OXFORD IB PREPARED

# MATHEMATICS: APPLICATIONS AND INTERPRETATION

IB DIPLOMA PROGRAMME

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Sample student answer, **p8**: N16, MATSD, SP1, TZ0, Q1; Sample student answer, **p22**: M17, MATHL, HP2, TZ2, Q6; Sample student answer, **p14**: N16, MATSD, SP1, TZ0, Q13; Sample student answer, **p17**: N17, MATSD, SP1, TZ0, Q9; Sample student answer, **p23–4**: N18, MATHL, HP2, TZ0, Q1; Sample student answer, **p26**: M18, MATHL, HP1, TZ2, Q7; Sample student answer, **p32**: N17, MATHL, HP2, TZ0, Q1; Sample student answer, **p55**: N18, MATME, SP1, TZ0, Q2; Sample student answer, **p59**: M16, MATME, SP2, TZ2, Q3; Sample student answer, **p64**: M17, MATSD, SP1, TZ2, Q14; Sample student answer, **p65**: M16, MATSD, SP1, TZ2, Q13; Sample student answer, **p83**: M19, MATME, SP2, TZ2, Q5; Sample student answer, **p84–5**: M18, MATME, SP2, TZ2, Q6; Sample student answer, **p106**: M17, MATME, SP2, TZ2, Q9; Sample student answer, **p107**: N17, MATHL, HP2, TZ0, Q3; Sample student answer, **p115**: M13, MATHL, HP2, TZ1, Q5; Sample student answer, **p126–7**: M14, MATME, SP1, TZ2Q9; Sample student answer, **p138**: M17, MATHL, HP3, TZ0/DM, Q2; SL Practice questions, Paper 1, Q5, **p139**: N16, MATSD, SP1, TZ0, Q11; SL Practice questions, Paper 1, Q6, **p139**: M17, MATSD, SP1, TZ1, Q13; SL Practice questions, Paper 1, Q11, **p140**: M17, MATSD, SP1, TZ1, Q9; SL Practice questions, Paper 1, Q12 **p141**: M15, MATSD, SP1, TZ1, Q9; SL Practice questions, Paper 1, Q14, **p141**: M13, MATME, SP2, TZ2, Q7; SL Practice questions, Paper 1, Q16 **p141–2**: N17, MATHL, HP2, TZ0, Q5; SL Practice questions, Paper 1, Q18, **p142–3**: M17, MATSD, SP1, TZ2, Q12; SL Practice questions, Paper 2, Q2, **p143**: N17, MATSD, SP2, TZ0, Q3; SL Practice questions, Paper 2, Q3, **p143–4**: M17, MATSD, SP2, TZ1, Q2; HL Practice questions, Paper 1, Q6, **p145**: M09, MATHL, HP3, TZ0/DM, Q1; HL Practice questions, Paper 1 Q9, **p145**: N13, MATHL, HP2, TZ0, Q8; HL Practice questions, Paper 1, Q10, **p145**: M16, MATME, SP1, TZ2, Q7; HL Practice questions, Paper 1, Q13, **p146**: N18, MATHL, HP2, TZ0, Q7; Sample student answer, **p157**: N16, MATME, SP1, TZ0, Q5; Sample student answer, **p158**: M16, MATME, SP1, TZ2, Q8; Sample student answer, **p169**: M19, MATHL, HP2, TZ2, Q2; Sample student answer, **p170**: N16, MATME, SP2, TZ0, Q6; Sample student answer, **p175**: M16, MATHL, HP3, TZ0, Q1; Sample student answer, **p178–9**: M16, MATHL, HP2, TZ2, Q6; Sample student answer, **p183**: M18, MATME, SP1, TZ2, Q3; Sample student answer, **p189**: M19, MATME, SP2, TZ2, Q1; Sample student answer, **p195–6**: M17, MATSD, SP1, TZ1, Q6; Sample student answer, **p210**: M18, MATHL, HP3, TZ0, Q3; SL Practice questions, Paper 2, Q1, **p218–19**: N17, MATME, SP2, TZ0/SP, Q8; Sample student answer, **p224–5**: M16, MATSD, SP1, TZ2, Q15; Sample student answer, **p226**: M19, MATSD, SP1, TZ2, Q15; Sample student answer, **p238**: M19, MATME, SP2, TZ2, Q2; Sample student answer, **p245**: N12, MATHL, HP3, TZ0/SE, Q2(a); HL Practice questions, Paper 2, Q2, **p251**: M16, MATHL, HP1, TZ2, Q11.

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# Contents



Worked solutions to end-of-chapter practice questions and exam papers in this book can be found on your support website. Access the support website here:
www.oxfordsecondary.com/ib-prepared-support

# Introduction

This book provides full coverage of the new (first examination in 2021) IB diploma syllabus in Mathematics: Applications and Interpretations for both Higher Level and Standard Level. It complements the two Oxford University Press course companions *Mathematics: Applications and Interpretations, HL and SL*, ISBN 978-0-19-842704-9 and 978-0-19-842698-1. There is a sister IB Prepared exam guide for the new Mathematics: Analysis and Approaches course, which complements the two Oxford University Press course companions *Mathematics: Analysis and Approaches, HL and SL*, ISBN 978-0-19-842716-2 and 978-0-19-842710-0. This book offers support to students preparing for their examinations. It will help you revise the study material, learn the essential terms and concepts, strengthen your problem-solving skills and improve your approach to IB examinations. The book is packed with worked examples and exam tips that demonstrate best practices and warn against common errors. All topics are illustrated by annotated example student answers to questions informed by past examinations, which explain why marks may be scored or missed.

Practice questions and a complete set of IB-style examination papers provide further opportunities to check your knowledge and skills, boost your confidence and monitor the progress of your studies.

Full solutions to all questions and examination papers are given online at **www.oxfordsecondary.com/ib-prepared-support.**

As with any study guide, this book is not intended to replace your course materials, such as textbooks, past papers, specimen papers and markschemes, the IB Mathematics: Applications and Interpretations syllabus including the formula booklet, notation list, glossary of command terms and your own notes. To succeed in the examination, you will need to use a broad range of resources, many of which are available online. The authors hope that this book will navigate you through this critical part of your studies, making your preparation for the exam less stressful and more efficient.

## DP Mathematics: Applications and Interpretations assessment

All standard level (SL) and higher level (HL) students must complete the internal assessment and take two (SL) or three (HL) papers for their external assessment. Papers 1 and 2 are usually sat close to each other and Paper 3 a day or two later. The internal and external assessment marks are combined as shown in the table below to give your overall DP Mathematics grade, from 1 (lowest) to 7 (highest).

| Assessment | Description | Topics | SL | | HL | |
|---|---|---|---|---|---|---|
| | | | marks | weight | marks | weight |
| Internal | Mathematical Exploration | — | 20 | 20% | 20 | 20% |
| Paper 1 | Short-response questions. Technology required | 1, 2, 3, 4, 5 Including AHL topics for HL | 80 | 40% | 110 | 30% |
| Paper 2 | Extended-response questions. Technology required | | 80 | 40% | 110 | 30% |
| Paper 3 | Two Extended response, problem-solving questions. Technology required | All syllabus Especially AHL | – | – | 55 | 20% |

The final IB diploma score is calculated by combining grades for six subjects with up to three additional points from *Theory of knowledge* and *Extended essay* components.

## Overview of the book structure

The book is divided into sections that cover the internal assessment, standard level (SL) and additional higher level (AHL) material in five chapters with a set of questions at the end of each chapter, and a complete set of practice examination papers.

The **Internal assessment** section outlines the nature of the Mathematical Exploration that you will have to carry out and explains how to select a topic and present your Exploration in a suitable format to satisfy the marking criteria and achieve the highest grade.

The final section contains IB-style **practice examination Papers 1** and **2** for SL, and **Papers 1, 2** and **3** for HL, written exclusively for this book. These papers will give you an opportunity to test yourself before the actual exam and at the same time provide additional practice questions for every topic.

The answers and solutions to all practice questions and examination papers are given online at **www.oxfordsecondary.com/ib-prepared-support**.

## Command terms

*Command terms* are pre-defined words and phrases used in all IB Mathematics questions. Each command term specifies the type and depth of the response expected from you in a particular question.

| Command term | Definition |
|---|---|
| Calculate | Obtain a numerical answer, showing the relevant stages in the working. |
| Comment | Give a judgment based on a given statement or result of a calculation. |
| Compare | Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout. |
| Compare and contrast | Give an account of similarities and differences between two (or more) items or situations, referring to both (all) of them throughout. |
| Construct | Display information in a diagrammatic or logical form. |
| Contrast | Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout. |
| Deduce | Reach a conclusion from the information given. |
| Demonstrate | Make clear by reasoning or evidence, illustrating with examples or practical application. |
| Describe | Give a detailed account. |
| Determine | Obtain the only possible answer. |
| Differentiate | Obtain the derivative of a function. |
| Distinguish | Make clear the differences between two or more concepts or items. |
| Draw | Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve. |
| Estimate | Obtain an approximate value. |
| Explain | Give a detailed account, including reasons or causes. |
| Find | Obtain an answer showing relevant stages in the working. |
| Hence | Use the preceding work to obtain the required result. |
| Hence or otherwise | It is suggested that the preceding work is used, but other methods could also receive credit. |
| Identify | Provide an answer from a number of possibilities. |
| Integrate | Obtain the integral of a function. |
| Interpret | Use knowledge and understanding to recognize trends and draw conclusions from given information. |
| Investigate | Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions. |
| Justify | Give valid reasons or evidence to support an answer or conclusion. |
| Label | Add labels to a diagram. |
| List | Give a sequence of brief answers with no explanation. |
| Plot | Mark the position of points on a diagram. |
| Predict | Give an expected result. |
| Prove | Use a sequence of logical steps to obtain the required result in a formal way. |
| Show | Give the steps in a calculation or derivation. |
| Show that | Obtain the required result (possibly using information given) without the formality of proof. "Show that" questions do not generally require the use of a calculator. |
| Sketch | Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features. |
| Solve | Obtain the answer(s) using algebraic and/or numerical and/or graphical methods. |
| State | Give a specific name, value or other brief answer without explanation or calculation. |
| Suggest | Propose a solution, hypothesis or other possible answer. |
| Verify | Provide evidence that validates the result. |
| Write down | Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown. |

## Preparation and exam strategies

In addition to the above suggestions, there are some simple rules you should follow during your preparation study and the exam itself.

1. Get ready for study. Have enough sleep, eat well, drink plenty of water and reduce your stress by positive thinking and physical exercises. A good night's sleep is particularly important before the exam day, as it can significantly improve your score.
2. Organize your study environment. Find a comfortable place with adequate lighting, temperature and ventilation. Eliminate all possible distractions. Keep your papers and computer files organized. Bookmark useful online and offline material.
3. Plan your studies. Make a list of your tasks and arrange them by importance. Break up large tasks into smaller, easily manageable parts. Create an agenda for your studying time and make sure that you can complete each task before the deadline.
4. Use this book as your first point of reference. Work your way through the topics systematically and identify the gaps in your understanding and skills. Spend extra time on the topics where improvement is required. Check your textbook and online resources for more information.
5. Read actively. Focus on understanding rather than memorizing. Recite key points and definitions using your own words. Try to solve every worked example and practice problem before looking at the answer. Make notes for future reference.
6. Get ready for the exams.
   - Practise answering exam-style questions under a time constraint.
   - Learn how to use the Mathematics formula booklet (allowed in all exams) quickly and efficiently.
   - Solve as many questions from past papers as you can. Your school should give you a trial exam, but you can create another using the papers at the end of this book. Know what each paper will involve. All questions in all papers are compulsory: you should aim to complete all the questions. There will be 5 minutes reading time. Use this time to plan your approach and identify questions where you can use your GDC effectively.
     - SL Papers 1 and 2 are 1.5 hours long, HL papers 1 and 2 are 2 hours long, Paper 3 is 1 hour long.
     - Calculators are required for all papers. Make sure that your calculator is fully charged/has new batteries.
     - Paper 1 consists of short response questions to be answered on the exam paper underneath the question.
     - Paper 2 consists of extended response questions involving sustained reasoning to be answered in the answer booklets provided.
     - Paper 3 consists of two extended response problem-solving questions, also to be answered in the answer booklets provided.
     - For Papers 2 and 3, anything written on the exam paper will not be seen by the examiner.
     - The marks allocated to each part of a question are a good guide to how many minutes should be spent on that part.
   - Make sure that you have all the equipment required for the exam: pens, pencils, ruler, graphic display calculator (GDC), watch.
7. Optimize your exam approach. Read all questions carefully, paying extra attention to command terms. Keep your answers as short and clear as possible. Double-check all numerical values and units. Label axes in graphs and annotate diagrams. Use the exam tips from this book. Use the correct notation e.g. put arrows on vectors. If you introduce a variable explain what it stands for.
8. Do not panic. Take a positive attitude and concentrate on things you can improve. Set realistic goals and work systematically to achieve these goals. Be prepared to reflect on your performance and learn from your errors in order to improve your future results.

## Key features of the book

Each chapter typically covers one topic, and starts with **You should know** and **You should be able to** checklists. Chapters contain the following features:

### Note

Provide quick hints and explanations to help you better understand a concept.

Definitions to rules and concepts are given in a grey box like this one, and explained in the text.

### Assessment tip

This feature assists in answering particular questions, warns against common errors and shows how to maximize your score when answering particular questions.

### Example

**Examples** offer solutions to typical questions and demonstrate common problem-solving techniques.

**Sample student answers** show typical student responses to IB-style questions (many of which are taken from past examination papers). In each response, positive and negative feedback on the student's response is given in the green and red pull-out boxes. The correct answer will always be given. An example is shown below.

**SAMPLE STUDENT ANSWER**

Consider the distinct complex numbers $z = a + ib$, $w = c + id$, where $a, b, c, d \in \mathbb{R}$.

(a) Find the real part of $\frac{z+w}{z-w}$

(b) Find the value of the real part of $\frac{z+w}{z-w}$ when $|z| = |w|$

(a) $\cancel{\frac{a+c+i(b+d)}{a-c \times i(b \times d)}}$ $\quad a + bi - (c + di)$

$\frac{a+c}{a-c}$ $\quad \frac{a+c+i(b+d)}{a-c+i(b-d)}$

$\frac{a+c}{a-c+i(b-d)} + \frac{i(b+d)}{a-c+i(b-d)}$

(b) $\sqrt{a^2+b^2} = \sqrt{c^2+d^2}$ $\quad \frac{1}{a \times c} \times \frac{i(b+d)}{1 + \frac{i(b-d)}{a-c}} \times \frac{i}{i}$

$a^2 + b^2 = c^2 + d^2$

$\cancel{\frac{1}{A}} \cancel{\frac{C}{B}}$

$\frac{1}{a-c} \times \frac{-(b+d)}{i \times \frac{(b-d)}{a-c}}$

▼ Although $\frac{z+w}{z-w}$ is found correctly in terms of $a$ and $b$, the student does not recall the method of division of complex numbers. The correct approach is to multiply both the numerator and denominator by the conjugate of $(a-c) + \mathrm{i}(b-d)$.

▲ Despite difficulties in part (a), the student moves on to part (b) and is awarded a point for the knowledge and understanding he/she demonstrates about the modulus of a complex number.

**Links** provide a reference to relevant material, within another part of this book or the IB Mathematics: Applications and Interpretations syllabus, that relates to the text in question.

Questions not similar to past IB examinations will not have the exam paper icon.

**Practice questions** are given at the end of each chapter. The questions are headed SL (standard level) and AHL (additional higher level). Students following the higher level course should attempt all the questions.

The paper 1 questions are further divided into groups. Group 1 questions are set at a level of 1–3, group 2 at a level of 4–5 and group 3 at a level of 6–7.

Students following the SL course need to be aware that some sections are only in the higher level course. These sections will be indicated by (HL).

# 1 NUMBER AND ALGEBRA

## 1.1 REAL NUMBERS

### You should know:

- ✔ the meaning of significant figures and of decimal places
- ✔ that the exact value of a measurement lies in an interval centred on its rounded value
- ✔ the laws of exponents
- ✔ the definition of a logarithm
- ✔ the standard form of a number, also known as scientific notation, gives a convenient way to represent very large and very small numbers
- ✔ computer and GDC notation for standard form is not appropriate for you to write in any assessment.

### You should be able to:

- ✔ approximate a number to a given number of decimal places, significant figures or to the nearest unit
- ✔ apply the laws of exponents to simplify an expression
- ✔ find the upper and lower bounds of a rounded value
- ✔ apply the formula for percentage error
- ✔ apply the definition of logarithms to rearrange an exponential expression and solve equations with technology
- ✔ calculate with numbers written in standard form by hand and with technology
- ✔ write down numbers given by a calculator or GDC in appropriate notation.

### Approximation, errors and estimation

The processes of counting, measuring, estimating and approximating are fundamental in the application of mathematics. For example, Artem carries out an experiment in which he asks students to guess the number of sweets in a large jar. 131 students each make an estimate, ranging from 737 to 8378 sweets. Artem has read that finding the mean of all the guesses can provide a good **estimate** – a number as close as possible to the **exact** number. He calculated the mean as 1581.083 969 465 6 and was amazed to find that this was close to the exact value of 1627. Artem reflected that averaging the 131 numbers on a spreadsheet took far less time than counting all the sweets!

Artem went on to express his estimate by **rounding** it correct to two **significant figures** by applying these rules:

> Significant figures:
>
> 1,2,3,4,5,6,7,8 and 9 are always significant.
>
> 0 is significant if it is between two significant figures or at the end of the number after the decimal point.
>
> Rounding to a given number of decimal places or significant figures:
>
> 5,6,7,8,9 round up, whereas 0,1,2,3,4 round down

**Note**

Artem applied the idea of "The wisdom of the crowd", in which estimates of a quantity made by each member of a large group are collected and the average found. The theory is that the overestimates and underestimates balance out when the average is taken, leaving a good estimate of the size of the quantity.

To express his estimate to two significant figures, Artem examined the third significant figure in 1581.083 969 465 6, which is 8, and rounded up to write 1600. Similarly, for 1627 he rounded 2 down to 0 to get 1600. Hence the estimate and the exact values were the same when expressed correct to two significant figures.

**Example 1.1.1**

| | |
|---|---|
| (a) Write 0.041 792 m correct to: | (i) two decimal places<br>(ii) two significant figures |
| (b) Write 67.0812° correct to: | (i) the nearest degree<br>(ii) three significant figures |
| (c) Write 8307.59 km correct to: | (i) the nearest km<br>(ii) three significant figures |

**Solution**

| | |
|---|---|
| (a) (i) 0.041 792 m rounds to 0.04 m to 2 dp | Examine the third decimal place. Since it is 1, round the second decimal place to to 4. |
| (ii) 0.041 792 m rounds to 0.042 m to 2 sf | Examine the third significant figure. Since it is 7, round the second significant figure to 2. |
| (b) (i) 67.0812° rounds to 67° to the nearest degree | Examine the first decimal place. Since it is 0, round the units digit to 7. |
| (ii) 67.0812° rounds to 67.1° to 3 sf | Examine the fourth significant figure. Since it is 8, round the third significant figure to 1. |
| (c) (i) 8307.59 km rounds to 8308 km to the nearest km | Examine the first decimal place. Since it is 5, round the units digit to 8. |
| (ii) 8307.59 km rounds to 8310 km to 3 sf | Examine the fourth significant figure. Since it is 7, round the third significant figure to 1. |

Valentina repeats Artem's experiment since she finds it hard to imagine that his method could work. With a different jar containing 2210 sweets, Valentina asks 154 students, averages their guesses and finds an estimate of 2303.980 519 480 5. Artem says his estimate is better since it is closer to the exact value: the difference between his estimate and the exact value is 1627 – 1581.083 969 465 6 = 45.916 030 534 4 whereas for Valentina it is 2269.435 064 935 1 – 2210 = 59.435 064 935 1.

But in fact, Valentina's error is less when expressed as a percentage of the exact value. You find the percentage error $\varepsilon$ by applying the formula $\varepsilon = \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$, where $v_E$ is the exact value and $v_A$ is the approximate value of $v$.

Valentina:

$$\left|\frac{59.4350649351}{2210}\right| \times 100\% = 2.69\%$$

Artem:

$$\left|\frac{45.9160305344}{1627}\right| \times 100\% = 2.82\%$$

**Assessment tip**

Avoid premature rounding by using the unrounded answers on your GDC. By doing this you avoid accumulation of errors.

**Assessment tip**

Unless otherwise stated in the question, all final numerical answers must be given exactly or correct to three significant figures.

Whenever a measurement is approximated to the nearest unit $u$ as $v_A$, the exact value $v_E$ lies in an interval: $v_A - \frac{u}{2} \le v_E < v_A + \frac{u}{2}$.

**Example 1.1.2**

The base of the Cheops pyramid is a square. Each side measures 230 m to the nearest metre.

(a) Find the upper and lower bounds for the length of one side of the base.

(b) Hence find the interval in which the exact area $A$ m$^2$ of the base lies.

Regulations require that the dimensions of a football pitch for international fixtures must measure within 100–110 m long and 64–75 m wide.

(c) Calculate the smallest value of $\frac{A}{F}$ where $F$ is the area of the football pitch in m$^2$.

(d) Hence comment on the minimum number of football pitches that fit into the square base of the pyramid.

**Solution**

(a) The upper bound is $230 + 0.5 = 230.5$ m, and the lower bound is $230 - 0.5 = 229.5$ m

Add one half of the unit to the approximate value and then subtract one half of the unit to the approximate value.

(b) Hence if the area of the base is $A$ m$^2$, then $229.5^2 \text{ m}^2 \le A < 230.5^2 \text{ m}^2$ therefore $52\,670.25 \text{ m}^2 \le A < 53\,130.25 \text{ m}^2$

The upper bound of the area is the upper bound of the side length squared.

(c) The smallest value of $\frac{A}{F}$ is $\frac{52\,670.25}{110 \times 75} \approx 6.38$

Use the lower bound of the area of the base and the largest possible area of pitch.

(d) The area of the base of the Cheops pyramid is more than six times larger in area than an international football pitch.

**Note**

Don't confuse $2^3$ with $2 \times 3$

**Assessment tip**

The exponent is known as the power or index. The plural of index is indices. You can use these terms in your assessments, but you must understand the meaning of exponent.

## The laws of exponents and an introduction to logarithms

When you were younger you learned that $3 \times 2$ is a way to represent the repeated addition $2 + 2 + 2$. Similarly, the expression $2^3$ represents the repeated multiplication $2 \times 2 \times 2$. You call 2 the base and 3 the exponent. You say "2 raised to the exponent 3 is 8" or "2 to the power 3 is 8".

In SL, the exponent will always be an integer.

Exponential expressions are simplified according to the laws of exponents. These laws are not in the formula booklet so you must know them before your exam.

| | | |
|---|---|---|
| $x^m x^n = x^{m+n}$ | $(x^m)^n = x^{mn}$ | $x^0 = 1, \quad x^1 = x, \quad 0^m = 0$ |
| $\frac{x^m}{x^n} = x^{m-n}$ | $(xy)^m = x^m y^m$ | $x^{-m} = \frac{1}{x^m}$ |

### Common powers

The base can be any positive real number. However, the following powers are commonly encountered, and some of the commonly used decimal equivalents are shown in this table:

| $n$ | ... | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $2^n$ | ... | $\frac{1}{16}$ | $\frac{1}{8}=0.125$ | $\frac{1}{4}=0.25$ | $\frac{1}{2}=0.5$ | 1 | 2 | 4 | 8 | 16 | ... |
| $3^n$ | ... | $\frac{1}{81}$ | $\frac{1}{27}$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 | 27 | 81 | ... |
| $5^n$ | ... | $\frac{1}{625}$ | $\frac{1}{125}$ | $\frac{1}{25}=0.04$ | $\frac{1}{5}=0.2$ | 1 | 5 | 25 | 125 | 625 | ... |
| $10^n$ | ... | $\frac{1}{10\,000}=0.0001$ | $\frac{1}{1000}=0.001$ | $\frac{1}{100}=0.01$ | $\frac{1}{10}=0.1$ | 1 | 10 | 100 | 1000 | 10 000 | ... |

**Example 1.1.3**

Use the laws of exponents to write each expression as the power of one integer and hence find its value.

(a) $3 \times 3^3$

(b) $\frac{5^{12}}{5^9}$

(c) $\frac{2^2}{5^{-2}}$

**Solution**

(a) $3 \times 3^3 = 3^1 \times 3^3 = 3^4 = 81$ Apply the law $x^m x^n = x^{m+n}$

(b) $\frac{5^{12}}{5^9} = 5^{12-9} = 5^3 = 125$ Apply the law $\frac{x^m}{x^n} = x^{m-n}$

(c) $\frac{2^2}{5^{-2}} = \frac{2^2 \times 5^0}{5^{-2}} = 2^2 \times 5^{0-(-2)}$ Apply the laws $\frac{x^m}{x^n} = x^{m-n}$

$= 2^2 \times 5^2 = (2 \times 5)^2 = 10^2 = 100$ and $(xy)^m = x^m y^m$

**Assessment tip**

In calculations, it is best to use exact values, for example a fraction or a terminating decimal.

You learned to solve equations like $x + 2 = 64$, $2x = 64$ and $x^2 = 64$ in previous years. But how do you solve an equation like $2^x = 64$?

The solution to this equation is written as $x = \log_2 64$ and you read it as "$x$ equals log base 2 of 64". This just means you ask yourself, "What exponent do I need to raise 2 by in order to get 64?"

The answer is 6. The expressions $6 = \log_2 64$ and $2^6 = 64$ are just rearrangements of each other, so are equivalent, just as $2 \times 32 = 64$ and $\frac{64}{2} = 32$ are equivalent.

The general definition of a logarithm is

$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0$, $b > 0$ and $a \neq 1$

You say, "$x$ equals log base $a$ of $b$"

You can calculate logarithms on your GDC for any positive base.

Two of the most commonly used bases for logarithms are 10 and e. The number e ≈ 2.718 281 828 459... is an irrational number with many applications in the sciences. This is why $\log_e x$ is referred to as the "natural log" of $x$ and this is written $\ln x$ in the examination and on your GDC.

### Example 1.1.4

Solve the following equations:

(a) $10^x = 13.7$ (b) $10^{3y} = 45$ (c) $112.8e^{0.25z} = 745.03$ (d) $3.1 \log_{10} v = -0.651$

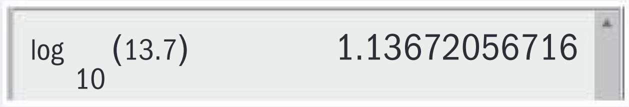
**Solution**

(a) $10^x = 13.7$

$\Rightarrow x = \log_{10} 13.7$

$x = 1.14$

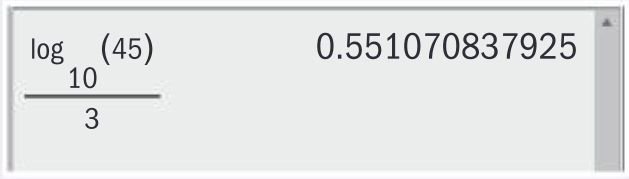
Apply the definition of a logarithm, use your GDC and give the answer to three significant figures.




(b) $10^{3y} = 45 \Rightarrow 3y = \log_{10} 45$

$y = \dfrac{\log_{10} 45}{3} = 0.551$

Use your GDC only at the final step in order to be efficient and accurate with your working.



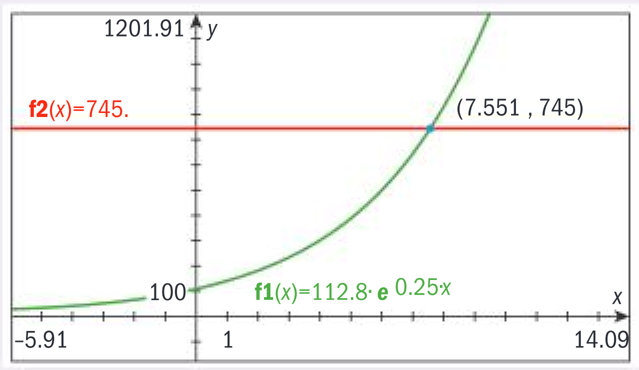

(c) $112.8e^{0.25z} = 745.03$

$\Rightarrow e^{0.25z} = \dfrac{745.03}{112.8}$

$\Rightarrow 0.25z = \ln\left(\dfrac{745.03}{112.8}\right)$

$\Rightarrow z = 4\ln\left(\dfrac{745.03}{112.8}\right) = 7.55$

Note that exponential and logarithmic equations can also be solved with graphs:

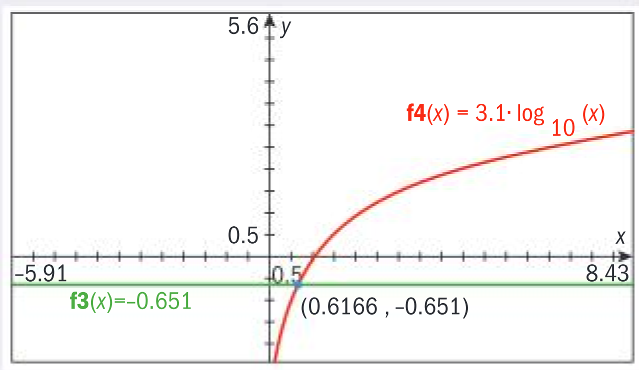



(d) $3.1 \log_{10} v = -0.651$

$\Rightarrow \log_{10} v = \dfrac{-0.651}{3.1}$

$\Rightarrow v = 10^{\frac{-0.651}{3.1}} = 0.617$

Rearrange the equation then apply the definition of a logarithm, or solve with a graph.




## Standard form

A number expressed in the form $a \times 10^k$, where $1 \le a < 10$ and $k$ is an integer is written in **standard form**, also known as **scientific notation**. Positive values of $k$ give numbers which are at least 10, whereas negative values of $k$ give positive numbers less than 1. Most often in applications, the numbers expressed in standard form are either very large or very small. The value of $a$ tells you how many $10^k$ units your number has.

For example, Carolina wants to compare the distance from the Earth to the Sun with the number $2^{44}$ to find out which is larger. She finds in an astronomy book that the distance from the Earth to the Sun is 14 959 800 000 000 cm. She writes this number in standard form by counting the number of decimal places (13) from the right until the second significant digit (4), and writes $1.49598 \times 10^{13}$ cm.

Carolina types $2^{44}$ into her GDC which gives her 1.7592186044416E13. She correctly writes this as $1.7592186044416 \times 10^{13}$ by replacing "E13" with "$\times 10^{13}$". Carolina concludes that $2^{44}$ is the larger number since the power of 10 is the same for both numbers.

### Example 1.1.5

(a) Neptune is located $4.3514 \times 10^9$ km from Earth. Write this number as a decimal.

(b) Nicole reads the number 8.03E–5 from her GDC. Write down this number in standard form then express it as a decimal number.

(c) One grain of pollen weighs 0.0000025 mg. Write this weight in standard form.

(d) (i) One Astronomical Unit (AU) is 149598000 km. The Voyager 1 spacecraft reached a distance of 141 AU from the Earth by the year 1998. Write this distance in kilometres in standard form.

(ii) Hence justify the estimate that the Voyager 1 spacecraft reached approximately 5 times as far from the Earth as Neptune by 1998.

**Solution**

(a) To find $4.3514 \times 10^9$ you move the decimal point 9 places to the right and write 4351400000

(b) 8.03E–5 means $8.03 \times 10^{-5}$. To express this as a decimal, you move the decimal point 5 places to the left and write 0.0000803. Writing the answer as $80.3 \times 10^{-6}$ would not be awarded any marks because it is not expressed in standard form.

(c) 0.0000025 mg can be written as $2.5 \times 10^{-6}$ mg

(d) (i) $141 \times 149598000 = 21093318000$. So the distance be written as $2.1093318 \times 10^{10}$ km.

(ii) $\frac{2.1093318 \times 10^{10}}{4.3514 \times 10^9} \approx \frac{2 \times 10^{10}}{4 \times 10^9} = \frac{20 \times 10^9}{4 \times 10^9} = 5$. Hence the distance reached by Voyager 1 from Earth is approximately 5 times as far as the distance from Earth to Neptune.

### Example 1.1.6

You are given the following information:

| | |
|---|---|
| Number of cells in the human body = 37.2 trillion | Average length of a cell is 10000 nanometres |
| One trillion = 1000000000000 | One nanometre (nm) = $10^{-9}$ metre |

(a) Write down the number of cells in the human body and the average length of a cell in standard form.

(b) Hence calculate the length if all the cells are placed in a straight line, expressing your answer in km.

**Solution**

(a) 37.2 trillion $= 37.2 \times 1000000000000 = 37.2 \times 10^{12} = 3.72 \times 10^{13}$

10000 nanometres $= 10000 \times 10^{-9}$ m $= (10^4 \times 10^{-9})$ m $= 1 \times 10^{-5}$ m

(b) The length of all the cells is therefore $(3.72 \times 10^{13}) \times (1 \times 10^{-5}$ m$)$ $= 3.72 \times 10^8$ m

Since 1 km = 1000 m, the distance is $3.72 \times 10^5$ km

**Assessment tip**

$37.2 \times 10^{12}$ is not written in standard form because $37.2 > 10$. You must write this number as $3.72 \times 10^{13}$

SAMPLE STUDENT ANSWER

Let $p = \frac{\cos x + \sin y}{\sqrt{w^2 - z}}$, where $x = 36°$, $y = 18°$, $w = 29$ and $z = 21.8$

(a) Calculate the value of $p$. Write down your full calculator display.

(b) Write your answer to part (a):

(i) correct to two decimal places

(ii) correct to three significant figures.

(c) Write your answer to part (b) (ii) in the form $a \times 10^k$, where $1 \le a < 10, k \in \mathbb{Z}$.

$\frac{\cos 36° + \sin 18°}{\sqrt{29^2 - 21.8}} = \frac{\cos 36° + \sin 18°}{28.62167011} = 0.0390625$

(a) 0.0390625

(b) (i) 0.04

(ii) 0.0391

(c) $3.91 \times 10^{-3}$

▼ Introducing a separate calculation for the denominator could reduce accuracy and it is more efficient to just enter the entire expression in one calculation on the GDC.

▲ Student correctly writes down the full calculator display, then correctly gives the number to the levels of accuracy required.

▼ Counting the decimal places is a common source of error. Typing this expression into the GDC and comparing with 0.0391 would have identified the error.

# 1.2 SEQUENCES AND SERIES

## You should know:

- ✔ the meaning of a sequence
- ✔ the meaning of a series
- ✔ the notation for sequences and sigma notation
- ✔ what defines an arithmetic sequence and a geometric sequence
- ✔ that there are two equivalent formulae for the sum of an arithmetic series
- ✔ that there are two equivalent formulae for the sum of a geometric series
- ✔ the meaning of the terms depreciation, inflation, annuity and amortization.

## You should be able to:

- ✔ identify if a sequence is arithmetic or geometric
- ✔ write and understand sigma notation
- ✔ apply formulae to calculate and solve problems in context, including financial applications
- ✔ use approximation when an arithmetic sequence is not an exact model
- ✔ apply technology such as a graph, spreadsheet or finance package to solve problems.

It's often said that mathematics is the science of patterns. Sequences and series enable you to represent number patterns seen in real-life contexts and apply formulae to solve problems.

A **sequence** is an ordered list of numbers, each of which is called a **term**. A commonly used notation for a sequence is $u_1, u_2, u_3, \dots, u_n$ meaning the first, second, third terms and so on until the $n$th term. Each term in the sequence is calculated with a formula.

A **series** is formed by adding the terms of a sequence; the series $S_n = u_1 + u_2 + u_3 + \ldots + u_n$ is the sum of the first $n$ terms of a sequence.

**Sigma notation** gives you a concise way to write this:

$$S_n = \sum_{i=1}^{n} u_i$$

This can be read as "The sum of all the terms $u_i$, where $i$ takes the values 1, 2, 3, and so on, until $n$. Similarly, $\sum_{i=k}^{n} u_i = u_k + u_{k+1} + u_{k+2} + \ldots + u_n$

### Example 1.2.1

The $n$th term of a sequence is found with the formula $u_n = 23 - 2n$

(a) Calculate the first five terms of the sequence.

(b) Hence find $S_5 = \sum_{i=1}^{5} u_i$

(c) Find $\sum_{i=73}^{75} u_i$

**Solution**

(a) $u_1 = 23 - 2(1)$, $u_2 = 23 - 2(2)$, $u_3 = 23 - 2(3)$, $u_4 = 23 - 2(4)$, $u_5 = 23 - 2(5)$.
So the first five terms are 21, 19, 17, 15 and 13.

Replace $n$ with the values 1, 2, 3, 4 and 5 in the formula.

Find the answers and list them.

(b) $\sum_{i=1}^{5} u_i = u_1 + u_2 + u_3 + u_4 + u_5$
$= 21 + 19 + 17 + 15 + 13 = 85$

The command term "Hence" is an explicit instruction to apply the previous result.

(c) $\sum_{i=73}^{75} u_i = u_{73} + u_{74} + u_{75} =$
$(23 - 2(73)) + (23 - 2(74)) + (23 - 2(75)) = -375$

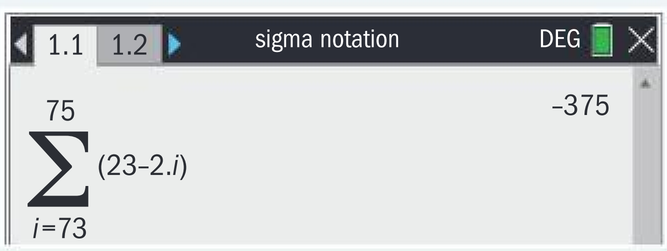
Replace $n$ with the values 73, 74 and 75 in the formula for the sequence and add these terms to find the sum.

(Finding $S_{75} - S_{72}$ is another way to find the sum required.)

$$S_{75} - S_{72} = (u_{75} + u_{74} + u_{73} + u_{72} + u_{71} + \ldots + u_2 + u_1) - (u_{72} + u_{71} + \ldots + u_2 + u_1)$$
$$= u_{75} + u_{74} + u_{73}$$

### Assessment tip

The previous example can be calculated on your GDC, however you must be able to calculate expressions given in sigma notation by hand too.



