OXFORD IB DIPLOMA PROGRAMME

MIXED REVIEW

MATHEMATICS:
ANALYSIS AND APPROACHES

STANDARD LEVEL

COURSE COMPANION

ENHANCED ONLINE

Natasha Awada
Laurie Buchanan
Jennifer Chang Wathall
Ed Kemp
Paul La Rondie
Jill Stevens
Ellen Thompson

OXFORD

2 Representing relationships: introducing functions

1 We consider the function $f(x) = \frac{e^x - 1}{e^x + 1}$.

a Write down the domain of the above function.

b Work out the inverse of the above function.

c Work out the domain of the inverse function.

d Hence, write down the range of f.

e Prove that $f(f^{-1}(x)) = x = f^{-1}(f(x))$, for every x in the domain of f.

2 We consider the functions $f(x) = 2kx^2 + 3(k+1)x - 7, k \in \mathbb{R}$ and $g(x) = x + 2$. Prove that there is no value for the parameter $k \in \mathbb{R}$ such that the function $f(g(x))$ is a perfect square.

3 **a** if $f(x) = x + \sqrt{x^2 + 1}$, $x \in \mathbb{R}$, prove that $f(x) \times f(-x) = 1$, $\forall x \in \mathbb{R}$.

b We consider function $f(x) = \frac{ax + 3}{2 - x}, \forall x \neq 2$. Find the value of $a \in \mathbb{R}$ such that $f(f(x)) = x, \forall x \neq 2$.

4 Given the functions $f(x) = \ln(x + \sqrt{x^2 + 1})$ and $g(x) = \frac{e^x - e^{-x}}{2}$, prove that $f(g(x)) = g(f(x))$.

5 We consider the function $f(x) = x + \ln(x), x > 0$.

a Prove that f is a 1 -1 function.

b Work out the value of $f^{-1}(e + 1)$.

c Solve the equation: $\ln(\frac{2\lambda^2 + 1}{\lambda^2 + 5}) = 4 - \lambda^2$

Exam-style questions

6 It is given that $f(x) = \frac{1}{\sqrt{x}} - 1$, $x > 0$, $x \in \mathbb{R}$.

a Determine the exact value of $ff(0.01)$. (4)

b Find an expression for $f^{-1}(x)$. (3)

c State the domain and range of $f^{-1}(x)$. (2)

7 **a** Find the expansion of $\left(\frac{1}{4}+x\right)^4$ in ascending powers of x. (3)

b Find the expansion of $\left(2x-\frac{1}{x}\right)^6$ in ascending powers of x. (3)

c Hence, or otherwise, find the term independent of x in the expansion of $\left(2x-\frac{1}{x}\right)^6\left(\frac{1}{4}+x\right)^4$. (3)

8 A convergent geometric series has sum to infinity of 65.

Given that the first term in the series is 52, find the number of terms required for the sum of the series to exceed 64.99. (7)

9 **a** Find the value of a, given that $f(x)=\frac{4x-1}{2x+a}$ is a self-inverse function. (6)

b Determine the largest possible domain for $f(x)$. (2)

c Solve the equation $fff(x)=\frac{1}{4}$. (3)

10 It is given that $\frac{x^2-36}{2x^2-11x-6}\times\frac{Ax^2+Bx+C}{x-4}\times\frac{2x^2-11x+12}{x^2+9x+18}\equiv\frac{2x^2+x-5}{2x+1}$

Prove that $A=1, B=5$ and $C=6$ (7)

Answers

1 **a** Domain $= \mathbb{R}$

b $y = \dfrac{e^x - 1}{e^x + 1}$

$y(e^x + 1) = e^x - 1$

$ye^x + y = e^x - 1$

$e^x(y - 1) = -y - 1$

$e^x = \dfrac{1 + y}{1 - y}$

$f^{-1}(x) = \ln(\dfrac{1 + x}{1 - x})$

c The domain of the inverse is $D_{f^{-1}} = \{x \in \mathbb{R} \,/\, \dfrac{1 + x}{1 - x} > 0\}$.

We obtain $D_{f^{-1}} = \{x \in \mathbb{R} \,/\, (1 + x)(1 - x) > 0\} = \{x \in \mathbb{R} \,/\, -1 < x < 1\}$.

d We know that if $f : D_f \to R_f$, where D_f, R_f are domain and range respectively, then $f^{-1} : R_f \to D_f$.

Hence, $R_f = \{y \in \mathbb{R} \,/\, -1 < y < 1\}$.

e $f(f^{-1}(x)) = \dfrac{e^{\ln(\frac{1+x}{1-x})} - 1}{e^{\ln(\frac{1+x}{1-x})} + 1} = \dfrac{\frac{1 + x}{1 - x} - 1}{\frac{1 + x}{1 - x} + 1} = \dfrac{\frac{1 + x - 1 + x}{1 - x}}{\frac{1 + x + 1 - x}{1 - x}} = \dfrac{2x}{2} = x$

$f^{-1}(f(x)) = \ln(\dfrac{1 + \frac{e^x - 1}{e^x + 1}}{1 - \frac{e^x - 1}{e^x + 1}}) = \ln(\dfrac{\frac{e^x + 1 + e^x - 1}{e^x + 1}}{\frac{e^x + 1 - e^x + 1}{e^x + 1}}) = \ln(\dfrac{2e^x}{2}) = \ln(e^x) = x$

2 $f(g(x)) = 2k(x + 2)^2 + 3(k + 1)(x + 2) - 7 =$

$= 2k(x^2 + 4x + 4) + 3(kx + 2k + x + 2) - 7 =$

$2kx^2 + x(3 + 11k) + 14k - 1$.

Expression A is a perfect square if there is another expression B such that $B^2 = A$.
In this case $f(g(x))$ is a perfect square if the discriminant of it, in terms of the parameter k, is zero.
We are to prove this by contradiction:
We assume that there is at least one value for the parameter k such that the $f(g(x))$ is a perfect square.

We calculate the discriminant of the $f(g(x))$ in terms of k: $(11k + 3)^2 - 4 \times 2k \times (14k - 1) =$

$= 121k^2 - 38k + 9$.

We equal the above quantity to zero: $121k^2 - 38k + 9 = 0$.

Solving the above the equation: $D = 38^2 - 4 \times 121 \times 9 = -2912 < 0$ - contradiction.

Therefore, the is no value for real k.

3 **a** Initially, we find $f(-x) = -x + \sqrt{(-x)^2 + 1} = -x + \sqrt{x^2 + 1}$.

We calculate: $f(x) \times f(-x) = (x + \sqrt{x^2 + 1}) \times (-x + \sqrt{x^2 + 1}) = (\sqrt{x^2 + 1})^2 - x^2 = x^2 + 1 - x^2 = 1$.

b We work out the $f(f(x))$.

$$f(f(x)) = \frac{a\times(\frac{ax+3}{2-x})+3}{2-(\frac{ax+3}{2-x})} = \frac{\frac{a^2x+3a+3(2-x)}{2-x}}{\frac{2(2-x)-(ax+3)}{2-x}} = \frac{a^2x-3x+3a+6}{x(-a-2)+1}.$$

However, $f(f(x)) = x$.

Therefore, $\frac{a^2x-3x+3a+6}{x(-a-2)+1} = x$.

We obtain: $a^2x-3x+3a+6 = x^2(-a-2)+x$

$x^2(a+2)+x(a^2-4)+3a+6=0$.

Hence, $a+2=0$

$$\begin{array}{l} a^2-4=0 \\ 3a+6=0 \end{array},$$

from where we obtain that $a=-2$.

4 $f(g(x)) = \ln(\frac{e^x-e^{-x}}{2}+\sqrt{\frac{(e^x-e^{-x})^2+4}{4}}) = \ln(\frac{e^x-e^{-x}}{2}+\sqrt{\frac{e^{2x}+e^{-2x}+2}{4}}) = \ln(\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}) =$

$= \ln(\frac{e^x-e^{-x}+e^x+e^{-x}}{2}) = \ln(\frac{2e^x}{2}) = \ln(e^x) = x$

$$g(f(x)) = \frac{e^{\ln(x+\sqrt{x^2+1})}-e^{-\ln(x+\sqrt{x^2+1})}}{2} = \frac{e^{\ln(x+\sqrt{x^2+1})}-e^{\ln(\frac{1}{x+\sqrt{x^2+1}})}}{2} = \frac{x+\sqrt{x^2+1}-(\frac{1}{x+\sqrt{x^2+1}})}{2} =$$

$$= \frac{(x+\sqrt{x^2+1})^2-1}{2(x+\sqrt{x^2+1})} = \frac{x^2+2x\sqrt{x^2+1}+x^2+1-1}{2(x+\sqrt{x^2+1})} = \frac{x^2+x\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \frac{x(x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}} = x$$

Therefore, $f(g(x)) = g(f(x))$.

5 **a** We find the derivative of f: $f'(x) = 1+\frac{1}{x} > 0, \forall x \in \mathbb{R}$.

Hence, f is an increasing function. Therefore, f is 1 – 1.

b We know that $f(a)=b \Rightarrow a = f^{-1}(b)$.

We observe that $f(e) = e+\ln(e) = e+1$.

Therefore, $f^{-1}(e+1) = e$.

c By using the Laws of Logarithms, we obtain:

$\ln(2\lambda^2+1)-\ln(\lambda^2+5) = 4-\lambda^2$.

$\ln(2\lambda^2+1)+(2\lambda^2+1)-\ln(\lambda^2+5)-(\lambda^2+5) = 4-\lambda^2+(2\lambda^2+1)-(\lambda^2+5)$

$f(2\lambda^2+1)-f(\lambda^2+5)=0$

$f(2\lambda^2+1)=f(\lambda^2+5)$

Since f is 1 – 1 function, we can write:

$2\lambda^2+1=\lambda^2+5$

$\lambda^2=4$

$\lambda=\pm 2$.

6 **a** $f(0.01) = \frac{1}{\sqrt{0.01}}-1 = 9$ M1A1

$ff(0.1) = f(9) = \frac{1}{\sqrt{9}}-1 = -\frac{2}{3}$ M1A1

b Let $y = \frac{1}{\sqrt{x}} - 1$

Attempt to make x the subject M1

$y + 1 = \frac{1}{\sqrt{x}}$

$\sqrt{x} = \frac{1}{y+1}$

$x = \frac{1}{(y+1)^2}$ A1

$f^{-1}(x) = \frac{1}{(x+1)^2}$ A1

c Domain of $f^{-1}(x)$ is $x > -1$, $(x \in \mathbb{R})$ A1

Range of $f^{-1}(x)$ is $f^{-1}(x) > 0$, $(f^{-1}(x) \in \mathbb{R})$ A1

7 **a** $\left(\frac{1}{4} + x\right)^4 = \sum_{r=0}^{4} {}^4C_r x^{4-r} \left(\frac{1}{4}\right)^r$ M1

$= x^4 + x^3 + \frac{3x^2}{8} + \frac{x}{16} + \frac{1}{256}$ A1A1

b $\left(2x - \frac{1}{x}\right)^6 = \sum_{r=0}^{6} {}^6C_r (2x)^{6-r} \left(-\frac{1}{x}\right)^r$ M1

$= 64x^6 - 129x^4 + 240x^2 - 160 + \frac{60}{x^2} - \frac{12}{x^4} + \frac{1}{x^6}$ A1A1

c Require $\left(-160 \times \frac{1}{256}\right) + \left(60 \times \frac{3}{8}\right) + (-12 \times 1)$ M1A1

$= \frac{79}{8} (= 9.875)$ A1

8 $65 = \frac{52}{1-r}$ M1A1

$1 - r = \frac{52}{65}$

$r = \frac{13}{65} = 0.2$ A1

Let n be the required number of terms for the sum to be equal to 64.99.

$64.99 = \frac{52(1 - 0.2^n)}{1 - 0.2}$ M1A1

$\frac{51.992}{52} = 1 - 0.2^n$

$0.2^n = 1 - \frac{51.992}{52}$

$n = 5.46$ A1

So require 6 terms A1

9 a Let $y = \dfrac{4x-1}{2x+a}$

Attempt to make x the subject M1

$2xy + ay = 4x - 1$ A1

$1 + ay = 4x - 2xy$

$1 + ay = x(4 - 2y)$ A1

$x = \dfrac{1+ay}{4-2y}$ A1

$f^{-1}(x) = \dfrac{1+ax}{4-2x} \quad \left(\text{or } f^{-1}(x) = \dfrac{-ax-1}{2x-4}\right)$ A1

Comparison with $f(x) = \dfrac{4x-1}{2x+a}$ yields $a = -4$ A1

So $f(x) = \dfrac{4x-1}{2x-4}$

b Domain is $x \in \mathbb{R},\ x \neq 2$ A2

c f is self-inverse, so $fff(x) = f(x)$ R1

$f(x) = \dfrac{1}{4} \Rightarrow \dfrac{4x-1}{2x-4} = \dfrac{1}{4}$ M1

$16x - 4 = 2x - 4$

$14x = 0$

$x = 0$ A1

10 Attempt to factorise relevant expressions M1

$\dfrac{(x+6)(x-6)}{(2x+1)(x-6)} \times \dfrac{Ax^2+Bx+C}{x-4} \times \dfrac{(2x-3)(x-4)}{(x+6)(x+3)} \equiv \dfrac{(2x-3)(x+2)}{2x+1}$ A3

Attempt to cancel terms: M1

$\dfrac{Ax^2+Bx+C}{(x+3)} \equiv (x+2)$ A1

$Ax^2 + Bx + C \equiv (x+2)(x+3)$

$Ax^2 + Bx + C \equiv x^2 + 5x + 6$

Equating coefficients gives $A = 1, B = 5$ and $C = 6$ A1

3 Modelling relationships: linear and quadratic functions

1 The terms in an arithmetic sequence are given by $u_n = 3 + (n-1)(2)$.

a Write down

i the first term in the sequence;

ii the common difference.

b Find the first five terms in the sequence.

c Graph the points (n, u_n), for the five terms found in part b.

d Find the gradient-intercept form of the equation of the line that contains the points graphed in part c.

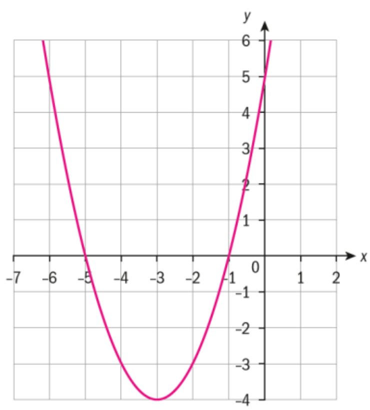
e Show that $u_n = 3 + (n-1)(2)$ is equivalent to the equation found in part d.

2 A linear function, f, passes through the points $(2, 4)$ and $(6, 3)$.

a Find the gradient-intercept form of the equation of the function.

b Find f^{-1}.

3 Part of the graph of the quadratic function f is shown in the diagram. The domain of f is all real numbers.



a Write down the range of f.

b Determine whether the inverse of f exists. Explain your reasoning.

c Write down

i the coordinates of the y-intercept

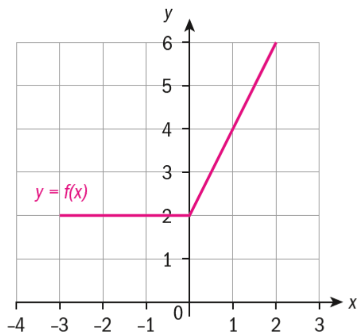
ii the coordinates of the x-intercepts

iii the vertex of the graph of f

iv the equation of the axis of symmetry of the graph of f.

d Find an equation for the function f.

4 The graph of $y = f(x)$, where $-3 \le x \le 2$, is shown in the diagram.



a Graph f and f^{-1} on the same axes.

b Given that $g(x) = 2f(x-3) - 5$, graph f and g on the same axes.

5 Let $f(x) = x^2 - 2x - 3$ and $g(x) = x - 2$.

a Let $h(x) = (f \circ g)(x)$. Show that $h(x) = x^2 - 6x + 5$.

b Find the equation of the axis of symmetry for the graph of h.

c Find the coordinates of the vertex of the graph of h.

d Find an equation for h in the form $h(x) = (x-p)(x-q)$, where p and q are integers.

e Sketch a graph of $y = -h(x)$, where $1 \le x \le 5$.

Exam-style questions

6 The first term of an arithmetic sequence is 0 and the common difference is 8.

a Find the value of the 32nd term in the sequence. (2)

The first term of a geometric sequence is 6. The 11th term of this sequence is equal to the 25th term of the arithmetic sequence above.

b Find the value of r, the common ratio of the geometric sequence. (4)

c Determine whether the sum to infinity of the geometric sequence exists. (2)

7 In increasing powers of x, the fourth term in the binomial expansion of $(3x - a)^9$ is $145152x^3$.

Determine the value of a. (5)

8 Find the range of values of x that satisfy both $2x^2 + 3x - 9 < 0$ and $2x^2 - 9x - 35 < 0$. (9)

9 **a** Determine the range of values of p for which the equation $px^2 + px + 1 = p$ has two distinct real solutions. (6)

Suppose that $f(x) = px^2 + px + 1 - p$, $x \in \mathbb{R}$.

b Show that the line of symmetry of the graph of $y = f(x)$ is independent of p. (2)

c In the case when $p = -4$, express $f(x)$ in the form $f(x) = a(x-h)^2 + k$ where a, h and k are integers. Hence solve the equation $f(x) = 0$. (6)

10 a Describe a series of transformations that maps the graph of $y = x^2$ onto the graph of $y = 3 - (x+5)^2$. (6)

Consider the function $f(x) = 3 - (x+5)^2$

b Find the largest possible domain of $f(x)$ such that f^{-1} exists, and justify your answer.

