$	$\Sigma y = 1112$	$\Sigma x^2 = 3881$	$\Sigma y^2 = 139398$	$\Sigma xy = 14877$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 3881 - \frac{169^2}{10} = 1024.9$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 139398 - \frac{1112^2}{10} = 15743.6$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 14877 - \frac{169 \times 1112}{10} = -3915.8$$

$y = a + bx$, where

$b = \frac{S_{xy}}{S_{xx}} = \frac{-3915.8}{1024.9} = -3.821$ and

$a = \bar{y} - b\bar{x} = \frac{1112}{10} + 3.821 \times \frac{169}{10} = 175.775$

so $y = 175.775 - 3.821x$,

$y = 175.775 - 3.821 \times 7 = 149.028 = 149$

b $x = a + by$, where

$b = \frac{S_{xy}}{S_{yy}} = \frac{-3915.8}{15743.6} = -0.249$ and

$a = \bar{x} - b\bar{y} = \frac{169}{10} + 0.249 \times \frac{1112}{10} = 44.589$

so $x = 44.589 - 0.249y$,

$x = 44.589 - 0.249 \times 100 = 19.7$ km

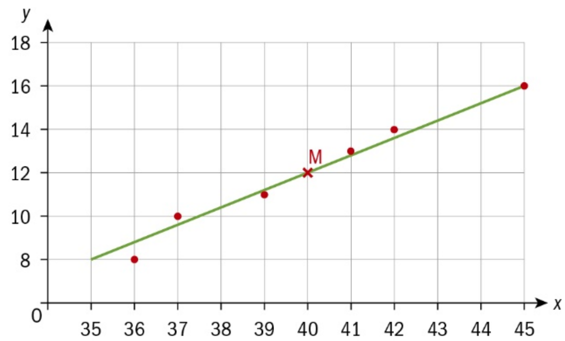
Chapter Review

1 a The PMCC lies between -1 and 1

b A -0.6, B 0.9, C 0.5, D 0, E -0.96

c Strong negative, linear

2 a,d & f



b mean temp

$$= \frac{39 + 36 + 45 + 41 + 42 + 37}{6} = \frac{240}{6}$$

$= 40\,°\mathrm{C}$

c mean cost

$$= \frac{11 + 8 + 16 + 13 + 14 + 10}{6} = \frac{72}{6} = 12$$

so 1200 Dirhams

e $y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$

$y - 12 = \frac{12 - 16}{40 - 45}(x - 40)$

$y = 0.8x - 32 + 12$

$y = 0.8x - 20$

3 a

t	e	t^2	e^2	te
0	29	0	841	0
2	38	4	1444	76
4	27	16	729	108
6	19	36	361	114
8	12	64	144	96
$\Sigma t = 20$	$\Sigma e = 125$	$\Sigma t^2 = 120$	$\Sigma e^2 = 3519$	$\Sigma te = 394$

$$S_{tt} = \Sigma t^2 - \frac{(\Sigma t)^2}{n}$$

$$S_{tt} = 120 - \frac{20^2}{5} = 40$$

$$S_{ee} = \Sigma e^2 - \frac{(\Sigma e)^2}{n}$$

$$S_{ee} = 3519 - \frac{125^2}{5} = 394$$

$$S_{te} = \Sigma te - \frac{(\Sigma t)(\Sigma e)}{n}$$

$$S_{te} = 394 - \frac{20 \times 125}{5} = -106$$

$e = at + b$, where

$a = \frac{S_{te}}{S_{tt}} = \frac{-106}{40} = -2.65$ and

$b = \bar{e} - a\bar{t} = \frac{125}{5} + 2.65 \times \frac{20}{5} = 35.6$

b $e = 35.6 - 2.65 \times 5 = 22.35 \Rightarrow 22$ eggs

c Because $t = 40$ is too far outside the domain

4 a

x	y	x^2	y^2	xy
28	3600	784	12960000	100800
46	5200	2116	27040000	239200
38	4400	1444	19360000	167200
34	3800	1156	14440000	129200
52	6000	2704	36000000	312000
50	5900	2500	34810000	295000
$\Sigma x = 248$	$\Sigma y = 28900$	$\Sigma x^2 = 10704$	$\Sigma y^2 = 144610000$	$\Sigma xy = 1243400$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 10704 - \frac{248^2}{6} = 453.333$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 144610000 - \frac{28900^2}{6} = 5408333$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 1243400 - \frac{248 \times 28900}{6} = 48866.7$$

$y = a + bx$, where

$b = \frac{S_{xy}}{S_{xx}} = \frac{48866.7}{453.333} = 107.794$ and

$a = \bar{y} - b\bar{x} = \frac{28900}{6} - 107.794 \times \frac{248}{6} = 361.181$

so $y = 108x + 361$

b Need to find the smallest x such that $120x > 107.794x + 361.181$,

$$120x > 107.794x + 361.181$$
$$(120 - 107.794)x > 361.181$$
$$x > \frac{361.181}{12.206}$$
$$x > 29.59$$

So the smallest number of chairs is 30

5 a $\frac{24 + 23.5 + 23 + 22 + 21 + 20.3 + 20 + 18.2 + 17 + 26}{10}$

$= \frac{215}{10} = 21.5$

b

x	y	x^2	y^2	xy
24	260	576	67600	6240
23.5	199	552.25	39601	4676.5
23	174	529	30276	4002
22	162	484	26244	3564
21	149	441	22201	3129
20.3	135	412.09	18225	2740.5
20	118	400	13924	2360
18.2	115	331.24	13225	2093
17	102	289	10404	1734
26	246	676	60516	6396
$\Sigma x = 215$	$\Sigma y = 1660$	$\Sigma x^2 = 4690.58$	$\Sigma y^2 = 302216$	$\Sigma xy = 36935$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 4690.58 - \frac{215^2}{10} = 68.08$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 302216 - \frac{1660^2}{10} = 26656$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 36935 - \frac{215 \times 1660}{10} = 1245$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{1245}{\sqrt{68.08 \times 26656}} = 0.924$$

c There is a strong positive correlation. The hotter the day, the more bottles sold.

d $y = a + bx$, where

$b = \frac{S_{xy}}{S_{xx}} = \frac{1245}{68.08} = 18.29$ and

$a = \bar{y} - b\bar{x} = \frac{1660}{10} - 18.29 \times \frac{215}{10} = -227.235$

so $y = 18.3x - 227$

e $y = 18.29 \times 19.6 - 227.235 = 131.249$

$\Rightarrow 131$ bottles

f 36 is far outside the domain that we have

6 a

x	y	x^2	y^2	xy
3500	110000	12250000	12100000000	385000000
2000	65000	4000000	4225000000	130000000
5000	100000	25000000	10000000000	500000000
6000	135000	36000000	18225000000	810000000
5000	120000	25000000	14400000000	600000000
3000	90000	9000000	8100000000	270000000
4000	100000	16000000	10000000000	400000000
8000	140000	64000000	19600000000	1120000000
$\Sigma x =$ 36500	$\Sigma y =$ 860000	$\Sigma x^2 =$ 191250000	$\Sigma y^2 =$ 96650000000	$\Sigma xy =$ 4215000000

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 191250000 - \frac{36500^2}{8}$$

$$= 24718750$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 96650000000 - \frac{860000^2}{8}$$

$$= 4200000000$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 4215000000 - \frac{36500 \times 860000}{8}$$

$$= 291250000$$

$y = a + bx$, where

$$b = \frac{S_{xy}}{S_{xx}} = \frac{291250000}{24718750} = 11.78 \text{ and}$$

$$a = \bar{y} - b\bar{x} = \frac{860000}{8} - 11.78 \times \frac{36500}{8} = 53753.8$$

so $y = 11.8x + 53754$

b $y = 11.8 \times 7000 + 53754 = \$136354$

c i and ii would change, iii would remain the same

7 a

x	y	x^2	y^2	xy
30	3.2	900	10.24	96
65	7.5	4225	56.25	487.5
110	8.4	12100	70.56	924
140	15.1	19600	228.01	2114
185	16.5	34225	272.25	3052.5
Σx = 530	Σy = 50.7	Σx^2 = 71050	Σy^2 = 637.31	Σxy = 6674

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 71050 - \frac{530^2}{5} = 14870$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 637.31 - \frac{50.7^2}{5} = 123.212$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 6674 - \frac{530 \times 50.7}{5} = 1299.8$$

$y = ax + b$, where

$$a = \frac{S_{xy}}{S_{xx}} = \frac{1299.8}{14870} = 0.0874 \text{ and}$$

$$b = \bar{y} - a\bar{x} = \frac{50.7}{5} - 0.0874 \times \frac{530}{5}$$

$$= 0.876$$

b The gradient indicates that a car travelling one additional mile uses 0.0874 litres of fuel

c $y = 0.876 + 0.0874 \times 160 = 14.9$ litres

d Not reliable as 5 is outside the domain of the original data

8 a

x	y	x^2	y^2	xy
1	6	1	36	6
1.5	7	2.25	49	10.5
2	10	4	100	20
2.5	15	6.25	225	37.5
3	9	9	81	27
3.5	17	12.25	289	59.5
4	20	16	400	80
4.5	18	20.25	324	81
Σx = 22	Σy = 102	Σx^2 = 71	Σy^2 = 1504	Σxy = 321.5

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 71 - \frac{22^2}{8} = 10.5$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 1504 - \frac{102^2}{8} = 203.5$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 321.5 - \frac{22 \times 102}{8} = 41$$

$y = ax + b$, where

$a = \frac{S_{xy}}{S_{xx}} = \frac{41}{10.5} = 3.90$ and

$b = \bar{y} - a\bar{x} = \frac{102}{8} - 3.90 \times \frac{22}{8} = 2.01$

b An increase in one gram of hormone leads to just under 4 extra flowers

c A plant with no growth hormone will produce 2 flowers

d $y = 2.01 + 3.905 \times 1.75 = 8.84$

e $12 = 2.01 + 3.905x$

$3.905x = 12 - 2.01 \Rightarrow x = \frac{9.99}{3.905} = 2.56$ g

f Not appropriate as 1000 is far outside the domain of the data provided

9 a

x	y	x^2	y^2	xy
100	204	10000	41616	20400
200	257	40000	66049	51400
300	292	90000	85264	87600
400	315	160000	99225	126000
500	330	250000	108900	165000
600	355	360000	126025	213000
700	370	490000	136900	259000
$\Sigma x = 2800$	$\Sigma y = 2123$	$\Sigma x^2 = 1400000$	$\Sigma y^2 = 663979$	$\Sigma xy = 922400$

$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$

$S_{xx} = 1400000 - \frac{2800^2}{7} = 280000$

$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$

$S_{yy} = 663979 - \frac{2123^2}{7} = 20103.4$

$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$

$S_{xy} = 922400 - \frac{2800 \times 2123}{7} = 73200$

$y = ax + b$, where

$a = \frac{S_{xy}}{S_{xx}} = \frac{73200}{280000} = 0.261$ and

$b = \bar{y} - a\bar{x} = \frac{2123}{7} - 0.261 \times \frac{2800}{7} = 199$

b Each additional gram increases the length of the spring by 0.261 mm

c The spring was 199 mm long before any weight was added

d $y = 199 + 0.261 \times 550 = 343$ mm

e 2 kg is outside the domain of the data, so extrapolation is unreliable

f $x = ay + b$, where

$a = \frac{S_{xy}}{S_{yy}} = \frac{73200}{20103.4} = 3.641$ and

$b = \bar{x} - a\bar{y} = \frac{2800}{7} - 3.641 \times \frac{2123}{7} = -704.263$

so $x = 3.641y - 704.263$,

$x = 3.641 \times 300 - 704.263 = 388$ g

10 a

x	y	x^2	y^2	xy
15	26	225	676	390
25	30	625	900	750
35	25	1225	625	875
45	26	2025	676	1170
55	20	3025	400	1100
65	14	4225	196	910
$\Sigma x = 240$	$\Sigma y = 141$	$\Sigma x^2 = 11350$	$\Sigma y^2 = 3473$	$\Sigma xy = 5195$

$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$

$S_{xx} = 11350 - \frac{240^2}{6} = 1750$

$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$

$S_{yy} = 3473 - \frac{141^2}{6} = 159.5$

$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$

$S_{xy} = 5195 - \frac{240 \times 141}{6} = -445$

$y = ax + b$, where

$a = \frac{S_{xy}}{S_{xx}} = \frac{-445}{1750} = -0.254$ and

$b = \bar{y} - a\bar{x} = \frac{141}{6} + 0.254 \times \frac{240}{6} = 33.7$,

$y = 33.7 - 0.254x$

b $y = 33.7 - 0.254 \times 50$

$= 21$ decimal places

c $r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$

$r = \frac{-445}{\sqrt{1750 \times 159.5}} = -0.842$

d There is a strong, negative correlation

11 a $0.51 \times 120 + 7.5 = 68.7$ M1A1

b The line of best fit goes through $(\bar{x}, \bar{y})$ R1

$\bar{y} = 0.51 \times 100 + 7.5 = 58.5$ A1

c Strong, positive A1A1

d x on y A1

12 i perfect positive A1

ii strong negative A1

iii weak positive A1

iv weak negative A1

v zero A1

13 a $r = 0.979$ (3sf) A2

b Strong, positive A1A1

c i $y = 1.23x - 21.3$ A1A1

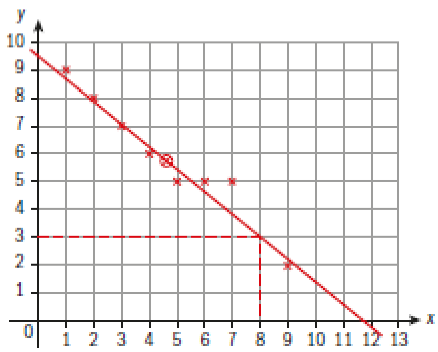
ii $x = 0.776y + 20.8$ A1A1

d $1.23 \times 105 - 21.3 = 110$ A1

e $0.776 \times 95 + 20.8 = 95$ A1

f It is extrapolation R1

14 a



(scales: A1; 3 points plotted correctly: A2; all points plotted correctly: award a further A1)

b strong, negative A1A1

c i $\bar{x} = 4.625$

ii $\bar{y} = 5.875$

iii see above A2A2A1

d see above M1

line passes through the mean A1

e 3.2 see above for lines drawn on A1A1

15 a $100 = 70m + c$

$140 = 100m + c$

$40 = 30m$

$m = \frac{4}{3}$

$c = \frac{20}{3}$ (M1)A1A1

b Positive A1

c Line goes through $(\bar{x}, \bar{y})$ (R1)

$\bar{y} = \frac{4}{3}90 + 6\frac{2}{3} = 126\frac{2}{3}$ (M1)A1

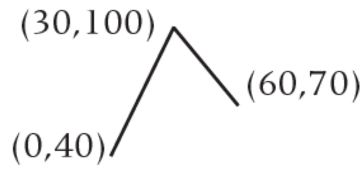
d Estimate is $\frac{4}{3}60 + 6\frac{2}{3} = 86\frac{2}{3}$ (M1)A1

16 a 40℃ A1

b 70℃ A1

c 100℃ A1

d i



A1

ii $T \geq 80$

$40 + 2t = 80 \Rightarrow t = 20$

$130 - t = 80 \Rightarrow t = 50$ M1

Interval is $20 \leq t \leq 50$. A1A1

17 a

x	13	14	15	16	16	17	18	18	19	19
y	2	0	3	1	4	1	1	2	1	2

A3 (A2 for 5 A1 for 3)

b $r = -0.0695(3sf)$ A2

c Very weak (negative) correlation so line of best fit is meaningless R1

25-year-old would be extrapolation R1

18 i Gradient $m = \frac{0.6}{3} = 0.2$ M1A1

ii $l = 0.6$ A1

iii $k = 3$ A1

iv $a = 5$ A1

v $b = 0.6$ A1

vi Gradient $p = \frac{0.9 - 0.6}{8 - 5} = 0.1$ M1A1

vii $0.6 = 0.1 \times 5 + q \Rightarrow q = 0.1$ M1A1

viii $r = 8$ A1

19 a i 0.849 (3sf) A2

ii strong, positive A1A1

iii $y = 0.937x + 0.242$ A1A1

b i 0.267 (3sf) A2

ii weak, positive A1A1

iii the r value is too small for this to be particularly meaningful R1

20 a i no change

$r = 0.87$ A1

ii no change

15 A1

iii The scatter diagram has just been moved down by 4 and to the right by 5. R1

iv Strong, positive A1A1

b i no change

$r = 0.87$ A1

ii $2 \times 15 = 30$ A1

iii the scatter diagram has been stretched vertically. R1

c **i** $r = -0.87$ A1

ii $\frac{15}{-3} = -5$ A1

iii The scatter diagram has been stretched horizontally and reflected in the y-axis. R1R1

iv Strong, negative A1A1

8 Quantifying randomness: probability

Skills Check

1 a $1-\frac{3}{7}=\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$

b $\frac{2}{5}+\frac{5}{7}=\frac{14}{35}+\frac{25}{35}=\frac{39}{35}$

c $\frac{2}{5}\times\frac{2}{3}=\frac{2\times 2}{5\times 3}=\frac{4}{15}$

d $1-\left(\frac{1}{7}\times\frac{3}{8}\right)=1-\frac{3}{56}=\frac{53}{56}$

e $\dfrac{\frac{3}{20}}{\frac{7}{20}}=\frac{3\div 20}{7\div 20}=\frac{3}{7}$

2 a $1-0.375=0.625$

b $0.65+0.05=0.7$

c $0.7\times 0.6=\frac{7\times 6}{10^2}=\frac{42}{100}=0.42$

d $0.25\times 0.64=0.64\div 4=0.16$

e $0.5\times 30=30\div 2=15$

f $0.22\times 0.22=\frac{22^2}{100^2}=\frac{484}{10000}=0.0484$

Exercise 8A

1 $P(\text{odd})=\frac{n(\{1,3,5,7,9\})}{n(\{1,2,3,4,5,6,7,8,9,10\})}$

$=\frac{5}{10}=\frac{1}{2}$

2 $P(\text{defective})=\frac{30}{150}=\frac{1}{5}$

3 $P(\text{chorus})=\frac{20}{20+10+5}=\frac{20}{35}=\frac{4}{7}$

4 a $P(\text{even})=\frac{n(\{2,4,6,8\})}{n(\{1,2,3,4,5,6,7,8\})}=\frac{4}{8}=\frac{1}{2}$

b $P(\text{multiple of 3})=\frac{n(\{3,6\})}{n(\{1,2,3,4,5,6,7,8\})}$

$=\frac{2}{8}=\frac{1}{4}$

c $P(\text{multiple of 4})=\frac{n(\{4,8\})}{n(\{1,2,3,4,5,6,7,8\})}$

$=\frac{2}{8}=\frac{1}{4}$

d $P(\text{not a multiple of 4})$

$=1-P(\text{multiple of 4})=1-\frac{1}{4}=\frac{3}{4}$

e $P(\text{less than 4})=\frac{n(\{1,2,3\})}{n(\{1,2,3,4,5,6,7,8\})}$

$=\frac{3}{8}$

f As there is no 9 on an 8 sided dice, $P(9)=0$

5 a $P(C)=\frac{n(\{C\})}{n(\{S,T,A,T,I,S,T,I,C,S\})}=\frac{1}{10}$

b As there is no P in the word "STATISTICS", $P(P)=0$

c $P(\text{vowel})=\frac{n(\{A,I,I\})}{n(\{S,T,A,T,I,S,T,I,C,S\})}$

$=\frac{3}{10}$

6 a Every other number is even, so $P(\text{even})=\frac{1}{2}$

b $P(\text{contains digit 1})$

$=\frac{n(\{1,10-19,21,31,41\})}{50}=\frac{14}{50}=\frac{7}{25}$

7 Let x be the number of seats on a minibus, then

$P(\text{coach})=\frac{3x}{x+x+x+x+3x}=\frac{3x}{7x}=\frac{3}{7}$

8 Using $P(\text{green})=2P(\text{yellow})$ and $1=P(\text{red})+P(\text{yellow})+P(\text{blue})+P(\text{green})$, we see that

$1=P(\text{red})+P(\text{yellow})+P(\text{blue})+P(\text{green})$

$1=0.4+P(\text{yellow})+0.3+P(\text{green})$

$1=0.7+P(\text{yellow})+P(\text{green})$

$1-0.7=P(\text{yellow})+2P(\text{yellow})$

$0.3=3P(\text{yellow})$

$\frac{0.3}{3}=P(\text{yellow})$

$P(\text{yellow})=0.1$

$P(\text{green})=2P(\text{yellow})=0.2$

9 Number of people who buy raffle tickets $=\frac{360}{2}=180$, of these half bought 2 tickets and the other half bought one, so there were $2\times\frac{180}{2}+180=360$ raffle tickets sold. Therefore

$P(\text{win})=\frac{1}{360}$

Exercise 8B

1 a i $P(\text{age 15})=0.18$

ii $P(\text{age 16 or higher})$

$=P(\text{age 16})+P(\text{age 17})+P(\text{age 18})$

$=0.22+0.27+0.13=0.62$

b Number of 15 year old students $=1200\times P(\text{age 15})=1200\times 0.18=216$

2 a The relative frequency of getting a 1 is

$\frac{\text{frequency of 1}}{\text{total spins}} = \frac{27}{100} = 0.27$

b The spinner is probably not fair because the relative frequencies are not close to each other, a 1 occurred nearly 4 times more than a 6

c Estimated number of 4s $= 3000 \times \frac{15}{100} = 450$

3 a 10 of each

b

Relative frequency	0.1	0.1	0.15	0.138	0.138	0.15	0.138	0.0875

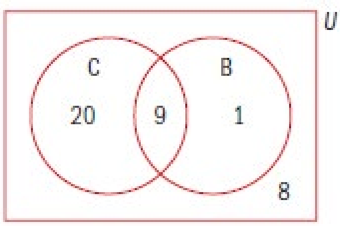
c

Relative frequency	0.0925	0.1225	0.1375	0.125	0.14	0.145	0.1075	0.13

d There is a big difference between relative frequency of getting a 1 and getting a 6. This suggests that the dice is not fair.

Exercise 8C

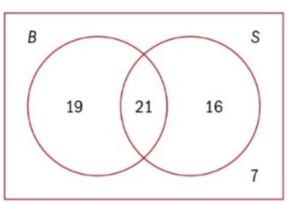
1 a



b From the Venn diagram,

$P(\text{neither}) = \frac{8}{38} = \frac{4}{19}$

2 a



b From the Venn diagram,
$19 + 21 + 16 + 7 = 63$

c i $P(\text{badminton}) = \frac{40}{63}$

ii $P(\text{both}) = \frac{21}{63} = \frac{1}{3}$

iii $P(\text{neither}) = \frac{7}{63} = \frac{1}{9}$

iv $P(\text{at least one}) = 1 - P(\text{neither})$

$= 1 - \frac{1}{9} = \frac{8}{9}$

3 Let A = gave a card and B = gave a present

a $P(\text{card or present}) = \frac{31 + 40 - 25}{50} = \frac{23}{25}$

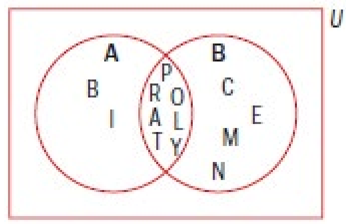
b $P(\text{card but no present}) = \frac{31 - 25}{50} = \frac{3}{25}$

c $P(\text{neither card nor present})$

$= 1 - P(\text{card or present}) = 1 - \frac{23}{25} = \frac{2}{25}$

4 $A = \{P, R, O, B, A, I, L, T, Y\}$ and
$B = \{C, O, M, P, L, E, N, T, A, R, Y\}$

a



b $A \cap B = \{P, R, O, A, L, T, Y\}$

c $A \cup B = \{P, R, O, B, A, I, L, T, Y, C, M, E, N\}$

5 a $A \cap B = \{6\}$

b $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$

c $A' = \{1, 3, 5, 7, 9\}$

d $A' \cap B = \{1, 3, 5, 7, 9\} \cap \{3, 6, 9\} = \{3, 9\}$

e $A \cup B'$

$= \{2, 4, 6, 8, 10\} \cup \{1, 2, 4, 5, 7, 8, 10\}$

$= \{1, 2, 4, 5, 6, 7, 8, 10\}$

f $A' \cup B'$

$= \{1, 3, 5, 7, 9\} \cup \{1, 2, 4, 5, 7, 8, 10\}$

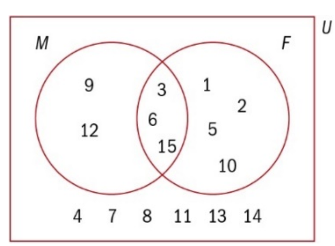
$= \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

6 $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

a i $M = \{3, 6, 9, 12, 15\}$

ii $F = \{1, 2, 3, 5, 6, 10, 15\}$

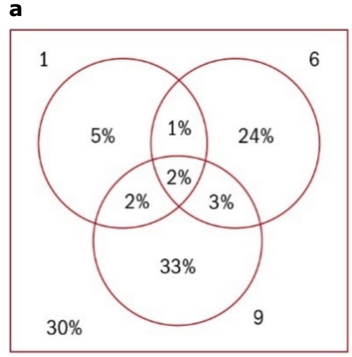
b



c i $P(M \cap F) = \frac{3}{15} = \frac{1}{5}$

ii $P((M \cup F)') = \frac{6}{15} = \frac{2}{5}$

7 a



b i $P(\text{only 9 pm}) = 33\%$

ii $P(\text{only 6 pm}) = 24\%$

iii $P(\text{no news}) = 30\%$

Exercise 8D

1 a $P(\text{prime}) = \frac{n(\{2,3,5,7\})}{10} = \frac{4}{10} = \frac{2}{5}$

b $P(\text{prime or multiple of 3})$

$= \frac{n(\{2,3,5,7\}) + n(\{3,6,9\}) - n(\{3\})}{10}$

$= \frac{4+3-1}{10} = \frac{6}{10} = \frac{3}{5}$

c $P(\text{multiple of 3 or 4})$

$= \frac{n(\{3,6,9\}) + n(\{4,8\})}{n(\{10\})}$

$= \frac{3+2}{10} = \frac{5}{10} = \frac{1}{2}$

2 $P(\text{camera owner or female})$

$= \frac{n(\text{cam}) + n(\text{fem}) - n(\text{fem and cam})}{n(U)}$

$= \frac{30+25-18}{55} = \frac{37}{55}$

3 Let $A = \{M, A, T, H, E, I, C, S\}$ and $B = \{T, R, I, G, O, N, M, E, Y\}$

a $P(A) = \frac{n(A)}{n(U)} = \frac{8}{26} = \frac{4}{13}$

b $P(B) = \frac{n(B)}{n(U)} = \frac{9}{26}$

c $P(A \cap B) = \frac{n(\{E, I, M, T\})}{n(U)} = \frac{4}{26} = \frac{2}{13}$

d $P(A \cup B) = \frac{n(A) + n(B) - n(A \cap B)}{n(U)}$

$= \frac{8+9-4}{26} = \frac{13}{26} = \frac{1}{2}$

4 a $P(\text{fiction or non-fiction})$

$= P(\text{fiction}) + P(\text{non-fiction}) - P(\text{both})$

$= 0.4 + 0.3 - 0.2 = 0.5$

b $P(\text{no book})$

$= 1 - P(\text{fiction or non-fiction})$

$= 1 - 0.5 = 0.5$

5 a $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= \frac{1}{4} + \frac{1}{8} - \frac{1}{8} = \frac{1}{4}$

b $P(X \cup Y)' = 1 - P(X \cup Y) = 1 - \frac{1}{4} = \frac{3}{4}$

6 a $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.2 + 0.4 - 0.5 = 0.1$

b $P(A' \cup B) = P(A') + P(B) - P(A' \cap B)$

$= 1 - P(A) + P(B) - (P(B) - P(A \cap B))$

$= 1 - 0.2 + 0.4 - (0.4 - 0.1) = 0.9$

7 a $3P(A \cap B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{3}{16} + \frac{3}{8} - P(A \cap B)$

so $4P(A \cap B) = \frac{9}{16}$

$\Rightarrow P(A \cap B) = \frac{9}{64} \Rightarrow P(A \cup B) = \frac{27}{64}$

b $P(A \cup B)' = 1 - P(A \cup B) = 1 - \frac{27}{64} = \frac{37}{64}$

c $P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{3}{16} - \frac{9}{64} = \frac{3}{64}$

Exercise 8E

1 a No **b** Yes **c** No **d** Yes

e No **f** No **g** No

2 $P(N \cap M) = P(N) + P(M) - P(N \cup M)$

$= \frac{1}{5} + \frac{1}{10} - \frac{3}{10} = 0$

so N and M are mutually exclusive

3 $P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$ because only one school can win

a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{4} - 0 = \frac{7}{12}$

b $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

c Yes, because the probability of A, B or C winning is not equal to 1.

Exercise 8F

1 $U = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

a $P(\text{more heads than tails})$

$= \frac{n(\{HHH, HHT, HTH, THH\})}{n(U)} = \frac{4}{8} = \frac{1}{2}$

b $P(\text{at least two heads consecutively})$

$= \frac{n(\{HHH, HHT, THH\})}{n(U)} = \frac{3}{8}$

c $P(\text{heads and tails alternately})$

$= \frac{n(\{HTH, THT\})}{n(U)} = \frac{2}{8} = \frac{1}{4}$

2 a

	1	2	3	4
1	1, 1	1, 2	1, 3	1, 4
2	2, 1	2, 2	2, 3	2, 4
3	3, 1	3, 2	3, 3	3, 4
4	4, 1	4, 2	4, 3	4, 4

b i $P(\text{red is higher than blue})$

$= \frac{n(\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\})}{n(U)}$

$= \frac{6}{16} = \frac{3}{8}$

ii P(difference between numbers is 1)

$= \frac{n(\{(1,2),(2,3),(3,4),(2,1),(3,2),(4,3)\})}{n(U)}$

$= \frac{6}{16} = \frac{3}{8}$

iii P(red is odd and blue is even)

$= \frac{n(\{(1,2),(1,4),(3,2),(3,4)\})}{n(U)}$

$= \frac{4}{16} = \frac{1}{4}$

iv P(sum is prime)

$= \frac{n(\{(1,1),(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\})}{n(U)}$

$= \frac{9}{16}$

3 a

	1	2	3
2	2, 1	2, 2	2, 3
3	3, 1	3, 2	3, 3
4	4, 1	4, 2	4, 3
5	5, 1	5, 2	5, 3

b i P(cards have same number)

$= \frac{n(\{(2,2),(3,3)\})}{n(U)} = \frac{2}{12} = \frac{1}{6}$

ii P(largest number is 3)

$= \frac{n(\{(2,3),(3,1),(3,2),(3,3)\})}{n(U)}$

$= \frac{4}{16} = \frac{1}{4}$

iii P(sum is less than 7)

$\frac{n(\{(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(5,1)\})}{n(U)}$

$= \frac{9}{12} = \frac{3}{4}$

iv P(product is at least 8)

$= \frac{n(\{(3,3),(4,2),(4,3),(5,2),(5,3)\})}{n(U)}$

$= \frac{5}{12}$

v P(at least one even number)

$= \frac{n(\{(2,1),(2,2),(2,3),(3,2),(4,1),(4,2),(4,3),(5,2)\})}{n(U)}$

$= \frac{8}{12} = \frac{2}{3}$

4 a P(at start after 2 rolls)

$= \frac{n(\{(1,3),(2,4),(3,1),(4,2),(5,6),(6,5),(5,5),(6,6)\})}{n(U)}$

$= \frac{8}{36} = \frac{2}{9}$

b P(2 meters from start after 2 rolls)

$= \frac{n(\{(1,1),(2,2),(3,3),(4,4)\})}{n(U)} = \frac{4}{36} = \frac{1}{9}$

c To be more than 1 but less than 2 meters away, he must go to a corner

P(between 1 and 2 meters after 2 rolls)

$= \frac{n(\{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,1),(1,4)\})}{n(U)}$

$= \frac{8}{36} = \frac{2}{9}$

Exercise 8G

1 $P(\text{both purple}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

2 $P(\text{all 3 like pasta}) = \left(\frac{4}{5}\right)^3 = \frac{64}{125}$

3 $P(\text{loses both}) = (1 - 0.75) \times (1 - 0.85)$

$= 0.0375$

4 a $P(B) = P(A \cap B) + P(A \cup B) - P(A)$

$= 0 + 0.4 - 0.2 = 0.2$

$P(B \cap C) = P(B) + P(C) - P(B \cup C)$

$= 0.2 + 0.3 - 0.34$

$= 0.16$

b Not independent as $P(B \cap C) \neq 0$

5 $P(\text{head and not 6}) = \frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$

6 P(not hitting with 4 missiles)

$= \left(\frac{1}{9}\right)^4 = \frac{1}{6561}$

7 a $P(E) = 1 - P(E') = 1 - 0.6 = 0.4$

b i Because

$P(E) \times P(F) = 0.24 = P(E \cap F)$

ii Because $P(E \cap F) \neq 0$

c $P(E \cup F') = P(E) + P(F') - P(E \cap F')$

$= P(E) + 1 - P(F) - (P(E) - P(E \cap F))$

$= 0.4 + 1 - 0.6 - 0.4 + 0.24 = 0.64$

8 The only possible way to have a sum of 6 is if all dice show 2. $P(\text{sum to 6}) = \left(\frac{2}{6}\right)^3 = \frac{1}{27}$

9 a $P(A \cap B) = P(A) \times P(B) = 0.9 \times 0.3 = 0.27$

b $P(A \cap B') = P(A) - P(A \cap B)$

$= 0.9 - 0.27 = 0.63$

c $P(A \cup B') = P(A) + P(B') - P(A \cap B')$

$= 0.9 + 0.3 - 0.63 = 0.97$

Exercise 8H

1 a $n(\text{both subjects})$

$= n(\text{film}) + n(\text{theatre}) - n(\text{either})$

$= 15 + 20 - (27 - 4) = 12$

b i $P(\text{theatre and not film})$

$= P(\text{theatre}) - P(\text{theatre and film})$

$= \frac{20}{27} - \frac{12}{27} = \frac{8}{27}$

ii $P(\text{theatre or film})$

$= P(\text{theatre}) + P(\text{film})$
$\quad - P(\text{theatre and film})$

$= \frac{20}{27} + \frac{15}{27} - \frac{12}{27} = \frac{23}{27}$

iii $P(\text{theatre} \mid \text{film})$

$= \frac{P(\text{theatre and film})}{P(\text{film})}$

$= \frac{\frac{12}{27}}{\frac{15}{27}} = \frac{12}{15} = \frac{4}{5}$

2 a $P(\text{even} \mid \text{not multiple of 4})$

$= \frac{P(\text{even and not multiple of 4})}{P(\text{not multiple of 4})}$

$= \frac{\frac{n(\{2,6,14\})}{8}}{\frac{n(\{1,2,6,7,11,14,29\})}{8}} = \frac{3}{7}$

b $P(<15 \mid >5)$

$= \frac{P(\text{less than 15 and greater than 5})}{P(\text{greater than 5})}$

$= \frac{\frac{n(\{6,7,11,14\})}{8}}{\frac{n(\{6,7,11,14,24,29\})}{8}} = \frac{4}{6} = \frac{2}{3}$

c $P(\text{less than 5} \mid \text{less than 15})$

$= \frac{P(\text{less than 5 and less than 15})}{P(\text{less than 15})}$

$= \frac{\frac{n(\{1,2\})}{8}}{\frac{n(\{1,2,6,7,11,14\})}{8}} = \frac{2}{6} = \frac{1}{3}$

d $P(1 \leftrightarrow 10 \mid 5 \leftrightarrow 25)$

$= \frac{P(1 \leftrightarrow 15 \text{ and } 5 \leftrightarrow 25)}{P(5 \leftrightarrow 25)}$

$= \frac{\frac{n(\{6,7,11,14\})}{8}}{\frac{n(\{6,7,11,14,24\})}{8}} = \frac{4}{5}$

3 a $P(V \cap W) = 0$ because they are mutually exclusive

b $P(V \mid W) = 0$ because they are mutually exclusive

c $P(V \cup W) = P(V) + P(W) - P(V \cap W)$

$= 0.26 + 0.37 - 0 = 0.63$

4 a $P(\text{male and left-handed}) = \frac{5}{50} = \frac{1}{10}$

b $P(\text{right-handed}) = \frac{43}{50}$

c $P(\text{right-handed} \mid \text{female})$

$= \frac{P(\text{right-handed and female})}{P(\text{female})} = \frac{\frac{11}{50}}{\frac{13}{50}} = \frac{11}{13}$

5 $P(J \mid K) = \frac{P(J \cap K)}{P(K)} = \frac{P(J) \times P(K)}{P(K)} = P(J) = 0.3$

6 $P(\text{two boys} \mid \text{one is a boy})$

$= \frac{P(\text{two boys and one is a boy})}{P(\text{one is a boy})}$

$= \frac{\frac{n(\{BB\})}{n(\{BB,BG,GB,GG\})}}{\frac{n(\{BB,BG,GB\})}{n(\{BB,BG,GB,GG\})}} = \frac{1}{3}$

Exercise 8I

1 a $P(\text{three picture cards}) = \frac{12}{52} \times \frac{11}{51} \times \frac{10}{50} = \frac{11}{1105}$

b $P(\text{two picture cards}) = 3 \times \frac{12}{52} \times \frac{11}{51} \times \frac{40}{50}$

$= \frac{132}{1105}$

2 a $P(\text{two broken pens}) = \frac{5}{14} \times \frac{4}{13} = \frac{20}{182} = \frac{10}{91}$

b $P(\text{at least one broken pen})$

$= P(\text{one broken pen}) + P(\text{two broken pens})$

$= 2 \times \frac{9}{14} \times \frac{5}{13} + \frac{10}{91} = \frac{55}{91}$

c $P(\text{girl picks broken pen}) = \frac{1}{4}$

3 a $P(\text{male}) = \frac{3}{10}$

b $P(\text{one male and one female})$

$= 2 \times \frac{3}{10} \times \frac{7}{9} = \frac{7}{15}$

4 a $P(\text{at least one answers correctly})$

$= P(\text{one answers correctly})$
$\quad + P(\text{both answer correctly})$

$= \frac{5}{7} \times \frac{4}{9} + \frac{2}{7} \times \frac{5}{9} + \frac{5}{7} \times \frac{5}{9} = \frac{55}{63}$

b $P(\text{Luca correct} \mid \text{at least one correct})$

$= \frac{P(\text{Luca correct})}{P(\text{at least one correct})} = \frac{\frac{5}{7}}{\frac{55}{63}} = \frac{9}{11}$

c $P(\text{Ian correct} \mid \text{at least one correct})$

$$= \frac{P(\text{Ian correct})}{P(\text{at least one correct})} = \frac{\frac{5}{9}}{\frac{55}{63}} = \frac{7}{11}$$

d $P(\text{two correct} \mid \text{at least one correct})$

$$= \frac{P(\text{two correct})}{P(\text{at least one correct})} = \frac{\frac{5}{7} \times \frac{5}{9}}{\frac{55}{63}} = \frac{5}{11}$$

Chapter review

1 a $P(\text{divisible by 5})$

$$= \frac{n(\{10, 15, 20, \ldots 85, 90, 95\})}{n(\{10, 11, \ldots, 98, 99\})}$$

$$= \frac{18}{90} = \frac{1}{5}$$

b $P(\text{divisible by 3})$

$$= \frac{n(\{12, 15, 18, \ldots 93, 96, 99\})}{n(\{10, 11, \ldots, 98, 99\})} = \frac{30}{90} = \frac{1}{3}$$

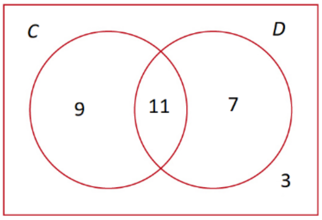
c $P(\text{greater than 50})$

$$= \frac{n(\{51, 52, 53, \ldots, 98, 99\})}{n(\{10, 11, \ldots, 98, 99\})} = \frac{49}{90}$$

d $P(\text{a square number})$

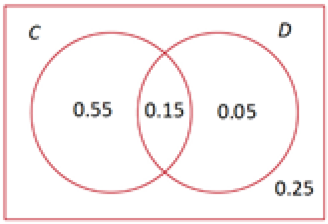
$$= \frac{n(\{16, 25, 36, 49, 64, 81\})}{n(\{10, 11, \ldots, 98, 99\})} = \frac{6}{90} = \frac{1}{15}$$

2



From Venn diagram $P(\text{Cat and dog}) = \frac{11}{30}$

3 a



From Venn diagram $P(C \cap D') = 0.55$

b $P(C \cap D) = 0.15$

$P(C) \times P(D) = 0.7 \times 0.2 = 0.14$

$P(C \cap D) \neq P(C) \times P(D)$

Therefore C and D are not independent events.

4 a $P(A \cap B) = P(B) \times P(A \mid B)$

$= 0.2 \times 0.1 = 0.02$

b $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.2 - 0.02$

$= 0.78$

c $P(A \cup B) - P(A \cap B)$

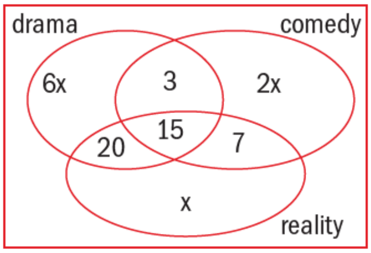
$= 0.78 - 0.02 = 0.76$

d $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$

$$= \frac{0.02}{0.6} = \frac{2}{60} = \frac{1}{30}$$

5 a $6x$

b



c $15 + 3 + 7 + 20 + 6x + 2x + x = 90$

$45 + 9x = 90$

$9x = 45$

$x = 5$

6 a $P(C \cap D) = P(D) \times P(C \mid D)$

$= 0.5 \times 0.6 = 0.3$

b Not mutually exclusive as $P(C \cap D) \neq 0$

c $P(C) \times P(D) = 0.4 \times 0.5 = 0.2$

$P(C \cap D) \neq P(C) \times P(D)$

Therefore C and D are not independent events.

d $P(C \cup D) = P(C) + P(D) - P(C \cap D)$

$= 0.4 + 0.5 - 0.3$

$= 0.6$

e $P(D \mid C) = \frac{P(C \cap D)}{P(C)}$

$$= \frac{0.3}{0.4} = \frac{3}{4} = 0.75$$

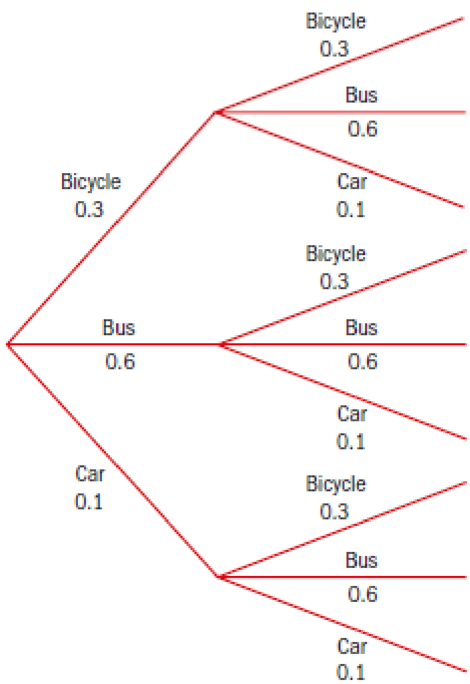
7 a $P(\text{properly}) = \frac{3}{5} \times 0.35 + \frac{2}{5} \times 0.55$

$= 0.21 + 0.22 = 0.43$

b $P(\text{Jill} \mid \text{Properly}') = \frac{P(\text{Jill} \cap \text{Properly}')}{P(\text{Properly}')}$

$$= \frac{\frac{2}{5} \times 0.45}{0.57} = \frac{0.18}{0.57} = 0.316$$

8 a



b **i** $0.3 \times 0.3 = 0.09$

ii $0.3 \times 0.6 = 0.18$

iii $0.3 \times 0.3 + 0.6 \times 0.6 + 0.1 \times 0.1$

$= 0.09 + 0.36 + 0.01 = 0.46$

c $0.7 \times 0.7 \times 0.7 = 0.343$

d $3 \times 0.1^2 \times 0.6 + 3 \times 0.3^2 \times 0.1$

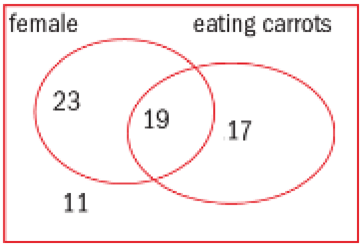
$= 0.018 + 0.027 = 0.045$

9 a $\frac{6}{16} = \frac{3}{8}$

b $\frac{10}{15} = \frac{2}{3}$

c $\frac{5}{15} \times \frac{4}{14} = \frac{1}{3} \times \frac{2}{7} = \frac{2}{21}$

10



Both female and eating carrots = 19.

a $\frac{11}{70}$

b $P(F|C) = \frac{P(F \cap C)}{P(C)} = \frac{19}{36}$

c $P(F) \times P(C) = \frac{42}{70} \times \frac{36}{70}$

$= \frac{54}{175}$

$\neq \frac{19}{70} = P(F \cap C)$

Therefore not independent.