An **arithmetic sequence** is one in which the difference $d$ between any two consecutive terms is constant. This is called the **common difference**. For example, the common difference of the sequence in Example 1.2.1 is –2 because $u_{n+1} - u_n = -2$ is always true for this sequence. Informally, you can say, "Each term of the sequence is 2 less than the previous term". If the common difference is positive, then the terms of the sequence will increase.

The general formula for the $n$th term of an arithmetic sequence with first term $u_1$ and common difference $d$ can be found by investigating the terms of this sequence:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| $u_n$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | … |
| $u_n =$ | $u_1$ | $u_1 + d$ | $u_1 + d + d$ $= u_1 + 2d$ | $u_1 + 2d + d$ $= u_1 + 3d$ | $u_1 + 3d + d$ $= u_1 + 4d$ | $u_1 + 4d + d$ $= u_1 + 5d$ | … |

These formulae are in section 1.2 of the formula book.

The pattern gives you the general formula $u_n = u_1 + (n - 1)d$.

The formula for the sum of $n$ terms of this sequence is $S_n = \frac{n}{2}(u_1 + u_n)$.

By substituting the formula for $u_n$ you can also write the equivalent formula $S_n = \frac{n}{2}(2u_1 + (n - 1)d)$.

### Example 1.2.2

Adrian's swimming coach is planning his training schedule over ten days. He tells Adrian to swim 50 lengths of the swimming pool on day one, and then increase the number of lengths by 5 each day.

(a) Find how many lengths Adrian will swim on day ten.

(b) Hence or otherwise find how many lengths Adrian will swim in total over the ten days.

**Solution**

(a) The number of lengths Adrian swims on day ten is $u_{10} = 50 + (10 - 1)5 = 95$ lengths

Represent the sequence with first term 50 and common difference 5 and apply the formula $u_n = u_1 + (n - 1)d$.

(b) The total number of lengths can be found by $S_{10} = \frac{10}{2}(50 + 95) = 725$ lengths

Since the question asks for the total number of lengths over the ten days, apply the formula for the sum of the arithmetic series. The other formula could be applied instead: $S_{10} = \frac{10}{2}(2(50) + (10 - 1)5) = 725$. But the form used in the solution is more direct.

**Assessment tip**

A common mistake for Example 1.2.2 part (a) is to answer $50 + (10)5 = 100$.

This is incorrect because on the first day Adrian swims $50 + (0)5$ lengths, hence $50 + (10)5$ would be the 11th term of the sequence.

The formulae for arithmetic sequences and series can be used to solve problems by hand and with technology.

### Example 1.2.3

Adrian plans a training schedule to raise money for charity through online sponsorships and wants to know some facts to publish online. Use the information in Example 1.2.2 to find:

(a) the first day when Adrian swims more than 250 lengths

(b) the number of days needed for Adrian to have swum more than 5000 lengths in total.

**Solution**

(a) $50 + 5(n - 1) > 250 \Rightarrow$
$5(n - 1) > 200 \Rightarrow (n - 1) > 40,$
Hence $n > 41$

Use $u_n = u_1 + (n - 1)\,d$ to write the inequality needed.

So, on day 42 of training, Adrian will first swim more than 250 lengths.

It is a good idea to check your answer: On day 41, Adrian swims 250 lengths. On day 42 he swims 255 lengths, hence 42 is correct.

(b) $\frac{n}{2}(2(50) + 5(n - 1)) > 5000 \Rightarrow$
$n > 36.21925...$

You must apply this formula to write the inequality because $n$ is not known.

Hence after 37 days training Adrian will have swum more than 5000 lengths.

Use your GDC to create a graph or a table to solve the inequality. Notice that $S_{36} = 4950$ and $S_{37} = 5180$. The final answer must be an integer and, in this case, the nearest integer does not satisfy the inequality.

A **geometric sequence** is one in which the ratio $r$ between any two consecutive terms is constant. You call this quantity the **common ratio**. For example, the common ratio of the sequence 4, 6, 9, 13.5, … is 1.5 because $\frac{u_{n+1}}{u_n} = 1.5$. Informally, you can say, “Each term of the sequence is 1.5 times larger than the previous term”. If $r > 1$, then the size of the terms of the sequence will increase.

The general formula for the $n$th term of a geometric sequence with first term $u_1$ and common ratio $r$ can be found by investigating the terms of this sequence:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| $u_n$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | … |
| $u_n =$ | $u_1$ | $u_1(r)$ | $u_1(r)(r)$ | $u_1(r)(r)(r) = u_1r^3$ | $u_1r^4$ | $u_1r^5$ | … |

The pattern gives you the general formula $u_n = u_1r^{n-1}$.

### Example 1.2.4

For each of these geometric sequences, find:

(i) the first term $u_1$ (ii) the common ratio $r$ (iii) $u_7$

(a) 5, 10, 20, 40, … (b) 24, 12, 6, 3, … (c) 1.5, –3, 6, –12, …

**Solution**

(a)

(i) $u_1 = 5$

(ii) $r = \frac{10}{5} = 2$

(iii) $u_7 = 5(2)^6 = 320$

The common ratio is equal to $r = \frac{u_{n+1}}{u_n}$, so you have a choice of how to find $r$:

$\frac{20}{10}$, $\frac{40}{20}$ or $\frac{10}{5}$ are equivalent ways to find $r = 2$.

(b)

(i) $u_1 = 24$

(ii) $r = \frac{12}{24} = \frac{1}{2}$

(iii) $u_7 = 24\left(\frac{1}{2}\right)^6 = 0.375$

It is important to remember that the 7th term means the common ratio is raised to the 6th power, since $u_7 = u_1 r^{7-1} = u_1 r^6$. This is an application of the formula $u_n = u_1 r^{n-1}$.

(c)

(i) $u_1 = 1.5$

(ii) $\frac{-3}{1.5} = -2$

(iii) $u_7 = 1.5(-2)^6 = 96$

The formula for the sum of $n$ terms of this sequence is $S_n = \frac{u_1(r^n - 1)}{r - 1}$. If $r$ is greater than 1, you may find this form more convenient to use.

By multiplying the numerator and denominator of this formula by $-1$, you can also write the equivalent form $S_n = \frac{u_1(1 - r^n)}{1 - r}$, which is more convenient to use if $0 < r < 1$.

You can solve problems with sequences by first interpreting a context and then applying the appropriate formulae.

### Example 1.2.5

A space probe in a protective pod is released from a spacecraft and falls to the surface of a planet 20 m below. On impact with the surface of the planet, the pod bounces upwards. Engineers estimate the pod will bounce to 65% of its previous height after each bounce.

(a) Find, to the nearest metre, the total distance travelled by the pod at the moment of the fourth impact.

(b) The physical condition of the probe is measured in safety units. Each bounce costs the probe 2 safety units and the probe begins with 112 safety units. If the number of safety units falls below 84, the probe will fail.

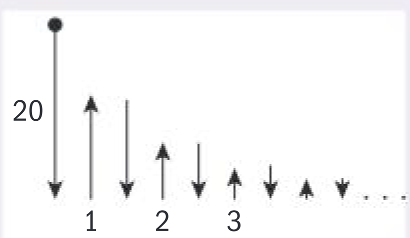
If the probe bounces 13 times, predict if the probe will fail.

**Assessment tip**

There are other equivalent ways to solve this problem.

**Solution**

(a)




The height that the pod reaches after 1, 2, and 3 bounces is $20 \times 0.65$, $20 \times 0.65^2$, $20 \times 0.65^3$

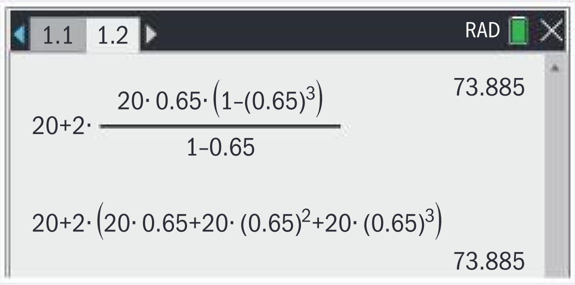
Sketching a diagram is a good problem-solving strategy.

Therefore the answer required is

$20 + 2(20 \times 0.65) + 2(20 \times 0.65^2) + 2(20 \times 0.65^3)$

$= 20 + 2(20 \times 0.65 + 20 \times 0.65^2 + 20 \times 0.65^3)$

$= 20 + 2\left(\dfrac{20 \times 0.65(1 - 0.65^3)}{1 - 0.65}\right)$




The probe travels 74 m to the nearest metre.

Use the sketch to interpret the context and identify the sequence is geometric by representing "65% of" as a common ratio of 0.65

Apply the formula for the sum of three terms of a geometric sequence, or just add up the three terms. Write all your working out clearly.

(b) The sequence of safety units after each bounce is 110, 108, 106, The number of safety units $u_n$ remaining after 13 bounces is

$u_{13} = 110 + (13 - 1)(-2) = 86$

Since $86 > 84$, the probe is not predicted to fail.

Since the number of safety units decreases by the same amount each bounce, the number of safety units remaining is an arithmetic sequence.

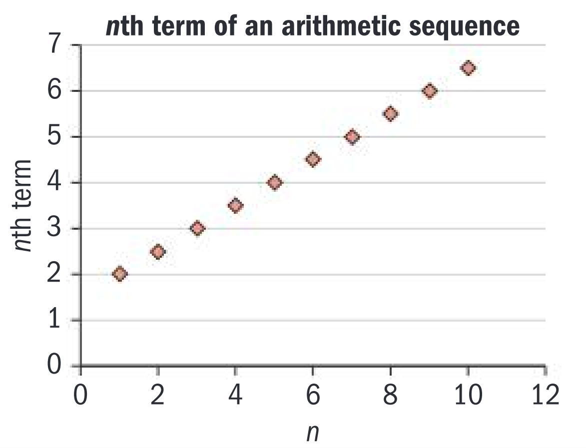
**Assessment tip**

The GDC screen should correspond to what you have written on your exam paper. This helps you check your working.

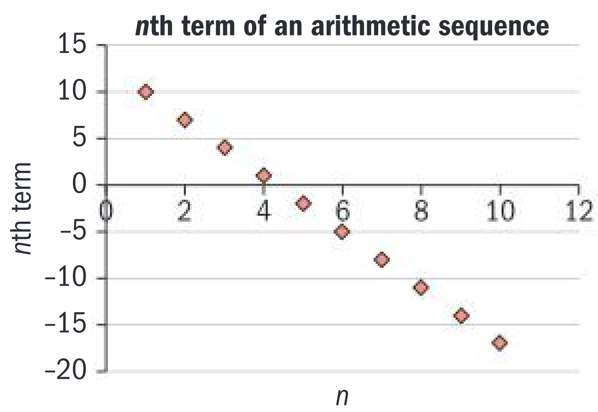
Arithmetic sequences can be modelled by linear functions with domain $\mathbb{Z}^+$.

Arithmetic and geometric sequences are linked to linear and exponential models respectively in section 2.3 of this book.

First term 2, common difference 0.5

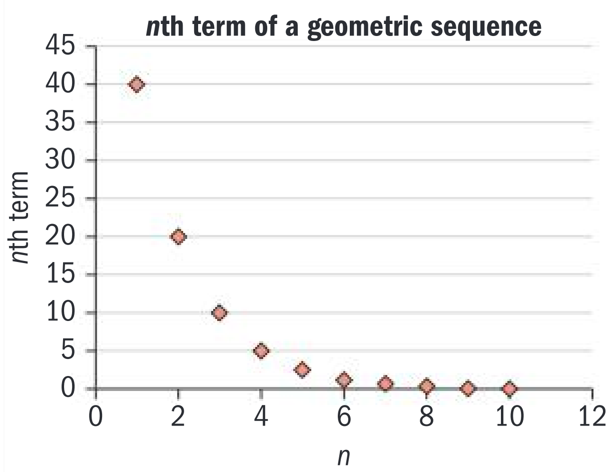



First term 10, common difference −3

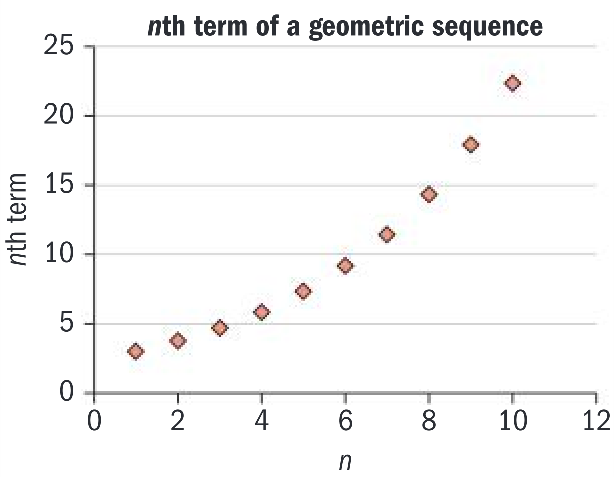



Geometric sequences with positive common ratio can be modelled by exponential functions with domain $\mathbb{Z}^+$.

First term 40, common ratio 0.5




First term 3, common ratio 1.25



SAMPLE STUDENT ANSWER

A comet orbits the Sun and is seen from Earth every 37 years. The comet was first seen from Earth in the year 1064.

(a) Find the year in which the comet was seen from Earth for the fifth time.

(b) Determine how many times the comet has been seen from Earth up to the year 2014.

▼ The student uses a list as a problem-solving method for (a), which gives the correct answer. However, applying the formula for the fifth term of the sequence would be more efficient.

▼ The student successfully extends the list and scores maximum marks for (b), after writing down an incorrect application of the formula for the sum of an arithmetic series. However it is a very time-consuming and error-prone method: applying the formula for the $n$th term is more efficient. Use of lists is not recommended, and full marks may not be awarded for method.

| | | | | | |
|---|---|---|---|---|---|
| (a) 1 | 1064 | 10 | 1397 | 19 | 1730 |
| 2 | 1101 | 11 | 1434 | 20 | 1767 |
| 3 | 1138 | 12 | 1431 | 21 | 1804 |
| 4 | 1175 | 13 | 1508 | 22 | 1841 |
| 5 | 1212 | 14 | 1545 | 23 | 1873 |
| 6 | 1249 | 15 | 1582 | 24 | 1905 |
| 7 | 1286 | 16 | 1619 | 25 | 1952 |
| 8 | 1323 | 17 | 1656 | 26 | 1989 |
| 9 | 1360 | 18 | 1693 | 27 | 2026 |

(b) $2014 = \frac{x}{2}(2(1064)(x-1)37)$

(a) 1212

(b) 26 times

## Financial applications of arithmetic and geometric sequences

The formulae for arithmetic sequences and geometric sequences have many applications, including in financial mathematics, using the following terms and formulae:

**Compound interest:** the amount added at the end of each interest (compounding) period is a percentage of the previous value of the investment.

| Term | Symbol | Meaning of term |
|---|---|---|
| Future value | $FV$ | The value of the investment or loan at a specified time in the future. |
| Present value | $PV$ | The value of the investment or loan now. |
| Number of compounding periods per year | $k$ | The interest can be added on (compounded) yearly, half-yearly, quarterly or monthly, meaning $k$ is 1, 2, 4 or 12 respectively. |
| Nominal annual rate of interest | $r$ | The percentage interest per year that is divided equally over the $k$ compounding periods per year.<br>For example, a nominal annual interest rate of 12% compounded quarterly means that 3% interest is added at the end of each quarter. |
| Number of years | $n$ | The total number of compounding periods per year is therefore $kn$.<br>Note that the first value of $n$ is zero. |

The formula for the future value under compound interest is $FV = PV\left(1+\frac{r}{k\times 100}\right)^{kn}$, an example of a geometric sequence with first term $PV$ and common ratio $1+\frac{r}{k\times 100}$

| **Simple interest:** the amount added at the end of each interest period is constant | | |
|---|---|---|
| **Term** | **Symbol** | **Meaning of term** |
| Capital | $C$ | The amount invested (or borrowed) at the beginning. |
| Interest rate | $r$ | The percentage of $C$ added to $C$ at the end of each interest period. |
| Number of interest periods | $n$ | The total duration of the investment or loan is $n$ multiplied by the length of one interest period.<br>Note that the first value of $n$ is zero. |
| Interest | $I$ | The amount added to the investment or loan. |

The formula for the future value $A$ under simple interest is $A = C + \frac{Crn}{100}$, an example of an arithmetic sequence with first term $C$ and common difference $\frac{Cr}{100}$

### Example 1.2.6

Steven invests \$4,350 in a savings scheme paying simple interest at a rate of 1.7% per annum, and Chimdi invests the same amount in a different scheme with nominal annual interest rate 1.15% compounded quarterly. They both invest their money for 7 years. Steven predicts that his investment will be worth more. Determine if Steven is correct.

**Solution**

Steven's investment is worth
$4350 + 4350 \times 0.017 \times 7 = \$4867.65$

You leave your answer correct to two decimal places in financial questions unless otherwise specified.

Chimdi's investment is worth
$4350\left(1+\frac{1.15}{4\times 100}\right)^{7\times 4} = \$4714.11$

You apply the formula from the formula booklet or use your GDC finance app.

Steven is correct.

You can use the laws of exponents, logarithms or the finance app on your GDC to solve problems.

### Example 1.2.7

Nicole invests €16,500 in a savings scheme paying compound interest at a rate of 2.1% per annum compounded monthly.

Calculate how long it will take for Nicole's money to double in value.

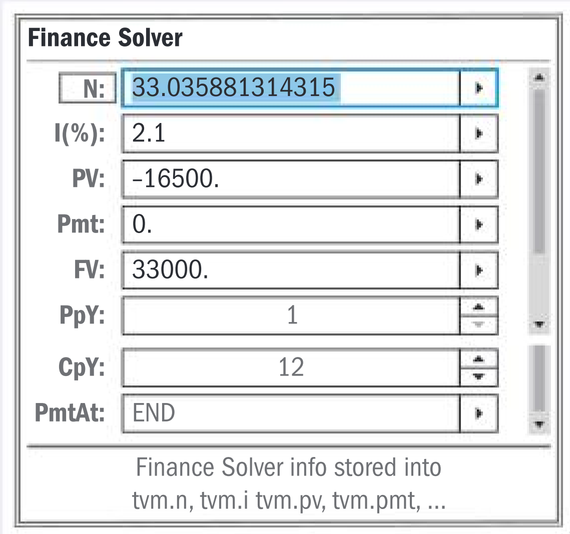
>> **Assessment tip**

Get to know your finance app well in advance of the exam but also make sure you know how to use algebra to solve financial problems.

>> **Assessment tip**

Write down the formula with the information substituted in, or the data you enter in your finance app. This helps to clarify your thoughts and shows your method to the examiner.

**Solution**




This question can be solved on your finance app.

- The amount €16,500 is entered as –16 500 since the money is being *given* to the investment scheme.
- The number of payments per year is PpY, which is 1
- The number of compounding periods per year is CpY, which is 12

For your working, write down: N = ?, I = 2.1, PV = –16 500, FV = 33 000, C/Y = 12

After 33 years and one month, Nicole's money will have doubled in value.

This is the correct interpretation, since 33 years is not sufficient for the investment to double in value when the interest is compounded monthly.

Depreciation is a reduction in the value of an asset over time.

Inflation is the fall in the purchasing value of money. For example, if the inflation rate is 1.7% per year, prices increase by 1.7% by the end of the year.

**Example 1.2.8**

Alexander buys a new car for $65,000. He learns that his car will depreciate by 20% per year. Cornelia buys a vintage car for $5500, which will increase in value by 17% each year and claims that her car will one day have the same value as Alexander's car.

(a) Show that the value $A(x)$ of Alexander's car after $x$ years can be written as

$$A(x) = \$65{,}000(0.8)^x$$

(b) Hence or otherwise find the time in years by which the two cars will have the same value and find this value to the nearest $100.

**Solution**

(a) $A(x)$ is the future value of Alexander's car.

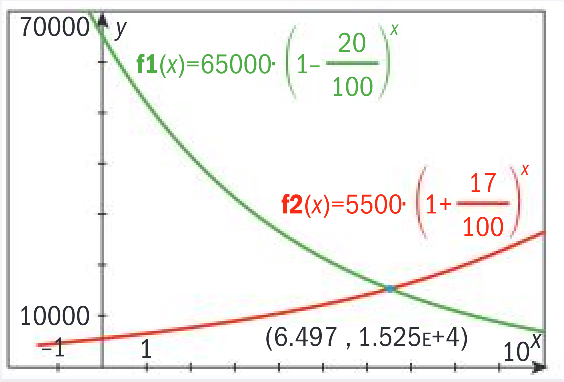
So $A(x) = \$65{,}000\left(1 - \frac{20}{100}\right)^x$

which simplifies to $A(x) = \$65{,}000(0.8)^x$

Replace the information given in the formula for the future value, $FV = PV\left(1 + \frac{r}{k \times 100}\right)^{kn}$

The value is depreciating hence $r = -20\%$

(b) Similarly, the value of Cornelia's car is
$B(x) = \$5{,}500(1.17)^x$




Graph both functions and find their intersection point.

The graphs intersect at (6.497,15 250).

Hence after 6.50 years, both cars have value $15,300 to the nearest $100.

The years are given correct to three significant figures and the value to the level of accuracy required.

**SAMPLE STUDENT ANSWER**

Juan buys a bicycle in a sale. He gets a discount of 30% off the original price and pays $560.

(a) Calculate the original price of the bicycle.

To buy the bicycle, Juan takes a loan of $560 for 6 months at a nominal annual interest rate of 75%, **compounded monthly**. Juan believes that the total amount he will pay will be less than the original price of the bicycle.

(b) Calculate the difference between the original price of the bicycle and the total amount Juan will pay.

FV = 96
PV = 560
N = 6
K = 12
% = 75

P = 560

$\frac{560}{100} + 30x$

$= 35.6$

$560 + 35.6 = 595.6$

$x = 560 \times \left(1 + \frac{75}{100 \times 12}\right)^{12 \times 6}$

$x = 44042.46471$

$= \frac{-595.6}{43446.86471}$

▲ The student correctly identifies the compound interest formula, gaining a mark for trying to substitute the values.

Compound interest is linked to exponential models in section 2.3 of this book.

▼ The student fails to apply percentages, which is prior learning. The correct approach is original price $= \frac{560}{0.7}$.

▼ The student does not apply the formula correctly, reading 6 months as 6 years. Crucial information such as units should be noticed in the 5 minute reading time. The correct approach is to use $12 \times 0.5$ as the exponent, giving a final answer of $805.68

▼ The student should notice that the answer is unreasonable in the context of the question.

## Amortization and annuities

**Annuity:** A sum of money $P$ (an investment) is paid to a bank by an investor.

Interest is regularly added to $P$ and payments are regularly made from the bank to the investor over an agreed period of time until the investment is used up. Example: retirement planning.

> **Assessment tip**
>
> In examinations the payments will be made at the end of the period.

**Amortization:** A sum of money $L$ (a loan) is paid to a borrower by a bank.

Interest is regularly added to $L$ and payments are regularly made to the bank over an agreed period of time until the loan is paid off. Example: buying a house.

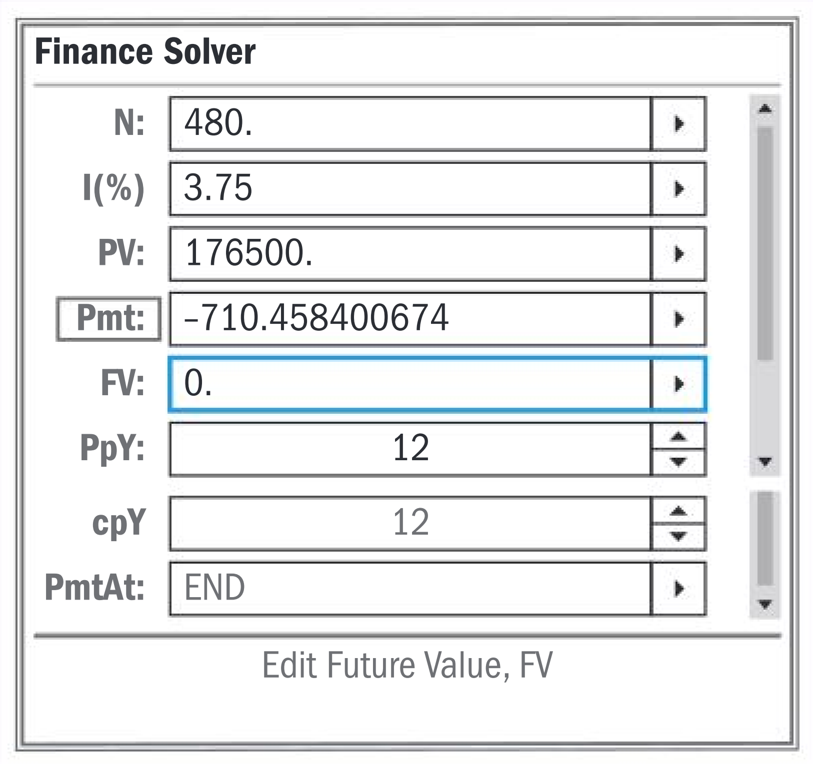
### Example 1.2.9

Mats takes out a 20 year loan of £176,500 to buy an apartment. The nominal annual interest rate is 3.75% compounded monthly. Mats make his repayments monthly.

(a) Find the repayment that Mats makes each month.

(b) Hence find the total interest Mats pays over the 20 years of his loan.

Solution




N is the number of monthly payments that Mats makes over the 20 years, so is equal to $20 \times 12 = 240$

P/Y and C/Y are both 12, since Mats makes repayments monthly and the interest is calculated monthly.

PV = 176 500 has a positive sign since this sum is added to Mats's account at the start of the loan.

Pmt is negative because this is the payment Mats makes, so this amount is subtracted from his account each month.

FV = 0 because when the loan is paid off, the value of the loan is zero.

(a) The repayment each month is £710.46

(b) The total paid by Mats is 480 × £710.458.....
= £341,020.03
Hence the total interest Mats pays is
£341,020.03 – £176,500 = £164,520.03

### Example 1.2.10

Yimo wishes to invest in a savings scheme for her retirement. Her financial advisor finds two options for her to choose from, each are for a period of 25 years.

**Option A:** Deposit equal amounts of €350 every month. The nominal annual interest rate is 4% compounded monthly.

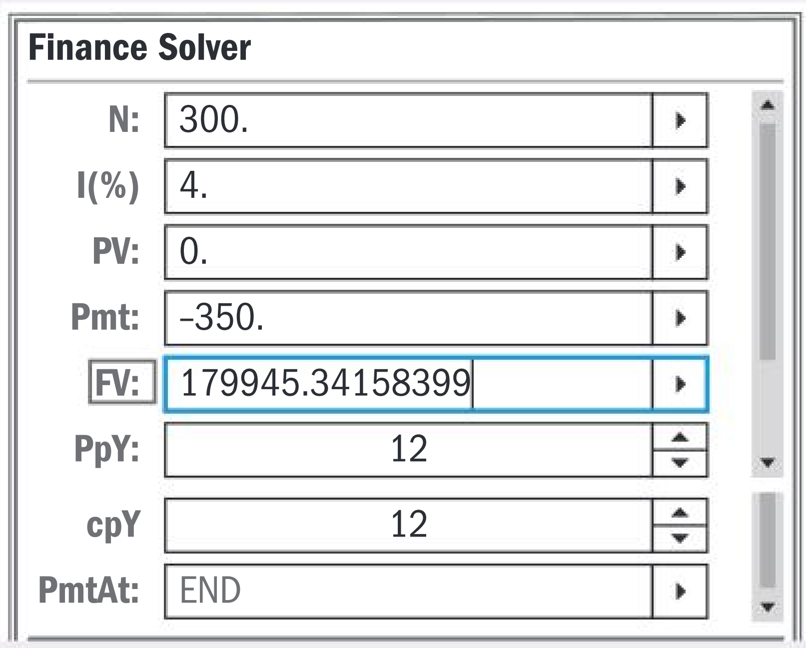
**Option B:** Deposit equal amounts of €450 every month. The nominal annual interest rate is 2.1% compounded monthly.

(a) Find the value of each investment after 25 years.

(b) Hence state which option Yimo should choose and justify your answer.

Solution

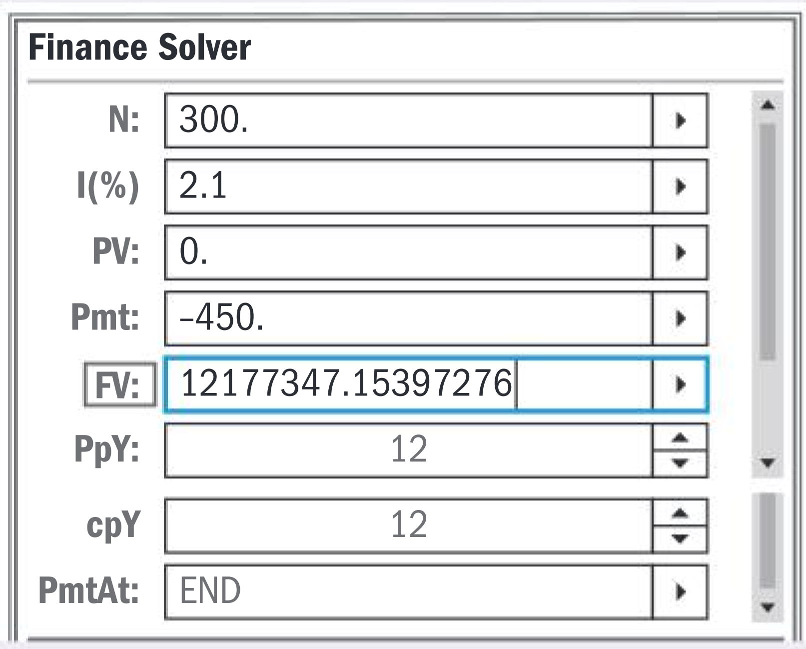
**Option A:**




N is the number of payments, which is $25 \times 12 = 300$

Pmt is negative because this is the payment Yimo makes, so this amount is subtracted from her account each month.

**Option B:**




The future value (FV) is positive because at the end of 25 years the investment moves from the bank to Yimo's account.

The interest is compounded monthly so the CpY value is 12.

(a) Option A is worth €179,945.34 and option B €177,347.15

Write a sum of money correct to two decimal places.

(b) Option A is better not just because it pays slightly more, but because the monthly deposits are significantly less. In option A, a total of $300 \times €350 = €105{,}000$ is paid in by Yimo which is far less than the total invested in option B: $300 \times €450 = €135{,}000$, meaning that option A earns more interest: €74,945.34 compared to €42,347.15 under option B.

Write full sentences to answer the question in context.

# 1.3 ALGEBRA (AHL)

## You should know:

- ✔ the laws of exponents for rational exponents
- ✔ the laws of logarithms are a consequence of the laws of exponents
- ✔ when an infinite geometric series converges.

## You should be able to:

- ✔ simplify numerical and algebraic expressions that involve rational exponents
- ✔ apply the laws of logarithms to simply numerical and algebraic expressions
- ✔ find the sum to infinity of a convergent geometric series.

In this section you extend and build on your knowledge and understanding of exponents, logarithms and geometric series.

### Rational exponents

The rules of rational exponents

$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$, $m, n \in \mathbb{Z}$, $n \neq 0$

enable you to find exact values efficiently and rearrange exponential formulae.

For example, $343^{-\frac{2}{3}} = \frac{1}{343^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{343})^2} = \frac{1}{(7)^2} = \frac{1}{49}$

**Note**

This calculation is more efficient than finding $\frac{1}{343^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{343^2}} = \dots$ .

**Example 1.3.1**

You are given that the volume of a hemisphere is $V = \sqrt{\frac{4S^3}{243\pi}}$, where $S$ is the total surface area of the hemisphere.

(a) If $S = 15$ m², find $V$:

(i) as an exact value in the form $\frac{ab^{\frac{3}{2}}}{c\sqrt{\pi}}$ where $a, b, c \in \mathbb{Z}^+$

(ii) correct to three significant figures.

Teodora is designing a fountain installation at an airport in the shape of a hemisphere.

She explores themes of harmony in her installation and wants to know if it is possible to have the same value for the volume as for the surface area of a hemisphere.

(b) Find $S$ in terms of $V$.

(c) Hence find the radius in metres of the hemisphere for which the values of $S$ and $V$ are equal.

The budget for the installation requires that the hemisphere must fit in a square of area 100 m² and contain less than 200 m³.

(d) Determine if the hemisphere found in (c) fits these requirements.

**Solution**

(a) (i) $V = \sqrt{\frac{4(15^3)}{243\pi}} = \sqrt{\frac{4(5^3)(3^3)}{3^5\pi}} = \sqrt{\frac{4(5^3)}{3^2\pi}} = \frac{2(5^{\frac{3}{2}})}{3\sqrt{\pi}}$

(ii) $V = 4.21$ m²

Apply the laws of exponents to find an exact value, then your GDC to find the approximate value.

(b) $V = \sqrt{\frac{4S^3}{243\pi}} \Rightarrow V^2 = \frac{4S^3}{243\pi} \Rightarrow 243V^2\pi = 4S^3$

$\Rightarrow S^3 = \frac{243V^2\pi}{4} \Rightarrow S = \left(\frac{243V^2\pi}{4}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{243\pi}{4}}V^{\frac{2}{3}}$

Apply the laws of exponents to rearrange the formula.

(c) $S = \sqrt[3]{\frac{243\pi}{4}}V^{\frac{2}{3}}$. Hence, if $V = S$,

$V = \sqrt[3]{\frac{243\pi}{4}}V^{\frac{2}{3}} \Rightarrow V^{\frac{1}{3}} = \sqrt[3]{\frac{243\pi}{4}} \Rightarrow V = \frac{243\pi}{4}$

The volume of a hemisphere in terms of $r$ is $V = \frac{2}{3}\pi r^3$ hence $\frac{2}{3}\pi r^3 = \frac{243\pi}{4} \Rightarrow$

$r^3 = \frac{729}{8} \Rightarrow r = \frac{9}{2}$ m

Apply the laws of exponents to solve the equation $V = S$.

(d) Since $4.5 < 5$, the hemisphere will fit in a square of area 100 m$^2$.

Since $\frac{243\pi}{4} = 191$ to three significant figures, the volume of the hemisphere also meets the requirement for volume since the volume is less than 200 m$^3$.

You find exact values and approximate values to solve the problem in context.

## Laws of logarithms

The laws of logarithms are equivalent forms of the laws of exponents. For example, if $a^m = x$ and $a^n = y$, then $xy = a^{m+n}$ follows from the laws of exponents. Also, $\log_a x = m$ and $\log_a y = n$ follow from the definition of logarithms, as does $\log_a xy = m + n$.

This establishes the law $\log_a xy = \log_a x + \log_a y$. The other laws are $\log_a \frac{x}{y} = \log_a x - \log_a y$ and $\log_a x^m = m\log_a x$. All three laws are found in the formula booklet.

### Example 1.3.2

(a) Write $3\log_{10} x + 0.5 \log_{10} y - 6 \log_{10} z$ as a single logarithm.

(b) The equation $y = ax^b$ can be written in the form $\ln y = m \ln x + c$. Apply the laws of logarithms to find $m$ and $c$ in terms of $a$ and $b$.

**Solution**

(a) $3\log_{10} x + 0.5 \log_{10} y - 6 \log_{10} z$

$= \log_{10} x^3 + \log_{10} y^{0.5} - \log_{10} z^6$

$= \log_{10} x^3y^{0.5} - \log_{10} z^6$

$= \log_{10} \frac{x^3y^{0.5}}{z^6}$

Apply the law $\log_a x^m = m\log_a x$

Work from left to right to apply the laws $\log_a \frac{x}{y} = \log_a x - \log_a y$ and $\log_a x^m = m\log_a x$.

(b) $y = ax^b \Rightarrow \ln y = \ln ax^b$

$= \ln a + \ln x^b$

$= \ln a + b \ln x$

Hence $m = b$ and $c = \ln a$

**SAMPLE STUDENT ANSWER**

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p+2q)\right)=\frac{1}{2}(\log_{10}p+\log_{10}q)$, $p>0$, $q>0$, find $p$ in terms of $q$.

$$\log_{10}\left[\frac{1}{2\sqrt{2}}(p+2q)\right]=\frac{1}{2}\log_{10}(pq)$$
$$\log_{10}\left[\frac{1}{2\sqrt{2}}(p+2q)\right]^2=\log_{10}(pq)$$
$$\left[\frac{1}{2\sqrt{2}}(p+2q)\right]^2=pq$$

▲ The student correctly applies the laws of logarithms, setting out the working methodically and gaining 2 marks.

$$\frac{1}{8}(p+2q)^2=pq$$
$$\frac{1}{8}(p^2+4pq+4q^2)=pq$$
$$p^2+4pq+4q^2=8pq$$
$$p^2+4q^2=4pq$$

$$\cancel{p+\frac{4q^2}{p}=4q} \qquad \cancel{p=\frac{p^2+4q^2}{4q}}$$
$$\cancel{p=p4q} \qquad \cancel{4pq+4}$$
$$\cancel{p-4q=4q^2} \qquad \cancel{4pq=p^2+4q^2}$$

▼ The student spends valuable time on complicated and unsuccessful algebraic manipulations because the wrong approach has been chosen. Applying factorization (which is prior learning) to $p^2-4pq+4q^2$ quickly leads to the answer.

## Infinite geometric series

You know that the formula for the sum of $n$ terms of a geometric series is $S_n=\frac{u_1(1-r^n)}{1-r}$

When $|r|<1$, $r^n$ decreases in size as $n$ increases so that $\lim_{n\to\infty} r^n=0$.

A consequence of this is that $\lim_{n\to\infty} S_n=S_\infty=\frac{u_1}{1-r}$. This fact can be observed in making a link between the rational number $\frac{1}{3}$ and its decimal representation $0.3333\ldots = 0.3+0.03+0.003+0.0003+\ldots$.

The latter expression can be written as an infinite geometric series with first term 0.3 and common ratio 0.1:

$$\sum_{n=1}^{\infty}0.3(0.1)^{n-1}=\frac{0.3}{1-0.1}=\frac{0.3}{0.9}=\frac{1}{3}$$

**Example 1.3.3**

A space probe sets out on a mission in which it travels 120 m north, then 60 m east, 30 m south, then 15 m west and so on, collecting data every time it changes direction. The initial position of the probe is represented by the origin and the distances travelled between each change of direction follow a geometric sequence.