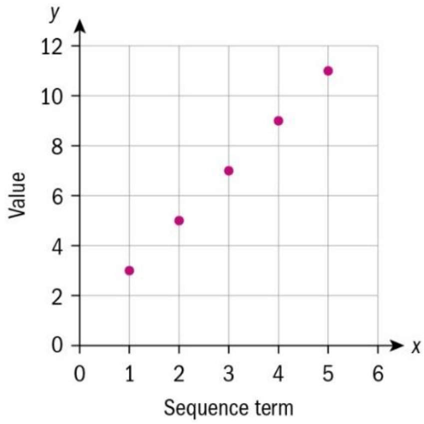
On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. (5)

Answers

1 **a** **i** $u_1 = 3$ **ii** 2

b 3, 5, 7, 9, 11

c



d $y = 2x + 1$

e let $n = x$, $y = u_n$

$y = 3 + (x - 1)(2)$

$y = 3 + 2x - 2 = 2x + 1$

2 **a** $y = \frac{-x}{4} + \frac{9}{2}$ **b** $f^{-1}(x) = -4x + 18$

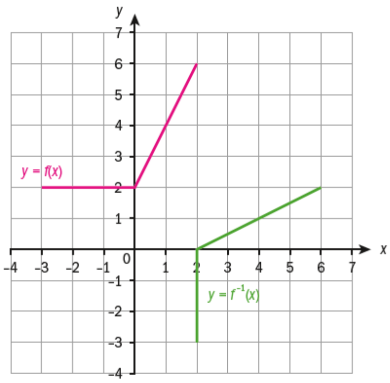
3 **a** $y \geq -4$

b The inverse does not exist, because f is not one-to-one.

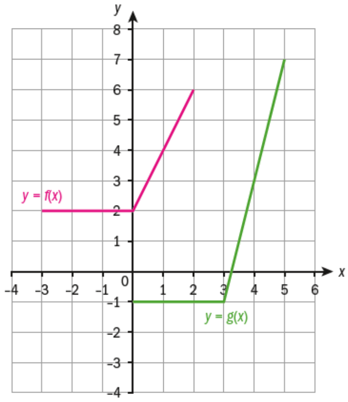
c **i** (0, 5) **ii** (−5, 0), (−1, 0) **iii** (−3, −4) **iv** $x = -3$

d $f(x) = x^2 + 6x + 5$ or $f(x) = (x + 5)(x + 1)$ or $f(x) = (x + 3)^2 - 4$

4 **a**



b



5 **a** $(f \cdot g)(x) = (x - 2)2 - 2(x - 2) - 3$

$= x^2 - 4x + 4 - 2x + 4 - 3$

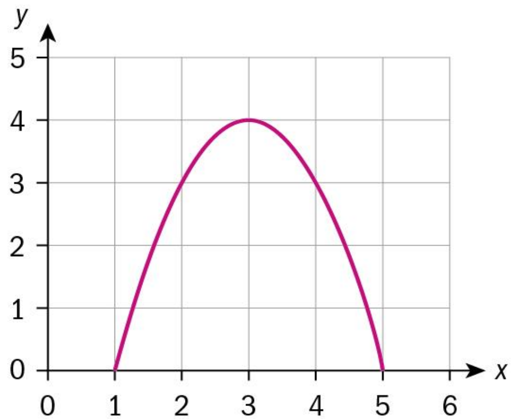
$= x^2 - 6x + 5$

b $x = 3$

c $(3, -4)$

d $h(x) = (x - 1)(x - 5)$

e



6 **a** $u_{32} = u_1 + 31d$ M1

$= 0 + 31 \times 8$

$= 248$ A1

b $6r^{10} = 24 \times 8$ M1A1

$r^{10} = 32$ A1

$r = \sqrt{2}$ A1

c $r = \sqrt{2} > 1$ R1

therefore S_∞ does not exist. A1

7 $\binom{9}{3} 3^3 (-a)^6 = 145152$ M1A1

$84 \times 27 \times a^6 = 145152$ A1

$a^6 = 64$ A1

$a = 2$ A1

8 $2x^2 + 3x - 9 < 0$

$(2x - 3)(x + 3) < 0$ M1

Critical values $x = -3,\ x = \frac{3}{2}$ A1A1

Solution $-3 < x < \frac{3}{2}$ A1

$2x^2 - 9x - 35 < 0$

$(2x + 5)(x - 7) < 0$ M1

Critical values $x = -\frac{5}{2},\ x = 7$ A1A1

Solution $-\frac{5}{2} < x < 7$ A1

So, solution required is $-\frac{5}{2} < x < \frac{3}{2}$ A1

9 a $px^2 + px + (1-p) = 0$

Require $b^2 - 4ac > 0$ M1

$p^2 - 4p(1-p) > 0$ A1

$5p^2 - 4p > 0$

$p(5p-4) > 0$ A1

Critical values $p = 0,\ p = \frac{4}{5}$ A1

Solution is $p < 0$ and $p > \frac{4}{5}$ A1A1

b Line of symmetry is $x = -\frac{b}{2a}$ M1

$x = -\frac{p}{2p} = -\frac{1}{2}$, so independent of p . R1

c $f(x) = -4x^2 - 4x + 5$

$= -4\left(x^2 + x - \frac{5}{4}\right)$ M1

$= -4\left(\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{5}{4}\right)$ A1

$= -4\left(\left(x + \frac{1}{2}\right)^2 - \frac{3}{2}\right)$

$= 6 - 4\left(x + \frac{1}{2}\right)^2$ A1

$f(x) = 0 \Rightarrow 6 - 4\left(x + \frac{1}{2}\right)^2 = 0$ M1

$\left(x + \frac{1}{2}\right)^2 = \frac{3}{2}$

$x + \frac{1}{2} = \pm\sqrt{\frac{3}{2}}$

$x = -\frac{1}{2} \pm \sqrt{\frac{3}{2}}$ A2

10 a Horizontal translation left 5 A1A1

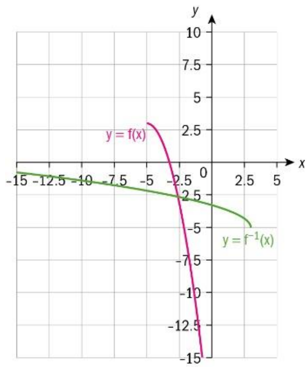
Reflection in the x-axis A1A1

Vertical translation up 3 A1A1

b For f^{-1} to exist, we require the domain of $y = 3 - (x+5)^2$ for which the function is one-to-one. R1

$y = 3 - (x+5)^2$ is maximized when $x = -5$, and hence function has line of symmetry when $x = -5$. R1

So function is one-to-one and f^{-1} exists when $x \geq -5,\ (x \in \mathbb{R})$. A1



Graph of $y = f(x)$	A1 for shape, A1 for domain
Graph of $y = f^{-1}(x)$	A1 for shape, A1 for domain

Note: There is a second possible solution where the domain of f is $x \le -5$

4 Equivalent representations: rational functions

1 In an arithmetic sequence, the first term is 6 and the fifth term is 22. Find the third term.

2 If $f(x) = 3x + 4$ and $g(x) = \frac{x+1}{x-1}$, find

a $f^{-1}(2)$

b $(g \circ f)(x)$

3 Find the equation of the line passing through the points $(-1,-1)$ and $(3,5)$.

4 Find the asymptotes, domain and range of $y = \frac{8x}{4x-1}$.

5 Find the term containing x^5 in the expansion of $(x+y)^{12}$.

6 Solve $x^2 + 2x - 3 = \frac{2x+1}{x+2}$.

7 Find the equation of the line perpendicular to and passing through the midpoint of *AB* when *A* is the point $(-8,-3)$ and *B* is $(12,-7)$.

8 Find the possible values of *k* in the quadratic equation $4x^2 + kx = -9$ when there are two identical solutions for *x*.

9 Find the sum of the infinite geometric series $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \ldots$

10 The rational function $y = \frac{5}{x-a} + b$ has a vertical asymptote at $x = 6$.

a Write down the value of *a*.

The function passes through $(1,7)$.

b Find the value of *b*.

c Write down the equation of the horizontal asymptote.

Exam-style questions

11 Consider the function given by $f(x) = \frac{2-3x}{x+4}$.

a State the equation of the vertical asymptote. (1)

b State the equation of the horizontal asymptote. (1)

c Sketch the graph of $y = f(x)$. On your sketch, show clearly the asymptotes from parts a and b as dashed lines. State the coordinates of the points where $y = f(x)$ intersects the coordinate axes. (5)

12 Consider the functions $f(x) = 3 + \frac{4}{x-5}$ $(x \neq 5,\ x \in \mathbb{R})$ and $g(x) = \frac{1}{x}$ $(x \neq 0,\ x \in \mathbb{R})$.

a The function $f(x)$ may be obtained from $g(x)$ through a sequence of transformations. State the required transformations (in order). (6)

b State the range of $f(x)$. (2)

c Find an expression for $f^{-1}(x)$. (4)

d State the domain and range of $f^{-1}(x)$. (2)

13 Line L_1 is perpendicular to the line $4x - 12y = 1$ and goes through the point $(-1, 8)$.

Line L_2 goes through the points $(-2, -8)$ and $(12, 3)$.

Find the exact coordinates of the point of intersection of L_1 and L_2 (9)

14 Consider the functions $f(x) = \frac{1}{x+1}$, $x \in \mathbb{R},\ x \neq -1$ and $g(x) = \frac{x}{4} - 1$, $x \in \mathbb{R}$.

a Given $p(x) = g \circ f(x)$, express $p(x)$ in the form $\frac{ax+b}{cx+d}$, where a, b, c, d are integers (3)

b Hence find an expression for $p^{-1}(x)$ (3)

c Hence find $pppppp(1)$, justifying your answer. (2)

15 Find the sum of the integers between 1000 and 2000 that are divisible by 7. (5)

Answers

1 $u_3 = 14$

2 **a** $-\frac{2}{3}$

b $\frac{3x+5}{3x+3}$

3 $y = \frac{3x+1}{2}$

4 Asymptote: $x = 0.25$

Domain: $x \in \mathbb{R} : x \neq 0.25$

Range: $y \in \mathbb{R} : y \neq 2$

5 $792x^5y^7$

6 $x = -2, -\frac{3}{2} \mp \frac{\sqrt{29}}{2}$

7 $y = 5x - 15$

8 $k = \pm 12$

9 $S = \frac{2}{5}$

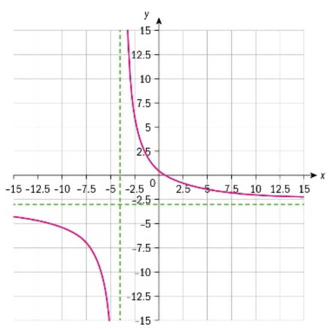
10 **a** $a = 6$

b $b = 8$

c $y = 8$

11 **a** $x = -4$ A1

b $y = -3$ A1



A1A1 for each branch

A1 for both asymptotes

Intercepts $\left(\frac{2}{3}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ A1A1

12 **a** Translation through $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ A1A1

Stretch by a scale factor 4, parallel to the y-axis. A1A1

Translation through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ A1A1

(Note: The first two transformations may be seen in any order)

b $f(x) \neq 3,\ f(x) \in \mathbb{R}$ A2

c $x = 3 + \dfrac{4}{y-5}$ M1

$x - 3 = \dfrac{4}{y-5}$ A1

$y - 5 = \dfrac{4}{x-3}$

$y = 5 + \dfrac{4}{x-3}$ A1

$f^{-1}(x) = 5 + \dfrac{4}{x-3}$ A1

d Domain is $x \neq 3,\ x \in \mathbb{R}$ A1

Range is $f^{-1}(x) \neq 5,\ f(x) \in \mathbb{R}$ A1

13 $4x - 12y = 1 \Rightarrow y = \dfrac{x}{3} - \dfrac{1}{12}$ M1

So L_1 has gradient of -3 A1

$y = -3x + c$

Substituting $(-1, 8)$ gives $8 = 3 + c$, so $c = 5$ M1A1

$L_1 : y = -3x + 5$

Equation of L_2 is $\dfrac{y-3}{x-12} = \dfrac{-8-3}{-2-12}$ (or equivalent) M1

$\dfrac{y-3}{x-12} = \dfrac{-11}{-14}$

$-14y + 42 = -11x + 132$

$L_2 : 11x - 14y - 90 = 0$ A1

Attempt to solve simultaneously: M1

$11x - 14(5 - 3x) - 90 = 0$

$11x - 70 + 42x - 90 = 0$

$53x = 160$

$x = \dfrac{160}{53}$ A1

$y = 5 - \dfrac{480}{53} = -\dfrac{215}{53}$ A1

$\left(\dfrac{160}{53}, -\dfrac{215}{53}\right)$ AG

14 a $p(x) = \dfrac{\left(\dfrac{1}{x+1}\right)}{4} - 1$ M1

$= \dfrac{1}{4x+4} - 1$ A1

$= \dfrac{1-(4x+4)}{4x+4}$

$= \dfrac{-4x-3}{4x+4}$ A1

b $x = \dfrac{-4y-3}{4y+4}$ M1

$x(4y+4) = -4y-3$ A1

$4xy + 4x = -4y - 3$

$4xy + 4y = -3 - 4x$

$4y(x+1) = -3 - 4x$

$y = \dfrac{-3-4x}{4(x+1)}$ A1

$f^{-1}(x) = \dfrac{-3-4x}{4x+4}$ AG

c $p(x)$ is a self-inverse function. R1

Therefore $pppppp(1) = 1$ A1

15 Require the summation of $1001 + 1008 + \ldots + 1995$ which is an AP with first term 1001, difference 7, and final term 1995. R1

$1995 = 1001 + 7(n-1)$ (or similar method to determine number of terms) M1

So require $n = 143$ terms A1

$S_{143} = \dfrac{143}{2}[2 \times 1001 + 142 \times 7]$ M1

$= 214\ 214$ A1

5 Measuring change: differentiation

1 Let $f(x) = x^2 - 7x + 6$

a Find the equation of the axis of symmetry of the graph of *f*.

$f(x)$ can also be expressed in the form $f(x) = (x - h)^2 + k$ for some constants *h* and *k*.

b Write down the exact value of *h*

c Find the value of *k*.

d By using the second derivative test, prove that (*h*, *k*) is a minimum point of $f(x)$.

2 Consider the functions $f(x) = 2x - 5$ and $g(x) = \frac{3x-8}{x-3}, x \neq 3$

a Find $g^{-1}(2)$.

b What do you deduce about the function $g(x)$?

c Find *x* such that $(f \circ g^{-1})(x) = 0$.

d State the equation of any asymptotes for $(f \circ g^{-1})(x)$.

3 An arithmetic sequence has r as its common difference, and a geometric sequence has r as its common ratio. For both sequences, $a_1 = 1$.

a Write down the first four terms of both sequences, in terms of r.

b If the sum of the third and fourth terms of the arithmetic sequence is equal to the sum of the third and four terms of the geometric sequence, find the three possible values of r.

c Find the value of r from part b for which S_∞ exists.

d For the value of r that you found in part **c**, find the sum of the first 20 terms of the arithmetic sequence.

4 The graph of $f(x) = a(x - b)(x - 2)$ has axis of symmetry $x = -0.5$ and *y*-intercept at (0,-6).

a Find the value of *b*.

b Find the value of *a*.

c Find the value(s) of *k* such that the line $y = kx - 10$ is tangent to the curve $y = f(x)$.

5 A function *f* has its derivative given by $f'(x) = 2x^2 - 2kx - 7$, where *k* is constant.

a The function has an inflexion point at $x=-1$. Find the value of *k*.

b Find the equation of the tangent to the curve *f* at (1,−5).

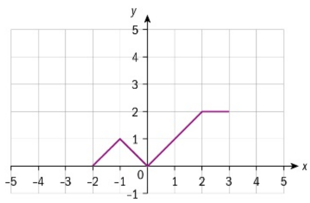
6 Consider the function $f(x) = \frac{8}{3}x^3 + x^2 - 3x - \frac{1}{3}$

a Find the coordinates of the local maximum and local minimum points of $y = f(x)$.

b The graph of f is translated to the graph of g by the translation $\begin{pmatrix} 0 \\ k \end{pmatrix}$. Find the value(s) of k such that $g(x) = 0$ has exactly one root.

Exam-style questions

7



Find

a the domain of f (1)

b the zeros of f (1)

c the range of f (1)

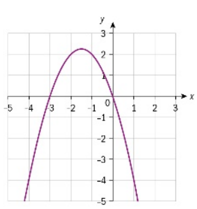
d the intervals where the derivative of f is positive (2)

e the coordinates of a point on the graph where the tangent to the graph is horizontal (1)

f the points in the domain of f where the derivative is not defined. (1)

8 **a** Determine which linear function (1), (2) or (3) is the graph of the derivative of each quadratic curve (A), (B) and (C). (3)

1



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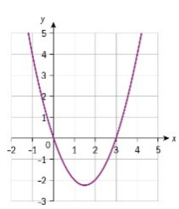

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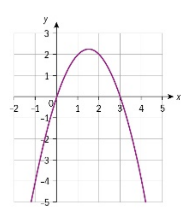
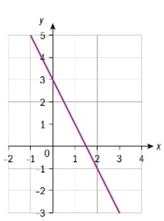
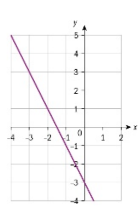


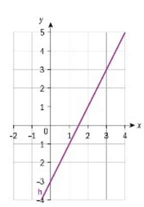
A



B



C



b State the zeros of the quadratic function in (A); (1)

c Write down the gradient of the tangent to the quadratic graph in (B) at $x = 0$. (1)

9 The following table shows the annual profits (in thousands of dollars) for a company during a 5-year period.

Year	2000	2001	2002	2003	2004
Profit	98.5	100.9	101.1	101.2	102.3

a Find

i the average rate at which the profits varied between 2000 and 2002;

ii the average rate at which the profits varied between 2002 and 2004. (3)

b Compare the values obtained in (a) and state their meaning in the context of the question. (2)

10 Consider the graph of the function defined by $f(x) = x^3 + 2x^2 - 4x - 5$.

a Show that $f'(x) = (3x-2)(x+2)$. (2)

b Find the coordinates of the points A and B on the graph of f where the tangents to the graph are horizontal. (3)

c Find the equation of the normal to the curve at $x = 2$. (6)

d Hence find the exact coordinates of the points at which the normal found in part (c) meets the tangents found in part (b). (4)

11 The graph of $y = ax^2 + bx$ contains the point $A(1,5)$. The tangent L to $y = ax^2 + bx$ at A has gradient 7.

a Find the values of the real parameters a and b. (6)

b Write down an equation for the tangent L. (1)

c Find the coordinates of the point B on the graph where the tangent to the graph is horizontal. (3)

d State the geometrical significance of the point B on the graph of $y = ax^2 + bx$. (1)

Answers

1 **a** $x = \frac{-7}{2}$ **b** $h = \frac{7}{2}$ **c** $k = 6.25$ **d** (h, k) is a minimum point.