11 a $\frac{1}{6}$ A1

b 2,4 or 6: $\frac{3}{6} = \frac{1}{2}$ A1

c Primes are 2,3,5: $\frac{3}{6} = \frac{1}{2}$ (M1)A1

d 4 or 5: $\frac{2}{6} = \frac{1}{3}$ (M1)A1

e Impossible: 0 A1

12 a $\frac{1}{36}$ A1

b $\frac{6}{36} = \frac{1}{6}$ A1

c $\frac{1}{36}$ A1

d $2 \times \frac{1}{36} = \frac{1}{18}$ A1

e $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$

or using a lattice diagram $\frac{6}{36} = \frac{1}{6}$

(M1)A1

f $\frac{1}{6}$ since independent A1

g $P(R5 \cup B5) = P(R5) + P(B5) - P(R5 \cap B5)$

$= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$ or using a lattice diagram

(M1)A1

h Considering the list in (e) $\frac{2}{6} = \frac{1}{3}$

or using conditional probability formula

(M1)A1

13 a Independent $\Leftrightarrow P(F \cap R) = P(F) \times P(R)$

R1

$\frac{1}{6} \neq \frac{1}{3} \times \frac{1}{4}$ so not independent A1

b $P(F \cup R) = P(F) + P(R) - P(F \cap R)$ M1

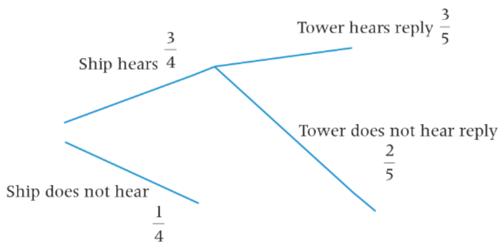
$= \frac{1}{3} + \frac{1}{4} - \frac{1}{6} = \frac{5}{12}$ A1

c P(exactly one team)$= \frac{5}{12} - \frac{1}{6} = \frac{1}{4}$ (M1)A1

Could also use a Venn diagram in (b) and (c)

d $P([F \cap R]|F) = \frac{P(F \cap R)}{P(F)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$ M1A1

14 a



A2

b $\frac{3}{4}\times\frac{3}{5}=\frac{9}{20}$ M1A1

c $1-\frac{9}{20}=\frac{11}{20}$ M1A1

d $P(\textit{ship not hear} \mid \textit{tower has no reply})$

$=\frac{P(\textit{ship not hear} \cap \textit{tower has no reply})}{P(\textit{tower has no reply})}$

$=\frac{\frac{1}{4}}{\frac{11}{20}}=\frac{5}{11}$ M1A1

e $P(A\cap B)=0$ so events are mutually exclusive. R1A1

15 a $30\times\frac{3}{5}=18$ (M1)A1

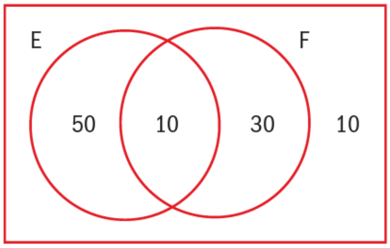
b $50\times\frac{2}{5}=20$ (M1)A1

c $T\times\frac{3}{5}=30\Rightarrow T=50$ (M1)A1

16 a Let x be the number speaking both English and French. $(60-x)+x+(40-x)+10=100$

$\Rightarrow 110-x=100\Rightarrow x=10$ (M1)A1

b



A3 (A1 shape A2 numbers)

c i $\frac{50}{100}=\frac{1}{2}$ A1

ii $\frac{90}{100}=\frac{9}{10}$ A1

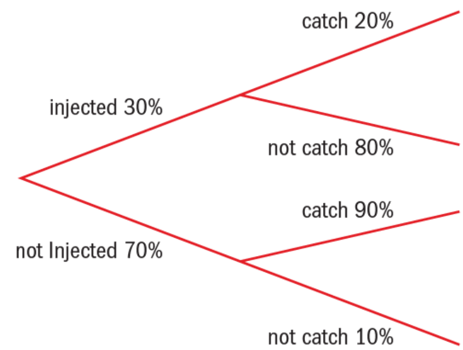
iii $\frac{40}{100}=\frac{2}{5}$ A1

d $P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{10}{40}=\frac{1}{4}$ M1A1

e If independent then $P(E|F)=P(E)$ R1

$\frac{1}{4}\neq\frac{60}{100}$ so not independent A1

17 a



A4 (A2 layout A2 numbers)

b $\frac{70}{100}\times\frac{90}{100}=\frac{63}{100}$ A1

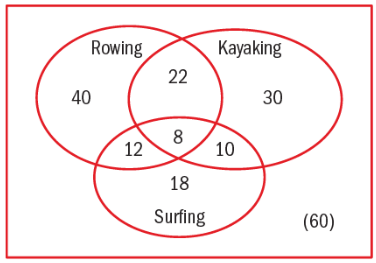
c $\frac{30}{100}\times\frac{80}{100}=\frac{6}{25}$ A1

d $\frac{30}{100}\times\frac{20}{100}+\frac{70}{100}\times\frac{90}{100}=\frac{69}{100}$ M1A1

e $P(I'|C)=\frac{P(I'\cap C)}{P(C)}=\frac{\frac{63}{100}}{\frac{69}{100}}=\frac{21}{23}$ M1A1

f $P(I|C')=\frac{P(I\cap C')}{P(C')}=\frac{\frac{24}{100}}{\frac{31}{100}}=\frac{24}{31}$ M1A1

18 a



A4 (A1 shape, A3, 7 numbers, A2, 4 numbers, A1 2 numbers)

b $200-140=60$ (M1)A1

c i $\frac{30}{200}=\frac{3}{20}$ A1

ii $\frac{122}{200}=\frac{61}{100}$ A1

iii $\frac{92}{200}=\frac{23}{50}$ A1

iv $1-\frac{82}{200}=\frac{118}{200}=\frac{59}{100}$ (M1)A1

d $\frac{20}{48}=\frac{5}{12}$ or by using the formula A2

19 a i $\frac{5}{8}\times\frac{4}{7}=\frac{5}{14}$ (M1)A1

ii RG or GR

$\frac{5}{8}\times\frac{3}{7}+\frac{3}{8}\times\frac{5}{7}=\frac{15}{28}$ (M1)A1

b **i** $\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$ (M1)A1

ii RG or GR

$\frac{5}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{5}{8} = \frac{15}{32}$ (M1)A1

20 a **i** $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow 0.4 = \frac{P(A \cap B)}{0.5} \Rightarrow P(A \cap B) = 0.2$

M1A1

ii $P(A) = P(A \cap B) + P(A \cap B')$

$= 0.2 + 0.4 = 0.6$ M1A1

iii $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.5 - 0.2 = 0.9$ M1A1

iv $P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{0.4}{0.5} = 0.8$

M1A1

b $P(A|B) \neq P(A|B')$ so not independent

R1A1

9 Representing equivalent quantities: exponentials and logarithms

Skills Check

1 **a** $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

b $10^3 = 10 \times 10 \times 10 = 1000$

c $\left(\frac{1}{3}\right)^5 = \frac{1^5}{3^5} = \frac{1}{243}$

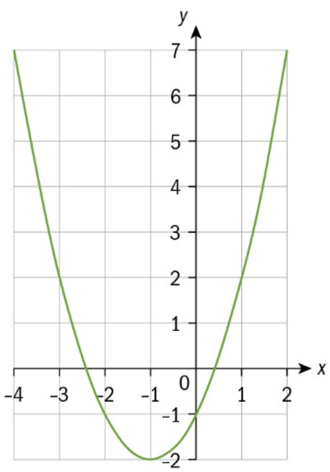
d $\left(\frac{5}{6}\right)^3 = \frac{5^3}{6^3} = \frac{125}{216}$

2 **a** $2^3 = 8$, $x = 3$

b $10^4 = 10000$, $x = 4$

c $4^4 = 456$, $x = 4$

3



Exercise 9A

1 $a^5 \cdot a^3 \cdot a^7 = a^{5+3} \cdot a^7$

$= a^8 \cdot a^7$

$= a^{8+7}$

$= a^{15}$

2 $2x^3y^2 \cdot 7x^4y^6 = 2 \cdot 7 \cdot x^3 \cdot x^4 \cdot y^2 \cdot y^6$

$= 14 \cdot x^{3+4} \cdot y^{2+6}$

$= 14x^7y^8$

3 $4ab^3 \cdot 0.5a^6c = 4 \cdot 0.5 \cdot a \cdot a^6 \cdot b^3 \cdot c$

$= 2 \cdot a^{1+6} \cdot b^3 \cdot c$

$= 2a^7b^3c$

4 $\frac{8m^5}{4m^3} = \frac{8}{4} \cdot m^{5-3} = 2 \cdot m^2$

5 $\frac{6u^5v^2}{9u^3v^3} = \frac{6}{9} \cdot u^{5-3}v^{2-3}$

$= \frac{2}{3} \cdot u^2 \cdot v^{-1}$

$= \frac{2u^2}{3v}$

6 $\left(3 \cdot rs^3\right)^3 = 3^3 \cdot r^3 \cdot \left(s^3\right)^3$

$= 27r^3 \cdot s^{3 \cdot 3}$

$= 27r^3s^9$

7 $\left(-2x^4yz^5\right)^3 = \left(-2\right)^3 \cdot \left(x^4\right)^3 y^3 \left(z^5\right)^3$

$= -8 \cdot x^{4 \cdot 3} \cdot y^3 \cdot z^{5 \cdot 3}$

$= -8x^{12}y^3z^{15}$

8 $(\frac{x^{12}y^8}{x^5y^6})^2 = (\frac{x^{12}}{x^5} \cdot \frac{y^8}{y^6})^2$

$= (x^{12-5}y^{8-6})^2$

$= (x^7y^2)^2$

$= x^{7 \cdot 2}y^{2 \cdot 2}$

$= x^{14}y^4$

9 $\frac{(5x)^2(5y^3)}{(5x^3y^4)^3} = \frac{5^2x^25y^3}{5^3(x^3)^3(y^4)^3}$

$= \frac{125x^2y^3}{125x^{3 \cdot 3}y^{4 \cdot 3}}$

$= \frac{x^2y^3}{x^9y^{12}}$

$= \frac{x^2}{x^9} \cdot \frac{y^3}{y^{12}}$

$= x^{2-9} \cdot y^{3-12}$

$= x^{-7} \cdot y^{-9}$

$= \frac{1}{x^7y^9}$

10 $\frac{9x^3(y^3)^3}{-81(x^{-2})^4y^{11}} = \frac{9x^3y^{3 \cdot 3}}{-81x^{-2 \cdot 4}y^{11}}$

$= \frac{9x^3y^9}{-81x^{-8}y^{11}}$

$= -\frac{1}{9}x^{3-(-8)}y^{9-11}$

$= -\frac{1}{9}x^{11}y^{-2}$

$= -\frac{x^{11}}{9y^2}$

11 The area of a square of length l is l^2. Therefore the area of a square with side length $3x^2y$ is $(3x^2y)^2$.

$(3x^2y)^2 = 3^2(x^2)^2y^2$

$= 9x^{2 \cdot 2}y^2$

$= 9x^4y^2$

12 The area of a rectangle with width w and length l is $w \cdot l$. Then the area of this rectangle is

$$4a^3b^2 \cdot \frac{5a}{2b^3} = \frac{4\cdot 5}{2}\cdot a^{3+1}\cdot b^2 \cdot \frac{1}{b^3}$$
$$= 10\cdot a^4 \cdot b^{2-3}$$
$$= 10\cdot a^4\cdot b^{-1}$$
$$= 10\frac{a^4}{b}$$

Exercise 9B

1 a $7^{\frac{1}{2}} = \sqrt[2]{7} = \sqrt{7}$

b $2^{\frac{3}{5}} = (2^{\frac{1}{5}})^3 = (\sqrt[5]{2})^3 = \sqrt[5]{2^3}$

c $6^{\frac{3}{2}} = (6^{\frac{1}{2}})^3 = (\sqrt[2]{6})^3 = \sqrt[2]{6^3}$

d $2^{\frac{5}{4}} = (2^{\frac{1}{4}})^5 = (\sqrt[4]{2})^5 = \sqrt[4]{2^5}$

e $5^{\frac{-1}{2}} = (5^{\frac{1}{2}})^{-1} = (\sqrt[2]{5})^{-1} = \sqrt[2]{5^{-1}} = \sqrt[2]{\frac{1}{5}}$

f $(3x)^{\frac{-3}{2}} = [(3x)^{\frac{1}{2}}]^{-3} = \sqrt[2]{3x}^{-3}$

$= \sqrt[2]{(3x)^{-3}} = \sqrt{\frac{1}{(3x)^3}}$

g $3x^{\frac{-3}{2}} = 3(x^{\frac{1}{2}})^{-3} = 3\sqrt[2]{x}^{-3} = 3\sqrt[2]{x^{-3}}$

$= 3\sqrt[2]{\frac{1}{x^3}} = \frac{3}{\sqrt{x^3}}$

2 a $\sqrt{10}^3 = \left(10^{\frac{1}{2}}\right)^3 = 10^{\frac{3}{2}}$

b $\sqrt[5]{a^6} = \left(\sqrt[5]{a}\right)^6 = \left(a^{\frac{1}{5}}\right)^6 = a^{\frac{6}{5}}$

c $\sqrt[3]{m^7} = (\sqrt[3]{m})^7 = (m^{\frac{1}{3}})^7 = m^{\frac{7}{3}}$

d $\frac{1}{\sqrt{5x}} = \sqrt{5x}^{-1} = [(5x)^{\frac{1}{2}}]^{-1} = (5x)^{\frac{-1}{2}}$

e $\frac{1}{\sqrt[4]{(2d)^5}} = \sqrt[4]{(2d)^5}^{-1} = ((2d)^5)^{\frac{-1}{4}} = (2d)^{\frac{-5}{4}}$

f $3\sqrt{x} = 3x^{\frac{1}{2}}$

g $\frac{3}{\sqrt{x}} = 3\cdot\sqrt{x}^{-1} = 3\cdot(x^{\frac{1}{2}})^{-1} = 3x^{\frac{-1}{2}}$

Exercise 9C

1 a $2^x = 16 = 2^4 \Rightarrow x = 4$

b $10^x = 1000000 = 10^6 \Rightarrow x = 6$

c $2^{x+1} = 64 = 2^6 \Rightarrow x+1 = 6 \Rightarrow x = 5$

d $3^{2x-1} = 27 = 3^3 \Rightarrow 2x-1 = 3$

$\Rightarrow 2x = 4 \Rightarrow x = 2$

e $3^{1-2x} = 1 = 3^0 \Rightarrow 1-2x = 0$

$\Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

f $3\cdot 2^x = 48 = 3\cdot 16 = 3\cdot 2^4 \Rightarrow x = 4$

g $4^{x+2} = \frac{1}{64} = \frac{1}{2^6} = 2^{-6}$

$4 = 2^2$

$(2^2)^{x+2} = 2^{2(x+2)} = 2^{2x+4} = 2^{-6}$

$2x + 4 = -6$

$x = -5$

h $\sqrt[4]{3} = 9^x = (3^2)^x = 3^{2x}$

$3^{\frac{1}{4}} = 3^{2x}$

$\frac{1}{4} = 2x$

$x = \frac{1}{8}$

i $(\frac{1}{5})^x = 25 = 5^2$

$(5^{-1})^x = 5^{-x}$

$5^{-x} = 5^2$

$x = -2$

j $2^x = 2\sqrt{2} = 2\cdot 2^{\frac{1}{2}} = 2^{1+\frac{1}{2}} = 2^{\frac{3}{2}} \Rightarrow x = \frac{3}{2}$

2 a $2^{x+3} = 4^{x-2} = (2^2)^{x-2} = 2^{2(x-2)}$

$x + 3 = 2(x-2) = 2x - 4$

$x = 7$

b $5^{x-3} = 25^{x-4} = (5^2)^{x-4} = 5^{2(x-4)}$

$x - 3 = 2(x-4) = 2x - 8$

$x = 5$

c $6^{2x-6} = 36^{3x-4} = (6^2)^{3x-4} = 6^{2(3x-4)}$

$2x - 6 = 2(3x-4) = 6x - 8$

$2 = 4x$

$x = \frac{1}{2}$

d $9^{5x+2} = (\frac{1}{3})^{11-x} = (3^{-1})^{11-x} = 3^{x-11}$

$9^{5x+2} = (3^2)^{5x+2} = 3^{2(5x+2)}$

$2(5x+2) = x - 11$

$10x + 4 = x - 11$

$9x = -15$

$x = \frac{-5}{3}$

Exercise 9D

1 a $y = 10^x$ has exponential growth as $10 > 0$ and $10 \neq 1$

b $y = 6^{-x}$ has exponential decay as $6 > 0$, $6 \neq 1$ and the coefficient of x is negative.

c $y = (\frac{3}{5})^x$ has exponential decay as

$\frac{3}{5} < 1$

d $y = (0.45)^x$ has exponential decay as $0.45 < 1$

e $y = (1.5)^x$ has exponential growth as $1.5 > 1$

2 $f(0) = h(0) = 1$ So f and h are either A or D.

h has exponential decay so it must be line A.

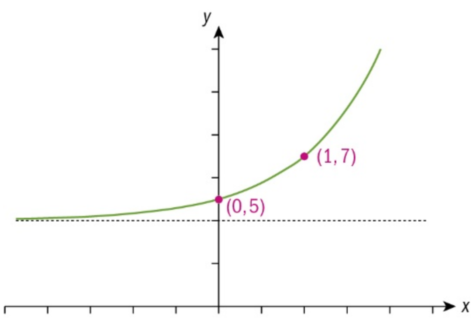
Thus f is line D.

$g(0) = 2$

$j(0) = 2^{-2} = \frac{1}{4}$

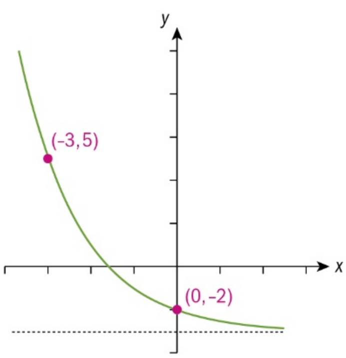
Thus g is C, j is E and finally i is B.

3 a i



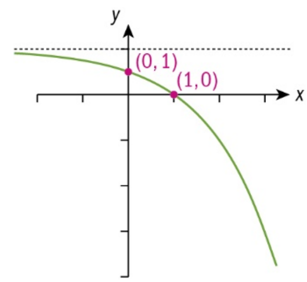
ii y= 4 **iii** $x \in \mathbb{R}, y > 4$

b i



ii y = − 3 **iii** $x \in \mathbb{R}, y > -3$

c i



ii y = 2 **iii** $x \in \mathbb{R}, y < 2$

4 $H = (65)2^{-\frac{t}{2}} + 25$

a

$t = 0 \Rightarrow H = 65 \cdot 2^0 + 25 = 65 + 25 = 90°C$

b $t = 3 \Rightarrow H = 65 \cdot 2^{-\frac{3}{2}} + 25 = 48°C$

c $H \leq 40$

$65 \cdot 2^{-\frac{t}{2}} + 25 \leq 40$

$65 \cdot 2^{-\frac{t}{2}} \leq 15$

$2^{-\frac{t}{2}} \leq \frac{15}{65} = \frac{3}{13}$

$2^{-\frac{t}{2}} \leq \frac{1}{4}$

$2^{-\frac{t}{2}} \leq 2^{-2}$

$-\frac{t}{2} \leq -2$

$t \geq 4$

d $25°C$ is the temperature of the room as that is the only constant in the equation.

5 a $y = 30(0.9)^x$

$x = 0$

$y = 30(0.9)^0 = 30$

The value of a new car is \$30 000.

b $x = 3$

$y = 30(0.9)^3$

$y = 21.87$

The value of a 3 year old car is \$21 870.

c $y = 30(0.9)^x = \frac{30}{2} = 15$

$(0.9)^x = \frac{15}{30} = \frac{1}{2}$

Using GDC we find $x = 6.58$.

6 a $P = 40(1.5)^t$

$t = 0$

$P = 40(1.5)^0 = 40$

There were initially 40 squirrels.

b $t = 2$

$P = 40(1.5)^2 = 90$

c $P = 40(1.5)^x = 200$

$(1.5)^x = \frac{200}{40} = 5$

Using GDC we find $x = 3.97$.

7 a $A = A_0(2)^{-\frac{t}{5730}}$

$A_0 = 100$

$t = 1000$

$A = 100(2)^{-\frac{1000}{5730}}$

$A = 88.6$

b Use GDC to sketch the graph.

i $A = 100(2)^{-\frac{t}{5730}} = 75$

$2^{-\frac{t}{5730}} = \frac{75}{100} = \frac{3}{4}$

That gives $t = 2378$ years.

ii $A = 100(2)^{-\frac{t}{5730}} = \frac{100}{2} = 50$

From the graph $t = 5730$ years.

Exercise 9E

1 a $e = 2.718$ **b** $e^2 = 7.389$

c $e^{-2} = 0.135$ **d** $3e = 8.155$

e $\frac{e}{2} = 1.359$ **f** $5\sqrt{e} = 8.244$

g $4e - 5 = 5.873$

2 a $y = f(x) = 1^x = 1$ is a line.

b The graph of $h(x)$ is between the graphs of $g(x)$ and $i(x)$. They are all exponential graphs.

3 The transformation that maps $f(x)$ onto $g(x)$ is a reflection in the y – axis.

4 a $t = 0$

$G(0) = 4500e^{0.3\cdot 0} = 4500e^0 = 4500cm^2$

b $t = 10$

$G(10) = 4500e^{0.3\cdot 10} = 4500e^3$

$= 90\,400 \text{ cm}^2$ (3 s.f.)

5 $PV = 5000$

$r = 0.05$

$t = 6$

$FV = 5000 \cdot e^{0.05\cdot 6} = 5000e^{0.3} = \$6749$

6 a The population a= 7 billion was growing at a rate of r= 1.1% so the exponential growth formula gives

$y = 7(1+0.011)^t$ billion.

b $t = 2025 - 2011 = 14$

$y = 7(1+0.011)^{14} = 8.16$ billion.

c $y = 7(1+0.011)^t = 10$

$(1+0.011)^t = \frac{10}{7}$

From the graph $t = 32.6$ so in the year 2043.

7 The value of a car decreases by 15% every year. Tatiana buys a new car for \$25 000.

Use the formula for exponential decay with $a = \$25000$ and $r = 0.15$.

a $y = 25000(1-0.15)^t$ in thousands of dollars

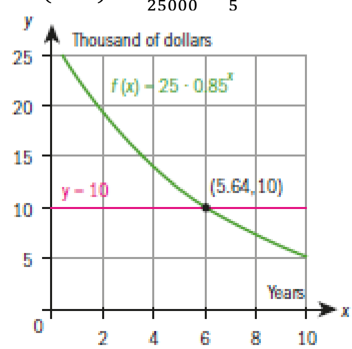
b $t = 3$

$y = 25000(1-0.15)^3$

$= 25000(0.85)^3 = \$15400$

c $y = 10\,000$

$(0.85)^t = \frac{10000}{25000} = \frac{2}{5}$



From the graph $t = 5.64$ years.

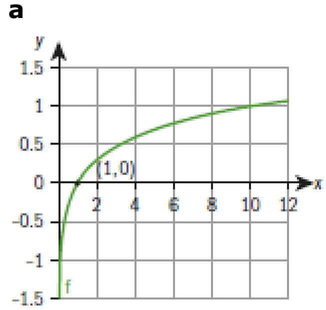
Exercise 9F

1 a The graph of $g(x)$ is a vertical translation of 4 units up

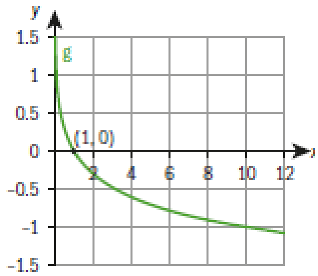
b The graph of $h(x)$ is a horizontal translation of 3 units right.

c The graph of $i(x)$ is a vertical stretch of scale factor 2.

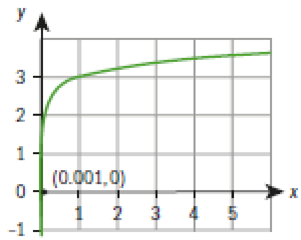
2 a



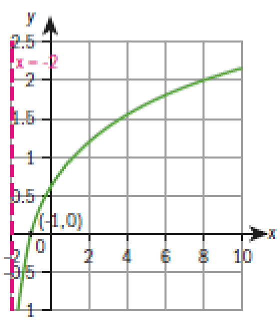
b



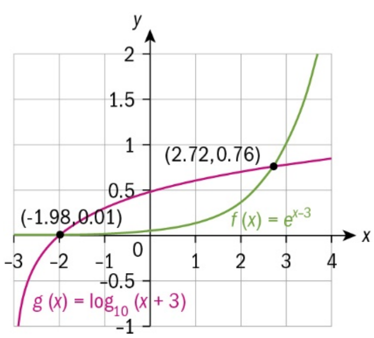
c



d



3



$x = -1.98,\ 2.72$

Exercise 9G

1 **a** $\log_p q = r \Rightarrow p^r = q$

b $\log_3 5 = r \Rightarrow 3^r = 5$

c $\log_7 q = 6 \Rightarrow 7^6 = q$

d $\log_p 5 = 3 \Rightarrow p^3 = 5$

e $\log 11 = x \Rightarrow 10^x = 11$

2 **a** $r^s = t \Rightarrow \log_r t = s$

b $8^2 = 64 \Rightarrow \log_8 64 = 2$

c $10^x = 25 \Rightarrow \log 25 = x$

d $3^{-2} = \frac{1}{9} \Rightarrow \log_3 \frac{1}{9} = -2$

e $27^{\frac{2}{3}} = 9 \Rightarrow \log_{27} 9 = \frac{2}{3}$

3 **a** $\log_7 1 \Rightarrow 7^x = 1$

$7^x = 7^0$

$x = 0$

$\log_7 1 = 0$

b $\log_8 1 \Rightarrow 8^x = 1$

$8^x = 8^0$

$x = 0$

$\log_8 1 = 0$

c $\log_9 1 \Rightarrow 9^x = 1$

$9^x = 9^0$

$x = 0$

$\log_9 1 = 0$

d $\log_x 1 \Rightarrow$

$x^y = 1$

$x^y = x^0$

$y = 0$

$\log_x 1 = 0$

4 **a** $\log_3 3 \Rightarrow 3^x = 3$

$3^x = 3^1$

$x = 1$

$\log_3 3 = 1$

b $\log_4 4 \Rightarrow 4^x = 4$

$4^x = 4^1$

$x = 1$

$\log_4 4 = 1$

c $\log_5 5 \Rightarrow 5^x = 5$

$5^x = 5^1$

$x = 1$

$\log_5 5 = 1$

d $\log_x x \Rightarrow$

$x^y = x$

$x^y = x^1$

$y = 1$

$\log_x x = 1$

5 **a** $\log_3 9 \Rightarrow 3^x = 9$

$3^x = 3^2$

$x = 2$

$\log_3 9 = 2$

b $\log_2 32 \Rightarrow 2^x = 32$

$2^x = 2^5$

$x = 5$

$\log_2 32 = 5$

c $\log_5 125 \Rightarrow 5^x = 125$

$5^x = 5^3$

$x = 3$

$\log_5 125 = 3$

d $\log_4 256 \Rightarrow 4^x = 256$

$4^x = 4^4$

$x = 4$

$\log_4 256 = 4$

e $\log_{25} 5 \Rightarrow$

$25^x = 5$

$25^x = 25^{\frac{1}{2}}$

$x = \frac{1}{2}$

$\log_{25} 5 = \frac{1}{2}$

f $\log_8 2 \Rightarrow$

$8^x = 2$

$8^x = 8^{\frac{1}{3}}$

$x = \frac{1}{3}$

$\log_8 2 = \frac{1}{3}$

g $\log_3 \frac{1}{27} \Rightarrow$

$3^x = \frac{1}{27}$

$3^x = 3^{-3}$

$x = -3$

$\log_3 \frac{1}{27} = -3$

Exercise 9H

1 a $\log 3 + \log 5 = \log 3 \cdot 5 = \log 15$

b $\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

c $3\log 5 = \log 5^3 = \log 125$

d $3\log 4 - 4\log 3 = \log 4^3 - \log 3^4$

$= \log 64 - \log 81$

$= \log \frac{64}{81}$

e $\log x + \log 1 = \log x \cdot 1 = \log x$

f $\log 236 - \log 1 = \log \frac{236}{1} = \log 236$

g $5\log 2 + 2\log 5 = \log 2^5 + \log 5^2$

$= \log 32 + \log 25 = \log 32 \times 25$

$= \log 800$

h $\log 128 - 6\log 2 = \log 128 - \log 2^6$

$= \log 128 - \log 64$

$= \log \frac{128}{64}$

$= \log 2$

i $\log 2 + \log 3 + \log 4 = \log 2 \cdot 3 + \log 4$

$= \log 6 + \log 4$

$= \log 6 \cdot 4$

$= \log 24$

j

$\log 12 - 2\log 2 + \log 3 = \log 12 - \log 2^2 + \log 3$

$= \log 12 - \log 4 + \log 3$

$= \log \frac{12}{4} + \log 3$

$= \log 3 + \log 3$

$= \log 3 \cdot 3$

$= \log 9$

k $5\log 2 + 4\log 3 = \log 2^5 + \log 3^4$

$= \log 32 + \log 81$

$= \log 32 \cdot 81$

$= \log 2592$

l

$\log 6 + 3\log 3 - \log 2 = \log 6 + \log 3^3 - \log 2$

$= \log 6 + \log 27 - \log 2$

$= \log 6 \cdot 27 - \log 2$

$= \log 162 - \log 2$

$= \log \frac{162}{2}$

$= \log 81$

Exercise 9I

1 a $\log 18 = \log 3 \cdot 6$

$= \log 3 + \log 6$

$= x + y$

b $\log 2 = \log \frac{6}{3}$

$= \log 6 - \log 3$

$= y - x$

c $\log 9 = \log 3^2$

$= 2\log 3$

$= 2x$

d $\log 27 = \log 3^3$

$= 3\log 3$
$= 3x$

e $\log 36 = \log 6^2$
$= 2\log 6$
$= 2y$

f $\log \frac{1}{2} = \log 1 - \log 2$
$= 0 - \log \frac{6}{3}$
$= -(\log 6 - \log 3)$
$= -(y - x)$
$= x - y$

2 a $\log_5 28 = \log_5 7 \cdot 4$
$= \log_5 7 + \log_5 4$
$= m + n$

b $\log_5 \frac{7}{4} = \log_5 7 - \log_5 4$
$= m - n$

c $\log_5 49 = \log_5 7^2$
$= 2\log_5 7$
$= 2m$

d $\log_5 64 = \log_5 4^3$
$= 3\log_5 4$
$= 3n$

e $\log_5 \frac{49}{4} = \log_5 49 - \log_5 4$
$= \log_5 7^2 - \log_5 4$
$= 2\log_5 7 - \log_5 4$
$= 2m - n$

f $\log_5 \frac{7}{16} = \log_5 7 - \log_5 16$
$= m - \log_5 4^2$
$= m - 2\log_5 4$
$= m - 2n$

g $\log_5 112 = \log_5 7 \cdot 4^2$
$= \log_5 7 + \log_5 4^2$
$= \log_5 7 + 2\log_5 4$
$= m + 2n$

h $\frac{\log_5 7}{\log_5 4} = \frac{m}{n}$

i $\frac{\log_5 49}{\log_5 64} = \frac{\log_5 7^2}{\log_5 4^3} = \frac{2\log_5 7}{3\log_5 4} = \frac{2m}{3n}$

j $\log_5 100 = \log_5 25 \cdot 4$
$= \log_5 5^2 \cdot 4$
$= \log_5 5^2 + \log_5 4$
$= 2\log_5 5 + n$
$= 2 + n$

3 $x = \log_2 A$
$y = \log_2 B$
$z = \log_2 C$

$\log_2 (\frac{A}{BC^3})^4 = 4\log_2 (\frac{A}{BC^3})$
$= 4(\log_2 A - \log_2 (BC^3))$
$= 4(x - (\log_2 B + \log_2 C^3))$
$= 4(x - (y + 3\log_2 C))$
$= 4(x - y - 3z)$

4 $\log_3 P = x, \log_3 Q = y$

a $\log_3 P^3 Q = \log_3 P^3 + \log_3 Q$
$= 3\log_3 P + y$
$= 3x + y$

b $\log_3 \frac{\sqrt{P}}{Q} = \log_3 \sqrt{P} - \log_3 Q$
$= \log_3 P^{\frac{1}{2}} - \log_3 Q$
$= \frac{1}{2}\log_3 P - \log_3 Q$
$= \frac{1}{2}x - y$

5 a $\log x - \log(x - 5) = \log M$
$\log \frac{x}{x-5} = \log M$
$\frac{x}{x-5} = M$

b $\log x - \log(x - 5) = 1$
$= \log 10$
$\log \frac{x}{x-5} = \log 10$
$\frac{x}{x-5} = 10$
$x = 10(x - 5)$
$= 10x - 50$
$50 = 9x$
$x = \frac{50}{9}$

Exercise 9J

1 a $\ln e^3 = 3\ln e = 3$

b $\ln e^4 = 4\ln e = 4$

c $\ln\sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}\ln e = \frac{1}{2}$

d $\ln\sqrt[3]{e} = \ln e^{\frac{1}{3}} = \frac{1}{3}\ln e = \frac{1}{3}$

e $\ln\frac{1}{e} = \ln 1 - \ln e = 0 - \ln e = 0 - 1 = -1$

f $\ln\frac{1}{e^2} = \ln 1 - \ln e^2 = 0 - 2\ln e = -2$

2 a $e^{\ln 2} = 2$

b $e^{\ln 3} = 3$

c $e^{\ln x} = x$

d $e^{2\ln 4} = e^{\ln 4^2} = 4^2 = 16$

e $e^{3\ln x} = e^{\ln x^3} = x^3$

f $e^{-\ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$

3 a $\ln x = 2.7$

$x = e^{2.7} = 14.9$

b $\ln(x+1) = 1.86$

$x + 1 = e^{1.86}$

$x = e^{1.86} - 1 = 5.42$

4 $e^{x\ln a} = e^{\ln a^x} = a^x$

Exercise 9K

1 a $\log_3 8 = 1.89$ **b** $\log_6 24 = 1.77$

c $\log_5 8 = 1.29$ **d**` $\log_3 30 = 3.10$

e $\log_7\frac{1}{4} = -0.712$ **f** $\log_2\frac{3}{5} = -0.737$

2 $p = \log_3 A$

$q = \log_3 B$

$\log_A B = \frac{\log_3 B}{\log_3 A} = \frac{q}{p}$

3 a $\log_3 6 = \frac{\log_x 6}{\log_x 3} = \frac{s}{r}$

b $\log_6 3 = \frac{\log_x 3}{\log_x 6} = \frac{r}{s}$

c

$\log_3 36 = \log_3 6^2 = 2\log_3 6 = 2\frac{\log_x 6}{\log_x 3} = 2\frac{s}{r}$

d $\log_3 54 = \frac{\log_x 54}{\log_x 3}$

$= \frac{\log_x(6\cdot 9)}{r}$

$= \frac{\log_x 6 + \log_x 3^2}{r}$

$= \frac{s + 2\log_x 3}{r}$

$= \frac{s + 2r}{r}$

e $\log_9 6 = \frac{\log_x 6}{\log_x 9} = \frac{s}{\log_x 3^2} = \frac{s}{2\log_x 3} = \frac{s}{2r}$

f $\log_6 18 = \frac{\log_x 18}{\log_x 6} = \frac{\log_x(6\cdot 3)}{\log_x 6}$

$= \frac{\log_x 6 + \log_x 3}{\log_x 6} = \frac{s + r}{r}$

g $\log_3 2 = \frac{\log_x 2}{\log_x 3} = \frac{\log_x \frac{6}{3}}{\log_x 3}$

$= \frac{\log_x 6 - \log_x 3}{\log_x 3} = \frac{s - r}{r}$

4 $\log_x y = \frac{\log_y y}{\log_y x} = \frac{1}{\log_y x}$

5 $\ln 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$

6 "Show that" is to use numbers to demonstrate a certain property and that it works for the numbers that you are using. To prove it to use variables to prove that the system works for all numbers.

Exercise 9L

1 a $2^x = 5$

$\log 2^x = \log 5$

$x\log 2 = \log 5$

$x = \frac{\log 5}{\log 2}$

$x = 2.32$

b $3^x = 17$

$\log 3^x = \log 17$

$x\log 3 = \log 17$

$x = \frac{\log 17}{\log 3}$

$x = 2.58$

c $9^x = 49$

$\log 9^x = \log 49$

$x\log 9 = \log 49$

$x = \frac{\log 49}{\log 9}$

$x = 1.77$

d $3^x = 69$

$$\log 3^x = \log 69$$
$$x\log 3 = \log 69$$
$$x = \frac{\log 69}{\log 3}$$
$$x = 3.85$$

e $16^x = 67$

$$\log 16^x = \log 67$$
$$x\log 16 = \log 67$$
$$x = \frac{\log 67}{\log 16}$$
$$x = 1.52$$

f $12^x = 5$

$$\log 12^x = \log 5$$
$$x\log 12 = \log 5$$
$$x = \frac{\log 5}{\log 12}$$
$$x = 0.65$$

g $7^x = 4$

$$\log 7^x = \log 4$$
$$x\log 7 = \log 4$$
$$x = \frac{\log 4}{\log 7}$$
$$x = 0.712$$

h $19^x = 2$

$$\log 19^x = \log 2$$
$$x\log 19 = \log 2$$
$$x = \frac{\log 2}{\log 19}$$
$$x = 0.235$$

i $e^x = 5$

$$\log e^x = \log 5$$
$$x\log e = \log 5$$
$$x = \frac{\log 5}{\log e}$$
$$x = 1.61$$

j $e^x = 10$

$$\log e^x = \log 10$$
$$x\log e = 1$$
$$x = \frac{1}{\log e}$$
$$x = 2.30$$

2 **a** $2^{4x} = 9$

$$\log 2^{4x} = \log 9$$
$$(4x)\log 2 = \log 9$$
$$x = \frac{\log 9}{4\log 2}$$
$$x = 0.792$$

b $6^{3x} = 4$

$$\log 6^{3x} = \log 4$$
$$(3x)\log 6 = \log 4$$
$$x = \frac{\log 4}{3\log 6}$$
$$x = 0.258$$

c $5^{\frac{1}{2}x} = 79$

$$\log 5^{\frac{1}{2}x} = \log 79$$
$$\frac{1}{2}x\log 5 = \log 79$$
$$x = \frac{2\log 79}{\log 5}$$
$$x = 5.43$$

d $2^{x+1} = 15$

$$\log 2^{x+1} = \log 15$$
$$(x+1)\log 2 = \log 15$$
$$x + 1 = \frac{\log 15}{\log 2}$$
$$x = \frac{\log 15}{\log 2} - 1$$
$$x = 2.91$$

e $6^{x-2} = 4$

$$\log 6^{x-2} = \log 4$$
$$(x-2)\log 6 = \log 4$$
$$x - 2 = \frac{\log 4}{\log 6}$$
$$x = \frac{\log 4}{\log 6} + 2$$
$$x = 2.77$$

f $e^{x-1} - 4 = 6$

$$e^{x-1} = 10$$
$$\log e^{x-1} = \log 10$$
$$(x-1)\log e = 1$$
$$x - 1 = \frac{1}{\log e}$$
$$x = 1 + \frac{1}{\log e}$$
$$x = 3.30$$

g $2^{3x-2} = 53$

$$\log 2^{3x-2} = \log 53$$
$$(3x-2)\log 2 = \log 53$$
$$3x - 2 = \frac{\log 53}{\log 2}$$
$$3x = 2 + \frac{\log 53}{\log 2}$$
$$x = \frac{1}{3}(2 + \frac{\log 53}{\log 2})$$
$$x = 2.58$$

h $4^{2x+1} = 10$

$$\log 4^{2x+1} = \log 10$$
$$(2x+1)\log 4 = 1$$
$$2x + 1 = \frac{1}{\log 4}$$
$$2x = \frac{1}{\log 4} - 1$$
$$x = \frac{1}{2}(\frac{1}{\log 4} - 1)$$
$$x = 0.330$$

i $11^{x-8} - 11 = 48$

$$11^{x-8} = 59$$
$$\log 11^{x-8} = \log 59$$
$$(x-8)\log 11 = \log 59$$
$$x - 8 = \frac{\log 59}{\log 11}$$
$$x = 8 + \frac{\log 59}{\log 11}$$
$$x = 9.70$$

j $9^{x+10} + 22 = 100$

$$9^{x+10} = 78$$
$$\log 9^{x+10} = \log 78$$
$$(x+10)\log 9 = \log 78$$
$$x + 10 = \frac{\log 78}{\log 9}$$
$$x = \frac{\log 78}{\log 9} - 10$$
$$x = -8.02$$

3 a $6 \times 2^x = 14$

$$2^x = \frac{14}{6}$$
$$\log 2^x = \log \frac{14}{6}$$
$$x \log 2 = \log \frac{14}{6}$$
$$x = \frac{\log \frac{14}{6}}{\log 2}$$
$$x = 1.22$$

b $4 \times 6^{3x} = 16$

$$6^{3x} = \frac{16}{4} = 4$$
$$\log 6^{3x} = \log 4$$
$$3x \log 6 = \log 4$$
$$x = \frac{\log 4}{3\log 6}$$
$$x = 0.258$$

c $3 \times 4e^{2-2x} + 1 = 4$

$$3 \times 4e^{2-2x} = 3$$
$$e^{2-2x} = \frac{3}{12}$$
$$e^{2-2x} = \frac{1}{4}$$
$$\log e^{2-2x} = \log \frac{1}{4}$$
$$(2-2x)\log e = \log \frac{1}{4}$$
$$2 - 2x = \frac{\log \frac{1}{4}}{\log e}$$
$$2x = 2 - \frac{\log \frac{1}{4}}{\log e}$$
$$x = \frac{1}{2}(2 - \frac{\log \frac{1}{4}}{\log e})$$
$$x = 1.69$$

d $10 - 2e^{7x+5} = 3$

$$2e^{7x+5} = 7$$
$$e^{7x+5} = \frac{7}{2}$$
$$\log e^{7x+5} = \log \frac{7}{2}$$
$$(7x+5)\log e = \log \frac{7}{2}$$
$$7x + 5 = \frac{\log \frac{7}{2}}{\log e}$$
$$7x = \frac{\log \frac{7}{2}}{\log e} - 5$$
$$x = \frac{1}{7}(\frac{\log \frac{7}{2}}{\log e} - 5)$$
$$x = -\ 0.535$$

e $2^{x-1} = 3^{x+1}$

$$\log 2^{x-1} = \log 3^{x+1}$$
$$(x-1)\log 2 = (x+1)\log 3$$
$$x\log 2 - \log 2 = x\log 3 + \log 3$$
$$x(\log 2 - \log 3) = \log 3 + \log 2$$
$$x = \frac{\log 3 + \log 2}{\log 2 - \log 3}$$
$$x = -4.42$$

f $3^{2x-1} = 5^x$

$$\log 3^{2x-1} = \log 5^x$$
$$(2x-1)\log 3 = x\log 5$$
$$2x\log 3 - \log 3 = x\log 5$$
$$x(2\log 3 - \log 5) = \log 3$$
$$x = \frac{\log 3}{2\log 3 - \log 5}$$
$$x = 1.87$$

g $4^{3x+1} = 6^{1-2x}$

$$\log 4^{3x+1} = \log 6^{1-2x}$$
$$(3x+1)\log 4 = (1-2x)\log 6$$
$$3x\log 4 + \log 4 = \log 6 - 2x\log 6$$
$$x(3\log 4 + 2\log 6) = \log 6 - \log 4$$
$$x = \frac{\log 6 - \log 4}{3\log 4 + 2\log 6}$$
$$x = 0.0524$$

h $e^{x+1} = 5^{x-2}$

$$\log e^{x+1} = \log 5^{x-2}$$
$$(x+1)\log e = (x-2)\log 5$$
$$x\log e + \log e = x\log 5 - 2\log 5$$
$$x(\log e - \log 5) = -\log e - 2\log 5$$
$$x = \frac{-\log e - 2\log 5}{\log e - \log 5}$$
$$x = 6.92$$

4 a $3\ln 2 + \ln 3 = \ln 2^3 + \ln 3$

$= \ln 8 + \ln 3 = \ln(8 \cdot 3) = \ln 24$

b $6\ln 2 - \ln 4 = -\ln x$

$$\ln 2^6 - \ln 4 = -\ln x$$
$$\ln\frac{2^6}{4} = -\ln x$$
$$\ln\frac{64}{4} = -\ln x$$
$$\ln 16 = -\ln x$$
$$\ln x = -\ln 16$$
$$x = e^{-\ln 16}$$
$$x = \frac{1}{16}$$

5 a $e^{2x} - 5e^x + 4 = 0$

$$e^{2x} - 4e^x - e^x + 4 = 0$$
$$e^x(e^x - 4) - (e^x - 4) = 0$$
$$(e^x - 4)(e^x - 1) = 0$$

Hence $e^x = 4$ or $e^x = 1$. Thus

$$e^x = 4$$
$$\log e^x = \log 4$$
$$x\log e = \log 4$$
$$x = \frac{\log 4}{\log e} = \ln 4$$

or

$$e^x = 1$$
$$\log e^x = \log 1$$
$$x\log e = 0$$
$$x = 0.$$

Hence $x = 0, \ln 4$.

b $e^{2x} - 2e^x - 3 = 0$

$$e^{2x} - 3e^x + e^x - 3 = 0$$
$$e^x(e^x - 3) + (e^x - 3) = 0$$
$$(e^x - 3)(e^x + 1) = 0$$

Therefore, $e^x - 3 = 0$ or $e^x + 1 = 0$. Hence

$$e^x - 3 = 0$$
$$e^x = 3$$
$$\ln e^x = \ln 3$$
$$x\ln 3 = \ln 3$$
$$x = \ln 3$$

Or $e^x + 1 = 0$ which has no real solutions. Therefore $x = \ln 3$.

c $e^{4x} + 4e^{2x} - 12 = 0$

$$e^{4x} + 6e^{2x} - 2e^{2x} - 12 = 0$$
$$e^{2x}(e^{2x} + 6) - 2(e^{2x} + 6) = 0$$
$$(e^{2x} + 6)(e^{2x} - 2) = 0$$

Hence either $e^{2x} + 6 = 0$ or $e^{2x} - 2 = 0$.