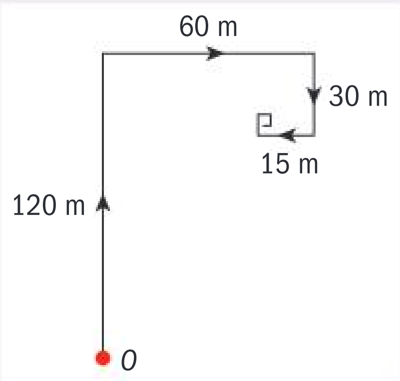
(a) Write down the position of the probe as it changes direction for the fourth time.

(b) Represent the path of the probe as a sum of vectors.

(c) Hence predict the coordinates of the position $P$ that the probe will converge to.

**Solution**

(a)




Representing the context with a sketch is a good problem solving strategy.

The position of the probe as it changes direction for the fourth time is (45, 90)

The position can be read from the diagram.

(b) The path of the probe is represented by the infinite sum

$$\begin{pmatrix}0\\120\end{pmatrix}+\begin{pmatrix}60\\0\end{pmatrix}+\begin{pmatrix}0\\-30\end{pmatrix}+\begin{pmatrix}-15\\0\end{pmatrix}+\begin{pmatrix}0\\7.5\end{pmatrix}+\begin{pmatrix}3.75\\0\end{pmatrix}+\ldots$$

The first terms and the common ratios for the horizontal and vertical translations can be identified from the sum.

The probe gets closer and closer to $P$. This is an application of the concept of a limit.

(c) The $y$-coordinate of $P$ can be modelled by an infinite geometric series with first term 120 and common ratio –0.25, hence the limit of the $y$-coordinate is $\frac{120}{1-(-0.25)} = 96$.

Similarly, the $x$-coordinate is found by $\frac{60}{1-(-0.25)} = 48$.

Hence $P$ is (96, 48)

**SAMPLE STUDENT ANSWER**

Consider a geometric sequence with a first term of 4 and a fourth term of –2.916

(a) Find the common ratio of this sequence.

(b) Find the sum to infinity of this sequence.

▼ The student incorrectly writes down the sum of the first four terms instead of the fourth term. This should have been noticed in the 5 minute reading time. The correct approach is to solve $-2.916 = 4r^3$ to find $r = -0.9$.

(a) $a = 4$

$a + ar + ar^2 + ar^3 = -2.916$

$a(1 + r + r^2 + r^3) = -2.916$

$4(1 + r + r^2 + r^3) = -2.916$

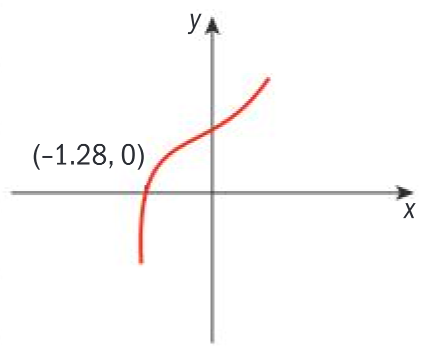
$r^3 + r^2 + r + 1 = \frac{-2.916}{4}$

$r^3 + r^2 + r + 1 = -0.729$

$r^3 + r^2 + r + 1.729 = 0$

using GDC graph of $r^3 + r^2 + r + 1.729 = 0$

▼ The student has demonstrated the correct use of technology, but no marks are awarded because the equation is incorrect.




$r = -1.28$

▼ The student selects the correct formula and sets out the working well; however the formula for the infinite sum cannot be applied if the common ratio is less than $-1$.

(b) $S_\infty = \frac{u_1}{1-r}$

$= \frac{4}{1-(-1.2771)}$

$= 1.75662$

$= 1.76$

# 1.4 COMPLEX NUMBERS (AHL)

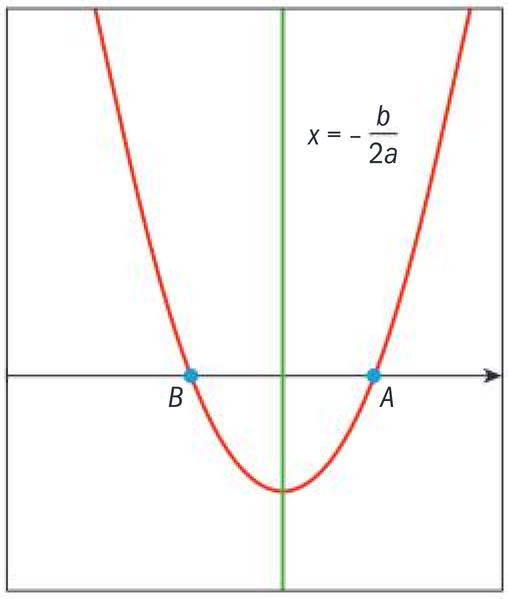
## You should know:

- ✔ that defining $i^2 = -1$ extends the real numbers to the set of complex numbers
- ✔ the meaning of the complex plane (Argand diagram)
- ✔ the terms real part, imaginary part, conjugate, modulus and argument of a complex number
- ✔ the definitions of Cartesian form, polar form and Euler form of a complex number
- ✔ that adding and subtracting two complex numbers can be represented as vector addition
- ✔ that multiplying two complex numbers can be represented as a rotation and a stretch.

## You should be able to:

- ✔ write down the real part, imaginary part and conjugate of a given complex number
- ✔ represent a complex number in the complex plane
- ✔ calculate the conjugate, modulus and argument of a given complex number
- ✔ calculate sums, differences, products and quotients of complex numbers in Cartesian form by hand and with technology
- ✔ calculate powers of complex numbers in Cartesian form using technology
- ✔ convert between the different forms of a complex number by hand and by technology
- ✔ calculate powers, quotients and integer powers in polar or exponential forms.
- ✔ apply complex numbers to find the amplitude and phase shift of the addition of sinusoidal functions with different phase shift angles.

**Complex numbers** provide a logical and useful extension to the set of real numbers.



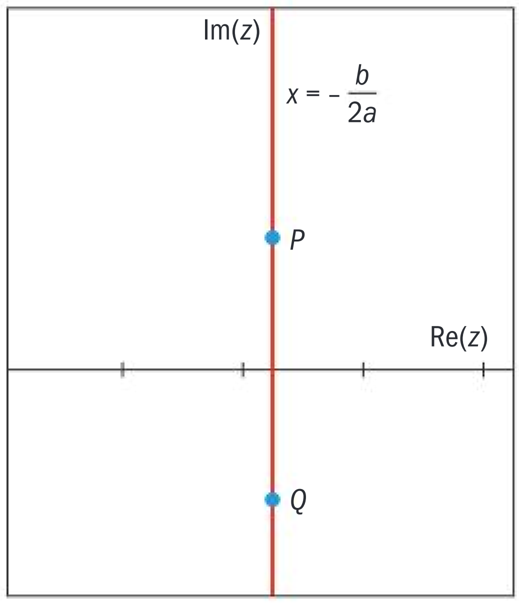

For example, you know that the real roots of $ax^2 + bx + c = 0$ are $x_1, x_2 = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ whenever $b^2 - 4ac \geq 0$

Hence the zeros of $y = ax^2 + bx + c$ are reflections of each other in the line $x = -\frac{b}{2a}$ and can be represented as the points $A\left(-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, 0\right)$ and $B\left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}, 0\right)$ as shown.

A complex number written in Cartesian form as $z = x + y\mathrm{i}$ where $x, y \in \mathbb{R}$ and $\mathrm{i}^2 = -1$ has real part $\mathrm{Re}(z) = x$ and imaginary part $\mathrm{Im}(z) = y$.

The conjugate of $z$ is $z^* = x - y\mathrm{i}$

Complex numbers are represented on the complex plane (Argand diagram) on which the $x$-axis is called the real axis and the $y$-axis the imaginary axis.




When $b^2 - 4ac < 0$, no real roots exist. However, by writing $\mathrm{i}^2 = -1$, you can write the roots as $z_1, z_2 = -\frac{b}{2a} \pm \frac{\sqrt{\mathrm{i}^2(4ac - b^2)}}{2a} = -\frac{b}{2a} \pm \frac{\mathrm{i}\sqrt{(4ac - b^2)}}{2a}$ using the laws of exponents.

Hence $\mathrm{Re}(z_1) = -\frac{b}{2a}$ and $\mathrm{Im}(z_1) = \frac{\sqrt{4ac - b^2}}{2a}$

Since $z_1{}^* = z_2 = -\frac{b}{2a} - \frac{\mathrm{i}\sqrt{4ac - b^2}}{2a}$, the zeros of $y = ax^2 + bx + c$ are reflections of each other in the real axis and can be represented as the points $P\left(-\frac{b}{2a}, \frac{\sqrt{4ac - b^2}}{2a}\right)$ and $Q\left(-\frac{b}{2a}, \frac{\sqrt{4ac - b^2}}{2a}\right)$ as shown.

Given $z = x + y\mathrm{i}$ and $w = a + b\mathrm{i}$,

- If $z = w$ then $x = a$ and $y = b$ (this is known as equating real and imaginary parts)
- $\lambda z = \lambda x + \lambda y\mathrm{i}$ for $\lambda \in \mathbb{R}$
- Addition and subtraction of complex numbers written in Cartesian form is defined by adding like terms: $z + w = (x + a) + (y + b)\mathrm{i}$ and $z - w = (x - a) + (y - b)\mathrm{i}$
- Hence $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$ and $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$.
- Multiplication is defined by expanding brackets and simplifying:
  $zw = (x + y\mathrm{i})(a + b\mathrm{i}) = (xa - yb) + (ya + xb)\mathrm{i}$
- To divide complex numbers, multiply top and bottom by the conjugate of the denominator and simplify:
  $$\frac{z}{w} = \frac{zw^*}{ww^*} = \frac{(xa + yb) + (ya - xb)\mathrm{i}}{a^2 + b^2}$$

### Example 1.4.1

(a) Find the roots of the equation $x^2 - 10x + 26 = 0$ using technology.

(b) Find the roots of the equation $x^2 - 4x + 7 = 0$ by hand.

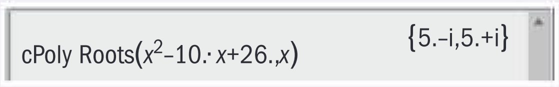
(c) Let your answers to (a) and (b) with positive imaginary parts be $z$ and $w$ respectively. Find $z + w$.

(d) Represent $z$, $w$ and $z + w$ as vectors on an Argand diagram. Comment on the statement "It does not matter in which order you add complex numbers."

**Solution**

(a) The roots are $5 + \text{i}$ and $5 - \text{i}$

Use the polynomial solver on your GDC.




(b) The roots are $\dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$

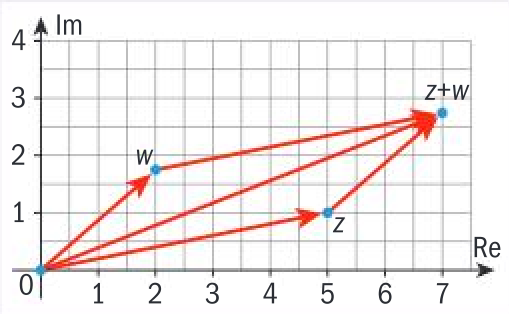
$= \dfrac{4 \pm \sqrt{-12}}{2(1)} = 2 + \sqrt{3}\text{i}$ and $2 - \sqrt{3}\text{i}$

Apply the quadratic formula.

(c) Let $z = 5 + \text{i}$ and $w = 2 + \sqrt{3}\text{i}$

$z + w = 7 + (1 + \sqrt{3})\text{i}$

(d)




Clearly label your diagram.

The diagram confirms that addition of complex numbers is commutative: the order of the addition does not matter.

The vector diagram shows that adding the complex numbers gives the same result no matter the order.

**SAMPLE STUDENT ANSWER**

Consider the distinct complex numbers $z = a + \text{i}b$, $w = c + \text{i}d$, where $a, b, c, d \in \mathbb{R}$.

(a) Find the real part of $\dfrac{z + w}{z - w}$

(b) Find the value of the real part of $\dfrac{z + w}{z - w}$ when $|z| = |w|$

(a) $\cancel{\dfrac{a + c + i(b + d)}{a - c \times i(b \times d)}}$

$\dfrac{a + c}{a - c}$

$a + bi - (c + di)$

$\dfrac{a + c + i(b + d)}{a - c + i(b - d)}$

$\dfrac{a + c}{a - c + i(b - d)} + \dfrac{i(b + d)}{a - c + i(b - d)}$

$\dfrac{1}{a \times c} \times \dfrac{i(b + d)}{1 + \dfrac{i(b - d)}{a - c}} \times \dfrac{i}{i}$

$\cancel{\dfrac{1}{A}} \times \cancel{\dfrac{C}{B}}$

$\dfrac{1}{a - c} \times \dfrac{-(b + d)}{i \times \dfrac{(b - d)}{a - c}}$

(b) $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$

$a^2 + b^2 = c^2 + d^2$

▼ Although $\dfrac{z + w}{z - w}$ is found correctly in terms of $a$ and $b$, the student does not recall the method of division of complex numbers. The correct approach is to multiply both the numerator and denominator by the conjugate of $(a - c) + \text{i}(b - d)$.

▲ Despite difficulties in part (a), the student moves on to part (b) and is awarded a point for the knowledge and understanding he/she demonstrates about the modulus of a complex number.

As with real numbers, you can carry out operations with complex numbers by hand and with technology as appropriate.

**Assessment tip**

Assessment tip: make sure you know where to find *i* on your GDC.

### Example 1.4.2

Given $z_1 = 2 - \mathrm{i}$, $z_2 = 0.7 + \sqrt{13}\mathrm{i}$ and $z_3 = 1 + \mathrm{i}$, find

(a) $z_1 + {z_3}^2$

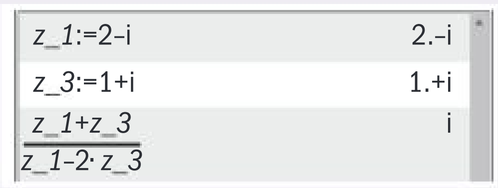
(b) $\dfrac{z_1 + z_3}{z_1 - 2z_3}$

(c) $(z_2 + 0.81z_1)^6$ giving the real and imaginary parts correct to the nearest unit.

**Solution**

(a) $z_1 + {z_3}^2 = (2 - \mathrm{i}) + (1 + \mathrm{i})^2$
$= (2 - \mathrm{i}) + 1 + 2\mathrm{i} + \mathrm{i}^2 = 2 + \mathrm{i}$

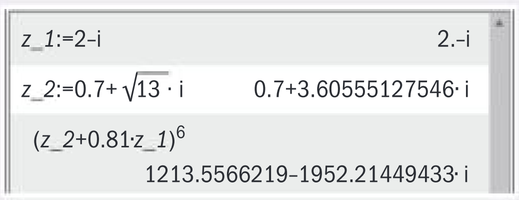
This could be done more efficiently by finding the solution directly from the GDC.

(b) 


$\dfrac{z_1 + z_3}{z_1 - 3z_3} = \mathrm{i}$ from GDC

An alternative method is

$$\frac{z_1 + z_3}{z_1 - 2z_3} = \frac{3}{-3\mathrm{i}} = \frac{3(3\mathrm{i})}{(-3\mathrm{i})(3\mathrm{i})} = \frac{9\mathrm{i}}{9} = \mathrm{i}$$

(c) 

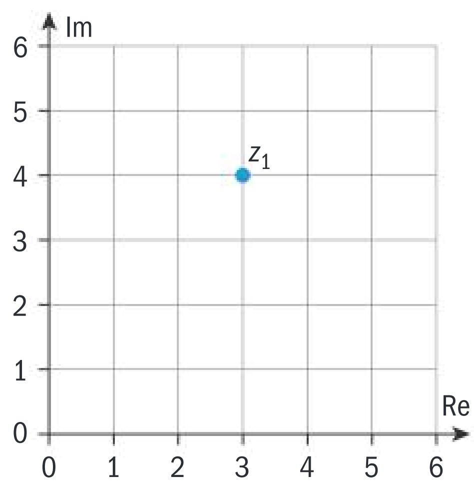

$(z_2 + 0.81z_1)^6 = 1214 - 1952\mathrm{i}$

Use of technology is by far the most efficient method. Use of the memory facility of your GDC helps you check your answer.

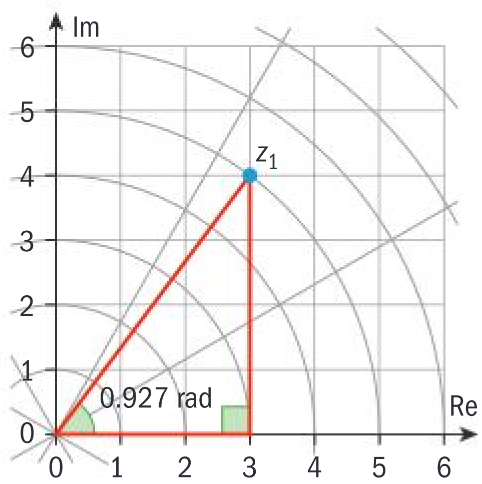
Since complex numbers can be represented as vectors, we can find their magnitude and direction.

For a complex number $z = x + yi$, its modulus $|z| = \sqrt{x^2 + y^2}$ is the length of the position vector defined by $z$. The argument $\arg(z)$, the angle between the positive real axis and the position vector defined by $z$ where $-\pi < \arg(z) \le \pi$, is found with the equation $\arg(z) = \arctan\left(\dfrac{y}{x}\right)$.

For example, $z = 3 + 4\mathrm{i}$, represented in Cartesian form, can be plotted on an Argand diagram by the coordinate pair (3, 4) as shown on the right.




$|z| = \sqrt{3^2 + 4^2} = 5$ and $\arg(z) = \arctan\left(\dfrac{4}{3}\right) = 0.927$. Hence $z$ is 5 units from the origin in the direction given by 0.927 radians measured from the real axis as shown on the right.




Notice that the Cartesian coordinate pair (3, 4) and the polar coordinate pair (5, 0.927) each give the information needed to locate $z$ on the Argand diagram.

Therefore, you have two different ways to represent a complex number: $3 + 4\mathrm{i}$ and $5(\cos(0.927) + \sin(0.927)\mathrm{i})$ are equivalent.

The generalised form of this result is given by:

If $z = x + yi$, then $z = r(\cos(\theta) + i\sin(\theta))$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \arg(z)$.

You write this as $z = r\text{cis}(\theta)$ for short. It can be shown that $r\text{cis}(\theta) = re^{i\theta}$, which is known as the exponential (or the Euler form) of a complex number.

### Example 1.4.3

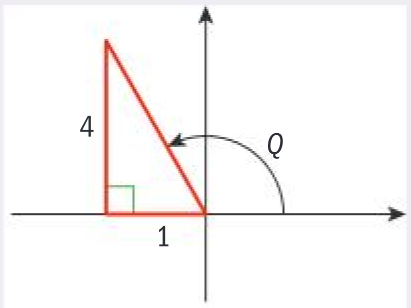
(a) Represent $z = -1 + 4i$ in exponential form.

(b) Represent $w = 3.1e^{-0.871i}$ in cartesian form.

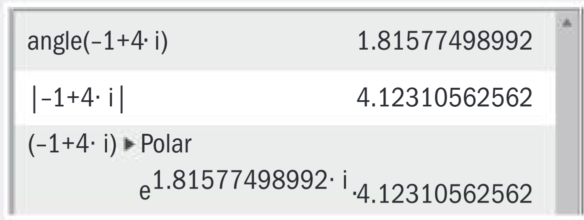
**Solution**

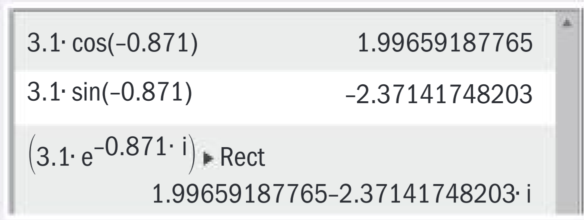
(a) $r = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$

A sketch shows that $z$ is in the second quadrant.
So, $\arg(z) = \pi - \arctan(4) \approx 1.82$




Your GDC can confirm:




Hence $z = \sqrt{17}e^{1.82i}$

(b) $3.1e^{-0.871i} = 3.1(\cos(-0.871) + i\sin(-0.871))$

Hence $w = 2.00 - 2.37i$

**Assessment tip**

Make sure you know how your GDC converts between Cartesian, polar and exponential forms of complex number.

Why bother with three representations of complex numbers? One reason is to visualize and understand patterns, as seen in the next example.

### Example 1.4.4

(a) Given $z_1 = 1.5e^{\frac{\pi}{6}i}$, find $z_n = (z_1)^n$ for $n = 0, 2, 3, 4, 5$

(b) Represent the sequence $\{z\} = \{z_0, z_1, z_2, z_3, z_4, z_5\}$ on an Argand diagram.

(c) Describe the patterns in $\{z\}$ (i) algebraically (ii) graphically

**Solution**

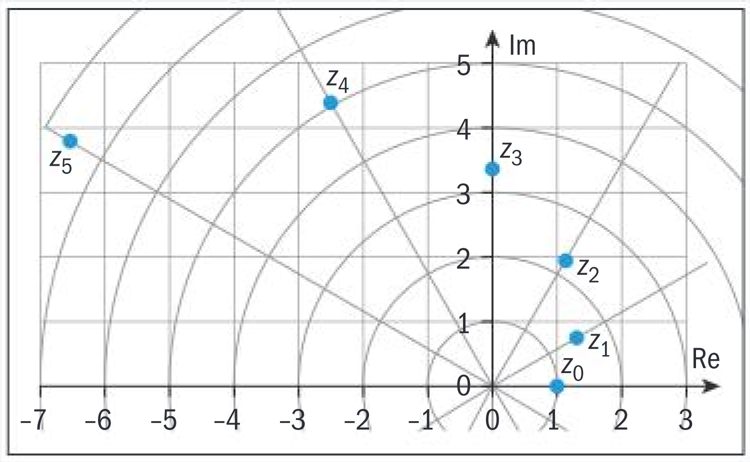
(a) $z_0 = 1$, $z_2 = \left(1.5e^{\frac{\pi}{6}i}\right)^2 = 2.25e^{\frac{\pi}{3}i}$

$z_3 = 3.375e^{\frac{\pi}{2}i}$, $z_4 = 5.0625e^{\frac{2\pi}{3}i}$,

$z_5 = 7.593\,75e^{\frac{5\pi}{6}i}$

Apply the laws of exponents to the Euler form.

(b)




(c) (i) The moduli of $\{z\}$ are a geometric sequence with first term 1, common ratio 1.5. The arguments are an arithmetic sequence with first term 0 and common difference $\frac{\pi}{6}$.

(ii) The sequence shows a stretch factor 1.5 and a rotation of $\frac{\pi}{6}$ each time the power is increased by 1.

A consequence of the equivalence $r(\cos(\theta) + \mathrm{i}\sin(\theta)) = re^{\mathrm{i}\theta}$ is that the laws of exponents can be used to show that if $z = r(\cos(\theta) + \mathrm{i}\sin(\theta))$ and $w = \rho(\cos(\alpha) + \mathrm{i}\sin(\alpha))$ then:

- $zw = r\rho(\cos(\theta + \alpha) + \mathrm{i}\sin(\theta + \alpha))$ and $\frac{z}{w} = \frac{r}{\rho}(\cos(\theta - \alpha) + \mathrm{i}\sin(\theta - \alpha))$
- Also, patterns as seen in the previous example can be generalized as $z^n = (re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta} = r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta)),\ n \in \mathbb{Z}$.
  This is known as De Moivre's theorem.

These formulae give convenient ways to represent, manipulate and calculate with complex numbers. For example, powers can be found efficiently without needing to expand brackets.

## Example 1.4.5

Given $z = Ae^{B\mathrm{i}}$ and $w = 2e^{-\mathrm{i}}$, find $\mathrm{Re}\left(\frac{z^6}{w^4}\right)$

**Solution**

$\frac{z^6}{w^4} = \frac{(Ae^{B\mathrm{i}})^6}{(2e^{-\mathrm{i}})^4} = \frac{A^6e^{6B\mathrm{i}}}{2^4e^{-4\mathrm{i}}} = \frac{A^6}{16}e^{(6B+4)\mathrm{i}}$

Apply the laws of exponents.

Hence $Re\left(\frac{z^6}{w^4}\right) = \frac{A^6}{16}\cos(6B + 4)$

Write down the real part of the polar form of $\frac{A^6}{16}e^{(6B+4)\mathrm{i}}$, $\frac{A^6}{16}(\cos(6B + 4) + \mathrm{i}\sin(6B + 4))$

Despite being named "imaginary", there are many applications of complex numbers which give you useful results in the real world: for example, finding the amplitude and phase shift of a sum of sinusoidal functions with the same period is useful in electrical engineering.

## Example 1.4.6

Two electricity sources $V_1$ and $V_2$ are modelled with the functions $V_1 = 5\sin(3t + \frac{\pi}{4})$ and $V_2 = 2\sin(3t + \frac{\pi}{6})$ respectively.

(a) Graph $V_1$, $V_2$ and $V = V_1 + V_2$ on the same axes.

(b) Hence compare and contrast the features of the three graphs.

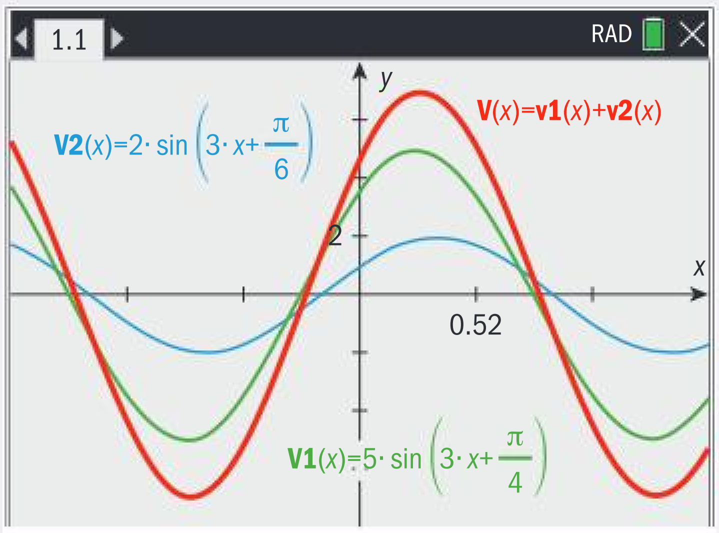
(c) By applying $a\sin(bt + c) = \mathrm{Im}(ae^{(bt+c)\mathrm{i}})$, calculate the maximum value of $V$ and its phase shift.

**Assessment tip**

When referring to the adding of sinusoidal functions of the form $a\sin(bt + c)$, the phase shift is $c$. Elsewhere, the phase shift refers to $c$ in the form of the equation $a\sin(b(x - c))$. The question will always make it clear what is meant by the term in the context.

Solution

(a)




Choose a viewing window in radians which shows the details of all three graphs clearly.

(b) The period of both $V_1$ and $V_2$ is $\frac{2\pi}{3}$ and the period of $V$ looks the same. The amplitudes of $V_1$ and $V_2$ are 5 and 2 respectively and the period of $V$ is approximately 7. The phase shifts of $V_1$ and $V_2$ are $\frac{\pi}{4}$ and $\frac{\pi}{6}$ respectively and the phase shift of $V$ appears between these values.

(c) $V = 5\sin\left(3t+\frac{\pi}{4}\right) + 2\sin\left(3t+\frac{\pi}{6}\right)$

Apply $V = V_1 + V_2$ then $a\sin(bt+c) = \mathrm{Im}(ae^{(bt+c)i})$

$= \mathrm{Im}\left(5e^{\left(3t+\frac{\pi}{4}\right)i}\right) + \mathrm{Im}\left(2e^{\left(3t+\frac{\pi}{6}\right)i}\right)$

Apply $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$

$= \mathrm{Im}\left(5e^{\left(3t+\frac{\pi}{4}\right)i} + 2e^{\left(3t+\frac{\pi}{6}\right)i}\right)$

Factorize by $e^{3ti}$ and use your GDC to find $5e^{\frac{\pi}{4}i} + 2e^{\frac{\pi}{6}i}$ in exponential form:


$= \mathrm{Im}\left(e^{3ti}\left(5e^{\frac{\pi}{4}i} + 2e^{\frac{\pi}{6}i}\right)\right)$

$= \mathrm{Im}(e^{3ti}\,(6.95e^{0.711i})) = \mathrm{Im}(6.95e^{(3t+0.711)i})$

$= 6.95\sin(3t + 0.711)$

Hence the maximum value of $V$ is 6.95 and the phase shift is 0.711

## Assessment tip

$a\sin(bx+c) = \mathrm{Im}(ae^{(bx+c)i})$ is a consequence of the formula $r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$ which is in the formula book.

# 1.5 SYSTEMS OF EQUATIONS

## You should know:

- ✔ the meaning of the terms zero, root and intercept
- ✔ that a system of linear equations or a polynomial equation can be solved by hand or with technology, however technology is the most efficient method.

## You should be able to:

- ✔ use technology to solve systems of linear equations in up to three variables
- ✔ use technology to solve polynomial equations.

A polynomial function has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_2x^2 + a_1x + a_0$$

where $n \in \mathbb{N}$ and each $a_i \in \mathbb{R}$ for $0 \le i \le n$. Solving polynomial equations by hand is often time-consuming and error-prone, so it is often better to use technology.

The zeros, roots and intercepts of a function are terms that describe the following features:

The zeros of a **function** $f(x)$ are solutions of the equation $f(x) = 0$.

The solutions of the **equation** $f(x) = 0$ are its roots.

The point where the graph of $y = f(x)$ crosses the $y$-axis is the $y$-intercept $(0, f(0))$.

The point(s) where the graph of $y = f(x)$ crosses the $x$-axis are the $x$-intercepts $(\beta_1, 0)$, $(\beta_2, 0)$, $(\beta_3, 0)$, …where $\beta_1, \beta_2, \beta_3, \dots$ are the zeros of $f(x)$.

### Assessment tip

These terms are easy to mix up. Make sure you know them so you can understand exam questions and use the correct terminology in your internal assessment.

### Example 1.5.1

(a) Find the coordinates of all the axes intercepts of $g(x) = 2.1x^2 - 1.3x - 9$

(b) Find the zeros of the function $h(x) = -0.3x^4 + x^3 - 0.8x + 2$

**Solution**

(a) The $y$-intercept is $(0, -9)$

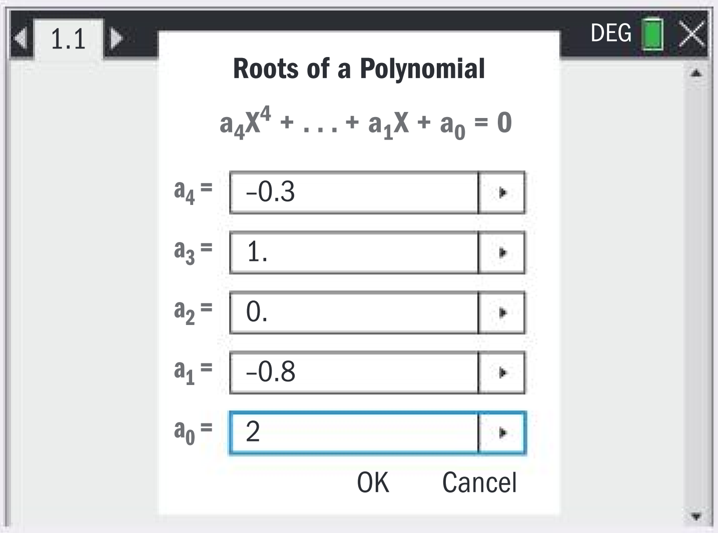
The $x$-intercepts are $(-1.78, 0)$ and $(2.40, 0)$

The $y$-intercept can be written down immediately but the $x$-intercepts are best found with technology:

```
PolyRoots(2.1·x²-1.3·x-9.,x)
        {-1.78368408134,2.40273170039}
```

(b) The zeros of the function are $-1.30$ and $3.27$

Take care to enter the coefficient of $x^2$ as 0.



You should also use technology to solve systems of equations in the examination. Systems can be solved by hand using methods of substitution or elimination, however using technology is a more reliable and efficient method.

**Assessment tip**

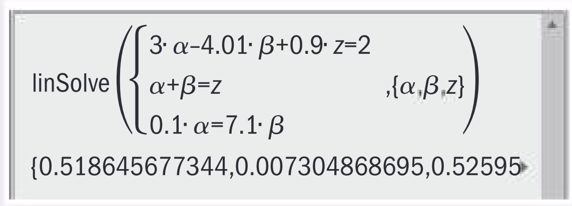
Make sure you know how to enter the information to your GDC in order to solve a polynomial equation or to solve a system of equations, and how to interpret the output.

### Example 1.5.2

Solve the system of equations:

$$3\alpha - 4.01\beta + 0.9z = 2$$
$$\alpha + \beta = z$$
$$0.1\alpha = 7.1\beta$$

**Solution**




$\alpha = 0.519,\ \beta = 0.00730,\ z = 0.526$

Entering the system exactly as it is written in the examination question can make it easier for you to check you have entered everything correctly. This app shows that this GDC gives the solution in the order $\alpha$, $\beta$, $z$.

Write the answers in the correct order correct to three significant figures, unless told otherwise.

**Assessment tip**

In the examination there will always be a unique solution to a system of equations. So if your GDC says otherwise, you will have entered the system incorrectly.

**SAMPLE STUDENT ANSWER**

Boxes of mixed fruit are on sale at a local supermarket.

Box A contains 2 bananas, 3 kiwis and 4 melons, and costs $6.58.

Box B contains 5 bananas, 2 kiwis and 8 melons, and costs $12.32.

Box C contains 5 bananas and 4 kiwis and costs $3.00.

Find the cost of each type of fruit.

$$2B + 3K + 4M = 6.58$$
$$5B + 2K + 8M = 12.32$$
$$5B + 4K + 0M = 3.00$$

$$\left.\begin{array}{l}\text{Banana}: \$0.36\\ \text{Kiwi}: \$0.30\\ \text{Melon}: \$1.24\end{array}\right\}\text{GDC}$$

▲ The student carefully sets up the equations, ready for entry into a GDC. This is the best method since it is fast and accurate; much better than the error-prone and laborious method of solving by hand.

▲ The student is able to begin the exam with the knowledge that 4 marks have been scored quickly.

**Assessment tip**

Use the five-minute reading time before the examination begins to look for opportunities like this question in which use of the GDC is the most efficient method.

# 1.6 MATRICES AND MATRIX ALGEBRA (AHL)

## You should know:

- ✔ the terms element, row, column and order of a matrix. An $m \times n$ matrix has $m$ rows and $n$ columns
- ✔ that the matrix product **AB** is defined only if the number of columns in **A** is equal to the number of rows in **B**
- ✔ the terms identity matrix, zero matrix, determinant of a matrix and inverse matrix
- ✔ the ways in which matrix algebra is different from that of algebraic expressions and formulae
- ✔ the definition of an eigenvector and its associated eigenvalues.

## You should be able to:

- ✔ identify the meaning of each element of a matrix in the context of a problem
- ✔ add, subtract and multiply matrices by hand and with technology
- ✔ find inverses and determinants for $2 \times 2$ matrices by hand, and for $n \times n$ matrices with technology
- ✔ solve systems of equations using an inverse matrix
- ✔ find the eigenvalues and eigenvectors of a $2 \times 2$ matrix **A**
- ✔ write a matrix **A** in diagonalized form and apply it to find powers of **A**.

Matrices give you a powerful way to organize and manipulate data. Some of the basic definitions of matrices can be easily made sense of in a practical context. For example, the management of a café collects data about total sales one day in two branches, Street and Mall, and presents this data in a $2 \times 3$ sales matrix **S**:

$$\mathbf{S} = \begin{array}{c} \\ \text{Street} \\ \text{Mall} \end{array} \begin{array}{c} \begin{array}{ccc} \text{coffee} & \text{water} & \text{snack} \end{array} \\ \begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix} \end{array}$$

We see that the matrix is just a convenient table setting out the information. For example, in the Street branch 76 snacks were sold: this **element** is denoted $\mathbf{S}_{1,3}$ because it is in the first row, third column. $2 \times 3$ is the **order** of the matrix. If the management wanted to write down a sales target in which all sales rise by 10%, they could calculate this **scalar multiple** of **S** by multiplying each element by 1.1:

$$1.1\mathbf{S} = 1.1 \times \begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix} = \begin{pmatrix} 1.1 \times 212 & 1.1 \times 145 & 1.1 \times 76 \\ 1.1 \times 311 & 1.1 \times 170 & 1.1 \times 112 \end{pmatrix}$$

$$= \begin{pmatrix} 233.20 & 159.50 & 83.6 \\ 342.10 & 187 & 123.20 \end{pmatrix}$$

The management expresses the current prices in € of their products $P_1$ and the proposed new prices $P_2$ in a $3 \times 2$ price matrix **P** and wishes to investigate what effect these changes would have on total earnings:

$$\mathbf{P} = \begin{array}{c} \\ \text{coffee} \\ \text{water} \\ \text{snack} \end{array} \begin{array}{c} \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{pmatrix} 2.3 & 2.4 \\ 2.1 & 2.1 \\ 3.4 & 3.1 \end{pmatrix} \end{array}$$

>> **Assessment tip**

Writing column and row headings is not compulsory, but it may help you plan your approach to a problem in context.

From these matrices, the management can find the matrix **SP** because the number of columns of **S** is equal to the number of rows of **P**, so that the rows of **S** and the columns of **P** can be combined as follows in the process of **matrix multiplication**:

$$\mathbf{SP} = \begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix} \begin{pmatrix} 2.3 & 2.4 \\ 2.1 & 2.1 \\ 3.4 & 3.1 \end{pmatrix}$$

$$= \begin{pmatrix} 212\times2.3+145\times2.1+76\times3.4 & 212\times2.4+145\times2.1+76\times3.1 \\ 311\times2.3+170\times2.1+112\times3.4 & 311\times2.4+170\times2.1+112\times3.1 \end{pmatrix}$$

$$= \begin{pmatrix} 1050.50 & 1048.90 \\ 1453.10 & 1450.60 \end{pmatrix}$$

From this, the management can find out about their proposed changes. For example, by comparing elements $\mathbf{SP}_{1,1}$ = €1050 and $\mathbf{SP}_{1,2}$ = €1048.90 to see that the total earnings for the street branch are predicted to be virtually unchanged under the proposed changes.