2 **a** $g^{-1}(x) = 2$ **b** $g(x)$ is self-inverse **c** $x = 1$ **d** $x = 3$

3 **a** Arithmetic: 1, $1 + r$, $1 + 2r$, $1 + 3r$.

Geometric: 1, r, r^2, r^3

b $r = 2, -\frac{3}{2} - \frac{\sqrt{5}}{2}, -\frac{3}{2} + \frac{\sqrt{5}}{2}$

c $r = -\frac{3}{2} + \frac{\sqrt{5}}{2} \approx -0.382$

d $S_{20} = -52.57$ (2 d.p.)

4 **a** $b = -1$

b $a = 3$

c $k = \frac{12}{\sqrt{3}} - 9$ at $\left(\frac{2}{\sqrt{3}}, -2 - 6\sqrt{3}\right), k = \frac{-12}{\sqrt{3}} - 9$ at $\left(\frac{-2}{\sqrt{3}}, -2 + 6\sqrt{3}\right)$

5 **a** $k = 2.5$ **b** $y = -10x + 5$

6 **a** $\left(\frac{1}{2}, -\frac{5}{4}\right)$ is a local minimum.

$\left(-\frac{3}{4}, \frac{65}{48}\right)$, is local maximum.

b $k > \frac{5}{4}$

7 **a** $-2 < x < 3$ A1

b -2 and 0 A1

c $0 \le y \le 2$ A1

d $-2 < x < -1, 0 < x < 2$ A1A1

e Any point of the form $(x, 2)$ where $2 \le x \le 3$ A1

f -2, -1, 0, 2 and 3 A1

8 **a** 1C, 2A and 3B A1A1A1

b -3 and 0 A1

c -3 A1

9 **a** **i** $\dfrac{101.1 - 98.5}{2002 - 2000} = 1.3$ M1A1

ii $\dfrac{102.3 - 101.1}{2004 - 2002} = 0.6$ A1

b The average increase in annual profits was greater between 2000 and 2002 ($1300 per year) than it was between 2002 and 2004 ($600 per year). R1A1

10 **a** $f'(x) = 3x^2 + 4x - 4$ M1A1

$= (3x - 2)(x + 2)$ AG

b	$f'(x)=(3x-2)(x+2)=0$ for horizontal tangent	M1
	$x=\frac{2}{3}\Rightarrow y=-\frac{175}{27}$ so point is $\left(\frac{2}{3},-\frac{175}{27}\right)$	A1
	$x=-2\Rightarrow y=3$ so point is $(-2,3)$	A1
c	$f'(2)=16$	A1
	$m_{\text{tangent}}m_{\text{normal}}=-1\Rightarrow$ Gradient of normal: $-\frac{1}{16}$	M1A1
	$f(2)=3$	A1
	$y-3=-\frac{1}{16}(x-2)\Rightarrow\ x+16y=50$ (or equivalent)	M1A1
	Tangent $y=-\frac{175}{27}$ meets normal $x+16y=50$ when $x=\frac{4150}{27}$	M1
	Tangent $y=3$ meets normal $x+16y=50$ when $x=2$	M1
	So points are $\left(-\frac{175}{27},\frac{4150}{27}\right)$ and $(2,3)$	A2
11 a	$y'=2ax+b$	A1
	Attempt to form two simultaneous equations in *a* and *b*	M1
	$\begin{cases}7=2a+b\\5=a+b\end{cases}$	A1A1
	Solving gives $a=2$ and $b=3$	A1A1
b	$y-5=7(x-1)\Rightarrow y=7x-2$	A1
c	$y'(x)=4x+3=0\Rightarrow x=-\frac{3}{4}$	M1
	$y\left(-\frac{3}{4}\right)=-\frac{9}{8}$	M1
	So coordinates are $B\left(-\frac{3}{4},-\frac{9}{8}\right)$	A1
d	B is the minimum point (accept 'vertex') of the parabola.	R1

6 Representing data: statistics for univariate data

1 Consider the curve $f(x) = \frac{2}{x-3} + 1$.

a Write down the equation of the vertical asymptote.

b Find the value at which the curve crosses the x-axis.

2 If $f(x) = 2x^2 - 8x + 5$,

a Find the values of a, h and k, when $f(x)$ is written in the form $a(x-h)^2 + k$

b Write down the equation of the axis of symmetry.

3 Find the coefficient of x^3 in the expansion of $(2-x)^5$.

4 Find three positive integers a, b and c that have a mean of 9, a median of 11 and a range of 10.

5 Solve $e^x = x^2 - 2$

6 **a** Write an expression for the n^{th} term of the arithmetic sequence 14, 11, 8, 5,

b Find the 50^{th} term.

7 Find the coordinates of the maximum point of the curve $f(x) = x^3 - 10x^2 + 12x - 2$

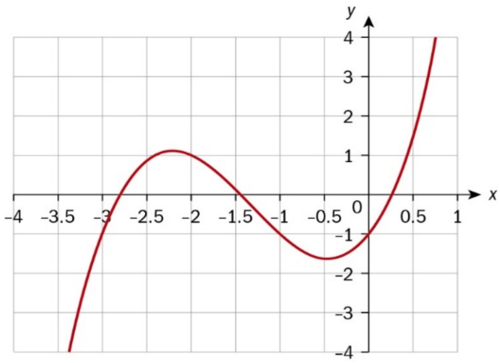
8 If $f(x) = 4x - 4$ and $g(x) = 3 - \frac{1}{2}x$

a Find g^{-1}.

b Solve the equation $(f \circ g)(x) = -20$

9 If Jo invests \$1000 at 15% interest per annum, compounded monthly, how many months will it take for Jo's investment to exceed \$3000?

10 This diagram shows the graph of $f(x)$. Make a copy of and sketch the graph of $f'(x)$ on the same axes.



Exam-style questions

11 A maths textbook states that if data is skewed to the right then it has the following properties.

i The mean is always greater than the mode.

ii The median is always greater than the mode.

iii The mean is usually greater than the median.

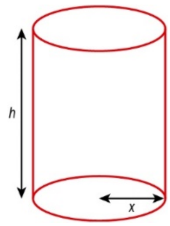
a For the following data:

1, 3, 4, 4, 6, 8, 10, 12

Find the mode, the median and mean. (3)

b Determine which of the statements **i**, **ii** and **iii** hold for this data. Hence, deduce whether this book would classify this data as skewed to the right. (2)

12 A canned food manufacturer wants to cut production costs and reduce the amount of aluminium used to make the cans. The company requires cylindrical cans with volume 500 cm^3. Let h represent the height of a can, and x represent the radius of the base, as shown.



a Show that $h = \frac{500}{\pi x^2}$. (2)

b Show that the total surface area of the can is given by $S(x) = 2\pi x^2 + \frac{1000}{x}$. (2)

c Hence find the values of radius and height that minimize the total surface area of the can. (4)

13 a A set of data is made up of the first seven terms of an Arithmetic Progression, with first term 4 and common difference of 3.

For this data, find

i the median

ii the mean

iii the interquartile range. (6)

b A set of data is made up of the first seven terms of a Geometric Progression, with first term 1 and common ratio 2.

For this data, find

i the median

ii the mean (giving your answer as a fraction)

iii the interquartile range. (6)

c State which of these two data sets has mean equal to its median. Suggest a reason for this. (2)

14 A set of data has a mean of 16, a mode of 14, a median of 18, an interquartile range of 20 and a variance of 36.

a All the values in this data set have 5 added to them. Write down the new value of the

i mean **ii** mode **iii** median **iv** interquartile range
v variance. (5)

b A particular value x was an outlier in the original set of data. Determine whether the new value of x + 5 will still be an outlier in the new set. Justify your answer. (3)

c All the values in the original set of data are multiplied by -2. Write down the new value of the

i mean **ii** mode **iii** median **iv** interquartile range
v variance **vi** standard deviation. (6)

15 For a set containing data for an entire population, the mean is defined by $\mu = \frac{\sum x}{n}$ and the variance by $\sigma^2 = \frac{\sum x^2}{n} - \mu^2$.

A population data set with five values is: 1, 6, 8, x, y. For this data,

$y > x$

the mean is 4.4

the variance is 6.64.

Use the definitions of mean and variance to find the values of x and y.

Answers

1 a $x = 3$ **b** 1

2 a $a = 2,\ h = 2,\ k = -3$ **b** $x = 2$

3 -40

4 3, 11 and 13

5 -1.49

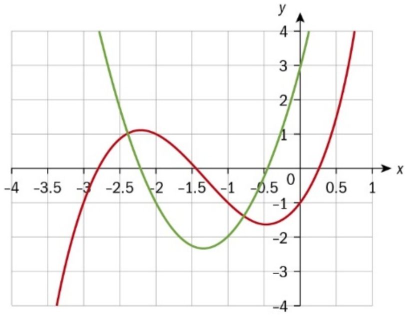
6 a $17 - 3n$ **b** -133

7 $\left(\frac{2}{3}, \frac{50}{27}\right)$

8 a $g^{-1}(x) = 6 - 2x$ **b** 14

9 89 months

10



11 a mode = 4, median = 5, mean $= \frac{48}{8} = 6$ A1A1A1

b All three statements hold for this data, so the book would classify this data as being skewed to the right. A1R1

12 a $V = 500 \Rightarrow \pi h x^2 = 500 \Rightarrow h = \frac{500}{\pi x^2}$ M1A1AG

b $S = 2\pi x^2 + 2\pi x h$ M1

$S(x) = 2\pi x^2 + 2\pi x \frac{500}{\pi x^2}$ M1

$S(x) = 2\pi x^2 + \frac{1000}{x}$ AG

c Use of GDC to maximize S M1

$x = 4.30$ cm A1

$h = 8.60$ cm M1A1

13 a **i** The median is the middle term which is $u_4 = 4 + (4-1)3 = 13$ M1A1

ii $\bar{x} = \dfrac{\sum_{i=1}^{7} x_i}{n} = \dfrac{\frac{7}{2}(2 \times 4 + (7-1)3)}{7} = 13$ M1A1

iii IQR$= Q_3 - Q_1 = u_6 - u_2 = 4d = 12$ M1A1

b **i** The median is the middle term which is $u_4 = 1 \times 2^{4-1} = 8$ M1A1

ii $\bar{x} = \dfrac{\sum_{i=1}^{7} x_i}{n} = \dfrac{\left[\dfrac{1 \times (2^7 - 1)}{(2-1)}\right]}{7} = \dfrac{127}{7}$ M1A1

iii IQR$= Q_3 - Q_1 = u_6 - u_2 = 2^5 - 2 = 30$ M1A1

c The arithmetic progression A1

For this data set, a graph of n against u_n would yield a straight line (or equivalent) R1

14 a **i** 21 **ii** 19 **iii** 23 **iv** 20
v 36 A1A1A1A1A1

b Q_1 and Q_3 both increase by 5 R1

$1.5 \times$ IQR remains the same R1

Distance from new point to a new quartile will be the same as from the old point to the old quartile, so it will remain as an outlier. R1

OR

Recognition that the box and whisker plot is just moved 5 units to the right so

outliers will be unaffected.

c **i** −32 **ii** −28 **iii** −36 **iv** 40
v $36 \times (-2)^2 = 144$ **vi** 12 A1A1A1A1A1A1

15 $\dfrac{1+6+8+x+y}{5} = 4.4 \Rightarrow 15 + x + y = 22 \Rightarrow x + y = 7$ M1A1

$\dfrac{1+36+64+x^2+y^2}{5} - 19.36 = 6.64 \Rightarrow 101 + x^2 + y^2 = 130 \Rightarrow x^2 + y^2 = 29$ M1A1

Solving simultaneous equations, e.g.

$x^2 + (7-x)^2 = 29$ M1

$2x^2 - 14x + 49 = 29 \Rightarrow x^2 - 7x + 10 = 0$ A1

$(x-2)(x-5) = 0$ M1

$x = 2$ or $x = 5$, $x = 2 \Rightarrow y = 5$ and $x = 5 \Rightarrow y = 2$

So $x = 2, y = 5$ R1A1A1

7 Modelling data between two data sets: statistics for bivariate data

1 If $f(x) = x^2 + 1$, and $g(x) = x + 2$, find

a $f(-3)$

b $(f \circ g)(x)$

c Solve $(f \circ g)(x) = 2$

2 The price of fried rice at 15 stalls in Hong Kong was
$3.00, $2.82, $2.75, $2.55, $2.98, $2.53, $2.40, $2.80, $2.50, $2.65, $2.48, $2.57, $2.30, $2.79, $2.25.

Find

a the median

b the interquartile range.

3 The graph of the function $f(x) = 2x - 8$ intersects the x-axis at A and the y-axis at B.

Find the coordinates of

a A

b B

c Find the area of triangle OAB, where O is the origin.

4 The equation $ax^2 + 3x + 1 = 0$ has exactly one solution. Find the value of a.

5 Find the coefficient of x^4 in the expansion of $(2x - 3)^{12}$.

6 The mass M kg of a substance at time t years is given by $M = 8e^{-0.2t}$

a Write down the initial mass.

b How long does it take for the mass to be reduced to 6 kg?

7 Gardener George notices that the more birds in his garden, the more bees appear. He counts the number of birds and bees in his garden over a seven-hour period and tabulates the data.

Birds	2	3	4	5	6	7	8
Bees	1	7	18	14	19	24	25

a Find a linear regression line in the form $y = mx + b$ for his data.

b Find Pearson's product-moment correlation coefficient (r) for this data.

c How many bees can he expect when there are 9 birds?

d Explain whether George could use this model to estimate the number of bees when there are 1000 birds.

8 $f(x) = \dfrac{2x - 3}{3x - 4}$

a Find the equations of the horizontal and vertical asymptotes.

b State the domain and range of *f*.

c Solve $f(x) = g(x)$ where $g(x) = 3 - 2x$.

9 Solve $6^x = 3^{x+2}$ giving your answer in the form $\dfrac{\ln a}{\ln b}$, where *a* and *b* are integers.

10 If $f(x) = \dfrac{1}{1+x^2},$

a Write down the equation of the horizontal asymptote of *f*.

b Find $f'(x)$

c Find the coordinates of the maximum point of *f*.

d Find $f''(x)$

e Find the *x*-coordinate of the point on the curve of *f* where the gradient of the tangent is a maximum

Exam-style questions

11 A swimming pool is 25 m long. The depth, *d* m, of water x m away from the edge at the shallow end is given by

$$d(x) = \begin{cases} 0.5 + 0.1x & 0 \le x \le 15 \\ -1 + 0.2x & 15 \le x \le 25 \end{cases}$$

a Write down the depth of water at the shallow end, when $x = 0$. (1)

b Find the depth when $x = 15$. (1)

c Find the maximum depth of the pool. (1)

d Swimmers are only allowed to dive into the pool when the depth is greater than, or equal to, 3 m. Calculate the minimum distance from the shallow end at which diving is allowed. (2)

e Josie can only stand up in the pool if the depth is less than 1 m. Calculate the maximum distance from the shallow end that Josie can be, if she is still able to stand up. (2)

f **i** Sketch a cross-section of the pool showing x against $d(x)$ for $0 \le x \le 25$.