$e^{2x} + 6 = 0$ has no real solutions thus it remains $e^{2x} - 2 = 0$ which gives

$$e^{2x} - 2 = 0$$
$$e^{2x} = 2$$
$$\ln e^{2x} = \ln 2$$
$$2x\ln e = \ln 2$$
$$2x = \ln 2$$
$$x = \frac{\ln 2}{2}.$$

6 **a** $t = 0 \Rightarrow h(0) = 10(1.075)^{k\cdot 0}$

$= 10(1.075)^0 = 10$ cm

b $h(4) = 12$

$h(4) = 10(1.075)^{k\cdot 4} = 10(1.075)^{4k}$

$12 = 10(1.075)^{4k}$

$(1.075)^{4k} = \frac{12}{10}$

$\log(1.075)^{4k} = \log\frac{12}{10}$

$4k\log(1.075) = \log\frac{12}{10}$

$k = \frac{1}{4}\frac{\log\frac{12}{10}}{\log(1.075)}$

$k = 0.630$

c $h(t) = 2\cdot h(0) = 20$

$20 = 10(1.075)^{t\cdot k} = 10(1.075)^{t\cdot 0.630}$

$(1.075)^{t\cdot 0.630} = \frac{20}{10} = 2$

$\log(1.075)^{t\cdot 0.630} = \log 2$

$t\cdot 0.630\log(1.075) = \log 2$

$t = \frac{\log 2}{0.630\log(1.075)} = 15.2$

7 **a** $t = 0$

$\Rightarrow P(0) = 20000(0.9)^{k\cdot 0} + 1000$

$= 20000(0.9)^0 + 1000$

$= 20000 + 1000 = 21000$

b $P(3) = 16000$

$P(3) = 20000(0.9)^{3k} + 1000 = 16000$

$20000(0.9)^{3k} = 15000$

$(0.9)^{3k} = \frac{15000}{20000} = 0.75$

$\log(0.9)^{3k} = \log 0.75$

$3k\log(0.9) = \log 0.75$

$k = \frac{\log 0.75}{3\log 0.9} = 0.910$

c $P(t) = 5000$

$20000(0.9)^{t\cdot 0.910} + 1000 = 5000$

$20000(0.9)^{t\cdot 0.910} = 4000$

$(0.9)^{t\cdot 0.910} = \frac{4000}{20000} = 0.2$

$\log(0.9)^{t\cdot 0.910} = \log 0.2$

$0.910t\log 0.9 = \log 0.2$

$t = \frac{\log 0.2}{0.910\log 0.9} = 16.8$

8 **a** $t = 0$

$\Rightarrow W(0) = 84 - 10\ln(0+1)$

$= 84 - 10\ln 1$

$= 84 - 10\cdot 0 = 84$

b $t = 10$

$\Rightarrow W(10) = 84 - 10\ln(10+1)$

$= 84 - 10\ln 11 = 60$

c $W(t) = \frac{100}{2} = 50$

$84 - 10\ln(t+1) = 50$

$10\ln(t+1) = 84 - 50 = 34$

$\ln(t+1) = \frac{34}{10}$

$t + 1 = e^{3.4}$

$t = e^{3.4} - 1$

$t = 29$

9 **a** $t = 100$

$\Rightarrow A(100) = A_0(0.5)^{\frac{100}{25000}}$

$= 500(0.5)^{\frac{1}{250}} = 499$

b $A(t) = 500(0.5)^{\frac{t}{25000}} = 100$

$0.5^{\frac{t}{25000}} = \frac{100}{500} = \frac{1}{5}$

$\log 0.5^{\frac{t}{25000}} = \log 0.2$

$\frac{t}{25000}\log 0.5 = \log 0.2$

$t = \frac{25000\log 0.2}{\log 0.5}$

$t = 58048$

10 $2^x = 3e^{4x}$

$\ln 2^x = \ln 3e^{4x}$

$x\ln 2 = \ln 3 + \ln e^{4x}$

$x\ln 2 = \ln 3 + 4x\ln e$

$x\ln 2 = \ln 3 + 4x$

$x(\ln 2 - 4) = \ln 3$

$x = \frac{\ln 3}{\ln 2 - 4}$

Exercise 9M

1 **a** $(7e^x)' = 7e^x$

b $(-\frac{1}{4}e^x)' = -\frac{1}{4}e^x$

c $(9\ln x)' = 9\frac{1}{x} = \frac{9}{x}$

d $(\pi\ln x)' = \frac{\pi}{x}$

e $(\ln 5x)' = \frac{1}{5x}(5x)' = \frac{5}{5x} = \frac{1}{x}$

f $(\ln 6x)' = \frac{(6x)'}{6x} = \frac{6}{6x} = \frac{1}{x}$

g $(\ln 7x)' = \frac{(7x)'}{7x} = \frac{7}{7x} = \frac{1}{x}$

h $(e^{2x})' = e^{2x}(2x)' = 2e^{2x}$

i $(e^{4x})' = e^{4x}(4x)' = 4e^{4x}$

j $(e^{5x})' = e^{5x}(5x)' = 5e^{5x}$

2 a $(5\ln x - 2e^{x})' = \frac{5}{x} - 2e^{x}$

b $(x^2 - e^{-\frac{1}{2}x} + \ln x)' = 2x - e^{-\frac{1}{2}x}(-\frac{1}{2}) + \frac{1}{x}$

$= 2x + \frac{1}{2}e^{-\frac{1}{2}x} + \frac{1}{x}$

c $(4 - \ln 9x + e^{-5x} + x^3)'$

$= -\frac{(9x)'}{9x} + e^{-5x}(-5x)' + 3x^2$

$= -\frac{1}{x} - 5e^{-5x} + 3x^2$

d $(\ln 7x + \ln 7 + e^{7x} - 7x)'$

$= \frac{(7x)'}{7x} + e^{7x}(7x)' - 7$

$= \frac{1}{x} + 7e^{7x} - 7$

e $(e^{10} - 5\ln x + 6e^{4x})' = -\frac{5}{x} + 6e^{4x}(4x)'$

$= -\frac{5}{x} + 24e^{4x}$

f $(\ln(e^{x}x^3))' = \frac{(e^{x}x^3)'}{e^{x}x^3} = \frac{e^{x}x^3 + 3x^2e^{x}}{e^{x}x^3}$

$= \frac{x+3}{x} = 1 + \frac{3}{x}$

g $(\ln(\frac{x^2+1}{x^3-x}))' = \frac{x^3-x}{x^2+1}(\frac{x^2+1}{x^3-x})'$

$= \frac{x^3-x}{x^2+1}\frac{(x^2+1)'(x^3-x)-(x^2+1)(x^3-x)'}{(x^3-x)^2}$

$= \frac{1}{x^2+1}\frac{2x(x^3-x)-(x^2+1)(3x^2-1)}{x^3-x}$

$= \frac{1}{x^2+1}\frac{2x^4-2x^2-3x^4+x^2-3x^2+1}{x^3-x}$

$= \frac{1}{x^2+1}\frac{-x^4-4x^2+1}{x^3-x}$

$= \frac{-x^4-4x^2+1}{(x^2+1)(x^3-x)}$

$= \frac{2x(x^3-x)-(x^2+1)(3x^2-1)}{(x^2+1)(x^3-x)}$

$= \frac{2x}{x^2+1} - \frac{3x^2-1}{x^3-x}$

3 a $(e^{2x^3})' = e^{2x^3}(2x^3)' = e^{2x^3}(2\cdot 3x^2)$

$= 6x^2e^{2x^3}$

b $(e^{(4x^3+5)^2})' = e^{(4x^3+5)^2}((4x^3+5)^2)'$

$= e^{(4x^3+5)^2}(2(4x^3+5)(4\cdot 3x^2))$

$= 24x^2(4x^3+5)e^{(4x^3+5)^2}$

c $(\ln(3x^5))' = \frac{(3x^5)'}{3x^5} = \frac{15x^4}{3x^5} = \frac{5x^4}{x^5} = \frac{5}{x}$

d $((\ln x)^3)' = 3(\ln x)^2(\ln x)' = \frac{3}{x}(\ln x)^2$

e $(xe^{2x})' = x'e^{2x} + x(e^{2x})'$

$= e^{2x} + 2xe^{2x} = (1+2x)e^{2x}$

f $(2x^3e^{-3x})' = 2(x^3)'e^{-3x} + 2x^3(e^{-3x})'$

$= 6x^2e^{-3x} + 2x^3(-3x)'e^{-3x}$

$= 6x^2e^{-3x} - 6x^3e^{-3x}$

$= 6x^2(1-x)e^{-3x}$

g $((x^2+1)e^{3x})'$

$= (x^2+1)'e^{3x} + (x^2+1)(e^{3x})'$

$= 2xe^{3x} + 3(x^2+1)e^{3x}$

h $(xe^{ax^2+1})'$

$= e^{ax^2+1} + xe^{ax^2+1}(ax^2+1)'$

$= e^{ax^2+1} + 2ax^2e^{ax^2+1}$

i $(x\ln x)' = x'\ln x + x(\ln x)'$

$= \ln x + x\frac{1}{x} = \ln x + 1$

j $(x^3\ln x)' = (x^3)'\ln x + x^3(\ln x)'$

$= 3x^2\ln x + \frac{x^3}{x}$

$= 3x^2\ln x + x^2$

$= x^2(3\ln x + 1)$

k $(x^2\ln(2x+3))'$

$= (x^2)'\ln(2x+3) + x^2(\ln(2x+3))'$

$= 2x\ln(2x+3) + x^2\frac{(2x+3)'}{2x+3}$

$= 2x\ln(2x+3) + \frac{2x^2}{2x+3}$

l $(\frac{e^{3x}}{x^2})' = \frac{(e^{3x})'x^2 - e^{3x}(x^2)'}{x^4}$

$= \frac{3e^{3x}x^2 - 2xe^{3x}}{x^4} = \frac{3xe^{3x} - 2e^{3x}}{x^3}$

$= \frac{e^{3x}(3x-2)}{x^3}$

m $\left(\frac{2e^{4x}}{1-e^{x}}\right)' = \frac{(2e^{4x})'(1-e^{x}) - 2e^{4x}(1-e^{x})'}{(1-e^{x})^2}$

$$= \frac{8e^{4x}(1-e^x)+2e^{4x}e^x}{(1-e^x)^2} = \frac{2e^{4x}(4-3e^x)}{(1-e^x)^2}$$

n $(\frac{e^x+1}{e^x-1})'$

$$= \frac{(e^x+1)'(e^x-1)-(e^x+1)(e^x-1)'}{(e^x-1)^2}$$

$$= \frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}$$

$$= \frac{e^{2x}-e^x-e^{2x}-e^x}{(e^x-1)^2}$$

$$= \frac{-2e^x}{(e^x-1)^2}$$

o $(\frac{x}{\ln x})' = \frac{x'\ln x - x(\ln x)'}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

p $(\frac{2-\ln x}{x})' = \frac{(2-\ln x)'x-(2-\ln x)x'}{x^2}$

$$= \frac{-\frac{1}{x}x-2+\ln x}{x^2} = \frac{-3+\ln x}{x^2}$$

q $(\frac{1+\ln x}{x^2})' = \frac{(1+\ln x)'x^2-(1+\ln x)(x^2)'}{x^4}$

$$= \frac{\frac{1}{x}x^2-2x(1+\ln x)}{x^4}$$

$$= \frac{x-2x(1+\ln x)}{x^4}$$

$$= \frac{1-2(1+\ln x)}{x^3}$$

4 A turning point has the first derivative equal to 0.

$y = \ln x - x$

$y' = \frac{1}{x} - 1$

$y'(x) = 0 \Leftrightarrow \frac{1}{x} - 1 = 0 \Leftrightarrow x = 1$

$y(1) = -1$

Therefore $(1,-1)$ is a turning point. To check if it is a maximum or minimum we need the second derivative.

$y'' = (\frac{1}{x}-1)' = -\frac{1}{x^2} < 0$. Hence $(1,-1)$ is a maximum.

5 $f(x) = 2e^{2x}$

$f'(x) = (2e^{2x})' = 2e^{2x}(2x)' = 4e^{2x}$

$f(0) = 2e^{2\cdot 0} = 2$

$f'(0) = 4e^{2\cdot 0} = 4$

Equation of the tangent at the point $(0,2)$ is:

$y - 2 = 4(x-0)$

$y - 2 = 4x$

$y = 4x + 2$

6 $y = \ln x$

$y(1) = 0$

$y' = \frac{1}{x}$

$y'(1) = 1$

The gradient at the point $(1,0)$ is 1. Hence the equation of the tangent at the point $(1,0)$ is:

$y - 0 = 1(x-1) = x - 1$

$y = x - 1$

The product of the tangent and the normal is -1. Thus the gradient of the normal at $(1,0)$ is -1 Therefore the equation of the normal at the point $(1,0)$ is:

$y - 0 = -1(x-1)$

$y = -x + 1$

7 The point where $x = 6$ is $f(6) = e^{-6} + 4$.

$f'(x) = (e^{-x}+4)' = -e^{-x}$

$f'(6) = -e^{-6}$

Hence we get that the value of the gradient of the tangent of $f(x)$ at $(6, e^{-6}+4)$ is $-e^{-6}$.

8 $f(x) = x^2 + \ln x$

$f'(x) = 2x + \frac{1}{x}$

$f'(x) = 3 \Leftrightarrow 2x + \frac{1}{x} = 3 \Leftrightarrow$

$2x^2 + 1 = 3x$

$2x^2 - 3x + 1 = 0$

$2(x^2 - x) - x + 1 = 0$

$2x(x-1) - (x-1) = 0$

$(x-1)(2x-1) = 0$

$(x-1)(2x-1) = 0$

Hence the values for which the derivative is 3 are $x = 1$ and $x = \frac{1}{2}$.

The y – coordinates are:

$f(1) = 1 + \ln 1 = 1$ and

$f(\frac{1}{2}) = \frac{1}{4} + \ln\frac{1}{2} = \frac{1}{4} - \ln 2$. Thus the points are $(1,1)$ and $(\frac{1}{2}, \frac{1}{4} - \ln 2)$.

9 $f(x) = \ln(e^x + e^{-x})$

$f'(x) = \dfrac{(e^x + e^{-x})'}{(e^x + e^{-x})} = \dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}$

$f'(x) = 0.6$

$\dfrac{(e^x - e^{-x})}{(e^x + e^{-x})} = 0.6$

$e^x - e^{-x} = 0.6(e^x + e^{-x})$

$0.4e^x - 1.6e^{-x} = 0$

$0.4e^{2x} - 1.6 = 0$

$e^{2x} = \dfrac{1.6}{0.4} = 4$

$\ln e^{2x} = \ln 4$

$2x \ln e = \ln 2^2$

$2x = 2\ln 2$

$x = \ln 2$

$f(\ln 2) = \ln(e^{\ln 2} + e^{-\ln 2}) = \ln(2 + \dfrac{1}{2}) = \ln\dfrac{5}{2}$

Thus the point is $(\ln 2, \ln\dfrac{5}{2})$.

10 $f(x) = xe^x - e^x$

$f'(x) = e^x + xe^x - e^x = xe^x$

The gradient of the tangent at $x = 1$ is $f'(1) = e$ while the gradient of the normal is $-\dfrac{1}{e}$. Hence the equation of the tangent is:

$y - 0 = e(x - 1)$

$y = e(x - 1) = ex - e.$

The equation of the normal is:

$y - 0 = -\dfrac{1}{e}(x - 1) = -\dfrac{1}{e}x + \dfrac{1}{e}.$

Chapter review

1 **a** $\log_2 16 = 4 \Rightarrow 2^4 = 16$

b $\log_5 125 = 3 \Rightarrow 5^3 = 125$

c $\log_9 81 = 2 \Rightarrow 9^2 = 81$

d $\log_{12} 144 = 2 \Rightarrow 12^2 = 144$

e $\log 10000 = 4 \Rightarrow 10^4 = 10000$

2 **a** $3^4 = 81 \Rightarrow \log_3 81 = 4$

b $15^2 = 225 \Rightarrow \log_{15} 225 = 2$

c $81^{\frac{1}{2}} = 9 \Rightarrow \log_{81} 9 = \dfrac{1}{2}$

d $a^{14} = c \Rightarrow \log_a c = 14$

e $e^4 = x \Rightarrow \ln x = 4$

3 **a** $2^{2x-2} = 16$

$2^{2x-2} = 2^4$

$2x - 2 = 4$

$2x = 6$

$x = 3$

b $9^{3x-1} = \dfrac{1}{27} = 3^{-3}$

$3^{2(3x-1)} = 3^{-3}$

$2(3x - 1) = -3$

$6x - 2 = -3$

$6x = -1$

$x = -\dfrac{1}{6}$

c $3^{2-x} = 243$

$3^{2-x} = 3^5$

$2 - x = 5$

$x = -3$

d $4^{1-2x} = \dfrac{1}{64}$

$4^{1-2x} = 4^{-3}$

$1 - 2x = -3$

$2x = 4$

$x = 2$

4 **a** $15\log 6x = -15$

$\log 6x = \dfrac{-15}{15} = -1$

$\log 6x = \log\dfrac{1}{10}$

$6x = \dfrac{1}{10}$

$x = \dfrac{1}{60}$

b $\log(-7x) = 3$

$\log(-7x) = \log 10^3$

$-7x = 1000$

$x = \dfrac{-1000}{7}$

c $3\log 10x = -6$

$\log 10x = -2$

$\log 10x = \log 10^{-2}$

$10x = \dfrac{1}{100}$

$x = \dfrac{1}{1000}$

d $-\log 4x = -2$

$\log 4x = 2$

$\log 4x = \log 10^2$

$4x = 100$

$x = 25$

5 **a** $\log_5 25 = \log_5 5^2 = 2$

b $\log_2 128 = \log_2 2^7 = 7$

c $\log_{21} 21 = 1$

d $\log_a a = 1$

e $\log_6 1 = \log_6 6^0 = 0$

f $3^{\log_3(79)} = 79$

g $\ln e^{19} = 19$

h $e^{\ln 7} = 7$

6 **a** $2^x = 17$

$x = \log_2 17 = 4.09$

b $6^{6x+3} = 19$

$6x + 3 = \log_6 19$

$x = \dfrac{\log_6 19 - 3}{6}$

$x = -0.226$

c $2 \times 12^{3x} = 11$

$12^{3x} = \dfrac{11}{2}$

$\log 12^{3x} = \log \dfrac{11}{2}$

$3x \log 12 = \log \dfrac{11}{2}$

$x = \dfrac{\log \frac{11}{2}}{3\log 12}$

$x = 0.229$

d $6 \times 8^{-5x} = 18$

$8^{-5x} = 3$

$\log 8^{-5x} = \log 3$

$-5x \log 8 = \log 3$

$x = \dfrac{-\log 3}{5\log 8}$

$x = -0.106$

e $e^{x+5} = 13$

$\log e^{x+5} = \log 13$

$(x+5)\log e = \log 13$

$x + 5 = \dfrac{\log 13}{\log e}$

$x = \dfrac{\log 13}{\log e} - 5$

$x = -2.44$

f $2e^{6x+8} = 10$

$e^{6x+8} = 5$

$\log e^{6x+8} = \log 5$

$(6x+8)\log e = \log 5$

$6x + 8 = \dfrac{\log 5}{\log e}$

$x = \dfrac{\frac{\log 5}{\log e} - 8}{6}$

$x = -1.07$

g $4e^{6x+9} - 3 = 30$

$4e^{6x+9} = 33$

$e^{6x+9} = \dfrac{33}{4}$

$\log e^{6x+9} = \log \dfrac{33}{4}$

$(6x+9)\log e = \log \dfrac{33}{4}$

$x = \dfrac{\frac{\log \frac{33}{4}}{\log e} - 9}{6}$

$x = -1.15$

h $5 - 4e^{-5x-4} = -16$

$4e^{-5x-4} = 21$

$e^{-5x-4} = \dfrac{21}{4}$

$\log e^{-5x-4} = \log \dfrac{21}{4}$

$(-5x-4)\log e = \log \dfrac{21}{4}$

$-5x - 4 = \dfrac{\log \frac{21}{4}}{\log e}$

$x = -\dfrac{\frac{\log \frac{21}{4}}{\log e} + 4}{5}$

$x = -1.13$

7 **a** $\log 3 + \log 4 = \log 3 \cdot 4 = \log 12$

b $\log 15 - \log 5 = \log \dfrac{15}{5} = \log 3$

c $2\log x - 5\log y = \log x^2 - \log y^5 = \log \dfrac{x^2}{y^5}$

d $8\log_5 x + 2\log_5 y = \log_5 x^8 + \log_5 y^2$

$= \log_5 x^8 y^2$

e $\ln x + \dfrac{1}{2}\ln y + \ln \dfrac{1}{2} z = \ln x + \ln y^{\frac{1}{2}} + \ln \dfrac{1}{2} z$

$= \ln x\sqrt{y}\,\dfrac{1}{2} z$

f $4\ln x - 3\ln y - 2\ln z = \ln x^4 - \ln y^3 - \ln z^2$

$= \ln \dfrac{x^4}{y^3 z^2}$

8 **a** $\ln ab = \ln a + \ln b = p + q$

b $\ln a^3 = 3\ln a = 3p$

c $\ln a^2 b^3 = \ln a^2 + \ln b^3 = 2\ln a + 3\ln b$

$= 2p + 3q$

d $\ln \dfrac{b^5}{a^4} = \ln b^5 - \ln a^4 = 5\ln b - 4\ln a$

$= 5q - 4p$

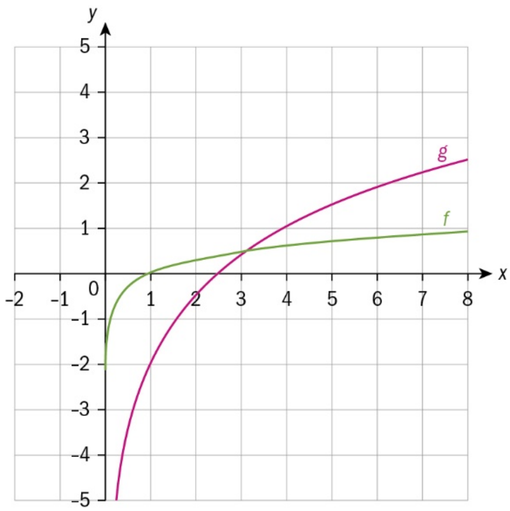
9 **a** $\log_3 17 = \dfrac{\log 17}{\log 3} = 2.58$

b $\log_5 0.5 = \dfrac{\log 0.5}{\log 5} = -0.431$

c $\log_8 200 = \dfrac{\log 200}{\log 8} = 2.55$

10 1.02, 5.65

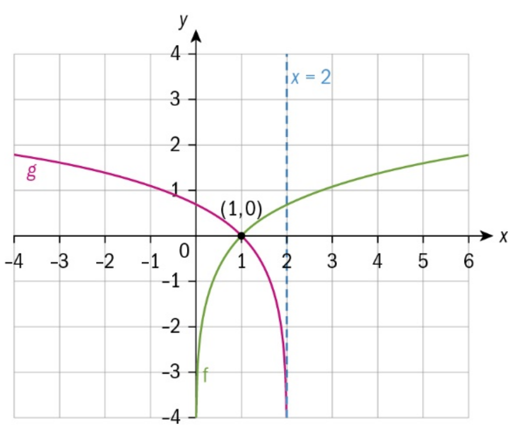
11 **a**



b $x = 0$

c Vertical stretch of scale factor 5 and a vertical translation of 2 units down.

12 **a**



b $x = 2$

c A reflection in the y-axis and a horizontal translation of 2 units to the right.

d $x = 1$

13 $f(x) = 2^x$

$g(x) = -(2^{-x}) - 2$

14 $f(x) = \ln x$

$g(x) = 3\ln(x+5)$

15 **a** $(8e^x + 7\ln x)' = 8e^x + \dfrac{7}{x}$

b $(e^{3x})' = (3x)'e^{3x} = 3e^{3x}$

c $(x\ln x - x)' = \ln x + \dfrac{x}{x} - 1$

$= \ln x + 1 - 1 = \ln x$

d $(e^{6x^2+5x})' = e^{6x^2+5x}(6x^2+5x)'$

$= (12x+5)e^{6x^2+5x}$

e $(\ln(x^2+8))' = \dfrac{(x^2+8)'}{(x^2+8)} = \dfrac{2x}{(x^2+8)}$

f $\left(\dfrac{9e^x+1}{2e^x+1}\right)'$

$= \dfrac{(9e^x+1)'(2e^x+1) - (9e^x+1)(2e^x+1)'}{(2e^x+1)^2}$

$= \dfrac{9e^x 2e^x + 9e^x - 9e^x 2e^x - 2e^x}{(2e^x+1)^2}$

$= \dfrac{7e^x}{(2e^x+1)^2}$

g $(\ln\sqrt{3x-2})' = \dfrac{\sqrt{3x-2}\,'}{\sqrt{3x-2}} = \dfrac{(3x-2)'}{2(\sqrt{3x-2})^2}$

$= \dfrac{3}{2(3x-2)}$

h $(e^x \ln x)' = e^x \ln x + \dfrac{e^x}{x}$

16 $f(x) = 4xe^{x^2-1}$

$f'(x) = 4e^{x^2-1} + 4xe^{x^2-1}(x^2-1)'$

$= 4e^{x^2-1} + 8x^2e^{x^2-1}$

At point $(1,4)$ the tangent has gradient $f'(1) = 4 + 8 = 12$ so the normal has gradient $-\dfrac{1}{12}$.

Therefore the equation of the normal is

$y - 4 = \dfrac{-1}{12}(x-1)$

$12y - 48 = -x + 1$

$x + 12y - 49 = 0$

17 $f(x) = x^2 + \ln x$

$f'(x) = 2x + \frac{1}{x}$

$f'(2) = 4 + \frac{1}{2} = \frac{9}{2} = 4.5$

18 a $t = 0 \Rightarrow P(0) = 30e^{0.032 \cdot 0} = 30e^0 = 30$

Hence the population in 2020 is 30000.

b $t = 5 \Rightarrow P(5) = 30e^{0.032 \cdot 5} = 35.205$

Hence the population in 2025 is 35205.

c $P(t) = 30e^{0.032 \cdot t} = 40$

$e^{0.032 \cdot t} = \frac{40}{30} = \frac{4}{3}$

$0.032 \cdot t = \ln\frac{4}{3}$

$t = \frac{\ln\frac{4}{3}}{0.032} = 9$

The population is expected to reach 40000 in 2029.

19 $V(t) = 150\,000e^{0.05875t} = 200\,000$

$e^{0.05875t} = \frac{200\,000}{150\,000} = \frac{4}{3}$

$0.05875t = \ln\frac{4}{3}$

$t = \frac{\ln\frac{4}{3}}{0.05875} = 4.90$

Hence the predicted year is 2025.

20 $f(t) = 500(0.75)^t$

$f(24) = 500(0.75)^{24}$

$f(24) = 0.502$

21 a i A1

b -2 A1

c 12 A1

d $y < 16$ A1

e $x > -2$ A1

f $y = 16$ A1

22 a $N(0) = 35$ A1

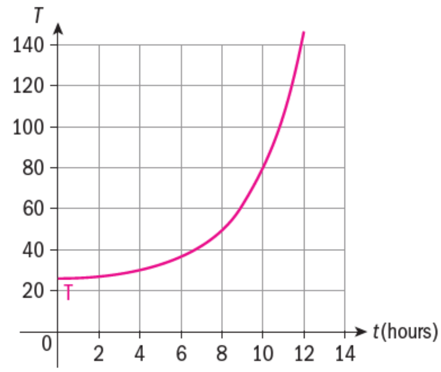
b $N(4) = 410$ M1A1

c $N(t) > 1000$ M1

$t > 5.449...$ A1

d $0 \le t < 5.45$ A1

23 a



A1 shape A1 domain

b $T(6) = 25 + e^{0.4\times6}$

36.0 (1 d.p.) A1

c Solve $25 + e^{0.4t} = 100$ M1

$t = 10.793...$ A1

10 hours 48 minutes A1

24 a $x > 2$ A1

b $x = 2$ A1

c $3\ln(x-2)+1 = 0 \Rightarrow x = 2 + e^{-\frac{1}{3}}$ M1A1

d $f'(x) = \frac{3}{x-2}$ A1

$f'(x) = 1 \Rightarrow \frac{3}{x-2} = 1 \Rightarrow x = 5$ M1

$f(5) = 3\ln 3 + 1$ M1

$y - (3\ln 3 + 1) = x - 5$ A1

(or $y = x + 3\ln 3 - 4$)

25 a $\log_2 3x - \log_2(x-3)$ A1

$\log_2 \frac{3x}{x-3}$ A1

b $\ln x^3 - \ln(x-1)^2 + \ln e$ A1A1A1

$\ln\frac{ex^3}{(x-1)^2}$ A1

26 a i $T_5 = 73.205$ thousand taxis M1A1

ii $T_n = 100 \Rightarrow n = 9$ M1

2019 A1

b $P_{10} = 2.1873705...$ M1A1

2.187 million people A1

c Adjusting units in (i) or (ii) A1

i $\frac{P_5 \times 10^6}{T_5 \times 10^3} = 28.4$ people per taxi M1A1

ii $\dfrac{P_{10}\times 10^6}{T_{10}\times 10^3} = 18.6$ people per taxi A1

d The model predicts a reduction in the number of people per taxi, which may mean that the taxis are in use for fewer hours or fewer taxis are used every day. R1

27 a For example, for $x = e$ M1

$\ln\left(x^2\right) = \ln\left(e^2\right) = 2 \neq (\ln\ e)^2 = 1$ A1R1

b $(\ln x)^2 - \ln\left(x^2\right) - 15 = 0$

$\Rightarrow (\ln x)^2 - 2\ln(x) - 15 = 0$ M1

$(\ln x - 5)(\ln x + 3) = 0$ M1

$\ln x - 5 = 0, \ln x + 3 = 0$

$\ln x = 5, \ln x = -3$ A1

$x = e^5, x = e^{-3}$ A1A1

28 a i $T(0) = 94 \Rightarrow 25 + a = 94$ M1

$a = 69$ A1

ii $T(20) = 29$

$\Rightarrow 25 + 69e^{20b} = 29$ M1

$b = -0.142$ (3 s.f.) A1

b $T(30) = 26.0$ (3 s.f.) M1A1

c $y = 25$ A1

d The temperature of the room. R1

e $\dfrac{dT}{dt} = -9.82\ldots e^{-0.142\ldots t}$ M1A1

f $\dfrac{d^2T}{dt^2} = 1.399\ldots e^{-0.142\ldots t}$ M1A1

g The rate of change is always negative which means the temperature is decreasing; as the second derivative is always positive, the temperature will not have a minimum but will approach the value 25 given by the horizontal asymptote. A2

29 a i $f(0) = \left(\dfrac{3}{2}\right)^{-1} + 2 = 2\dfrac{2}{3}$

$= 2.67$ (3 s.f) M1A1

ii $(2.67, 0)$ A1

b $x = \left(\dfrac{3}{2}\right)^{y-1} + 2$ M1

$\left(\dfrac{3}{2}\right)^{y-1} = x - 2$ M1

$y = \log_{\frac{3}{2}}(x-2) + 1$ M1A1

$g(x) = \log_{\frac{3}{2}}(x-2) + 1$

c i $y = 2$ A1

ii $x = 2$ A1

d $x > 2$ A1

e Use GDC solver or intersection of graphs M1

$x = 2.16$ A1

30 a $f(4) = \log_2 4 + \log_2(15) - \log_2(5)$ M1

$= \log_2 \dfrac{4\times 15}{5} = \log_2 12$ A1AG

b $f(x) = \log_2 x + \log_2 (x-1)(x+1)$

$-\log_2(x+1)$ M1

$f(x) = \log_2 \dfrac{x(x-1)(x+1)}{x+1}$ M1A1

$f(x) = \log_2 x(x-1)$ M1

$f(x) = \log_2\left(x^2 - x\right)$ AG

10 From approximation to generalization: integration

Skills Check

1 **a** 30 × 60 = 1800 cm²
b 0.5 × 4 × 9 = 18 m²
c 0.5 × 4² × π = 8π mm²

2 **a** $3x^2 + 15x$ **b** $x^2 - 25$
c $9x^2 + 6x + 1$ **d** $2x^2 - 9x - 5$

3 **a** $x^{\frac{1}{3}}$ **b** $x^{\frac{4}{7}}$
c $(2x+5)^{\frac{1}{2}}$ **d** $(x-3)^{\frac{2}{3}}$

Note: throughout this chapter, C denotes an arbitrary constant.

Exercise 10A

1 $F(x) = \frac{1}{10+1}x^{10+1} + C = \frac{1}{11}x^{11} + C$

2 $F(x) = \frac{1}{5+1}x^{5+1} + C = \frac{1}{6}x^{6} + C$

3 $F(x) = \frac{1}{25+1}x^{25+1} + C = \frac{1}{26}x^{26} + C$

4 $F(x) = \frac{1}{-6+1}x^{-6+1} + C = -\frac{1}{5}x^{-5} + C$

5 $f(x) = \frac{1}{x^8} = x^{-8}$

$F(x) = \frac{1}{-8+1}x^{-8+1} + C$

$= -\frac{1}{7}x^{-7} + C = -\frac{1}{7x^7} + C$

6 $f(x) = \frac{1}{x^2} = x^{-2}$

$F(x) = \frac{1}{-2+1}x^{-2+1} + C = -\frac{1}{x} + C$

7 $F(x) = \frac{1}{\frac{2}{3}+1}x^{\frac{2}{3}+1} + C = \frac{3}{5}x^{\frac{5}{3}} + C$

8 $F(x) = \frac{1}{\frac{1}{10}+1}x^{\frac{1}{10}+1} + C = \frac{10}{11}x^{\frac{11}{10}} + C$

9 $F(x) = \frac{1}{-\frac{1}{4}+1}x^{-\frac{1}{4}+1} + C = \frac{4}{3}x^{\frac{3}{4}} + C$

10 $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$F(x) = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C = \frac{2}{3}x^{\frac{3}{2}} + C$

11 $f(x) = \sqrt[4]{x^3} = x^{\frac{3}{4}}$

$F(x) = \frac{1}{\frac{3}{4}+1}x^{\frac{3}{4}+1} + C = \frac{4}{7}x^{\frac{7}{4}} + C$

12 $f(x) = \frac{1}{\sqrt[7]{x}} = x^{-\frac{1}{7}}$

$F(x) = \frac{1}{-\frac{1}{7}+1}x^{-\frac{1}{7}+1} + C = \frac{7}{6}x^{\frac{6}{7}} + C$

Exercise 10B

1 $\int x^4 dx = \frac{1}{4+1}x^{4+1} + C = \frac{1}{5}x^5 + C$

2 $\int (6x^2 + 4x + 5)\,dx$

$= 6\int x^2\,dx + 4\int x\,dx + 5\int dx$

$= 2x^3 + 2x^2 + 5x + C$

3 $\int (15t^4 + 12t^3 + 2t + 5)\,dt$

$= 15\int t^4\,dt + 12\int t^3\,dt + 2\int t\,dt + 5\int dt$

$= 3t^5 + 3t^4 + t^2 + 5t + C$

4 $\int 8\,dx = 8x + C$

5 $\int \frac{1}{u^7}du = \int u^{-7}\,du = -\frac{1}{6}u^{-6} + C = -\frac{1}{6u^6} + C$

6 $\int \frac{2}{x^5}dx = 2\int x^{-5}\,dx = -\frac{1}{2}x^{-4} + C = -\frac{1}{2x^4} + C$

7 $\int (w^3 + \sqrt[3]{w})\,dw$

$\int w^3\,dw + \int w^{\frac{1}{3}}\,dw = \frac{1}{4}w^4 + \frac{3}{4}w^{\frac{4}{3}} + C$

8 $\int (4\sqrt{x} + 3)\,dx = 4\int x^{\frac{1}{2}}\,dx + 3\int dx$

$= \frac{8}{3}x^{\frac{3}{2}} + 3x + C$

9 $\int \sqrt[9]{x^5}\,dx = \int x^{\frac{5}{9}}\,dx = \frac{9}{14}x^{\frac{14}{9}} + C$

10 $\int du = u + C$

11 **a** $f(x) = x^5 + \frac{3}{x^2} = x^5 + 3x^{-2}$

$f'(x) = 5x^4 - 6x^{-3} = 5x^4 - \frac{6}{x^3}$

b $\int \left(x^5 + \frac{3}{x^2}\right)dx = \int x^5\,dx + 3\int x^{-2}\,dx$

$= \frac{1}{6}x^6 - 3x^{-1} + C = \frac{x^6}{6} - \frac{3}{x} + C$

12 $\int(3x^2+px+q)\,dx = x^3+\frac{p}{2}x^2+qx+C$

$= x^3+8x^2+7x+C$

So comparing the coefficients,

$\frac{p}{2}=8 \Rightarrow p=16$

$q=7$

Exercise 10C

1 $\int\frac{6}{x}\,dx = 6\int\frac{1}{x}\,dx = 6\ln|x|+c$

2 $\int 5e^u\,du = 5\int e^u\,du = 5e^u+C$

3 $\int\frac{1}{2x}\,dx = \frac{1}{2}\int\frac{1}{x}\,dx = \frac{1}{2}\ln|x|+C$

4 $\int\frac{e^x}{3}\,dx = \frac{1}{3}\int e^x\,dx = \frac{1}{3}e^x+C$

5 $\int(3x+2)^2\,dx = \int(9x^2+12x+4)\,dx$

$= 9\int x^2\,dx + 12\int x\,dx + 4\int dx$

$= 3x^3+6x^2+4x+C$

6 $\int\ln(e^{x+1})\,dx = \int(x+1)\,dx = \frac{1}{2}x^2+x+C$

7 $\int t^2(t+3)\,dt = \int(t^3+3t^2)\,dt$

$= \int t^3\,dt + 3\int t^2\,dt = \frac{1}{4}t^4+t^3+C$

8 $\int e^{\ln(3x)}\,dx = 3\int x\,dx = \frac{3}{2}x^2+C$

9 $\int\frac{x^4+3x^2+2x}{x}\,dx = \int(x^3+3x+2)\,dx$

$= \int x^3\,dx + 3\int x\,dx + 2\int dx$

$= \frac{1}{4}x^4+\frac{3}{2}x^2+2x+C$

10 $\int\frac{e^u-4}{2}\,du = \frac{1}{2}\int e^u\,du - 2\int du$

$= \frac{1}{2}e^u-2u+C$

Exercise 10D

1 $\int(7x-5)^4\,dx = \frac{1}{7\cdot5}(7x-5)^5+C$

$= \frac{1}{35}(7x-5)^5+C$

2 $\int(-3x+7)^6\,dx = -\frac{1}{3\cdot7}(-3x+7)^7+C$

$= -\frac{1}{21}(-3x+7)^7+C$

3 $\int\frac{1}{10x+13}\,dx = \frac{1}{10}\ln|10x+13|+C$

4 $\int e^{-4x+3}\,dx = -\frac{1}{4}e^{-4x+3}+C$

5 $4\int(5x+1)^3\,dx = \frac{4}{5\cdot4}(5x+1)^4+C$

$= \frac{1}{5}(5x+1)^4+C$

6 $\int\frac{2}{3x+8}\,dx = \frac{2}{3}\ln|3x+8|+C$

7 $\int 3e^{4-2x}\,dx = -\frac{3}{2}e^{4-2x}+C$

8 $\int 7(2x-9)^4\,dx = \frac{7}{2\cdot5}(2x-9)^5+C$

$= \frac{7}{10}(2x-9)^5+C$

9 $\int\frac{1}{(4x+3)^2}\,dx = \int(4x+3)^{-2}\,dx$

$= -\frac{1}{4}(4x+3)^{-1}+C$

10 $\int(2x+1)^{\frac{1}{3}}\,dx = \frac{1}{\frac{4}{3}\cdot2}(2x+1)^{\frac{4}{3}}+C$