Finding $\mathbf{PS} = \begin{pmatrix} 1234 & 741.50 & 443.60 \\ 1098.30 & 661.50 & 394.80 \\ 1684.90 & 1020 & 605.60 \end{pmatrix}$ illustrates that in general, changing the order of multiplication of matrices gives a different answer: this means that matrix multiplication is not **commutative**. In fact, not only does **PS** have different elements and different order when compared to **SP**, it has little meaning in this context. So **PS** ≠ **SP** is true in three different ways!

The following results are true in general for matrix addition:

- Matrix addition **A** + **B** is defined only if **A** and **B** have the same orders
- Matrix addition is commutative: **A** + **B** = **B** + **A**
- Matrix addition is associative: (**A** + **B**) + **C** = **A** + (**B** + **C**)

The following results are true in general for matrix multiplication:

- Matrix multiplication **AB** is defined only if the number of columns of **A** is equal to the number of rows of **B**.
- Matrix multiplication is not commutative: in general, **AB** ≠ **BA**
- Matrix multiplication is distributive over addition: **A**(**B** + **C**) = **AB** + **AC** and (**B** + **C**)**A** = **BA** + **CA**
- Matrix multiplication is associative: (**AB**)**C** = **A**(**BC**)

You can use technology to calculate with matrices.

### Example 1.6.1

(a) Given $\mathbf{A} = (3 \quad -4)$, $\mathbf{B} = \begin{pmatrix} 0 & 3 \\ -1 & 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0.5 & 0 \end{pmatrix}$, and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, find:

(i) **AB** + **AC** (ii) **A**(**B** + **C**) (iii) (**B** + **C**)**A**

(b) Comment on your results.

(c) Find:

(i) **B**(**CD**) (ii) (**BC**)**D**

(d) Comment on your results.

**Solution**

(a)

| | |
|---|---|
| $a \cdot b + a \cdot c$ | [5. −4.] |
| $a \cdot (b+c)$ | [5. −4.] |
| $(b+c) \cdot a$ | "Error: Dimension error" |

Hence $\mathbf{AB} + \mathbf{AC} = (5 \quad -4) = \mathbf{A}(\mathbf{B} + \mathbf{C})$. However, $(\mathbf{B} + \mathbf{C})\mathbf{A}$ is not defined because the number of columns of $\mathbf{B} + \mathbf{C}$ is 2, which is not the same as the number of rows of $\mathbf{A}$ which is 1.

(b) This illustrates that matrix multiplication is distributive but not commutative.

(c) $\mathbf{B}(\mathbf{CD}) = (\mathbf{BC})\mathbf{D} = \begin{pmatrix} 1.5 & 0 \\ -4 & -1 \end{pmatrix}$

(d) This illustrates that matrix multiplication is associative.

Enter the matrices in your GDC:

| | |
|---|---|
| $[3 \quad -4] \rightarrow a$ | $[3. \quad -4.]$ |
| $\begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix} \rightarrow b$ | $\begin{bmatrix} 0. & 3. \\ -1. & 4. \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 1 \\ 0.5 & 0 \end{bmatrix} \rightarrow c$ | $\begin{bmatrix} 1. & 1. \\ 0.5 & 0. \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \rightarrow d$ | $\begin{bmatrix} 1. & 0. \\ 5. & 1. \end{bmatrix}$ |

This can save time as well as make your calculations clear for you to read on the GDC screen.

In order to know how to carry out algebra with matrices, the concepts of zero, identity and inverse are essential.

- The $m \times n$ zero matrix $\mathbf{0}$ has all its elements equal to zero, and is such that $\mathbf{0} + \mathbf{A} = \mathbf{A} + \mathbf{0}$ for all $m \times n$ matrices $\mathbf{A}$. The zero matrix of order $2 \times 3$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For example:

$$\begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 212 & 145 & 76 \\ 311 & 170 & 112 \end{pmatrix}$$

- The $n \times n$ identity matrix $\mathbf{I}$ is has all its elements equal to zero except those in the main diagonal which are all equal to 1, and is such that $\mathbf{IB} = \mathbf{BI}$ for all $n \times n$ matrices $\mathbf{B}$.

  The identity matrix of order $3 \times 3$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 7 & 4 \\ 1 & 6 & 3 \\ 3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 4 \\ 1 & 6 & 3 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 4 \\ 1 & 6 & 3 \\ 3 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{0}$ and $-\mathbf{A}$ enable you to solve matrix equations as follows:

| | | |
|---|---|---|
| Equation with $\mathbf{A}$ and $\mathbf{B}$ given, $\mathbf{X}$ to be found. | $\mathbf{X} + \mathbf{A} = \mathbf{B}$ | |
| Step by step re-arrangement of the equation using matrix algebra | $\mathbf{X} + \mathbf{A} + (-\mathbf{A}) = \mathbf{B} + (-\mathbf{A})$ | Add $-\mathbf{A}$ to both sides of the equation |
| | $\mathbf{X} + (\mathbf{A} + (-\mathbf{A})) = \mathbf{B} + (-\mathbf{A})$ | Matrix addition is associative |
| | $\mathbf{X} + (\mathbf{0}) = \mathbf{B} + (-\mathbf{A})$ | $\mathbf{A} + (-\mathbf{A})$ gives the zero matrix $\mathbf{0}$. |
| Solution | $\mathbf{X} = \mathbf{B} + (-\mathbf{A})$,<br>which is equal $(-\mathbf{A}) + \mathbf{B}$ to since matrix addition is commutative. | |

The addition of $-\mathbf{A}$ carries out the inverse process of adding $\mathbf{A}$. Similarly, to solve more types of equations, you need to be able to carry out the inverse of multiplying by a matrix $\mathbf{A}$.

For example, for all $2 \times 2$ matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse of $\mathbf{A}$ is:

$\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$ad - bc$ is the determinant of $\mathbf{A}$ and is denoted det $\mathbf{A}$ or $|\mathbf{A}|$.

For $\mathbf{A}^{-1}$ to exist, det $\mathbf{A} \neq 0$ hence $ad \neq bc$.

$\mathbf{I}$ and $\mathbf{A}^{-1}$ enable you to solve matrix equations as follows:

**Note**

Do not confuse $\mathbf{A}^{-1}$ with the reciprocal of a number.

**Note**

Because matrix multiplication is not commutative, the order of multiplication matters. Hence if you post/pre-multiply by a matrix on one side of an equation, you must do the same to the other side.

| | | |
|---|---|---|
| Equation with **A** and **B** given, **X** to be found. | $\mathbf{XA} = \mathbf{B}$ | |
| Step by step re-arrangement of the equation using matrix algebra | $\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$<br>$\mathbf{X}(\mathbf{AA}^{-1}) = \mathbf{BA}^{-1}$<br>$\mathbf{X}(\mathbf{I}) = \mathbf{BA}^{-1}$ | Post multiply **both** sides of the equation by $\mathbf{A}^{-1}$<br>Matrix multiplication is associative<br>$\mathbf{AA}^{-1}$ gives the identity matrix $\mathbf{I}$ |
| Solution | $\mathbf{X} = \mathbf{BA}^{-1}$, which is not equal $\mathbf{A}^{-1}\mathbf{B}$ to since matrix multiplication is not commutative.<br>In fact, $\mathbf{A}^{-1}\mathbf{B}$ is the solution to a different equation: $\mathbf{AX} = \mathbf{B}$ which can be derived in the same way as above, but by pre-multiplying both sides by $\mathbf{A}^{-1}$ at the start of the re-arrangement. | |

### Example 1.6.2

(a) Solve the equation $\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 2 & 0.5 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}$ by hand.

(b) Solve the equation $\begin{pmatrix} 1 & 2 & 0 \\ 0.7 & 4 & -2 \\ 1 & 1 & 7 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \\ 0 \end{pmatrix}$ using technology.

**Solution**

(a) $\begin{vmatrix} 2 & 0.5 \\ 4 & -3 \end{vmatrix} = 2(-3) - 0.5(4) = -8$

First, find the determinant of $\begin{pmatrix} 2 & 0.5 \\ 4 & -3 \end{pmatrix}$

$\begin{pmatrix} 2 & 0.5 \\ 4 & -3 \end{pmatrix}^{-1} = -\dfrac{1}{8}\begin{pmatrix} -3 & -0.5 \\ -4 & 2 \end{pmatrix}$ so

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix} \times \left(-\dfrac{1}{8}\right)\begin{pmatrix} -3 & -0.5 \\ -4 & 2 \end{pmatrix}$

Post-multiply both sides of the equation by the inverse of $\begin{pmatrix} 2 & 0.5 \\ 4 & -3 \end{pmatrix}$

$= -\dfrac{1}{8}\begin{pmatrix} 4 & -2 \\ -17 & 2.5 \end{pmatrix} = \begin{pmatrix} \dfrac{-1}{2} & \dfrac{1}{4} \\ \dfrac{17}{8} & \dfrac{-5}{16} \end{pmatrix}$

so $x = -\dfrac{1}{2}$, $y = \dfrac{1}{4}$, $z = \dfrac{17}{8}$ and $t = -\dfrac{5}{16}$

(b) $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0.7 & 4 & -2 \\ 1 & 1 & 7 \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ 11 \\ 0 \end{pmatrix}$

Write out the solution by pre-multiplying both sides by the inverse matrix. Use technology to find the answer.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0.7 & 4 & -2 \\ 1 & 1 & 7 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -4 \\ 11 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -16.9135802469 \\ 6.45679012346 \\ 1.49382716049 \end{bmatrix}$$

so $p = -16.9$, $q = 6.46$, $r = 1.49$

## Eigenvalues and eigenvectors

For a $2 \times 2$ matrix $\mathbf{A}$ and a $2 \times 1$ vector $\boldsymbol{x}$, if $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ then the effect of multiplying $\boldsymbol{x}$ by $\mathbf{A}$ is just to stretch $\boldsymbol{x}$ by a factor of $\lambda$. In this case, $\boldsymbol{x}$ is called an eigenvector of $\mathbf{A}$ and $\lambda$ is its associated eigenvalue.

($\boldsymbol{x} = \mathbf{0}$ is always a solution of $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, but this trivial solution is not useful in applications.)

The procedure to find the eigenvalues and eigenvectors is:

- Rearrange $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ to the form $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$
- Solve the characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}| = 0$ to find the eigenvalues $\lambda_1$ and $\lambda_2$
- Find the solution sets of the systems $\mathbf{A}\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $\mathbf{A}\boldsymbol{x} = \lambda_2\boldsymbol{x}$
- Identify a particular solution from each solution set to form the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

This procedure is applied below. You will not be able to use technology to do this in the examination.

### Example 1.6.3

Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

**Solution**

| | |
|---|---|
| $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ | Apply the definition $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. |
| $\Rightarrow \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ | Apply $\lambda\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ |
| $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ | Since matrix multiplication is distributive. |
| $\left[\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ | Carry out the matrix subtraction. |
| $\begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$ | The only unique solution is $\begin{pmatrix} x \\ y \end{pmatrix} = 0$. |
| $\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-\lambda) - 2(-1) = 0$ $\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 2)(\lambda - 1)$ $= 0 \Rightarrow \lambda = 1$ or $\lambda = 2$. | Normally, your solution in an exam would begin at this line. For other solutions to the system to exist there must be an infinity of solutions, so that $\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$. |
| If $\lambda = 1$ then $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $3x + 2y = x \Rightarrow y = -x$ or, from multiplication of the second row, $-x = y$ | Each system does not have a unique solution. However, the relationship between the components of the eigenvector can be found from the system. |
| If $\lambda = 2$ then $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $3x + 2y = 2x \Rightarrow 2y = -x$ or, from multiplication of the second row, $-x = 2y$ | |
| For $\lambda = 1$, the solutions are of the form $\begin{pmatrix} x \\ -x \end{pmatrix}$ since $-x = y$. | You describe the general solution for each system. |

For $\lambda = 2$, the solutions are of the form $\begin{pmatrix} -2y \\ y \end{pmatrix}$ since $x = -2y$.

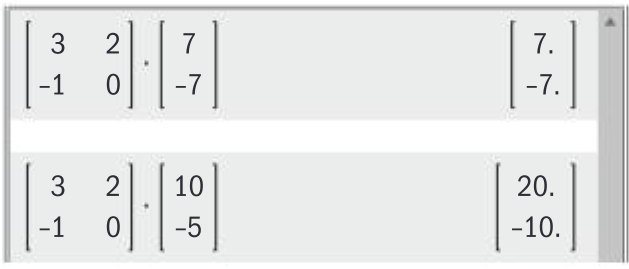
The eigenvectors are therefore $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, associated with eigenvalues $\lambda = 1$ and $\lambda = 2$ respectively.

Choosing particular solutions for the eigenvectors.

**Assessment tip**

You can confirm your eigenvectors are correct using technology in the examination using matrix multiplication. But you will not be able to *find* eigenvalues nor eigenvectors with technology in the examination.

Note that in Example 1.6.3, you could choose $\begin{pmatrix} 7 \\ -7 \end{pmatrix}$ instead of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, or $\begin{pmatrix} 10 \\ -5 \end{pmatrix}$ instead of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.




You can confirm with technology that these are eigenvectors associated with 1 and 2 respectively.

The procedure to diagonalize **A** is:

- Find the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ with eigenvalues $\lambda_1$ and $\lambda_2$ of **A**.
- Form the matrices $\mathbf{P} = (\boldsymbol{x}_1 \quad \boldsymbol{x}_2)$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
- Write $\mathbf{A} = \mathbf{PDP}^{-1}$, the diagonalized form of **A**.

Now since $\mathbf{D}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$, it can be shown that $\mathbf{D}^n = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$.

Also, $\mathbf{A}^2 = (\mathbf{PDP}^{-1})^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\,\mathbf{DP}^{-1} = \mathbf{P}(\mathbf{DD})\mathbf{P}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$.

Similarly, it can be shown that $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$. You can apply this formula to find $\mathbf{A}^n$ very efficiently.

### Example 1.6.4

Diagonalize the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ and hence find $\mathbf{A}^7$

**Solution**

**A** has eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, associated with eigenvalues $\lambda = 1$ and $\lambda = 2$ respectively.

Form $\mathbf{P} = (x_1 \quad x_2)$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Therefore $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}^{-1}$

Apply $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

Hence $\mathbf{A}^7 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^7\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}^{-1} \Rightarrow$

$$\mathbf{A}^7 = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^7 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2^8 \\ -1 & 2^7 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 255 & 254 \\ -127 & -126 \end{pmatrix}$$

Finding $\mathbf{A}^7$ with diagonalization takes two matrix mulitplications wheras finding $\mathbf{A}^7$ by hand would take six.

# SL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

1. A cuboid has width $n$, length $n+4$ and height $2n+1$, where $n \in \mathbb{Z}^+$

   a. Show that the volume $V$ of the cuboid can be written as $V = 2n^3 + 9n^2 + 4n$

   b. Hence or otherwise, find the value of $n$ for which $V = 12\,495$

2. A bowl is in the shape of a hemisphere. Its radius is measured as 6.3 cm, correct to 1 decimal place.

   a. Determine the upper and lower bounds of the radius of the hemisphere.

   b. Hence, find the maximum percentage error of the volume of the hemisphere.

3. When David joins a gym in January 2021, the monthly fee is £34. The fee increases by 2% every **two** years.

   a. Calculate how much the monthly fee will be in February 2026.

   b. Find the **total amount** David will pay for his gym membership if he stays for exactly 7 years.

4. A sequence of 80 cylindrical concrete columns is planned in the garden of a new university campus. The height of each column is 10 m and the radii are in an arithmetic sequence with first term 0.5 m and common difference 0.05 m.

   a. Find the first column to have volume greater than 100 $m^3$.

   b. Find the total amount of concrete needed for the 80 columns, to the nearest $m^3$.

5. Erin measures the height, width and length of a cuboid-shaped food container as 254 mm, 65 mm and 154 mm respectively.

   a. Using these measurements, find the volume of the container in $mm^3$, giving your answer correct to two significant figures.

   b. Write down your answer to part **a** in the form $a \times 10^n$, $1 \le a < 10$, $n \in \mathbb{Z}$

   c. Given that the measurements of height, width and length are correct to the nearest mm, determine if the container will definitely be able to hold 2.5 litres of washing liquid.

6. Callum is saving money to buy a digital camera. He invests $p$ Euros (€) in an account that pays an annual interest rate of 4.3%, compounded monthly.
   After 6 years he has €2069.99 in his account.

   a. Calculate the value of $p$. Give your answer correct to 2 decimal places.

   Callum bought a digital camera for €2000 and sold it six years later for €900.

   b. Find the rate at which the value of camera depreciated each year.

7. In this question, assume that the distance from Earth to Mars is 252.65 million km and that the space probe travels in a straight line.

   a. Find the distance to Mars in metres and express your answer in standard form.

   A space probe leaves Earth at a speed of 60 000 kilometres per hour.

   b. Find the speed of the space probe in metres per second, and express your answer in the form $a \times 10^n$, $1 \le a < 10$, $n \in \mathbb{Z}$.

   c. Use your answers to parts **a** and **b** to find the time taken in seconds for the probe to travel the distance to Mars.

## GROUP 2

8. Pilar deposits $17,000 in a bank account which pays a nominal interest rate of 7%, compounded yearly.

   a. Find how much interest Pilar has earned after 6 years.

   Ximena deposited $20,000 in a bank account. Her account pays a nominal annual interest rate of $r$ %, compounded monthly. After five years, the total amount in Ximena's account is $24,351.19

   b. Find $r$, correct to 2 decimal places.

9. Eito is exploring the mathematics of earthquakes. The magnitude $R$ of an earthquake can be modelled by the formula $R = \log_{10}\left(\frac{x}{0.001}\right)$, where $x$ is a reading from a seismograph in mm which represents the size of the movement in the Earth caused by the earthquake.

a. Find the magnitude of an earthquake for which $x = 52\,098$ mm.

b. An earthquake has a magnitude of 8.9. Find the size of the movement in the Earth, $x$, correct to the nearest mm.

**10.** The distance from the Earth to the Sun is approximately 149 600 000 km.

a. Find the distance from the Earth to the Sun in cm.

b. Write your answer to part **a** in the form $a \times 10^n$, $1 \le a < 10$, $n \in \mathbb{Z}$

Madita draws a graph of the function $f(x) = 2^x$, $-2 \le x \le p$ with a scale 1 cm = 1 unit on each axis.

c. The distance from the Earth to the Sun is equal to $f(p)$. Find $p$ correct to the nearest centimetre.

**11.** Magzhan is going to train for a marathon. He plans to run 4 km in his first week of training, then increase the amount he runs by 0.5 km every week.

a. Find how many km Magzhan will run in his 12th week of training.

b. Using sigma notation, write down an expression for the total distance Magzhan has run after 20 weeks in training, and calculate this distance.

c. A marathon is approximately 42 km long. How many weeks of training must Magzhan complete before he runs at least 42 km in one week of training?

**12.** Bicycle sharing app Ebike is analysing its business performance in a city.

In 2015, the number of bicycles hired by Ebike customers was 1530. In 2018, the figure was 1624.

a. Assuming that the number of hires follows a geometric sequence, predict the number of hires in 2025 to the nearest whole number.

b. Predict the year in which the number of hires first exceeds 2000.

**13.** A new café opens in Galoisville. The owners invest €17,500 in the kitchen equipment. The value of the equipment decreases by 5% each year.

a. Find the value of the kitchen equipment after seven years.

b. The managers decide to re-fit the kitchen when the value of the equipment falls below 40% of its original value. Find how many complete years will pass before the kitchen is re-fitted.

## GROUP 3

**14.** Three functions $g$, $f$ and $h$ are defined for $n \in \mathbb{Z}^+$ as:

$g(n) = \sum_{k=1}^{n}(5.7 + 3k)$, $f(n) = \sum_{k=1}^{n}(-0.5 + 0.7^k)$ and

$h(n) = \sum_{k=1}^{n}(9.3 \times 1.1^k)$

a. Identify which function represents an arithmetic series and justify your answer.

b. Hence write down the first term and common difference of the arithmetic sequence.

c. Identify which function represents a geometric series and justify your answer.

d. Hence write down the first term and common ratio of the geometric sequence.

e. Find the least value of $q$ for which $h(n) > g(n)$ for all $n > q$.

**15.** A trip to a remote jungle region is planned. 68 people and 80 cases must be transported on narrow roads. The organizers of the trip can hire cars or vans. Each car carries 4 people and 3 cases, and each van can carry 7 people and 10 cases.

a. If $v$ is the number of vans hired, and $c$ the number of cars hired, write down a system of two equations that represent this context.

b. Use technology to solve the system.

c. Hence find the number of vans and the number of cars that should be hired, given that a van costs more to hire than a car.

**16.** Emilia is comparing loan deals. She wishes to buy an apartment for £250,500.

**Bank A** requires a 10% deposit, then a loan for 20 years at a nominal interest rate of 1.3%, compounded monthly. Repayments are made each month.

**Bank B** requires a 15% deposit, then a loan for 25 years at a nominal interest rate of 1.1%, compounded monthly. Repayments are made each month.

a. Calculate the monthly repayments for each loan.

b. Hence find the total repayment for each loan.

c. Hence describe the advantages and disadvantages of Emilia choosing the deal from Bank A in preference to the deal from Bank B.

17. Teodora is planning a rock garden in the shape of a triangle $ABC$. To find side $c$ she uses the formula $c = \sqrt{a^2 + b^2 - 2ab\cos C}$.

a. If $a = 10$ m, $b = \sqrt{7}$ m and $C = 60°$, find the exact value of $c$.

b. If Teodora approximates $\sqrt{7}$ m to three significant figures and uses this value instead of $\sqrt{7}$, find the percentage error in the value of $c$.

c. If $a = 10$ m and $b = \sqrt{7}$ m are exact, but $C = 60°$ is accurate only to the nearest degree, find the maximum percentage error in the value of $c$.

18. Chimdi invests \$9500 in a savings account that pays a nominal annual interest rate of 5%, compounded quarterly.

a. Find how many years Chimdi must wait until the value of her investment doubles.

Nicole invests \$9500 in a saving account which pays a nominal interest rate of $r$%, compounded monthly.

b. Find the minimum nominal annual interest rate needed for Nicole to double her money in 12 years.

## PAPER 2

1. Denise works for a car sharing subscription scheme. On her first day at work, she sells $y$ subscriptions. Her manager asks her to increase the number of subscriptions she sells by the same amount, $x$, each working day.

Denise sells 27 subscriptions on the sixth working day, and 41 on the thirteenth working day.

a. Write down a system of equations, in terms of $y$ and $x$, for the number of subscriptions Denise sells on the sixth day and for the number of subscriptions Denise sells on the thirteenth day, stating any assumptions you make.

b. Hence, predict the value of $y$ and the value of $x$.

Assume that on each of her first 32 working days, Denise increases the number of subscriptions she sells by at least 1 but no more than 3 each day and that she sells $y$ subscriptions on her first day.

c. Determine if Denise will sell at least 1000 subscriptions in total in her first 32 working days.

2. Alex has €21,000 to invest. He considers three options:

**Option A:** Invest in a savings account paying an annual interest rate of 4%, compounded yearly

**Option B:** Invest in a savings account paying a nominal annual interest rate of 3.8%, compounded monthly

**Option C:** Invest in a fund that guarantees a payment of €1100 per year.

a. To the nearest €, calculate the value of Alex's investment after one year if he chooses:

i. Option A

ii. Option B

iii. Option C

b. Write down an expression for the value $V$ of Alex's investment after $n$ years if he chooses:

i. Option A

ii. Option B

iii. Option C

Alex is told by his friend Beth that his investment will grow in value faster in Option B than in Option A, since the compounding periods are more frequent.

c. Determine if Beth's claim is correct and justify your answer.

d. Find the number of complete years for which Alex's investment would be worth more in **both** Options A and B than in Option C.

e. Find, to the nearest €, the difference in the total interest Alex will earn after 20 years by choosing option A instead of Option C.

# HL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

1. a. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{C}=\begin{pmatrix}2 & 4\\ 1.5 & 1\end{pmatrix}$

   b. Hence, write down matrices **P** and **D** such that $\mathbf{C}=\mathbf{PDP}^{-1}$

   c. Find a general expression for $\mathbf{C}^n$ in terms of $n$.

2. The magnitude $R$ of an earthquake on the Richter scale is measured by the formula $R=\log_{10}\left(\frac{I}{I_0}\right)$ where $I$ is the intensity of the earthquake and $I_0$ is the intensity of ground movement on a normal day.

   a. The San Francisco, California earthquake of 1906 measured 8.25 on the Richter scale, and the earthquake near Dashur, Egypt in 1992 measured 5.9 on the Richter scale. Calculate how many times more intense the San Francisco earthquake was in comparison to the Dashur earthquake.

   b. The earthquake near Japan's north-east coast in 2011 was 7.08 times more intense than the San Francisco earthquake of 1906. Calculate the measure of the Japanese earthquake on the Richter scale.

3. a. Show that the number 0.151 515 151 515 can be written as

$$\sum_{r=1}^{6} 0.15\times 10^{2-2n}$$

   b. Show that the number $0.\overline{51}$ can be written as

$$\sum_{r=1}^{\infty} 0.51\times(0.01)^{r-1}$$

   c. Hence find the exact value of $0.\overline{51}$

4. Solve the equation $\frac{z}{z+2\text{i}}=4-7\text{i}$

5. a. Find the exact values of the roots of $3x^2+x-7=0$

   b. Find the exact values of the roots of $3x^2+x+7=0$

   c. Verify that for each equation, the conjecture below is true:

   The roots of $ax^2+bx+c=0$ add to $\frac{-b}{a}$

6. Simplify these algebraic expressions:

   a. $\sqrt[3]{9x^{14}\times 375x^{10}\times\left(2x^2\right)^3}$

   b. $\left(\frac{729x^{12}}{64y^3}\right)^{\frac{1}{3}}$

   c. $\frac{\left(5y^{-1}x^2\right)^3}{\left(3y^2x^{-4}\right)^5}$

## GROUP 2

7. Given $\omega_1=2\text{cis}\left(\frac{\pi}{4}\right)$ and $\omega_2=3\text{cis}\left(\frac{-\pi}{6}\right)$, find expressions in Euler form for the following:

   a. $\omega_1\omega_2$

   b. $\left(\frac{\omega_1}{\omega_2}\right)^4$

   c. $\left(\omega_1^*\right)^3$

8. Two voltage sources $V_1$ and $V_2$ are connected in a circuit so that the total voltage is $V=V_1+V_2$. If $V_1=7\sin(20t+5)$ and $V_2=3\sin(20t+13)$, find an expression for the total voltage in the form $V=A\sin(20t+B)$.

9. $\mathbf{A}=\begin{pmatrix}2 & -2\\ 3 & 0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & 5\\ 2 & 1\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}0 & \pi\\ -3 & 0\end{pmatrix}$, and $\mathbf{D}=\begin{pmatrix}4 & 0\\ -3 & \sqrt{3}\end{pmatrix}$

   a. Find the $2\times 2$ matrix **X** that satisfies the equation $\mathbf{XA}+\mathbf{XB}=\mathbf{C}-3\mathbf{D}$.

   b. Solve the equation $\mathbf{BXA}=\mathbf{D}$, expressing all answers as matrices in the form $q\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, where $q\in\mathbb{Q}$ and $a,b,c,d\in\mathbb{R}$ are all exact values.

10. A geometric series has common ratio $1.5\left(2^x\right)$

   a. Find the values of $x$ for which the sum to infinity of the series exists.

   b. If the first term of the sequence is 13, and the sum to infinity of the series is 16, find the value of $x$.

11. $\log_4 a$, $\log_4(a+9)$ and $\log_4(a+20)$ are the first three terms of an arithmetic sequence.

   a. Find the value of $a$.

   b. Hence, find the sum of the first 20 terms of the sequence.

**12.** Consider $\omega = \text{cis}\left(\frac{\pi}{3}\right)$

**a.** Given that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, express $w$ in Cartesian form.

**b.** Find $\omega^2$ in Cartesian form and in modulus-argument form. Hence, find exact values for $\cos\left(\frac{2\pi}{3}\right)$ and $\sin\left(\frac{2\pi}{3}\right)$.

**c.** Express each element of the set $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\}$ in modulus-argument form.

**d.** Hence, with reference to an Argand diagram or otherwise, show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$

## GROUP 3

**13. a.** Given that $\log_3(ax+b) = 2 + 2\log_3(ax-b)$, use the laws of logarithms to find $p$, $q$ and $r$ in terms of $a$ and $b$ in the quadratic equation $px^2 + qx + c = 0$.

**b.** Find the exact values of $r$ and $s$ if the piecewise function $f(x)$ is continuous:

$$f(x) = \begin{cases} e^{sx} & x < 1 \\ 3 + 2x^3 & 1 \le x < 2 \\ \ln(x-r) & x \ge 2 \end{cases}$$

**c.** Find the exact solution of the equation $4^{2x+1} = 3^{1-x}$. Express the answer in the form $\frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Q}$.

**14. a.** Given $z = 2 + 2\sqrt{3}\text{i}$ and $\omega = \sqrt{2} - \sqrt{2}\text{i}$, find $zw$ and $\frac{z}{w}$ in Cartesian form.

**b.** Express $z$ and $\omega$ in Euler form. Hence, find $zw$ and $\frac{z}{w}$ in Euler form.

**c.** By comparing the two forms of $zw$, find the exact value of $\cos\left(\frac{\pi}{12}\right)$

**15.** The organizers of a sports league are investigating a proposed change to how points are awarded.

Three teams $T_1$, $T_2$ and $T_3$ have their results (win, draw or lose) represented in the matrix **R**:

$$\mathbf{R} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{win} & \text{draw} & \text{lose} \end{array} \\ \begin{pmatrix} 8 & 3 & 1 \\ 4 & 5 & 3 \\ 2 & 4 & 6 \end{pmatrix} \end{array}$$

The proposed changes, $p_1$ and $p_2$, to the points awarded are represented in the matrix **P**:

$$\mathbf{P} = \begin{array}{c} \\ \text{win} \\ \text{draw} \\ \text{lose} \end{array} \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ -1 & 0 \end{pmatrix} \end{array}$$

**a.** Find the matrix **RP** and interpret the elements of the matrix.

The proposed changes, $p_1$ and $p_2$, to the points awarded have sponsorship payment plans $s_1$ and $s_2$ respectively, based on the number of points scored.

Matrix **E** shows the amount in £ that $T_1$, $T_2$ and $T_3$ would like to win under sponsorship plans $s_1$ and $s_2$.

$$\mathbf{E} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 12{,}000 & 15{,}600 \\ 10{,}000 & 20{,}000 \\ 8000 & 14{,}000 \end{pmatrix} \end{array}$$

The sponsorship payments for each point awarded under changes $p_1$ and $p_2$ are represented by the matrix **S**.

$$\mathbf{S} = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 70 & 65 \\ 40 & 100 \end{pmatrix} \end{array}$$

**b.** Solve the equation $\mathbf{XS} = \mathbf{E}$ for a $3 \times 2$ matrix **X** and interpret its elements in context.

**16.** Solve the simultaneous equations

$$1 + 2\log_{10} y = \log_{10} 10x$$

$$1 + \log_{10} x = \log_{10}(7 - 20y)$$

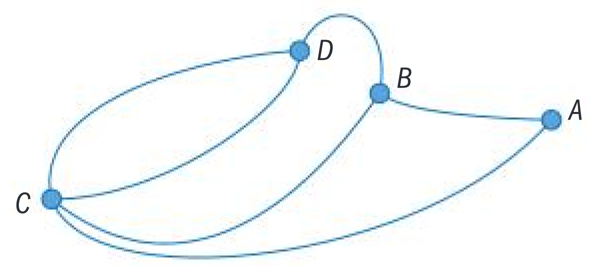
**17.** Solve the simultaneous equations

$$\ln\left(\frac{y^2 z}{x}\right) = 3$$

$$\ln(x^2) + \ln(y^3) = 10$$

$$\ln\left(\sqrt{xyz}\right) = -1$$

**18.** The sketch below shows the roads connecting four villages $A$, $B$, $C$ and $D$.



The matrix **P** shows the number of one-stage journeys between the four villages:

$$\mathbf{P} = \text{from} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \quad\quad\text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \end{array}$$

**a.** Complete the matrix **P**.

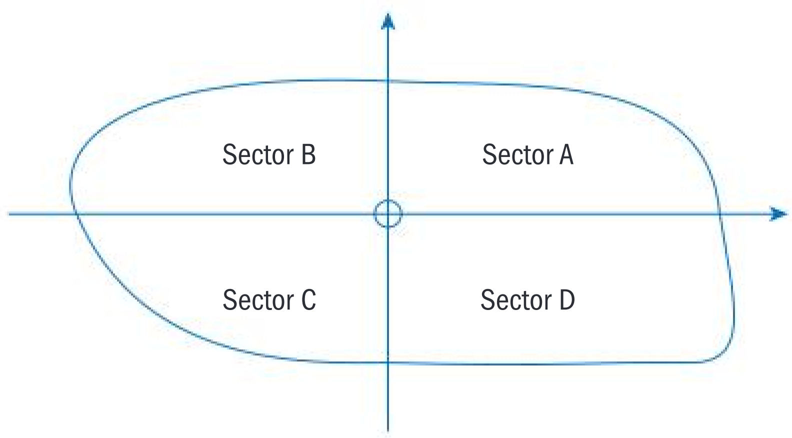
**b.** Find $\mathbf{P}^2$

**c.** Find the number of two-stage journeys from $C$ to $B$ and identify this number in the elements of $\mathbf{P}^2$.

**d.** Interpret the elements of $\mathbf{P}^2$.

## PAPER 2

**1.** An Unmanned Aerial Vehicle (UAV) is surveying a large forested area, divided into sectors as shown.

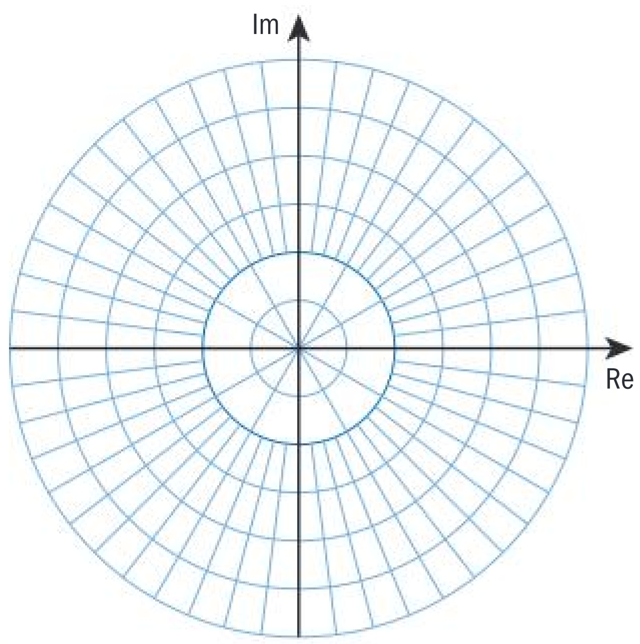



The position $z$ after $t$ hours of the UAV named *Zappy* is given by $z(t) = \left(1.5\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^t$

Distances are measured in km.

**a.** Find the values of $z(0)$, $z(1)$, $z(2)$, $z(3)$, $z(4)$.

**b.** Hence, sketch the positions of *Zappy* at $t = 0$, 1, 2, 3 and 4 hours on this Argand diagram:




**c.** Determine after how many hours *Zappy* is first more than 50 km from its initial position **and** is in sector D.

**d.** If, at $t = 13$ hours, *Zappy* is programmed to return directly to its starting position, find the components of the vector that represents this journey.

**e.** A second UAV named *Alpha* takes off at the same time as *Zappy*. The position of *Alpha* after $t$ hours is given by $a(t) = \left(\sqrt{1.51\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}\right)^{2t}$. Flight engineer Nico states that *Alpha* and *Zappy* are in danger of a near miss after 6 hours – meaning that they will be within 100 m of each other. Flight engineer Zank disagrees. Determine which flight engineer is correct.

**2.** The *BetaGamma* Information Technology store has two megastores, one in Cairo, Egypt and the other in Toulouse, France. Each megastore has three types of employee who have three different salaries, one for each type of employee. For the financial year of 2020, employees who work on the IT desk earn \$900 per week, those working in the sales room earn \$700 per week and those who work in the office earn \$500 per week.

The number of workers at each site is given in the following table:

| | IT desk | Sales room | Office |
|---|---|---|---|
| Cairo | 52 | 15 | 19 |
| Toulouse | 34 | 9 | 12 |

**a.** Construct and label a $3 \times 2$ matrix **E** to represent the number of employees of each type in each megastore.

**b. i.** What would the matrix $\mathbf{S} = (900 \quad 700 \quad 500)$ represent?

**ii.** What are the dimensions of **S**?

**c.** If $\mathbf{SE} = (66\,800 \quad x)$, calculate the value of $x$.

**d.** Describe what the value of $x$ represents.

**e.** Find the total amount of money that *BetaGamma* must pay in one year to its employees in both megastores in the financial year of 2020.

The managers of *BetaGamma* predict that each year after 2020, all salaries will increase by 1% and one of each type of employee will retire in each megastore.

**f.** Predict the total amount of money that *BetaGamma* must pay in one year to its employees in both megastores in the financial year of 2022.

## PAPER 3

**1.** *The aim of this question is to explore the long-term population of a colony of seals using matrices.*

The female population of a colony of seals can be divided into three groups: pups, young seals and adults. *Pups* are seals aged 0-1 years, *young seals* are those aged 1-2 years and an *adult seal* is any seal over 2 years old.

If, in year $n$, the population of female pups is $p_n$, female young seals is $y_n$ and female adults is $a_n$, then the population of females in year $n+1$ can be found from:

$$\begin{pmatrix} p_{n+1} \\ y_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} p_n \\ y_n \\ a_n \end{pmatrix}$$

The matrix $\mathbf{L} = \begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$ is called the Leslie matrix for the population. It is used to model the average change in the number of females in a population.