The width of the pool is 10 m.

ii Hence calculate the volume of water in the pool. Give your answer in m^3 correct to 1 decimal place. (6)

12 Ten pairs of paired bivariate data, (x, y), are given in the table below.

x	1	2	2.5	3	4	4.5	5	6	6.5	8
y	0.2	0.4	0.5	0.8	1.6	2.2	3.3	6.3	9	26

a For this data

i calculate the Pearson product-moment correlation coefficient

ii write down the equation of the y on x regression line. (4)

b For each data pair, list the value of x against the value of $\log y$. Construct a table like the one below to show your results.

x	
log y	

(2)

c For the paired bivariate data showing x against the value of $\log y$,

i calculate the Pearson product-moment correlation coefficient, giving your answer to 4 decimal places.

ii write down the equation of the $\log y$ on x regression line. (4)

d For $x = 5.5$, find an estimate for y using

i the y on x regression line found in **a ii**

ii the $\log y$ on x regression line found in **c ii**. (4)

e Suggest, with a reason, which of the two estimates found in part **d** is the better one. (2)

13 Thirteen pairs of paired bivariate data are given in the table below

x	1	2	2	3	4	5	6	7	8	9	10	11	12
y	1	1	2	4	5	5.5	5	4.5	4	4	3.5	3	2.5

a Draw a scatter diagram to represent this data. (3)

b **i** Calculate the Pearson product-moment correlation coefficient for the above data.

ii Using two words, describe the linear correlation between x and y. (2)

c **i** For the six data pairs where $x \le 5$, calculate the Pearson product-moment correlation coefficient.

ii Using two words, describe the linear correlation between x and y for the six data pairs from part **i**.

iii Find the equation of the y on x regression line for these six data pairs. (4)

d **i** For the seven data pairs where $x \ge 6$, calculate the Pearson product-moment correlation coefficient.

ii Using two words, describe the linear correlation between x and y for the seven data pairs from part **i**.

iii Find the equation of the y on x regression line for these seven data pairs. (4)

e Find the intersection point of the two lines found in parts **c iii** and **d iii**. (2)

f State two reasons why it is better to split the data into two halves in order to model it. (2)

14 a **i** Two values of paired bivariate data x and y are measured and recorded as (x_1, y_1) and (x_2, y_2). Explain why the Pearson product-moment correlation coefficient, r, for this data must be either +1 or −1.

ii Suppose $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, 6)$. Determine whether $r = 1$ or $r = -1$.

iii Comment on whether it would be appropriate to draw a line of best fit for the data given in part **ii** to help calculate the strength of correlation between x and y. (4)

b 100 pairs of paired bivariate data have perfect correlation (either positive or negative). If (x_0, y_0) is one of these data pairs, where x_0 is an outlier for the x-values, state (with a brief geometrical explanation) whether or not y_0 will be an outlier for the y-values. (3)

15 a For paired bivariate data (x, y), which satisfies

$$y = -3x + 2, \quad -4 \le x \le 4, \qquad x, y \in \mathbb{Z}$$

i describe geometrically the shape of the scatter diagram

ii sketch (very roughly) the scatter diagram

iii from the list below, select the description that would best describe the linear correlation:

Perfect positive, strong positive, weak positive, no correlation, weak negative, strong negative, perfect negative. (4)

b Repeat all of part (a) but this time for paired data which satisfies

$$x^2 + y^2 \le 16 \qquad x, y \in \mathbb{Z}.$$ (4)

c Repeat all of part **a** but this time for paired data which satisfies

$$|y - 2x| \le 1, \quad -4 \le x \le 4, \ x, y \in \mathbb{Z}.$$ (5)

Answers

1 **a** 10 **b** $x^2 + 4x + 5$ **c** $-3, -1$

2 **a** \$2.57 **b** \$0.32

3 **a** $(4, 0)$ **b** $(0, -8)$ **c** 16 units^2

4 2.25

5 51963120

6 **a** 8kg **b** 1.44yrs

7 **a** $y = 3.82x - 3.68$ **b** 0.937 **c** 31 **d** No. 1000 is too far outside of the domain.

8 **a** $y = \frac{2}{3},\ x = \frac{3}{4}$ **b** $x \in \mathbb{R}, x \neq \frac{3}{4},\ y \in \mathbb{R}, y \neq \frac{2}{3}$ **c** $x = 1, 1.5$

9 $a = 9,\ b = 2$

10 **a** $y = 0$ **b** $f'(x) = -\frac{2x}{\left(1 + x^2\right)^2}$ **c** $(0, 1)$ **d** $f''(x) = \frac{6x^2 - 2}{\left(1 + x^2\right)^3}$ **e** -0.577

11 **a** 0.5 m A1

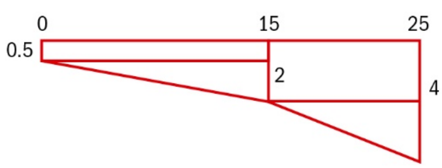
b 2 m A1

c 4 m A1

d $d \geq 3 \Rightarrow -1 + 0.2x \geq 3 \Rightarrow x \geq 20$, so minimum distance is 20 m M1A1

e $d < 1 \Rightarrow 0.5 + 0.1x < 1 \Rightarrow x < 5$, so maximum distance is 5 m M1A1

f **i**



A2

ii Cross-sectional area is 2 rectangles plus two triangles M1

$0.5 \times 15 + 2 \times 10 + \frac{1}{2} \times 15 \times 1.5 + \frac{1}{2} \times 10 \times 2 = 48.75$ M1A1

Volume is $10 \times 48.75 = 487.5$ m^3 A1

12 **a** **i** $r = 0.821(3sf)$ A2

ii $y = 2.97x - 7.58(3sf)$ A1A1

b

x	1	2	2.5	3	4	4.5	5	6	6.5	8
log y	-0.699	-0.398	-0.301	-0.0969	0.204	0.342	0.519	0.799	0.954	1.42

A2

c **i** $r = 0.9996(4dp)$ A2

ii $\log y = 0.304x - 1.02(3sf)$ A1A1

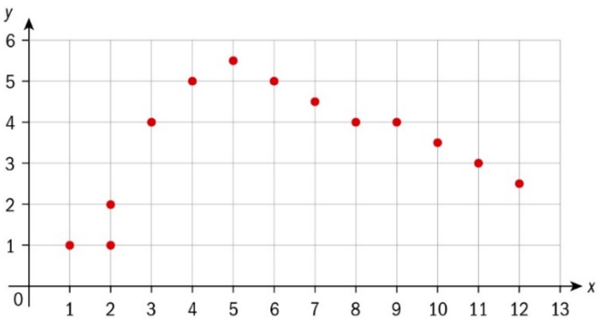
d **i** $y = 2.97 \times 5.5 - 7.58 = 8.76(3sf)$ M1A1

ii $\log y = 0.304 \times 5.5 - 1.02 = 0.652 \Rightarrow y = 4.49(3sf)$ M1A1

e The estimate using the $\log y$ on x regression line will be the better estimate A1

because there was almost perfect correlation between $\log y$ and x. R1

13 a



A3

b **i** $r = 0.262(3sf)$ A1

ii weak, positive A1

c **i** $r = 0.952(3sf)$ A1

ii strong, positive A1

iii $y = 1.3x - 0.6$ A1A1

d **i** $r = -0.988(3sf)$ A1

ii strong, negative A1

iii $y = -0.393x + 7.32(3sf)$ A1A1

e intersect at $(4.7, 5.5)$ $(1dp)$ A1A1

f There was only minor linear correlation for the whole data. R1

The scatter diagram suggests that a better model is two lines of best fit, each showing strong linear correlation by their PMCC values. R1

14 a **i** As the two points must lie on a straight line, there will be perfect correlation. R1R1

Perfect correlation in Pearson's PMCC is denoted by $r = \pm 1$. AG

ii Since the straight line has positive gradient, $r = 1$. R1

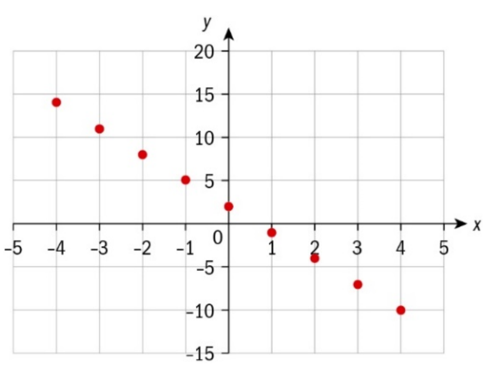
iii Since the 2 points *must* lie on a straight line, it is impossible to determine the strength of correlation between x and y using only these values. Hence, a line of best fit would not be appropriate. R1

b Perfect correlation $\Rightarrow$ All the points lie on a straight line. R1

Since x_0 is an outlier, (x_0, y_0) must be sufficiently separated from the majority of the other points and thus y_0 is also an outlier. R1A1

15 a **i** all points lie on a straight line A1

ii

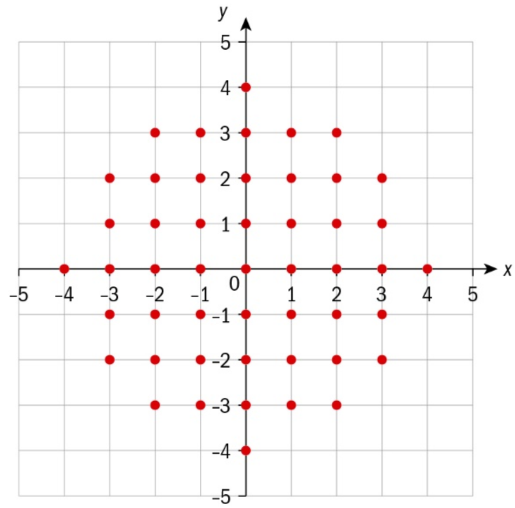


A2

iii Perfect negative A1

b **i** Points lie on or inside a circle of radius 4. A1

ii



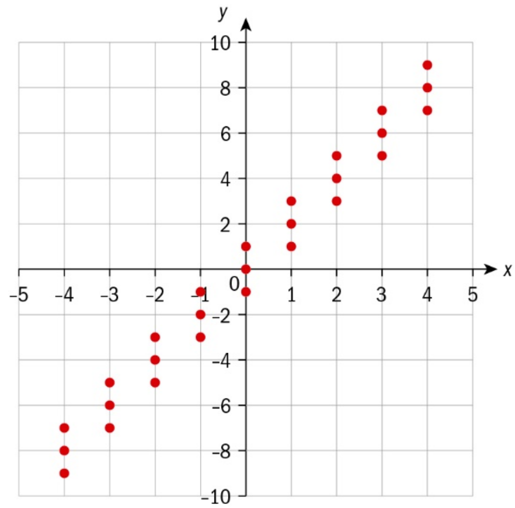
A2

iii no correlation A1

c **i** $|y-2x| \le 1 \Rightarrow -1 \le y-2x \le 1 \Rightarrow y-2x = -1, 0 \text{ or } 1$ R1

So points lie on 3 lines: $y = 2x - 1$, $y = 2x$, $y = 2x + 1$ A1

ii



A2

iii strong positive A1

8 Quantifying randomness: probability

1 **a** I pick 2 cards from a well-shuffled pack. Work out the probability that the 2nd card is an ace given that the first card is an ace.

b In a box there are 12 white cards and 18 black cards. I pick up two cards randomly. Find the probability that I pick a black card as a second card.

2 **a** I pick two cards from a well-shuffled pack at random. Find the probability they are both aces.

b There are 6 boys and 8 girls. I pick randomly three people. What is the probability that all three people are girls?

3 In a town, 15% of the houses do not have a smart TV, 40% of the houses do not have a video game console and 10% of the houses have neither a smart TV nor a video game console. I choose one house randomly. Find the probability that it has smart TV and a video game console.

4 If $\frac{P(A)}{P(A')} = \frac{7}{8}$, for an event A, find $P(A)$ and $P(A')$.

5 We consider $\Omega = \{w \in \mathbb{N} \,/\, 10 \le w \le 20\}$, A the subset of Ω which contains the multiples of 3 and B the subset of Ω which contains the multiples of 4. I choose one number from Ω randomly. Work out the following probabilities:

a $P(A)$ **b** $P(B')$ **c** $P(A \cap B)$

d $P(A \cup B)$ **e** $P(A \mid B)$ **f** $P(B \mid A)$

Exam-style questions

6 Eight cards have the following numbers written on them:

$0, \quad 1, \quad \pi, \quad \sqrt{2}, \quad 1.5, \quad -3, \quad \frac{22}{7}, \quad 1000.$

A card is chosen at random. Find the probability that the number written on it is

a a natural number

b an integer

c a rational number

d an irrational number

e greater than 3

f a real number

g smaller than zero. (7)

7 **a** If one of the following quadratics is chosen at random, find the probability that it is concave upwards. Justify your answer.

$y = x^2$, $y = 2x^2 - 3x + 4$, $y = -3x^2 + 4$, $y = 4(x-1)^2$, $y = \pi x^2 + \sqrt{2}x + 4$ (2)

b If the derivative of the following functions is evaluated at $x = 0$, find the probability that the answer is zero. You must show all your working to gain full marks.

$y = x^2$, $y = \sin x$, $y = \cos x$, $y = e^x$, $y = 4(x-1)^2$ (4)

8 Rachel, a librarian, has 10 books on her desk. These 10 books consist of 6 thrillers, 3 autobiographies and 1 science fiction book. She chooses 5 books at random to place back on the shelves. Calculate the probability that

a she chooses all thrillers

b her choice includes the science fiction book.

c her choice does not include an autobiography. (6)

9 Martin and Rob are fighting a duel to the death with pistols (over the affections of a lady).

They fire at each other at the same time. The probability that Martin's bullet hits and kills Rob is always $\frac{2}{3}$. The probability that Rob's bullet hits and kills Martin is always $\frac{3}{5}$. All probabilities are independent. If at the end of the first round they are both alive they continue the process. They continue in this way until at least one of them is dead.

a Find the probability that they are both alive at the end of the first round. (2)

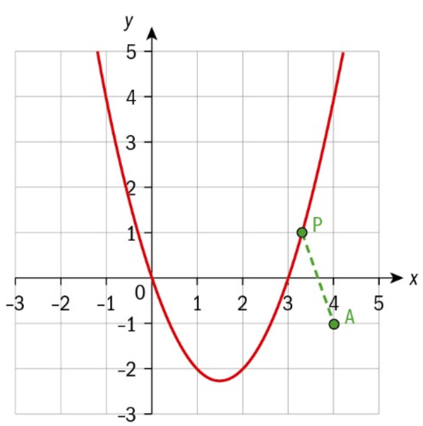
b Find the probability that they are both dead at the end of the first round. (2)

c Find the probability that Martin is alive and Rob is dead at the end of the first round. (2)

d Find the probability that Martin is alive and Rob is dead after the second round. (3)

e Find the probability that Martin is alive and Rob is dead at the end of the whole duel. (6)

10 Consider the graph of the quadratic function defined by $f(x) = x^2 - 3x$ and the point $A(4, -1)$. Let P be a point on the graph of f with x-coordinate x.



a State the coordinates of P in terms of x. (1)

b Find an expression for the distance between

i the origin O and the point P

ii the point A and the point P. (4)

c Determine the coordinates of P such that the perimeter of triangle OAP is minimum, and find the length of the perimeter in this case. (6)

Answers

1 **a** We denote as A the event that the 1st pick is an ace and as B the event that the 2nd pick is an ace.

Since these two events are independent we obtain that $P(A \cap B) = P(A) \times P(B)$

Obviously, we have that $P(A) = \frac{4}{52}, P(B) = \frac{3}{51}$

$P(A \cap B) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$

We are required to find $P(B \mid A)$.

We know that $P(B \mid A) = \frac{P(B \cap A)}{P(A)}$.

Hence, we obtain that $P(B \mid A) = \frac{\frac{1}{221}}{\frac{4}{52}} = \frac{3}{51}$.

b We denote as w_1, b_1, b_2 the events of picking white as a first card, black as a first card and black as a second card.

Then obviously, $P(w_1) = \frac{12}{30}, P(b_2) = \frac{18}{29}, P(b_1) = \frac{18}{30}$

By denoting as X the required event, we get that:

$\mathrm{P}(X) = \frac{12}{30} \times \frac{18}{29} + \frac{18}{30} \times \frac{17}{29} = \frac{3}{5}$

2 **a** We denote by A the event of picking one ace and by X the required event of picking two aces.

Obviously, $P(A) = \frac{4}{52}$.

The events of picking one ace first and then one other ace at the second time are independent. Hence, we can say that $P(X) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$.

b We denote by A the event of picking the first person to be a girl and by X the required event of picking three girls

Obviously, $P(A) = \frac{8}{14}$.

The events of picking three people to be all girls are independent. Hence, we can say that $P(X) = \frac{8}{14} \times \frac{7}{13} \times \frac{6}{12} = \frac{2}{13}$.

3 We denote T, V the event of having a smart TV and the event of having a video game console respectively.

We are required to find $P(T \cap V)$.

Hence, $P(T) = 1 - 15\% = 1 - \frac{15}{100} = \frac{85}{100}$

$P(V) = 1 - 40\% = 1 - \frac{40}{100} = \frac{60}{100}$

$P(T' \cap V') = \frac{10}{100}$

We know from theory that $P(T \cup V) + P(T \cap V) = P(T) + P(V)$.

We also know that $P(T' \cap V') = P((T \cup V)') = 1 - P(T \cup V)$.

Hence, $P(T \cup V) = 1 - \frac{10}{100} = \frac{90}{100}$.

Therefore, $P(T \cap V) = \frac{85}{100} + \frac{60}{100} - \frac{90}{100} = \frac{55}{100}$.

4 We know that $P(A)+P(A')=1$.

So, $P(A')=1-P(A)$.

In the given equation, we obtain: $\dfrac{P(A)}{1-P(A')}=\dfrac{7}{8}$

$8P(A)=7(1-P(A))$

$15P(A)=7$

$P(A)=\dfrac{7}{15}$

$P(A')=1-\dfrac{7}{15}=\dfrac{8}{15}$

5 **a** Let's analyse the subsets A and B.

Hence, $A=\{12,15,18\}$ and $B=\{12,16,20\}$.

Note that $\Omega=\{10,11,12,13,14,15,16,17,18,19,20\}$.

Obviously then, $P(A)=\dfrac{3}{11}$

b We know that $P(B')=1-P(B)$.

But, similarly as above, we have that $P(B)=\dfrac{3}{11}$.

Therefore, $P(B')=\dfrac{8}{11}$.

c Intersection of two sets means the common elements.

Hence, $P(A\cap B)=\dfrac{1}{11}$, as A,B just have "12" in common.

d We know that $P(A\cup B)+P(A\cap B)=P(A)+P(B)$.

Therefore, $P(A\cup B)=\dfrac{3}{11}+\dfrac{3}{11}-\dfrac{1}{11}=\dfrac{5}{11}$.

e We also know that $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}$.

Therefore, $P(A\mid B)=\dfrac{\frac{1}{11}}{\frac{3}{11}}=\dfrac{1}{3}$.

f Similarly, we get that $P(B\mid A)=\dfrac{P(B\cap A)}{P(A)}$.