$= \frac{3}{8}(2x+1)^{\frac{4}{3}}+C$

11 $\int\left(e^{5x}+\frac{8}{5x-3}\right)dx = \int e^{5x}\,dx + \int\frac{8}{5x-3}\,dx$

$= \frac{1}{5}e^{5x}+\frac{8}{5}\ln|5x-3|+C$

12 $\int\frac{1}{\sqrt[3]{4x+7}}\,dx = \int(4x+7)^{-\frac{1}{3}}\,dx$

$= \frac{1}{4\cdot\frac{2}{3}}(4x+7)^{\frac{2}{3}}+C = \frac{3}{8}(4x+7)^{\frac{2}{3}}+C$

13 a Using the chain rule,

$f'(x) = 5\cdot3(3x+10)^4 = 15(3x+10)^4$

b $\int(3x+10)^5\,dx = \frac{1}{3\cdot6}(3x+10)^6+C$

$= \frac{1}{18}(3x+10)^6+C$

14 a $f(x) = (12x+7)^{-1}$

Using the chain rule,

$f'(x) = -12(12x+7)^{-2}$

b $\int\frac{1}{12x+7}\,dx = \frac{1}{12}\ln|12x+7|+C$

Exercise 10E

1 $h(t) = \int (6t^2 + 1)dt = 2t^3 + t + C$

$h(2) = 18 + C = 8 \Rightarrow C = -10$

$\therefore h(t) = 2t^3 + t - 10$

2 $y(x) = \int 8(2x-3)^3\, dx = (2x-3)^4 + C$

$y(2) = 6 = C + 1 \Rightarrow C = 5$

$\therefore y(x) = (2x-3)^4 + 5$

3 **a** $v(t) = \int a(t)dt = \int (4t+1)dt$

$= 2t^2 + t + C$

$v(0) = 2 = C \Rightarrow C = 2$

$\therefore v(t) = 2t^2 + t + 2$

b $s(t) = \int v(t)dt = \int (2t^2 + t + 2)dt$

$= \frac{2}{3}t^3 + \frac{1}{2}t^2 + 2t + C_1$

$s(0) = 8 = C_1 \Rightarrow C_1 = 8$

$\therefore s(t) = \frac{2}{3}t^3 + \frac{1}{2}t^2 + 2t + 8$

4 **a** $a(t) = \frac{dv}{dt} = 8e^{2t} + 1$

$\therefore a(3) = 8e^6 + 1$

b $s(t) = \int (4e^{2t} + t)dt = 2e^{2t} + \frac{t^2}{2} + C$

$s(0) = 4 = 2 + C \Rightarrow C = 2$

$\therefore s(t) = 2e^{2t} + \frac{t^2}{2} + 2$

5 $f(x) = \int f(x)dx = \int \frac{1}{8x-7}\, dx$

$= \frac{1}{8}\ln(8x-7) + C$

$f(1) = \frac{7e}{8} = \ln 1 + C = C \Rightarrow C = \frac{7e}{8}$

$\therefore f(x) = \ln|8x-7| + \frac{7e}{8}$

Exercise 10F

1 **a** $A = \int_0^6 \frac{2x}{3}\, dx = 12$

b $A = \frac{1}{2}bh = \frac{1}{2}(6)(4) = 12$

2 **a** $A = \int_{-1}^{2} 5\, dx = 15$

b $A = bh = 3(5) = 15$

3 **a** $A = \int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{9\pi}{2}$

b $A = \frac{\pi r^2}{2} = \frac{\pi(3)^2}{2} = \frac{9\pi}{2}$

4 **a** $A = \int_0^6 \left(\frac{x}{3} + 3\right) dx = 24$ □

b $A = \left(\frac{a+b}{2}\right)h = \left(\frac{3+5}{2}\right)(6) = 24$

5 **a** $A = \int_0^5 \sqrt{25 - x^2}\, dx = \frac{25\pi}{4}$

b $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(5)^2 = \frac{25\pi}{4}$

6 **a** $A = \int_{-4}^{4} |x|\, dx = 16$

b $A = 2 \cdot \frac{bh}{2} = bh = 4(4) = 16$

Exercise 10G

1 Geometric: $\left(\frac{a+b}{2}\right)h = \left(\frac{3+2}{2}\right)(3) = \frac{15}{2}$

Integration:

$\int_0^2 3dx + \int_2^3 (9 - 3x)dx = 6 + \left[9x - \frac{3x^2}{2}\right]_2^3$

$= 6 + \left(\frac{27}{2} - 12\right) = \frac{15}{2}$

2 Geometric: $-\frac{1}{2}bh = -\frac{1}{2}(2)(3) = -3$

Integration:

$\int_3^4 (9 - 3x)\, dx + \int_4^5 (3x - 15)dx$

$= \left[9x - \frac{3x^2}{2}\right]_3^4 + \left[\frac{3x^2}{2} - 15x\right]_4^5$

$= \left(12 - \frac{27}{2}\right) + \left(-\frac{75}{2} - (-36)\right) = -3$

3 Geometric: $\frac{15}{2} - 3 + \frac{1}{2}(2)(6) = \frac{21}{2}$

Integration:

$\int_0^7 f(x)\, dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx + \int_5^7 f(x)dx$

$= \frac{15}{2} - 3 + \int_5^7 (3x - 15)dx = \frac{9}{2} + \left[\frac{3x^2}{2} - 15x\right]_5^7$

$= \frac{21}{2}$

4 $\int_1^6 \left(\frac{1}{2}f(x) - g(x)\right) dx$

$= \frac{1}{2}\int_1^6 f(x)\, dx - \int_1^6 g(x)\, dx$

$= \frac{1}{2}(-4) - 6 = -8$

5 $\int_{10}^1 f(x)\, dx = -\int_1^{10} f(x)\, dx = -12$

6 $\int_1^{10} g(x) dx = \int_1^6 g(x) dx + \int_6^{10} g(x) dx$

$= 6 + 14 = 20$

7 $\int_6^6 g(x) dx = 0$

8 $\int_6^{10} f(x) dx = \int_1^{10} f(x) dx - \int_1^6 f(x) dx$

$= 12 - (-4) = 16$

9 $\int_{-2}^7 f(x+3) dx = \int_1^{10} f(x) dx = 12$

10 $\int_3^{12} \left(\frac{1}{2}g(x-2)\right) dx = \frac{1}{2}\int_1^{10} g(x) dx$

$= \frac{1}{2}(20) = 10$

11 $\int_1^{10} (f(x)+4) dx = \int_1^{10} f(x) dx + [4x]_1^{10}$

$= 12 + (40 - 4) = 48$

12 a $\int_{-3}^3 \frac{1}{2}f(x) dx = \frac{1}{2}\int_{-3}^3 f(x) dx = \frac{1}{2}(20) = 10$

b $\int_1^3 f(x) dx = \int_{-3}^3 f(x) dx - \int_{-3}^1 f(x) dx$

$= 20 - 6 = 14$

c $\int_a^b f(x-4) dx = \int_{a-4}^{b-4} f(x) dx = 20$

So a possible pair of values for a and b is

$b - 4 = 3 \Rightarrow b = 7$

$a - 4 = -3 \Rightarrow a = 1$

d $\int_{-3}^3 (f(x)+k) dx = 20 + 6k = 32 \Rightarrow k = 2$

Exercise 10H

1 $\int_{-2}^3 6x\, dx = \left[3x^2\right]_{-2}^3 = 27 - 12 = 15$

2 $\int_1^4 \frac{e^x}{2} dx = \left[\frac{e^x}{2}\right]_1^4 = \frac{e^4 - e}{2}$

3 $\int_3^4 \frac{5}{u} du = [5\ln u]_3^4 = 5\ln 4 - 5\ln 3 = 5\ln\frac{4}{3}$

4 $\int_{-1}^2 (3x^2 + 4x - 2) dx = \left[x^3 + 2x^2 - 2x\right]_{-1}^2$

$= 12 - 3 = 9$

5 $\int_1^2 \left(\frac{4}{x^2} + 1\right) dx = \left[-\frac{4}{x} + x\right]_1^2 = 0 - (-3) = 3$

6 $\int_1^3 \frac{1}{x^3} dx = \left[-\frac{1}{2x^2}\right]_1^3 = -\frac{1}{18} - \left(-\frac{1}{2}\right) = \frac{4}{9}$

7 $\int_0^{16} \left(\frac{1}{t^{\frac{1}{4}}} - t^{\frac{1}{2}}\right) dt = \int_0^{16} \left(t^{-\frac{1}{4}} - t^{\frac{1}{2}}\right) dt = \left[\frac{4}{3}t^{\frac{3}{4}} - \frac{2}{3}t^{\frac{3}{2}}\right]_0^{16}$

$= \frac{32}{3} - \frac{128}{3} = -32$

8 $\int_{e^2}^{e^5} \frac{1}{x}\, dx = [\ln x]_{e^2}^{e^5} = 5 - 2 = 3$

9 a $A = \int_0^2 (-x^2 + 2x) dx = \left[-\frac{x^3}{3} + x^2\right]_0^2$

$= -\frac{8}{3} + 4 = \frac{4}{3}$

b $A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$

10 a $\int_0^4 2f(x) dx = 2\int_0^4 f(x) dx = 2(10) = 20$

b $\int_0^4 (2f(x) + x) dx = \int_0^4 2f(x) dx + \int_0^4 x\, dx$

$= 20 + \left[\frac{x^2}{2}\right]_0^4 = 28$

11 a $\int \frac{2}{x} dx = 2\ln|x| + C$

b $\int_2^k \frac{2}{x}\, dx = [2\ln x]_2^k = 2\ln\frac{k}{2} = \ln\frac{k^2}{4} = \ln 9$

$\therefore \frac{k^2}{4} = 9 \Rightarrow k = 6 \quad (k > 0)$

Exercise 10I

1 $\int_1^4 \frac{1}{4x-2} dx = \left[\frac{1}{4}\ln(4x-2)\right]_1^4 = \frac{1}{4}\ln 7$

2 $\int_0^1 (2t-1)^2\, dt = \left[\frac{1}{6}(2t-1)^3\right]_0^1 = \frac{1}{6}(1-(-1)) = \frac{1}{3}$

3 $\int_{-1}^2 e^{3x+4} dx = \left[\frac{1}{3}e^{3x+4}\right]_{-1}^2 = \frac{1}{3}\left(e^{10} - e\right)$

4 $\int_0^2 (x+2)(x-1)\,dx = \int_0^2 (x^2+x-2)\,dx$

$= \left[\frac{x^3}{3}+\frac{x^2}{2}-2x\right]_0^2 = \frac{8}{3}+2-4 = \frac{2}{3}$

5 $\int_1^2 (3x-3)^3\,dx = 27\int_1^2 (x-1)^3\,dx$

$= 27\int_1^2 (x^3-3x^2+3x-1)\,dx$

$= 27\left[\frac{x^4}{4}-x^3+\frac{3x^2}{2}-x\right]_1^2$

$= 27\left(0-\left(-\frac{1}{4}\right)\right) = \frac{27}{4}$

6 $\int_0^4 \sqrt{4x+9}\,dx = \int_0^4 (4x+9)^{\frac{1}{2}}\,dx$

$= \left[\frac{1}{\frac{3}{2}\cdot 4}(4x+9)^{\frac{3}{2}}\right]_0^4$

$= \frac{1}{6}\left(25^{\frac{3}{2}}-9^{\frac{3}{2}}\right) = \frac{1}{6}(125-27) = \frac{49}{3}$

7 $\int_{-2}^2 (e^t - e^{-t})\,dt = \left[e^t + e^{-t}\right]_{-2}^2$

$= (e^2+e^{-2})-(e^{-2}+e^2) = 0$

8 $\int_4^9 \frac{3\sqrt{t}+2}{\sqrt{t}}\,dt = \int_4^9 \left(3+2t^{-\frac{1}{2}}\right)dt = \left[3t+4t^{\frac{1}{2}}\right]_4^9$

$= (27+12)-(12+8) = 19$

9 a $f(x) = -2x(x^2-4) = 0$

$\Rightarrow x = 0$ or $x = \pm 2$

So $(0,0)$, $(-2,0)$, $(2,0)$

b $\int_0^2 2x(4-x^2)\,dx$

c $\int_0^2 2x(4-x^2)\,dx = \int_0^2 (8x-2x^3)\,dx$

$= \left[4x^2-\frac{x^4}{2}\right]_0^2 = 8$

10 a $\int_1^k \frac{1}{2x+1}\,dx$

b $\int_1^k \frac{1}{2x+1}\,dx = \frac{1}{2}\left[\ln(2x+1)\right]_1^k$

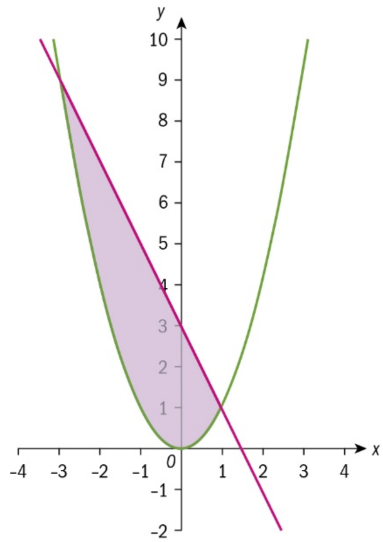
$= \frac{1}{2}\ln\left(\frac{2k+1}{3}\right) = \ln 3$

$\Rightarrow \ln\left(\frac{2k+1}{3}\right) = 2\ln 3 = \ln 9$

$\Rightarrow \frac{2k+1}{3} = 9 \Rightarrow k = 13$

Exercise 10J

1



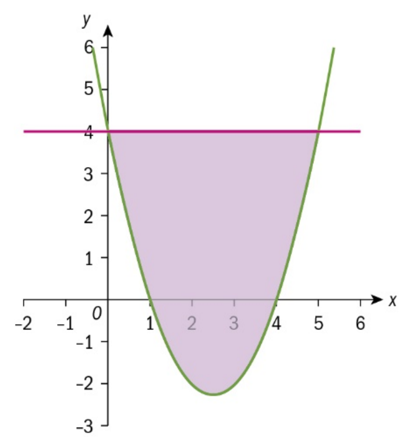
$f(x) = g(x) \Rightarrow x^2+2x-3$

$= (x+3)(x-1) = 0 \Rightarrow x = -3,\quad x = 1$

$\therefore A = \int_{-3}^1 (-2x+3-x^2)\,dx$

$= \left[-x^2+3x-\frac{x^3}{3}\right]_{-3}^1 = \frac{5}{3}-(-9) = \frac{32}{3}$

2



$x^2-5x+4 = 4 \Rightarrow x(x-5) = 0$

$x = 0$ or $x = 5$

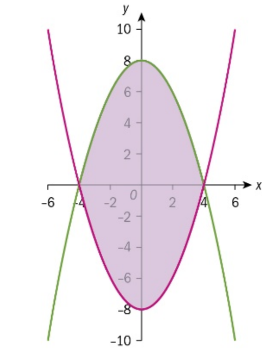
$A = \int_0^5 (4-4+5x-x^2)\,dx$

$= \int_0^5 (5x-x^2)\,dx = \left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^5$

$= \frac{125}{6}$

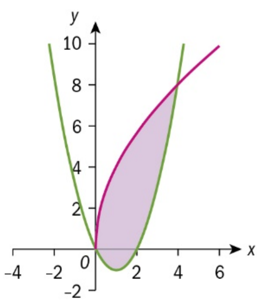
3 $-0.5x^2+8=0.5x^2-8 \Rightarrow x=\pm 4$

$\int_{-4}^{4}\left(\left(-0.5x^2+8\right)-\left(0.5x^2-8\right)\right)\mathrm{d}x=85.3$



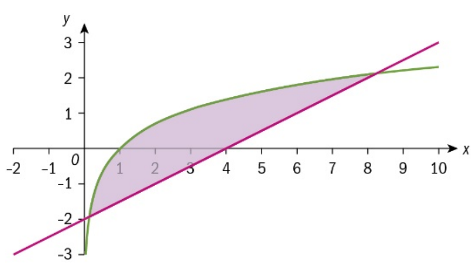
4 $x^2-2x=4\sqrt{x} \Rightarrow x=0,\ 4$

$\int_{0}^{4}\left(4\sqrt{x}-\left(x^2-2x\right)\right)\mathrm{d}x=16$



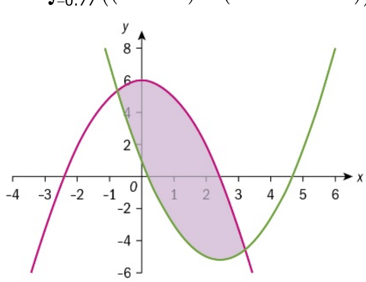
5 $\ln x=\frac{1}{2}x-2 \Rightarrow x=0.15,\ 8.21$

$\int_{0.15}^{8.21}\left(\ln x-\left(\frac{1}{2}x-2\right)\right)\mathrm{d}x=8.78$



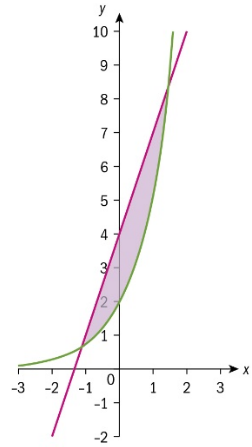
6 $x^2-5x+1=6-x^2 \Rightarrow x=-0.77,\ 3.27$

$\int_{-0.77}^{3.27}\left(\left(6-x^2\right)-\left(x^2-5x+1\right)\right)\mathrm{d}x=21.8$



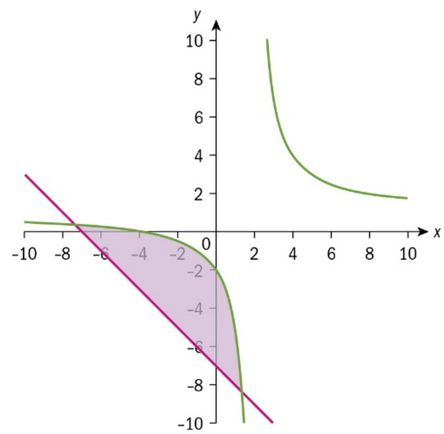
7 $2e^x=3x+4 \Rightarrow x=-1.11,\ 2$

$\int_{-1.11}^{2}\left(\left(3x+4\right)-2e^x\right)\mathrm{d}x=3.68$



8 $\frac{x+4}{x-2}=-x-7 \Rightarrow x=-7.36,\ 1.36$

$\int_{-7.36}^{1.36}\left(\frac{x+4}{x-2}-\left(-x-7\right)\right)\mathrm{d}x=27.5$



9 The area enclosed is given by the integral

$$\int_1^4\left(\frac{2}{x}-(-2)\right)\mathrm{d}x=\int_1^4\left(\frac{2}{x}+2\right)\mathrm{d}x$$

$$=\left[2\ln x+2x\right]_1^4=\left(2\ln 4+8\right)-\left(2\right)$$

$$=4\ln 2+6$$

$$\Rightarrow p=4,\ q=6$$

10 a $f(x)=g(x) \Rightarrow \sqrt{x}=\frac{x}{2}$

$$\therefore \sqrt{x}\left(1-\frac{\sqrt{x}}{2}\right)=0 \Rightarrow x=0 \quad \text{or} \quad x=4$$

$$\therefore (0,0) \quad \text{and} \quad (4,2)$$

b i $\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)\mathrm{d}x$

ii $\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)\mathrm{d}x=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{x^2}{4}\right]_0^4=\frac{4}{3}$

c i $\int_0^k\left(\sqrt{x}-\frac{x}{2}\right)\mathrm{d}x$

ii $\frac{2}{3}k^{\frac{3}{2}} - \frac{k^2}{4}$

iii $\frac{2}{3}k^{\frac{3}{2}} - \frac{k^2}{4} = \frac{2}{3} \Rightarrow k = 1.510 \quad (3\text{d.p.})$

Exercise 10K

1 $\int_{-2}^{0}\left(\left(x^2-2x\right)-\left(10x+x^2-3x^3\right)\right)dx$

$+\int_{0}^{2}\left(\left(10x+x^2-3x^3\right)-\left(x^2-2x\right)\right)dx$

$=24$

2 $\int_{0}^{1}\left(\left(x^3-3x^2+3x+1\right)-\left(x+1\right)\right)dx$

$+\int_{1}^{2}\left(\left(x+1\right)-\left(x^3-3x^2+3x+1\right)\right)dx$

$=0.5$

3 $\int_{-1.51677}^{0}\left(\left(x^3-2x\right)-3xe^{-x^2}\right)dx$

$+\int_{0}^{1.51677}\left(3xe^{-x^2}-\left(x^3-2x\right)\right)dx$

≈ 4.65

4 $\int_{-4}^{-1.41421}\left(\left(-x^4+16x^2\right)-\left(x^4-20x^2+64\right)\right)dx$

$+\int_{-1.41421}^{1.41421}\left(\left(x^4-20x^2+64\right)-\left(-x^4+16x^2\right)\right)dx$

$+\int_{1.41421}^{4}\left(\left(-x^4+16x^2\right)-\left(x^4-20x^2+64\right)\right)dx$

≈ 440

5 a $f(x)=h(x) \Rightarrow \frac{x^2}{2}-1=-3x-5.5$

$\Rightarrow x^2+6x+9=(x+3)^2=0$

$\Rightarrow x=-3$

$h(-3)=-3(-3)-5.5=3.5$

$\therefore P(-3,3.5)$

b The gradient of h is -3

$f'(x)=x \Rightarrow m=f'(-3)=-3$

$y-3.5=-3(x-(-3))$

$y-3.5=-3x-9$

$h(x)=-3x-5.5$

c $g(x)=h(x) \Rightarrow -x^2-1=-3x-5.5$

Using GDC $\Rightarrow Q(-1.10,-2.21)$

d i $\int_{-3}^{\frac{3}{2}-\frac{3\sqrt{3}}{2}}\left(\frac{x^2}{2}-1-(-3x-5.5)\right)dx$

$+\int_{\frac{3}{2}-\frac{3\sqrt{3}}{2}}^{0}\left(\frac{x^2}{2}-1-\left(-x^2-1\right)\right)dx$

ii 1.81 (2 d.p.)

Chapter Review

1 a $\int x^8\,dx = \frac{x^9}{9}+C$

b $\int\left(5x^4-6x^2+7\right)dx = x^5-2x^3+7x+C$

c $\int \sqrt[10]{x^3}\,dx = \int x^{\frac{3}{10}}\,dx = \frac{10}{13}x^{\frac{13}{10}}+C$

d $\int \frac{4}{x^9}\,dx = \int 4x^{-9}\,dx = -\frac{1}{2}x^{-8}+C$

$= -\frac{1}{2x^8}+C$

e $\int \frac{8x^5+4x}{2x^2}\,dx = \int\left(4x^3+\frac{2}{x}\right)dx$

$= x^4+2\ln|x|+C$

f $\int 4e^x\,dx = 4e^x+C$

g $\int\left(6\sqrt{x}+2\right)dx = \int\left(6x^{\frac{1}{2}}+2\right)dx$

$=4x^{\frac{3}{2}}+2x+C$

h $\int\left(x^2+3\right)^2 dx = \int\left(x^4+6x^2+9\right)dx$

$=\frac{x^5}{5}+2x^3+9x+C$

i $\int (4x+5)^6\,dx = \frac{1}{4\cdot 7}(4x+5)^7+C$

$=\frac{1}{28}(4x+5)^7+C$

j $\int 6e^{3x+2}\,dx = \frac{6}{3}e^{3x+2}+C = 2e^{3x+2}+C$

k $\int \frac{1}{6x-7}\,dx = \frac{1}{6}\ln|6x-7|+C$

l $\int \ln\left(e^{3x}\right)dx = \int 3x\,dx = \frac{3x^2}{2}+C$

2 a $\int_{-2}^{3}(6x-1)\,dx = \left[3x^2-x\right]_{-2}^{3}$

$=(24-14)=10$

b $\int_{-1}^{3}x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{3} = 9-\left(-\frac{1}{3}\right)=\frac{28}{3}$

c $\int_{9}^{25}\frac{3}{\sqrt{x}}\,dx = 6\left[\sqrt{x}\right]_{9}^{25} = 6(5-3)=12$

d $\int_{1}^{e^4}\frac{5}{x}\,dx = \left[5\ln x\right]_{1}^{e^4} = 5\ln e^4 = 20$

e $\int_{-2}^{0}8(2x+3)^3\,dx = \left[(2x+3)^4\right]_{-2}^{0}$

$=81-1=80$

f $\int_{3}^{5}e^{4x}\,dx = \left[\frac{1}{4}e^{4x}\right]_{3}^{5} = \frac{1}{4}\left(e^{20}-e^{12}\right)$

3 a $\int_1^4 2f(x)\,\mathrm{d}x = 2\int_1^4 f(x)\,\mathrm{d}x = 2(10) = 20$

b $\int_3^4 f(x)\,\mathrm{d}x = \int_1^4 f(x)\,\mathrm{d}x - \int_1^3 f(x)\,\mathrm{d}x$

$= 10 - 6 = 4$

c $\int_1^4 (f(x)+4)\,\mathrm{d}x = \int_1^4 f(x)\,\mathrm{d}x + \int_1^4 4\,\mathrm{d}x$

$= 10 + [4x]_1^4 = 10 + (16-4) = 22$

4 $f(x) = \int (4x^3+2)\,\mathrm{d}x = x^4 + 2x + C$

$f(2) = 20 + C = 24 \Rightarrow C = 4$

$\therefore f(x) = x^4 + 2x + 4$

5 a $\square \int_{-8}^0 f(x)\,\mathrm{d}x = -\frac{1}{2}\pi(4^2) = -8\pi$

b $|-8\pi| + \int_0^8 f(x)\,\mathrm{d}x = 21\pi$

$\Rightarrow \int_0^8 f(x)\,\mathrm{d}x = 13\pi$

$\therefore \int_{-8}^8 f(x)\,\mathrm{d}x = \int_{-8}^0 f(x)\,\mathrm{d}x + \int_0^8 f(x)\,\mathrm{d}x$

$= -8\pi + 13\pi = 5\pi$

6 $f(x) = -x(x+4) = 0 \Rightarrow x = -4$ or $x = 0$

$\int_{-4}^0 (-x^2-4x)\,\mathrm{d}x = -\left[\frac{x^3}{3} + 2x^2\right]_{-4}^0$

$= 0 + \frac{(-4)^3}{3} + 2(-4)^2$

$= 32 - \frac{64}{3} = \frac{32}{3}$

7 a Lines intersect at $x = -2.2808$, 2.4765 and 9.7467

$\int_{-2.2808}^{2.4765} \left(0.5\mathrm{e}^x + 1 - (x^4 - 5x^2)\right)\mathrm{d}x$

$+ \int_{2.4765}^{9.7467} \left(x^4 - 5x^2 - (0.5\mathrm{e}^x + 1)\right)\mathrm{d}x$

$= 7530.19$

≈ 7530

b Lines intersect at $x = -2$, 0 and 3

$\int_{-2}^0 \left(x^3 - 9x - (x^2 - 3x)\right)\,\mathrm{d}x$

$+ \int_0^3 \left(x^2 - 3x - (x^3 - 9x)\right)\,\mathrm{d}x$

$= \frac{253}{12} = 21.1$

8 a $y = f(1) = 1$

$f'(x) = 3x^2 \Rightarrow f'(1) = 3$

$\therefore y - 1 = 3(x-1) \Rightarrow y = 3x - 2$

b $x^3 = 3x - 2$

$\Rightarrow x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$

$= (x-1)^2(x+2) = 0$

$\therefore x = -2$

$y = 3(-2) - 2 = -8$

$\therefore (-2, -8)$

c $\int_{-2}^1 \left(x^3 - (3x-2)\right)\,\mathrm{d}x = \int_{-2}^1 (x^3 - 3x + 2)\,\mathrm{d}x$

$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^1 = \frac{3}{4} - (-6)$

$= \frac{27}{4} = 6.75$

9 a $f'(x) = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$ M1A1

$= \frac{2 - 2x^2}{(x^2+1)^2}$ A1

b $\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + C$ M1A1A1

10 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = -3\ ms^{-2}$ M1A1

b $s(t) = \int (40 - 3t)\,\mathrm{d}t = 40t - \frac{3t^2}{2} + C$

M1A1

$s(1) = 10 \Rightarrow 40 - \frac{3}{2} + C = 10$

$\Rightarrow C = -\frac{57}{2}$ M1

$s(t) = 40t - \frac{3t^2}{2} - \frac{57}{2}$

A1

11 a i $A_{region1} = \frac{2\times 2}{2} = 2$ M1A1

ii $A_{region2} = \frac{2\times 3}{2} = 3$ M1A1

b $A_{region3} = 3 - 2 = 1$ A1

12 a Use GDC to obtain value of definite integral M1

$A = \int_0^1 \mathrm{f}(x)\,\mathrm{d}x = 1.1202$ A1

b i 2.24 (3 s.f.) A1

ii $\int_1^2 2\mathrm{f}(x-1)\,\mathrm{d}x = 2\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = 2.24$

2.24 (3 s.f.) M1A1

13 a $x = 0, x = \pm 1$ A1A1

b Either:

$f'(x) = 1 \cdot (x^2 - 1) + 2x^2 = 3x^2 - 1$ M1A1

Or:

$f(x) = x^3 - x$ M1

$f'(x) = 3x^2 - 1$ A1

c $f'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{1}{3}} = \pm 0.577$ M1A1

d $\int_{-1}^{1} f(x)\, dx = 0$ M1A1

e The function changes sign in the interval $[-1, 1]$, so the areas above and below the x-axis cancel out. R1

f $\int_{-1}^{1} |f(x)|\, dx = 0.5$ M1A1

14 a $f(1) = 1^2 = 3 - 2 \times 1 = g(1)$ M1A1

$f(-3) = (-3)^2 = 3 - 2 \times (-3) = g(-3)$

A1

b $\int_{-3}^{1} (3 - 2x - x^2) dx = \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$

M1A1A1

$= \left(3 - 1 - \frac{1}{3}\right) - \left(-9 - 9 + \frac{27}{3}\right) = \frac{32}{3}$ A1

15 a $(x-2)^4$

$= x^4 + 4 \times (-2) x^3 + 6 \times (-2)^2 x^2$

$+ 4 \times (-2)^3 x + (-2)^4$ M1A1

$= x^4 - 8x^3 + 24x^2 - 32x + 16$ A1

b $\int (x-2)^4 dx$

$= \int x^4 - 8x^3 + 24x^2 - 32x + 16\ dx$

M1

$= \frac{x^5}{5} - 2x^4 + 8x^3 - 16x^2 + 16x + C$

A1A1

16 Use GDC to obtain graph of $y = |x|$

M1A1

Attempt to calculate area of both triangles

M1

$\int_{-1}^{2} |x|\ dx = \frac{1 \times 1}{2} + \frac{2 \times 2}{2} = 2.5$ A1AG

OR:

$\int_{-1}^{2} |x|\, dx = \int_{-1}^{0} (-x)\, dx + \int_{0}^{2} x\, dx$ M1A1A1

$= \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{2}$ A1

$= \left(0 + \frac{1}{2}\right) + (2 - 0) = \frac{5}{2}$ AG

17 a

A1 for shape; A1 for domain; A1 for end-points coordinates; A1 for maximum point and its coordinates.

b $0 \le y \le 1.104$ A1

c $m = \frac{0.1011}{5} = 0.0202\ldots$ M1

AB contains the origin R1

$y = 0.0202x$ A1

d $f'(x) = 3e^{-x} + 3x(-e^{-x}) = (3 - 3x)e^{-x}$

M1A1AG

e Solve $f'(x) = 0.0202\ldots \Rightarrow x = 0.98201\ldots$

A1

$f(0.98201\ldots) = 1.10345\ldots$ A1

$y - 1.10 = 0.0202(x - 0.982)$ A1

f Use of GDC to calculate M1

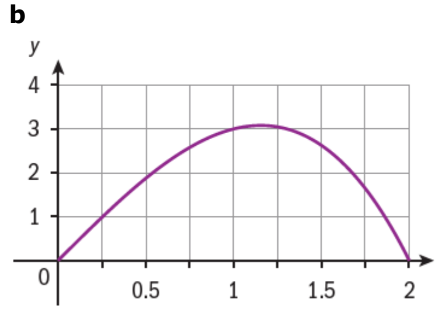
$\int_{0}^{5} 3xe^{x} - 0.0202x\ dx = 2.63$ A2

18 a

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$f(x)$	$\frac{15}{8}$	3	$\frac{21}{8}$

1 correct: A1; all correct: A2

b



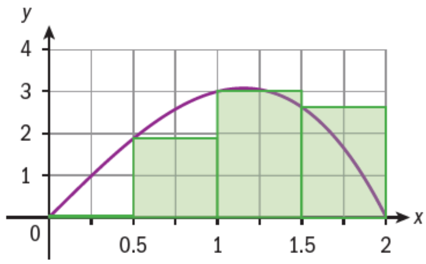
A1 for shape; A1 for domain; A1 for intercepts

c Either:

$$A = \left(0 \times \frac{1}{2}\right) + \left(\frac{15}{8} \times \frac{1}{2}\right) + \left(3 \times \frac{1}{2}\right) + \left(\frac{21}{8} \times \frac{1}{2}\right)$$

M1A1A1

$= \frac{15}{4} (= 3.75)$ A1

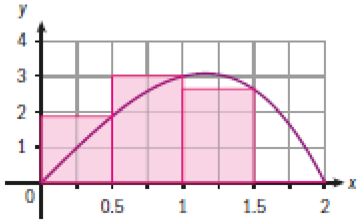


Or:

$$A = \left(\frac{15}{8} \times \frac{1}{2}\right) + \left(3 \times \frac{1}{2}\right) + \left(\frac{21}{8} \times \frac{1}{2}\right) + \left(0 \times \frac{1}{2}\right)$$

M1A1A1

$= \frac{15}{4} (= 3.75)$ A1



d $\int_0^2 (4x - x^3)\, dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2$ M1A1

$= 8 - 4 = 4$ A1

11 Relationships in space: geometry and trigonometry in 2D and 3D

Skills check

1 **a** $5\sqrt{10} \approx 15.8$ **b** $2\sqrt{6} \approx 4.9$

2 **a** **i** 12600cm^2 **ii** 1.26m^2

b $4\pi \approx 12.6\text{m}^3 \approx 12566\,l$

Exercise 11A

1 **a** $(3,0,0)$ **b** $(3,4,0)$

c $(3,0,2)$ **d** $(3,4,2)$

e Midpoint of OF

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{0+3}{2},\frac{0+4}{2},\frac{0+2}{2}\right)=(1.5,2,1)$$

f Distance of OF

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(3-0)^2+(4-0)^2+(2-0)^2}$$
$$=\sqrt{9+16+4}=\sqrt{29}\approx 5.4$$

2 **a** $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-4+5}{2},\frac{4-1}{2},\frac{3+3}{2}\right)=(0.5,1.5,3)$$

b $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-4-2}{2},\frac{4+2}{2},\frac{5+9}{2}\right)=(-3,3,7)$$

c $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{5-4}{2},\frac{2-3}{2},\frac{-4-8}{2}\right)=(0.5,-0.5,-6)$$

d $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-5.1+1.4}{2},\frac{-2+1.7}{2},\frac{9+11}{2}\right)$$
$$=(-1.85,-0.15,10)$$

3 **a** $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(4-2)^2+(3-3)^2+(1-5)^2}$$
$$=\sqrt{4+0+16}=\sqrt{20}\approx 4.47$$

b $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(2+3)^2+(4-7)^2+(-1-2)^2}$$
$$=\sqrt{25+9+9}=\sqrt{43}\approx 6.56$$

c $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$$
$$=\sqrt{4+36+64}=\sqrt{104}\approx 10.2$$

d $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(-2-2)^2+(1+1)^2+(3-3)^2}$$
$$=\sqrt{16+4+0}=\sqrt{20}\approx 4.47$$

4 **a** $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(-5-1)^2+(-6-2)^2+(-7-3)^2}$$
$$=\sqrt{36+64+100}=\sqrt{200}\approx 14.1$$

b $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(4-0)^2+(0+4)^2+(5-2)^2}$$
$$=\sqrt{16+16+9}=\sqrt{41}\approx 6.4$$

c $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1+1)^2+(2+1)^2+(3+1)^2}$$
$$=\sqrt{4+9+16}=\sqrt{29}\approx 5.39$$

d $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1-4)^2+(1-1)^2+(1+3)^2}$$
$$=\sqrt{9+0+16}=\sqrt{25}=5$$

Exercise 11B

1 **a** $\text{SA}=x^2+2xl=20^2+2\times20\times26$

$=1440\text{cm}^2$

b $\text{SA}=x^2+2xl=4^2+2\times4\times6.3=66.4\text{cm}^2$

c $\text{SA}=x^2+2xl=5^2+2\times5\times13=155\text{cm}^2$

d $\text{SA}=\pi r^2+\pi rl=\pi\times5^2+\pi\times5\times13$

$=283\text{cm}^2$

e $\text{SA}=\pi r^2+\pi rl=\pi\times6^2+\pi\times6\times14$

$=377\text{cm}^2$

f $\text{SA}=\pi r^2+\pi rl=\pi\times4^2+\pi\times4\times12$

$\approx 201\text{cm}^2$

2 **a** $SA = 4\pi r^2 = 4\pi \times 5^2 \approx 314\text{cm}^2$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 5^3 = 524\text{cm}^3$

b $SA = 4\pi r^2 = 4\pi \times \left(\frac{3}{2}\right)^2 = 28.3\text{cm}^2$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{3}{2}\right)^3 = 14.1\text{cm}^3$

3 **a** $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}(4 \times 4 \times 12) = 64\text{cm}^3$

b $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}\left(\frac{10 \times 13.1}{2} \times 11\right) = 240\text{cm}^3$

c $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}(9 \times 7 \times 5) = 105\text{cm}^3$

4 SA_c =curved surface area,

SA_T = total surface area

a $l = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

$SA_c = \pi rl = \pi \times 5 \times 13 \approx 204\text{cm}^2$

b $SA_T = \pi r^2 + \pi rl = \pi \times 5^2 + 204.2$

$\approx 283\text{cm}^2$

c $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 5^2 \times 12 \approx 314\text{cm}^2$

5 **a** $SA_c = \frac{4\pi r^2}{2} = \frac{4\pi \times 3^2}{2} = 56.5\text{cm}^2$

b $SA_T = \frac{4\pi r^2}{2} + \pi r^2 = 56.5 + \pi \times 3^2$

$=84.8\text{cm}^2$

c $V = \frac{4}{6}\pi r^3 = \frac{2}{3}\pi \times 3^3 = 56.5\text{cm}^3$

6 **a** $V_T = V_C + V_{HS} = \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{1}{3}\pi \times 4^2 \times 10 + \frac{1}{2}\left(\frac{4}{3}\pi \times 4^3\right) = 302\text{cm}^3$

b Sum of the curved surface area of the cone, SA_{cc}, and curved surface area of the hemisphere, SA_{CHS}

$l = \sqrt{10^2 + 4^2} \approx 10.8cm$

$SA_T = SA_{CC} + SA_{CHS} = \pi rl + 2\pi r^2$

$= \pi \times 4 \times 10.8 + 2\pi \times 4^2 = 236\text{cm}^2$

7 Volume of the water tank, V_T

$V_T = \pi r^2 h_{cyl} + \frac{1}{3}\pi r^2 h_{cone}$

$= \pi \times 1^2 \times (13 - 2) + \frac{1}{3}\pi \times 1^2 \times 2$

$= 36.7\text{m}^3$

Conversion to litres

$36.7\text{m}^3 \times \frac{1000000\text{cm}^3}{1\text{m}^3} \times \frac{1\text{L}}{1000\text{cm}^3}$

$= 36700\text{L}$

8 Volume of each ball,

$V_{ball} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (3.35)^3 = 157.5\text{ cm}^3$.