**a.** By considering the entries in **L**, write down:

**i.** the average number of young born to a young female

**ii.** the probability a pup survives and moves into the young seals group

**iii.** the fraction of adults that die each year.

Scientists begin measuring the population and their initial data is $p_0 = 20,\ y_0 = 15$ and $a_0 = 40$

**b.** Find the number of seals in each group one year later. Give your answers to the nearest integer.

**c.** Explain why the population after 3 years can be found from:

$$\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}^3 \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix}$$

Let $T_n$ be the **total** female population after $n$ years.

**d.** Use an extension of the above result to find:

**i.** $T_{20}$

**ii.** $T_{30}$

It is given that for $n \geq 20$ the sequence $T_n$ is approximately geometric.

**e. i.** Use your answer to part **d** to find the common ratio for the sequence $T_n$ for $n \geq 20$

**ii.** Hence, find when the total female population will first pass 50 000.

The vector $\boldsymbol{q}_n$ gives the proportion of the total population in each of the groups in year $n$.

**f.** Find the vector $\boldsymbol{q}_{30}$

It is given that each element of $\boldsymbol{q}_{30}$ is approximately equal to the corresponding element in $\boldsymbol{q}$, an eigenvector for **L**.

**g. i.** Use your answer to part **f** and the relation $\mathbf{L}\boldsymbol{q} = \lambda \boldsymbol{q}$ to find $\lambda$ the corresponding eigenvalue.

**ii.** Comment on your answer.

# 2 FUNCTIONS

## 2.1 LINES AND FUNCTIONS

### You should know:

- ✔ the gradient of a straight line expresses its steepness and direction
- ✔ that parallel lines have equal gradients and that if two lines are perpendicular the product of their gradients is –1
- ✔ the definition of a function and of domain and range
- ✔ an inverse function reverses or undoes the effect of a function
- ✔ that $f^{-1}(x)$ denotes the inverse function of $f(x)$, and exists only if $f(x)$ is one-to-one
- ✔ solving $f(x) = a$ is equivalent to finding $f^{-1}(a)$
- ✔ the graph of $y = f^{-1}(x)$ is found by reflecting the graph of $y = f(x)$ in the line $y = x$
- ✔ the domain of $f^{-1}(x)$ is equal to the range of $f(x)$.

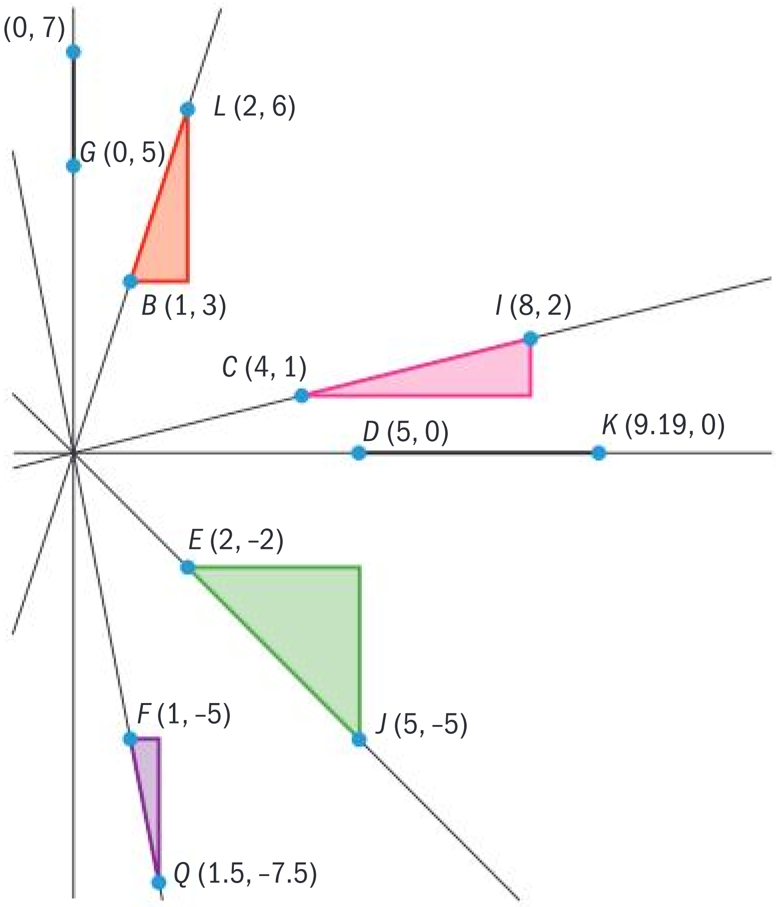
### You should be able to:

- ✔ calculate the gradient of a straight line in a context and interpret its meaning
- ✔ rearrange any of the three forms of the equation of a straight line into any other
- ✔ find the equation of a straight line from a context
- ✔ find the $x$-intercept and the $y$-intercept of a straight line from its equation
- ✔ represent a function as a table, graph or equation as appropriate
- ✔ use technology to determine the range of a function from its graph
- ✔ sketch the graph of $y = f^{-1}(x)$ by reflecting the graph of $y = f(x)$ in the line $y = x$
- ✔ apply $y = f(x)$ by replacing an $x$ value in the equation to predict the value of $y$, using technology if appropriate
- ✔ apply $y = f(x)$ by replacing a $y$ value in the equation to predict the value of $x$, using technology or the inverse function if appropriate
- ✔ apply and interpret $y = f^{-1}(x)$ in context.

Steepness is a simple yet important quantity to measure. The design of any modern building with stairs, escalators or ramps must include calculations of steepness. The formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ finds the gradient of the line segment [$AB$] with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$.

> When using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of the line segment [$AB$] with end points $A$ and $B$, it is not important which point you call $(x_1, y_1)$ and which you call $(x_2, y_2)$. The gradient will be the same, provided you apply the formula correctly.

- Any vertical line segment - such as [*GM*]- has **no defined** gradient since the denominator of the formula is zero.
- [*BL*] has gradient $m = \frac{6-3}{2-1} = 3$, showing that [*BL*] is steeper than [*CI*] which has gradient $m = \frac{2-1}{8-4} = \frac{1}{4}$ since $3 > \frac{1}{4}$.
- Any horizontal line segment - such as [*DK*] - has gradient 0 since the numerator of the formula is zero.
- Informally, [*EJ*] and [*FQ*] "go down" from left to right and hence have negative gradient.
- [*EJ*] has gradient –1, showing that the amount by which $y$ decreases is in a 1 : 1 ratio with the amount by which $x$ increases on [*EJ*].
- [*FQ*] has gradient $m = \frac{-7.5-(-5)}{1.5-1} = -5$, showing that it is the steepest of all the line segments shown.

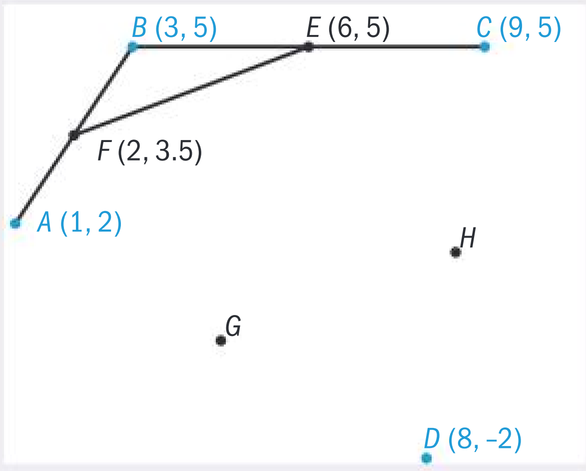



Two important ways to compare gradients $m_1$ and $m_2$ are:

- If $m_1 = m_2$ then the lines are parallel
- If $m_1 \times m_2 = -1$ then the lines are perpendicular

## Example 2.1.1

The diagram below shows part of a plan for a section of a bridge structure. The coordinates of points *A*, *B*, *C* and *D* are given. *E* is the midpoint of [*BC*] and *F* is the midpoint of [*AB*].




(a) Find the coordinates of *G* and of *H*, the midpoints of [*AD*] and [*DC*] respectively.

(b) The engineers predict that [*FE*] and [*GH*] will be parallel, and that [*FH*] and [*EG*] will be perpendicular. Determine if either prediction is correct. Justify your reasoning.

### Solution

(a) $G = \left(\frac{1+8}{2}, \frac{2+(-2)}{2}\right) = (4.5, 0)$

$H = \left(\frac{9+8}{2}, \frac{5+(-2)}{2}\right) = (8.5, 1.5)$

You apply the formula for the coordinates of the midpoint of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$, which is prior learning: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

(b) [*FE*] has gradient $\frac{5-3.5}{6-2} = \frac{1.5}{4} = \frac{3}{8}$ and [*GH*] has gradient $\frac{0-1.5}{4.5-8.5} = \frac{-1.5}{-4} = \frac{3}{8}$ so it is true to say that [*FE*] and [*GH*] are parallel.

For final answers, the numerator and denominator of a fraction should be integer values.

$[FH]$ has gradient $\frac{3.5-1.5}{2-8.5} = -\frac{2}{6.5} = -\frac{4}{13}$ and

$[EG]$ has gradient $\frac{5-0}{6-4.5} = \frac{5}{1.5} = \frac{10}{3}$

So $[FH]$ and $[EG]$ are not perpendicular because the product of their gradients is $\frac{-4}{13} \times \frac{10}{3} = \frac{-40}{39} \neq -1$

The line segments look perpendicular and in fact the angle between them is very close to 90°, but this calculation proves that the second prediction is false. Note that the result is determined without unnecessary decimal approximations of the fractions. The exact value of $\frac{-4}{13}$ is more convenient and easier to use than its decimal approximation –0.307 692…

**Assessment tip**

It is easier and more accurate to use exact values when working with gradients.

**Assessment tip**

When applying the point-gradient form, you can choose any point $(x_1, y_1)$ on the line. The answer will be the same.

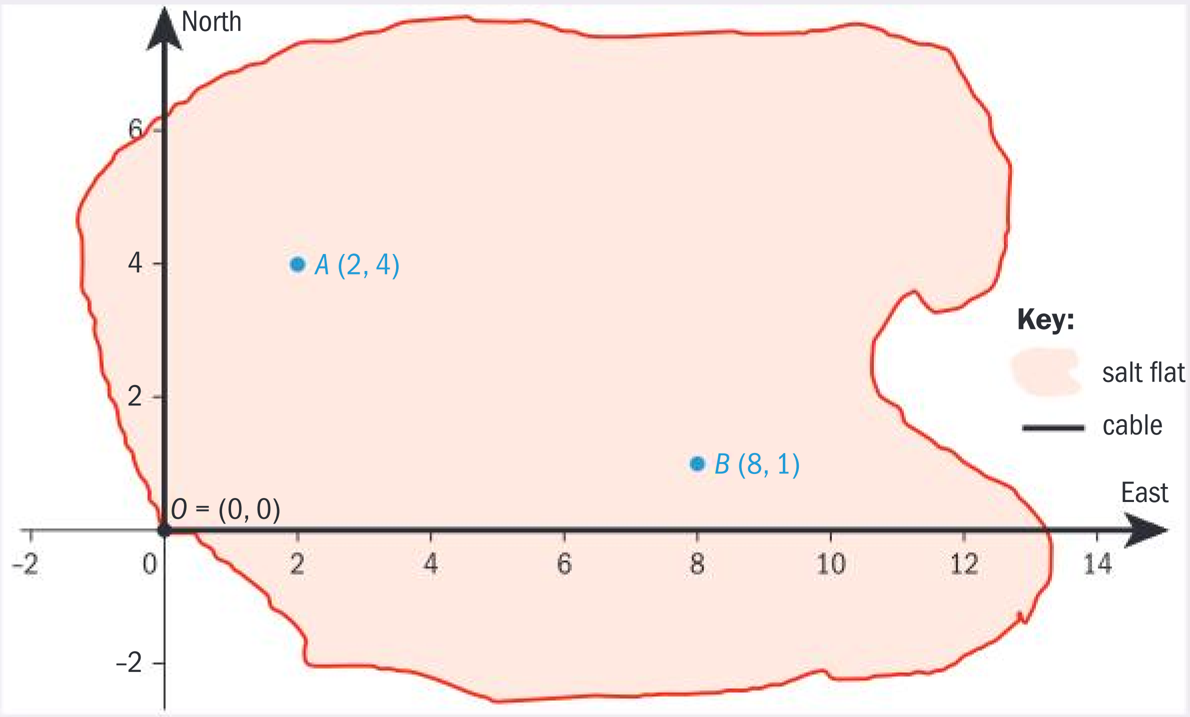
Geometric properties such as parallel and perpendicular lines were studied by Euclid in 300 BC. But it was in 17th century France that René Descartes made the powerful link between geometry and algebra for the first time. This means that you can apply your knowledge and understanding of algebra to solve problems in geometry – and with graphs.

For example, the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ for the gradient of the line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$, can be adapted to the **equation** $m = \frac{y - y_1}{x - x_1}$, which is satisfied by **all** points $(x, y)$ which lie on a straight line through $(x_1, y_1)$ with gradient $m$.

This can be written as $y - y_1 = m(x - x_1)$, the **point-gradient form** of the equation of a straight line. When this is rearranged to the form $y = mx + c$, this form is called the **gradient-intercept** form because the coordinates of the $y$-intercept can be written down as $(0, c)$.

## Example 2.1.2

A solar electricity power station is planned on a desert salt flat. The entrance to the salt flat is at $O(0, 0)$ from where electricity supply cables run due north and east from $O$ as shown. Two solar panel units are installed at points $A(2, 4)$ and $B(8, 1)$. A straight line cable passing through $A$ and $B$ is planned to connect each unit with each supply cable.



(a) Find the gradient of [$AB$]

(b) Hence find the equation of the straight line $l$ passing through points $A$ and $B$ in the form $y = mx + c$

(c) Hence write down the coordinates of the axis intercepts of $l$.

(d) Two further solar panel units are planned to be at points $C$ (0.9, 4.65) and $D$ (6.4, 1.8). Show that exactly one of these points lies on $l$.

**Solution**

(a) [$AB$] has gradient $\frac{4-1}{2-8} = \frac{3}{-6} = -\frac{1}{2}$

Write the gradient in its simplest form.

(b) $l$ has equation $y - 1 = -\frac{1}{2}(x - 8)$, which can be simplified to $y = -\frac{1}{2}x + 5$

Apply the point-gradient form since you know the gradient and the coordinates of two points on $l$.

(c) The $y$-intercept of $l$ is (0, 5). The $x$-intercept is found by solving $-\frac{1}{2}x + 5 = 0$, which gives $x = 10$, so the coordinates of the $x$-intercept of $l$ are (10, 0)

The question asks explicitly for the coordinates.

(d) For $C$, $y = -\frac{1}{2}(0.9) + 5 = 4.55 \neq 4.65$ hence $C$ is not on $l$.

For $D$, $y = -\frac{1}{2}(6.4) + 5 = 1.8$, hence $D$ is on $l$.

For a point to be on $l$, both the $x$-coordinate and the $y$-coordinate **must** satisfy the equation.

A third form of the equation of a straight line is the **general form** $ax + by + d = 0$. Often, this is a convenient way to express the equation, especially if $a$, $b$ and $d$ are integers. For example, the equation for $l$, which is $y = -\frac{1}{2}x + 5$, can be written as $2y = -x + 10$, hence $l$ is $x + 2y - 10 = 0$ in general form. Again, a point is on $l$ only if its coordinates satisfy the equation.

For $D(6.4, 1.8)$, $6.4 + 2(1.8) - 10 = 6.4 + 3.6 - 10 = 0$, confirming that $D$ is on $l$.

**Assessment tip**

Your answer to the question "Find the equation of the line $l$" should never begin "$l =$". This is incorrect use of notation because $l$ is not a variable. You write "$l$ has equation" and then write the equation in one of the three forms.

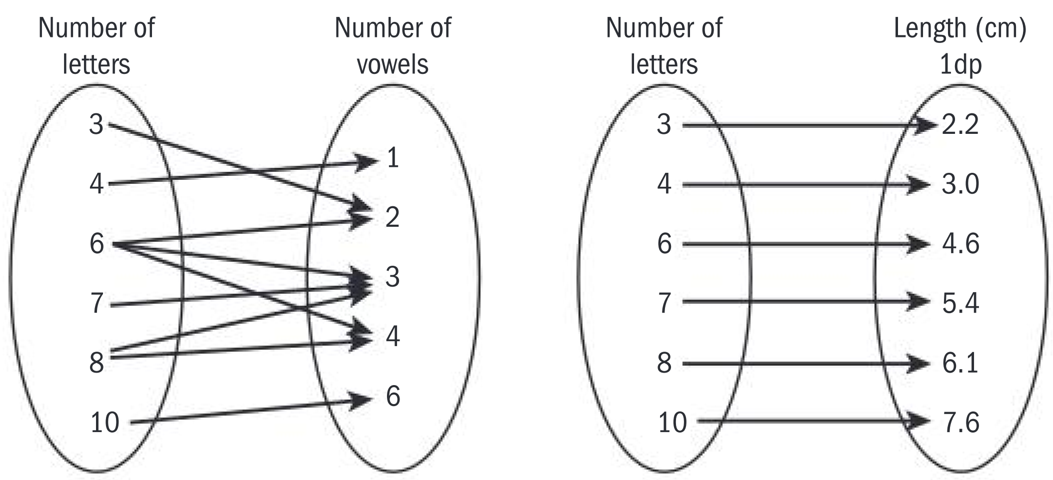
## Concept of a function

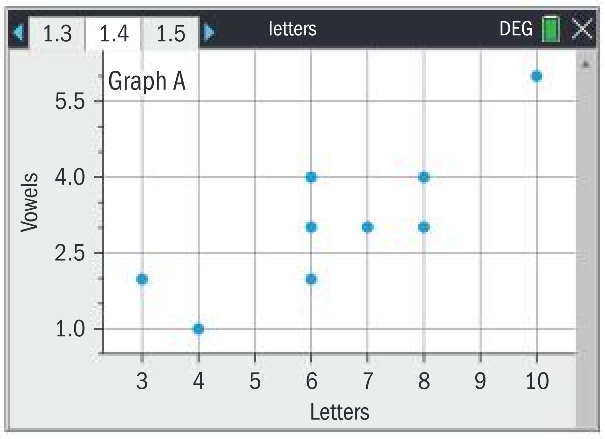
The concept of a function gives you a powerful way to communicate relationships and make predictions. For example, Acacia collects the first names of students in her class. She wants to know if relationships exist between the number of vowels, the number of letters and the length to the nearest mm of each name when printed in `Courier New` font (size 36). Acacia presents her data in this table:

| Name | Letters | Vowels | Length |
|---|---|---|---|
| `Adrian` | 6 | 3 | 4.6 |
| `Anastasiia` | 10 | 6 | 7.6 |
| `Barbora` | 7 | 3 | 5.4 |
| `Catarina` | 8 | 4 | 6.1 |
| `Nicole` | 6 | 3 | 4.6 |
| `Carolina` | 8 | 4 | 6.1 |

| Name | Letters | Vowels | Length |
|---|---|---|---|
| `Jingyuan` | 8 | 3 | 6.1 |
| `Steven` | 6 | 2 | 4.6 |
| `Rea` | 3 | 2 | 2.2 |
| `Zank` | 4 | 1 | 3.0 |
| `Madita` | 6 | 3 | 4.6 |
| `Acacia` | 6 | 4 | 4.6 |

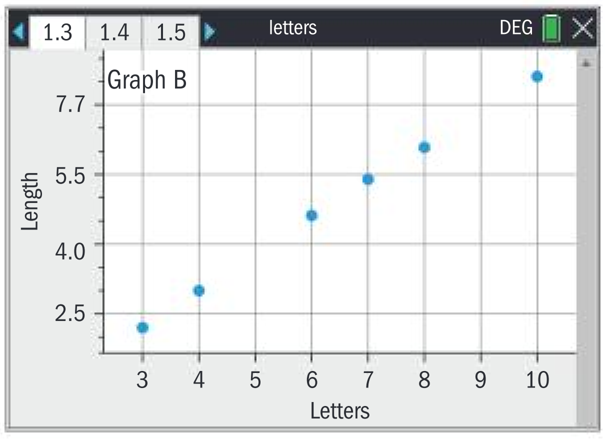
Acacia sees from the table that the greater the number of letters in the name, the longer the name in cm. The relationship between the numbers of letters and numbers of vowels is not so clear, so she draws mapping diagrams to explore further, and starts to perceive the differences in the relationships:



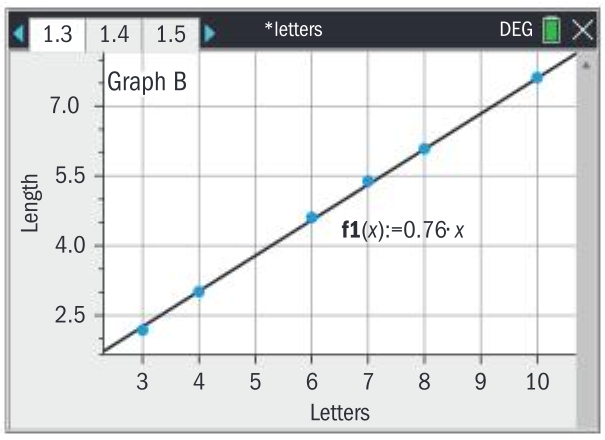




Acacia can see now that the relationship between the number of letters and the length is **one-to-one** because each different number of letters maps to exactly one length.

It was Descartes who had the idea to adapt the mapping diagram by forming a coordinate system, drawing perpendicular axes and plotting points to show the relative sizes of each variable on a grid. This forms a graph. Acacia uses her GDC to create graphs.




Graph B is the graph of a **function** because no two different points have the same $x$-coordinate. Graph A is the graph of a **relation.** This means that the variables are related, but there are at least two different points with the same $x$-coordinate. The set of $x$-coordinates (representing the number of letters in the word) is the **domain** $D = \{3, 4, 6, 7, 8, 10\}$ of the function $L$ (where $L$ represents length of word). The set of $y$-coordinates $\{2.2, 3.0, 4.6, 5.4, 6.1, 7.6\}$ is the **range** of $L$.




Acacia researches the context. She finds that `Courier New` is a *monospaced* font, meaning that each character occupies the *same* horizontal space. Hence $L$ is directly proportional to $n$. Acacia applies the fact that a name with 10 letters has length 7.6 cm to write down the function $L(n) = 0.76n$, where $L(n)$ is the length of a name with $n$ letters. This function is defined for all real numbers, however the domain of $L$ is $D = \{3, 4, 6, 7, 8, 10\}$ in the context of Acacia's class. Acacia can now predict the length using $L$ as a **mathematical model** of the context she is exploring: that is, she uses **mathematics to represent the real-world context.**

> A function $f$ gives a unique value $y = f(x)$ for each $x$ in the domain of $f$. This enables you to make predictions. $x$ is the independent variable and $y$ the dependent variable.

## Assessment tip

The domain of a function will be the largest possible domain for which the function is defined, unless otherwise stated.

### Example 2.1.3

Acacia widens her research of the lengths of first names to a global context, finding names as short as one letter and as long as 20. She models this context with $G(n) = 0.76n$ which gives the length in cm of a name with $n$ letters printed in `Courier New` font (size 36).

The domain is $\{1 \le n \le 20 \mid n \in \mathbb{N}\}$, meaning that $n$ is restricted to a natural number.

(a) (i) Write down the range of $G$.

(ii) Write down the length of the longest possible name.

(b) Predict the length of the name `Oluwapamilerinayo`

(c) A silver competition cup has spaces for winners' names to be engraved. The length of the space is limited to 10.8 cm. What is the maximum number of letters that the space can contain if the engraving machine uses `Courier New` font (size 36)?

**Solution**

(a) (i) The range of $G$ is $\{0.76n \mid 1 \le n \le 20, n \in \mathbb{N}\}$

The range of $G$ is the set of "outputs" of the function $L$.

(ii) $L(20) = 15.2$ cm

You replace the value of the independent variable in the equation to find the value of the function.

(b) The name has 17 letters, hence the length is $L(17) = 12.92$ cm

(c) $L(n) = 10.8$ has solution

$$n = \frac{10.8}{0.76} \approx 14.2$$

The maximum number of letters the space can contain is 14.

14.2 is not in the domain of $L$ but you interpret this number in the context of the problem.

With every operation you learn in mathematics, it is extremely useful to know how to find its inverse. You learned at an early age that addition and subtraction are opposite operations, as are multiplication and division. When you make an error on a computer, Ctrl+Z will "undo" your error. Similarly, **inverse functions** reverse or undo the effect of a function.

**Note**

Always use correct notation for the equation of a function. For example, $L(n) = 10.8$ is correct but $L = 10.8$ is not.

The inverse of the function $f(x)$ is denoted $f^{-1}(x)$. Do not confuse this with the reciprocal of $f(x)$, which is $(f(x))^{-1} = \frac{1}{f(x)}$.

**Assessment tip**

In internal assessment, use of a convenient letter to denote a function can help your mathematical communication – for example $v(t)$ for velocity in terms of time, or $C(n)$ for the cost of manufacturing $n$ products.

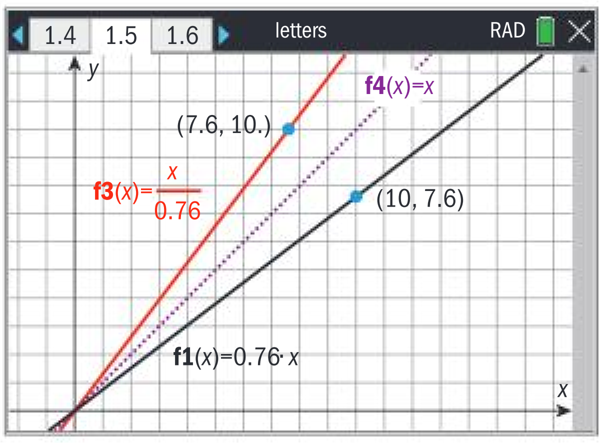
You can apply $L$ to find the length for a given number of letters. But imagine you were asked instead to find the number of letters in a name of length 7.6 cm.

When you write $L(n) = 7.6$, this is equivalent to

$$0.76n = 7.6 \Rightarrow n = \frac{7.6}{0.76} = 10 \text{ letters.}$$

This shows that the expressions $L(10) = 7.6$ and $10 = L^{-1}(7.6)$ are equivalent, where $L^{-1}$ is the inverse function of $L$.

Therefore, (10, 7.6) is on the graph of $L$ and (7.6, 10) would be on the graph of $L^{-1}$ as shown. These points are reflections of each other in the line $y = x$.

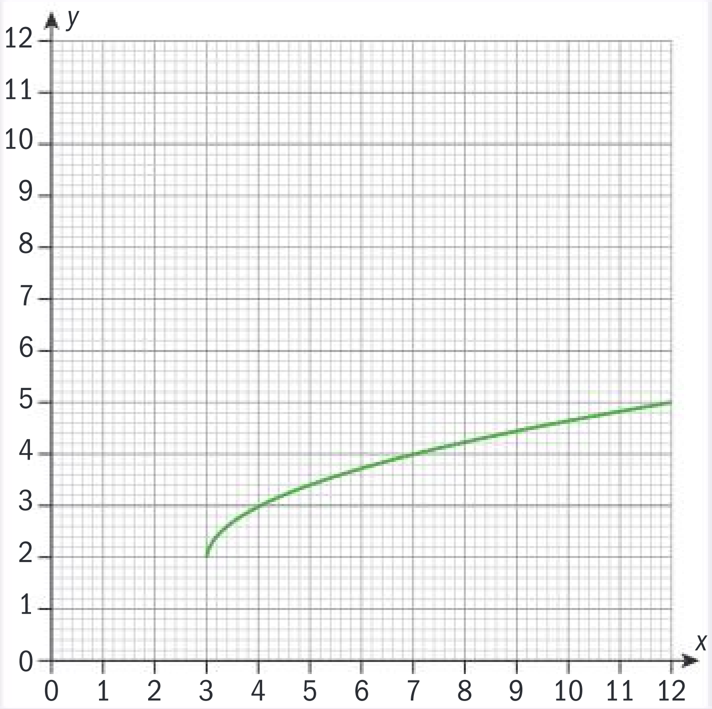


In general, if $(n, L(n))$ is on the graph of $L$, then $(L(n), n)$ is on the graph of $L^{-1}$.

Hence, the graphs of $L^{-1}$ and $L$ are reflections of each other in the line $y = x$.

### Example 2.1.4

Part of the graph of $f(x) = \sqrt{x-3} + 2$ is shown below.



(a) Use the graph to explain why the domain of $f$ is $x \geq 3$ and the range is $f(x) \geq 2$

(b) Draw the graph of $f^{-1}$ on the grid.

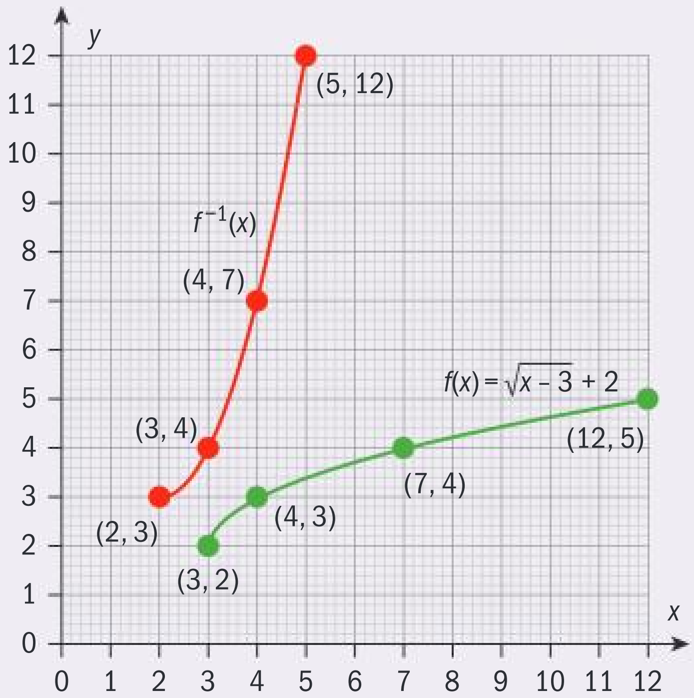
(c) Hence or otherwise determine the domain and the range of $f^{-1}$

**Solution**

(a) $f$ is only defined for $x \geq 3$, as shown by the end point (3, 2) on the graph of $f$. The graph continues to increase outside the given window, hence the range is $f(x) \geq 2$.

Predict what the function looks like outside the viewing window. You can change the window to check your prediction. Logically, $x - 3$ must be greater than zero for $f$ to have a real value.

(b)




The command term is “Draw”, so points must be plotted accurately and joined up with a smooth curve.

(c) The domain of $f^{-1}$ is $x \geq 2$ and the range is $f^{-1}(x) \geq 3$.

This can be seen from the graph or written down directly from (a) since the domain of $f^{-1}$ is always the range of $f$, and the range of $f^{-1}$ is always the domain of $f$.

Note that if $f(x)$ was not one-to-one then its reflection in the line $y = x$ would not be the graph of a function. For the inverse of $f(x)$ to be a function, $f(x)$ must be one-to-one.

The domain of $f^{-1}(x)$ is always equal to the range of $f(x)$.

# 2.2 COMPOSITE FUNCTIONS AND INVERSE FUNCTIONS (AHL)

## You should know:

- ✔ the definition of composite functions
- ✔ the notation $(f \circ g)(x) = f(g(x))$
- ✔ that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

## You should be able to:

- ✔ find the inverse of a function, restricting the domain if necessary
- ✔ apply composite functions in context.

Just as two or more numbers can be combined in many ways to create another number, functions can be combined to make another function, for example by adding, subtracting or multiplying two or more functions together. **Composite functions** express when one function is applied *after* another: when $a = g(x)$ is found, and then $y = f(a)$ is found.

This process defines a new function $f(g(x))$, and defines the operation "$\circ$" with the equation $(f \circ g)(x) = f(g(x))$. It is helpful to understand $g$ as the "inside" function and $f$ as the "outside" function.

> $g$ is applied first in finding $(f \circ g)(x)$

### Example 2.2.1

(a) For the functions $f(x) = 65x$ and $g(x) = 2.3x - 315$, determine if $f \circ g = g \circ f$.

Erin owns a café in which the number of coffees sold is modelled by the function $c(t) = 65t$, where $t$ is in hours her café is open and $0 \le t \le 10$. The profit earned is modelled by the function $p(c) = 2.3c - 315$, where $c$ is the number of coffees sold.

(b) Identify which of the functions $c \circ p$ or $p \circ c$ models Erin's profit as a function of time.

(c) Find the smallest number of hours $h$, where $h \in \mathbb{Z}^+$, for which Erin's café needs to open in order to make a profit.

**Solution**

(a) $(f \circ g)(x) = f(2.3x - 315) = 65(2.3x - 315)$
$= 149.5x - 20\,475$
$(g \circ f)(x) = g(65x) = 2.3(65x) - 315$
$= 149.5x - 315$

Hence $f \circ g \neq g \circ f$

Apply the definition $(f \circ g)(x) = f(g(x))$ carefully, starting with the inside function.

This fact is true in general. Function composition is not commutative.

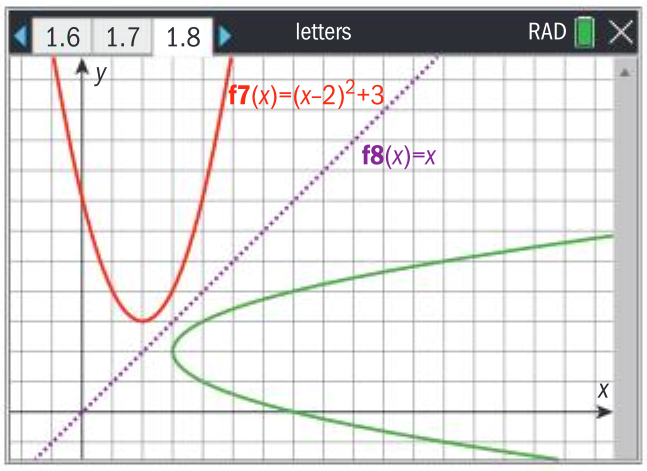
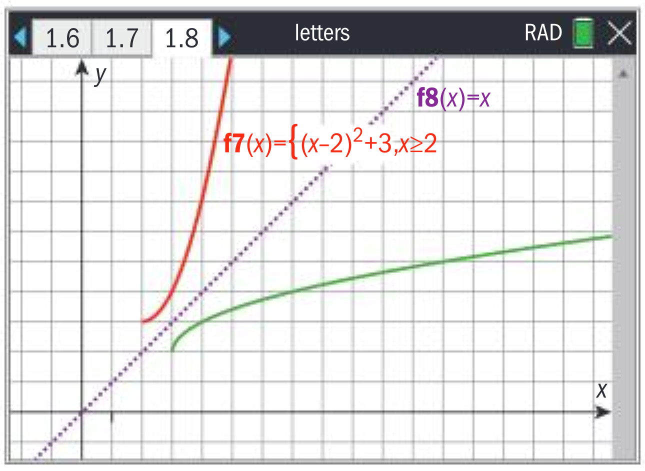
(b) $p \circ c = 149.5t - 315$ models Erin's profit because the independent variable is time.

In fact, $c \circ p$ does not have an application in this context.

(c) $149.5t - 315 > 0 \Rightarrow t > 2.11$ hours. Erin will need to be open for 3 hours in order to make a profit. Hence $h = 3$.

Being open for 2 hours would mean making a loss.

To find the inverse of a function, you first examine its graph.

| | |
|---|---|
| For example, the graph of $f(x) = (x - 2)^2 + 3$ shows that when reflected in $y = x$ the graph is that of a relation: |   |
| In order to find the inverse function, the domain of $f$ must be *restricted* so that it is one-to-one.<br>For example, if you restrict the domain of $f$ to $x \geq 2$, you now see the graphs of two functions: $f$ and $f^{-1}$.<br>(The restriction $x \leq 2$ would have the same effect)<br>The final step is to find the equation of $f^{-1}$. |   |
| To find the inverse of a function: | |
| (a) Write down the equation $y = f(x)$ | $y = (x - 2)^2 + 3$ |
| (b) Transpose $x$ for $y$ | $x = (y - 2)^2 + 3$ |
| (c) Solve the equation for $y$ | $x - 3 = (y - 2)^2$<br>$\sqrt{x-3} = y - 2$<br>$\sqrt{x-3} + 2 = y$ |
| (d) Hence write down the equation $f^{-1}(x) = y$ | $f^{-1}(x) = \sqrt{x-3} + 2$ |

**Assessment tip**

You can transpose $x$ for $y$ at the start of the process or at the end and still be awarded full marks.

Note that if the restriction $x \leq 2$ was made, you would take the negative root and find $f^{-1}(x) = -\sqrt{x-3} + 2$. In both cases, the identity $(f \circ f^{-1})(x) = x$ is true.

For example,

$$\begin{aligned}(f \circ f^{-1})(x) = f(\sqrt{x-3} + 2) &= (\sqrt{x-3} + 2 - 2)^2 \\ &= (\sqrt{x-3})^2 + 3 \\ &= x - 3 + 3 = x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) = f^{-1}((x-2)^2 + 3) &= \sqrt{(x-2)^2 + 3 - 3} + 2 \\ &= \sqrt{(x-2)^2} + 2 \\ &= x - 2 + 2 = x\end{aligned}$$

This is consistent with the definition of an inverse function.

Although it is not mentioned in the syllabus, the concept of the identity function is useful here: $i(x) = x$. This function has no effect at all on the value of $x$. Note that $(f^{-1} \circ f)(x) = i(x)$ reflects the fact that $f^{-1}$ must reverse or "undo" the effect of $f$: hence the composition has no effect.

**SAMPLE STUDENT ANSWER**

Two functions, $f$ and $g$, are defined in the following table.

| $x$ | –2 | 1 | 3 | 6 |
|---|---|---|---|---|
| $f(x)$ | 6 | 3 | 1 | –2 |
| $g(x)$ | –7 | –2 | 5 | 9 |

(a) Write down the value of $f(1)$

(b) Find the value of $(g \circ f)(1)$

(c) Find the value of $g^{-1}(-2)$

(a) $f(1) = 3$

(b) $g(f(x)) = g(f(1)) = 5$

(c) $g(1) = -2$

hence: $g^{-1}(-2) = 1$

▲ Because the student correctly applies the notation of composite functions, the value of $(g \circ f)(1)$ can be found easily from the table.