Therefore, $P(B\mid A)=\dfrac{\frac{1}{11}}{\frac{3}{11}}=\dfrac{1}{3}$

6 **a** $\dfrac{3}{8}$ A1

b $\dfrac{4}{8}=\dfrac{1}{2}$ A1

c $\dfrac{6}{8}=\dfrac{3}{4}$ A1

d $\dfrac{2}{8}=\dfrac{1}{4}$ A1

e $\dfrac{3}{8}$ A1

f 1 A1

g $\dfrac{1}{8}$ A1

7 a $ax^2 + bx + c$ is concave up if $a > 0$ R1

So probability is $\frac{4}{5}$ A1

b Derivatives are

$2x, \cos x, -\sin x, e^x, 8(x-1)$ M1A1

Derivatives evaluated at $x = 0$ are 0, 1, 0, 1 and 8 A1

So probability is $\frac{2}{5}$. A1

8 a $\frac{C_5^6}{C_5^{10}} = \frac{6}{252} = \frac{1}{42}$ M1A1

b $\frac{C_4^9}{C_5^{10}} = \frac{126}{252} = \frac{1}{2}$ M1A1

c $\frac{C_5^7}{C_5^{10}} = \frac{21}{252} = \frac{1}{12}$ M1A1

9 a $\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$ M1A1

b $\frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$ M1A1

c $\frac{2}{2} \times \frac{2}{5} = \frac{4}{15}$ M1A1

d must both miss 1st round; then in 2nd round Martin hits and Rob misses R1

$\frac{2}{15} \times \frac{2}{3} \times \frac{2}{5} = \frac{8}{225}$ M1A1

e Martin could win in the 1st round, or the 2nd or the 3rd… R1

$\frac{4}{15} + \frac{2}{15} \times \frac{4}{15} + \left(\frac{2}{15}\right)^2 \times \frac{4}{15} + \ldots$ M1A1

Infinite GP with $a = \frac{4}{15}$ and $r = \frac{2}{15}$ R1

$S_\infty = \frac{\frac{4}{15}}{1 - \frac{2}{15}} = \frac{4}{13}$ M1A1

10 a $P(x, x^2 - 3x)$ A1

b i $\sqrt{x^2 + (x^2 - 3x)^2}$ M1A1

ii $\sqrt{(x-4)^2 + (x^2 - 3x + 1)^2}$ M1A1

c perimeter OAP = OP + AP + OA

$= \sqrt{x^2 + (x^2 - 3x)^2} + \sqrt{(x-4)^2 + (x^2 - 3x + 1)^2} + \sqrt{4^2 + 1}$ M1A1

Use GDC to find minimum M1

$x = 1.5$ A1

$\Rightarrow y = (1.5)^2 - 3(1.5) = -2.25$ A1

Perimeter = 6.35 A1

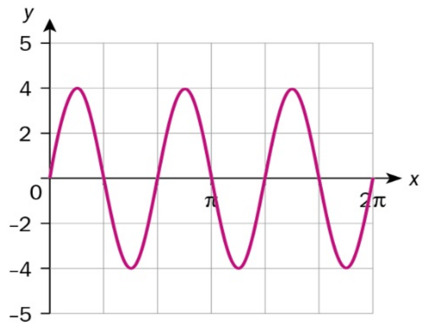
9 Representing equivalent quantities: exponentials and logarithms

1 The graph of $y = 3 - x^2$ intersects the graph of $y = 2x$ at the point (1, 2) and at one other point A. Find the coordinates of A.

2 A sum of \$4000 is invested at a compound interest rate of 3.5% per annum. To the nearest dollar, find the total value of the investment at the end of five years.

3 The graph of $y = a \sin bx$, for constants a and b, is shown below.

Find the values of a and b.



4 A data set has a mean of 10 and a standard deviation of 3.

Each value in the data set has 6 added to it. Write down the values of

a the new mean

b the new standard deviation.

Each value in the *original* data set is then multiplied by 5. Find

c the new mean

d the new variance.

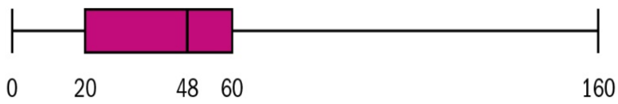
5 Solve $e^x = 5 - 2x$.

6 A cinema has 40 rows of seats. There are 25 seats in the first row, 30 seats in the second row, and each successive row of seats has two more seats in it than the previous row. Find the number of seats

a in the 20th row

b the cinema has in total.

7 A cricket team has 11 players. The number of runs scored, per player, over three games is summarized by this box plot.



a Find the median number of runs scored per player.

b Find the percentage of players that scored between 20 and 60 runs.

c State the interquartile range for this data.

d Explain why 160 is an outlier.

8 The amount of a drug, A mg, which remains in a patient's bloodstream at time t hours after the drug was injected is modelled by $(t)=500e^{-kt}$.

a Determine the amount of drug that was injected into the patient.

It is found that 250 mg of the drug remains after 5 hours have passed.

b Determine the value of k.

c Find the value of t at which 50 mg of the drug remains in the patient's bloodstream. Give your answer correct to 3 s.f.

9 Of the final year students at a school, 36 students study Biology and 18 students study Spanish. There are 53 students in the final year, and 10 do not study either Biology or Spanish.

a Determine how many students study both Biology and Spanish.

If a student is chosen at random, find the probability that they:

b study Biology but do not study Spanish

c study Spanish, given that they study Biology.

10 A stone is dropped vertically from the top of a cliff. Its velocity, v ms^{-1}, is modelled by

$V(t) = 50 - 50e^{-0.2t}$, where t is the number of seconds since the stone was dropped.

a Find the velocity of the stone when

i $t=0$

ii $t=10$.

b Find an expression for the acceleration of the stone, a ms^{-2}, as a function of t.

c State the value of a when $t=0$.

d Determine the value which v approaches as t becomes very large.

e Determine the value which a approaches as t becomes very large.

f Explain what your answers to parts d and e tell you about the motion of the stone.

Exam-style questions

11 A consumer report suggests that the probability of a six-year-old car having faulty lights is 0.32, and that the probability of a six-year-old car having faulty brakes is 0.21.

Find the probability that

a A six-year-old car chosen at random has faulty lights and faulty brakes. (2)

b The car has neither faulty lights, nor faulty brakes. (2)

c The car has either faulty lights, or faulty brakes, but not both. (2)

12 Katharina and Carolina go to a swimming pool. They both swim the first length of the pool in 2 minutes 6 seconds.

The time that Katharina takes to swim each subsequent length is 5 seconds more than the time she took to swim the previous length.

The time Carolina takes to swim a length is 1.04 multiplied by the time that she took to swim the previous length.

a **i** State the time Katharina takes to swim the third length.

ii Show that Carolina takes 2 min 16 seconds, to the nearest second, to swim the third length. (5)

Katharina and Carolina both swim a total of 10 lengths of the pool. They start at exactly at the same time.

b Show that Katharina completes the 10 lengths before Carolina, clearly showing your calculations and reasoning. (6)

c Hence, state the time that passes between Katharina finishing and Carolina finishing. Give your answer correct to the nearest second. (2)

13 Consider the function defined by $f(x) = a\ln(x-b)+c$, where a, b and c are parameters to be determined.

The graph of f has a vertical asymptote $x=2$, intercepts the $x-$axis at $x=3$, and the tangent to its graph at $x=1$ has gradient 2.

a Determine the value of each parameter a, b and c. (6)

b Hence sketch the graph of f. (2)

14 The velocity v, in metres per second, of a particle after t seconds is given by

$v(t) = 2(0.2t+0.3)^t - 1$, $0 \le t \le 4$.

a Sketch the graph of $v = v(t)$, showing clearly the axes intercepts. (3)

b State the value(s) of t when the particle is at rest. (2)

c Find the value of t when the acceleration of the particle is 0. (3)

15 Consider the four numbers $a, b, c, d \in \mathbb{Z}$ with $a \le b \le c \le d$.

The mean of these four numbers is 8.

The mode and the median are both equal to 6.

The range is 10.

Find the values of a, b, c and d, clearly showing your reasoning. (8)

Answers

1 $(-3,-6)$

2 $4751

3 $a=4,\ b=3$

4 **a** 16 **b** 3 **c** 50 **d** 225

5 1.06

6 **a** 66 **b** 490

7 **a** 48 **b** 50% **c** 40

d An outlier is greater than 1.5 IQR above Q3.

$Q_3+1.5IQR=60+1.5(60-20)=60+60=120$

160>120

8 **a** 500 **b** 0.139 **c** 16.6

9 **a** 11 **b** $\frac{25}{53}$ **c** $\frac{11}{36}$

10 a **i** 0 **ii** 43.2

b $10e^{-0.2t}$ **c** 10 **d** 50 **e** 0

f The terminal velocity of the stone is $50ms^{-1}$

11 a $P(L\cap B)=0.32\times0.21=0.0672$ M1A1

b $P(L'\cap B')=(1-0.32)\times(1-0.21)=0.5372$ M1A1

c $P(L\cup B)-P(L\cap B)=\left[1-P(L'\cap B')\right]-P(L\cap B)$ M1

$=1-0.5372-0.0672$

$=0.3956$ A1

12 a **i** 2 min 16 sec A1

ii 2 minutes 6 seconds = 126 seconds A1

$126\times1.04^2=136.28\ldots\text{sec}=2\text{min}16\text{sec}$ M1A1A1

b Katharina´s times form an AP. R1

K's total time $=(126\times2+9\times5)\times5=1485\text{sec}=24\text{min}45\text{sec}$ M1A1

Carolina´s times form a GP. R1

C´s total time= $126\times\frac{1.04^{10}-1}{1.04-1}=1512.77\text{sec}=25\text{min}13\text{sec}$ M1A1

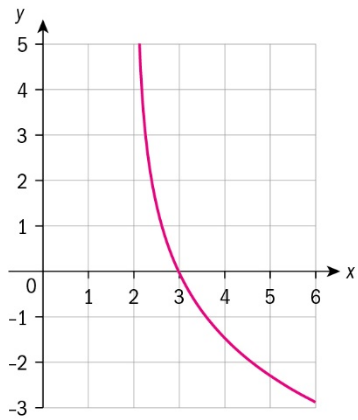
c 1513 − 1485 = 28 seconds M1A1

13 a vertical asymptote $x = 2 \Rightarrow b = 2$ R1

$f(3) = 0 \Rightarrow a\ln(3-2) + c = 0 \Rightarrow c = 0$ M1A1

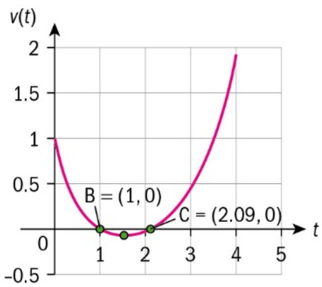
$f'(x) = \dfrac{a}{x-2}$ M1

$f'(1) = 2 \Rightarrow \dfrac{a}{1-2} = 2 \Rightarrow a = -2$ M1A1



A1 Shape; A1 Domain and zero

14 a



Shape A1, domain A1, Intercepts A1

b b $v(t) = 0 \Rightarrow t = 1, t = 2.09$ seconds A1A1

c c $a(t) = 0 \Rightarrow t = 1.52$ seconds M1A1

15 Mode & median = 6 $\Rightarrow b = c = 6$ R1A1A1

Range = $d - a = 10 \Rightarrow d = a + 10$ M1A1

Mean = 8 $\Rightarrow \dfrac{a + 6 + 6 + (a+10)}{4} = 8 \Rightarrow a = 5$ M1A1

$d = a + 10 \Rightarrow d = 15$ A1

10 From approximation to generalization: integration

1 **a** Find $\int (2x+5)^2 \, dx$, using the fact that it is of the form $\int (ax+b)^n \, dx$.

b Find $\int (2x+5)^2 \, dx$ by first expanding $(2x+5)^2$.

c Use the binomial theorem to expand your answer to part a.

d Explain why the answers to parts a and b are both correct.

2 A particle moves along a horizontal line with velocity $v(t) = t^3 - 2t$, for $t \geq 0$, where v is in centimetres per second and t is in seconds.

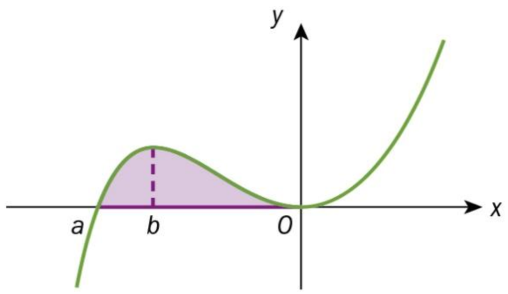
a Find the values of *t* for which the particle is moving left.

b Find the acceleration of the particle when $t = 3$.

c When $t = 2$, the displacement, s, of the particle is 5 centimetres.

Find an expression for s in terms of t.

3 Part of the graph of $f(x) = x^2 \ln(x+4)$ is shown in the following diagram.



The x-intercepts are at $x = 0$ and $x = a$.

a Find the value of *a*.

The graph has a maximum when $x = b$.

b Use a GDC to find the value of *b*.

Let R be the region under the graph of *f* between $x = a$ and $x = b$.

c Use a GDC to find the area of R.

4 Let $f(x) = \frac{1}{x}$ and $g(x) = 5x - 3$.

a Given that $h(x) = (f \circ g)(x)$, for $x > \frac{5}{3}$ find *h*.

b Find $\int h(x) \, dx$.

5 Consider the function f with second derivative $f''(x) = -3x + 4$. The graph of f has a minimum point at $\mathrm{A}\left(-\dfrac{1}{3}, -\dfrac{358}{27}\right)$ and a maximum point at $\mathrm{B}(3, -4)$.

a Use the second derivative to justify that A is a minimum point.

b Given that $f'(x) = -\dfrac{3}{2}x^2 + 4x + p$, find p.

c Find $f(x)$.

Exam-style questions

6 Consider the functions f and g defined by $f(x) = x^2 + x + 1$ and $g(x) = 2x + 1$.

a Find expressions for

i $(g \circ f)(x)$

ii $f'(x)$. (4)

b Find $\displaystyle\int \frac{2g(x)}{f(x)}\,dx$. (2)

7 A geometric sequence has first term e and third term e^2.

a Find the common ratio. (2)

b Hence show that all the terms of the sequence lie on the graph of $f(x) = \sqrt{e^x}$. (3)

c Show that the area of the region enclosed by the graph $y = f(x)$, the x-axis and the lines $x = 1$ and $x = 3$ is given by $2e^{\frac{1}{2}}(e - 1)$. (3)

8 Consider the function f defined by $f(x) = \ln(x + 1)$.

a State the largest possible domain D of f. (1)

b Find an expression for $f'(x)$. (2)

c Hence show that f is increasing on D. (2)

d Find an expression for the inverse function $f^{-1}(x)$. (3)

e Show that the graphs of f and f^{-1} have a common tangent t at $x = 0$. (3)

f State the equation of t. (1)

g Hence determine the area of the region enclosed by the tangent t, the graph of f^{-1} and the line $x = 1$ (4)

9 A particle moves along a straight line passing through the origin.

At time t s the particle's velocity, measured in cm s^{-1}, is given by $v(t) = (0.1t + 0.2)^t - 3$, for $0 \le t \le 10$. Use a GDC to answer the following.

a Find the value of t for which

i the acceleration of the article is zero (2)

ii the particle is at rest. (2)

b Find the total distance travelled by the particle in the first 10 seconds. (2)

10 The magnitudes of earthquakes are measured on a logarithmic scale.

The magnitudes of n earthquakes were recorded as: $\ln k, \ln k^2, \ln k^3, \ln k^4, \ldots, \ln k^n$

where k is a constant greater than one.

Show that the mean of this data is $\dfrac{(1+n)\ln k}{2}$. (7)

Answers

1 a $\frac{1}{6}(2x+5)^3+C$

b $\int(4x^2+20x+25)dx=\frac{4}{3}x^3+10x^2+25x+C$

c $\frac{4}{3}x^3+10x^2+25x+\frac{125}{6}+C$

d $\frac{4}{3}x^3+10x^2+25x+C_1=\frac{4}{3}x^3+10x^2+25x+\left(\frac{125}{6}+C_2\right)$, where $C_1=\frac{125}{6}+C_2$.