$h_{cyl} = 3 \times 6.7 = 20.1\text{cm}$

$V_{cyl} = \pi r^2 h = \pi \times 3.35^2 \times 20.1 = 708.7\text{cm}^3$

total free space is

$V_T = V_{cyl} - 3V_B \approx 708.7 - 3 \times 157.5 = 236\text{ cm}^3$

Exercise 11C

1 **a** $\sin\theta = \frac{19}{27}, \theta = \sin^{-1}\frac{19}{27} = 44.7°$

b $\tan\theta = \frac{33}{56}, \theta = \tan^{-1}\frac{33}{56} = 30.5°$

c $\tan\theta = \frac{12}{5}, \theta = \tan^{-1}\frac{12}{5} = 67.4°$

d $\cos\theta = \frac{11}{20}, \theta = \cos^{-1}\frac{11}{20} = 56.6°$

2 **a** $\cos 22° = \frac{x}{27}, x = 27\cos 22° = 25.0$

b $\tan 46° = \frac{44}{x}, x = \frac{44}{\tan 46°} = 42.5$

c $\tan 46° = \frac{7}{x}, x = \frac{7}{\tan 46°} = 6.76$

d $\sin 43° = \frac{x}{22}, x = 22\sin 43° = 15.0$

3 $\tan 25° = \frac{h}{12}, h = 12\tan 25° = 5.60$

$\tan a = \frac{5.6}{10}, a = \tan^{-1}\frac{5.6}{10} = 29.2°$

4 $\tan 30° = \frac{q}{55}$,

$q = 55\tan 30° = \frac{55}{\sqrt{3}} = 31.75\ldots \approx 31.8$

$\tan 50° = \frac{q}{(55 - p)} = \frac{31.75}{(55 - p)}$

$(55 - p)\tan 50° = 31.75$

$(55 - p) = \frac{31.75}{\tan 50°}$

$p = 55 - \dfrac{31.75}{\tan 50°} \approx 28.4$

5 $\cos 20° = \dfrac{7}{x}$, $x = \dfrac{7}{\cos 20°} = 7.45$

6 $\tan 30° = \dfrac{6}{h}$, $h = \dfrac{6}{\tan 30°} = 10.4$

7 **a** $\dfrac{1}{3} \times 20 \times 15 \times 30 = 3000\text{cm}^3$

b $AC = 2NC = \sqrt{20^2 + 15^2} = 25$. Then $NC = 12.5\text{cm}$

Then $TC = \sqrt{30^2 + NC^2} = \sqrt{30^2 + 12.5^2}$
$= \sqrt{1056.25} = 32.5\text{cm}$

c $\sin\theta = \dfrac{30}{32.5}$, $\theta = \sin^{-1}\dfrac{30}{32.5} = 67.4°$

d $\text{TM} = \sqrt{\text{TC}^2 - \text{MC}^2} = \sqrt{32.5^2 - 7.5^2} = 31.6$

e $\sin \text{TMN} = \dfrac{30}{\text{TM}} = \dfrac{30}{31.6}$,

$\text{TMN} = \sin^{-1}\dfrac{30}{31.6} = 71.7°$

8 **a** $\text{V} = \dfrac{1}{3} \times 230.4 \times 230.4 \times 146.5$

$= 2592276.5\text{cm}^3$

b $AB = 2OM$,

$EM = \sqrt{146.5^2 + OM^2}$

$= \sqrt{146.5^2 + 115.2^2} \approx 186\text{cm}$

c $\tan EMO = \dfrac{EO}{OM} = \dfrac{146.5}{115.2}$,

$EMO = \tan^{-1}\dfrac{146.5}{115.2} = 51.8° \approx 52°$

d $A_T = 4 \times A_F + A_B$, $A_F = \dfrac{1}{2} \times AB \times EM$

$A_T = 2 \times 230.4 \times 186.4 + 230.4 \times 230.4$
$= 139000\text{cm}^2$

e $EB = \sqrt{EM^2 + BM^2}$

$= \sqrt{186.4^2 + 115.2^2} = 219.1$

$\sin EBO = \dfrac{EO}{EB} = \dfrac{146.5}{219.1}$

$\sin^{-1}\dfrac{146.5}{219.1} = 42.0°$

Exercise 11D

1 $\tan 25° = \dfrac{h}{12}$

$h = 12\tan 25° = 5.60\text{m}$

2 $\sin 55° = \dfrac{h}{50}$

$h = 50\sin 55° = 41\text{m}$

3 **a** $\sin\theta = \dfrac{0.8}{3}$

$\theta = \sin^{-1}\dfrac{0.8}{3} = 15.5°$

b $h = \sqrt{3^2 - 0.8^2} = 2.89\text{m}$

4 $\tan 40° = \dfrac{81.5}{d}$

$d = \dfrac{81.5}{\tan 40°} = 97.1\text{m}$

5 R and J are the same height, so it cancels out. Then the calculation is

$\tan 70° = \dfrac{H}{3}$

$H = 3\tan 70° = 8.24\text{m}$

6 **a** $\cos 36° = \dfrac{N}{25}$

$N = 25\cos 36° = 20.2\text{km}$

b $\sin 36° = \dfrac{W}{25}$

$W = 25\sin 36° = 14.7\,km$

7 $\sin 68° = \dfrac{W}{51}$

$W = 51\sin 68° = 47.3\text{km}$

8 $H = H_E + H_D$

$\tan 23° = \dfrac{H_E}{300}$

$H_E = 300\ \tan 23° = 127.3\text{m}$

$\tan 30° = \dfrac{H_D}{300}$

$H_D = 300\tan 30° = 173.2\text{m}$

$H = H_E + H_D = 301\text{m}$

9 Let C be the bottom of the Eiffel tower. Then

$\text{AB} = \text{ABC} - \text{BC}$

$40° + 32° + \theta = 90°$

$\theta = 18°$

$\tan(32° + 18°) = \dfrac{\text{ABC}}{300}$

$\text{ABC} = 300\tan 50° = 357.53$

$\tan 18° = \dfrac{\text{BC}}{300}$

$\text{BC} = 300\tan 18° = 97.48$

$\text{AB} = \text{ABC} - \text{BC} = 260\text{m}$

10 $\tan 75° = \dfrac{\text{D} + 1.5}{498}$

$D = 498\tan 75° - 1.5 = 1857\text{m}$

Exercise 11 E

1 Area $= \frac{1}{2}ab\sin C$

a Area $= \frac{1}{2}\times 8\times 6\times \sin 80°$

Area $= 24\sin 80° = 23.6\text{cm}^2$

b Area $= \frac{1}{2}\times 10\times 15\times \sin 125°$

Area $= 75\sin 125° = 61.4\text{cm}^2$

c Area $= \frac{1}{2}\times 2.5\times 3.9\times \sin 34°$

Area $= 4.875\sin 34° = 2.73\text{cm}^2$

d Area $= \frac{1}{2}\times 4\times 7\times \sin 96°$

Area $= 14\sin 96° = 13.9\text{cm}^2$

e Area $= \frac{1}{2}\times 12\times 20\times \sin(180° - 80° - 40°)$

Area $= 120\sin 60° = 104\text{cm}^2$

f Area $= \frac{1}{2}\times 14\times 18\times \sin(180° - 78° - 60°)$

Area $= 126\sin 42° = 84.3\text{cm}^2$

g Area $= \frac{1}{2}\times 12\times 8\times \sin(180° - 30° - 67°)$

Area $= 48\sin 83° = 47.6\text{cm}^2$

2 Area $= \frac{1}{2}ab\sin C$

a Area $= \frac{1}{2}\times 8\times 5\times \sin 39°$

Area $= 20\sin 39° = 12.6\text{cm}^2$

b $16 = \frac{1}{2}\times 8\times 8\times \sin C$

$\sin C = \frac{1}{2}$

$C = \sin^{-1}\frac{1}{2} = 30°$

3 Area $= 2\times\frac{1}{2}ab\sin C$

Area $= 20\times 12\times \sin 60°$

Area $= 240\sin 60° = 208\text{cm}^2$

4 4 faces, so area is multiplied by 4

3 angles in an equilateral triangle are $60°$

Area $= 4\times\frac{1}{2}ab\sin C$

Area $= 2\times 10\times 10\times \sin 60°$

Area $= 173\text{cm}^2$

5 Area $= 5\times\frac{1}{2}ab\sin C$

Area $= 5\times\frac{1}{2}\times 4\times 4\times \sin 72° = 38.0\text{m}^2$

6 $\frac{1}{2}(x+2)(2x+1)\sin 60° = 5\sqrt{3}$

$(x+2)(2x+1) = \dfrac{10\sqrt{3}}{\frac{\sqrt{3}}{2}} = 20$

$2x^2 + x + 4x + 2 = 20$

$2x^2 + 5x - 18 = 0$

Either factorise, or use the quadratic formula:

$$x_{1,2} = \frac{-5\pm\sqrt{5^2 - 4\times 2\times(-18)}}{2\times 2} = \frac{-5\pm\sqrt{25+144}}{4}$$

$$x_{1,2} = \frac{-5\pm 13}{4}$$

We take the positive value, as distances cannot be negative.

Then $x = 2$

Exercise 11F

1 a $\frac{\sin\theta}{23} = \frac{\sin 35°}{45}$

$\sin\theta = \frac{23\sin 35°}{45}$

$\theta = \sin^{-1}\frac{23\sin 35°}{45} = 17.0°$

b $\frac{\sin\theta}{4} = \frac{\sin 66°}{8}$

$\sin\theta = \frac{4\sin 66°}{8}$

$\theta = \sin^{-1}\frac{4\sin 66°}{8} = 27.2°$

c $\frac{\sin\theta}{6} = \frac{\sin 75°}{18}$

$\sin\theta = \frac{6\sin 75°}{18}$

$\theta = \sin^{-1}\frac{6\sin 75°}{18} = 18.8°$

d $\frac{\sin\theta}{22} = \frac{\sin 48°}{63}$

$\sin\theta = \frac{22\sin 48°}{63}$

$$\theta = \sin^{-1}\frac{22\sin 48°}{63} = 15.0°$$

e $\frac{\sin\theta}{20} = \frac{\sin 82°}{29}$

$$\sin\theta = \frac{20\sin 82°}{29}$$

$$\theta = \sin^{-1}\frac{20\sin 82°}{29} = 43.1°$$

f $\frac{\sin\theta}{18} = \frac{\sin 78°}{34}$

$$\sin\theta = \frac{18\sin 78°}{34}$$

$$\theta = \sin^{-1}\frac{18\sin 78°}{34} = 31.2°$$

2 **a** $\frac{\sin 99°}{37} = \frac{\sin(180° - 99° - 18°)}{x}$

$$x = 37\frac{\sin 63°}{\sin 99°} = 33.4\text{ cm}$$

b $\frac{\sin 53°}{x} = \frac{\sin 44°}{7}$

$$x = 7\frac{\sin 53°}{\sin 44°} = 8.05$$

c $\frac{\sin 23°}{x} = \frac{\sin 77°}{15}$

$$x = 15\frac{\sin 23°}{\sin 77°} = 6.02$$

d $\frac{\sin 33°}{x} = \frac{\sin 108°}{24}$

$$x = 24\frac{\sin 33°}{\sin 108°} = 13.7$$

e $\frac{\sin(180° - 100° - 22°)}{x} = \frac{\sin 22°}{10}$

$$x = 10\frac{\sin 58°}{\sin 22°} = 22.6$$

f $\frac{\sin(180° - 52° - 56°)}{x} = \frac{\sin 52°}{6}$

$$x = 6\frac{\sin 72°}{\sin 52°} = 7.24$$

3 base $= 15 + x$

We have that $\frac{\sin 70°}{h} = \frac{\sin 20°}{x}$

and $\frac{\sin 40°}{h} = \frac{\sin 50°}{15 + x}$

Then $15 + x = \frac{\sin 50°}{\sin 40°}h$

$$x = \frac{\sin 50°}{\sin 40°}h - 15$$

We substitute back into the first equation

$$h = \frac{\sin 70°}{\sin 20°}x$$

so $h = \frac{\sin 70°}{\sin 20°}\left(\frac{\sin 50°}{\sin 40°}h - 15\right)$

$$\left(1 - \frac{\sin 70°}{\sin 20°}\left(\frac{\sin 50°}{\sin 40°}\right)\right)h = -15 \times \frac{\sin 70°}{\sin 20°}$$

$$h = \frac{-15 \times \frac{\sin 70°}{\sin 20°}}{1 - \frac{\sin 70°}{\sin 20°}\left(\frac{\sin 50°}{\sin 40°}\right)} = 18.1$$

4 $\frac{\sin 74°}{10} = \frac{\sin 64°}{x}$

$$x = \frac{10\sin 64°}{\sin 74°} = 9.35\text{km}$$

5 We have $\frac{\sin 49°}{d} = \frac{\sin 41°}{20 - b}$

$$20 - b = d\frac{\sin 41°}{\sin 49°}$$

$$b = 20 - d\frac{\sin 41°}{\sin 49°}$$

and $\frac{\sin 38°}{d} = \frac{\sin 52°}{b}$

$$b = d\frac{\sin 52°}{\sin 38°}$$

We equate both expressions for b

$$20 - d\frac{\sin 41°}{\sin 49°} = d\frac{\sin 52°}{\sin 38°}$$

$$20 = d\left(\frac{\sin 41°}{\sin 49°} + \frac{\sin 52°}{\sin 38°}\right)$$

$$d = \frac{20}{\frac{\sin 41°}{\sin 49°} + \frac{\sin 52°}{\sin 38°}} = 9.31\text{km}$$

6 We have $\frac{\sin 74°}{16 + AD} = \frac{\sin 16°}{DC}$

$$16 + AD = DC\frac{\sin 74°}{\sin 16°}$$

$$AD = DC\frac{\sin 74°}{\sin 16°} - 16$$

and $\frac{\sin 62°}{AD} = \frac{\sin 28°}{DC}$

$$AD = DC\frac{\sin 62°}{\sin 28°}$$

We equate both expressions for AD

$$DC\frac{\sin 74°}{\sin 16°} - 16 = DC\frac{\sin 62°}{\sin 28°}$$

$$DC\left(\frac{\sin 74°}{\sin 16°} - \frac{\sin 62°}{\sin 28°}\right) = 16$$

$$DC = \frac{16}{\frac{\sin 74°}{\sin 16°} - \frac{\sin 62°}{\sin 28°}} = 9.96\text{km}$$

7 We have $\frac{\sin 15°}{h} = \frac{\sin 75°}{10+d}$

$10 + d = h\frac{\sin 75°}{\sin 15°} = 3.73h$

$d = 3.73h - 10$

and $\frac{\sin 18°}{h} = \frac{\sin 72°}{d}$

$d = h\frac{\sin 72°}{\sin 18°} = 3.08h$

Then $3.08h = 3.73h - 10$

$3.73h - 3.08h = 10$

$0.65h = 10$

$h = 15.3$

8 We have $\frac{\sin 55°}{h} = \frac{\sin 90°}{4}$

$h = 4\sin 55° = 3.27660..\text{mm}$

and $\frac{\sin 78°}{4} = \frac{\sin 47°}{b}$

$b = 4\frac{\sin 47°}{\sin 78°} = 2.99077..\text{mm}$

Then

$$Area = \frac{1}{2}bh = \frac{1}{2} \times 3.27660 \times 2.99077 = 4.90 \text{ mm}^2$$

Exercise 11G

1 $\frac{\sin 64°}{10} = \frac{\sin A}{8}$

$\sin A = \frac{8}{10}\sin 64°$

$A = \sin^{-1}\left(\frac{8}{10}\sin 64°\right) = 46.0°$

and $180° - 46.0° = 134°$

2 $\frac{\sin 20°}{3} = \frac{\sin A}{5}$

$\sin A = \frac{5}{3}\sin 20°$

$C = \sin^{-1}\left(\frac{5}{3}\sin 20°\right) = 34.8°$

and $180° - 34.8° = 145.2°$

3 $\frac{\sin 45°}{8} = \frac{\sin B}{10}$

$\sin B = \frac{10}{8}\sin 45°$

$B = \sin^{-1}\left(\frac{10}{8}\sin 45°\right) = 62.1°$

and $180° - 62.1° = 117.9°$

4 $\frac{\sin 40°}{24} = \frac{\sin C}{30}$

$\sin C = \frac{30}{24}\sin 40°$

$C = \sin^{-1}\left(\frac{30}{24}\sin 40°\right) = 53.5°$

and $180° - 53.5° = 126.5°$

5 $Area = \frac{1}{2} \times AB \times BC \times \sin B = 20$

$\frac{1}{2} \times 8 \times 10 \times \sin B = 20$

$\sin B = \frac{20}{40}$

$B = \sin^{-1}\frac{20}{40} = 30°$

The obtuse angle is $180° - 30° = 150°$

Exercise 11H

1 a $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \times \cos 62°$

$a = 11.1\text{cm}$

b $b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 15^2 + 28^2 - 2 \times 15 \times 28 \times \cos 112°$

$b = 36.4\text{cm}$

c $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 14^2 + 22^2 - 2 \times 14 \times 22 \times \cos 80°$

$a = 23.9\text{m}$

d $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 10^2 + 9^2 - 2 \times 10 \times 9 \times \cos 66°$

$c = 10.4\text{m}$

e $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 40^2 + 25^2 - 2 \times 40 \times 25 \times \cos 20°$

$a = 18.6\text{cm}$

f $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 21^2 + 30^2 - 2 \times 21 \times 30 \times \cos 123°$

$a = 45.0\text{cm}$

2 a $\cos\theta = \frac{10.4^2 + 18^2 - 21.9^2}{2 \times 10.4 \times 18}$

$\theta = \cos^{-1}(-0.1267) = 97.3°$

b $\cos\theta = \frac{8.6^2 + 3.1^2 - 9.7^2}{2 \times 8.6 \times 3.1}$

$\theta = \cos^{-1}(-0.197299) = 101°$

c $\cos\theta = \dfrac{65^2 + 55^2 - 118^2}{2 \times 65 \times 55}$

$\theta = \cos^{-1}\dfrac{-3337}{3575} = 159°$

d $\cos\theta = \dfrac{5^2 + 5^2 - 3^2}{2 \times 5 \times 5}$

$\theta = \cos^{-1} 0.82 = 34.9°$

e $\cos\theta = \dfrac{24^2 + 22^2 - 20^2}{2 \times 24 \times 22}$

$\theta = \cos^{-1} 0.625 = 51.3°$

f $\cos\theta = \dfrac{3.8^2 + 7^2 - 4^2}{2 \times 3.8 \times 7}$

$\theta = \cos^{-1} 0.891729 = 26.9°$

3 **a** $\cos\theta = \dfrac{9^2 + 12^2 - 6^2}{2 \times 9 \times 12}$

$\theta = \cos^{-1} 0.875 = 29.0°$

b $A = \dfrac{1}{2} ab \sin C$

$A = \dfrac{1}{2} \times 12 \times 9 \times \sin 29.0° = 26.1\text{cm}^2$

4 $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 60^2 + 30^2 - 2 \times 60 \times 30 \times \cos 160°$

$c = 89\text{km}$

5 **a** $\tan 33° = \dfrac{h_1}{46}$

$h_1 = 46\tan 33° = 29.9\text{m}$

and $\tan 17° = \dfrac{h_2}{28}$

$h_2 = 28\tan 17° = 8.56\text{m}$

b $A = 180° - 33° - 17° = 130°$

$b = \sqrt{46^2 + 29.87^2} = 54.9$

$c = \sqrt{28^2 + 8.56^2} = 29.3$

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 54.9^2 + 29.3^2$
$\quad - 2 \times 54.9 \times 29.3 \times \cos 130°$

$a = 77.0\text{m}$

6 $C = 210° - 70° = 140°$

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 9^2 + 15^2 - 2 \times 9 \times 15 \times \cos 140°$

$c = 22.6\text{km}$

Exercise 11I

1 $\cos A = \dfrac{20^2 + 12^2 - 14^2}{2 \times 20 \times 12}$

$A = \cos^{-1} 0.725 = 43.53°$

$Area = \dfrac{1}{2} bc\sin A$

$= \dfrac{1}{2} \times 20 \times 12 \times \sin 43.5° = 82.6\text{cm}^2$

2 $\sin 7° = \dfrac{150 \times 10^6}{d}$

$d = \dfrac{150 \times 10^6}{\sin 7°} = 1230.8$ million km

3 **a** $PR^2 = PS^2 + RS^2 - 2 \times PS \times RS \times \cos S$

$PR^2 = 14^2 + 11^2 - 2 \times 14 \times 11 \times \cos 55°$

$PR = 11.8\text{m}$

b $\dfrac{\sin PSR}{PR} = \dfrac{\sin PRS}{PS}$

$\dfrac{\sin 55°}{11.84..} = \dfrac{\sin PRS}{14}$

$\sin PRS = 14 \times \dfrac{\sin 55°}{11.84..}$

$P\hat{R}S = \sin^{-1}\left(14 \times \dfrac{\sin 55°}{11.84..}\right) = 75.4809..°$

$P\hat{R}Q = 180° - 75.4809..° = 104.519...°$

$Q\hat{P}R = 180° - 50° - 104.519..° = 25.4809...°$

$\dfrac{\sin P\hat{Q}R}{PR} = \dfrac{\sin Q\hat{P}R}{QR}$

$\dfrac{\sin 50°}{11.84...} = \dfrac{\sin 25.4809..°}{QR}$

$QR = 11.84.. \times \dfrac{\sin 25.4809..°}{\sin 50°} = 6.65\text{m}$

c $A = \dfrac{1}{2} \times QS \times PS \times \sin S$

$= \dfrac{1}{2} \times (11 + 6.68) \times 14 \times \sin 50° = 94.81\text{m}^2$

4 **a** $\cos ADB = \dfrac{DB^2 + DA^2 - BA^2}{2 \times DB \times DA}$

$\cos ADB = \dfrac{12^2 + 20^2 - 28^2}{2 \times 12 \times 20}$

$ADB = \cos^{-1} - 0.5 = 120°$

b $Area = \dfrac{1}{2} \times BD \times DA \times \sin ADB$

$Area = \dfrac{1}{2} \times 12 \times 20 \times \sin 120° = 104m^2$.

c $\dfrac{\sin DCB}{BD} = \dfrac{\sin BDC}{BC}$

$$\frac{\sin DCB}{12} = \frac{\sin 60°}{13}$$

$$\sin DCB = \frac{12}{13}\sin 60°$$

$$DCB = \sin^{-1}\frac{12}{13}\sin 60° = 53.1°$$

d $CBD = 180° - BCD - BDC$

$$= 180° - 53.1° - 60° = 66.9°$$

$$\frac{\sin BAD}{BD} = \frac{\sin ADB}{AB}$$

$$\frac{\sin BAD}{12} = \frac{\sin 120°}{28}$$

$$\sin BAD = \frac{12}{28}\sin 120°$$

$$BAD = \sin^{-1}\frac{12}{28}\sin 120° = 21.79°$$

Then

$ABD = 180° - ADB - BAD$

$= 180° - 120° - 21.79° = 38.2°$

and so

$ABC = CBD + ABD = 66.9° + 38.2°$

$= 105.1° \neq 90°$

5 a $$\frac{\sin ABC}{AC} = \frac{\sin ACB}{AB}$$

$$\frac{\sin 46°}{48} = \frac{\sin ACB}{22.5}$$

$$\sin ACB = \frac{22.5}{48}\sin 46°$$

$$ACB = \sin^{-1}\frac{22.5}{48}\sin 46° = 19.7°$$

b $BAC = 180° - ABC - ACB$

$= 180° - 46° - 19.71° = 114.3°$

$BC^2 = AC^2 + AB^2$
$- 2 \times AC \times AB \times \cos BAC$

$BC^2 = 22.5^2 + 48^2$
$- 2 \times 22.5 \times 48 \times \cos 114.3°$

$BC = 60.8\text{m}$

6 a Let $AC=b$

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 95°$

$b = 6.67\text{cm}$

b Let BAC = A. Then

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 95°}{6.67} = \frac{\sin A}{5}$$

$$\sin A = \frac{5}{6.67}\sin 95°$$

$$A = \sin^{-1}\left(\frac{5}{6.67}\sin 95°\right) = 48.3°$$

c $ACD = 180° - 32° - 48.3° = 99.7°$

d Let C=AD. Then

$$\frac{\sin D}{d} = \frac{\sin C}{c}$$

$$\frac{\sin 48.3..°}{6.67..} = \frac{\sin 99.7..°}{c}$$

$$c = 6.67..\frac{\sin 99.7..°}{\sin 48.3..°} = 8.8039..$$

e $$A_{ABC} = \frac{1}{2} \times 4 \times 5 \times \sin 95° = 9.96\text{cm}^2$$

$$A_{ACD} = \frac{1}{2} \times 6.67.. \times 8.80.. \times \sin 32° = 15.55..\text{cm}^2$$

$$A_{ABCD} = A_{ABC} + A_{ACD} = 9.96.. + 15.55.. = 25.5\text{cm}^2$$
(3 s.f.)

7 $$\tan 50° = \frac{h}{x}$$

$$x = \frac{h}{\tan 50°}$$

and $$\tan 60° = \frac{h}{10 - x}$$

$$(10 - x)\tan 60° = h$$

$$10 - x = \frac{h}{\tan 60°}$$

$$x = 10 - \frac{h}{\tan 60°}$$

We equate both expressions for x and get

$$\frac{h}{\tan 50°} = 10 - \frac{h}{\tan 60°}$$

$$h\left(\frac{1}{\tan 50°} + \frac{1}{\tan 60°}\right) = 10$$

$$h = \frac{10}{\left(\frac{1}{\tan 50°} + \frac{1}{\tan 60°}\right)} = 7.06\text{m}$$

8 a $180° - 67° = 113°$

$113° + 123° + ABC = 360°$

$ABC = 124°$

b $b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 80^2 + 120^2 - 2 \times 80 \times 120 \times \cos 124°$

$b = 178\text{km}$

c $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{120^2 + 178^2 - 80^2}{2 \times 120 \times 178} = 0.9289$

$\cos^{-1} 0.9289 = 21.7°$

complement to $123°$ is $57°$, then

$360° - 57° - 21.73° = 281°$

9 We use lowercase letters for sides opposite capital letter angles. Find p

Complementary angle to $84°$:

$180° - 84° = 96°$.

Then $H\hat{P}Q = 360° - 210° - 96° = 54°$

We have two sides and one angle

a $p^2 = q^2 + h^2 - 2hq\cos H\hat{P}Q$

$p^2 = 340^2 + 160^2 - 2 \times 340 \times 160 \times \cos 54°$

$p = 278\text{km}$

b We find H as

$\cos H = \dfrac{p^2 + q^2 - h^2}{2pq}$

$= \dfrac{278^2 + 340^2 - 160^2}{2 \times 278 \times 340}$

$= 0.884913$

$H = \cos^{-1} 0.884913 = 27.8°$

Then $84° + 27.8° = 111.8°$ is B^c, the complement of the angle complementary to the bearing B.

$B^c = 180° - 111.8° = 68.2°$

Then $B = 360° - 68.2° = 292°$

10 Triangle ABC with sides a, b, c

$B = 360° - (180° - 30°) - 100° = 110°$

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 320^2 + 500^2 - 2 \times 320 \times 500 \times \cos 110°$

$b = 680\text{km}$

Then

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{680^2 + 500^2 - 320^2}{2 \times 680 \times 500}$

$= 0.897$

$A = \cos^{-1} 0.897 = 26.2°$

Bearing: $360° - (180° - 30° - A) = 236°$

11 a Let M be the midpoint of AD. Then triangle OMD is right-angled with OM=5, MD=5, then

$OD = \sqrt{OM^2 + MD^2} = 7.07\text{cm}$

Then

$VD = \sqrt{OD^2 + VO^2}$

$= \sqrt{7.07^2 + 20^2} = 21.2\text{cm}$

b $\tan\alpha = \dfrac{VO}{OM}$

$\alpha = \tan^{-1}\dfrac{20}{5} = 76.0°$

c Let K be the point that connects A with OA perpendicularly. Let M denote the midpoint of BA

$\cos OAM = \dfrac{OA^2 + MA^2 - OM^2}{2 \times OA \times MA}$

$= \dfrac{21.2^2 + 5^2 - 20.6^2}{2 \times 21.2 \times 5}$

$OAM = \cos^{-1} 0.236226 = 76.3°$

Then $\sin BAK = \dfrac{BK}{AB}$

$BK = AB\sin BAK = 10 \times \sin 76.3° = 9.72$

The angle between two sloping edges, β is formed by two sides of length BK and the diagonal of the base

$\cos\beta = \dfrac{9.72^2 + 9.72^2 - 14.4^2}{2 \times 9.72 \times 9.72}$

$\beta = \cos^{-1}\dfrac{9.72^2 + 9.72^2 - 14.4^2}{2 \times 9.72 \times 9.72} = 95.6°$

12 a $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{6^2 + 8^2 - 12^2}{2 \times 6 \times 8} = \dfrac{-11}{24}$

b The cosine of the angle is negative, so $ABC > 90°$, i.e. we have an obtuse angle.

Chapter Review

1 $a = \dfrac{1}{3}(\textit{base area} \times \textit{height})$

$= \dfrac{1}{3}(8 \times 8 \times 3) = 64\ \text{m}^3$

Slant height l is the hypotenuse of the triangle formed by the pyramid height and the distance from the origin O to the midpoint of a side of the base.

$l = \sqrt{3^2 + 4^2} = 5$

$b = x^2 + 2xl = 8^2 + 2 \times 8 \times 5 = 144\text{m}^2$

2 $a = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi \times 6^2 \times 8 = 96\pi\,\text{cm}^3$

slant height l is the hypotenuse of the triangle formed by the cone height and the cone radius

$l = \sqrt{8^2 + 6^2} = 10$

$b = \pi r^2 + \pi rl = \pi \times 6^2 + \pi \times 6 \times 10 = 96\pi\,\text{cm}^2$

3 $V = \frac{4}{3}\pi r^3 = \frac{32}{3}\pi$

$\frac{4}{3}r^3 = \frac{32}{3}$

$4r^3 = 32$

$r^3 = 8$

$r = 2$

Then $SA = 4\pi r^2 = 4\pi \times 2^2 = 16\pi \text{m}^2$

4 a $V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 4^2 \times 10$

$= \frac{160\pi}{3} = 168\text{cm}^3$

b $V_{tr} = V_{cone} - V_{cut}$

$V_{cut} = \frac{1}{3}\pi r_{cut}^2 h_{cut} = \frac{1}{3}\pi \times 2^2 \times (10-6)$

$= \frac{16\pi}{3} = 16.8\text{cm}^3$

$V_{tr} = 168 - 16.8 = 151\text{cm}^3$

5 a $d = 2r$

$65 = 2r$

$r = \frac{65}{2}$

$r = 32.5\text{mm} = 3.25 \text{ cm}$

$V = \pi r^2 h = \pi \times 3.25^2 \times 39 = 1294.14\text{cm}^3$

b Each ball has a diameter of $2 \times 3.25 = 6.5\text{cm}$

$\frac{h}{6.5} = \frac{39}{6.5} = 6$

6 tennis balls fit in the cylinder

c $V_{air} = V_{cyl} - 6V_{ball}$

$V_{air} = 1294.14 - 6\left(\frac{4}{3}\pi \times 3.25^3\right)$

$= 431.38.. = 431\text{cm}^3$ (3 s.f.)

d $431.38..\text{cm}^3 \times \frac{1\text{m}^3}{1\,000\,000\text{cm}^3}$

$= 0.431 \times 10^{-3} \text{ m}^3$

6 a $V_T = V_{cyl} + V_{sph}$

$r = \frac{d}{2} = \frac{3}{2} = 1.5\text{m}$

$V_T = \pi r^2 h + \frac{4}{3}\pi r^3$

$= \pi \times 1.5^2 \times 8.5 + \frac{4}{3}\pi \times 1.5^3 = 74.2\text{m}^3$

b $SA = h \times 2\pi r + 4\pi r^2$

$= 8.5 \times 2\pi \times 1.5 + 4\pi \times 1.5^2 = 108\text{cm}^2$

7 a $V = \frac{1}{3}(\textit{base area} \times \textit{height})$

$= \frac{1}{3} \times 6 \times 6 \times 3 = 36 \text{ cm}^3$

b $W = 12 \times V = 12 \times 36 = 432\text{grams}$

c $OC = \frac{1}{2}AC$

$AC = \sqrt{AB^2 + BC^2} = \sqrt{6^2 + 6^2} = 8.49\text{cm}$

Then $OC = \frac{1}{2}AC = \frac{1}{2} \times 8.49 = 4.24\text{cm}$

Then

$VC = \sqrt{VO^2 + OC^2}$

$= \sqrt{3^2 + 4.24^2} = 5.20\text{cm}$

as required.

d We split the triangle BVC into two right-angles triangles, BVM and MVC, M is the midpoint of BC.

Then $\frac{\sin BMV}{VB} = \frac{\sin BVM}{BM}$

$\frac{\sin 90°}{5.20} = \frac{\sin BVM}{3}$

$\sin BVM = 3\frac{\sin 90°}{5.20}$

$BVM = \sin^{-1} 3\frac{\sin 90°}{5.20} = 35.2°$

$2BMV = BVC$

so $BVC = 2 \times 35.2 = 70.5°$

e Slant height

$VM = \sqrt{VB^2 - BM^2}$

$= \sqrt{5.20^2 - 3^2} = 4.25\text{cm}$

$SA = x^2 + 2xl$

$= 6^2 + 2 \times 6 \times 4.25 = 87.0\text{cm}^2$

8 a $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$d = \sqrt{(1-1)^2 + (5-0)^2 + (3-3)^2}$

$d = 5$

b Midpoint

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{1+1}{2}, \frac{5+0}{2}, \frac{3+3}{2}\right) = (1{,}2.5{,}3)$

c $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$d = \sqrt{(7-1)^2 + \left(\sqrt{15} - 0\right)^2 + (10-3)^2}$

$d = \sqrt{6^2 + 15 + 7^2} = 10$

9 $Area = \frac{1}{2}ab\sin C$

$Area = \frac{1}{2} \times 6 \times 4\sin 30°$

$= 12\sin 30° = \frac{12}{2} = 6\,\text{cm}^2$

10 a $r = \frac{d}{2} = \frac{16}{2} = 8\,\text{cm}$

$V_B = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 8^3 = 2144.66\,\text{cm}^3$

Then

$V_T = 80V_B = 80 \times 2144.66$

$= 171573\,\text{cm}^3$

b $V_{cone} = \frac{1}{3}\pi r^2 h = 171573\,\text{cm}^3$

Then $\frac{1}{3}\pi \times 40^2 h = 171573$

$h = 3\frac{171573}{1600\pi} = 102\,\text{cm}$

11 a $\tan 60° = \frac{CD}{10}$

$CD = 10\tan 60° = 10\sqrt{3}$ m

b $\tan 30° = \frac{CD}{OA}$

$OA = \frac{10\sqrt{3}}{\tan 30°} = 30\text{m}$

$AB = OA - OB = 30 - 10 = 20\text{m}$

12 4 faces, equilateral triangles

$h = \sqrt{6^2 - 3^2} = 3\sqrt{3}$

$A = \frac{1}{2}bh = \frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}$

$A_T = 4 \times 9\sqrt{3} = 36\sqrt{3}\,\text{cm}^2$

13 a $A = 2(x)(3x) + 2xh + 2(3x)h$

$= 6x^2 + 8xh$

as required.

b $6x^2 + 8xh = 600$

$8xh = 600 - 6x^2$

$h = \frac{600 - 6x^2}{8x} = \frac{300 - 3x^2}{4x}$

c $V = base\,area \times height = (3x)(x)h$

$V = 3x^2\frac{300 - 3x^2}{4x}$

$= \frac{3}{4}x\left(300 - 3x^2\right)$

$= \frac{9}{4}x\left(100 - x^2\right)$

14 Distance at angle $45°$

$\tan 45° = \frac{70}{OA}$

$OA = \frac{70}{\tan 45°} = 70\text{m}$

distance at angle $10°$

$\tan 10° = \frac{70}{OB}$

$OB = \frac{70}{\tan 10°} = 397\text{m}$

$\frac{\Delta d}{\Delta t} = \frac{397 - 70}{5} = 65.4 \text{ m / min}$

$65.4\frac{\text{m}}{\text{min}} \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{60 \text{ min}}{1 \text{ h}} = 3.92 \text{ km / h}$

15 We can form a triangle with the bottom of the building, O

$ACO = 90° - 32° = 58°$

$\tan 15° = \frac{x}{CO}$

$CO = \frac{x}{\tan 15°}$

and $\tan 58° = \frac{CO}{80 + x}$

$\tan 58° = \frac{\frac{x}{\tan 15°}}{80 + x} = \frac{x}{(80 + x)\tan 15°}$

$\tan 58°\tan 15°(80 + x) = x$

$x(\tan 58°\tan 15° - 1) = -80\tan 58°\tan 15°$

$x = \frac{-80\tan 58°\tan 15°}{\tan 58°\tan 15° - 1} = 60.1\text{m}$

16 a $\cos\theta = \frac{8^2 + 7^2 - 6^2}{2 \times 8 \times 7} = 0.6875$

$\theta = \cos^{-1} 0.6875 = 46.6°$

b $A = \frac{1}{2}ab\sin C = \frac{1}{2} \times 8 \times 7 \times \sin 46.6°$

$= 20.3\,\text{cm}^2$

17 a $CB^2 = AC^2 + AB^2 - 2 \times AC \times AB \times \cos BAC$

$CB^2 = 15^2 + 34^2 - 2 \times 15 \times 34 \times \cos 25°$

$CB = 21.4\,\text{m}$

b $ACB = 180° - 25° - 85° = 70°$

Then $\dfrac{\sin ACB}{AB} = \dfrac{\sin ABC}{AC}$

$\dfrac{\sin 70^\circ}{34} = \dfrac{\sin 85^\circ}{AC}$

$AC = 34\dfrac{\sin 85^\circ}{\sin 70^\circ} = 36.0\text{m}$

c $A = \dfrac{1}{2} \times AB \times AC \times \sin BAC$

$= \dfrac{1}{2} \times 34 \times 36.0 \times \sin 25^\circ = 259\text{m}^2$

d $ACB = 180^\circ - 70^\circ = 110^\circ$

and so $ABC = 180^\circ - 110^\circ - 25^\circ = 45^\circ$

$\dfrac{\sin ACB}{AB} = \dfrac{\sin ABC}{AC}$

$\dfrac{\sin 110^\circ}{34} = \dfrac{\sin 45^\circ}{AC}$

$AC = 34\dfrac{\sin 45^\circ}{\sin 110^\circ} = 25.6\text{m}$

18 a $\dfrac{\sin QRS}{QS} = \dfrac{\sin QSR}{QR}$

$\dfrac{\sin 42^\circ}{QS} = \dfrac{\sin 85^\circ}{30}$

$QS = 30\dfrac{\sin 42^\circ}{\sin 85^\circ} = 20.2\text{m}$

b $SQR = 180^\circ - 85^\circ - 42^\circ = 53^\circ$

$Area = \dfrac{1}{2} \times QS \times QR \times \sin SQR$

$Area = \dfrac{1}{2} \times 20.2 \times 30 \times \sin 53^\circ = 242\text{m}^2$

c $Area = \dfrac{1}{2} \times PQ \times QS \sin\theta = 141$

$\sin\theta = \dfrac{242}{12 \times 20.2}$

$\theta = \sin^{-1}\dfrac{242}{12 \times 20.2} = 86.7^\circ$

and the obtuse angle,
$180^\circ - 86.7^\circ = 93.3^\circ$

d We choose the obtuse angle, $\theta = 93.3^\circ$

$PS^2 = PQ^2 + QS^2 - 2 \times PQ \times QS \times \cos\theta$

$PS^2 = 12^2 + 20.2^2$
$\quad - 2 \times 12 \times 20.2 \times \cos 93.3^\circ$

$PS = 24.1\text{m}$

19 a $ABC = 360^\circ - (180^\circ - 100^\circ) - 150^\circ$

$= 130^\circ$

b $\dfrac{\sin ABC}{AC} = \dfrac{\sin BCA}{AB}$

$\dfrac{\sin 130^\circ}{100} = \dfrac{\sin BCA}{70}$

$\sin BCA = 70\dfrac{\sin 130^\circ}{100}$

$BCA = \sin^{-1} 70\dfrac{\sin 130^\circ}{100} = 32.4^\circ$

Then the bearing is given by

$360^\circ - 32.4^\circ - (180^\circ - 150^\circ) = 298^\circ$

c $CAB = 180^\circ - 130^\circ - 32.4^\circ = 17.6^\circ$

$BC^2 = AC^2 + AB^2 - 2 \times AC \times AB \times \cos CAB$

$BC^2 = 100^2 + 70^2$
$\quad - 2 \times 100 \times 70 \times \cos 17.6^\circ$

$BC = 39.4\text{km}$

20 a $PQA = 180^\circ - 95^\circ = 85^\circ$

$QPA = 180^\circ - 85^\circ - 26.5^\circ = 68.5^\circ$

b $\dfrac{\sin PAQ}{PQ} = \dfrac{\sin QPA}{QA}$

$\dfrac{\sin 26.5^\circ}{PQ} = \dfrac{\sin 68.5^\circ}{119}$

$PQ = 119\dfrac{\sin 26.5^\circ}{\sin 68.5^\circ} = 57.1\text{m}$

c $\dfrac{\sin QPA}{QA} = \dfrac{\sin PQA}{PA}$

$\dfrac{\sin 68.5^\circ}{119} = \dfrac{\sin 85^\circ}{PA}$

$PA = 119\dfrac{\sin 85^\circ}{\sin 68.5^\circ} = 127.4\text{m}$

$\sin PAG = \dfrac{PG}{PA}$

$\sin 26.5^\circ = \dfrac{PG}{127.4}$

$PG = 127.4\sin 26.5^\circ = 56.8\text{m}$

21 a $S = 4 \times \dfrac{5 \times 7}{2} + 5^2 = 95\ \text{cm}^2$ M1A1

b $h = \sqrt{7^2 - 2.5^2} = 6.54\ \text{cm}$ M1A1

$V = \dfrac{1}{3} \times 5^2 \times 6.538\ldots = 54.5\ \text{cm}^3$

M1A1

22 a $l = \sqrt{10^2 - 3^2} = 9.54\ m$ M1A1

b $S = 4 \times \dfrac{6 \times 9.538\ldots}{2}$

$= 114.47\ldots = 114\ \text{m}^2$ M1A1

c $h = 42 + \sqrt{9.538\ldots^2 - 3^2}$

$= 51.1\ \text{m}$ M1A1

d $\arccos\left(\dfrac{3}{10}\right) = 72.5^\circ$ M1A1

e $CP = 6\tan 60^\circ = 10.4$ m M1A1

23 a $l = \sqrt{5^2 + 3^2} = 5.83$ cm M1A1

$S = 2\times(\pi \times 3 \times 5.83\ldots) = 110$ cm^2

M1A1

b $\dfrac{2\times\frac{1}{3}\times\pi\times 3^2\times 5}{\pi\times 3.05^2\times 10.1}\times 100\% = 31.9\%$

M1A1

24 a $x = \dfrac{22-12}{2} = 5$ A1

$h = \sqrt{13^2 - 5^2} = 12$ M1A1

b $A = \dfrac{22+12}{2}\times 12 = 204$ cm^2 M1A1

c $C = 90^\circ + \sin^{-1}\left(\dfrac{5}{13}\right) = 112.6^\circ$ M1A1

d $AC = \sqrt{17^2 + 12^2} = 20.8$ cm M1A1

25 a $A\hat{B}C = 135^\circ$ A1

$AC = \sqrt{20^2 + 25^2 - 2\times 20\times 25\times\cos 135^\circ}$

M1A1

$AC = 41.6$ km A1

b $\dfrac{\sin\hat{C}}{20} = \dfrac{\sin 135^\circ}{41.61\ldots}$ M1

$\hat{C} = 19.9^\circ$ A1

Therefore the bearing of A from point C is $360 - 105 - 19.9 = 235.1^\circ$ M1A1

26 a $FC = \sqrt{8^2 + 10^2 + 6^2} = 10\sqrt{2}$

M1A1AG

b $M\left(\dfrac{0+8}{2},\dfrac{0+0}{2},\dfrac{6+6}{2}\right) = (4,0,6)$ M1A1

c $FM = \sqrt{4^2 + 0^2 + 6^2} = 2\sqrt{13}$ M1A1

d $CM = \sqrt{4^2 + 10^2 + 0^2} = 2\sqrt{29}$ A1

$p = 2\sqrt{13} + 2\sqrt{29} + 10\sqrt{2}$ cm M1A1

e $\tan M = \frac{6}{4} = \frac{3}{2}$ M1

$\cos M = \dfrac{1}{\sqrt{\frac{9}{4}+1}} = \dfrac{2\sqrt{13}}{13}$ M1A1AG

27 a $A = \dfrac{1}{2}\times 5\times 10\sin 30^\circ = \dfrac{25}{2}$ M1A1

b $BD^2 = 5^2 + 10^2 - 2\times 5\times 10\cos 30^\circ$

M1A1

$BD = \sqrt{125 - 50\sqrt{3}}$ A1

$BD = \sqrt{25\left(5 - 2\sqrt{3}\right)}$ M1

$BD = 5\sqrt{5 - 2\sqrt{3}}$ AG

c $\dfrac{\sin C\hat{D}B}{13} = \dfrac{\sin 45^\circ}{5\sqrt{5-2\sqrt{3}}}$ M1A1

$\sin C\hat{D}B = \dfrac{13\sqrt{2}}{10\sqrt{5-2\sqrt{3}}}$ A1

d The angle $C\hat{D}B$ can either be acute or obtuse A1

and the two possible values add up to 180°. A1

28 a $V = \dfrac{1}{2}\times\dfrac{4\pi}{3}\times 3^3 + \pi\times 3^2\times 7$ M1A1A1

$= 81\pi = 254$ cm^3 (3 s.f.) A1

b $S = \dfrac{1}{2}\times 2\pi\times 3^2 + 2\pi\times 3\times 7 + \pi\times 3^2$

M1A1

$= 60\pi = 188$ cm^2 (3 s.f.) A1

29 a $\tan 32^\circ = \dfrac{30}{x}$

$\Rightarrow x = \dfrac{30}{\tan 32^\circ} = 48.0$ metres M1A1A1

b $AB = y = \sqrt{(3+48.0)^2 + 30^2}$

$= 59.2$ metres M1A1

c $\arctan\left(\frac{30}{51.0}\right) = 30.5^\circ$ M1A1

30 a $BC = \sqrt{48^2 + 57^2 - 2\times 48\times 57\cos 117^\circ}$

$= 89.7$ metres M1A1A1

b $A = \dfrac{1}{2}\times 48\times 57\sin(117^\circ)$

$= 1219$ sq metres (3 s.f.) M1A1

c $\frac{\sin B}{48} = \frac{\sin 117^\circ}{89.7} \Rightarrow B = 28.5^\circ$ M1A1A1