▲ Knowledge that solving $g(x) = -2$ is equivalent to finding $g^{-1}(-2)$ is applied correctly.

# 2.3 GRAPHS AND FEATURES OF MODELS

## You should know:

- ✔ the difference between the command terms "draw" and "sketch"
- ✔ the meaning of the key features: maximum and minimum values, intercepts, vertex, zeros of functions or roots of equations; vertical and horizontal asymptotes
- ✔ the names, equations and features of linear, quadratic, exponential, cubic and sinusoidal models
- ✔ the difference between direct and inverse variation
- ✔ the formulae for direct and for inverse variation
- ✔ the stages of the modelling process.

## You should be able to:

- ✔ use technology to graph functions including their sums and differences
- ✔ use technology to find points of intersection
- ✔ determine the key features of a graph
- ✔ transfer a graph from a screen to paper
- ✔ create a sketch of a graph from information given, labelling axes and key features to give a general idea of the shape
- ✔ find the parameters of a model by setting up and solving a system of equations using technology or by substitution of points into a given function
- ✔ apply the stages of the modelling process.

When asked to "Draw" the graph of a function, to gain maximum marks you must represent the graph accurately, with points correctly plotted and joined in a straight line or smooth curve. Axes must be labelled to show the scales and independent and dependent variables. Axes and straight lines should be drawn with a ruler. You draw the graph on graph paper.

**Assessment tip**

Using the scales given in the question is essential so that the graph fits on the graph paper.

When asked to **sketch** the graph of a function, you represent the general idea of the shape of the relationship. Relevant features such as maximum or minimum points, asymptotes and axes intercepts should be approximately in the correct location. An indication of scale must be given. There is not the same requirement of accuracy as in **draw** however you should take care to make the sketch shows relevant features in their correct places.

### Example 2.3.1

Two paths are planned in a forest.

Path A is modelled by the equation $y = 0$, running east from the point (0, 0) to (8, 0). Path B is modelled by $f(d) = -e^{-0.1d} \sin(90d)$, $0 \le d \le 8$. Distances are measured in km and $d$ is the distance east from (0, 0).

The planners of the garden want to install a water fountain at each point where the paths cross and at each point where the second path turns.

(a) Sketch the graphs of $y = 0$ and $y = f(d)$ for $0 \le d \le 8$, showing all the maximum points, minimum points and zeros of $f$.

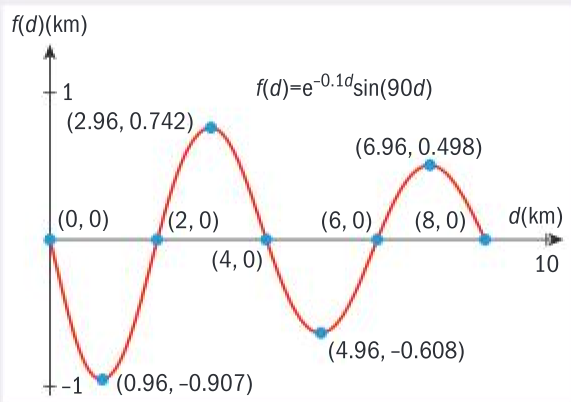
(b) Hence find the number of water fountains planned and the coordinates of the water fountain that is furthest from path A.

**Assessment tip**

Use the domain and range given in the question to determine the viewing window on your GDC.

**Solution**

(a)




Use the given domain for the maximum and minimum $d$ values. Adjust the maximum and minimum values on the $y$-axis so that you can see the graph in detail.

Show the relevant points in their correct places and include points at each end of the domain if they are relevant to the question. Give a sense of scale on the axes.

**Assessment tip**

You can display the graph with a grid background in order to make transferring the graph from your GDC screen to paper easier.

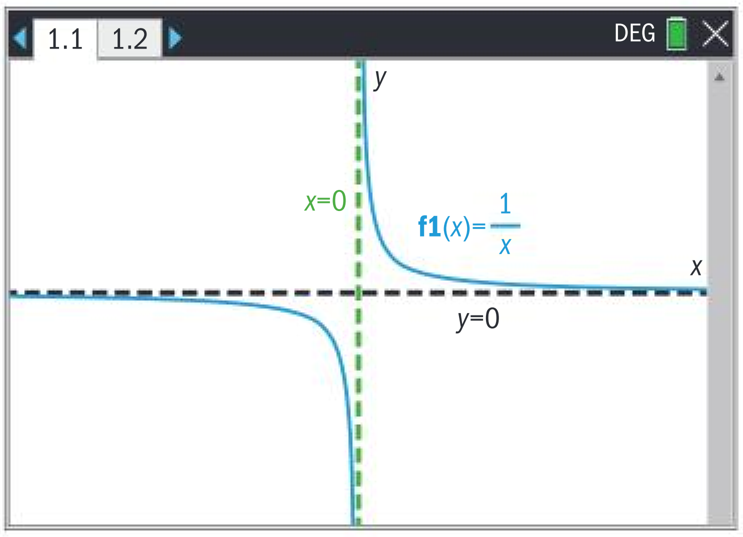
(b) Hence the number of fountains planned is 9. The fountain furthest from path A has coordinates (0.96, –0.907).

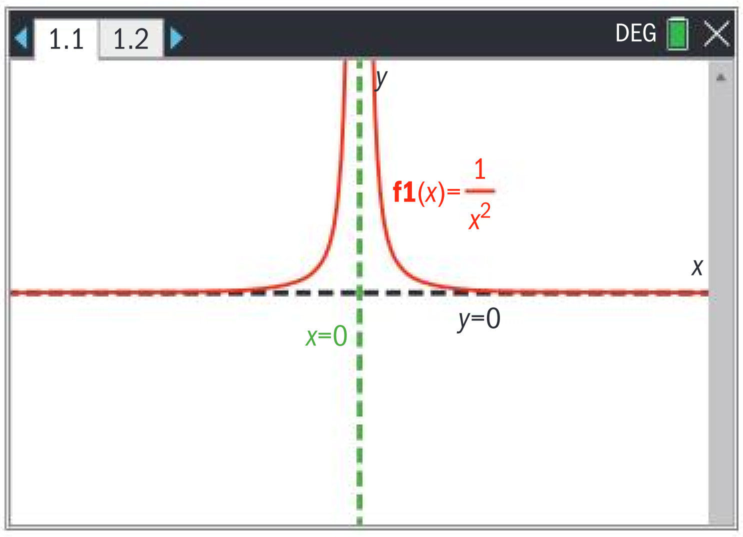
There are other key features of a graph that you determine with technology as follows:

| **Feature of the graph of $y = f(x)$** | **How to determine** |
|---|---|
| $y$-intercept | Find $f(0)$ |
| $x$-intercept(s), zeros of $f(x)$, roots of the equation $f(x) = 0$ | Solve $f(x) = 0$ or use your GDC's graphing app. |
| Vertex, maximum and minimum points | Use your GDC's graphing app or apply your knowledge and understanding of calculus. |

**Asymptotes:** Functions of the form $f(x) = \frac{a}{x^n}$, $n \in \mathbb{Z}^+$ have a vertical asymptote with equation $x = 0$. This is a consequence of the fact that the domain of $f(x)$ is $x \neq 0$.

The horizontal asymptote is $y = 0$ because the range of $f(x)$ is $y \neq 0$. You represent asymptotes with dashed lines because they are not part of the graph of the function $y = f(x)$.




A horizontal asymptote shows the long-term behaviour of the function, also known as the limit of $y = f(x)$ as $x$ increases in size. Asymptotic behaviour can be suggested by the shape of a graph, and determined by further exploration with your GDC as shown in Example 2.3.2

### Example 2.3.2

The temperature in a house $t$ hours after an electrical heating unit turns on is modelled by the function $c(t) = 23te^{-1.3t} + 19$

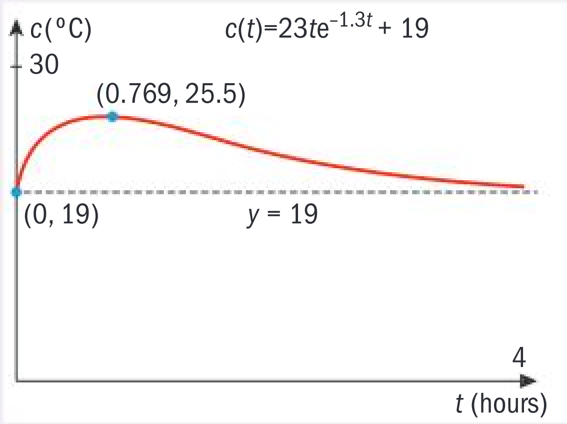
(a) Sketch the graph of $y = c(t)$ for $0 \leq t \leq 4$, showing the coordinates of the $y$-intercept, the maximum point and the horizontal asymptote.

(b) Hence describe how the temperature in the house changes in the four hours after the heating unit turns on.

(c) The heating unit switches off when the temperature reaches 24 °C but continues to emit heat as it cools. Find the value of $t$ when the heating unit switches off.

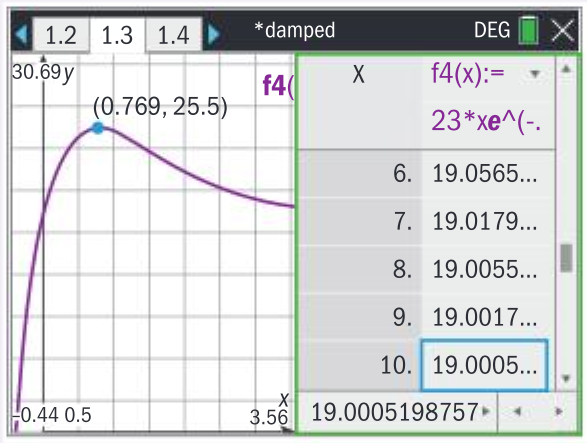
Solution

(a)




You can explore the behaviour of the function outside the given domain with a table or by widening the viewing window.

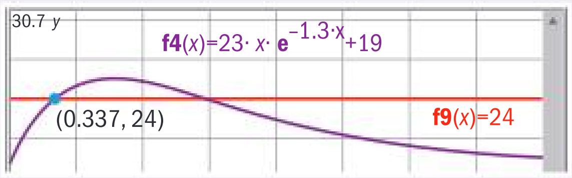
Both perspectives show that the limiting value is 19, hence the equation of the asymptote is $y = 19$.




(b) At $t = 0$, the temperature is 19 °C and the heater switches on. The temperature increases until 0.769 hours have passed, reaching a maximum of 25.5 °C before it starts to decrease, getting gradually closer to 19 °C.

Give a detailed account of the changes in temperature, using your answer to (a).

(c)




After 0.337 hours, the heating unit switches off.

Use your GDC to find the intersection of $y = c(t)$ and $y = 24$.

**Assessment tip**

Don't limit your view of the function to only what your GDC first shows you: exploration of the behaviour of the function outside the viewing window can give you valuable insights.

**Assessment tip**

When sketching or drawing asymptotic behaviour, take care to show the shape correctly. The graph must be seen to get gradually closer to an asymptote. It should not veer away from nor cross the asymptote.

Once you have entered two functions into your GDC, it can graph the sum of the functions, or their difference, without you having to type in the equations again, as shown in Example 2.3.3

## Example 2.3.3

Xtar Jewellery models the cost in £ of making $x$ pairs of limited edition earrings with the function $c(x) = 3500 + 26.5x$, and the revenue earned by selling the $x$ pairs of earrings with the function $r(x) = 77.6x - 0.08x^2$.

(a) Find the least number of pairs of earrings that must be sold in order for Xtar to make a profit.

(b) By considering the graph of $p(x) = r(x) - c(x)$ or otherwise, find the number of earrings that will maximize Xtar's profit.

**Solution**

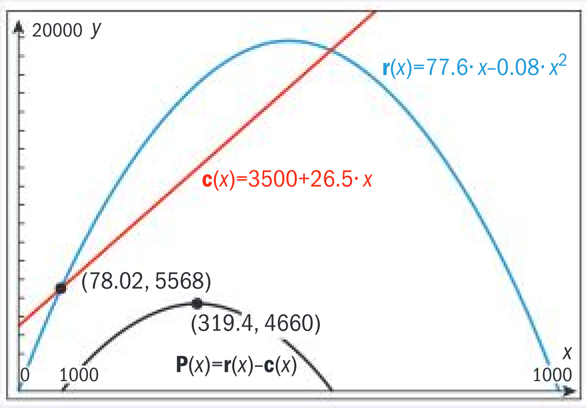
(a) The least number of pairs of earrings Xtar must sell is 79.

(b) Since 319.4 is not an integer, the profit for 319 and 320 must be found:

$p(319) = 4660.02$
$p(320) = 4600.00$

Xtar will make the most profit by making and selling 319 pairs of earrings.

In order to make a profit, revenue must be greater than the cost. The graphs show that cost and revenue are equal at $x = 78.02$




Since you have entered the functions already, you type the expression $p(x) = r(x) - c(x)$ and the calculator provides the graph required.

**Assessment tip**

In order to find the vertex of a quadratic function, you can apply the formula for the axis of symmetry, use calculus or technology. In the context of the problem, choose the most efficient method.

**SAMPLE STUDENT ANSWER**

Let $f(x) = e^{0.5x} - 2$, for $-4 \le x \le 4$

(a) For the graph of $f$:

(i) write down the $y$-intercept

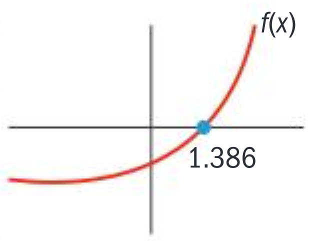
(ii) find the $x$-intercept

(iii) write down the equation of the horizontal asymptote.

(b) Sketch the graph of $f$.

(a) (i) $y = -1$ the $y$-intercept is $(0, -1)$

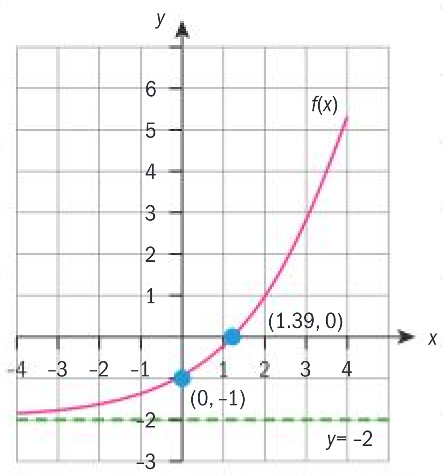
(ii) $f(x) = 0 = e^{0.5x} - 2$




$\therefore$ $x$-intercept is $(1.39, 0)$

(iii) $y = -2$

(b)

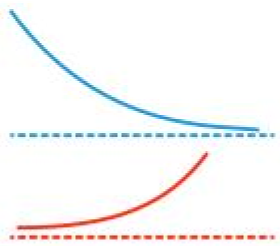
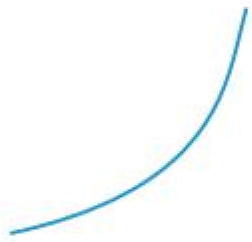
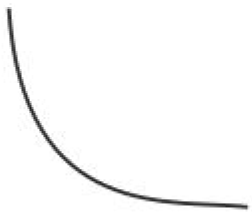
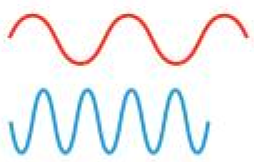



▲ The student expresses the intercepts and equation of the asymptote with the correct notation and to the level of accuracy required.

▲ The sketch shows the main features of the graph on the domain given, and the asymptotic behaviour is sketched well, hence full marks are awarded.

## Types of models

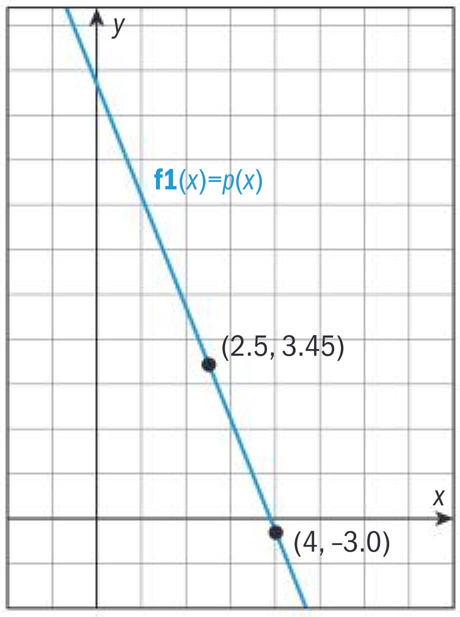
You have seen in previous sections examples of mathematics being used to represent real-world contexts by application of mathematical models. There are several examples of models you must know, summarized in the table here.

| Name of model | Equation | One example of application | Name of shape and sketches of examples of shape | One distinguishing feature |
|---|---|---|---|---|
| **Linear** | $f(x) = mx + c$ | Simple interest | Straight line  | Constant gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$ |
| **Quadratic** | $f(x) = ax^2 + bx + c$ | Projectile motion | Parabola  | Axis of symmetry $x = \frac{-b}{2a}$ |
| **Exponential** | $f(x) = ka^x + c$<br>$f(x) = ka^{-x} + c$<br>$f(x) = ke^{rx} + c$ | Compound interest, depreciation | Exponential growth/ decay  | Horizontal asymptote $y = c$ |
| **Direct variation** | $f(x) = ax^n$, $n \in \mathbb{Z}^+$ | In science, for example Charles' law |  | As $x$ increases, $f(x)$ increases |
| **Inverse variation** | $f(x) = ax^n$<br>where $n \in \mathbb{Z}$, $n < 0$ | In science, for example Boyle's law |  | As $x$ increases, $f(x)$ decreases |
| **Cubic** | $f(x) = ax^3 + bx^2 + cx + d$ | Volume of a box optimisation problems | Cubic  | Can have a local maximum **and** a local minumum. |
| **Sinusoidal** | $f(x) = a\sin(bx) + d$<br>$f(x) = a\cos(bx) + d$ | Average monthly temperatures | Sine wave  | Periodic with period $= \frac{360}{\lvert b \rvert}$ |

Each model represents a relationship between the $x$ and $y$ variables. Each model has parameters that determine the equation, and these parameters determine the features of the graph. For example, the model $f(x) = mx + c$ has parameters $m$ and $c$ which determine the gradient and the $y$-intercept respectively. For each model, there are ways to determine these parameters so that the model can be determined and then applied to make predictions.

### Linear models

Information given:




To determine the parameters:

(1) First find $m = \frac{y_2 - y_1}{x_2 - x_1}$ using the two points.

(2) Replace the $x$- and $y$-coordinates of any point on the line into the equation $p(x) = mx + c$ to find $c$.

Or:

(1) Replace the $x$- and $y$-coordinates of the two points in the equation $p(x) = mx + c$.

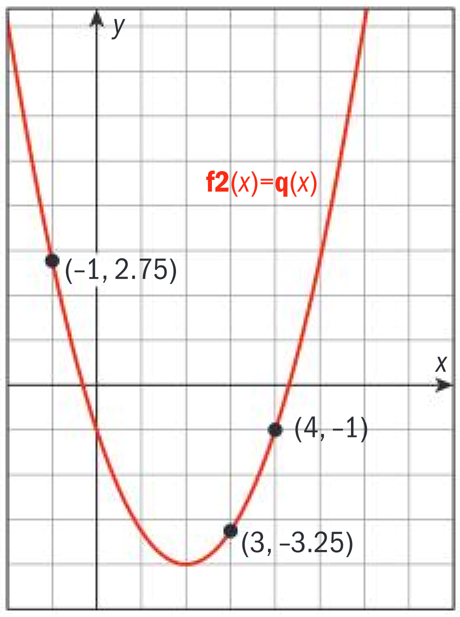
(2) Hence write down a system of two simultaneous equations in $m$ and $c$.

(3) Use your GDC to solve the system.

Or: If a data set is given, use the linear regression tool on your GDC.

### Quadratic models

Information given:




To determine the parameters:

(1) Replace the $x$- and $y$-coordinates of the three points in the equation $q(x) = ax^2 + bx + c$

(2) Hence write down a system of three simultaneous equations in $a$, $b$ and $c$.

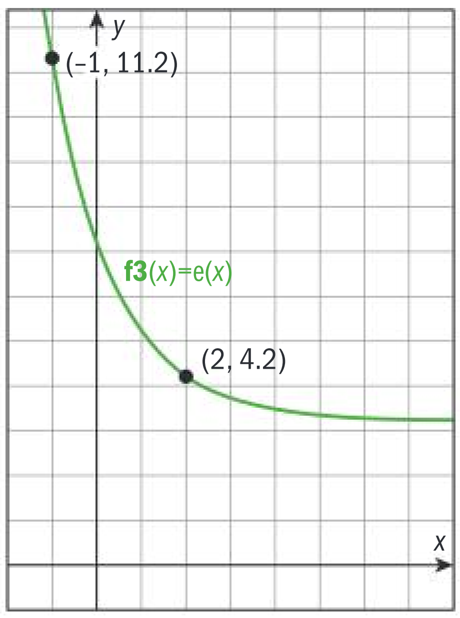
(3) Use your GDC to solve the system.

If one of the parameters is given, you write down and solve a system of two simultaneous equations.

(Note that for both the linear and the quadratic model, the coordinates of the $y$-intercept are $(0, c)$ and so if this point is given you just write down the parameter $c$ directly.)

### Exponential models

Information given:




To determine the parameters:

For the models $e(x) = ka^x + c$, $e(x) = ka^{-x} + c$ or $e(x) = ke^{rx} + c$, you will be given at least one of the parameters.

To find the remaining parameters:

(1) Replace the $x$- and $y$-coordinates of the point(s) in the equation for $e(x)$

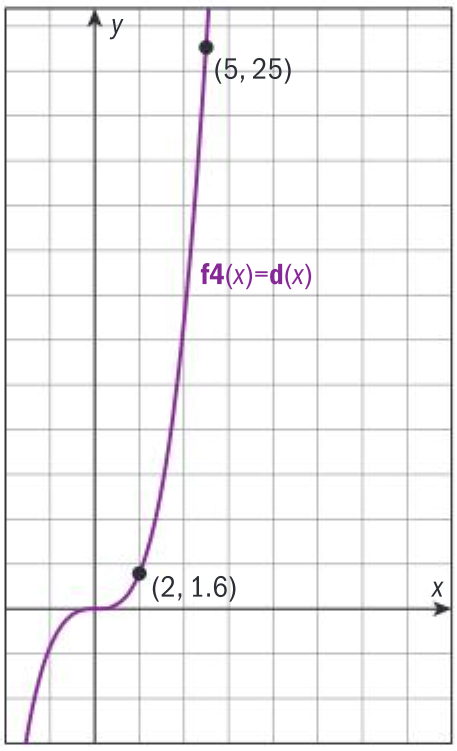
(2) Hence write down equation(s) in the remaining parameter(s)

(3) Solve by hand or with your GDC.

(Note that for the exponential model, the horizontal asymptote is $y = c$, and so if this is given you just write down the parameter $c$ directly. In this model, the horizontal asymptote of $e(x)$ is $y = 3.2$)

**Direct variation**

Information given:




To determine the parameters:

The direct variation model has equation $d(x) = ax^n$ where $n \in \mathbb{Z}^+$. This means that $d(x)$ varies directly as $x^n$.

(1) Replace the $x$- and $y$-coordinates of the points in the equation for $d(x)$

(2) Hence write down equations in the parameters $a$ and $n$.

(3) Solve the equations using the laws of indices and/or logarithms.

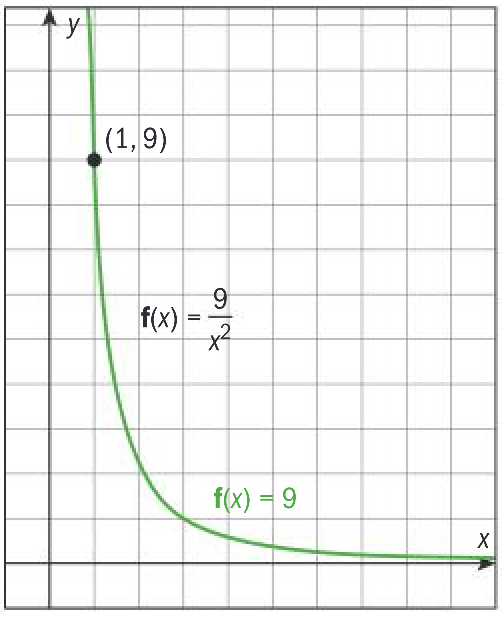
(Note that if one of the parameters is given, you write down and solve only one equation, as shown in the following example of inverse variation.)

**Assessment tip**

In exam questions, $y$ is proportional to $x^n$ is equivalent to stating that $y$ varies directly as $x^n$. Similarly, $y$ is inversely proportional to $x^n$ $(n > 0)$ is equivalent to stating that $y$ varies inversely as $x^n$.

**Inverse variation**

Information given:




To determine the parameters:

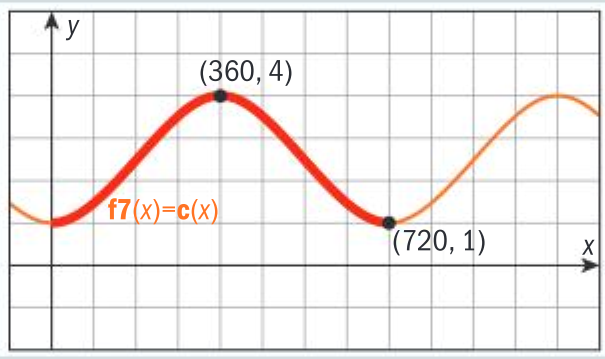
The inverse variation model has equation $i(x) = ax^n$ where $n \in \mathbb{Z}$, $n < 0$. This means $i(x)$ varies inversely as $x^n$.

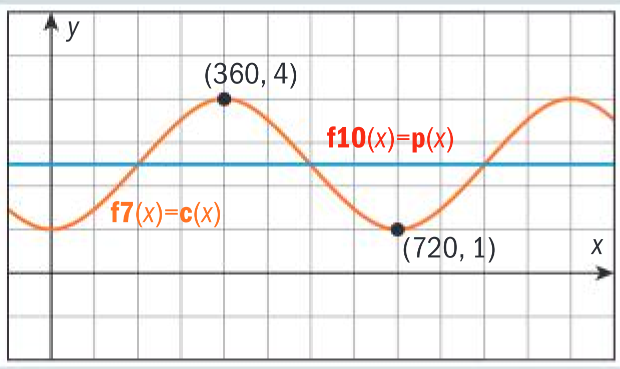
In this example, you are given that $n = -2$

(1) Replace the $x$- and $y$-coordinates of the point in the equation for $i(x)$

(2) Hence write down equations in the remaining parameter.

(3) Solve the equations.




The sinusoidal model has three features for which you must know the terminology:

Sinusoidal models are **periodic**, meaning that they have a repeating pattern.

The graph of $c(x) = -1.5\cos\left(\frac{x}{2}\right) + 2.5$ is shown.

The largest possible domain is $\mathbb{R}$ and the graph can be thought of as a repetition of the red shape. The shortest length of such a shape that can be used to make the whole graph is the **period** of the function.

Hence, the period of $c(x)$ is $720°$.

The horizontal line in the middle of the graph is the **principal axis**.

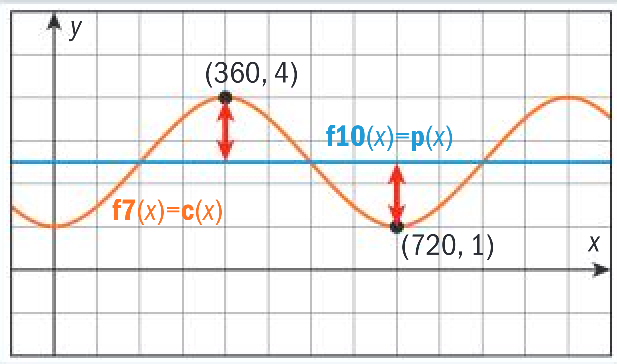
The equation of the principal axis is $y = d$, where $d$ is the average of the maximum and minimum values of the function.

Hence, the principal axis of $c(x)$ is $y = 2.5$ because $\frac{4+1}{2} = 2.5$

The distance from the principal axis to any maximum or minumum point of the sinusoidal function is the **amplitude** of the function.

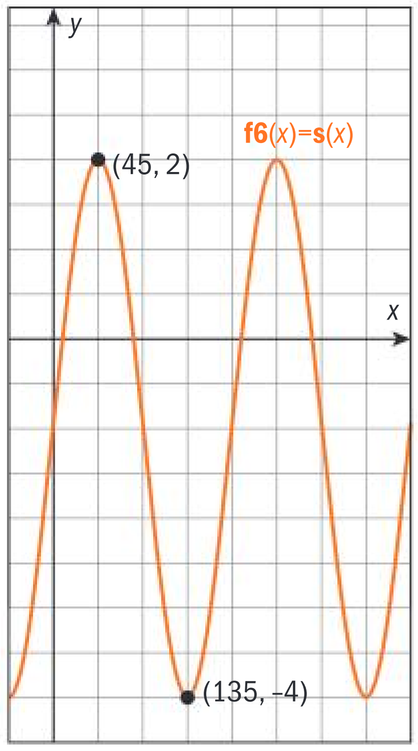
This can be found by subtracting the minimum value from the maximum value and dividing by 2.

Hence, the amplitude of $c(x)$ is 1.5 because $\frac{4-1}{2}=1.5$




**Sinusiodal models**

Information given:




To determine the parameters:

The method is the same for finding the parameters of $s(x) = a\sin(bx) + d$ as for $c(x) = a\cos(bx) + d$.

(1) Identify the maximum and minimum values of $s$ from the information given.

(2) Identify the period of $s$.

(3) Find $d = \frac{max + min}{2}$, which gives you the equation of the principal axis $y = d$.

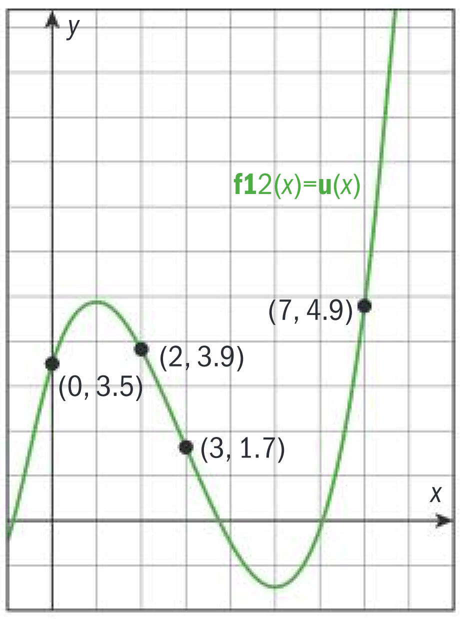
(4) Find $b = \frac{360}{period}$

(5) $|a| = \frac{max - min}{2}$, which gives you the amplitude of $s$.

(6) The sign of $a$ depends on the orientation of the graph. So use your values of $a$, $b$ and $d$ to check your model on your GDC.

**Cubic models**

Information given:




To determine the parameters:

For a cubic function $u(x) = ax^3 + bx^2 + cx + d$, you will always be given at least one of the parameters $a$, $b$, $c$ or $d$.

(1) Replace the $x$- and $y$-coordinates of the points given in the equation $u(x) = ax^3 + bx^2 + cx + d$

(2) Hence write down a system of simultaneous equations.

(3) Use your GDC to solve the system.

(Note that for the cubic model, the $y$-intercept is $(0, d)$ and so if this point is given you just write down the parameter $d$ directly.)

You can practice these methods of finding the parameters with the information given for each model above. Check your answers by graphing the function on your GDC and comparing with the graphs given. The answers are:

$p(x) = 9.7 - 2.5x$ $\qquad q(x) = 0.75x^2 - 3x - 1$

$e(x) = 3.2 + 4(0.5^x)$ $\qquad d(x) = 0.2x^3$

$i(x) = \frac{9}{x^2}$ $\qquad s(x) = 3\sin(2x) - 1$

$u(x) = 0.2x^3 - 1.8x^2 + 3x + 3.5$

SAMPLE STUDENT ANSWER

Jashanti is saving money to buy a car. The price of the car, in US Dollars (USD), can be modelled by the equation

$$P = 8500(0.95)^t$$

Jashanti's savings, in USD, can be modelled by the equation

$$S = 400t + 2000$$

In both equations, $t$ is the time in months since Jashanti started saving for the car.

(a) Write down the amount of money Jashanti saves per month.

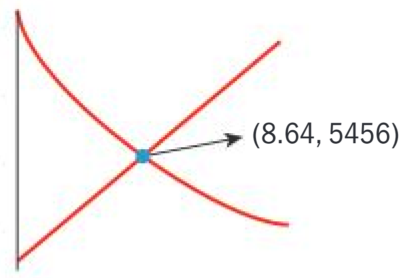
(b) Use your GDC to find how long it will take for Jashanti to have saved enough money to buy the car.

Jashanti does not want to wait too long and wants to buy the car two months after she started saving. She decides to ask her parents for the extra money that she needs.

(c) Calculate how much extra money Jashanti needs.

$y_1 = 8500(0.95)^t$

$y_2 = 400t + 2000$




$= 8.64$ months

$P = (8500)(0.95)^2$

$= 7671.3$

$S = 400(2) + 2000$

$= 2800$

$7671.3 - 2800$

$= 4871.3$ USD

(a) 400 USD

(b) 8.64 months

(c) 4871.3 USD

▲ The GDC is used to find the intersection between the two graphs and the working is shown clearly.

▲ The working is clearly written and correct, gaining full marks. However, writing $8500(0.95^2) - 400(2) - 2000$ then entering it into the GDC would be more efficient.

▲ The student correctly interprets the gradient of the linear function as the rate at which Jashanti saves.

**SAMPLE STUDENT ANSWER**

A population of mosquitoes decreases exponentially. The size of the population, $P$, after $t$ days is modelled by

$$P = 3200 \times 2^{-t} + 50, \text{ where } t \geq 0$$

(a) Write down the exact size of the initial population.

(b) Find the size of the population after 4 days.

(c) Calculate the time it will take for the size of the population to decrease to 60.

The population will stabilize when it reaches a size of $k$.

(d) Write down the value of $k$.

(a) $3200 \times 2 + 50 = 6450$

(b) $P = 3200 \times 2^{-4} + 50$

$P = 250$

(c) $3200 \times 2^{-5} + 50 = 150$

$3200 \times 2^{-7} + 50 = 75$

$3200 \times 2^{-8} + 50 = 62.5$

8 days

(d) ----

▼ The student has either misinterpreted what 'initial' means, or has incorrectly calculated $2^0$.

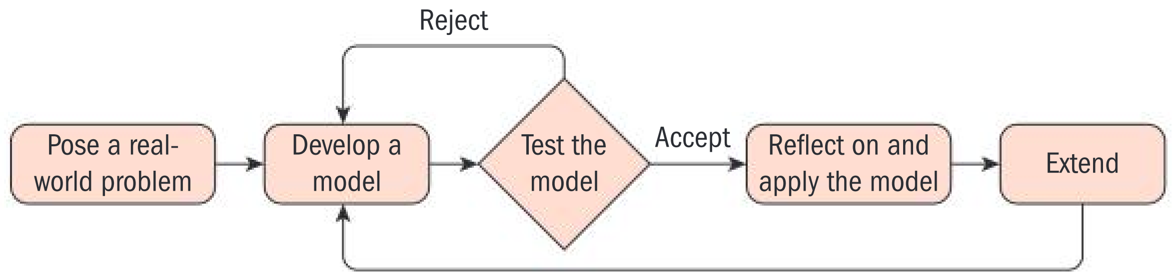
▲ The clear working and correct answer gain full marks.

▼ Use of a GDC to solve the equation $P = 60$ would give the correct answer of 8.32 days.

▼ Use of the GDC to graph the function would show the long-term stable population of 50. Or, this could be written down directly from the formula for $P$.

## The modelling process

The modelling process is described by this cycle of continuous improvement:




The stages of the modelling process can be unpacked into skills that help you in both paper 1 and paper 2, as well as your internal assessment:

(a) **Pose a real-world problem**: When given a set of data, can the shape of its graph be represented with a function? What predictions would you then want to make?

(b) **Develop a model**: The context of the problem and/or the shape of the data may suggest one or more of the models from this section as an appropriate representation. You then apply the skills covered in this section for finding the parameters of your model. Write a reasonable domain for your model.

(c) **Test the model**: Once you have calculated the parameters, how well does the graph of your model fit the graph of the data? If the fit is not good, you reject the model and go back to step (b). If the fit is good, you continue to step (d). If the fit is good but you can see improvements that can be made, go back to (b) and try to make these improvements, for example by altering the domain or the parameters of the model.

(d) **Reflect on and apply the model**: What is it about the context that makes your model appropriate? What predictions can be made? To what extent are your predictions reliable? Can your model be applied to other related contexts?

(e) **Extend**: You may be able to adapt your model to represent other contexts.

The following two examples include features of the modelling process.

### Example 2.3.4

A new drug is administered to a patient in a clinic to treat an infection. The nurse monitors the amount of the drug in the patient's bloodstream during the 10 hours after the drug was administered, writing down the following data set:

| Time in hours ($x$) | Amount of drug in $\mu g$ |
|---|---|
| 0 | 10 |
| 0.5 | 8.8 |
| 1 | 7.1 |
| 1.5 | 4.8 |
| 2 | 4.3 |
| 2.5 | 3.41 |
| 3 | 2.6 |
| 3.5 | 2.18 |

| | |
|---|---|
| 4 | 1.8 |
| 4.5 | 1.3 |
| 5 | 1.2 |
| 5.5 | 0.94 |
| 6 | 0.74 |
| 6.5 | 0.6 |
| 7 | 0.49 |
| 8 | 0.3 |
| 9 | 0.21 |
| 10 | 0.11 |

The nurse wishes to explore if either of the functions $f(x) = ax^n$ where $n \in \mathbb{Z}$, $n < 0$ and $g(x) = bc^x$ are appropriate models for the data set.

(a) Plot the data set using technology.