2 a $0<t<\sqrt{2}$ **b** 25 cm s^{-2} **c** $s(t)=\frac{1}{4}t^4-t^2+5$

3 a -3 **b** -2.18 **c** 1.65

4 a $h(x)=\frac{1}{5x-3}$ **b** $\frac{1}{5}\ln(5x-3)+C$

5 a $f''\left(-\frac{1}{3}\right)=5>0$ implies that A is a minimum point

b $p=\frac{3}{2}$

c $f(x)=-\frac{1}{2}x^3+2x^2+\frac{3}{2}x-13$

6 a i $g(f(x))=g(x^2+x+1)$ M1

$2(x^2+x+1)+1=2x^2+2x+3$ A1

ii $f'(x)=2x+1$ M1A1

b $\int\frac{2g(x)}{f(x)}\,\mathrm{d}x=2\int\frac{2x+1}{x^2+x+1}\,\mathrm{d}x$ M1

$=2\ln(x^2+x+1)+C$ A1

7 a $r^2=\frac{e^2}{e}=e\Rightarrow r=\sqrt{e}$ M1A1

b Any term u_n can be written $u_n=e\times\left(\sqrt{e}\right)^{n-1}=\left(\sqrt{e}\right)^{n+1}=\sqrt{e^{n+1}}$ M1A1

So $u_n=\sqrt{e^{n+1}}=f(n+1)$ and hence lies on graph of $f(x)=\sqrt{e^x}$ R1

c $\int_1^3 f(x)\,\mathrm{d}x=\int_1^3\sqrt{e^x}\,\mathrm{d}x=\int_1^3 e^{\frac{x}{2}}\,\mathrm{d}x=2\left[e^{\frac{x}{2}}\right]_1^3=2\left(e^{\frac{3}{2}}-e^{\frac{1}{2}}\right)$ M1A1A1

$=2e^{\frac{1}{2}}(e-1)$ AG

8 a $x>-1$ A1

b $f'(x)=\frac{1}{x+1}$ M1A1

c $x>-1\Rightarrow f'(x)=\frac{1}{x+1}>0$ M1R1

Therefore f is increasing in D. AG

d $x = \ln(y+1)$ M1

$e^x = y+1$ M1

$y = e^x - 1$

$f^{-1}(x) = e^x - 1$ A1

e $\left(f^{-1}\right)'(x) = e^x$ M1

$\left(f^{-1}\right)'(0) = 1 = f'(0)$ so gradient of tangent to $f(x)$ at $x = 0$ is equal to gradient of tangent to $f'(x)$ at $x = 0$. R1

Also, $f(0) = 0 = f^{-1}(0)$, so tangents pass through same point, and are hence the same line. R1

f $y = x$ A1

g $\int_0^1 e^x - 1 - x \, dx$ M1A1

$= \left[e^x - x - \frac{x^2}{2}\right]_0^1 = e - \frac{5}{2}$ A1A1

9 **a** **i** Use GDC to solve $\frac{dv}{dt} = 0$ M1

$t = 3.35$ seconds A1

ii Use GDC to solve $v(t) = 0$ M1

$t = 9.26$ seconds A1

b Use GDC to calculate $\int_0^{10} |v(t)| \, dt$ M1

$\int_0^{10} |v(t)| \, dt = 24.1$ cm. A1

10 $\bar{x} = \frac{\ln k + \ln k^2 + \ldots + \ln k^n}{n}$ M1

$= \frac{\ln k + 2\ln k + \ldots + n\ln k}{n}$ M1A1

$= \frac{\ln k}{n}(1 + 2 + \ldots + n)$ A1

$= \frac{\ln k}{n} \times \frac{n(n+1)}{2}$ M1A1

$\frac{(n+1)}{2}\ln k.$ A1

11 Relationships in space: geometry and trigonometry in 2D and 3D

1 If the first term in an arithmetic series is 2 and the 4th term is 14, find:

a the common difference

b the sum of the first 14 terms.

2 The function f is given by $f(x)=\sqrt{x-4}$. Sketch the function showing 2 points and state its domain and range.

3 **a** Write down the asymptotes of $f(x)=\dfrac{2x-1}{x-1}$

b Solve $\dfrac{2x-1}{x-1}=x^3+2$

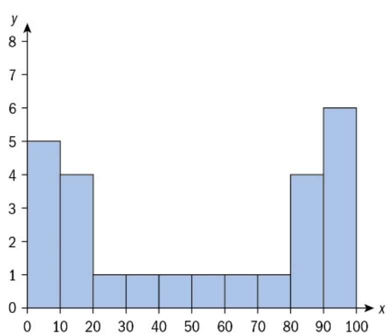
4 Match the following statistics with the histograms.

a Mean 49. Standard deviation 20

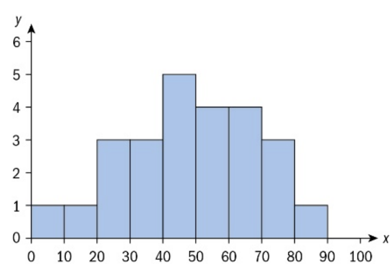
b Mean 52. Standard deviation 40

c Mean 68. Standard deviation 16

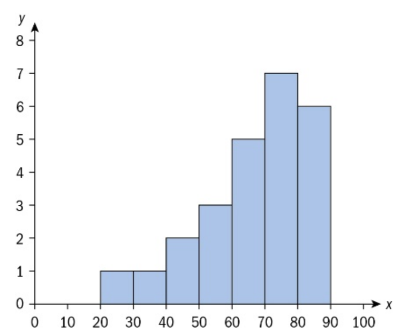
Histogram 1



Histogram 2



Histogram 3



5 If $\log_9 p = x$, $\log_9 q = y$ and $\log_9 r = z$, express $\log_9\left(\dfrac{p^2}{q^3 r}\right)^4$ in terms of x, y and z.

6 Consider the functions $f(x)=4x$ and $g(x)=\dfrac{1}{x-3}$

a Calculate $(f\circ g)(5)$

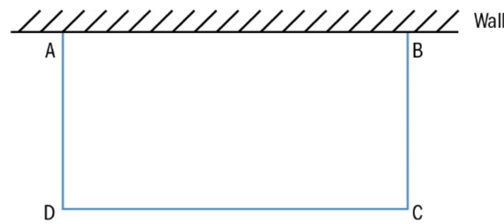
b Find $g^{-1}(x)$

c Write down the domain of g^{-1}.

7 The rate of change of the number of fish (f) in a pond is modelled by the equation $\dfrac{df}{dt}=2t+5t^{1.5}$, where t is the time in months. If there are 100 fish originally, how many will there be after 4 months? Round your answer to the nearest fish.

8 The diagram shows a rectangular play area ABCD enclosed on three sides by 120 m of fencing, and on the fourth by a wall AB.

Find the maximum play area.

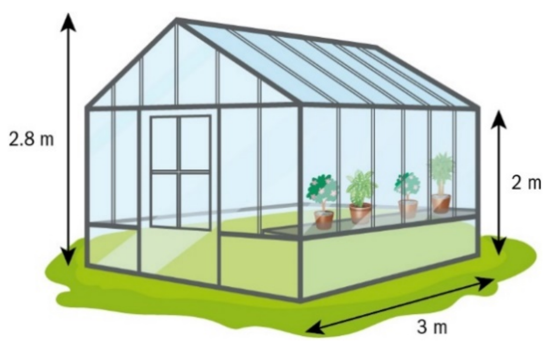


Exam-style questions

9 Mrs Smith´s greenhouse has a maximum height of 2.8 metres.

The two vertical, rectangular sides are 2 metres high and 3 metres long.

The roof consists of two equal rectangles, each of which has an area of 6 m^2.



a Find the dimensions of the rectangles that make the roof. (2)

b Hence find the width of the greenhouse. (3)

c Determine the total surface area of all the vertical sides of the greenhouse. (3)

d Calculate the volume of the greenhouse. (3)

Mrs Smith spent \$2350 on the construction of the greenhouse. She borrowed this amount from the local bank that offered her a special deal. Mrs Smith will pay a fixed amount of \$50 per month for 5 years.

e Calculate the annual interest rate the bank is charging Mrs Smith. (4)

10 John draws a rectangle and a square. Let x represent the length of the side of the square in centimetres. The length of the rectangle is 3 cm longer than the side of the square and the width of the rectangle is double the length of the side of the square.

a Write down an expression for the sum of the areas of the two shapes. (2)

The sum of the areas of the rectangle and square is 24 cm^2.

b Show that x is solution of the equation $x^2 + 2x - 8 = 0$. (2)

c Hence find the dimensions of the two shapes. (4)

11 A rock-climber slips off a rock-face and falls vertically. At first, he falls freely under gravity, but after 3 seconds a safety rope slows his fall.

The height, h metres, of the rock-climber t seconds after he falls is given by:

$$h(t) = \begin{cases} 150 - 5t^2, 0 \le t \le 3 \\ t^2 - 33t + 195, 3 < t \le 5 \end{cases}$$

a Find the height of the rock-climber when t = 2 s. (2)

b Sketch a graph of h against t for $0 \le t \le 5$. On your sketch, label the coordinates of the points at which $t = 0, t = 3$ and $t = 5$. (5)

c Find $\frac{dh}{dt}$ for

i $0 < t < 3$ **ii** $3 < t < 5$. (3)

d Find the velocity of the rock-climber when t = 2. (2)

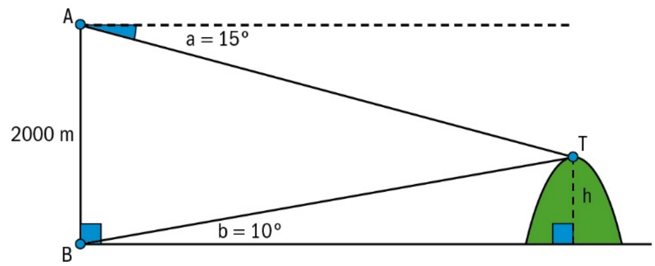
e Show that the velocity of the rock-climber is increasing in the interval $3 < t < 5$. (2)

f Gianni is 100 metres away from the base of the vertical rock when the climber reaches his minimum height whilst falling during the interval $0 \le t \le 5$. By assuming that Gianni is looking from ground level, estimate the angle of elevation, in degrees, at which Gianni looks at the climber. (3)

12 When the top T of a mountain is viewed from point A, which is 2000 m above the ground, the angle of depression is 15°.

When T is viewed from point B on the ground, the angle of elevation is 10°.

Given that the points A, B and T lie in the same vertical plane, find



a the size of the angle ATB (2)

b the height of the mountain, giving the answer correct to 1 decimal place. (4)

13 The arithmetic mean of two positive real numbers a and b is $\frac{a+b}{2}$.

Their geometric mean is $\sqrt{ab}$.

a For the numbers 1 and 4, find

i their arithmetic mean **ii** their geometric mean. (2)

b Use the fact that, for any real numbers a and b, $\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0$ to prove that the arithmetic mean of two real numbers is greater than, or equal to, the geometric mean. (4)

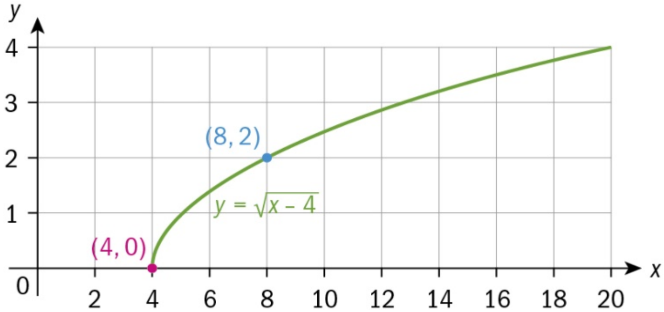
c **i** Given that a and b are positive real numbers, apply the result from part **b** to a^2 and b^2 to prove that $a^2 + b^2 - 2ab\cos C \ge 0$ for any angle C. You must fully justify your proof.

ii State the geometrical rule which is associated with the result you proved in part **i**. Explain how this geometrical rule, and the proved inequality, are consistent with one another. (7)

Answers

1 **a** 4 **b** 392

2



Domain $x \geq 4$

Range $y \geq 0$

3 **a** $x = 1,\ y = 2$ **b** $-0.819, 1.38$

4 a2, c1, b3.

5 $8x - 12y - 4z$

6 **a** 2 **b** $g^{-1}(x) = \frac{1}{x} + 3$ **c** $x \neq 0$

7 181

8 1800 m^2

9 **a** $6 \div 3 = 2$

2 metres by 3 metres A1A1

b $2\sqrt{2^2 - 0.8^2} = 3.67$ M1A1A1

c $2\left((3 \times 2) + \left(\frac{3.66.. \times 0.8}{2}\right) + 3.66.. \times 2\right)$ M1A1

$= 29.6\ m^2$ A1

d $3 \times \frac{3.66.. \times 0.8}{2} + 2 \times 3.66.. \times 3$ M1A1

$= 26.4\ m^3$ A1

e $2350\left(1 + \frac{R}{100}\right)^5 = 50 \times 60$ M1A1A1

5% A1

10 **a** $x^2 + 2x(x + 3)$ A1A1

b $x^2 + 2x(x + 3) = 24$ M1

$3x^2 + 6x - 24 = 0$ A1

$x^2 + 2x - 8 = 0$ AG

c $(x + 4)(x - 2) = 0$ M1

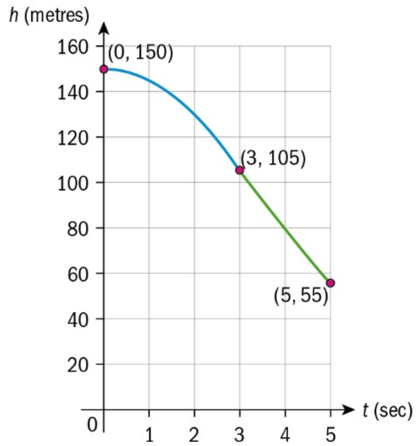
$x = 2$ A1

Square has dimensions 2×2 A1

Rectangle has dimensions 5×4 A1

11 a $h(2) = 150 - 5 \times 2^2 = 130$ metres M1A1

b



(Three points correctly labelled) A3

(correct shape of each branch of the graph) A1A1

c **i** $h'(t) = -10t, 0 < t < 3$ M1A1

ii $h'(t) = 2t - 33, \quad 3 < t < 5$ A1

d $h'(2) = -20 \text{ m s}^{-1}$ M1A1

e $h''(t) = 2 > 0, \quad 3 < t < 5$ A1R1

Hence the velocity is increasing AG

f $\tan\theta = \dfrac{55}{100}$ M1A1

$\theta = \tan^{-1}\left(\dfrac{55}{100}\right) = 28.8^{\circ}$ A1

12 a $10 + 15 = 25^{\circ}$ M1A1

b First find length BT: $\dfrac{\sin 25^{\circ}}{2000} = \dfrac{\sin 75^{\circ}}{\text{BT}}$ M1A1

$\text{BT} = 4571.15\ldots \text{metres}$ A1

Now find height of T: $h = 4571.15\ldots \sin 10^{\circ} = 794$ metres A1

13 a **i** 2.5 **ii** 2 A1A1

b $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$

$\Rightarrow a - 2\sqrt{a}\sqrt{b} + b \geq 0$ M1A1

$\Rightarrow a + b \geq 2\sqrt{a}\sqrt{b} \Rightarrow \dfrac{a+b}{2} \geq \sqrt{ab}$ M1A1

So the arithmetic mean is greater or equal to the geometric mean. AG

c **i** Part b $\Rightarrow \dfrac{a^2 + b^2}{2} \geq \sqrt{a^2 b^2}$ M1

$\Rightarrow a^2 + b^2 \geq 2ab$ A1

$\Rightarrow a^2 + b^2 \geq 2ab\cos C$ A1

for any angle C, since $-1 \leq \cos C \leq 1$ and $a, b > 0$ R2

So $a^2 + b^2 - 2ab\cos C \geq 0$ AG

ii For a triangle with sides a, b, c and opposite angles A, B, C, the cosine rule states that

$c^2 = a^2 + b^2 - 2ab\cos C$ A1

As $c^2 \geq 0$ this is consistent with the inequality in **c i** R1

12 Periodic relationships: trigonometric functions

1 Convert each angle measure from degrees to radians. Give exact answers in terms of π.

a 270° **b** 135° **c** 240°

2 Convert each angle measure from radians to degrees.

a $\frac{\pi}{2}$ **b** $\frac{8\pi}{3}$ **c** $\frac{7\pi}{2}$

3 Without using your GDC, solve each equation for $-\pi \le \theta \le 2\pi$.

a $\sin\left(\frac{\theta}{2}\right) = 0$ **b** $\sin^2\theta = \cos^2\theta$

4 The velocity (v) of a ball in a science experiment is modelled by $v(t) = 25 - 4t^2$ where t is the time measured in seconds.

a If the displacement (s) after 3 secs is 10m, find an expression for s in terms of t.

b At what time is the displacement a maximum?

c Find the distance travelled in the first second.

5 $f(x) = x$ and $g(x) = 2\sqrt{x}$

a Sketch f and g on the same axes

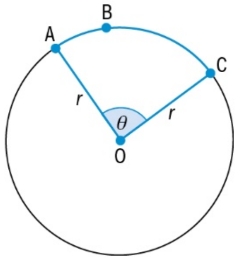
b Write down an expression for the area contained between f and g.

c Find the area.

d The line $x = a$ divides the area in half. Find the value of a.

Exam-style questions

6 The diagram shows a circle with centre O and radius r. The points A, B and C lie on the circumference of the circle. The area of the sector OABC is 4 cm^2 and the length of the arc ABC is 3 cm.



Let $\theta = A\hat{O}C$, where θ is measured in radians.

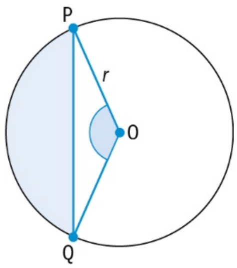
a Determine the exact values of

i r

ii θ (4)

b Hence find the area of the triangle AOC, giving your answer correct to three significant figures. (2)

7 The diagram shows a circle with centre O and radius r.



The points P and Q lie on the circumference of the circle.