12 Periodic relationships: trigonometric functions

Skills check

1 **a** $\frac{\sqrt{2}}{2}$ **b** $\sqrt{3}$ **c** $\frac{\sqrt{3}}{2}$

2 **a** $(-0.618,0),(1,0),(1.62,0)$

b $(0.633,0)$

3 **a** $(-1.61,0.199)$

b $(2.21,0.792)$

Exercise 12A

1 **a** $45° = \frac{45\pi}{180} = \frac{\pi}{4}$

b $60° = \frac{60\pi}{180} = \frac{\pi}{3}$

c $270° = \frac{270\pi}{180} = \frac{3\pi}{2}$

d $360° = \frac{360\pi}{180} = 2\pi$

e $18° = \frac{18\pi}{180} = \frac{\pi}{10}$

f $225° = \frac{225\pi}{180} = \frac{5\pi}{4}$

g $80° = \frac{80\pi}{180} = \frac{4\pi}{9}$

h $200° = \frac{200\pi}{180} = \frac{10\pi}{9}$

i $120° = \frac{120\pi}{180} = \frac{2\pi}{3}$

j $135° = \frac{135\pi}{180} = \frac{3\pi}{4}$

2 **a** $\frac{\pi}{6} = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$

b $\frac{\pi}{10} = \frac{\pi}{10} \times \frac{180°}{\pi} = 18°$

c $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$

d $3\pi = 3\pi \times \frac{180°}{\pi} = 540°$

e $\frac{7\pi}{20} = \frac{7\pi}{20} \times \frac{180°}{\pi} = 63°$

f $\frac{4\pi}{5} = \frac{4\pi}{5} \times \frac{180°}{\pi} = 144°$

g $\frac{7\pi}{4} = \frac{7\pi}{4} \times \frac{180°}{\pi} = 315°$

h $\frac{14\pi}{9} = \frac{14\pi}{9} \times \frac{180°}{\pi} = 280°$

i $\frac{5\pi}{3} = \frac{5\pi}{3} \times \frac{180°}{\pi} = 300°$

j $\frac{13\pi}{4} = \frac{13\pi}{4} \times \frac{180°}{\pi} = 585°$

3 **a** $10° = \frac{10\pi}{180} \approx 0.175$

b $40° = \frac{40\pi}{180} \approx 0.698$

c $25° = \frac{25\pi}{180} \approx 0.436$

d $300° = \frac{300\pi}{180} \approx 5.24$

e $110° = \frac{110\pi}{180} \approx 1.92$

f $75° = \frac{75\pi}{180} \approx 1.31$

g $85° = \frac{85\pi}{180} \approx 1.48$

h $12.8° = \frac{12.8\pi}{180} \approx 0.233$

i $37.5° = \frac{37.5\pi}{180} \approx 0.654$

j $1° = \frac{\pi}{180} \approx 0.0175$

4 **a** $1^c = 1 \times \frac{180°}{\pi} = 57.3°$

b $2^c = 2 \times \frac{180°}{\pi} = 115°$

c $0.63^c = 0.63 \times \frac{180°}{\pi} = 36.1°$

d $1.41^c = 1.41 \times \frac{180°}{\pi} = 80.8°$

e $1.55^c = 1.55 \times \frac{180°}{\pi} = 88.8°$

f $3^c = 3 \times \frac{180°}{\pi} = 172°$

g $0.36^c = 0.36 \times \frac{180°}{\pi} = 20.6°$

h $1.28^c = 1.28 \times \frac{180°}{\pi} = 73.3°$

i $0.01^c = 0.01 \times \frac{180°}{\pi} = 0.573°$

j $2.15^c = 2.15 \times \frac{180°}{\pi} = 123°$

Exercise 12B

1 a i $l = r\theta = 14 \times \frac{\pi}{2} = 7\pi\,\text{cm}$

ii $l = r\theta = 12 \times \frac{3\pi}{4} = 9\pi\,\text{cm}$

iii $l = r\theta = 3 \times \frac{5\pi}{6} = \frac{5\pi}{2}\text{cm}$

iv $l = r\theta = 15 \times \frac{14\pi}{9} = \frac{70\pi}{3}\text{cm}$

b i $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 14^2 \times \frac{\pi}{2} = 49\pi\,\text{cm}^2$

ii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 12^2 \times \frac{3\pi}{4} = 54\pi\,\text{cm}^2$

iii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{5\pi}{6} = \frac{15\pi}{4}\text{cm}^2$

iv $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 15^2 \times \frac{14\pi}{9} = 175\pi\,\text{cm}^2$

2 $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times r^2 \times \frac{\pi}{12} = 3\pi$

$r^2 = 3\pi \times 2 \times \frac{12}{\pi} = 72$

$r = \sqrt{72} = 6\sqrt{2}\,\text{cm}$

3 a $A = \frac{1}{2}r^2\theta = 36\pi\,\text{cm}^2$

$\frac{1}{2} \times 12^2 \times \theta = 36\pi$

$\theta = \frac{36\pi \times 2}{144} = \frac{\pi}{2}$

b $l = r\theta = 12 \times \frac{\pi}{2} = 6\pi$

Then $P = 2r + l = 2 \times 12 + 6\pi = 42.8\,\text{m}$

4 Area of sector:

$A_s = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times 1.5 = 75$

Area of triangle:

$A_t = \frac{1}{2}r^2 \sin\theta = \frac{1}{2}10^2 \sin 1.5 = 49.9$

note that the angle is in radians

Area of shaded region:

$A = A_s - A_t = 75 - 49.9 = 25.1\,\text{units}^2$

5 l per second: $l = r\theta = 4 \times \frac{\pi}{12} = \frac{\pi}{3}$

60 times in a minute:

$60l = 20\pi\,\text{m}$

6 We have $\frac{1}{60}{}^\circ$ per minute, which in radians is $\frac{1}{60}{}^\circ = \frac{\pi}{60 \times 180} = \frac{\pi}{10800}$

Then $l = r\theta = 6371 \times \frac{\pi}{10800} = 1.85\,\text{km}$

Exercise 12C

1 a 130° is obtuse, hence we have a negative cosine

b 320° is obtuse, hence we have a negative sine

c $\tan 225^\circ = \frac{\sin 225^\circ}{\cos 225^\circ}$ is negative sine divided by negative cosine hence we have a positive tangent

2 a $\sin 36^\circ = \sin(180^\circ - 36^\circ) = \sin 144^\circ$

b $\sin 50^\circ = \sin(180^\circ - 50^\circ) = \sin 130^\circ$

c $\sin 85^\circ = \sin(180^\circ - 85^\circ) = \sin 95^\circ$

d $\sin 460^\circ = \sin(460^\circ - 360^\circ) = \sin 100^\circ$

e $\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{2\pi}{3}$

f $\sin\frac{\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{4\pi}{5}$

g $\sin\frac{2\pi}{7} = \sin\left(\pi - \frac{2\pi}{7}\right) = \sin\frac{5\pi}{7}$

h $\sin\frac{8\pi}{3} = \sin\left(\frac{8\pi}{3} - 2\pi\right) = \sin\frac{2\pi}{3}$

3 a $\cos 40^\circ = \cos(360^\circ - 40^\circ) = \cos 320^\circ$

b $\cos 110^\circ = \cos(360^\circ - 110^\circ) = \cos 250^\circ$

c $\cos 300^\circ = \cos(360^\circ - 300^\circ) = \cos 60^\circ$

d $\cos 500^\circ = \cos(500^\circ - 360^\circ) = \cos 140^\circ$

e $\cos\frac{\pi}{8} = \cos\left(2\pi - \frac{\pi}{8}\right) = \cos\frac{15\pi}{8}$

f $\cos\frac{\pi}{10} = \cos\left(2\pi - \frac{\pi}{10}\right) = \cos\frac{19\pi}{10}$

g $\cos\frac{3\pi}{2} = \cos\left(2\pi - \frac{3\pi}{2}\right) = \cos\frac{\pi}{2}$

h $\cos\frac{9\pi}{4} = \cos\left(\frac{9\pi}{4} - 2\pi\right) = \cos\frac{\pi}{4}$

4 a $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{5\pi}{6}$

angles $\frac{\pi}{6}, \frac{5\pi}{6}$

b $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$\cos\frac{\pi}{4} = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\frac{7\pi}{4}$

angles $\frac{\pi}{4}, \frac{7\pi}{4}$

c $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\tan\frac{\pi}{3} = \tan\left(\frac{\pi}{3}+\pi\right) = \tan\frac{4\pi}{3}$

angles $\frac{\pi}{3}, \frac{4\pi}{3}$

5 **a** $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \left(\frac{8}{17}\right)^2$

$\sin\theta = \sqrt{1-\left(\frac{8}{17}\right)^2} = \pm\frac{15}{17}$

We take the positive value for θ acute

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{15}{17}}{\frac{8}{17}} = \frac{15}{8}$

Exercise 12D

1 **a** $\theta = \cos^{-1}0.6 = 53.1°$

and $360° - 53.1° = 306.9°$

b $\theta = \sin^{-1}0.15 = 8.63°$

and $180° - 8.63° = 171.4°$

c $\theta = \tan^{-1}0.2 = 11.3°$

and $180° + 11.3° = 191.3°$

d $\theta = \tan^{-1}-0.76 = 322.8°$

and $322.8° - 180° = 142.8°$

e $\theta = \cos^{-1}-0.43 = 115.5°$

and $360° - 115.5° = 244.5°$

2 **a** $\theta = \sin^{-1}0.82 = 0.96$

and $\pi - 0.96 = 2.18$

b $\theta = \tan^{-1}-0.94 = 5.53$

and $5.53 - \pi = 2.39$

c $\theta = \cos^{-1}-0.94 = 2.79$

and $2\pi - 2.79 = 3.49$

d $\theta = \cos^{-1}0.77 = 0.69$

and $2\pi - 0.69 = 5.59$

e $\theta = \sin^{-1}-0.23 = 6.05$

and $\pi - 6.05 = -2.91 = 2\pi - 2.91 = 3.37$

3 $2\sin^2\theta + 5\sin\theta = 3$

$(\sin\theta + 3)(2\sin\theta - 1) = 0$

Then $\sin\theta + 3 = 0$

$\sin\theta = -3$

$\theta = \sin^{-1}-3$

This is outside of the domain for the sine function. Second equation gives us

$2\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

4 **a** $4\cos x = 3\sin x$

$\frac{\sin x}{\cos x} = \frac{4}{3}$

$\tan x = \frac{4}{3}$

$x = \tan^{-1}\frac{4}{3} = 0.93$

and $0.93 + \pi = 4.07$

b $2\sin x + \cos x = 0$

$2\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -\frac{1}{2}$

$\tan x = -\frac{1}{2}$

$x = \tan^{-1}-\frac{1}{2} = 5.82$ and 2.68

c $\tan^2 x - \tan x - 2 = 0$

$(\tan x - 2)(\tan x + 1) = 0$

$\tan x = 2$ and $\tan x = -1$

Then $x = \tan^{-1}2 = 1.107.. = 1.11$ (3 s.f.)

and $1.107 + \pi = 4.249.. = 4.25$ (3 s.f.)

and $x = \tan^{-1}(-1) = 5.50$

and $5.50 - \pi = 2.36$

d $2\cos^2 x + \sin x = 1$

$2(1-\sin^2 x) + \sin x = 1$

$-2\sin^2 x + \sin x + 1 = 0$

$-(\sin x - 1)(2\sin x + 1) = 0$

so $\sin x = 1$, then $x = 1.57$

and $\sin x = -\frac{1}{2}$, then $x = 5.76$ and 3.67

5 **a** $\cos\theta = 0.3$

$\theta = \cos^{-1}0.3 = 1.27$

and $2\pi - 1.27 = 5.02$

For $2\pi \le \theta \le 3\pi$

$2\pi + 1.27 = 7.55$

For $-\pi \le \theta \le 0$

$5.02 - 2\pi = -1.27$

b $\tan\theta = 1.61$

$\theta = \tan^{-1}1.61 = 1.01$

and $1.01 + \pi = 4.16$

For $2\pi \le \theta \le 4\pi$

$2\pi + 1.01 = 7.30$

$4.16 + 2\pi = 10.4$

c $\sin\theta = -2\cos\theta$

$\frac{\sin\theta}{\cos\theta} = -2$

$\tan\theta = -2$

$\theta = \tan^{-1} - 2 = -1.11$

$\pi - 1.11 = 2.03$

d $2\tan^2\theta + 5\tan\theta = 3$

$(\tan\theta + 3)(2\tan\theta - 1) = 0$

$\tan\theta = -3 \text{and} \tan\theta = \frac{1}{2}$

$\theta = \tan^{-1} - 3 \text{and} \theta = \tan^{-1}\frac{1}{2}$

$\theta = -1.25$

and $\theta = 0.46 = 0.46 - 2\pi = -5.82$

as well as $-1.25 - \pi = -4.39$
and $-5.82 + \pi = -2.68$

6 $3\cos x = 5\sin x$

$\frac{\sin x}{\cos x} = \frac{3}{5}$

$\tan x = \frac{3}{5}$

$x = \tan^{-1}\frac{3}{5} = 31°$

as well as $31° + 180° = 211°$

Exercise 12E

1 a $\cos^{-1}\frac{\sqrt{3}}{2} = 2x$

Because we have $2x$, and $-180° \le x \le 180°$, then we use $-360° \le 2x \le 360°$.

$2x = 30°, -30°, 330°, -330°$

Then $x = 15°, -15°, 165°, -165°$

b $\cos^{-1}\frac{\sqrt{3}}{2} = 3x$

Because we have $3x$, and $-180° \le x \le 180°$, then we use $-540° \le 3x \le 540°$.

$3x = 30°, -30°, 330°, -330°, 390°, -390°$

Then $x = 10°, -10°, 110°, -110°,$
$130°, -130°$

c $2\cos 3x - 1 = 0$

$\cos 3x = \frac{1}{2}$

$3x = \cos^{-1}\frac{1}{2}$

Because we have $3x$, and $-180° \le x \le 180°$, then we use $-540° \le 3x \le 540°$.

$3x = 60°, -60°, 300°, -300°, 420°, -420°$

Then $x = 20°, -20°, 100°, -100°,$
$140°, -140°$

d $3\tan\frac{x}{2} + 3 = 0$

$\tan\frac{x}{2} = -\frac{3}{3} = -1$

Because we have $\frac{x}{2}$, and

$-180° \le x \le 180°$, then we use

$-90° \le \frac{x}{2} \le 90°$.

Then $\frac{x}{2} = -45°$ and so $x = -90°$

2 a $\sin^{-1}\frac{\sqrt{3}}{2} = 3\theta$

Because we have 3θ, and $0 \le \theta \le 2\pi$, then we use $0 \le 3\theta \le 6\pi$.

$3\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}$

then $\theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$

b $\cos 3\theta - 1 = 0$

$3\theta = \cos^{-1}1$

Because we have 3θ, and $0 \le \theta \le 2\pi$, then we use $0 \le 3\theta \le 6\pi$.

$3\theta = 0, 2\pi, 4\pi, 6\pi$ then $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

c $\sin\frac{\theta}{2} = \frac{\sqrt{2}}{2}$

$\frac{\theta}{2} = \sin^{-1}\frac{\sqrt{2}}{2}$

Because we have $\frac{\theta}{2}$, and $0 \le \theta \le 2\pi$,

then we use $0 \le \frac{\theta}{2} \le \pi$

$\frac{\theta}{2} = \frac{\pi}{4}, \frac{3\pi}{4}$

Then $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

d $\sin^2\frac{2\theta}{3}-1=0$

$\sin\frac{2\theta}{3}=\pm1$

Because we have $\frac{2\theta}{3}$, and $0\le\theta\le 2\pi$,

then we use $0\le\frac{2\theta}{3}\le\frac{4\pi}{3}$.

$\frac{2\theta}{3}=\sin^{-1}\pm1=\frac{\pi}{2}$

Then $\theta=\frac{\pi}{2}\times\frac{3}{2}=\frac{3\pi}{4}$

Exercise 12F

1 a $2\sin 5\cos 5=\sin 2\times 5=\sin 10$ by the double angle formula

b $2\sin\frac{\pi}{2}\cos\frac{\pi}{2}=\sin 2\times\frac{\pi}{2}=\sin\pi$ by the double angle formula

c $2\sin 4\pi\cos 4\pi=\sin 2\times 4\pi=\sin 8\pi$ by the double angle formula

d $\cos^2 0.4-\sin^2 0.4=\cos 2\times 0.4=\cos 0.8$ by the double angle formula

e $2\cos^2 6-1=\cos 2\times 6=\cos 12$ by the double angle formula

f $1-2\sin^2\frac{\pi}{4}=\cos 2\times\frac{\pi}{4}=\cos\frac{\pi}{2}$ by the double angle formula

2 a We use the Pythagorean identity

$\sin^2\theta+\cos^2\theta=1$

$\left(\frac{1}{3}\right)^2+\cos^2\theta=1$

$\cos^2\theta=1-\left(\frac{1}{3}\right)^2=1-\frac{1}{9}=\frac{8}{9}$

We take only the positive value as θ is acute $\cos\theta=\frac{2\sqrt{2}}{3}$

b $\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{1}{3}\times\frac{2\sqrt{2}}{3}=\frac{4\sqrt{2}}{9}$

c $\cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{8}{9}-\left(\frac{1}{3}\right)^2=\frac{7}{9}$

d $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{\frac{4\sqrt{2}}{9}}{\frac{7}{9}}=\frac{4\sqrt{2}}{7}$

3 a $\sin^2\theta+\cos^2\theta=1$

$\sin^2\theta+\left(-\frac{1}{2}\right)^2=1$

$\sin^2\theta=1-\frac{1}{4}=\frac{3}{4}$

$\sin\theta=\frac{\sqrt{3}}{2}$

b $\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{\sqrt{3}}{2}\times-\frac{1}{2}=-\frac{\sqrt{3}}{2}$

c $\cos 2\theta=\cos^2\theta-\sin^2\theta=\left(-\frac{1}{2}\right)^2-\frac{3}{4}=-\frac{1}{2}$

d $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=\sqrt{3}$

4 a θ is obtuse, so we take the negative value of the cosine

$\sin^2\theta+\cos^2\theta=1$

$\left(-\frac{1}{8}\right)^2+\cos^2\theta=1$

$\cos^2\theta=1-\frac{1}{64}=\frac{63}{64}$

$\cos\theta=-\frac{\sqrt{63}}{8}$

$\sin 2\theta=2\sin\theta\cos\theta$

$=2\times-\frac{1}{8}\times-\frac{\sqrt{63}}{8}=\frac{\sqrt{63}}{32}$

b $\cos 2\theta=\cos^2\theta-\sin^2\theta$

$=\left(-\frac{\sqrt{63}}{8}\right)^2-\left(-\frac{1}{8}\right)^2$

$=\frac{63}{64}-\frac{1}{64}=\frac{62}{64}=\frac{31}{32}$

c $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{\frac{\sqrt{63}}{32}}{\frac{31}{32}}=\frac{\sqrt{63}}{31}$

d $\sin 4\theta=2\sin 2\theta\cos 2\theta$

$=2\times\frac{\sqrt{63}}{32}\times\frac{31}{32}=\frac{31\sqrt{63}}{512}$

5 a $\sin 2\theta=\sin\theta$

this is true for $\theta=0,2\pi,\pi$

divide by $\sin\theta$

$2\cos\theta=1$

$\cos\theta=\frac{1}{2}$

$\theta=\cos^{-1}\frac{1}{2}=\frac{\pi}{3},\frac{5\pi}{3}$

b $\cos 2\theta + \sin\theta = 0$

$1 - 2\sin^2\theta + \sin\theta = 0$

$-(\sin\theta - 1)(2\sin\theta + 1) = 0$

$\sin\theta = 1$

$\theta = \sin^{-1} 1 = \frac{\pi}{2}$

and $\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1} - \frac{1}{2} = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

and $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

c $\sin 2\theta = \sqrt{3}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{3}\cos\theta$

this is true for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

divide by $\cos\theta$

$2\sin\theta = \sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

then $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

d $\cos\theta = \sin\theta\sin 2\theta$

$\cos\theta = \sin\theta 2\sin\theta\cos\theta$

true for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

divide by $\cos\theta$

$2\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

Then $\theta = \sin^{-1} \pm 1/\sqrt{2} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

e $\cos 2\theta = \cos\theta$

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(\cos\theta - 1)(2\cos\theta + 1) = 0$

$\cos\theta = 1$

$\cos^{-1} 1 = \theta = 0, 2\pi$

$2\cos\theta + 1 = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1} - \frac{1}{2} = \frac{2\pi}{3}, \frac{4\pi}{3}$

6 **a** $32\sin x\cos x$

$= 2 \times 16 \times \sin x\cos x = 16\sin 2x$

Then a=16, b=2

b $16\sin 2x = 8$

so $\sin 2x = \frac{8}{16} = \frac{1}{2}$

Note that $0 \le 2x \le 2\pi$, so

$\sin^{-1}\frac{1}{2} = 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

and so $x = \frac{\pi}{12}, \frac{5\pi}{12}$

7 $Area = \frac{1}{2} \times \sqrt{15}x\sin 2\theta = 10$

$\frac{1}{2}\sqrt{15}x \times 2 \times \sin\theta\cos\theta = 10$

$\sqrt{15}x\sin\theta\cos\theta = 10$

$\sqrt{15}x\sin\theta\left(\sqrt{1-\sin^2\theta}\right) = 10$

with $\sin\theta = \frac{1}{4}$ we have that

$\sqrt{15}x \times \frac{1}{4} \times \left(\sqrt{1-\left(\frac{1}{4}\right)^2}\right) = 10$

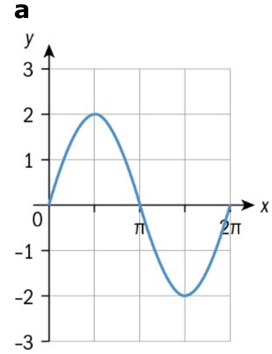
$\sqrt{15}x \times \frac{1}{4} \times \frac{\sqrt{15}}{4} = 10$

$\frac{15}{16}x = 10$

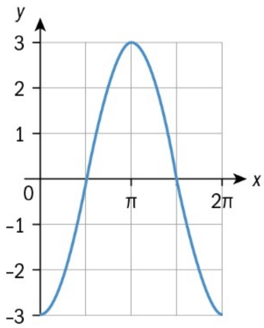
$x = \frac{160}{15} = \frac{32}{3}$

Exercise 12G

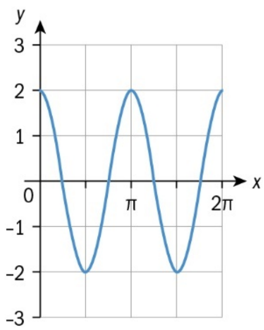
1 **a**



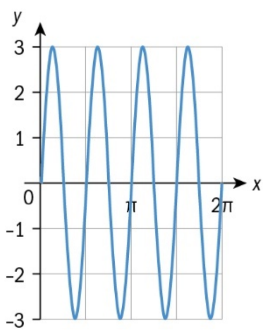
b



c



d



2 a The amplitude is $|1| = 1$ and the period is $\frac{2\pi}{3}$.

b The amplitude is $|0.5| = 0.5$ and the period is $\frac{2\pi}{2} = \pi$.

c The amplitude is $|-4| = 4$ and the period is $\frac{2\pi}{3}$.

d The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$ and the period is $\frac{2\pi}{\frac{1}{3}} = 6\pi$.

3 a period of π, amplitude of 1, then $y = \sin 2x$

b period of π, amplitude of 3, then $y = 3\cos 2x$

c period of 2π, amplitude of 2, then $y = -2\cos x$

d period of 2π, amplitude of 2, then $y = 2\sin(-x)$

4 a The amplitude is $|6| = 6$

b The period is $\frac{2\pi}{\pi/2} = 4$

Exercise 12H

1 a amplitude 3, period $\frac{2\pi}{2} = \pi$. Option iv

b amplitude 2, period 2π and vertical shift $+1$. Option ii

c amplitude $\frac{1}{2}$, horizontal shift $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ units to the right. Option i

d amplitude 1, period $\frac{2\pi}{1/2} = 4\pi$, vertical shift 2. Option iii

2 a period 2π, amplitude $\frac{max - min}{2} = \frac{5-1}{2} = 2$, vertical shift $+3$

, $y = 2\sin x + 3$

b period 2π, amplitude $\frac{max - min}{2} = \frac{2-0}{2} = 1$, vertical shift $+1$

horizontal shift π, $y = \cos(x - \pi) + 1$

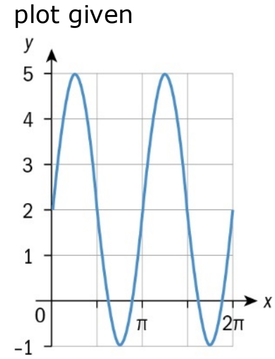
c period 2π, amplitude $\frac{max - min}{2} = \frac{0+4}{2} = 2$, vertical shift -2

$y = 2\cos x - 2$

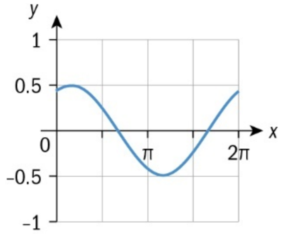
d period $\frac{2\pi}{3}$, amplitude $\frac{max - min}{2} = \frac{-0.5+1.5}{2} = 0.5$,

vertical shift -1, $y = 0.5\sin 3x - 1$

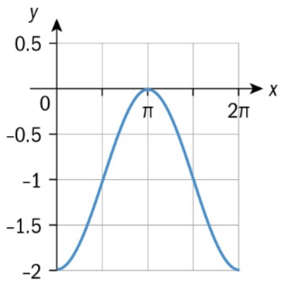
3 a vertical shift 2, amplitude 3, period π, plot given



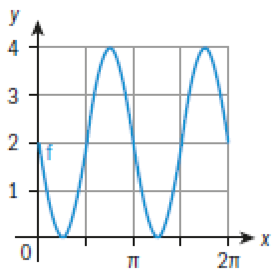
b horizontal shift $-\frac{\pi}{3}$, amplitude 0.5, period 2π, plot given



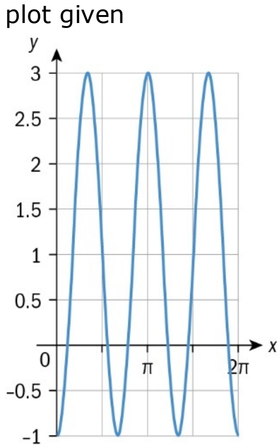
c vertical shift -1, amplitude 1, period 2π, horizontal shift $-\pi$, plot given



d vertical shift 2, amplitude 2, period π, plot given



e vertical shift 1, amplitude 2, period $\frac{2\pi}{3}$, horizontal shift $-\pi$, plot given



4 $(-0.824, 0), (0.824, 0)$

5 Plot $2\sin x - x - 1 = y$. Find zero at $(-2.38, 0)$

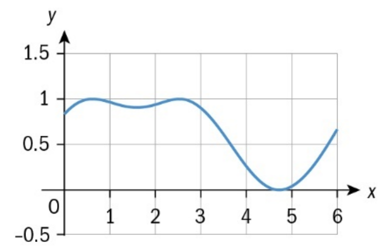
6 Plot $\sin x - \frac{1}{2} = y$, find zero at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

7 Plot $e^x - \cos x = y$ between -2 and -1. Find zero at $(-1.29, 0)$

8 **a** c is the horizontal shift: $\frac{\pi}{2}$

b The graph of $y = \cos x$ may be translated $\frac{\pi}{2}$ horizontally to the right to become the graph of $y = \sin x$

9 **a**



b $0.6075, 1.571, 2.534, 4.712$

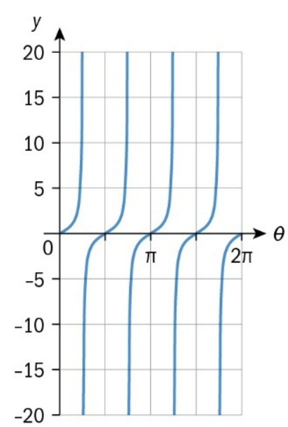
10 a $(2.36, -1)$

b $\frac{2\pi}{\frac{1}{2}} = 4\pi$

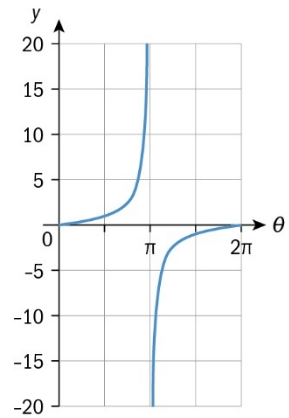
c $x = 0, 1.26, 3.77, 4.19$

Exercise 12I

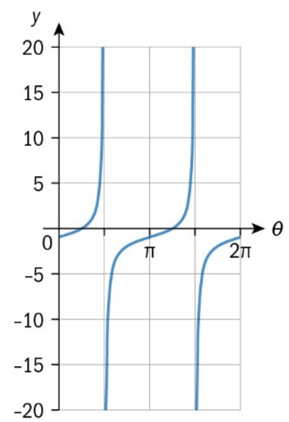
1 **a**



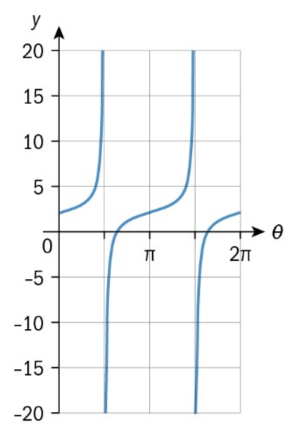
b



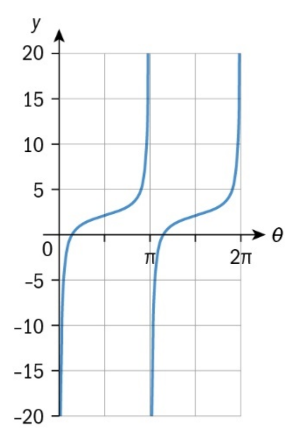
c



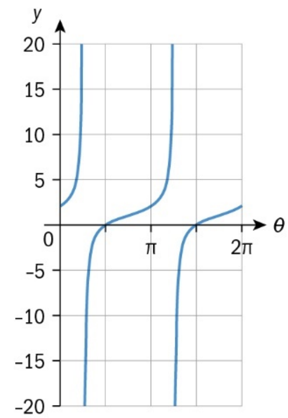
d



e



f



2 **a** $1.25, 4.39$

b $1.13, 4.53$

c $-\pi, 0, \pi$

d $-0.903, 0.677, 1.98, 2.61$

Exercise 12J

1 **a** Minimum when $\sin\left(\frac{\pi}{6}(t-5)\right) = -1$, i.e.

$h(t) = -5 + 7 = 2$ m

Maximum when $\sin\left(\frac{\pi}{6}(t-5)\right) = 1$, i.e.

$h(t) = 5 + 7 = 12$ m

b $\sin\left(\frac{\pi}{6}(t-5)\right) = 1$

$$\left(\frac{\pi}{6}(t-5)\right) = \frac{\pi}{2} \Rightarrow t = 8$$

High tide at 8 am

c $\sin\left(\frac{\pi}{6}(t-5)\right) = -1$

$$\left(\frac{\pi}{6}(t-5)\right) = -\frac{\pi}{2} \Rightarrow t = 2$$

Low tide at 2 am

d $h(9) = 5\sin\left(\frac{\pi}{6}(9-5)\right) + 7$

$$= 5\sin\frac{2\pi}{3} + 7$$

$$= \frac{5\sqrt{3}}{2} + 7 = 11.3 \text{ m}$$

e $h(t) = 5\sin\left(\frac{\pi}{6}(t-5)\right) + 7 = 3$

$\frac{\pi}{6}(t-5) = \sin^{-1}\left(-\frac{4}{5}\right)$

$\frac{\pi}{6}(t-5) = -\pi + \sin^{-1}\left(\frac{4}{5}\right), -\sin^{-1}\left(\frac{4}{5}\right),$

$\pi + \sin^{-1}\left(\frac{4}{5}\right), 2\pi - \sin^{-1}\left(\frac{4}{5}\right)$

$t = 0.771, 3.229, 12.771, 15.229$

0:46 am, 3:14 am, 12:46 am, 3:14 pm (multiply decimals by 60 to convert the decimal number of hours into minutes)

2 **a** 13000 on February 1st

b 7000 on August 9th

c $3000\cos(0.5(4-1)) + 10000 = 10212$

3 **a** 20

b 10

c amplitude $\frac{20-0}{2} = 10$, vertical shift 10

period $\approx 4\pi$. Then $y = 10\sin 0.5x + 10$

d 16 fish

4 **a** 35 m

b 5 m

c $\frac{35-5}{2} = 15\text{m}$

d $a = 15, c = \textit{vertical shift} = 20$

e horizontal distance between maximums: $4 - 0 = 4$ (period)

f b is calculated as $b = \frac{2\pi}{4} = \frac{\pi}{2}$

g plot $y = 15\cos\frac{\pi}{2}t + 20$. Find $y = 30$ at

$t = 0.535$

5 **a** radius=12 m, so diameter=24 m

maximum is at 2m + 24 m = 26 m above the ground

minimum is at 2 m above the ground

amplitude $\frac{26-2}{2} = 12$, increases then

decreases, so $a = -12$

$c = \frac{\max + \min}{2} = \frac{26+2}{2} = 14\text{m}$

$b = \frac{2\pi}{40} = \frac{\pi}{20}$

$h(t) = -12\cos\frac{\pi}{20}x + 14$

b At $t = p$, $-12\cos\frac{\pi}{20}p + 14 = 20$

so $-12\cos\frac{\pi}{20}p = 6$, i.e. $\cos\frac{\pi}{20}p = -\frac{1}{2}$.

Then the required angle is $\theta = \frac{2\pi}{3}$

c $\frac{\pi}{20}p = \frac{2\pi}{3}$ then $p = \frac{40}{3} = 13.3\text{s}$

6 **a** $\max = 20 \times 2 + 1 = 41\text{m}$, $\min = 1\text{m}$.
Then amplitude is 20, $a = -20$

vertical shift =

$\frac{\max + \min}{2} = \frac{41+1}{2} = 21\text{m}$

period $\frac{2\pi}{40} = \frac{\pi}{20}$

$h(t) = -20\cos\frac{\pi}{20}x + 21$

b for $h = 23m = -20\cos\frac{\pi}{20}x + 21$,

$\cos\frac{\pi}{20}x = \frac{23-21}{-20} = -\frac{1}{10}$

$\frac{20}{\pi}\cos^{-1} - \frac{1}{10} = x = 10.64s$

7 **a** amplitude $\frac{60-40}{2} = 10$

period $2 \times (1.8 - 0.3) = 3$, then $b = \frac{2\pi}{3}$

vertical shift $\frac{60+40}{2} = 50$

maximum at 0.3 instead of 0, so horizontal shift of 0.3

$y = 50 + 10\cos\left(\frac{2\pi}{3}(t-0.3)\right)$

b

$y = 50 + 10\cos\left(\frac{2\pi}{3}(17.2-0.3)\right) = 43.3\text{m}$

c Plotting the function and the line $y = 59$ gives $t = 0.0847$ s

Chapter review

1 **a** $30 \times \frac{\pi}{180} = \frac{\pi}{6}$

b $150 \times \frac{\pi}{180} = \frac{5\pi}{6}$

c $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$

d $120 \times \frac{\pi}{180} = \frac{2\pi}{3}$

e $-20 \times \frac{\pi}{180} = -\frac{\pi}{9}$

f $-240 \times \frac{\pi}{180} = -\frac{4\pi}{3}$

g $-270 \times \frac{\pi}{180} = -\frac{3\pi}{2}$

h $144 \times \frac{\pi}{180} = \frac{4\pi}{5}$

2 a $\frac{3\pi}{2} \times \frac{180}{\pi} = 270°$

b $\frac{7\pi}{6} \times \frac{180}{\pi} = 210°$

c $-\frac{7\pi}{12} \times \frac{180}{\pi} = -105°$

d $\frac{\pi}{9} \times \frac{180}{\pi} = 20°$

e $\frac{7\pi}{3} \times \frac{180}{\pi} = 420°$

f $-\frac{11\pi}{30} \times \frac{180}{\pi} = -66°$

g $\frac{11\pi}{6} \times \frac{180}{\pi} = 330°$

h $\frac{34\pi}{15} \times \frac{180}{\pi} = 408°$

3 a We have that $r\theta = l$ so

$6 = 5 \times \theta$

$\theta = \frac{6}{5}$ rad

b $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 25 \times \frac{6}{5} = 15$

4 a We use the Pythagorean identity

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + 0.6^2 = 1$

$\cos^2\theta = 1 - 0.36 = 0.64$

$\cos\theta = \pm 0.8$

We take the positive value for acute θ

$\cos\theta = 0.8$

and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75$

b $2\sin x = \tan x$

holds for $x = 0$. We divide by $\tan x$ to get

$2\cos x = 1$

$\cos x = \frac{1}{2}$

holds for $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$

5 a $\sin^{-1} -\frac{1}{2} = \pi + \frac{\pi}{6}$ and$2\pi - \frac{\pi}{6}$

So $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ and $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

c $\tan^{-1} 1 = \frac{\pi}{4}$ and $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

6 a $8\sin x\cos x = 4 \times 2\sin x\cos x = 4\sin 2x$

so $a = 4$ and $b = 2$

b $4\sin 2x = 2$

$2\sin 2x = 1$

$\sin 2x = \frac{1}{2}$

$\sin^{-1}\frac{1}{2} = 2x$

Note that $0 \le 2x \le 4\pi$, so

$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

so $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

7 a the angle is obtuse, so we need a positive sine

$\left(-\frac{12}{13}\right)^2 + \sin^2\theta = 1$

$\sin^2\theta = 1 - \frac{144}{169} = \frac{25}{169}$

$\sin\theta = \frac{5}{13}$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{119}{169}$

c $\sin(\theta + \pi) = -\sin\theta = -\frac{5}{13}$

8 a $2\sin^2\theta + \sin\theta - 1 = 0$

$(\sin\theta + 1)(2\sin\theta - 1)$

Then $\sin\theta = -1$

and $\sin\theta = \frac{1}{2}$

b $\sin^{-1} -1 = \frac{3\pi}{2}$

and $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$

9 a $\left(\frac{3}{4}\right)^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{9}{16} = \frac{7}{16}$

Obtuse angle, so we take the negative cosine $\cos\theta = \frac{-\sqrt{7}}{4}$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{\sqrt{7}}{4}\right)^2 - \left(\frac{3}{4}\right)^2 = -\frac{1}{8}$$

10 period: $\frac{\pi}{2}$, amplitude $= \frac{30+30}{2} = 30$.

Then $b = \frac{2\pi}{\frac{\pi}{2}} = 4$ and $a = 30$

11 a period is $\pi - 0 = \pi$

b 3

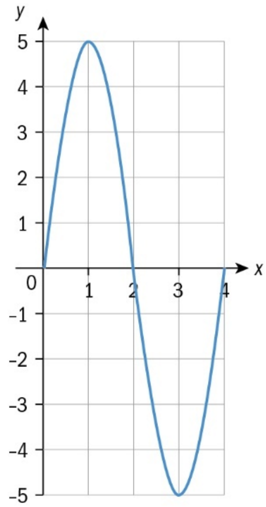
c 3

d $b = \frac{2\pi}{\pi} = 2$

12 a 5

b $\frac{2\pi}{\frac{\pi}{2}} = 4$

c



13 $A_{shaded} = A_{sector} - A_{triangle}$

$A_{shaded} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$

$A_{shaded} = \frac{1}{2} \times 8^2 \times \frac{\pi}{6} - \frac{1}{2} \times 8^2 \times \sin\frac{\pi}{6}$

$= 0.755$

14 a $A = \frac{1}{2}\sqrt{2} \times 4 \times \sin\theta = 2$

$2\sqrt{2}\sin\theta = 2$

$\sin\theta = \frac{1}{\sqrt{2}} = \frac{3\pi}{4}$

for obtuse θ

b Then $CBD = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

and so $A_{BDC} = \frac{1}{2} \times 4^2 \times \frac{\pi}{4} = 2\pi$

15 a Plot $y = \cos x - x^2$ and find zeros for $0 \le x \le \frac{\pi}{2}$, then $x = 0.824$

b Plot $y = 4\sin \pi x - 4e^{-x} + 3$ for $0.5 \le x \le 1.5$ and find zero at $x = 1.14$

16 $A = \frac{1}{2}ab\sin\theta = \frac{1}{2} \times 10 \times 8 \times \sin\theta = 10$

$\sin\theta = \frac{1}{4}$

$\theta = \sin^{-1}\frac{1}{4} = 0.25$

Obtuse angle and positive sine,

$\pi - 0.25 = 2.89$

17 Area of the sector is

$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times 0.8 = 40$

Then $O\hat{A}N = \pi - 0.8 = 0.77$

Using the sine rule $\frac{\sin A\hat{N}O}{AO} = \frac{\sin O\hat{A}N}{ON}$

$\frac{\sin\frac{\pi}{2}}{10} = \frac{\sin 0.77}{ON}$

$ON = 10\sin 0.77 = 6.96$

Then $A_{triangle} = \frac{1}{2} \times 10 \times 6.96 \times \sin 0.8 = 25$

Then

$A_{shaded} = A_{sect} - A_{triangle}$

$= 40 - 25 = 15\text{cm}^2$

18 a Using the sine rule $\frac{\sin ADO}{AO} = \frac{\sin AOD}{AD}$

$AD = AO\frac{\sin AOD}{\sin ADO} = 8\frac{\sin 0.8}{\sin 0.4} = 14.7\text{cm}$

b $DAO = \pi - ADO - AOD$

$= \pi - 0.8 - 0.4 = 1.94$

$\frac{\sin DAO}{OD} = \frac{\sin ADO}{AO}$

$OD = AO\frac{\sin DAO}{\sin ADO} = 8\frac{\sin 1.94..}{\sin 0.4} = 19.1\text{cm}$

c $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 8^2 \times 0.8 = 25.6\text{cm}^2$

d $A_{triangle} = \frac{1}{2} \times AD \times OD \times \sin ADO$

$= \frac{1}{2} \times 14.7 \times 19.1 \times \sin 0.4 = 54.9\text{cm}^2$

$A_{ABCD} = A_{triangle} - A_{sector}$

$= 54.9 - 25.6 = 29.3\text{cm}^2$

19 a $f(x) = 2\times 10\times \sin 3x\cos 3x = 10\sin 6x$

b $10\sin 6x = 0$

$\sin 6x = 0$

for $0 \le 6x \le 2\pi$. Then $6x = 0, \pi, 2\pi$

and $x = 0, \frac{\pi}{6}, \frac{\pi}{3}$

20 a $f(\theta) = (2\cos^2\theta - 1) + \cos\theta + 1$

$= 2\cos^2\theta + \cos\theta$

b This is a quadratic equation in $\cos\theta$ with positive discriminant, so there are 2 distinct values of $\cos\theta$ which satisfy it.

c $\cos\theta(2\cos\theta + 1) = 0$

$\cos\theta = 0$

$\Rightarrow \theta = -90°, 90°$

$2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = -120°, 120°$

21 a $l = r\theta = 15\times\frac{\pi}{6} = 7.85\text{m}$

b $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 15^2\times\frac{\pi}{6} = 58.9\ \text{m}^2$

c When wheel turns through $\frac{\pi}{2}$, A is 15 m above the ground.