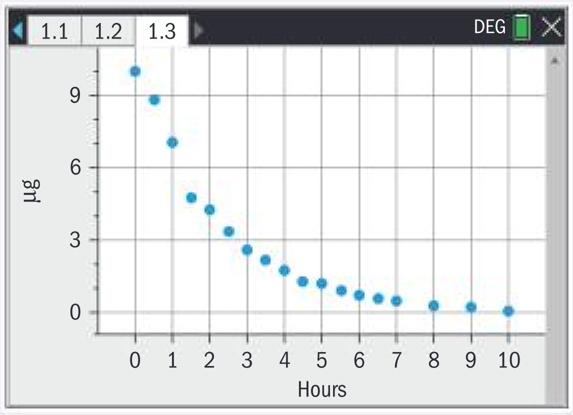
(b) Hence state why $f(x)$ and $g(x)$ are both more appropriate than a linear model with a negative gradient.

(c) State reasons why an inverse variation model or an exponential decay model may be appropriate to model the data.

(d) The nurse knows that when the amount of drug in the bloodstream falls below 3 $\mu g$, the drug does not have an effect. Hence write down a prediction that the nurse would wish to make in this context.

**Solution**

(a)




Examine and consider the key features of the shape of the data.

(b) The slope of the graph is not constant, hence a linear model is not appropriate.

(c) Both models have the feature that an increase in time gives a decrease in the amount of drug, and both models are curves with horizontal asymptote $y = 0$.

You give reasons based on your knowledge and understanding, using the correct terminology.

(d) The nurse would want to predict the time at which is the level of drug will fall below 3 $\mu g$.

You may be asked to find the parameters of a model by substituting given values in a given function, and then comment critcally on how well the model fits the data.

### Example 2.3.5

Use the same data and context as Example 2.3.4

The nurse uses the points (3.5, 2.18) and (6, 0.74) to find the parameters of $f$ and the points (2.5, 3.41) and (7, 0.49) to find the parameters of $g$.

(a) Find the parameters of $f$.

(b) Find the parameters of $g$.

(c) Plot each model on the same axes as the data, hence identify which of $f$ and $g$ is the more appropriate model, justifying your choice.

(d) Determine an appropriate domain for your model.

(e) Use your model to predict when the level of drug in the bloodstream will be below 3 $\mu g$.

Solution

(a) $f(3.5) = a3.5^n = 2.18$,

$f(6) = a6^n = 0.74$

Hence $\frac{3.5^n}{6^n} = \frac{2.18}{0.74} \Rightarrow \left(\frac{3.5}{6}\right)^n = \frac{2.18}{0.74}$

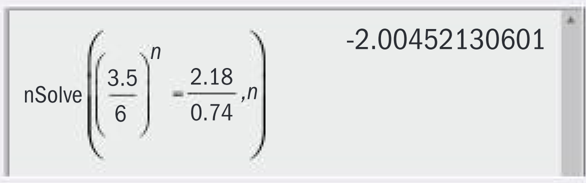
$n = \log_{\frac{3.5}{6}} \frac{2.18}{0.74}$

$= -2.004\,521\,306\,01$ hence

$a = \frac{2.18}{3.5^{-2.00452130601}} = 26.85..$, so

$f(x) = 26.9x^{-2.00}$

Use an equation solver to find the value of $n$.




(b) $g(2.5) = bc^{2.5} = 3.41$

$g(7) = bc^7 = 0.49$

Hence $\frac{c^7}{c^{2.5}} = \frac{0.49}{3.41}$

$\Rightarrow c^{4.5} = \frac{0.49}{3.41} \Rightarrow c = \sqrt[4.5]{\frac{0.49}{3.41}}$

$c = 0.649\,777\ldots$ hence

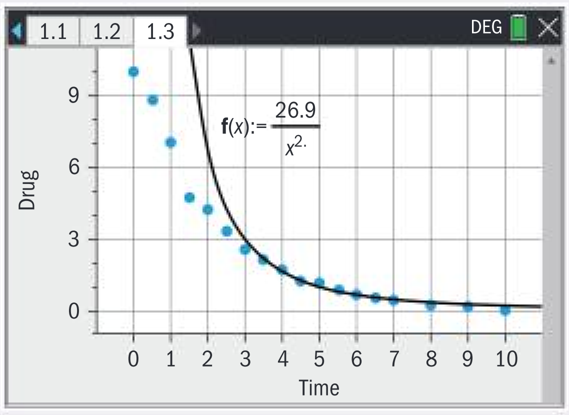
$b = \frac{3.41}{0.649777\ldots^{2.5}} = 10.019\,414\ldots$ so

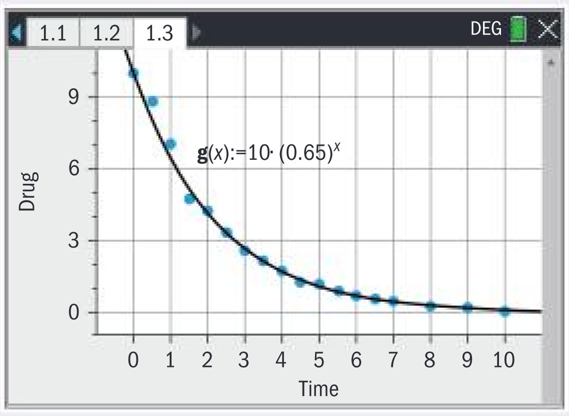
$g(x) = 10.0 \times 0.65^x$

Replace the values given into the model, and apply the laws of indices and the inverse function of $c^{4.5}$ to find $c$.

Use the unrounded vale of $c$ in the calculation of $b$.

(c)



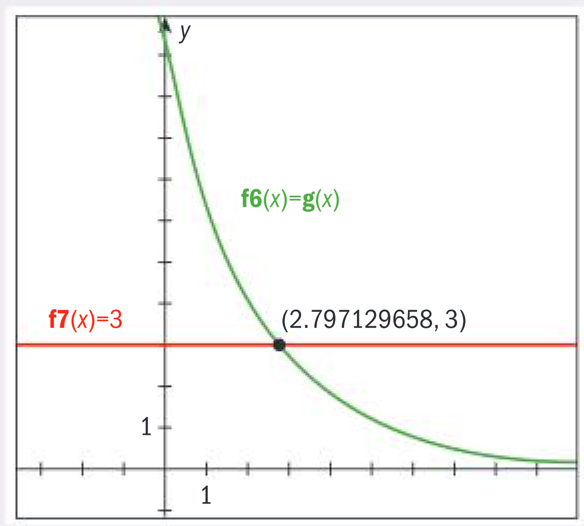

$g$ is the more appropriate model because it passes through more points than $f$, including the initial point (0, 10).

$f$ has a good fit for the second half of the data, but the vertical asymptote is not an appropriate feature of a model for this data.

(d) $0 \le x \le 10$

(e) The level of drug first falls below 3 $\mu g$ after 2.80 hours.

Comment critically on the goodness of fit, including the shape of the function.



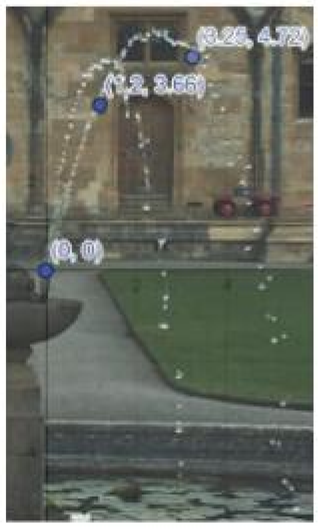
You can apply the modelling process in internal assessment as well as in the examination.

For example, Callum notices this fountain on his way to college every day and admires the shape of the water.

His physics teacher tells him the shape is likely to be parabolic, so Callum decides to explore if his physics teacher is correct by applying the modelling process.

Callum carefully places axes and three points as shown: and writes down the coordinates (0,0), (1.20,3.66) and (3.25,4.72) in order to find the parameters of a quadratic model.

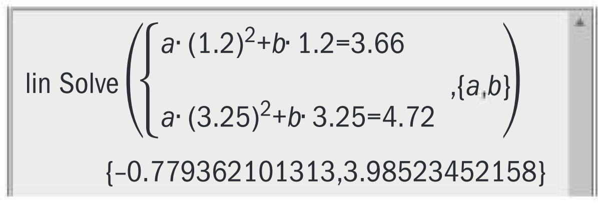
The quadratic model is $f(x) = ax^2 + bx + c$ but since Callum plans for the origin to be on his model, $c = 0$.




Callum substitutes the coordinates in the function to set up his system of simultaneous equations:

$f(1.20) = a(1.20)^2 + b(1.20) = 3.66$
$f(3.25) = a(3.25)^2 + b(3.25) = 4.72$

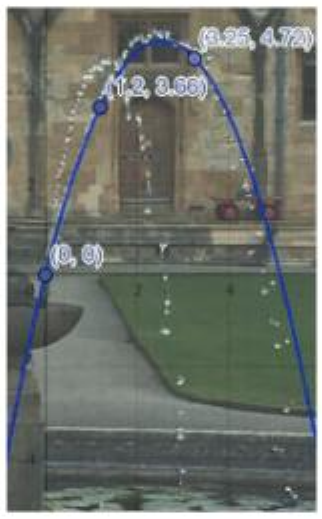
Callum then enters this system into his GDC to find the parameters:




Callum approximates the parameters:

$a \approx -0.779$ and $b \approx 3.99$ and then graphs his model.

Callum starts to reflect critically on what he has found. On the one hand, his model fits the three points perfectly. On the other hand, he is not pleased about how well the function models the fountain because he judges that the quadratic model is more symmetric than the path taken by the water actually is.

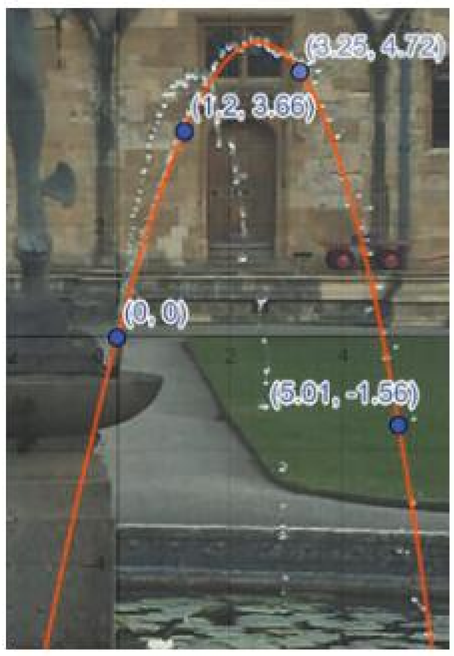



For this reason, Callum rejects the quadratic model and decides to explore if a cubic model is more appropriate.

Callum carries out the same process as above with $f(x) = ax^3 + bx^2 + cx$ and a third point (5.01,–1.56).

This gives him a system of three equations in three unknowns which he solves with his GDC.

He finds that $f(x) = -0.0585x^3 - 0.519x^2 + 3.76x$



## Assessment tip

Real-world data can be "messy" and may not present itself as "neat" nor "nice" – as seen in the water fountain example.

Callum reflects on what he has found. The cubic function fits his points perfectly. Also, it seems to model the path taken by the water better than his quadratic model did. Callum reflects on the context of his exploration – perhaps once the water has lost speed, the effect of gravity on its path is more significant? Callum wonders how he could extend his work to explore larger more powerful fountains and make predictions about their shapes.

Some contexts can be modelled only by forming a function from pieces of other functions. These **piecewise functions** can have two or more pieces, each with its own domain, as shown in the next example.

### Example 2.3.6

In a large city there are two workshops, AudioEx and Vintair which each offer restoration services of vintage electronic devices. AudioEx charges a \$50 "bench fee" which is a fixed charge just for the device to be put on the workshop bench for inspection. After this, restoration costs \$10 per hour.

(a) Model the cost of a restoration at AudioEx with a function $A(t)$ where $A$ is the cost in \$ of a restoration taking $t$ hours of work, and state a reasonable domain.

The cost of a restoration at Vintair is modelled by the function

$$V(t) = \begin{cases} 40 + 12t, & 0 \le t \le 8 \\ 9t + 64, & t > 8 \end{cases}$$

where $V$ is the cost in \$ for $t$ hours of work.

(b) For a restoration taking 15 hours of work, find how much more AudioEx would charge than Vintair.

(c) Graph $V$ and $A$ on the same set of axes and write down the coordinates of any intersections between the two graphs.

Vintair claim their charges are always less than those of AudioEx.

(d) Explain why Vintair's claim is false.

(e) Erin has a budget of \$300 to restore a rare radio. Determine the difference between the maximum times she can afford at each of the workshops.

**Solution**

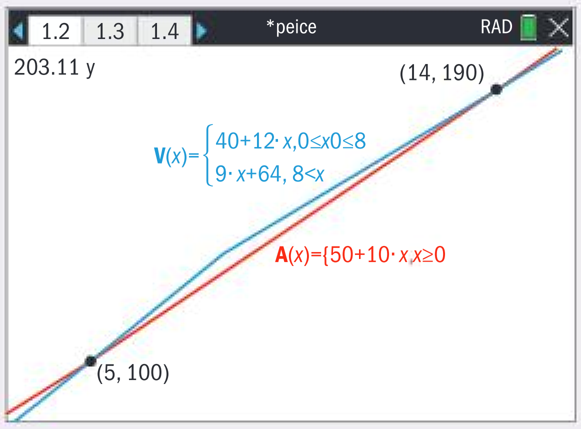
(a) $A(t) = 10t + 50$, where $t \ge 0$.

The cost for an AudioEx restoration increases at a constant rate of \$10 per hour so a linear model is appropriate. There is a fixed cost of \$50 so the parameters are 10 and 50.

(b) $A(15) = \$200$, $V(15) = \$199$ so the 15 hour restoration would cost \$1 more at AudioEx.

Find the cost by replacing the time in each function.

(c)



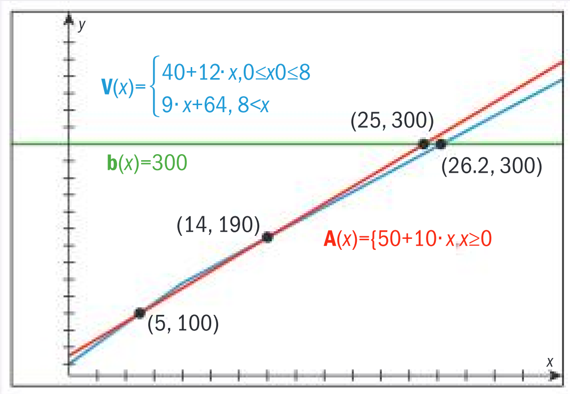

Take care to adjust the viewing window so that you can see the graphs clearly.

The coordinates of the points of intersection are (5, 100) and (14, 190)

(d) Vintair's claim is false because for $5 < t < 14$, its service costs more than that of AudioEx.

Interpret what the graphs show: the graph of $V$ is above that of $A$ on the interval $5 < t < 14$.

(e)




Find the difference in time by finding the two intersections with the line $y = 300$.

You then interpret the two $x$-coordinates 26.2 and 25 in order to solve the problem.

Erin can afford a maximum of 1.2 hours more at Vintair than at AudioEx on a budget of \$300.

When you carry out the modelling process you are involved in a "hands on" activity where you learn by actively "doing". Reflect on how this relates to the concepts of evidence and interpretation you learn in TOK.

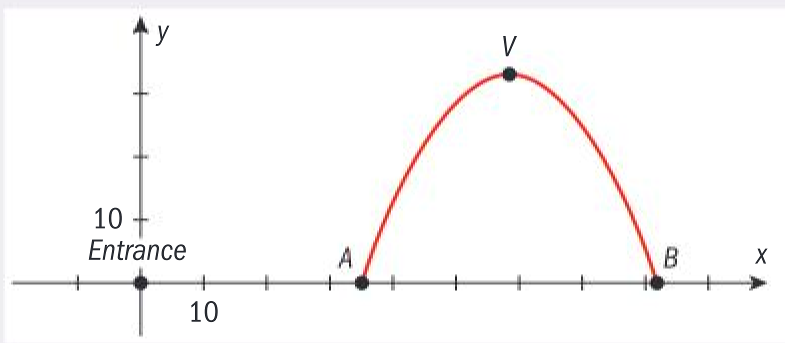
You might consider the quadratic function as simple and familiar. However, there is a lot of knowledge and understanding to apply with this function, much of which is a consequence of its symmetry. The quadratic function has three different forms and they each give you the following information about the features of the function.

| Form and equation ($a \neq 0$ in each case) | Features that can be **written down** from this form | Features that can be **calculated** from this form |
|---|---|---|
| Standard form $f(x) = ax^2 + bx + c$ | $y$-intercept $(0, c)$ | Axis of symmetry $x = \frac{-b}{2a}$ Roots of $f(x) = 0$ with the quadratic formula (HL only) |
| Factorized form $f(x) = a(x - \alpha)(x - \beta)$ | $x$-intercepts $(0, \alpha)$ and $(0, \beta)$ Roots of $f(x) = 0$ are $x = \alpha$ and $x = \beta$, as are the zeros of $f(x)$ | Axis of symmetry $x = \frac{\alpha+\beta}{2}$ $y$-intercept $(0, a\alpha\beta)$ |
| Vertex (completed square) form $f(x) = a(x - h)^2 + k$ | Vertex $(h, k)$ Axis of symmetry $x = h$ | Roots of $f(x) = 0$ by rearrangement $y$-intercept $(0, ah^2 + k)$ |

You apply the information given to you in a quadratic function if it is relevant in the context of a problem, and you use your GDC to carry out calculations.

### Example 2.3.7

A greenhouse is planned for a park in a large city. The height of the roof of the greenhouse in metres is modelled by the function $r(x) = -0.06x^2 + 7.02x - 172.2$ where $x$ is the horizontal distance from the park entrance at (0, 0) metres. Visitors walk from the entrance to point $A$ where they enter the greenhouse and they exit at point $B$. The vertex $V$ is the highest point of the greenhouse.



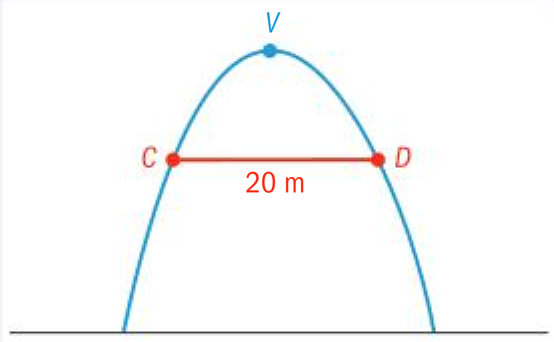
(a) Find the coordinates of $A$ and of $B$.

(b) Hence write down a reasonable domain for $r(x)$ and find the equation of the axis of symmetry.

(c) Hence find the height of the greenhouse.

(d) The architect sketches a plan for a horizontal walkway inside the greenhouse such that the distance $CD = 20$ m.

Calculate the vertical distance from the walkway to the ground.




**Solution**

(a) $A = (35, 0)$, $B = (82, 0)$

Use your GDC to find the zeros of $r(x) = 0$ and be sure to write the coordinates as was required in the question.

(b) The domain for $r(x)$ is $35 \le t \le 82$. The equation of the axis of symmetry is

$x = \frac{35+82}{2} = 58.5$

The equation of the axis of symmtery can also be found by replacing the values of $a$ and $b$ in the formula $x = \frac{-b}{2a}$

(c) The height of the greenhouse is $r(58.5) = 33.1$ m

The command term "Hence" is an explicit instruction to use the previous result. You use the equation of the axis of symmtery to find the maximum value directly.

(d) The height of the walkway is $r(68.5) = 27.1$ m

Apply the fact that the parabola is symmetric in order to solve the problem.

**Assessment tip**

Choose the most efficient method for finding the solution to a quadratic equation in the examination. Using your GDC is faster and less error-prone than using the quadratic formula.

Note that in the context of this problem, the $y$-intercept (0, –172.2) is not of any relevance. It is not always the case that you must use all of the information given.

Many natural phenomena such as monthly average temperatures can be modelled by sinusoidal functions since they show the repeating patterns that define periodic behaviour. Knowing the parameters of the sinusoidal function enable you to compare and contrast sets of periodic data as well as make predictions.

### Example 2.3.8

The average 24-hour temperature in °C for each month in Ushuaia, Argentina is given in this table:

| Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9.2 | 9.1 | 7.7 | 5.6 | 3.2 | 1.6 | 1.5 | 2.1 | 3.9 | 6.1 | 7.4 | 8.6 |

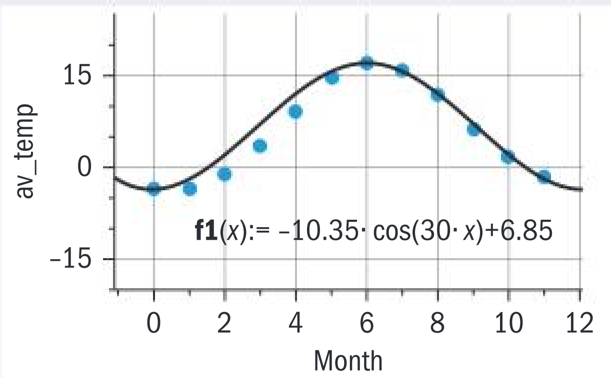
(a) Represent the twelve months from January to December by 0, 1, 2, …, 11 and hence plot temperature against months using technology.

The data can be modelled by one cycle of a sinusoidal function $T(t) = s + p\cos(qt)$, where $T$ is the average 24-hour temperature in °C and $t$ is the time in months.

(b) (i) Determine the values of the parameters $s$, $p$ and $q$.

(ii) Plot your model on the same axes as the points, and comment on the fit.

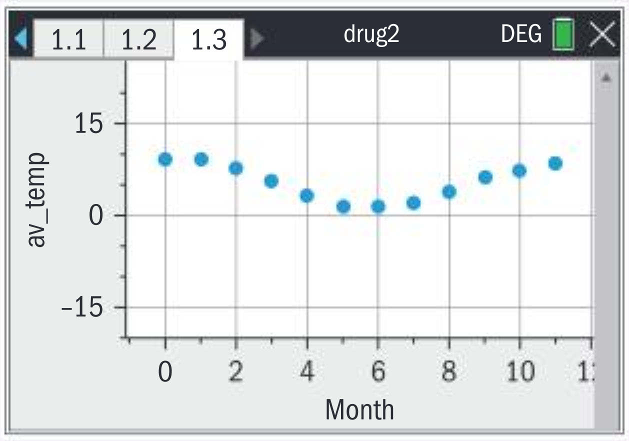
The average 24-hour temperature for Stockholm is represented by this graph and model:




(c) Compare and contrast the annual 24-hour temperature of Ushuaia with that of Stockholm.

Solution

(a)




Choose the viewing window so you can see the shape of the data.

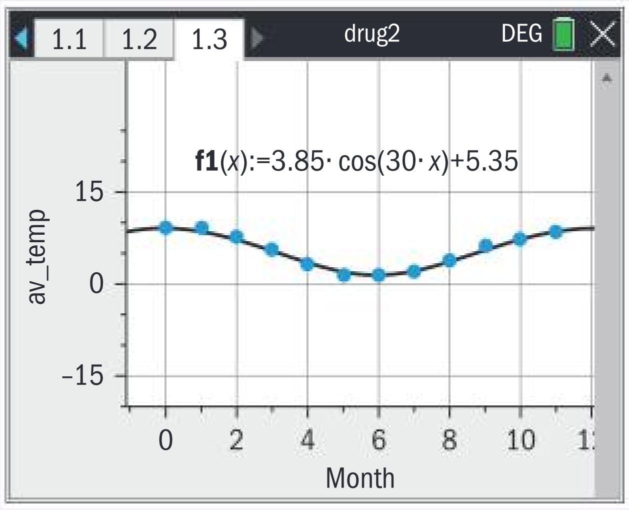
(b) (i) $|p| = \dfrac{9.2-1.5}{2} = 3.85$

$s = \dfrac{9.2+1.5}{2} = 5.35$

$q = \dfrac{360}{12} = 30$

Determine the parameters as shown above. Take care to intepret the parameters correctly if they appear in a different order to what you expect: for example, in $T$ the principal axis has equation $T = s$.

(ii)




$T(t) = 3.85\cos(30t) + 5.35$ fits the data well since the graph passes through all the points.

(c) Compare and contrast means give an account of the similarities and differences.

| Similarities | Differences |
|---|---|
| The data for Ushuaia and for Stockholm can be modelled with a sinusiodal function. | Stockholm has a greater range of temperatures, as shown by its greater amplitude, and a greater value on the principal axis. |
| Both models have period 12 months. | The orientation of Stockholm is the opposite of Ushuaia's: it has a minimum in January whereas Ushuaia has a maximum. |
| | The data fit for Ushuaia is better than that for Stockholm. |

**Assessment tip**

The order of parameters in any exam question on modelling might not follow the form you have studied. For example, the equation of a straight line can be written $y = mx + c$ or $y = a + bx$. Similarly, the parameters $s$, $p$ and $q$ in $s + p\cos(qt)$ determine the principal axis, amplitude and period respectively.

**Assessment tip**

Be aware of the dangers of extrapolation. For example, predicting average temperatures 20 years into the future may not be reliable due to increases in global temperatures due to climate change.

# 2.4 GRAPHS AND FEATURES OF MODELS (AHL)

## You should know:

- ✔ the formulae for translations, reflections and stretches of a graph
- ✔ that points on the $x$-axis are invariant under a vertical stretch and that points on the $y$-axis are invariant under a horizontal stretch
- ✔ the names, equations and features of natural logarithmic, logistic, piecewise and sinusoidal models (with phase shift)
- ✔ scaling very large or very small numbers using logarithms makes it easier to represent widely spread data
- ✔ linearizing data gives a valuable perspective on data or on models.

## You should be able to:

- ✔ apply and interpret transformations to any of the functions in both the SL and AHL sections of the guide
- ✔ apply and interpret composite transformations in the correct order
- ✔ apply linear, quadratic, cubic, exponential, power, sinusoidal and logistic regression tools on your GDC
- ✔ choose a manageable scale for data with a wide range of values
- ✔ plot and interpret log-log and semi-log graphs using technology
- ✔ use linearization to determine if data has an exponential or a power relationship
- ✔ apply the modelling process with sophistication and rigour.

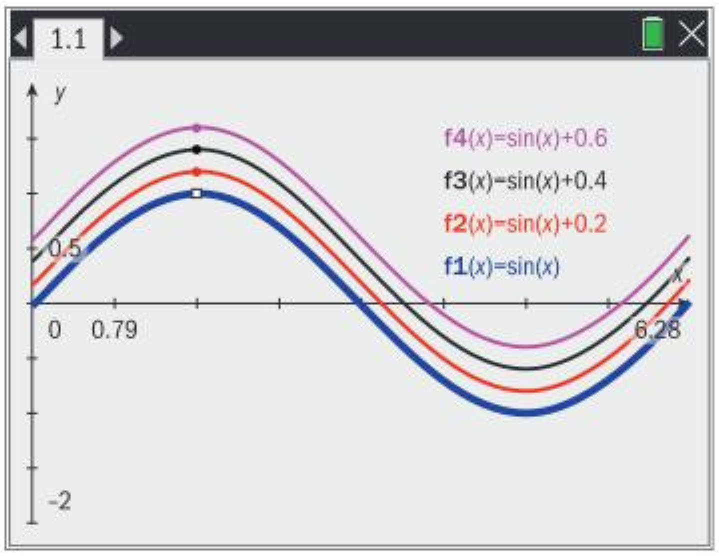
You have seen representations (mapping diagrams and relations) that help you understand the concept of a function. Another perspective is that of a "machine" with an "input" (the independent variable $x$) and an "output" (the dependent variable $y$). The point $(x, y)$ is on the graph of $f$.

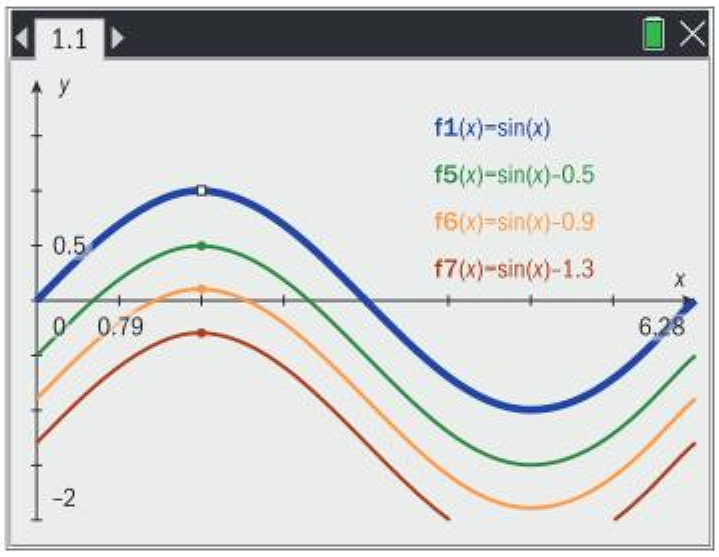
$$x \to \boxed{\quad f(x) \quad} \to y$$

Now imagine the function is transformed from $f(x)$ to $f(x) + b$ where $b \in \mathbb{R}$. What effect would that have on the graph of $f$?

$$x \to \boxed{\quad f(x)+b \quad} \to y+b$$

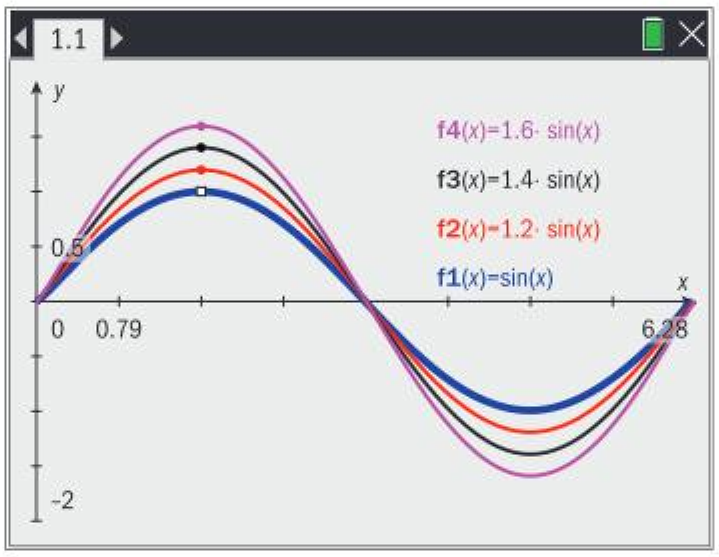
The point $(x, y)$ on the graph of $f$ has therefore been transformed to $(x, y + b)$. Applying this transformation to all the points on the graph of $f$ gives a vertical translation of the graph by $b$ units up if $b > 0$, or down if $b < 0$. In the diagram, $f(x) = \sin(x)$ has been transformed vertically:

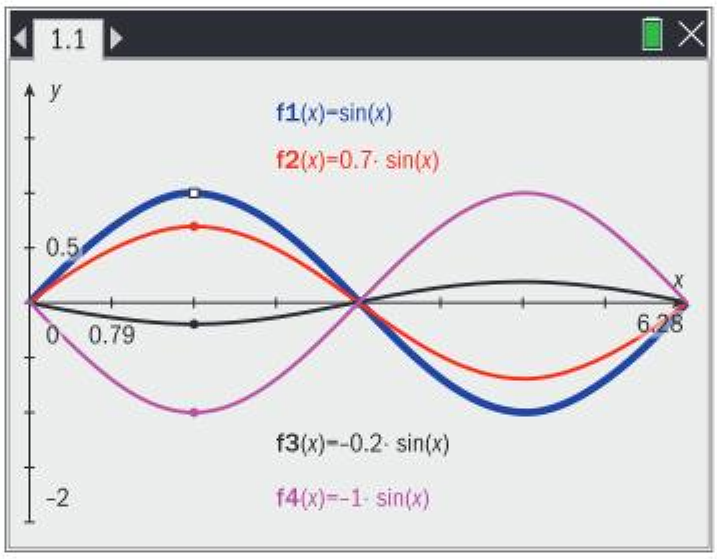


Similarly, the transformation $p \times f(x)$ can be represented by:

$$x \rightarrow \boxed{\quad pf(x) \quad} \rightarrow py$$




Each point $(x, y)$ on the graph of $f$ has been transformed to $(x, py)$. This is a vertical stretch of the graph of $f$ by a factor of $p$: if $p > 1$, the vertical distance of each point $(x, py)$ from the $x$-axis is $p$ times greater than that of $(x, y)$. Points on the $x$-axis are invariant: since the distance from the $x$-axis is zero, they remain where they are!

If $0 < p < 1$, each point $(x, py)$ is closer to the $x$-axis by a factor of $p$.

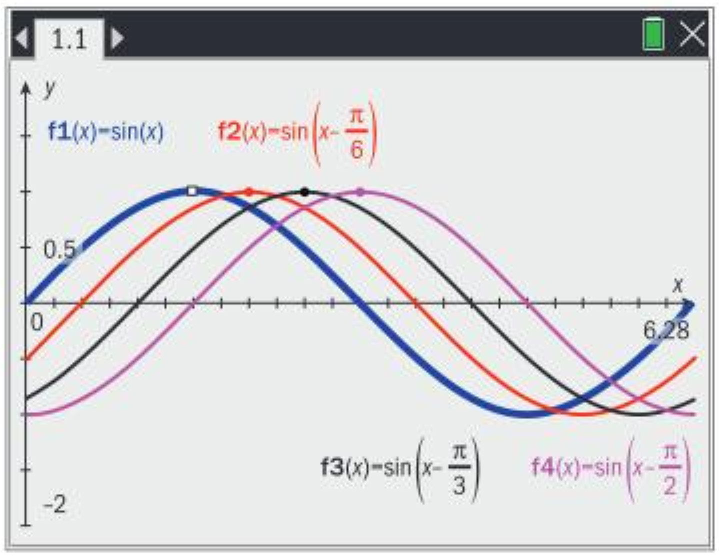
If $p = -1$, the graph of $f$ is transformed to that of $-f$. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the $x$-axis.

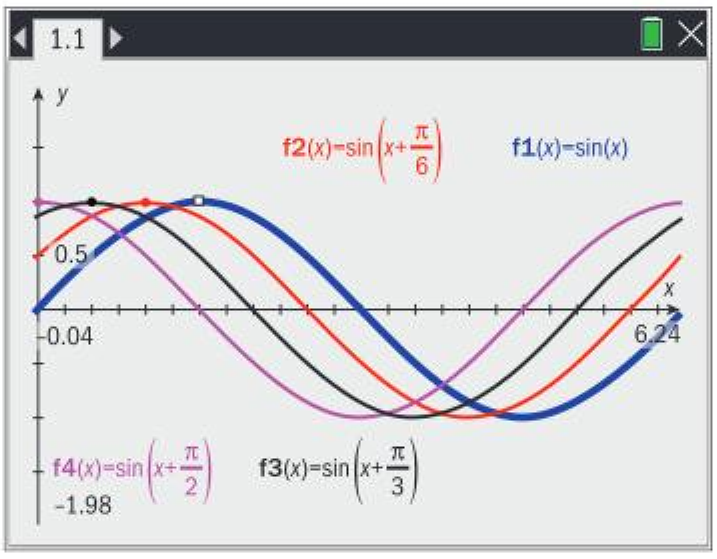
So far you considered transformations of $y$ given the same value of $x$. You can also consider transforms in $x$ given the same value of $y$.

The transformation $f(x - a)$ can be represented by:

$$x + a \rightarrow \boxed{\quad f(x-a) \quad} \rightarrow y$$

Hence for $f(x - a)$, $x + a$ must be the "input" in order to give $y$ as the "output" since $f((x + a) - a) = f(x) = y$. This means that the point $(x, y)$ on the graph of $f$ has been transformed to $(x + a, y)$. This is a horizontal translation by $a$ units to the right if $a > 0$, and to the left if $a < 0$.

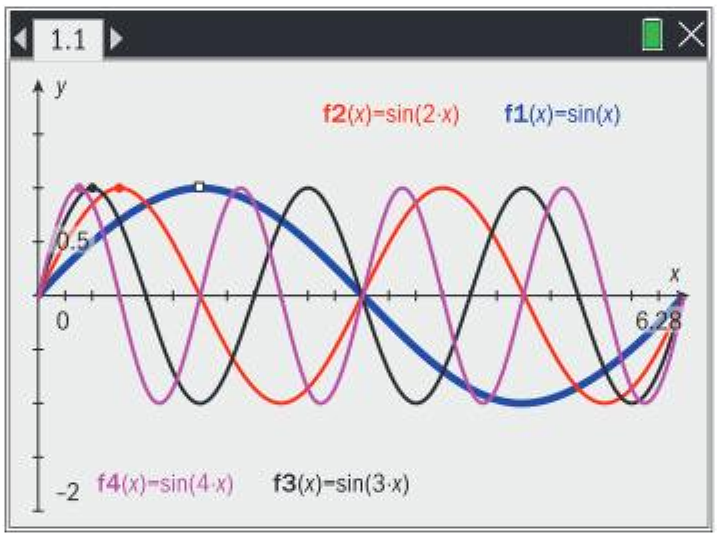


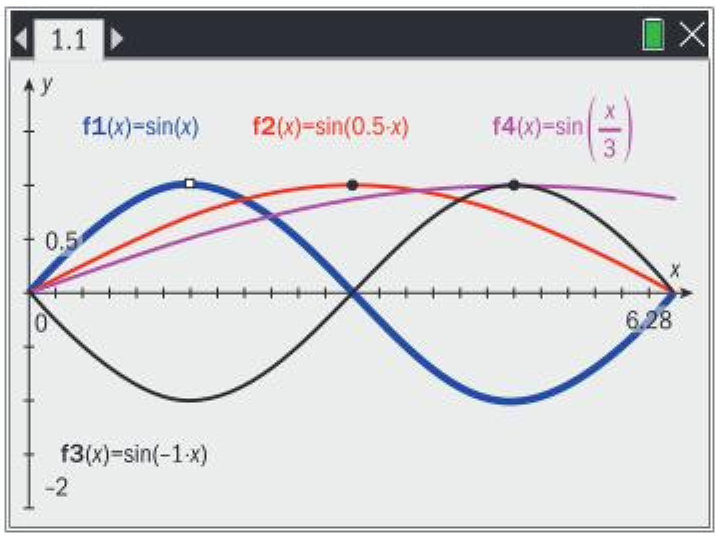




The transformation $f(qx)$ can be represented by:

$$\frac{x}{q} \rightarrow \boxed{\quad f(qx) \quad} \rightarrow y$$

Hence the point $(x, y)$ on the graph of $f$ has been transformed to $\left(\frac{x}{q}, y\right)$. This is a horizontal stretch by a factor of $\frac{1}{q}$.