Let $P\hat{O}Q = 2\alpha$ with $0 < \alpha < \frac{\pi}{2}$ and l be the length of the arc PQ.

a Show that $PQ = 2r\sin\alpha$. (4)

Consider the function defined by $f(x) = 2x - 3\sin x$, $0 < x < \frac{\pi}{2}$.

b Sketch the graph of f, showing clearly its zero. (2)

Given that $l = 1.5PQ$,

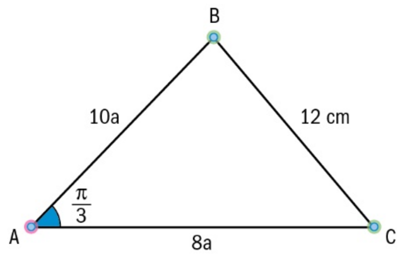
c Use the graph of f to find the value of α. (3)

d State the values of α for which $l < 1.5PQ$ (3)

The area shaded is 4.

e Find the value of r. (3)

8 The following diagram shows a triangle ABC, where $a > 0$ is a parameter to be determined.



$AB = 10a$, $AC = 8a$ and $B\hat{A}C = \frac{\pi}{3}$

a Show that $a = \sqrt{\frac{12}{7}}$. (3)

b Hence calculate the lengths of AB and AC, giving your answers as decimals to 3 significant figures. (2)

Let D be a point on the side AB such that $AD = x$.

c Find an expression for $d(x)$, the length of the line DC, given as a function of x. (2)

d Sketch the graph of $y = d(x)$, indicating clearly the coordinates of its minimum point. (3)

e For the value of x that minimizes $d(x)$,

i calculate the size of angle $A\hat{D}C$, showing your working or justifying your reasoning

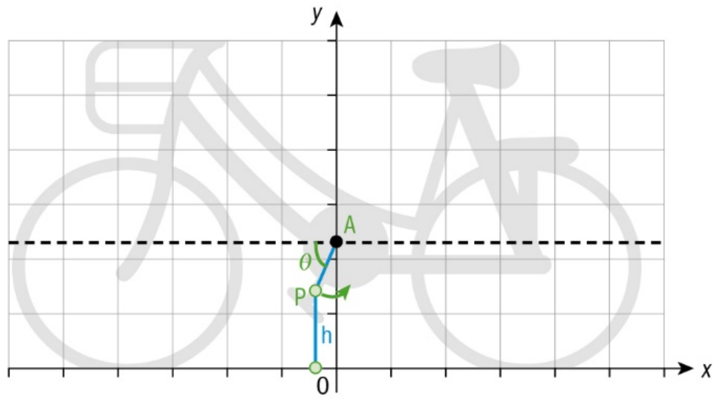
ii determine the perimeter of the triangle ADC. (4)

9 A quadratic sequence has an nth term of $n^2 - 4n + 21$.

a Find the 6th term of the sequence. (1)

b Prove that every term in the sequence is positive. (4)

10 The diagram shows the movement of a bicycle pedal. The position of the pedal is represented by the point P that rotates counter-clockwise about the point A, which is located 26 cm above the ground.



Let θ represent the angle which the line segment AP makes with the horizontal line through A, as shown in the diagram.

$\text{AP} = 15\,\text{cm}$ and $\theta = 2.1t$, where t represents the time, in seconds, after the start of the movement.

a State the coordinates of the initial position of the points A and P. (2)

b Show that the height of the pedal above the ground t seconds after the start of the movement is given by $h(t) = 26 - 15\sin(2.1t)$, $t \geq 0$. (3)

c Sketch the graph of h against t for $0 \leq t \leq 5$, indicating clearly the coordinates of its maximum and minimum points, correct to three significant figures. (3)

d Determine how long it takes for the pedal to complete a full turn about the point A. (2)

e Describe a sequence of transformations which are required to obtain the graph of h from the graph of $s(t) = \sin t$, indicating clearly the order in which transformations need to be performed. (4)

Answers

1 **a** $270°\left(\frac{\pi}{180°}\right)=\frac{270\pi}{180}=\frac{3\pi}{2}$ **b** $135°\left(\frac{\pi}{180°}\right)=\frac{135\pi}{180}=\frac{3\pi}{4}$ **c** $240°\left(\frac{\pi}{180°}\right)=\frac{240\pi}{180}=\frac{4\pi}{3}$

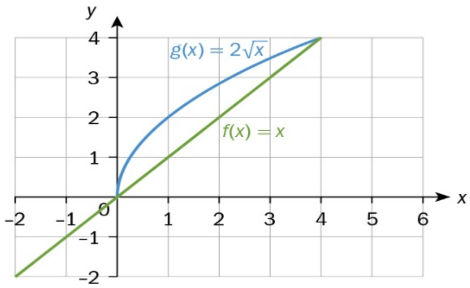
2 **a** $\frac{\pi}{2}\left(\frac{180°}{\pi}\right)=\frac{180°}{2}=90°$ **b** $\frac{8\pi}{3}\left(\frac{180°}{\pi}\right)=\frac{1440°}{3}=480°$ **c** $\frac{7\pi}{2}\left(\frac{180°}{\pi}\right)=\frac{1260°}{2}=630°$

3 **a** $\sin\left(\frac{\theta}{2}\right)=0\Rightarrow\frac{\theta}{2}=0,\pi\Rightarrow\theta=0,2\pi$

b $\sin^2\theta=\cos^2\theta\Rightarrow\frac{\sin^2\theta}{\cos^2\theta}=1\Rightarrow\tan^2\theta=1\Rightarrow\tan\theta=\pm1\Rightarrow\theta=\pm45°,\pm135°,\pm225°,\pm315°$

4 **a** $s(t)=25t-\frac{4}{3}t^2-29$ **b** 2.5secs **c** $23\frac{2}{3}$m

5 **a**



b $\int_0^4\left(2\sqrt{x}-x\right)dx$ **c** 2.67 **d** $a=1.51$

6 **a** **i** $r\theta=3$ A1

$r^2\frac{\theta}{2}=4$ A1

Solving simultaneously using an appropriate method M1

$r=\frac{8}{3}$ cm A1

ii $\theta=\frac{9}{8}$ radians A1

b $A=\frac{1}{2}\left(\frac{8}{3}\right)^2\sin\left(\frac{9}{8}\right)=3.21\text{ cm}^2$ M1A1

7 **a** Cosine rule gives

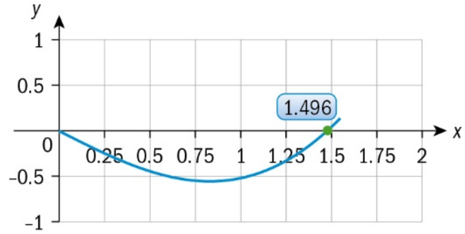
$PQ=\sqrt{r^2+r^2-2r^2\cos2\alpha}$ M1A1

$=\sqrt{2r^2-2r^2\left(1-2\sin^2\alpha\right)}$ M1

$=\sqrt{4r^2\sin^2\alpha}$ M1

$=2r\sin\alpha$ AG

b A1 for shape and domain, A1 for zero at 1.50 (3 sf)



c $l = 1.5PQ \Rightarrow 2r\alpha = 3r\sin\alpha \Rightarrow 2\alpha - 3\sin\alpha = 0$ M1

So $f(\alpha) = 2\alpha - 3\sin\alpha = 0$ R1

From part b, the zero of f is 1.50, so $\alpha = 1.50$ A1

d This is equivalent to values of α for which $f(x) < 0$, R1

that is when $0 < \alpha < 1.50$ A1A1

e $\left(\frac{1}{2} \times 2(1.496...) \times r^2\right) - \left(\frac{1}{2} r^2 \sin(2 \times 1.496...)\right) = 4$ A1

Solve for $r > 0$ using GDC M1

$r = 1.68$ A1

8 a $(10a)^2 + (8a)^2 - 2 \cdot 10a \cdot 8a\cos\frac{\pi}{3} = 12^2$ M1A1

$84a^2 = 144 \Rightarrow a^2 = \frac{12}{7}$ A1

$a = \sqrt{\frac{12}{7}}$ AG

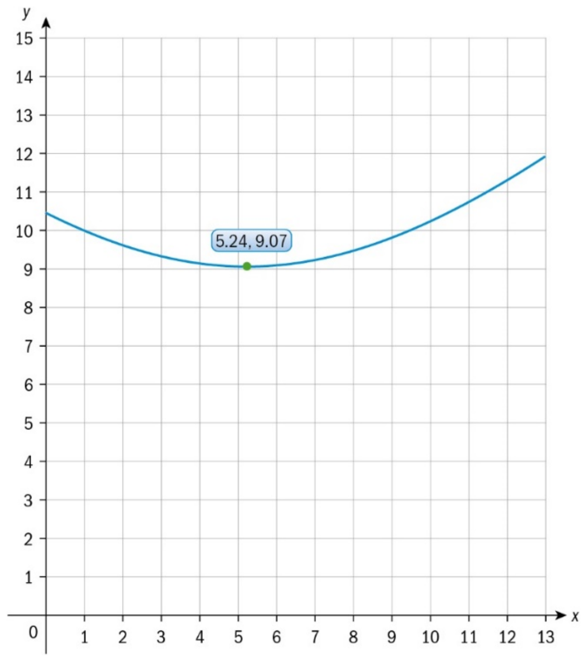
b $AB = 10\sqrt{\frac{12}{7}} = 13.1$ 3 s.f. and $AC = 8\sqrt{\frac{12}{7}} = 10.5$ A1A1

c Cosine rule gives

$[d(x)]^2 = x^2 + \left(8\sqrt{\frac{12}{7}}\right)^2 - 2 \times 8\sqrt{\frac{12}{7}} \times x\cos\frac{\pi}{3}$ M1

$d(x) = \sqrt{x^2 + \frac{768}{7} - 8\sqrt{\frac{12}{7}}x}$ A1

d



A1 for correct shape A1 for domain A1 for minimum point

e **i** $\dfrac{\sin\frac{\pi}{3}}{9.07} = \dfrac{\sin \text{A}\hat{\text{D}}\text{C}}{8\sqrt{\frac{12}{7}}}$ R1

(OR shortest distance from the point C to the line AB meets at right angles)

So $\text{A}\hat{\text{D}}\text{C} = \dfrac{\pi}{2}$ A1

ii $AC + CD + AD = 10.47\ldots + 9.071\ldots + 5.237\ldots = 24.8\,\text{cm}$ M1A1

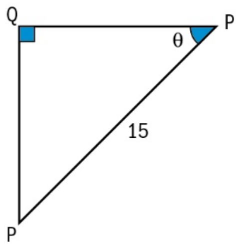
9 **a** $6^2 - 4(6) + 21 = 33$ A1

b Completing the square, $n^2 - 4n + 21 = (n - 2)^2 + 17$ M1A2

Since $(n - 2)^2 \geq 0$ for all n, $(n - 2)^2 + 17 > 0$ for all n. R1

10 **a** $A(0, 26)$ and $P(-15, 26)$ A1A1

b E.g. Consider a right-angled triangle PQA M1

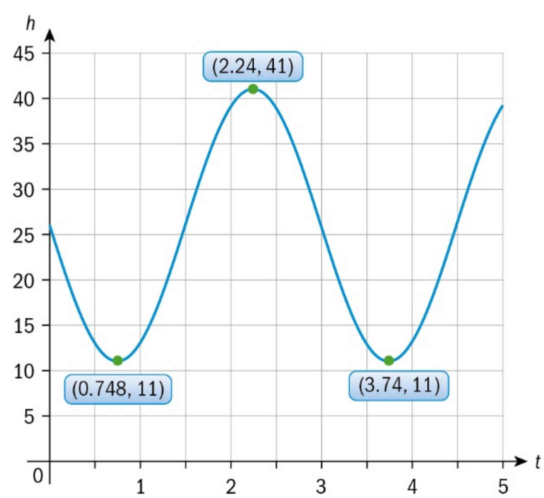


$PQ = 15\sin\theta = 15\sin 2.1t$ A1

$h = 26 - PQ$ M1

$h(t) = 26 - 15\sin 2.1t$ AG

c



A1 for shape and domain

A1 for maximum

A1 for minima

Period = time of full turn $= 3.740 - 0.748 = 2.99$ R1A1

Horizontal stretch with factor $\dfrac{1}{2.1}$; A1

Vertical stretch with factor 15; A1

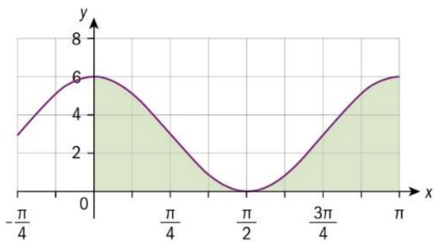
Reflection in the x – axis; A1

Vertical translation 26 units upwards A1

13 Modelling change: more calculus

1 A curve with equation $y = f(x)$ passes through the point $(0, 3)$. Its gradient function is $f'(x) = x - \cos x$. Find the equation of the curve.

2 The graph represents the function $f(x) = p\cos(qx) + r$, for $0 \le x \le \pi$, where $p, q, r \in \mathbb{N}$.



a Find the values of p, of q, and of r.

b Find the area of the shaded region.

3 A particle moves along a line such that its displacement from a fixed origin O is given by $s(t) = 2t - 10\ln(t + 1)$, for $t \ge 0$.

a Find the velocity function for s.

b Find when the particle is moving left.

c Show that the velocity of the particle is always increasing.

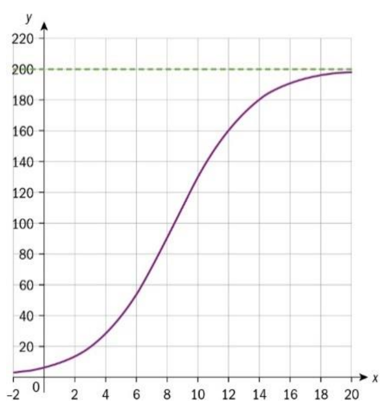
4 A rectangular plot of land is enclosed by 180 m of fencing material on three sides. The fourth side of the plot is bordered by a stone wall. Find the dimensions of the plot that encloses the maximum area. Find the maximum area.

5 A particle's velocity, in metres, is given by $v(t) = 0.4t\cos t$, for $0 \le t \le 6$, where t is time in seconds. Use a GDC to answer the following.

a Find the displacement of the particle after 6 seconds.

b Find the distance the particle travels in 6 seconds.

6 Part of the graph of $f(x) = \dfrac{200}{1 + 30e^{-0.4x}}$ is shown.



a Find the range of f.

b Show that $f'(x) = \dfrac{2400e^{-0.4x}}{\left(1 + 30e^{-0.4x}\right)^2}$.

c Use a GDC to find the maximum rate of change of f.

Exam-style questions

7 The point $A\left(\frac{1}{2}, a\right)$ lies on the graph of the curve of $y = \sin(2x - 1)$.

a Determine the value of a . (2)

b Find the gradient of the tangent to the curve at any point x on the domain. (2)

c Hence find the equation of the normal to the curve at A. (3)

8 Consider the functions defined by $f(x) = e^{2x}$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$.

a Find expressions for

i $f'(x)$

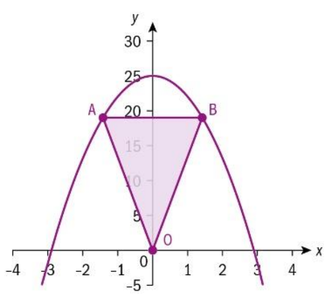
ii $g'(x)$. (2)

Consider the function $h(x) = f(x) \times g(x)$.

b Show that $h'\left(\frac{\pi}{3}\right) > 0$. (3)

9 The diagram shows an isosceles triangle with one of its vertices at the origin O and AB parallel to the x – axis. Points A and B both lie on the curve $y = 25 - 3x^2$.

Let the x-coordinate of B be x.



a State the coordinates of A and B in terms of x . (2)

b Show that an expression for the area of the triangle ABC is given by $25x - 3x^3$, $x > 0$. (1)

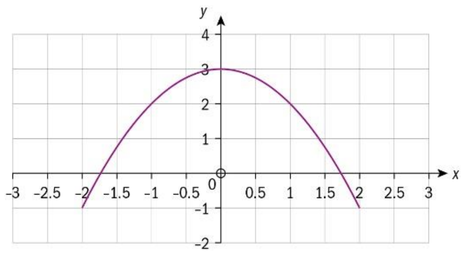
c Hence find the largest area that the triangle can have. Give your answer in exact form. (4)

d For the triangle with largest area, determine the following correct to three significant figures:

i its perimeter

ii the size of its internal angles in degrees. (6)

10 The diagram shows the graph of the function f defined by $f(x) = 3 - x^2$ for $-2 \le x \le 2$.



Consider the function g defined by $g(x) = \sin(e^x)$ for $-2 \le x \le 2$.

a On the same axes, sketch the graphs of f and g for $-2 \le x \le 2$. (2)

b Solve $f(x) = g(x)$ in this interval. (2)

c Hence state the solutions to $f(x) > g(x)$ in this interval. (1)

d Find the area of the region enclosed by the graphs of f and g. (2)

11 Consider the function f defined by $f(x) = \dfrac{1}{\sin x}$ for $-\pi < x < \pi, x \ne 0$.

a Use quotient rule to find an expression for $f'(x)$, for $-\pi < x < \pi, x \ne 0$. (2)

b Show that $f''(x) = \dfrac{1+\cos^2 x}{\sin^3 x}$. (3)

c Hence show that the graph has exactly one maximum and one minimum point in the domain given. (4)

d State the equations of the vertical asymptotes to the graph of f. (2)

e Hence sketch the graph of f. (3)

Answers

1 $f(x) = \frac{1}{2}x^2 - \sin x + 3$

2 **a** $p = 3$; $q = 2$; $r = 3$

b 3π

3 **a** $v(t) = 2 - \frac{10}{t+1}$

b $0 \le t < 4$

c $v'(t) = a(t) = \frac{10}{(t+1)^2}$ which is always greater than 0, therefore velocity is always increasing.