Wheel then turns a further $\frac{\pi}{6}$ radians to complete a total turn of $\frac{2\pi}{3}$.

$h_A = 15 + 15\sin\frac{\pi}{6} = 15\times\frac{3}{2} = 22.5\text{ m}$

d $h\left(\frac{\pi}{4}\right) = 15 - 15\cos 2\left(\frac{\pi}{4} + \frac{\pi}{8}\right) = 25.6\text{m}$

e $h(0) = 15 - 15\cos 2\left(\frac{\pi}{8}\right) = 4.39\text{m}$

f When $\cos 2\left(t + \frac{\pi}{8}\right)$ is -1, we have a maximum. Then

$\cos 2\left(t + \frac{\pi}{8}\right) = -1$

$2\left(t + \frac{\pi}{8}\right) = \pi$

$t = \frac{\pi}{2} - \frac{\pi}{8} = 1.18\text{s}$

22 a $p = 3$, $q = \frac{2\pi}{\frac{3\pi}{2}} = \frac{4}{3}$, $b = \frac{2\pi}{\pi} = 2$

$-d = -\frac{2}{2} = -1$, so $d = 1$

b $3\cos\frac{4}{3}x = \sin 2x - 1$

The intersection points can be found as $x = 1.30, 3.41, 6.19$

23 a 3

b $10 - 2 = 8 =$period, $b = \frac{2\pi}{8} = \frac{\pi}{4}$

c $\frac{8+2}{2} = 5$

d $f(x) = 3\sin\frac{\pi}{4}x + 5$

translated (3,0)

$\hat{f}(x) = 3\sin\frac{\pi}{4}(x-3) + 5$

reflected in the x axis

$\hat{f}(x) = -(3\sin\frac{\pi}{4}(x-3) + 5) = g(x)$

24 a $A(t) = (2\cos t - 1)(\cos t - 1)$ A1A1

b $(2\cos t - 1)(\cos t - 1) = 0$

$2\cos t - 1 = 0 \Rightarrow \cos t = \frac{1}{2}$

$\Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$ M1A1

$\cos t - 1 = 0 \Rightarrow \cos t = 1$

$t = 0, 2\pi$ M1A1

25 $2\cos^2 x = \sin 2x$

$\Rightarrow 2\cos^2 x - 2\sin x\cos x = 0$ M1

$2\cos x(\cos x - \sin x) = 0$ M1

$\cos x = 0, \cos x = \sin x$ M1

$x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ A1

$x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$ A1

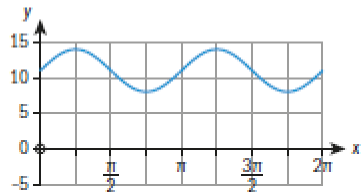
26 a Correct attempt to at least one parameter M1

$a = \frac{14-8}{2} = 3$ A1

$b = \frac{2\pi}{\pi} = 2$ A1

$c = \frac{14+8}{2} = 11$ A1

b



A1 for trigonometric scale and correct domain,

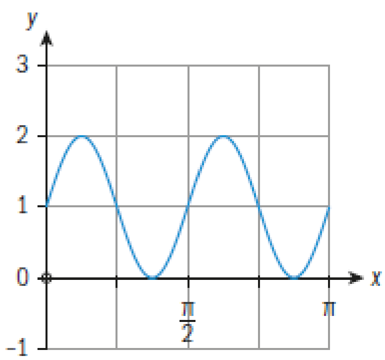
A1 for correct max/min,

A1 for two complete cycles

27 a $S(x) = \underbrace{\sin^2 2x + \cos^2 2x}_{} + \underbrace{2\sin 2x \cos 2x}_{}$ M1A1A1

$= 1 + \sin 4x$ AG

b A1 for correct shape, A1 for 2 cycles, A1 for correct max/min



c i $\frac{\pi}{2}$ A1

ii $0 \le y \le 2$ A1

d <Please insert the graph of $y = \cos(2x) - 1$ for $0 \le x \le 2\pi$ > A3

e i $\frac{1}{2}$ A1

ii $p = \pi$ (or any $\pi + 2n\pi, n \in ¢$)

A1

$q = -2$ A1

28 a i $A = \frac{1}{2} \times 2^2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$ A1

ii $l = 2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$ A1

b i $r\theta = \frac{\pi}{3}$ A1

$r^2 \frac{\theta}{2} = \pi$ A1

Solve simultaneously M1

$r = 6\ cm$ A1

ii $\theta = \frac{\pi}{18}$ A1

29 a i $-1 \le y \le 3$ A1

ii 2 A1

b $a = -2$ A1

$b = \frac{2\pi}{2} = \pi$ M1A1

$c = 1$ A1

c $-2\cos \pi x + 1 = 0 \Rightarrow \cos \pi x = \frac{1}{2}$ M1

$\pi x \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\right\}$ M1

$x \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right\}$ M1

30 a i $x = 0, x = \frac{\pi}{2}, x = \pi$ A1

ii $\frac{\pi}{2}$ A1

iii ¡ A1

b $b = 2$ A1

$d = 1$ A1

c The first point of inflexion occurs at $x = \frac{\pi}{4}$ R1

d $f\left(\frac{\pi}{8}\right) = -2$

$\Rightarrow f(x) = a\tan\left(2\left(\frac{\pi}{8} - \frac{\pi}{4}\right)\right) + 1 = -2$

M1

$a\underbrace{\tan\left(-\frac{\pi}{4}\right)}_{-1} = -3$ A1

$a = 3$ AG

e $\frac{\pi}{8}$ A1

$\frac{5\pi}{8}$ and $\frac{9\pi}{8}$ A1

31 a $-2\cos^2 x + \sin x + 3$

$= -2\left(1 - \sin^2 x\right) + \sin x + 3$ M1

$= 2\sin^2 x + \sin x + 1$ A1

b $-2\cos^2 x + \sin x + 3 = 2$

$\Rightarrow 2\sin^2 x + \sin x + 1 = 2$ M1

$2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$ M1

$\sin x = \dfrac{1}{2}, \sin x = -1$ A1

$x \in \left\{-\dfrac{11\pi}{6}, -\dfrac{\pi}{2}, -\dfrac{7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}\right\}$

A2

Award A1 for two correct solutions

13 Modelling change: more calculus

Skills Check

1 **a** $-\frac{\sqrt{2}}{2}$ **b** -1 **c** $-\frac{\sqrt{3}}{2}$

d $\frac{\sqrt{3}}{2}$ **e** 0 **f** $-\frac{1}{2}$

2 **a** $\cos^2 2x - \sin^2 2x = \cos^2 u - \sin^2 u$

$= \cos 2u = \cos 4x$

b $6\sin x\cos = 3\times 2\sin x\cos x = 3\sin 2x$

c $e^x\sin^2 x + e^x\cos^2 x = e^x(1) = e^x$

3 **a** $\frac{d}{dx}\left(\sqrt[3]{4x^3+7x}\right) = \frac{d}{dx}\left(\left(4x^3+7x\right)^{\frac{1}{3}}\right)$

$= \frac{1}{3}\left(12x^2+7\right)\left(4x^3+7x\right)^{-\frac{2}{3}} = \frac{12x^2+7}{3\left(4x^3+7x\right)^{\frac{2}{3}}}$

b $\frac{d}{dx}\left(3x^2 e^{2x}\right) = 6x\,e^{2x} + 6x^2 e^{2x}$

c $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{\frac{1}{x}x^2 - 2x\ln x}{x^4}$

$= \frac{x - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$

Exercise 13A

1 $f'(x) = 4(\cos x) + 3(-\sin x)$

$= 4\cos x - 3\sin x$

2 $\frac{dy}{dx} = 5\cos(5x)$

3 $\frac{dy}{dx} = \frac{1}{3}\cos\frac{x}{3} - (-3\sin(3x))$

$= \frac{1}{3}\cos\frac{x}{3} + 3\sin(3x)$

4 $f(x) = \frac{3}{\cos x} = 3(\cos x)^{-1}$

$f'(x) = 3(-\sin x)(-1)(\cos x)^{-2} = \frac{3\sin x}{\cos^2 x}$

$(= 3\sec x\tan x)$

5 $h(t) = \sin^3 t$

$h'(t) = (\cos t)\left(3\sin^2 t\right) = 3\cos t\sin^2 t$

6 $y = \sin\sqrt[3]{x} = \sin\left(x^{\frac{1}{3}}\right)$

$\frac{dy}{dx} = \left(\frac{1}{3}x^{-\frac{2}{3}}\right)\cos\sqrt[3]{x} = \frac{1}{3}x^{-\frac{2}{3}}\cos\sqrt[3]{x}$

7 $y = \cos\frac{\pi}{x} = \cos\left(\pi x^{-1}\right)$

$\frac{dy}{dx} = \left(-\pi x^{-2}\right)\left(-\sin\left(\pi x^{-1}\right)\right) = \frac{\pi}{x^2}\sin\frac{\pi}{x}$

8 $g(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ □□

$g'(x) = \frac{(\sin x)(-\sin x) - \cos x(\cos x)}{\sin^2 x}$

$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\text{cosec}^2 x$

9 $f(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow f'(x) = 0$

10 $y = \sin(2x)\cos(2x) = \frac{1}{2}\sin(4x)$

$\therefore \frac{dy}{dx} = 2\cos(4x)$

11 a $(g\,o f)(x) = \cos\left(4x^3\right)$

b $\frac{d}{dx}\left[(g\,o f)(x)\right] = \frac{d}{dx}\left(\cos\left(4x^3\right)\right)$

$= \left(12x^2\right)\left(-\sin\left(4x^3\right)\right) = -12x^2\sin\left(4x^3\right)$

c $r(x) = x^2\cos\left(4x^3\right)$

$r'(x) = 2x\cos\left(4x^3\right) + x^2\left(-12x^2\sin\left(4x^3\right)\right)$

$= 2x\cos\left(4x^3\right) - 12x^4\sin\left(4x^3\right)$

12 a **i** $f'(x) = \cos x$

ii $f''(x) = -\sin x$

iii $f'''(x) = -\cos x$

iv $f^{(4)}(x) = \sin x$

b $n = 4x$ where x is an integer, therefore $n = 4, 8, 12$

c **i** $f^{(80)}(x) = f^{(20\times 4)}(x) = \sin x$

ii $f^{(42)}(x) = f^{(10\times 4+2)}(x) = -\sin x$

Exercise 13B

1 $f(\pi) = -1$

$f'(x) = -\sin x + 2\cos x$

$\therefore f'(\pi) = -2$

Therefore at $(\pi, -1)$ the tangent has gradient -2 and the normal has gradient $\frac{1}{2}$

Tangent: $y - (-1) = -2(x - \pi)$

$\Rightarrow y = -2x + 2\pi - 1$

Normal: $y - (-1) = \frac{1}{2}(x - \pi)$

$\Rightarrow y = \frac{x}{2} - 1 - \frac{\pi}{2}$

2 $f\left(\frac{\pi}{3}\right) = 1$

$f'(x) = -6\sin(6x)$

$f'\left(\frac{\pi}{3}\right) = 0$

Therefore at $\left(\frac{\pi}{3}, 1\right)$ the tangent is parallel to the x – axis and the normal is parallel to the y – axis

Tangent: $y = 1$

Normal: $x = \frac{\pi}{3}$

3 a $f\left(\frac{1}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

b $f'(x) = \pi\cos(\pi x)$

c $f'\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}}$

$\therefore y - \frac{1}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}\left(x - \frac{1}{4}\right)$

$\Rightarrow y = \frac{\pi x}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$

4 $f'(x) = -2\sin x$

$f'(x) = -\sqrt{2} = -2\sin x$

$\Rightarrow \sin x = \frac{1}{\sqrt{2}}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

Exercise 13C

1 $f'(x) = 3(4)\cos\left(4x - \frac{\pi}{6}\right) + 5$

$= 12\cos\left(4x - \frac{\pi}{6}\right) + 5$

2 $\frac{dy}{dx} = (12x^2 + 4x + 1)e^{4x^3 + 2x^2 + x}$

3 $\frac{dy}{dx} = \frac{(1 - \cos x)(\cos x) - \sin x(\sin x)}{(1 - \cos x)^2}$

$= \frac{\cos x - 1}{(1 - \cos x)^2} = -\frac{1}{1 - \cos x}$

4 $f'(x) = 2xe^x + x^2e^x + e^x$

$= e^x(x^2 + 2x + 1) = e^x(x + 1)^2$

5 $h'(t) = -2e^{\cos t}\sin t$

6 $\frac{dy}{dx} = 5e^{5x}\sin(3x) + 3e^{5x}\cos(3x)$

$= e^{5x}(5\sin(3x) + 3\cos(3x))$

7 $\frac{dy}{dx} = -\sin x + \sin x + x\cos x = x\cos x$

8 $f'(x) = -2xe^{x^2}\sin\left(e^{x^2}\right)$

9 $f'(x) = \frac{1}{x}\sin(3x) + 3\ln(3x)\cos(3x)$

10 $f'(x) = 3\cos(3x) \cdot \frac{1}{\sin(3x)} = 3\cot(3x)$

11 a $f'(x) = -e^{\cos x}\sin x$

b $e^{\cos x}$ is always positive, so can just consider the behaviour of $-\sin x$

$\therefore$ Increasing for $\pi < x < 2\pi$

Decreasing for $0 < x < \pi$

c $f'(x) = 0 \Rightarrow x = 0,\ x = \pi$ or $x = 2\pi$

$f''(x) = e^{\cos x}(\sin^2 x - \cos x)$

$f''(0) = -e < 0$

$f''(\pi) = e^{-1} > 0$

$f''(2\pi) = f''(0) < 0$

Alternative method: Since the exponential function is a continuous increasing function, the minima and maxima of $\cos x$ will correspond directly to respective minima and maxima of $e^{\cos x}$

$\therefore$ Local maxima at $(0, e)$ and $(2\pi, e)$

Local minimum at (π, e^{-1})

12 a $f'(x) = 1 - \sin x$

$f''(x) = -\cos x$

b Concave up when $f''(x) > 0$

$\Rightarrow x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Concave down when $f''(x) < 0$

$\Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

c $f''(x) = 0$ when $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Concavity changes at both of these values of x, so $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ are both points of inflexion

13 a $f'(x) = -2\sin 2x + 2(-\sin x)(\cos x)$

$= -2\sin 2x - 2\sin x \cos x$

$= -2\sin 2x - \sin 2x$

$= -3\sin 2x$

b In the range $0 \le x \le \pi$,

$f'(x) = 0$ when

$x = 0,\ x = \frac{\pi}{2},\ \ x = \pi$

$\therefore (0, 2),\ \left(\frac{\pi}{2}, -1\right),\ (\pi, 2)$

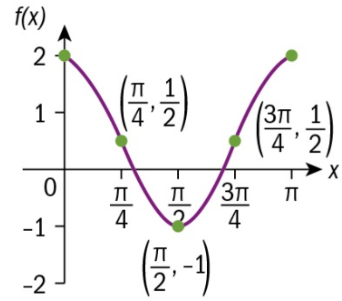
c $f''(x) = -6\cos 2x$

$f''(x) = 0$ in the interval when

$x = \frac{\pi}{4},\ \ x = \frac{3\pi}{4}$

Concavity changes at these points, so $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4}, \frac{1}{2}\right)$ are points of inflexion

d



Exercise 13D

1 $C'(x) = x - 50$

$C'(120) = 70$

This means it costs 70 Euros to produce the 121st table

2 a $v(t) = s'(t) = \dfrac{e^t(1) - t(e^t)}{e^{2t}}$

$= \dfrac{e^t(1-t)}{(e^t)^2} = \dfrac{1-t}{e^t}$

b $v(2) = \dfrac{1-2}{e^2} = -e^{-2}$

$\therefore$ Velocity is $-e^{-2}$ and speed is e^{-2}

c Look for change in sign of $v(t)$

This occurs at $t = 1$

3 $v(t) = d'(t) = 3\cos 18t + 6\sin 18t$

$v'\left(\frac{\pi}{27}\right) = 3\cos\left(\frac{2\pi}{3}\right) + 6\sin\left(\frac{2\pi}{3}\right)$

$= 3\left(-\frac{1}{2}\right) + 6\left(\frac{\sqrt{3}}{2}\right) = \frac{6\sqrt{3} - 3}{2}$

4 a $\dfrac{120\left(e^{0.2(10)} - e^0\right)}{10} = 12(e^2 - 1)$

b $P'(t) = 120(0.2)e^{0.2t} = 24e^{0.2t}$

c $P'(10) = 24e^2$

At day 10 the number of bacteria are increasing at a rate of 177 bacteria per day

5 a $P'(x) = -0.00015x^2 + 12$

$P'(200) = 6$

The profit gained by selling the 201^{st} unit of the chemical is 6 euros

b $C(x) = R(x) - P(x)$

$= 10x - 4 - \left(-0.00005x^3 + 12x - 200\right)$

$= 0.00005x^3 - 2x + 196$

c $C'(x) = 0.00015x^2 - 2$

$C'(200) = 0.00015(200)^2 - 2 = 4$

6 a $3.19s$ (use of GDC)

b $\dfrac{s(3.18533) - s(0)}{3.18533} = -0.408$ (3 s.f.)

so $-0.408\,\text{ms}^{-1}$

c $v(t) = s'(t) = -9.8t + 15.2$

d $v'(t) = 0 \Rightarrow t = \dfrac{15.2}{9.8} = 1.55$ (3s.f.)

This is the value of t at which the ball reaches its maximum height

(and changes direction)

Exercise 13E

1 $C'(x) = 1 - \dfrac{10000}{x^2}$

$C'(x) = 0 \Rightarrow x = 100$

$C''(x) = \dfrac{20000}{x^3}$

$\Rightarrow C''(100) > 0$ so minimum

2 a $|PQ| = |SR| = 8\cos\theta$

$|PS| = |QR| = 8\sin\theta$

$\therefore A = (8\cos\theta)(8\sin\theta)$

$= 64\sin\theta\cos\theta = 32(2\sin\theta\cos\theta)$

$= 32\sin 2\theta$

b $\dfrac{dA}{d\theta} = 64\cos 2\theta$

$\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = 0 \Rightarrow \theta = \dfrac{\pi}{4}$

c $\dfrac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -128\sin 2\theta$

At $\theta = \dfrac{\pi}{4}$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -128 < 0$

So this value of θ gives the maximum value of the area

3 a Let the angle between the downward vertical and the curved face of the cone be θ, then:

$\tan\theta = \dfrac{4}{6} = \dfrac{2}{3} = \dfrac{r}{6-h} \Rightarrow r = \dfrac{2}{3}(6-h)$

b $V = \pi hr^2 = \pi h\left(\dfrac{2}{3}(6-h)\right)^2$

$= \pi h\left(\dfrac{4}{9}(36 - 12h + h^2)\right)$

$= \dfrac{4\pi}{9}(36h - 12h^2 + h^3)$

c $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{4\pi}{9}(36 - 24h + 3h^2)$

$\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{4\pi}{9}(-24 + 6h)$

d $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 0 \Rightarrow 3h^2 - 24h + 36 = 0$

$\Rightarrow h^2 - 8h + 12 = (h-6)(h-2) = 0$

$h = 6$ would not make sense, so consider $h = 2$:

At $h = 2$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{4\pi}{9}(-12) = -\dfrac{16\pi}{3} < 0$

$\therefore h = 2,\ r = \dfrac{2}{3}(6-2) = \dfrac{8}{3}$

4 a $P(x) = R(x) - C(x) = 4\sqrt{x} - 2x^2$

b $\dfrac{\mathrm{d}P}{\mathrm{d}x} = \dfrac{2}{\sqrt{x}} - 4x$

c $\dfrac{\mathrm{d}P}{\mathrm{d}x} = 0 \Rightarrow 2 - 4x^{\frac{3}{2}} = 0 \Rightarrow x = 2^{-\frac{2}{3}} = 4^{-\frac{1}{3}}$

d Students should verify using their GDC that this does indeed give rise to the maximum profit

$P\left(4^{-\frac{1}{3}}\right) = 2.38110\ldots$

So maximum profit is \$2381.10

5 a Let $|AB| = |CD| = x \Rightarrow |BC| = |AD| = \dfrac{675}{x}$

$C(x) = 10x + 4x + 4(2)\left(\dfrac{675}{x}\right)$

$= 14x + \dfrac{5400}{x}$

b $C'(x) = 14 - \frac{5400}{x^2} = 0$

$\Rightarrow x = 19.64 \quad (2\text{d.p.})$

$C_{max} = C\left(\sqrt{\frac{5400}{14}}\right) = 549.91 \quad (2\text{d.p.})$

So minimum cost is \$550

Minimum to be verified by use of GDC

Exercise 13F

1 $\int 5\sin x\,dx = 5\int \sin x\,dx = -5\cos x + C$

2 $\int (4\cos x - 2\sin x)\,dx$

$= 4\int \cos x\,dx - 2\int \sin x\,dx$

$= 4\sin x + 2\cos x + C$

3 $\int \cos(7x)\,dx = \frac{1}{7}\sin(7x) + C$

4 $\int 6\cos(2x)\,dx = \frac{6}{2}\sin(2x) + C$

$= 3\sin(2x) + C$

5 $\int \sin(5x+3)\,dx = -\frac{1}{5}\cos(5x+3) + C$

6 $\int \left(x^3 + \sin(2x)\right)dx$

$= \int x^3\,dx + \int \sin(2x)\,dx$

$= \frac{x^4}{4} - \frac{1}{2}\cos(2x) + C$

7 $\int \cos\left(\frac{x}{2}\right)dx = 2\sin\left(\frac{x}{2}\right) + C$

8 $\int 2\pi\sin(2\pi x)\,dx$

$= -\frac{2\pi}{2\pi}\cos(2\pi x) + C$

$= -\cos(2\pi x) + C$

Exercise 13G

1 Let $u = 5x^3 + 4x \Rightarrow du = \left(15x^2 + 4\right)\,dx$

$\therefore \int \left(5x^3 + 4x\right)^2 \left(15x^2 + 4\right)dx$

$= \int u^2\,du = \frac{1}{3}u^3 + C$

$= \frac{1}{3}\left(5x^3 + 4x\right)^3 + C$

2 Let $u = 3x^5 \Rightarrow du = 15x^4\,dx$

$\therefore \int 15x^4 \sin\left(3x^5\right)dx = \int \sin u\,du$

$= -\cos u + C = -\cos\left(3x^5\right) + C$

3 Let $u = 2x^2 + 3x + 1 \Rightarrow du = (4x+3)\,dx$

$\therefore \int \frac{4x+3}{\left(2x^2+3x+1\right)^2}\,dx = \int u^{-2}\,du = -\frac{1}{u} + C$

$= -\frac{1}{2x^2+3x+1} + C$

4 Let $u = x^2 + 7x \Rightarrow du = (2x+7)\,dx$

$\therefore \int (2x+7)e^{x^2+7x}\,dx = \int e^u\,du = e^u + C$

$= e^{x^2+7x} + C$

5 Let $u = x^4 - 3x^2$

$\Rightarrow du = \left(4x^3 - 6x\right)\,dx = \frac{1}{2}\left(8x^3 - 12x\right)\,dx$

$\therefore \int \left(8x^3 - 12x\right)\left(x^4 - 3x^2\right)^3\,dx$

$= 2\int u^3\,du = \frac{u^4}{2} + C$

$= \frac{1}{2}\left(x^4 - 3x^2\right)^4 + C$

6 Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C$

$= 2e^{\sqrt{x}} + C$

7 Let $u = 2x^2 - 2x$

$\Rightarrow du = (4x-2)\,dx = 2(2x-1)\,dx$

$\therefore \int (2x-1)\cos\left(2x^2-2x\right)dx$

$= \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$

$= \frac{1}{2}\sin\left(2x^2 - 2x\right) + C$

8 Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$\therefore \int \sin x\cos^4 x\,dx = -\int u^4\,du = -\frac{u^5}{5} + C$

$= -\frac{1}{5}\cos^5 x + C$

9 Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$\therefore \int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C$

$= -\cos(\ln x) + C$

10 Let $u = e^{x^3} + 5 \Rightarrow du = 3x^2e^{x^3}\,dx$

$\therefore \int x^2e^{x^3}\sqrt{e^{x^3}+5}\,dx = \frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} + C$

$= \frac{2}{9}\left(e^{x^3}+5\right)^{\frac{3}{2}} + C$

11 $f(x) = \int e^{\sin x}\cos x\,dx$

$u = \sin x \Rightarrow du = \cos x\,dx$

$\therefore f(x) = \int e^u\,du = e^u + C = e^{\sin x} + C$

$f(\pi) = 1 + C = 12 \Rightarrow C = 11$

$\therefore f(x) = e^{\sin x} + 11$

12 $f(x) = \int \frac{4x}{2x^2+e^2}\,dx$

Let $u = 2x^2 + e^2 \Rightarrow du = 4x\,dx$

$\therefore f(x) = \int u^{-1}du = \ln u + C$

$= \ln\left(2x^2+e^2\right) + C$

$f(0) = 2 + C = 5 \Rightarrow C = 3$

$\therefore f(x) = \ln\left(2x^2+e^2\right) + 3$

Exercise 13H

1 $\int_0^{\frac{\pi}{4}}\cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2 $\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} 2\sin x\,dx = -2\left[\cos x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{6}}$

$= -2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \sqrt{3} + 1$

3 Let $u = x^2 + x \Rightarrow du = (2x+1)\,dx$

$x = 0 \Rightarrow u = 0, \quad x = 2 \Rightarrow u = 6$

$\therefore \int_0^2 \left(x^2+x\right)^3(2x+1)\,dx$

$= \int_0^6 u^3du = \left[\frac{u^4}{4}\right]_0^6 = 324$

4 Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$x = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$

$x = \frac{\pi}{6} \Rightarrow u = \frac{\sqrt{3}}{2}$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\cos^3 x\,dx = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} u^3du$

$= -\left[\frac{u^4}{4}\right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = \frac{1}{4}\left(\frac{9}{16} - \frac{1}{16}\right) = \frac{1}{8}$

5 Let $u = x^3 + 1 \Rightarrow du = 3x^2\,dx$

$x = 2 \Rightarrow u = 9$

$x = 1 \Rightarrow u = 2$

$\therefore \int_1^2 \frac{3x^2}{\sqrt{x^3+1}}\,dx = \int_2^9 u^{-\frac{1}{2}}du$

$= 2\left[u^{\frac{1}{2}}\right]_2^9 = 2\left(3 - \sqrt{2}\right)$

6 Let $u = e^x \Rightarrow du = e^x\,dx$

$x = \ln\frac{\pi}{3} \Rightarrow u = \frac{\pi}{3}$

$x = \ln\frac{\pi}{4} \Rightarrow u = \frac{\pi}{4}$

$\therefore \int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{3}} e^x\sin\left(e^x\right)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin u\,du = -\left[\cos u\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= -\left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}-1}{2}$

7 **a** As $e^x \neq 0$, consider $\sin x = 0$. $x = 0$ or $\pi \therefore k = \pi$

b Using GDC: $\int_0^{\pi} e^x\sin x\,dx = \frac{e^{\pi}+1}{2} \approx 12.1$

8 Limits of integration are $x = 0$, $x = 1.27531$ and $x = 4.06401$

Using GDC:

$\int_0^{1.27531} \sin x - \left(-x^3+5x^2-4x\right)\,dx$

$+ \int_{1.27531}^{4.06401} \left(-x^3+5x^2-4x\right) - \sin x\,dx$

≈ 11.4

Exercise 13I

1 **a** $v(t) = s'(t) = -t^2 + 8t - 12$

b $\int_0^9 v(t)\,dt = \int_0^9 \left(-t^2+8t-12\right)dt$

$= \left[-\frac{1}{3}t^3 + 4t^2 - 12t\right]_0^9 = -27\text{ m}$

c $\int_0^9 |v(t)|\,dt = \int_0^9 |-t^2+8t-12|\,dt = 48.3\text{ m}$

2 **a** $v(t) = s'(t) = 2t - 6$

b $\int_0^6 v(t)\,dt = \int_0^6 (2t-6)\,dt = \left[t^2-6t\right]_0^6 = 0\text{ m}$

c $\int_0^6 |v(t)|\,dt = \int_0^6 |2t-6|\,dt = 18\text{ m}$

3 a $v(t) = s'(t) = 3(t-1)^2$

b $\int_0^3 v(t)\,dt = \int_0^3 3(t-1)^2\,dt$

$= \left[(t-1)^3\right]_0^3 = 8-(-1) = 9\text{ m}$

c $\int_0^3 |v(t)|\,dt = \int_0^3 |3(t-1)^2|\,dt = 9\text{ m}$

4 a Displacement $= \int_0^8 v(t)\,dt = 22$ m

Distance $= \int_0^8 |v(t)|\,dt = 22$ m

b Displacement $= \int_2^{14} v(t)\,dt = 6$ m

Distance $= \int_2^{14} |v(t)|\,dt = 30$ m

c Displacement $= \int_0^{14} v(t)\,dt = 10$ m

Distance $= \int_0^{14} |v(t)|\,dt = 34$ m

5 a The acceleration is the gradient of this graph. The gradient at $t = 3$ is -3 so the acceleration is -3 ms^{-2}

b $t \in (0,3) \cup (5,7)$

c $\int_0^7 |v(t)|\,dt = 16.5\text{ m}$

6 a $a(t) = v'(t) = 2t$

$a(2) = 4\text{ ms}^{-1}$

b $s(t) = \int v(t)\,dt = \frac{1}{3}t^3 - 16t + C$

$s(0) = C = 10$

$\therefore s(t) = \frac{1}{3}t^3 - 16t + 10$

c $\int_2^6 |t^2 - 16|\,dt = 32$ m

Exercise 13J

1 a $v(t) = s'(t) = e^t(\cos t - \sin t)$

b $a(t) = v'(t) = -2e^t \sin t$

2 a $v(t) = s'(t) = -2\cos t$

$v(0) = -2\text{ ms}^{-1}$

b $v(t) = -2\cos t = 0 \Rightarrow t = \frac{\pi}{2}$

c $s\left(\frac{\pi}{2}\right) = 6 - 2 = 4\text{ m}$

3 a i $v(t) = 0$ whenever $\sin t = 0$

$\therefore t = 0,\ t = \pi,\ t = 2\pi$

ii $v(t) < 0$ whenever $\sin t < 0$

$\therefore t \in (\pi, 2\pi)$

b $a(t) = v'(t)$

$= e^{\cos t}\cos t + (-\sin t)e^{\cos t}\sin t$

$= e^{\cos t}(\cos t - \sin^2 t)$

c $s(t) = \int v(t)\,dt = \int e^{\cos t}\sin t\,dt$

Let $u = \cos t \Rightarrow du = -\sin t\ dt$

$\therefore s(t) = -\int e^u du = -e^u + C$

$= -e^{\cos t} + C$

$s(0) = 3e \Rightarrow -e + C = 3e \Rightarrow C = 4e$

$\therefore s(t) = 4e - e^{\cos t}$

4 a Assume initial displacement is 0.

$s(t) = \int v(t)\ dt = \int 5\sin t + 2\cos t\ dt$

$= 2\sin t - 5\cos t + C$

$s(0) = 2\sin 0 - 5\cos 0 + C = 0$

$= -5 + C = 0 \rightarrow C = 5$

$s(4) = 2\sin 4 - 5\cos 4 + 5 = 6.75$ m (3 s.f.)

b $\int_0^4 |5\sin t + 2\cos t|\,dt = 14.0$ m (3s.f.)

5 a i -2.37 ms^{-2} (use of GDC)

ii $v'(1.3) < 0$ so slowing down

b 3.54s and 5.01s (use of GDC)

c i -6.92 m (use of GDC)

ii At 6 seconds, the particle is 6.92m to the left of its inital position

d 18.6 m (use of GDC)

6 a -12.8 ms^{-2} (use of GDC)

b $t = 0.696,\ 5.59$ (use of GDC)

c 13.2 m (use of GDC)

d 24.8 m (use of GDC)

Exercise 13K

1 $\int_0^{1.5}\left(1025t^2 - t^3\right)dt = 1152$ spectators

2 $33.4 + \int_0^{8} 5.2te^{\left(-0.05t^3+2.3\right)}\,dt = 206$ cubic feet

3 $3800 + \int_0^{20} -150\left(1 - \frac{t}{80}\right)dt \approx 1175$ gallons

4 $\int_0^{10} 20.4e^{\frac{t}{18}}\,dt \approx 273$ billions of barrels

Chapter Review

1 a $f'(x) = 3\cos x - 4\sin x$

b $\frac{dy}{dx} = -3\sin(3x-4)$

c $h'(t) = 4\cos x \sin^3 x$

d $f(x) = (\cos x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(-\sin x)(\cos x)^{-\frac{1}{2}} = -\frac{\sin x}{2\sqrt{\cos x}}$

e $\frac{dy}{dx} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right)$

$= (\cos x)(\ln x) + \frac{\sin x}{x}$

f $\frac{dy}{dx} = \frac{1}{x}\cos(\ln x)$

g $s'(t) = \frac{e^t(-\sin t) - \cos t\left(e^t\right)}{e^{2t}}$

$= -\frac{e^t(\sin t + \cos t)}{\left(e^t\right)^2} = -\frac{\sin t + \cos t}{e^t}$

h $f'(x) = 2e^{2x}\sin 2x + 2e^{2x}\cos 2x$

$= 2e^{2x}(\sin 2x + \cos 2x)$

2 a $\int\left(3x^4 + \cos x\right)dx = 3\int x^4\,dx + \int \cos x\,dx$

$= \frac{3}{5}x^5 + \sin x + C$

b $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x + C$

c $\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$

d Let $u = 2x^3 + 5x \Rightarrow du = \left(6x^2+5\right)dx$

$\therefore \int\left(6x^2+5\right)\left(2x^3+5x\right)^4 dx$

$= \int u^4\,du = \frac{1}{5}u^5 + C$

$= \frac{1}{5}\left(2x^3+5x\right)^5 + C$

e Let $u = x^3 \Rightarrow du = 3x^2\,dx$

$\int 3x^2\cos\left(x^3\right)dx = \int \cos u\,du = \sin u + C$

$= \sin\left(x^3\right) + C$

f Let $u = x^2 + 5x$

$\Rightarrow du = (2x+5)\,dx = \frac{1}{2}(4x+10)\,dx$

$\int (4x+10)e^{x^2+5x}\,dx = 2\int e^u\,du = 2e^u + C$

$= 2e^{x^2+5x} + C$

g Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$\therefore \int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du = \sin u + C$

$= \sin(\ln x) + C$

h Let $u = e^{4x} + 5$

$\Rightarrow du = 4e^{4x}\,dx = 2\left(2e^{4x}\right)dx$

$\therefore \int \frac{2e^{4x}}{e^{4x}+5}\,dx = \frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln u + C$

$= \frac{1}{2}\ln\left(e^{4x}+5\right) + C$

3 a $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x\,dx = \left[-\cos x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

b Let $u = x^3 - 2x \Rightarrow du = \left(3x^2-2\right)dx$

$x = 2 \Rightarrow u = 4$

$x = -1 \Rightarrow u = 1$

$\int_{-1}^{2}\left(3x^2-2\right)\left(x^3-2x\right)^3 dx$

$= \int_1^4 u^3\,du = \frac{1}{4}\left[u^4\right]_1^4 = \frac{255}{4}$

c $\int_0^{\frac{\pi}{6}}(1+\cos x)\,dx = \left[x + \sin x\right]_0^{\frac{\pi}{6}} = \frac{\pi}{6} + \frac{1}{2}$

$= \frac{\pi+3}{6}$

d Let $u = 4x^2 + 1 \Rightarrow du = 8x\,dx$

$x = 1 \Rightarrow u = 5$

$x = 0 \Rightarrow u = 1$

$\therefore \int_0^1 8xe^{4x^2+1}\,dx = \int_1^5 e^u\,du = \left[e^u\right]_1^5 = e^5 - e$

4 a The surface area of the box is

$x^2 + 4xh = 432$

$\Rightarrow h = \dfrac{432 - x^2}{4x}$

b $V = x^2h = \dfrac{x\left(432 - x^2\right)}{4} = 108x - \dfrac{1}{4}x^3$

c $\dfrac{dV}{dx} = 108 - \dfrac{3}{4}x^2$

$\dfrac{dV}{dx} = 0 \Rightarrow x^2 = 144 \Rightarrow x = 12$

$\dfrac{d^2V}{dx^2} = -\dfrac{3}{2}x$ so the second derivative is negative at $x = 12$ □

$\therefore$ Maximum volume occurs when $x = 12$ and accordingly $h = 6$

5 a $v(t) = s'(t) = -3e^{\cos t}\sin t$

$a(t) = v'(t) = -3e^{\cos t}\cos t + 3e^{\cos t}\sin^2 t$

$= 3e^{\cos t}\left(\sin^2 t - \cos t\right)$

b $v(t) < 0$ when $\sin t > 0 \Rightarrow t \in (0, \pi)$

c Using a GDC, plot $a(t)$ to

see where it is positive

$t \in (0.905,\ 5.38)$

d $\int_0^{2\pi} |v(t)|\,dt = 14.1\text{ m}$

6 $4.5 + \int_0^{10} 7t^2e^{-1.2t}\,dt = 12.6$ cm (3 s.f.)

7 a $g'(x) = 2\sin x\cos x - 5$ A1A1

b $g'(x) = \sin 2x - 5 \le 1 - 5 = -4 < 0$

M1A1R1

Therefore g is decreasing on all its domain. AG

8 a $f'(x) = 5 - \sec^2 x$ M1A1

$f'\left(\dfrac{\pi}{4}\right) = 5 - \sec^2\left(\dfrac{\pi}{4}\right) = 4\dfrac{1}{2}$ A1

$f\left(\dfrac{\pi}{4}\right) = \dfrac{5\pi}{4} - 1$ A1

$y - \left(\dfrac{5\pi}{4} - 1\right) = 4\dfrac{1}{2}\left(x - \dfrac{\pi}{4}\right)$ A1

b the normal to the graph is vertical when the tangent is horizontal R1

$f'(x) = 0 \Rightarrow \sec^2 x = 5$ M1

$\cos^2 x = \dfrac{1}{5} \Rightarrow x = \arccos\dfrac{\sqrt{5}}{5}$ A1

$\cos^2 x = \dfrac{1}{5} \Rightarrow \tan x = \sqrt{5 - 1} = 2$ M1

$f\left(\arccos\dfrac{\sqrt{5}}{5}\right) = 5\arccos\dfrac{\sqrt{5}}{5} - 2$ A1

The coordinates of A are

$\left(\arccos\dfrac{\sqrt{5}}{5}, 5\arccos\dfrac{\sqrt{5}}{5} - 2\right)$.

9 a $d(t) = \left|\sin 2t - \sin(t - 0.24)\right|$ A1A1

b Use GDC to find the maximum M1

$t = 2.25$ A1

c Find intersection of graphs M1

$t = 1.13$ A1

10 a $\int (x - \sin x)\,dx = \dfrac{x^2}{2} + \cos x + C$

M1A1A1

b $\int_0^{\pi} (x - \sin x)\,dx = \left[\dfrac{x^2}{2} + \cos x\right]_0^{\pi} = \dfrac{\pi^2}{2} - 2$

M1A1A1

11 a $\dfrac{da}{dx} = \left(e^x\right)'\cos x + e^x(\cos x)'$

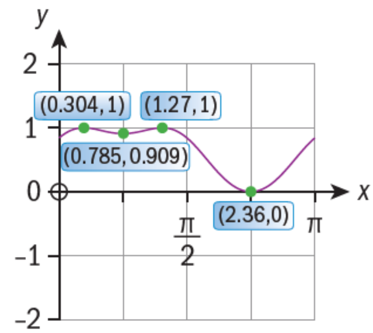
$= e^x\cos x - e^x\sin x$ M1A1A1

b $a(0) = 1$ A1

$a'(0) = 1$ A1

$y = x + 1$ M1A1

12 a A1 for shape, A1 for domain, A1 for scale on axes



b i Minimum points: (0.785,0.909) and (2.36,0) A1A1

Maximum points: (0.304,1) and (1.27,1) A1A1 ii $0 \le x \le 1$ A1

iii $\sin 1$ A1

c i $\int_{0}^{2.356...} \sin(1+\sin 2x)\ dx$ A1A1

ii Using GDC to evaluate the definite integral M1

area = 1.76 A1

13 a i -0.524 (or $-\frac{\pi}{6}$) A1

ii $-0.369 \le y \le 1.76$ A1A1

iii $f'(x) = 2\cos 2x - \sin x$ A1A1

b $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x + \cos x)\ dx = 2.25$ M1A1A1

14 a $s(0) = 2$ mm A1

b i $v = s' = 15\cos 3t + 2t$ M1A1

ii $a = v' = -45\sin 3t + 2$ M1A1

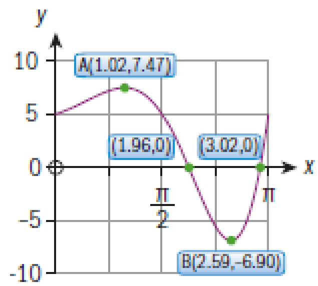
c $v = 0 \Rightarrow t = 0.548, t = 1.50, t = 2.74$ M1

$a < 0 \Rightarrow t = 0.548, t = 2.74$ R1A1

15 a i $f(0) = 5$ A1

ii $f(\pi) = 5$ A1

b A1 for coordinates of A, A1 for coordinates of B,
A1 for zeros, A1 for shape and domain



c $\int_{a}^{b} f(x)\ dx = 2.11$ (or 2.07 using 3 s.f. for a and b) A2

d The graph crosses the x-axis between a and b R1

e Either

$\int_{a}^{b} |f(x)|\ dx = 7.39$ M1A1

Or

$\int_{1.017...}^{1.961...} f(x)\ dx + \int_{1.9601...}^{2.588...} f(x)\ dx = 7.39$

M1A1

16 a $0 \le x \le 2\pi \Rightarrow 0 \le \frac{x}{2} \le \pi$ M1

In the 1st and 2nd quadrants sine is positive R1

Therefore $f(x) \ge 0$ for all

$0 \le x \le 2\pi$. AG

b $f(\pi - x) = \sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$

$= \sin\left(\pi - \left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$

$= \sin\left(\frac{\pi}{2} + \frac{x}{2}\right) = f(\pi + x)$ M1R1

Therefore $a = f(\pi - x) = f(\pi + x) = b$

A1

c $A(x) = 2x\sin\left(\frac{\pi - x}{2}\right)$ M1A1

Find maximum point (1.72,2.24)

M1A1

A(1.42,0.652) and B(4.86,0.652)

A1A1

d $p = 2AB + 2 \times 0.6520... = 8.19$ M1A1AG

14 Valid comparisons and informed decisions: probability distributions

Skills check

1 a $\dfrac{3\times3+5\times4+7\times5+9\times6+6\times7+2\times8}{3+5+7+9+6+2}$

$=\dfrac{176}{32}=5.5$

b $\dfrac{3\times10+10\times12+15\times15+9\times17+2\times20}{3+10+15+9+2}$

$=\dfrac{568}{39}\approx14.6$

2 a 5.5 **b** 14.6

3 a $\binom{6}{2}=15$ **b** $\binom{8}{5}=56$

c $\binom{9}{6}(0.3)^3(0.7)^6=0.267$

4 a $x=\dfrac{55}{32}=1.719$ **b** $x=\dfrac{149}{50}=2.98$

c $x=\dfrac{217}{25}=8.68$

Exercise 14A

1 a

s	2	3	4	5	6	7	8	9	10	11	12
$P(S=s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{35}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b

n	0	1	2
$P(N=n)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

c

n	1	2	3	4	5	6
$P(N=n)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

d

p	1	2	3	4	5	6	8	9	10
$P(P=p)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
p	12	15	16	18	20	24	25	30	36
$P(P=p)$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2 a

t	2	3	4	5	6
$P(T=t)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

b $P(T>4)=P(T=5)+P(T=6)$

$=\dfrac{12}{36}+\dfrac{9}{36}=\dfrac{7}{12}$

3 a

s	1	2	3	6	10
$P(S=s)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b $P(S>2)$

$=P(S=3)+P(S=6)+P(S=10)$

$=\dfrac{1}{6}+\dfrac{1}{6}+\dfrac{1}{6}=\dfrac{1}{2}$

4 a The sum of the probabilities in a probability distribution should be 1, so

$1=\dfrac{1}{3}+\dfrac{1}{3}+c+c$

$1=\dfrac{2}{3}+2c$

$1-\dfrac{2}{3}=2c$

$\dfrac{1}{3}=2c$

$\dfrac{1}{3}\div2=c$

$c=\dfrac{1}{6}$

b $P(1<X<4)=P(X=2)+P(X=3)$

$=\dfrac{1}{3}+\dfrac{1}{6}=\dfrac{1}{2}$

5 The sum of the probabilities must be equal to 1, so

$1=1^3c+2^3c+3^3c$

$1=c+8c+27c$

$1=36c$

$c=\dfrac{1}{36}$

6 As the sum of the probabilities must be 1, we have

$1 = 2k + 4k^2 + 6k^2 + k$

$1 = 3k + 10k^2$

$0 = 10k^2 + 3k - 1$

$0 = (2k+1)(5k-1)$

So the possible values for k are $k = -\frac{1}{2}$ and $k = \frac{1}{5}$ but as a probability must be non-negative, $k = \frac{1}{5}$ is the only possible solution

7 Using the fact that the sum of the probabilities must be equal to 1,

$1 = k\left(\frac{1}{3}\right)^0 + k\left(\frac{1}{3}\right)^1 + k\left(\frac{1}{3}\right)^2 + k\left(\frac{1}{3}\right)^3$

$1 = k + \frac{k}{3} + \frac{k}{9} + \frac{k}{27}$

$1 = \frac{40}{27}k$

$k = \frac{27}{40}$

8 **a** As the sum of the probabilities must equal 1,

$P(X \geq 2)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$\quad + P(X = 5)$

$= a + b + b + b$

$P(X < 2) = P(X = 0) + P(X = 1) = a + a$

$1 = a + a + a + b + b + b$

$1 = 3a + 3b$

$P(X \geq 2) = 3P(X < 2)$

$a + b + b + b = 3(a + a)$

$a + 3b = 6a$

$3b = 6a - a$

$3b = 5a$

$b = \frac{5}{3}a$

$1 = 3a + 3 \times \left(\frac{5}{3}a\right)$

$1 = 3a + 5a$

$1 = 8a$

$a = \frac{1}{8}$

$b = \frac{5}{3} \times \frac{1}{8}$

$b = \frac{5}{24}$

b $P(\text{sum} = 8) = P(3 \text{ and } 5) + P(4 \text{ and } 4)$

$= b^2 + b^2 = 2b^2$

$P(\text{sum} = 9) = P(4 \text{ and } 5) = b^2$

$P(\text{sum} = 10) = P(5 \text{ and } 5) = b^2$

$P(\text{sum} > 7) = P(\text{sum} = 8) + P(\text{sum} = 9)$

$\quad + P(\text{sum} = 10)$

$P(sum > 7) = 2b^2 + b^2 + b^2 = 4b^2$

$= \frac{25}{144}$

9 **a** $P(C = 3)$

$= P(A = 1 \text{ and } B = 2)$

$\quad + P(A = 2 \text{ and } B = 1)$

$= \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{6}$

$= \frac{5}{18}$

b

c	2	3	4	5	6
$P(C = c)$	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{6}{18}$	$\frac{5}{18}$	$\frac{1}{18}$

Exercise 14B

1 $E(X)$

$= \frac{1}{6} \times 1 + \frac{1}{6} \times 4 + \frac{1}{6} \times 9 + \frac{1}{6} \times 16$

$\quad + \frac{1}{6} \times 25 + \frac{1}{6} \times 36 = \frac{91}{6}$

≈ 15.2

2 $E(X)$

$= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 5 \times \frac{1}{6} + 8 \times \frac{1}{6} + 13 \times \frac{1}{6}$

$= \frac{16}{3}$

3 $E(X)$

$= 1 \times \frac{1}{36} + 2 \times \frac{2}{36} + 3 \times \frac{3}{36} + 4 \times \frac{4}{36} + 5 \times \frac{5}{36}$

$\quad + 6 \times \frac{6}{36} + 7 \times \frac{7}{36} + 8 \times \frac{8}{36}$

$= \frac{17}{3}$

4 **a** Using the fact that the sum of the probabilities must equal 1,

$1 = k + 2k + 3k + 4k + 5k + 4k$

$\quad + 3k + 2k + k$

$1 = 25k$

$k = \frac{1}{25}$

b $E(X)$

$$= 1\times\frac{1}{25}+2\times\frac{2}{25}+3\times\frac{3}{25}+4\times\frac{4}{25}+5\times\frac{5}{25}$$
$$+6\times\frac{4}{25}+7\times\frac{3}{25}+8\times\frac{2}{25}+9\times\frac{1}{25}$$
$$=5$$

5 a $0.2 \le k \le 1$

b $E(X)$

$$= 1\times 0.2+2\times(1-k)$$
$$+3\times(1-(1-k)-0.2)$$
$$=1.6+k$$

6 a If you pick the first red ball on the r th try, that means you have picked $r-1$ blue balls

$\text{P(R=1)} = \frac{1}{5}$

$\text{P(R=2)} = \frac{8}{45}$

$\text{P(R=3)} = \frac{7}{45}$

$\text{P(R=4)} = \frac{2}{15}$

$\text{P(R=5)} = \frac{1}{9}$

$\text{P(R=6)} = \frac{4}{45}$

$\text{P(R=7)} = \frac{1}{15}$

$\text{P(R=8)} = \frac{2}{45}$

b $E(R)$

$$= 1\times\frac{1}{5}+2\times\frac{8}{45}+3\times\frac{7}{45}+4\times\frac{2}{15}+5\times\frac{1}{9}$$
$$+6\times\frac{4}{45}+7\times\frac{1}{15}+8\times\frac{2}{45}+9\times\frac{1}{45}$$
$$=\frac{11}{3}$$

c 1

7 a $P(R=2)=P(\text{blue then red})$

$$=\frac{8}{10}\times\frac{2}{10}=\frac{4}{25}$$

b $P(R>3)=P(R=1)+P(R=2)+P(R=3)$

$$=\frac{2}{10}+\frac{8}{10}\times\frac{2}{10}+\frac{8}{10}\times\frac{8}{10}\times\frac{2}{10}$$
$$=\frac{61}{125}$$

$P(R=3)$

$=P(\text{blue then blue then red})$

$$=\frac{8}{10}\times\frac{8}{10}\times\frac{2}{10}=\frac{16}{125}$$

c For the first red to be drawn on the r th try, there are $r-1$ blues picked first so $P(R=r)=P(r \text{ blues then a red})$

$$=\left(\frac{8}{10}\right)^n\times\frac{2}{10}$$

d 1

8 a As the sum of the probabilities must equal 1, let $x=P(Z=0)$

$1=x+0.2+0.05+0.001+0.0001$

$1=x+0.2511$

$1-0.2511=x$

$x=0.7489$

b $E(Z)$

$$=0\times 0.7489+2\times 0.2+20\times 0.05$$
$$+200\times 0.001+1000\times 0.0001$$
$$=1.7$$

so the expected winnings on a ticket are \$1.70

c You expect to lose \$0.30 per ticket

Exercise 14C

1 $X \sim B\left(4,\frac{1}{2}\right)$

a $P(X=1)=\binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3$

$$=4\times\frac{1}{2}\times\frac{1}{8}=\frac{4}{16}=\frac{1}{4}$$

b $P(X<1)=P(X=0)$

$$=\binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4=1\times1\times\frac{1}{16}=\frac{1}{16}$$

c $P(X\le 1)=P(X=0)+P(X=1)$

$$=\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$$

d $P(X\ge 1)=1-P(X<1)=1-\frac{1}{16}=\frac{15}{16}$

2 $X \sim B\left(6,\frac{1}{3}\right)$

a $P(X=2)=\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4=0.329$

b $P(X<2)=\binom{6}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6+\binom{6}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5$

$=0.351$

c $P(X \le 2) = P(X < 2) + P(X = 2)$

$= 0.329 + 0.351 = 0.680$

d $P(X \ge 2) = 1 - P(X < 2)$

$= 1 - 0.351 = 0.649$

3 $X \sim B\left(8, \frac{2}{7}\right)$

a $P(X = 5) = \binom{8}{5}\left(\frac{2}{7}\right)^5\left(\frac{5}{7}\right)^3$

$= 0.0389$ (3 s.f.)

b $P(X < 5)$

$= \binom{8}{0}\left(\frac{2}{7}\right)^0\left(\frac{5}{7}\right)^8 + \binom{8}{1}\left(\frac{2}{7}\right)^1\left(\frac{5}{7}\right)^7$

$+ \binom{8}{2}\left(\frac{2}{7}\right)^2\left(\frac{5}{7}\right)^6 + \binom{8}{3}\left(\frac{2}{7}\right)^3\left(\frac{5}{7}\right)^5$

$+ \binom{8}{4}\left(\frac{2}{7}\right)^4\left(\frac{5}{7}\right)^4$

$= 0.952$ (3 s.f.)

c $P(X > 5) = 1 - P(X = 5) - P(X < 5)$

$= 1 - 0.038857.. - 0.9524..$

$= 0.00870$ (3 s.f.)

d $P(X \ge 1) = 1 - P(X = 0)$

$= 1 - \binom{8}{0}\left(\frac{2}{7}\right)^0\left(\frac{5}{7}\right)^8 = 0.932$

Exercise 14D

1 $R \sim B\left(4, \frac{1}{4}\right)$, then

r	0	1	2	3	4
$P(R = r)$	0.316	0.422	0.211	0.0469	0.00391

So the most likely number of times the red face shows is 1

2 $X \sim B(8, 0.55)$

a $P(X = 5) = \binom{8}{5}(0.55)^5(0.45)^3 = 0.257$

b If he misses at least 5 times then he hits at most 3 times,

$P(X \le 3) = \binom{8}{0}(0.55)^0(0.45)^8 + \binom{8}{1}(0.55)^1(0.45)^7$

$+ \binom{8}{2}(0.55)^2(0.45)^6 + \binom{8}{3}(0.55)^3(0.45)^5$

$= 0.260$

3 $X \sim B(16, 0.01)$

a $P(X = 0) = \binom{16}{0}(0.01)^0(0.99)^{16} = 0.851$

b If 13 are not faulty then 3 are faulty,

$P(X = 3) = \binom{16}{3}(0.01)^3(0.99)^{13}$

$= 0.000491$

c $P(X \ge 2) = 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$= 1 - \binom{16}{0}(0.01)^0(0.99)^{16}$

$- \binom{16}{1}(0.01)^1(0.99)^{15}$

$= 0.0109$

4 $X \sim B(10, 0.25)$

a $P(X = 5) = \binom{10}{5}(0.25)^5(0.75)^5 = 0.0584$

b $P(X \ge 3) = 1 - P(X < 3)$

$= 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$= 1 - \binom{10}{0}(0.25)^0(0.75)^{10} + \binom{10}{1}(0.25)^1(0.75)^9$

$+ \binom{10}{2}(0.25)^2(0.75)^8 = 0.474$

5 $X \sim B(5, 0.4)$,

$P(X \le 3)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$+ P(X = 3)$

$= \binom{5}{0}(0.4)^0(0.6)^5 + \binom{5}{1}(0.4)^1(0.6)^4$

$+ \binom{5}{2}(0.4)^2(0.6)^3 + \binom{5}{3}(0.4)^3(0.6)^2$

$= 0.913$

6 $X \sim B(6, 0.15)$

a $P(X > 1) = 1 - P(X = 0) - P(X = 1)$

$= 1 - \binom{6}{0}(0.15)^0(0.85)^6 - \binom{6}{1}(0.15)^1(0.85)^5$

$= 0.224$

b $P(X = 1) = \binom{6}{1}(0.15)^1(0.85)^5 = 0.399$

7 $X \sim B(15, 0.05)$

a i $P(X = 3) = \binom{15}{3}(0.05)^3(0.95)^{12}$

$= 0.0307$

ii $P(X = 0) = \binom{15}{0}(0.05)^0(0.95)^{15}$

$= 0.463$

iii $P(X \ge 2) = 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$$= 1 - \binom{15}{0}(0.05)^0(0.95)^{15}$$
$$- \binom{15}{1}(0.05)^1(0.95)^{14}$$
$$= 0.171$$

b **i** $(P(X = 0))^2 = 0.215$

ii $(P(X \geq 2))^2 = 0.0292$

iii $2 \times P(X = 0) \times P(X \geq 2) = 0.158$

Exercise 14E

1 $0.0256 = P(X < 1) = P(X = 0)$

$$= \binom{n}{0}(0.6)^0(0.4)^n = 1 \times 1 \times (0.4)^n = 0.4^n$$
$$\log 0.0256 = n \log 0.4$$
$$n = \frac{\log 0.0256}{\log 0.4}$$
$$n = 4$$

2 $X \sim B(n, 0.01)$,

$$0.5 < P(X = 0) = \binom{n}{0}(0.01)^0(0.99)^n$$
$$= 1 \times 1 \times (0.99)^n = 0.99^n$$
$$\log 0.5 < n \log 0.99$$
$$n < \frac{\log 0.5}{\log 0.99}$$
$$n < 68.968 \Rightarrow n = 68$$

3

$$0.25 > P(X < 1) = P(X = 0) = \binom{n}{0}(0.2)^0(0.8)^n$$
$$= 1 \times 1 \times (0.8)^n = 0.8^n$$
$$\log 0.25 > n \log 0.8$$
$$n > \frac{\log 0.25}{\log 0.8}$$
$$n > 6.213 \Rightarrow n = 7$$

4 $X \sim B(n, 0.3)$,

$$0.95 < P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - \binom{n}{0}(0.3)^0(0.7)^n = 1 - 0.7^n$$
$$0.7^n < 1 - 0.95$$
$$n \log 0.7 < \log 0.05$$
$$n > \frac{\log 0.05}{\log 0.7}$$
$$n > 8.399 \Rightarrow n = 9$$

5 $X \sim B(n, 0.5)$,

$$0.99 \leq P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - \binom{n}{0}(0.5)^0(0.5)^n = 1 - 0.5^n$$
$$0.5^n < 1 - 0.99$$
$$n \log 0.5 < \log 0.01$$
$$n > \frac{\log 0.01}{\log 0.5}$$
$$n > 6.644 \Rightarrow n = 7$$

Exercise 14F

1 **a** $X \sim B\left(40, \frac{1}{2}\right)$, $E(X) = np = 40 \times \frac{1}{2} = 20$

b $X \sim B\left(40, \frac{1}{6}\right)$, $E(X) = np = 40 \times \frac{1}{6} = \frac{20}{3}$

c $X \sim B\left(40, \frac{1}{4}\right)$, $E(X) = np = 40 \times \frac{1}{4} = 10$

2 $E(X) = np = 0.4n = 10 \Rightarrow n = \frac{10}{0.4} = 25$

3 **a** $X \sim B(15, 0.25)$

b $E(X) = np = 15 \times 0.25 = 3.75$

c $P(X \geq 10)$

$$= P(X = 10) + P(X = 11) + P(X = 12)$$
$$+ P(X = 13) + P(X = 14) + P(X = 15)$$
$$= \binom{15}{10}(0.25)^{10}(0.75)^5 + \binom{15}{11}(0.25)^{11}(0.75)^4$$
$$+ \binom{15}{12}(0.25)^{12}(0.75)^3 + \binom{15}{13}(0.25)^{13}(0.75)^2$$
$$+ \binom{15}{14}(0.25)^{14}(0.75)^1 + \binom{15}{15}(0.25)^{15}(0.75)^0$$
$$= 0.000\,795$$

4 **a** $P(\text{girl}) = \frac{0 \times 13 + 1 \times 34 + 2 \times 40 + 3 \times 13}{300}$

$$= \frac{153}{300} = 0.51$$

b $300 \times 0.51 \times 0.51 \times 0.49 = 38.2$

Exercise 14G

1 $E(X) = 12 = np$ and $\mathrm{Var}(X) = 3 = np(1 - p)$

Solving these simultaneously gives

$$12(1 - p) = 3$$
$$\frac{3}{12} = 1 - p$$
$$p = 1 - \frac{3}{12}$$
$$p = \frac{3}{4}$$
$$n \times \frac{3}{4} = 12$$
$$n = 12 \times \frac{4}{3}$$
$$n = 16$$

2 a $X \sim B(20, 0.2)$

b $E(X) = np = 20 \times 0.2 = 4$ and
$\text{Var}(X) = np(1-p) = 20 \times 0.2 \times 0.8 = 3.2$

c $P(X \geq 10) = 1 - P(X < 10)$

$= 1 - P(X = 0) - P(X = 1) - P(X = 2)$
$\quad - P(X = 3) - P(X = 4) - P(X = 5)$
$\quad - P(X = 6) - P(X = 7) - P(X = 8)$
$\quad - P(X = 9)$

$= 1 - \binom{20}{0}(0.2)^0(0.8)^{20} - \binom{20}{1}(0.2)^1(0.8)^{19}$
$- \binom{20}{2}(0.2)^2(0.8)^{18} - \binom{20}{3}(0.2)^3(0.8)^{17}$
$- \binom{20}{4}(0.2)^4(0.8)^{16} - \binom{20}{5}(0.2)^5(0.8)^{15}$
$- \binom{20}{6}(0.2)^6(0.8)^{14} - \binom{20}{7}(0.2)^7(0.8)^{13}$
$- \binom{20}{8}(0.2)^8(0.8)^{12} - \binom{20}{9}(0.2)^9(0.8)^{11}$
$= 0.00259$

3 We know that
$\text{Var}(X) = np(1-p) = 12 \times p(1-p) = 1.92$, so

$p(1-p) = \frac{1.92}{12}$
$0 = p^2 - p + 0.16$
$0 = (p - 0.8)(p - 0.2)$

Which gives us that $p = 0.2$ or $p = 0.8$.

Exercise 14H

1 a $P(-2 < Z < -1) + P(1 < Z < 2)$

$= 0.1359 + 0.1359 = 0.272$

b $P(-1.5 < z < -0.5) + P(0.5 < Z < 1.5)$

$= 0.2417 + 0.2417 = 0.483$

2 a $P(Z > 1) = 0.159$

b $P(Z > 2.4) = 0.0082$

c $P(Z < -1) = 0.159$

d $P(Z < -1.75) = 0.0401$

3 a $P(Z < 0.65) = 0.7422$

b $P(Z > 0.72) = 0.2358$

c $P(Z \geq 1.8) = 0.0359$

d $P(Z \leq -0.28) = 0.3897$

4 a $P(0.2 < Z < 1.2) = 0.3057$

b $P(-2 < Z \leq 0.3) = 0.5952$

c $P(-1.3 \leq X \leq -0.3) = 0.2853$

5 a $P(|Z| < 0.4)$

$= P(-0.4 < Z < 0.4) = 0.311$

b $P(|Z| > 1.24)$

$= P(Z > 1.24) + P(Z < -1.24) = 0.215$

Exercise 14I

1 $X \sim N(14, 5^2)$

a $P(X < 16) = P\left(Z < \frac{16-14}{5}\right)$

$= P(Z < 0.4) = 0.655$

b $P(X > 9) = P\left(Z > \frac{9-14}{5}\right)$

$= P(Z > -1) = 0.841$

c $P(9 \leq X < 12) = P\left(\frac{9-14}{5} \leq Z < \frac{12-14}{5}\right)$

$= P(-1 \leq Z < -0.4) = 0.186$

d As the mean is 14, $P(X < 14) = 0.5$

2 $X \sim N(48, 81)$

a $P(X < 52) = P\left(Z < \frac{52-48}{\sqrt{81}}\right)$

$= P(Z < 0.4444) = 0.672$

b $P(X \geq 42) = P\left(Z > \frac{42-48}{\sqrt{81}}\right)$

$= P(Z > -0.6667) = 0.748$

c $P(37 < X < 47) = P\left(\frac{37-48}{\sqrt{81}} < Z < \frac{47-48}{\sqrt{81}}\right)$

$= P(-1.2222 < Z < -0.1111) = 0.345$

3 $X \sim N(3.15, 0.02^2)$

a $P(X < 3.2) = P\left(Z < \frac{3.2-3.15}{0.02}\right)$

$= P(Z < 2.5) = 0.994$

b $P(X \geq 3.11)$

$= P\left(Z \geq \frac{3.11-3.15}{0.02}\right)$

$= P(Z > -2) = 0.977$

c $P(3.1 < X < 3.15)$

$= P\left(\frac{3.1-3.15}{0.02} < Z < \frac{3.15-3.15}{0.02}\right)$

$= P(-2.5 < Z < 0) = 0.494$

Exercise 14J

1 $X \sim N(100, 20^2)$

a $P(X < 130) = P\left(Z < \frac{130-100}{20}\right)$

$= P(Z < 1.5) = 0.933$

b $P(X > 90)$

$= P\left(Z > \frac{90-100}{20}\right)$

$= P(Z > -0.5) = 0.691$

c $P(80 < X < 125)$

$= P\left(\frac{80-100}{20} < Z < \frac{125-100}{20}\right)$

$= P(-1 < Z < 1.25) = 0.736$

2 $X \sim N(4, 0.25^2)$,

$P(3.5 < X < 4.5) = P\left(\frac{3.5-4}{0.25} < Z < \frac{4.5-4}{0.25}\right)$

$= P(-2 < Z < 2) = 0.9545$

, now $Y \sim B(500, 0.9545)$ and $E(Y) = np = 500 \times 0.9545 = 477.25$ so one would expect 477 to be accepted on average

3 $X \sim N(14, 4^2)$

a $P(X > 20) = P\left(Z > \frac{20-14}{4}\right)$

$= P(Z > 1.5) = 0.0668$

b $P(X < 10) = P\left(Z < \frac{10-14}{4}\right)$

$= P(Z < -1) = 0.1587 = 15.87\%$

4 $X \sim N(551.3, 15)$,

$P(X > 550) = P\left(Z > \frac{550-551.3}{15}\right)$

$= P(Z > -0.08667) = 0.5345 \approx 53.5\%$

5 $X \sim N(500, 20)$

a $P(X < 475) = P\left(Z < \frac{475-500}{20}\right)$

$= P(Z < -1.25) = 0.106$

b $P(3 \text{ packets less than } 475 \text{ g})$

$= (P(X < 475))^3 = 0.1056^3 = 0.00118$

Exercise 14K

1 **a** $P(Z < a) = 0.922 \therefore a = 1.42$

b $P(Z > a) = 0.342 \therefore a = 0.407$

c $P(Z > a) = 0.005 \therefore a = 2.58$

2 **a** $P(1 < Z < a) = 0.12$

$= P(Z < a) - P(Z < 1)$

$P(Z < a) - 0.8413 = 0.12$

$P(Z < a) = 0.12 + 0.8413$

$P(Z < a) = 0.9613$

$\therefore a = 1.77$

b $P(a < Z < 1.6) = 0.787$

$= P(Z < 1.6) - P(Z < a)$

$0.9452 - P(Z < a) = 0.787$

$P(Z < a) = 0.9452 - 0.787$

$P(Z < a) = 0.1582$

$\therefore a = -1.00$

c $P(a < Z < -0.3)$

$= 0.182 = P(Z < -0.3) - P(Z < a)$

$0.3821 - P(Z < a) = 0.182$

$P(Z < a) = 0.3821 - 0.182$

$P(Z < a) = 0.2001$

$\therefore a = -0.841$

3 **a** $\frac{1}{2}(1-0.3) = 0.35$, so we look for a such that $P(Z < a) = 1 - 0.35 = 0.65$,

$\therefore a = 0.385$

b $\frac{1}{2}0.1096 = 0.0548$, so we look for a such that

$P(Z < a) = 1 - 0.0548 = 0.9452$,

$\therefore a = 1.60$

4 **a** $P(Z < a) = 0.95 \therefore a = 1.64$

b $P(Z > a) = 0.2 \therefore a = 0.842$

Exercise 14L

1 $0.235 = P(X > a) = P\left(Z > \frac{a-5.5}{0.2}\right)$, this gives that $\frac{a-5.5}{0.2} = 0.722$

$\Rightarrow a = 0.722 \times 0.2 + 5.5 \Rightarrow a = 5.64$

2 $M \sim N(420, 10^2)$

a The first quartile equates to

$0.25 = P(M < a) = P\left(Z < \frac{a-420}{10}\right)$, so

$\frac{a-420}{10} = -0.674 \Rightarrow a = -0.674 \times 10 + 420$

$\Rightarrow a = 413$

b The 90th percentile is

$0.9 = P(M < a) = P\left(Z < \frac{a-420}{10}\right)$, giving

us $\frac{a-420}{10} = 1.282$

$\Rightarrow a = 1.282 \times 10 + 420 \Rightarrow a = 433$

3 $X \sim N(502, 1.6^2)$

a $P(X < 500)$

$= P\left(Z < \frac{500-502}{1.6}\right)$

$= P(Z < -1.25) = 0.106$

b $P(500 < X < 505)$

$= P\left(\frac{500-502}{1.6} < Z < \frac{505-502}{1.6}\right)$

$= P(-1.25 < z < 1.875)$

$= 0.864 = 86.4\%$

c $0.95 = P(b < X < a) = P(-a < X < a)$, so $\frac{1}{2}(1-0.95) = 0.025$, so we look for a' such that $P(Z < a') = 1 - 0.025 = 0.975$, $\therefore a' = 1.96$, so

$a = 1.6a' + 502 = 1.6 \times 1.96 + 502 = 505.1$

4 $X \sim N(550, 25^2)$

a $P(520 < X < 570)$

$= P\left(\frac{520-550}{25} < Z < \frac{570-550}{25}\right)$

$= P(-1.2 < Z < 0.8) = 0.673$

b $0.9 = P(X < a) = P\left(Z < \frac{a-550}{25}\right)$

$\frac{a-550}{25} = 1.282 \Rightarrow a = 1.282 \times 25 + 550$

$\Rightarrow a = 582$

5 $X \sim N(55, 15^2)$

a $0.95 = P(X < d) = P\left(Z < \frac{d-55}{15}\right)$, giving

$\frac{d-55}{15} = 1.645$

$\Rightarrow d = 1.645 \times 15 + 55 \Rightarrow d = 79.7$

b $0.1 = P(X < f) = P\left(Z < \frac{f-55}{15}\right)$, giving

$\frac{f-55}{15} = -1.282$

$\Rightarrow f = -1.282 \times 15 + 55 \Rightarrow f = 35.8$

Exercise 14M

1 $X \sim N(30, \sigma^2)$,

$0.115 = P(X > 40) = P\left(Z > \frac{40-30}{\sigma}\right)$, so

$\frac{40-30}{\sigma} = 1.2 \Rightarrow \sigma = \frac{40-30}{1.2} \Rightarrow \sigma = 8.33$

2 $X \sim N(\mu, 4^2)$,

$0.9 = P(X < 20.5) = P\left(Z < \frac{20.5-\mu}{4}\right)$, so

$\frac{20.5-\mu}{4} = 1.282 \Rightarrow \mu = 20.5 - 1.282 \times 4$

$\Rightarrow \mu = 15.4$

3 $X \sim N(\mu, \sigma^2)$,

$0.0217 = P(X > 58.39) = P\left(Z > \frac{58.39-\mu}{\sigma}\right)$

and

$0.0287 = P(X < 41.82) = P\left(Z < \frac{41.82-\mu}{\sigma}\right)$

, so $\frac{58.39-\mu}{\sigma} = 2.02 \Rightarrow 58.39 = 2.02\sigma + \mu$

and

$\frac{41.82-\mu}{\sigma} = -1.90 \Rightarrow -41.82 = 1.90\sigma - \mu$,

solving these simultaneously gives $\sigma = 4.23$ and $\mu = 49.9$

4 $X \sim N(\mu, \sigma^2)$,

$0.90 = P(X < 89) = P\left(Z < \frac{89-\mu}{\sigma}\right)$ and

$0.95 = P(X < 94) = P\left(Z < \frac{94-\mu}{\sigma}\right)$, so

$\frac{89-\mu}{\sigma} = 1.282 \Rightarrow 89 = 1.282\sigma + \mu$ and

$\frac{94-\mu}{\sigma} = 1.645 \Rightarrow 94 = 1.645\sigma + \mu$, solving these simultaneously gives $\sigma = 13.8$ and $\mu = 71.3$

5 $X \sim N(136, \sigma^2)$,

$0.12 = P(X > 145) = P\left(Z > \frac{145-136}{\sigma}\right)$, so

$\frac{145-136}{\sigma} = 1.175 \Rightarrow \sigma = \frac{145-136}{1.175}$

$\Rightarrow \sigma = 7.66$ cm

6 $X \sim N(\mu, 20^2)$,

$0.01 = P(X < 500) = P\left(Z < \frac{500-\mu}{20}\right)$, so

$\frac{500-\mu}{20} = -2.326$

$\Rightarrow \mu = 500 + 2.326 \times 20 \Rightarrow \mu = 546.5$ g

7 $X \sim N(0.85, \sigma^2)$

a $0.74 = P(X < 1.1) = P\left(Z > \frac{1.1-0.85}{\sigma}\right)$,

so $\frac{1.1-0.85}{\sigma} = 0.643$

$\Rightarrow \sigma = \frac{1.1-0.85}{0.643} \Rightarrow \sigma = 0.389$ kg

b $P(X>1)=P\left(Z>\frac{1-0.85}{0.389}\right)$

$=P(Z>0.386)=0.3497=35\%$

8 $X\sim N(\mu,7^2)$,

$0.025=P(X>68)=P\left(Z>\frac{68-\mu}{7}\right)$, so

$\frac{68-\mu}{7}=1.96\Rightarrow\mu=68-1.96\times7$

$\Rightarrow\mu=54.3$ cm

9 $X\sim N(2.9,\sigma^2)$,

$0.35=P(X>3)=P\left(Z>\frac{3-2.9}{\sigma}\right)$, so

$\frac{3-2.9}{\sigma}=0.385\Rightarrow\sigma=\frac{3-2.9}{0.385}$

$\Rightarrow\sigma=0.260$ m

10 $X\sim N(\mu,\sigma^2)$

a $0.30=P(X<108)=P\left(Z<\frac{108-\mu}{\sigma}\right)$

and

$0.20=P(X>154)=P\left(Z>\frac{154-\mu}{\sigma}\right)$, so

$\frac{108-\mu}{\sigma}=-0.524\Rightarrow108=-0.524\sigma+\mu$

and

$\frac{154-\mu}{\sigma}=0.842\Rightarrow154=0.842\sigma+\mu$,

solving these simultaneously gives

$\sigma=33.7$ and $\mu=125.66$

b $P(X>117)=P\left(Z>\frac{117-125.66}{33.7}\right)$

$=P(Z>-0.257)=0.601=60.1\%$,

so this is consistent with the normal distribution

11 $X\sim N(\mu,\sigma^2)$,

$0.95=P(X>495)=P\left(Z>\frac{495-\mu}{\sigma}\right)$ and

$0.99=P(X>490)=P\left(Z>\frac{490-\mu}{\sigma}\right)$, so

$\frac{495-\mu}{\sigma}=-1.645\Rightarrow495=-1.645\sigma+\mu$

and

$\frac{490-\mu}{\sigma}=-2.326\Rightarrow490=-2.326\sigma+\mu$,

solving these simultaneously gives

$\sigma=7.34$ and $\mu=507.1$

Chapter review

1 **a** $0.3+\frac{1}{k}+\frac{2}{k}+0.1+0.1=1$

$0.5+\frac{3}{k}=1$

$\frac{3}{k}=0.5$

$k=\frac{3}{0.5}=6$

b $E(X)$

$=-2\times0.3+-1\times\frac{1}{6}+0\times\frac{2}{6}$

$+1\times0.1+2\times0.1$

$=-\frac{6}{10}-\frac{1}{6}+0+\frac{1}{10}+\frac{2}{10}$

$=-\frac{3}{10}-\frac{1}{6}=-\frac{9}{30}-\frac{5}{30}=-\frac{14}{30}=-\frac{7}{15}$

2 **a** $1=c(6-1)+2c(6-2)+3c(6-3)$

$+4c(6-4)+5c(6-5)$

$1=5c+8c+9c+8c+5c$

$1=35c$

$c=\frac{1}{35}$

b $E(X)$

$=1\times\frac{5}{35}+2\times\frac{8}{35}+3\times\frac{9}{35}$

$+4\times\frac{8}{35}+5\times\frac{5}{35}$

$=\frac{5}{35}+\frac{16}{35}+\frac{27}{35}+\frac{32}{35}+\frac{25}{35}$

$=\frac{105}{35}$

$=3$

3 Find the value of x:

$\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+x=1$

$\frac{5}{8}+x=1$

$x=\frac{3}{8}$

$P(\text{total }6)=P(2,4)+P(3,3)+P(4,2)$

$=\frac{1}{4}\times\frac{3}{8}+\frac{1}{8}\times\frac{1}{8}+\frac{3}{8}\times\frac{1}{4}$

$=\frac{3}{32}+\frac{1}{64}+\frac{3}{32}$

$=\frac{6}{64}+\frac{1}{64}+\frac{6}{64}$

$=\frac{13}{64}$

4 **a** 2,4,6,8,12,16

b $\frac{1}{8}, \frac{2}{8}, \frac{1}{8}, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}$

c $E(X)$

$= 2 \times \frac{1}{8} + 4 \times \frac{2}{8} + 6 \times \frac{1}{8} + 8 \times \frac{2}{8}$

$+ 12 \times \frac{1}{8} + 16 \times \frac{1}{8}$

$= \frac{2}{8} + \frac{8}{8} + \frac{6}{8} + \frac{16}{8} + \frac{12}{8} + \frac{16}{8}$

$= \frac{60}{8}$

$= 7.5$

d E(Money per week)

$= 5 \times \frac{6}{8} + 10 \times \frac{2}{8} = \frac{30}{8} + \frac{20}{8} = \frac{50}{8}$

$= £6.25$

E(Money in 10 weeks)

$= 10 \times £6.25 = £62.50$

5 $X \sim B\left(5, \frac{1}{3}\right)$

$P(X = 3) = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}$

6 $P(X = 0) = 0.9 \times 0.9 = 0.81$

$P(X = 1) = 2 \times 0.1 \times 0.9 = 0.18$

$P(X = 2) = 0.1 \times 0.1 = 0.01$

$E(X) = 0 \times 0.81 + 1 \times 0.18 + 2 \times 0.01$

$= 0 + 0.18 + 0.02 = 0.2$

7 **a** $P(X < 65) = P(X > a)$

By symmetry $a = 75 + (75 - 65) = 85$

b $P(65 < X < a) = 0.954$

$P(X < a) - P(X < 65) = 0.954$

$P(X < a) - 0.023 = 0.954$

$P(X < a) = 0.977$

$\therefore a = 85$

$P(X > 85) = 0.023$

8 **a** $P(X = 1) = \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} + \left(\frac{2}{3}\right)^2 \times \frac{1}{3}$

$= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} = \frac{9}{27} + \frac{6}{27} + \frac{4}{27} = \frac{19}{27}$

b

x	$P(X = x)$
-5	$\frac{8}{27}$
1	$\frac{19}{27}$

c **i** $E(X) = -5 \times \frac{8}{27} + 1 \times \frac{19}{27}$

$= -\frac{40}{27} + \frac{19}{27} = -\frac{21}{27} = -\frac{7}{9}$

Expected loss of \$0.78

ii Expected loss of \$7

9 **a** $X \sim B(8, 0.3)$

$P(X = 3) = \binom{8}{3} 0.3^3 0.7^5 = 0.254$

b $P(X \geq 3) = 0.448$

10 X = no. of sixes when 6 dices are thrown

$X \sim B\left(6, \frac{1}{6}\right)$

$P(X = 3) = 0.0536$

Y = no. of times three sixes are seen

$Y \sim B(5, 0.0536)$

$P(Y = 2) = 0.0243$

11 **a** **i** $X \sim B(10, 0.2)$

$P(X = 4) = 0.0881$

ii $P(X > 5) = 0.00637$

b $E(X) = np = 10 \times 0.2 = 2$

c $Y \sim B(n, 0.2)$

$P(Y > 1) = 1 - P(Y = 0) = 1 - 0.8^n$

$1 - 0.8^n > 0.95$

$0.05 > 0.8^n$

$\log 0.05 > n \log 0.8$

$\frac{\log 0.05}{\log 0.8} < n$

$13.4 < n$

$\therefore n = 14$

12 $P(-a < Z < a) = 0.85$

By symmetry:

$P(Z < a) = 1 - \frac{0.85 - 0.5}{2} = 0.825$

$\therefore a = 0.935$

13 **a** $P(X < 80) = 0.85$

$P\left(Z < \frac{80 - 71}{\sigma}\right) = 0.85$

$\frac{80 - 71}{\sigma} = 1.036$

$\frac{9}{\sigma} = 1.036$

$\sigma = \frac{9}{1.036} = 8.68$

b $P\left(Z > \frac{65 - 71}{8.68}\right) = P(Z > -0.69) = 0.755$

14 $P\left(Z < \frac{30-\mu}{\sigma}\right) = 0.15$

$\Rightarrow \frac{30-\mu}{\sigma} = -1.036$

$\Rightarrow \mu = 30 + 1.036\sigma$

$P\left(Z > \frac{50-\mu}{\sigma}\right) = 0.10$

$\Rightarrow \frac{50-\mu}{\sigma} = 1.282$

$\Rightarrow \mu = 50 - 1.282\sigma$

$\therefore 30 + 1.036\sigma = 50 - 1.282\sigma$

$\Rightarrow 2.318\sigma = 20$

$\sigma = 8.63$

$\mu = 50 - 1.282(8.63) = 38.9$

15 a $P\left(Z > \frac{35-\mu}{2}\right) = 0.2$

$\Rightarrow \frac{35-\mu}{2} = 0.841..$

$\Rightarrow 35 - \mu = 1.683..$

$\Rightarrow \mu = 35 - 1.683.. = 33.3$

b $X \sim B(5, 0.8)$

$P(X = 5) = 0.328$

c $Y \sim B(5, 0.2)$

$P(X \geq 2) = 0.263$

16 a 0,1,2 A1

b $P(X = 2) = \frac{10}{18} \times \frac{10}{18} = \frac{25}{81}$ M1A1

c

x	0	1	2
$P(X = x)$	$\frac{16}{81}$	$\frac{40}{81}$	$\frac{25}{81}$

A2

17 a $0.2 + k + 0.25 + k - 0.05 + 0.3 = 1$

$\Rightarrow k = 0.15$ M1A1A1

b $E(X)$

$= 0 \times 0.2 + 1 \times 0.4 + 2 \times 0.1 + 3 \times 0.3 = 1.5$
M1A1

18 a $0.05 + 0.22 + 0.27 + a + b = 1$

$\Rightarrow a + b = 0.46$ M1A1

b $E(X) = 2.46$

$\Rightarrow 0 \times 0.05 + 1 \times 0.22 + 2 \times 0.27 + 3a + 4b$

$= 2.46$ M1A1

$3a + 4b = 1.7$

Solve simultaneously

$a + b = 0.46$ and $3a + 4b = 1.7$ M1

$a = 0.14$, $b = 0.32$ A1A1

19 a $X : B(10, 0.005)$ M1

$P(X = 1) = 0.0478$ (3 s.f.) A1

b $P(X \leq 1) = 0.999$ (3 s.f.) M1A1

c $P(X = 1 | X \leq 1) = \frac{P(X = 1)}{P(X \leq 1)}$

$= 0.0478$ M1A1A1

20 Let $X : B(n, p)$.

$np = 3$ and $npq = 1.2$ A1A1

Solve simultaneously M1

$q = 0.4 \Rightarrow p = 0.6$ A1

$n = 5$ A1

21 $X : N(50.1, 0.4^2)$

a $P(X < 49.5) = 0.0668$ (3 s.f.) M1A1

b $P(49.5 < X < 50.5) = 0.775$ (3 s.f.)
M1A1

c $P(X > 49 | X < 49.5)$

$= \frac{P(49 < X < 49.5)}{P(X < 49.5)} = 0.955$ M1A1A1

22 $X : N(\mu, 5^2)$

a $P(X < 5) = 0.754$

$\Rightarrow P\left(Z < \frac{5-\mu}{3}\right) = 0.754$ M1

$\frac{5-\mu}{3} = 0.6871... \Rightarrow \mu = 2.94$ M1A1

b $P(4 < X < 5) = 0.116$ M1A1

23 a i Let X be the number of correct answers in the 12 questions answered at random.

$X : B(12, 0.5)$ M1

$P(X = 2) = \binom{12}{2}(0.5)^{12} = 0.0161$

M1A1

ii $P(X = 12) = \binom{12}{12}(0.5)^{12} = 0.000244$

A1

b $E(X) = 12 \times 0.5 \times 0.5$

$= 3$ correct answers M1

3 correct random answers = 6 marks
A1

9 incorrect random answers =-9 marks
A1

8 answers known =16 marks A1

If the student answers all the question the expected number of marks is 13 marks which is 3 less than the total marks if he just answers the questions he knows the correct answer. R1

24 a i $W : N(\mu, \sigma^2)$

$P(W < 65) = 0.27$

$\Rightarrow P\left(Z < \frac{65-\mu}{\sigma}\right) = 0.27$ M1

$P(W > 96) = 0.25$

$\Rightarrow P\left(Z < \frac{96-\mu}{\sigma}\right) = 0.75$

$\frac{65-\mu}{\sigma} = -0.6128...$

$\frac{96-\mu}{\sigma} = 0.6744...$ A1A1

ii Solve simultaneously

$\frac{65-\mu}{\sigma} = -0.6128...,$ §

$\frac{96-\mu}{\sigma} = 0.6744...$ M1

$\mu = 79.8$ and $\sigma = 24.1$ A1A1

b $P(W > 100) = 0.20$ M1A1

c Let $Y : N(80.5, 10.1^2)$

$P(75 < Y < 85) = 0.379$. M1A1

d $630P(Y > 85) = 207$ M1A1

e $\frac{630 \times 80.5 + 370m}{1000} = 79.7573...$

M1

$m = 78.5\,\text{kg}$ A1

25 a i $T : N(45, 9^2)$

$P(T \geq 55) = 0.133$ M1A1

ii $P(T \geq 65 | T > 55) = \frac{P(T > 65)}{P(T \geq 55)}$

$= \frac{0.01313...}{0.13326...} = 0.0986$ M1A1A1

b $(0.133...)^3 = 0.00237$ M1A1

c $N : B(50, 0.133...)$

i $E(N) = 50 \times 0.133... = 6.66$ M1A1

ii $P(N \geq 5) = 1 - P(N \leq 4) = 0.814$

M1M1A1

26 $P(X > 82) = 0.1$

$\Rightarrow P(X < 82) = 0.9$

$\Rightarrow P\left(Z < \frac{82-\mu}{\sigma}\right) = 0.9$ M1A1

$P(X < 40) = 0.2$

$\Rightarrow P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.2$ A1

$\frac{82-\mu}{\sigma} = 1.28...,$

$\frac{40-\mu}{\sigma} = -0.841...$ M1

Solve simultaneously M1

$\mu = 56.6$ and $\sigma = 19.8$ A1A1