4 The dimensions are 45 m by 90 m with a maximum area of 4050 m^2.

5 **a** -0.687 m

b 4.34 m

6 **a** $0 < f(x) < 200$

b $f(x) = \frac{200}{1 + 30e^{-0.4x}} = 200\left(1 + 30e^{-0.4x}\right)^{-1}$

$f'(x) = -1 \cdot 200\left(1 + 30e^{-0.4x}\right)^{-2}\left(-0.4 \cdot 30e^{-0.4x}\right) = \frac{-200 \cdot -12e^{-0.4x}}{\left(1 + 30e^{-0.4x}\right)^2} = \frac{2400e^{-0.4x}}{\left(1 + 30e^{-0.4x}\right)^2}$

c 8.50

7 **a** $a = \sin\left(2 \times \frac{1}{2} - 1\right) = \sin 0 = 0$ M1A1

b $y'(x) = 2\cos(2x - 1)$ M1A1

c $y'\left(\frac{1}{2}\right) = 2\cos(0) = 2$ A1

$m = -\frac{1}{2}$ M1

$y = -\frac{1}{2}\left(x - \frac{1}{2}\right)$ A1

8 **a** **i** $f'(x) = 2e^{2x}$ A1

ii $g'(x) = \cos\left(x - \frac{\pi}{3}\right)$. A1

b $h'\left(\frac{\pi}{3}\right) = f'\left(\frac{\pi}{3}\right)g\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)g'\left(\frac{\pi}{3}\right)$ M1

$= 2e^{\frac{2\pi}{3}}\sin 0 + e^{\frac{2\pi}{3}}\cos 0$ A1

$= e^{\frac{2\pi}{3}} > 0$ A1AG

9 **a** $A\left(-x, 25-3x^2\right),\ B\left(x, 25-3x^2\right)$ A1A1

b $f(x) = \dfrac{2x\left(25-3x^2\right)}{2} = 25x - 3x^3, x > 0$ M1AG

c $f'(x) = 25 - 9x^2$ A1

$f'(x) = 0 \Rightarrow x = \pm\dfrac{5}{3}$ M1A1

$f\left(\dfrac{5}{3}\right) = \dfrac{125}{3} - \dfrac{125}{9} = \dfrac{250}{9}$ A1

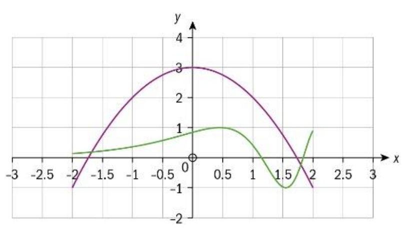
d **i** $OA = OB = \sqrt{\left(\dfrac{5}{3}\right)^2 + \left(\dfrac{50}{3}\right)^2} = 16.749\ldots$ A1

perimeter $= 2 \times \dfrac{5}{3} + 2 \times 16.749\ldots = 36.8$ M1A1

ii $A\hat{O}B = 2\arctan\dfrac{\frac{5}{3}}{\frac{50}{3}} = 2\arctan\dfrac{1}{10} = 11.4^{\mathrm{o}}$ M1A1

$O\hat{A}B = A\hat{B}O = \dfrac{180 - 11.4212..}{2} = 84.3^{\mathrm{o}}$ A1

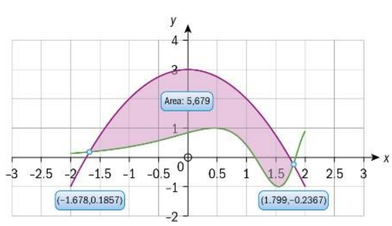
10 **a** A1 for shape of g, A1 for domain of g



b $f(x) = g(x) \Rightarrow x = -1.68, x = 1.80$ A1A1

c $-1.68 \le x \le 1.80$ A1

d $\displaystyle\int_{-1.678\ldots}^{1.799\ldots} \left(f(x) - g(x)\right)\ \mathrm{d}x = 5.68$ M1A1A1



11 a $f'(x) = \dfrac{1' \cdot \sin x - 1 \cdot (\sin x)'}{\sin^2 x} = -\dfrac{\cos x}{\sin^2 x}$ M1A1

b $f''(x) = \dfrac{(-\cos x)' \cdot \sin^2 x + \cos x \cdot (\sin^2 x)'}{(\sin^2 x)^2}$ M1

$= \dfrac{\sin x \cdot \sin^2 x + 2\sin x \cos^2 x}{\sin^4 x}$ A1A1

$= \dfrac{\sin^2 x + 2\cos^2 x}{\sin^3 x} = \dfrac{1+\cos^2 x}{\sin^3 x}$ A1AG

c $f'(x) = 0 \Rightarrow -\dfrac{\cos x}{\sin^2 x} = 0 \Rightarrow \cos x = 0$ M1

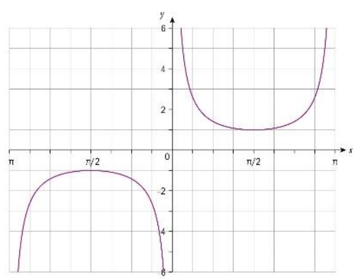
$x = \pm\dfrac{\pi}{2}$ A1

$f''\left(\dfrac{\pi}{2}\right) = \dfrac{1+0}{1} > 0 \Rightarrow f$ has a minimum at $\dfrac{\pi}{2}$ R1

$f''\left(-\dfrac{\pi}{2}\right) = \dfrac{1+0}{-1} < 0 \Rightarrow f$ has a maximum at $-\dfrac{\pi}{2}$ R1

d Vertical asymptotes: $x = \pm\pi, x = 0$ A1A1

e A1A1 - for each branch; A1 for domain and asymptotes



14 Valid comparisons and informed decisions: probability distributions

1 Consider a random variable $X \sim B(n, p)$.

Show that $P(X = x) = \frac{n - x + 1}{x} \times \frac{p}{1 - p} \times P(X = x - 1)$.

2 We consider X a discrete random variable such that $P(X = x_i) = \frac{3x_i - 2}{22}, i = 1, 2, 3, 4$.

a Complete the table:

x_i				
$P(X = x_i)$				

b Verify that $P(X = x_i)$ is a probability function.

c Find $E(X)$ and $Var(X)$, giving your answers in simplified fractions.

3 A factory supervisor checks some machines to check if they work efficiently. If a machine works efficiently, this machine "good", if it is "bad". The probability that a machine is "bad" is 0.1.

The supervisor checks a sample of four machines. Find the probability that:

a one machine is "bad"

b one machine is "good"

c at least one machine is "bad".

4 A sample of four flashlights are chosen at random. Let X be a random variable that counts the number of defunct flashlights in the sample. It is given that the random variable follows the binomial distribution. The probability of picking zero defunct flashlights equals the probability that all four are defunct.

a Find the probability that one flashlight will be defunct.

b Work out the probability that I will get 3 defunct flashlights out of the 4.

c Calculate $E(X)$ and $Var(X)$.

5 **a** The weight of each pupil in a school is measured. We consider as X the random variable that measures each weight. If the random variable X follows the normal distribution with mean $\mu = 60\text{kg}$ and variance $\sigma^2 = 25\text{kg}^2$, work out the probability $P(X \geq 70)$.

b Let X be the time of waiting in a doctor's office. We assume that $X \sim N(12.56,\ 3.75^2)$. Work out the probability $P(10 < X < 15)$.

Exam-style questions

6 Find the value of each of the following, giving your answer as an integer:

a $\log_{10} \frac{1}{100}$ (1)

b $\log_{10} 25 + 2\log_{10} 2$ (2)

c $\log_9 \left(3\log_{10} 5 + 3\log_{10} 2\right)$ (3)

7 Consider the rational function defined by $f(x) = \frac{2}{x-p} + q$, where $p, q \in \mathbb{Z}, x \in \mathbb{R}$.

a Write down the equations of the vertical and horizontal asymptotes of the graph of f in terms of p and q. (2)

The domain of f is $x \neq 1$ and its range is $y \neq 2$.

b Find the value of

i p

ii q (2)

c The point $\mathrm{A}(3, a)$ lies on the graph of f. Without finding an expression for the inverse of f, show that A also lies on the graph of f^{-1}. (3)

d Find an expression for $f^{-1}(x)$. (4)

8 The graph of a quadratic function has y-intercept 4 and one of its zeros is 2.

Given that $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Z}, x \in \mathbb{R}$,

a State the value of c. (1)

The equation of the axis of symmetry is $x = 3$.

b Justify that $a > 0$. (3)

c State the value of the other zero of f, and give a reason for your answer. (1)

d Hence find the value of each parameter a and b. (4)

9 Consider the strictly increasing function defined by $f(x) = \frac{2x}{x^2+1}$, $x \geq 0$.

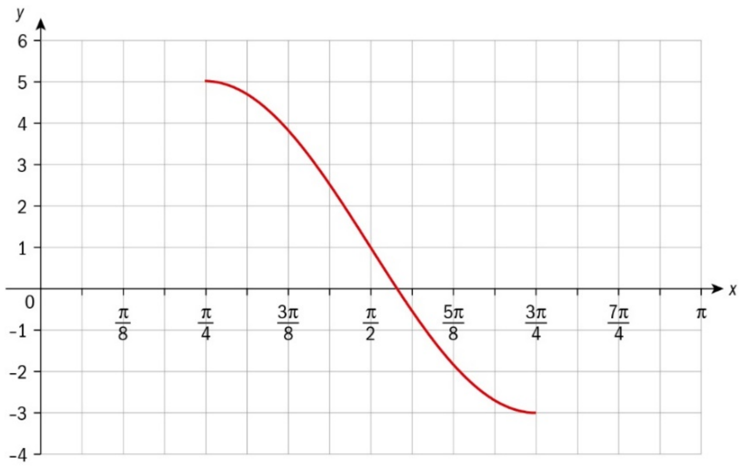
a Show that $f'(x) = \frac{2\left(1-x^2\right)}{\left(x^2+1\right)^2}$, $x \geq 0$. (4)

b Explain why the range of f is the interval $[0, 1]$ (4)

c Find $\int \frac{2x}{x^2+1}\, dx$ (2)

d Hence find the area enclosed by the graph of f and the x – axis between 0 and 1. (3)

10 The diagram shows part of the graph of $f(x) = A + B\sin(Cx)$ for real constants A, B and C, and x in radians.



a State the range of the function. (1)

b Using your answer to part a, justify that $A = 1$. (1)

c Given that $B > 0$, show that $B = 4$. (2)

d Justify that the function has period π. (2)

e Hence, find the value of C and justify your answer. (1)

Let $g(x) = \sin x$.

f The graph of g can be transformed onto the graph of f.

Describe the sequence of transformations required to obtain the graph of f from the graph of g, stating clearly the order in which they must be performed. (3)

Answers

1 $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$ since the random variable follows the binomial distribution.

$P(X=x-1)=\binom{n}{x-1}p^{x-1}(1-p)^{n-(x-1)}=\binom{n}{x-1}p^{x-1}(1-p)^{n-x+1}$.

$$\frac{n-x+1}{x}\times\frac{p}{1-p}\times\binom{n}{x-1}\times p^{x-1}\times(1-p)^{n-x+1}$$
$$=\frac{n-x+1}{x}\times\frac{p}{1-p}\times\frac{n!}{(x-1)!(n-x+1)!}\times p^{x-1}\times(1-p)^{n-x+1}$$
$$=\frac{(n-x+1)n!}{x(x-1)!(n-x+1)!}\times p^x\times(1-p)^{n-x}$$
$$=\frac{n!}{x!(n-x)!}\times p^x\times(1-p)^{n-x}$$
$$=\binom{n}{x}\times p^x\times(1-p)^{n-x}$$
$$=P(X=x)$$

2 **a** Apply the various values of i to $P(X=x_i)$ to get the results of the table.

$P(X=1)=\frac{3\times1-2}{22}=\frac{1}{22}$ $\quad P(X=2)=\frac{3\times2-2}{22}=\frac{4}{22}$

$P(X=3)=\frac{3\times3-2}{22}=\frac{7}{22}$ $\quad P(X=4)=\frac{3\times4-2}{22}=\frac{10}{22}$

x_i	1	2	3	4
$P(X=x_i)$	$\frac{1}{22}$	$\frac{4}{22}$	$\frac{7}{22}$	$\frac{10}{22}$

b Since, $P(X=1)+P(X=2)+P(X=3)+P(X=4)=\frac{1}{22}+\frac{4}{22}+\frac{7}{22}+\frac{10}{22}=1$,

then this is a probability function.

c $E(X)=\sum_{i=1}^{4}x_iP(X=x_i)$ and \$Var(X)=E(X^2)-(E(X))^2$

$E(X)=1\times\frac{1}{22}+2\times\frac{4}{22}+3\times\frac{7}{22}+4\times\frac{10}{22}=\frac{35}{11}$.

Also, $E(X^2)=\sum_{i=1}^{4}x_i^2P(X=x_i)=1\times\frac{1}{22}+4\times\frac{4}{22}+9\times\frac{7}{22}+16\times\frac{10}{22}=\frac{120}{11}$.

Therefore, $Var(X)=\frac{120}{11}-(\frac{35}{11})^2=\frac{95}{121}$.

3 $X\sim B(4,0.1)$.

a $P(X=1)=\binom{4}{1}\times0.1^1\times0.9^{4-1}=\frac{729}{2500}$. **b** $P(X=3)=\binom{4}{3}\times0.1^3\times0.9^{4-3}=\frac{9}{2500}$.

c $P(X\geq1)=1-P(X<1)=1-P(X=0)$ .Therefore, $P(X\geq1)=1-\binom{4}{0}\times0.1^0\times0.9^{4-0}=\frac{3439}{10000}$.

4 **a** $P(X=0)=P(X=4)$

$\binom{4}{0}p^0(1-p)^{4-0}=\binom{4}{4}p^4(1-p)^{4-4}\Rightarrow(1-p)^4=p^4\Rightarrow1-p=p\Rightarrow p=\frac{1}{2}$

b $P(X=x)=\binom{4}{x}(\frac{1}{2})^x(1-\frac{1}{2})^{4-x}$

Hence, $P(X=3)=\binom{4}{3}(\frac{1}{2})^3(1-\frac{1}{2})^{4-3}=\frac{1}{4}$.

c $E(X) = np$ and $Var(X) = np(1-p)$.

Hence, $E(X) = 4 \times \frac{1}{2} = 2$ and $Var(X) = 4 \times \frac{1}{2} \times (1 - \frac{1}{2}) = 1$.

5 a Use standardization $Z = \frac{X - \mu}{\sigma}$ and obtain that $Z \sim N(0,\ 1)$.

$P(X \geq 70) = 1 - P(X < 70) = 1 - P(Z < \frac{70 - \mu}{\sigma})$

$= 1 - P(Z < \frac{70 - 60}{5}) = 1 - P(Z < 2) = 1 - 0.977 = 0.023$

b Use standardization $Z = \frac{X - \mu}{\sigma}$ and obtain that $Z \sim N(0,\ 1)$.

$P(10 < X < 15) = P(\frac{10 - 12.56}{3.75} < Z < \frac{15 - 12.56}{3.75})$

$= P(-0.68 < Z < 0.65)$

$P(Z > -0.68) = 1 - P(Z < 0.68)$

$P(-0.68 < Z < 0.65) = P(Z < 0.65) - P(Z > -0.68) = P(Z < 0.65) - (1 - P(Z < 0.68))$

Therefore: $P(-0.68 < Z < 0.65) = P(Z < 0.65) + P(Z < 0.68) - 1 = 0.7422 + 0.7517 - 1 = 0.4939$

6 a $\log_{10} \frac{1}{100} = -2$ A1

b $\log_{10} 25 + 2\log_{10} 2 = \log_{10}\left(25 \times 2^2\right) = \log_{10} 100 = 2$ M1A1

c $\log_9\left(3\log_{10} 5 + 3\log_{10} 2\right) = \log_9\left(\log_{10} 125 + \log_{10} 8\right)$ M1

$= \log_9\left(\log_{10} 1000\right) = \log_9 3$ A1

$= \frac{1}{2}$ A1

7 a $x = p, y = q$ A1A1

b i $p = 1$ A1

ii $q = 2$ A1

c $a = f(3) = \frac{2}{3-1} + 2 = 3$ M1A1

A$(3,3)$ lies on $y = x$, and since graphs of f and f^{-1} are reflections in the line $y = x$, hence A also lies on the graph of f^{-1} R1AG

d $x = \frac{2}{y-1} + 2$ M1

Solve for y M1

$y = \frac{2}{x-2} + 1$ A1

$f^{-1}(x) = \frac{2}{x-2} + 1$ A1

8 a $c = 4$ A1

b A quadratic function has only one turning point at the axis of symmetry. So $f(x)$ has a turning point at $x = 3$ R1

As the function decreases from $x = 0$ (where $y = 4$) to $x = 2$ (where $y = 0$), it must also decrease until it reaches the turning point at $x = 3$. R1

So the turning point at x = 3 is a minimum, and hence $a > 0$ R1

c $x = 4$ A1

since the curve is symmetrical about $x = 3$ R1

d $f(2)=0 \Rightarrow 4a+2b+4=0$ and $f(4)=0 \Rightarrow 16a+4b+4=0$ A1

Solve simultaneously M1

$a=\dfrac{1}{2}$ and $b=-3$ A1A1

9 a $f'(x)=\dfrac{(2x)'(x^2+1)-2x(x^2+1)'}{(x^2+1)^2}=\dfrac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}$ M1A1A1

$=\dfrac{2-2x^2}{(x^2+1)^2}=\dfrac{2(1-x^2)}{(x^2+1)^2}$ A1

b $f(0)=0$ and $\lim\limits_{x\to\infty} f(x)=0$ R1

$f'(x)=0 \Rightarrow x=1$ A1

Since $f(x)>0$ for $x>0$, so f has a maximum at $x=1$ and $f(1)=1$ A1

As f is strictly increasing, $f'(x)>0 \Rightarrow x<1$ R1

Therefore, the range of f is the interval $[0,1]$ AG

c By inspection, $\int \dfrac{2x}{x^2+1}\,\mathrm{d}x=\ln(x^2+1)+C$ M1A1

d $\int_0^1 \dfrac{2x}{x^2+1}\,\mathrm{d}x=\left[\ln(x^2+1)\right]_0^1=\ln 2$ M1A1A1

10 a $-3\le y\le 5$ A1

b $A=\dfrac{5+(-3)}{2}=1$ R1AG

c $|B|=\dfrac{5-(-3)}{2}=4$ M1

As $B>0$, so $B=4$ R1AG

d The difference between the $x-$coordinates of the maximum and minimum is $\dfrac{3\pi}{4}-\dfrac{\pi}{4}=\dfrac{\pi}{2}$. M1

Therefore the period is $2\times\dfrac{\pi}{2}=\pi$ R1AG

e $C=\dfrac{2\pi}{\text{period of } f}$ M1

So $C=\dfrac{2\pi}{\pi}=2$ A1

f Vertical stretch by factor 4 and horizontal stretch by factor $\dfrac{1}{2}$ (in any order) A1A1

Followed by a vertical translation of 1 unit upwards. A1