Each point $(x, y)$ on the graph of $f$ has been transformed to $\left(\frac{x}{q}, y\right)$.

This is a horizontal stretch of the graph of $f$ by a factor of $\frac{1}{q}$: if $q > 1$, each point $\left(\frac{x}{q}, y\right)$ is $q$ times closer to the $y$-axis.

If $0 < q < 1$, each point $\left(\frac{x}{q}, y\right)$ is further from the $y$-axis by a factor of of $\frac{1}{q}$.

If $q = -1$, the graph of $y = f(x)$ is transformed to that of $y = f(-x)$, a reflection of the graph of $y = f(x)$ in the $y$-axis. Points on the $y$-axis are invariant.

These transformations can be summarized into one statement.

> The graph of $y = pf(q(x - a)) + b$ is a vertical stretch with scale factor $p$, a horizontal stretch with scale factor $\frac{1}{q}$, a horizontal translation by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and a vertical translation by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ of the graph of $y = f(x)$.

This statement is not in the formula booklet. Note that parameters which effect vertical transformations are outside $f$ and those which effect horizontal transformations are inside $f$.

It is important to be aware of the order in which these transformations are carried out, as shown in the following example. You can use technology and a consideration of composite functions to make sure you carry out transformations in the correct order.

### Example 2.4.1

(a) Given $f(x) = x^2$, $g(x) = 3x$ and $h(x) = x + 2$, express $y = 3x^2 + 2$ as a composition of functions.

(b) Hence describe the sequence of transformations which transform the graph of $y = f(x)$ to the graph of $y = 3x^2 + 2$.

**Solution**

(a) $3x^2 + 2 = h(3x^2) = h(g(x^2)) = h(g(f(x)))$

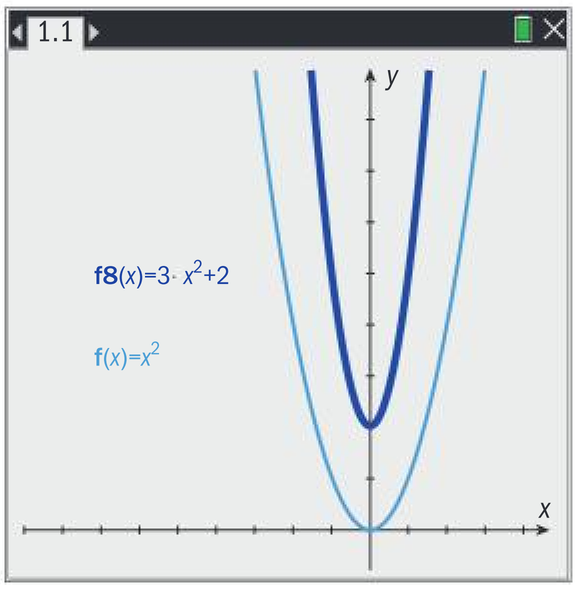
Apply your knowledge and understanding of composite functions.

(b) Hence the graph of $y = 3x^2 + 2$ is obtained from the graph of $y = f(x)$ by a vertical stretch of scale factor 3 followed by a vertical translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Write a detailed account by following the order of the composition from the inside: $g$ is applied first.

In the exam, should you have doubts regarding the correct order, verify the application of composite functions with technology. You define the functions in your GDC to save time, in a calculator page or your Y = editor, for example.


On your GDC, use colours or line attributes like bold, solid and dashed to distinguish between graphs. In what follows, the "target" of $y = 3x^2 + 2$ is shown in bold, for example.

<table>
<tr>
<td>Set up the problem:<br>You are aiming to transform the graph of $f(x) = x^2$ to the graph of $y = 3x^2 + 2$.</td>
<td>This sequence is incorrect because it does not transform the graph of $f(x) = x^2$ to the graph of $y = 3x^2 + 2$</td>
<td>This sequence is correct because it does! The vertical stretch of scale factor 3 is applied before the translation.</td>
</tr>
<tr>
<td>
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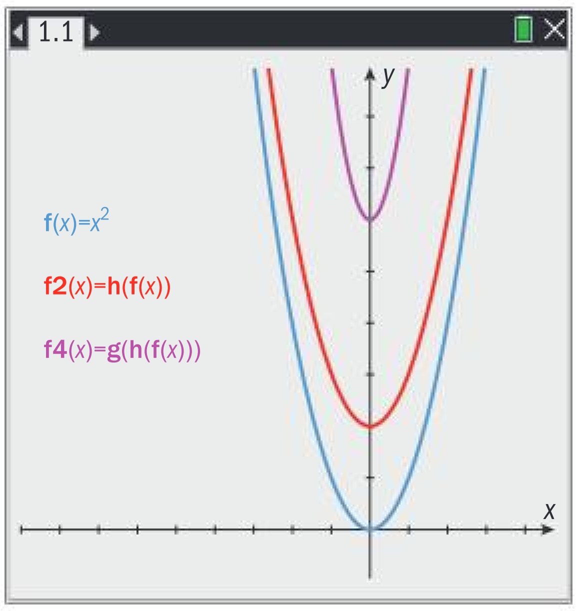
<td>
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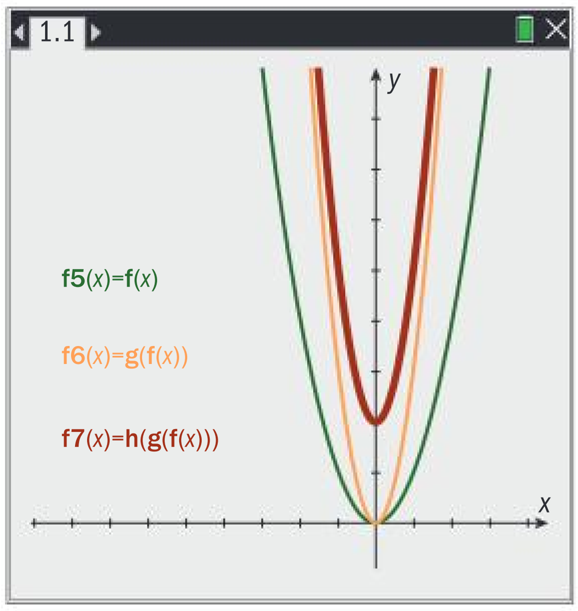
<td>
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</table>

This analysis is useful in helping you identify cases where the order in which the transformations take place does not matter. The composite function $h(g(f(x)))$ gives the correct sequence and this is not the same as $g(h(f(x)))$, a consequence of the fact that composition of functions is not commutative, as seen in example 2.2.1. This is why $h(g(f(x))) = 3x^2 + 2$ which is not the same function as $g(h(f(x))) = 3x^2 + 6$. Because the *positions* of $g$ and $h$ have been changed, the answers are different.

However, it is true that for any three functions, $f \circ (g \circ h) = (f \circ g) \circ h$.

Given $s(x) = \sin(x)$, $q(x) = 2x$ and $p(x) = 4x$, $y = 4\sin(2x)$ can be expressed as a composition of functions $4\sin(2x) = (p \circ s) \circ q(x) = p \circ (s \circ q)(x)$. This means that you get the correct answer by applying a horizontal stretch of scale factor $\frac{1}{2}$ to $s(x) = \sin(x)$ followed by a vertical stretch of scale factor 4 to the result, or by applying a vertical stretch of scale factor 4 to $s(x) = \sin(x)$ followed by a horizontal stretch of scale factor $\frac{1}{2}$ to the result.

### Assessment tip

Be aware that the order in which transformations are carried out can be verified by applying composite functions and/or use of technology.

### Assessment tip

Read the exam question carefully. If a question asks for "the" sequence of transformations – this means there is one correct answer, or "a" sequence of transformations – this means there is more than one correct answer.

**Example 2.4.2**

(a) Describe the sequence of transformations required to transform the graph of $y = \ln(x)$ onto the graph of $y = \ln\left(\frac{x}{3}\right) - 5$ by a vertical translation and a horizontal stretch.

(b) Determine a sequence of transformations required to transform the graph of $y = x^2$ onto the graph of $y = 4(x-1)^2$

**Solution**

(a) $\ln\left(\frac{x}{3}\right) - 5 = f(\ln(t(x)))$ where $f(x) = x - 5$ and $t(x) = \frac{1}{3}x$.

The sequence is a horizontal stretch with scale factor 3 followed by a vertical translation of $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$.

Give a full and detailed account using the correct notation and terminology.

(b) $4(x-1)^2 = f(s(h(x)))$ where $f(x) = 4x$, $h(x) = x - 1$ and $s(x) = x^2$.

A sequence is a horizontal translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ followed by a vertical stretch of scale factor 4.

Another correct answer is a vertical stretch of scale factor 4 followed by a horizontal translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

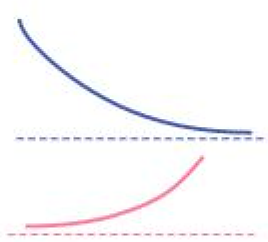
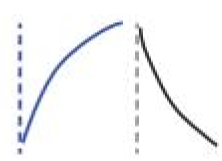
Note that there can be equivalent sequences of transformations involving different transformations.

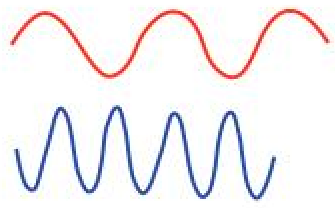
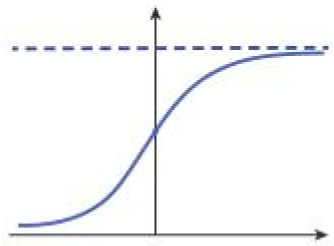
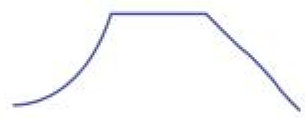
For example, since $\frac{81}{16}x^4 - 9.1 = \left(\frac{3}{2}x\right)^4 - 9.1$, either a horizontal or a vertical stretch can be applied.

Similarly, since $\ln(3x) - 1 = \ln(x) + (\ln(3) - 1)$, the transformation is just one vertical translation.

## Types of models

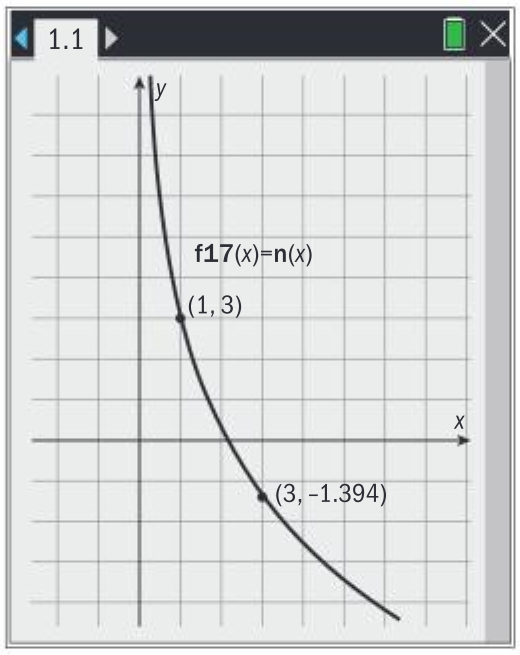
In addition to those already covered in SL, you must be able to carry out the modelling process with the following models, as well as apply them to make predictions. The exponential model has already been covered in SL but in HL it is further used to calculate half-life.

| Name of model | Equation | One example of application | Name of shape and sketches of examples of shape | One distinguishing feature |
|---|---|---|---|---|
| **Exponential** | $f(x) = ka^x + c$<br>$f(x) = ka^{-x} + c$<br>$f(x) = ke^{rx} + c$ | Calculation of half-life, for example of radioactive material. | Exponential growth/decay  | Horizontal asymptote $y = c$ |
| **Natural logarithmic** | $f(x) = a + b\ln(x)$ | The Richter magnitude of an earthquake | Logarithmic  | Vertical asymptote $x = 0$ |

| | | | | |
|---|---|---|---|---|
| **Sinusoidal** | $f(x) = a\sin(b(x - c)) + d$ | Sunrise times | Sine wave  | Phase shift $c$ |
| **Logistic** | $f(x) = \frac{L}{1 + Ce^{-kx}}$ <br> $L, C, k > 0$ | Increase in height of a plant. | Logistic curve  | Horizontal asymptotes $y = L$ and $y = 0$ |
| **Piecewise** | $f(x) = \begin{cases} a_1(x), & x_1 \le x < x_2 \\ \dots & \dots \\ a_n(x), & x_n \le x \le x_{n+1} \end{cases}$ | Velocity/ time graph of a journey with distinct parts. | Piecewise  | Continuous if $a_1(x_2) = a_2(x_2) \dots$ |

### Natural logarithmic models

Information given:




To determine the parameters of $n(x) = a + b\ln x$

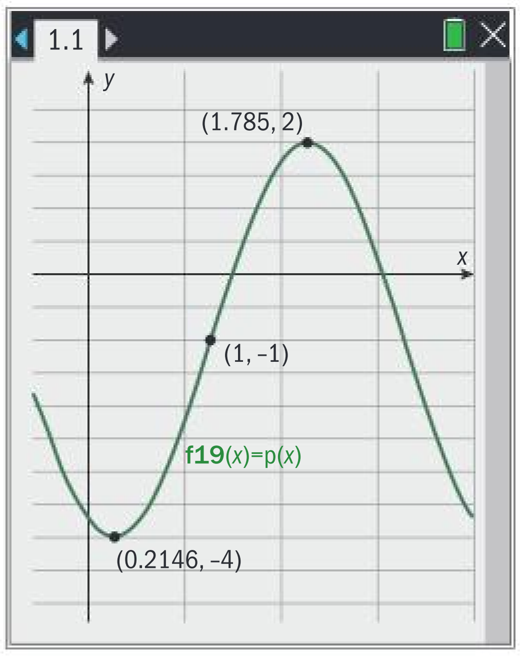
(1) Replace $x$ and $y$ in the equation $n(x) = a + b\ln x$ with the coordinates of two of the points on the curve.

(2) Hence write down a system of two simultaneous equations in $a$ and $b$.

(3) Use your GDC to solve the system.

### Sinusoidal models

Information given:




To determine the parameters of $p(x) = a\sin(b(x - c)) + d$:

(1) Carry out the steps for finding the parameters of $s(x) = a\sin(bx) + d$ as shown previously.

(2) You will need to know the coordinates of a point on the graph $(t, p(t))$ in order to find $c$.

Replace these coordinates in the equation and write down $p(t) = a\sin(b(t - c)) + d$.

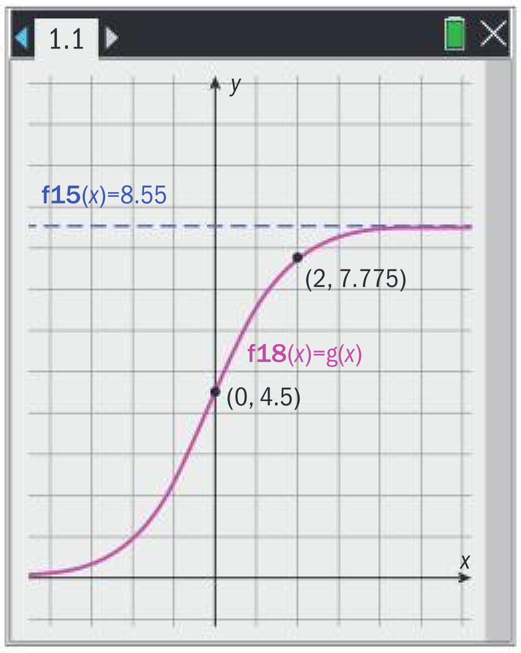
(3) Solve this equation for $c$.

Radian measure should be assumed unless the degree symbol is used for the unit of the independent variable.

$c$ can be referred to as the phase shift.

### Logistic models

Information given:




To determine the parameters of $g(x)=\frac{L}{1+Ce^{-kx}}$:

(1) The parameter $L$ can often be determined from the context of the problem – for example if the carrying capacity of a population is given, or the equation of the horizontal asymptote.

(2) If the initial condition $g(0)$ is given, solve $\frac{L}{1+C}=g(0)$ to determine $C$.

(3) Another point on the curve $(a, g(a))$ is required to determine $k$. Solve $\frac{L}{1+Ce^{-ka}}=g(a)$ to determine $k$.

This is not the only way to determine the parameters, as shown in Example 2.4.4.

### Piecewise models

Piecewise models could be described as "bespoke" insofar as they are uniquely designed to describe a context where behaviour changes from one model (piece) to another. A key requirement is that the pieces fit together, as seen in Example 2.4.6.

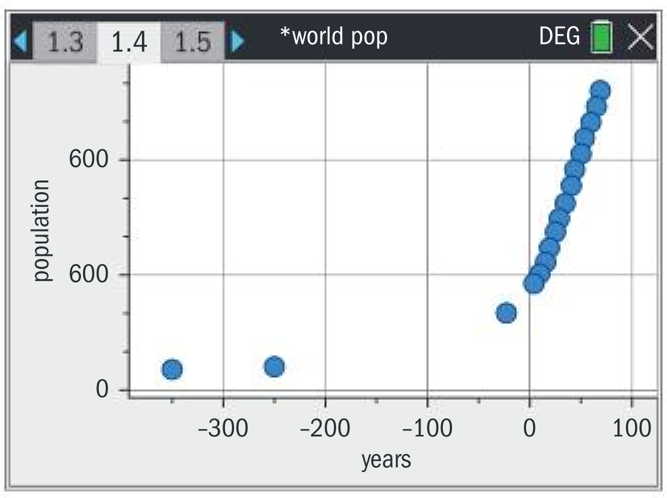
You can practice these methods of finding the parameters with the information given for each model above. Check your answers by graphing your function on your GDC and comparing with the graphs given.

The answers are: $n(x)=3-4\ln x$, $p(x)=3\sin(2(x-1))-1$, $g(x)=\frac{8.55}{1+0.9e^{-1.1x}}$

> If you are given sufficient points, you can create a model using the linear, quadratic, exponential, logarithmic, logistic or sinusoidal regression tool on your GDC as appropriate.

At HL, you are expected to apply the modelling process described in section 2.3 with sophistication, for example by applying knowledge and understanding from other topics. The following account shows how the modelling process can be applied at HL.

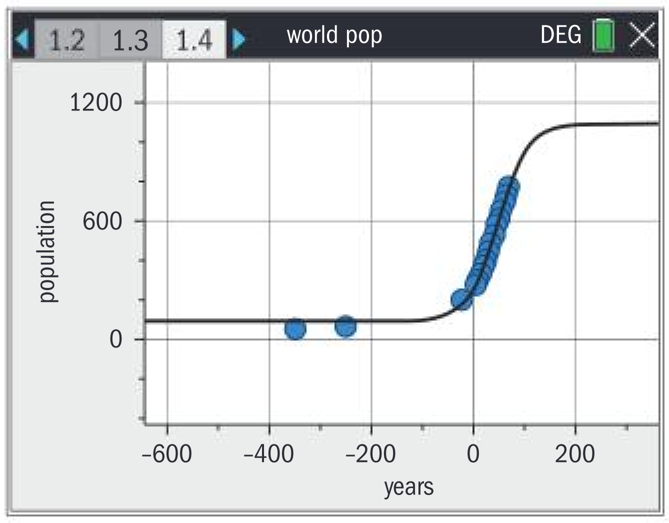
## World population modelling




Steve is given world popuation data in Geography class and he represents the data on a set of axes. On the $x$-axis, years represents the number of years after 1950 and on the $y$-axis, population is measured in tens of millions.

(a) **Pose a real-world problem:** What will the world population be in the year 2200?

(b) **Develop a model:** Steve examines the shape of the graph and decides that neither a linear function nor an exponential function are appropriate models, because the shape resembles part of a logistic function. This model is appropriate for the context since our planet can only support a finite carrying capacity, so he estimates the parameters of a logistic model with a regression tool.




(c) **Test the model:** Steve's model predicts a population of 10.94 billion in 2200, and a horizontal asymptote of $y = 1095$, meaning a population of almost 11 billion, greater than the figures for the carrying capacity of the world he has encountered in Geography class. Aware of the potential for error in such large numbers, Steve inspects the fit of his function and considers the following refinements one by one, going back to (b) after each:

| Knowledge | Possible refinement of the model |
|---|---|
| For the model $f(x) = \frac{L}{1+Ce^{-ka}}$, it can be shown by application of graph transforms and logarithms that changing the parameters have these effects on the graph of $y = f(x)$:<br>• $L$ - vertical stretch<br>• $C$ - horizontal translation<br>• $k$ - horizontal stretch. | Trying to improve the fit of the function to the data set through a seqeuence of graph transforms. |
| For $f(x) = \frac{L}{1+Ce^{-ka}}$, it can be shown with application of calculus that the point of inflexion is $\left(\frac{\ln C}{k}, \frac{L}{2}\right)$. | Estimating the point in the data where the rate of change is greatest, comparing with the point of inflexion of the model and adjusting the parameters if necessary. |
| Changing the exploration to include only the most reliable (recent) data will change the parameters of the model. Data points from 1600 and before can only be estimates. | Exploring if modelling a smaller sample of the data makes the model more representative of modern times. |

(d) **Reflect on and apply the model:** With every refinement of his model, Steve finds another prediction for the population in 2200. Given the complexity of the factors that contribute to world population, Steve wonders how a function could be able to predict world population with any degree of validity, but he knows that he has at least explored one model in some depth.

(e) **Extend:** Steve learns that the population of China and of India, the world's biggest countries, is projected to fall, and sets out to modify his model to take account of this in order to make modified predictions. He learns about the average carbon footprint of a human, and explores how population growth and carbon emissions targets relate to each other.

## Applications of models to problems

You have already seen an example of exponential decay in Example 2.3.4. Half-life is the time required for a substance (for example the amount of a drug in the bloodstream or the amount of a radioactive element) to reduce to half of its initial quantity. You can apply exponential models to calculate half-life.

### Example 2.4.3

Consider $g(x) = 10.0 \times 0.65^x$ from Example 2.3.5. This function models the amount of drug A in the bloodstream $x$ hours after 10 $\mu g$ of a drug was administered, $x \geq 0$.

(a) Find the half-life of drug A predicted by the model $g(x)$.

In a medical trial, 20 $\mu g$ of drug B is admistered to another patient at the same time as drug A is administered. It is found that the half-life is 1.2 hours. The amount of drug B in the bloodstream $x$ hours after 20 $\mu g$ of drug B was administered is modelled by $h(x) = De^{qx}$, $x \geq 0$.

(b) Find the parameters $D$ and $q$.

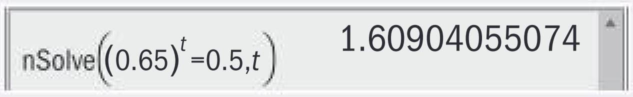
(c) Hence predict the time at which the level of drug in the bloodstream would be equal for both drug A and for drug B and how much of each drug remains at this time.

**Solution**

(a) $10.0 \times 0.65^x = 5 \Rightarrow 0.65^x = \frac{1}{2}$
$\Rightarrow x = \log_{0.65} 0.5 = 1.61$

The half-life is 1.61 hours

Write down and solve the equation or use an equation solver to find the value of $t$.




(b) $h(0) = 20 = De^{q(0)} \Rightarrow D = 20$

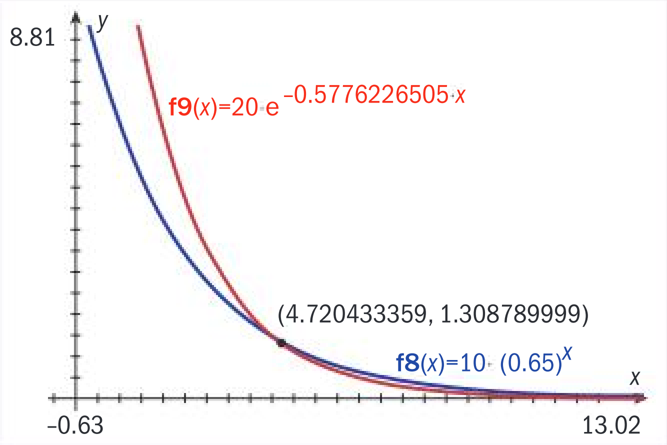
$20e^{q(1.2)} = 10 \Rightarrow 1.2q = \ln\left(\frac{1}{2}\right)$

Hence $q = \dfrac{\ln\left(\frac{1}{2}\right)}{1.2} = -0.578$

Interpret the initial value of 20.

(c) After 4.72 hours, the level of each drug will be same. 1.31 $\mu g$ remains.

Use the unrounded value −0.577 622 650 466 63... in part (c) to avoid unnecessary accumulation of error.




### Example 2.4.4

The marketing department of a school wants to model the time it takes for its social media posts to be viewed by the school community. The school community consists of 743 students, parents and alumni. The marketing department finds that 2 hours after posting, a post has been viewed by 20 members of the community and after 7 hours the post has been seen by 171 members. Assume that everyone will at some time look at the post.

(a) Model the number of views $V$ after $t$ hours with a logistic model $V(t) = \dfrac{L}{1 + Ce^{-kt}}$ and determine the parameters $L$, $C$ and $k$.

(b) The marketing department procedure is to stop monitoring social media posts when they have been viewed by 95% of the community. Predict for how many hours the post will be monitored.

**Solution**

(a) The carrying capacity is 743 so $L = 743$.

$$V(2) = \frac{743}{1+Ce^{-2k}} = 20$$

$$V(7) = \frac{743}{1+Ce^{-7k}} = 171$$

Hence,

$$20 + 20Ce^{-2k} = 743 = 171 + 171Ce^{-7k}$$

$$\Rightarrow \begin{cases} 20Ce^{-2k} = 723 \\ 171Ce^{-7k} = 572 \end{cases} \Rightarrow e^{-5k} = \frac{572}{723} \times \frac{20}{171}$$

$$\Rightarrow k = \frac{\ln\left(\frac{572}{723} \times \frac{20}{171}\right)}{-5} = 0.476$$

$$\Rightarrow C = \frac{723}{20e^{-2(0.476)}} = 93.7$$

Interpret the context in order to identify a parameter directly.

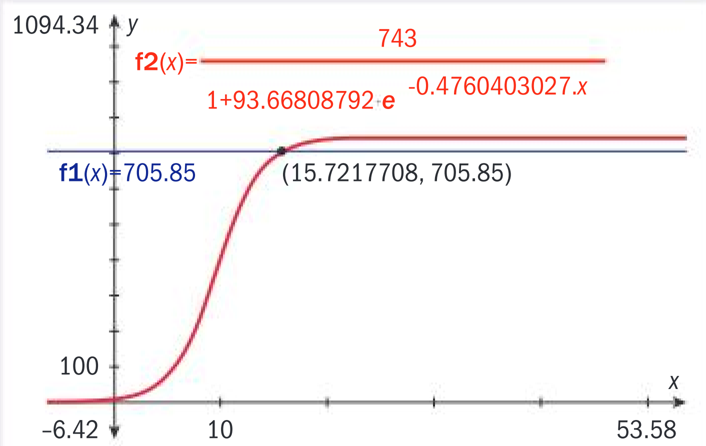
Set up and solve a system of equations using he laws of indices.

(b) 95% of 743 is 705.85

$$705.85 = \frac{743}{1+Ce^{-kt}} \Rightarrow t = 15.7217$$

After 15.7 hours the post will no longer be monitored.

Use the unrounded values $k = 0.476\,040\,302\,745\,52$ and $C = 93.668\,087\,922\,075$ to avoid unnecessary accumulation of error.




**SAMPLE STUDENT ANSWER**

The population of fish in a lake is modelled by the function

$$f(t) = \frac{1000}{1+24e^{-0.2t}},\ 0 \le t \le 30, \text{ where } t \text{ is measured in months.}$$

(a) Find the population of fish when $t = 10$.

(b) Find the rate at which the population of fish is increasing when $t = 10$.

(c) Find the value of $t$ for which the population of fish is increasing most rapidly.

(a) $f(10) = \dfrac{1000}{1+24e^{-0.2(10)}}$

$f(10) = 235$

(b) $\dfrac{d}{dx}\left(\dfrac{1000}{1+24e^{-0.2(10)}}\right) = 0$

$\cancel{f(k)}\ f'(10) = 0$

(c) ~~000~~ $t = 10$

▲ The student makes the correct substitution and correctly interprets the answer 235.402... as 235 in this context.

▲ The student recognizes that the rate of change is the derivative.

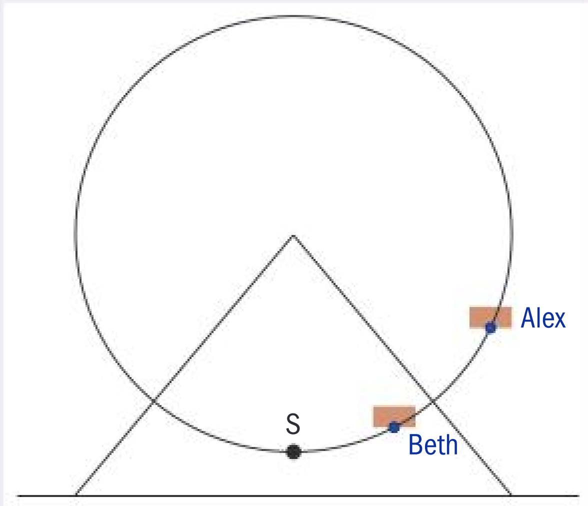
▼ The student could use a GDC to find the numerical derivative of $f$ at $t = 10$, which is 36.0 fish per month.

▼ Using the GDC to find the value of $t$ which maximizes $f'$ gives $t = 15.9$ months. This is the most efficient method.

### Example 2.4.5

A Ferris wheel in an amusement park has radius 30 m.

Passengers embark a viewing pod at S, 3 m from the ground. Alex and Beth reserve tickets for a ride on the Ferris wheel but depart at different times. Alex departs at 10:00.




The vertical distance of Alex's pod from the ground in metres is $a(t)$, where $t$ is the time in minutes after 10:00. One whole revolution of the pod takes 15 minutes and the pods rotate anti-clockwise.

(a) Determine the parameters of the model $a(t) = p + q\cos(rt)$, $t \geq 0$

(b) Beth's pod departs $c$ minutes after Alex. Given that 4.5 minutes after 10:00, Beth is 8.73 m above the ground, find the value of $c$ in the model $b(t) = p + q\cos(r(t - c))$, $t \geq c$.

(c) Explain how $b$ represents the fact that Beth departs $c$ minutes after Alex.

(d) Alex and Beth want to film each other for their social media when they will first be at the same vertical height. Find at what time they will photograph each other.

**Solution**

(a) The maximum and minimum heights are 3 m and 63 m hence $p = 33$ and $q = -30$.
The period is 15 so $r = \frac{2\pi}{15}$

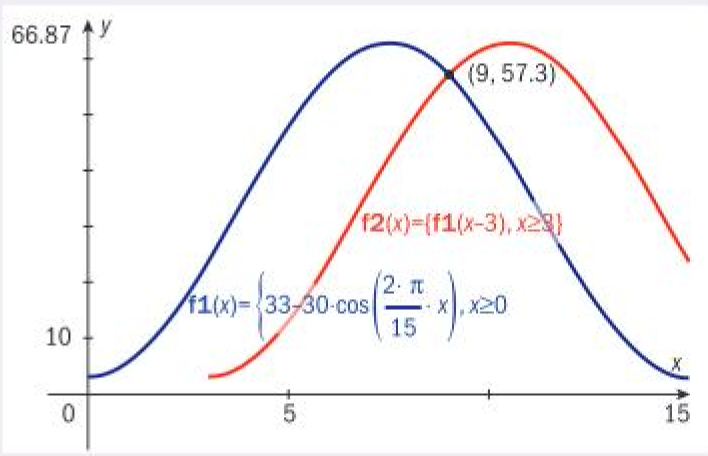
The value of $q$ could be either positive or negative. In this context, when $t = 0$ the height is 3 m, so $q$ must be $-30$

(b) $b\ (4.5) = 8.73$

$= 33 - 30\cos\left(\frac{2\pi}{15}(4.5 - c)\right)$

$\Rightarrow c = 45 - \frac{15}{2\pi}\cos^{-1}\left(\frac{8.73 - 33}{-30}\right) = 3.00$

Rearrange and only enter the expression in your calculator once in order to avoid the accumulation of error.



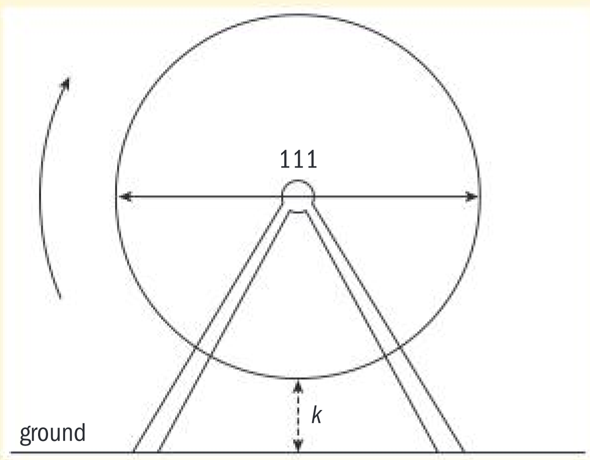

(c) The domain of $b$ and its equation represent the fact that after Alex has been moving for $c$ minutes, Beth is at the very start of her journey.

(d) Beth and Alex will both be 57.3 m above the ground at 10:09.

**SAMPLE STUDENT ANSWER**

At an amusement park, a Ferris wheel with diameter 111 metres rotates at a constant speed. The bottom of the wheel is $k$ metres above the ground. A seat starts at the bottom of the wheel.

**diagram not to scale**




The wheel completes one revolution in 16 minutes.

(a) After 8 minutes, the seat is 117 m above the ground. Find $k$.

After $t$ minutes, the height of the seat above ground is given by $h(t) = 61.5 + a\cos\left(\frac{\pi}{8}t\right)$, for $0 \le t \le 32$

(b) Find the value of $a$.

(c) Find when the seat is 30 m above the ground for the third time.

(a) 8 minutes = 117m above (half revolution)

$117 - 111 = 6$

$k = 6\text{m}$

(b) $h(t) = 61.5 + a\cos\left(\frac{\pi}{8}t\right)$

$h(8) = 61.5 + a\cos\left(\frac{\pi}{8}t\right)$

$117 = 61.5 + a\cos\left(\frac{\pi}{8}(8)\right)$

$55.5 = a\cos\left(\frac{\pi}{8}(8)\right)$ $\quad$ ~~cos = 180~~

$55.5 = a\cos(\pi)$ $\quad$ $\cos\pi = -1$

$55.5 = a(-1)$

$55.5 = -a$

$a = -55.5$

(c) $61.5 + (-55.5)\cos\left(\frac{\pi}{8}t\right) = 30\text{m}$

▲ The student correctly applies the information given in the context, showing clear reasoning.

▲ The student correctly replaces the correct information in the formula.

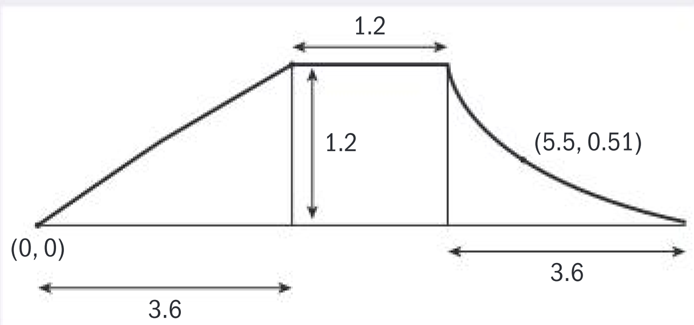
▼ The value of $a$ can be found more efficiently with the formula for amplitude: $|a| = \frac{(117 - 6)}{2}$ and use of a GDC to check.

▼ The correct equation is written down. The GDC is the most efficient way to solve the equation: $t = 18.5$ minutes.

Piecewise functions enable you to model phenomena that have distinct parts. The pieces of the function must be set up so that the function continuously moves from one piece to the other. In Example 2.4.6, the ramp at a skate park has three distinct pieces to model.

### Example 2.4.6

Daniel measures the dimensions of the ramp at a skate park and records his data on this sketch:




Model the cross-section of the ramp with a piecewise function.

### Solution

The first part of the ramp can be modelled by a linear function with slope $\frac{1.2}{3.6}$ and $y$-intercept at (0, 0), giving $l(x) = \frac{1}{3}x$.

The second part is a horizontal line $h(x) = 1.2$

The curved part is modelled by $c(x) = ka^x$.

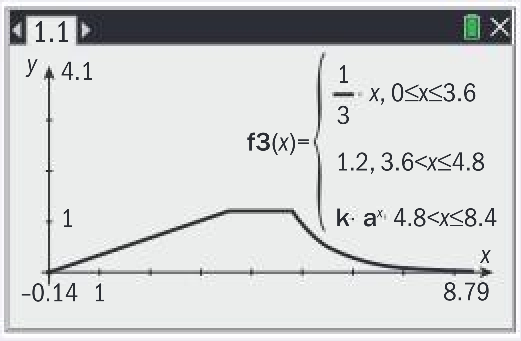
$$c(4.8) = ka^{4.8} = 1.2$$
$$c(5.5) = ka^{5.5} = 0.51$$

Hence $a = \left(\frac{0.51}{1.2}\right)^{\frac{1}{0.7}}$ and $k = \frac{1.2}{a^{4.8}}$

The model can be written as

$$r(x) = \begin{cases} \frac{1}{3}x & 0 \le x \le 3.6 \\ 1.2 & 3.6 < x \le 4.8 \\ ka^x & 4.8 < x \le 8.4 \end{cases}$$

The GDC confirms the shape and continuity:




The curved part of the ramp gets closer and closer to the horizontal in a smooth way so that skating is safer. Therefore, Daniel represents the curve with an exponential decay model.

Daniel stores the parameters in his GDC memory in order to ensure that the pieces $h$ and $c$ join at $x = 4.8$


**Assessment tip**

In the modelling process, giving distinct models distinct names is correct notation, and helps your communication.

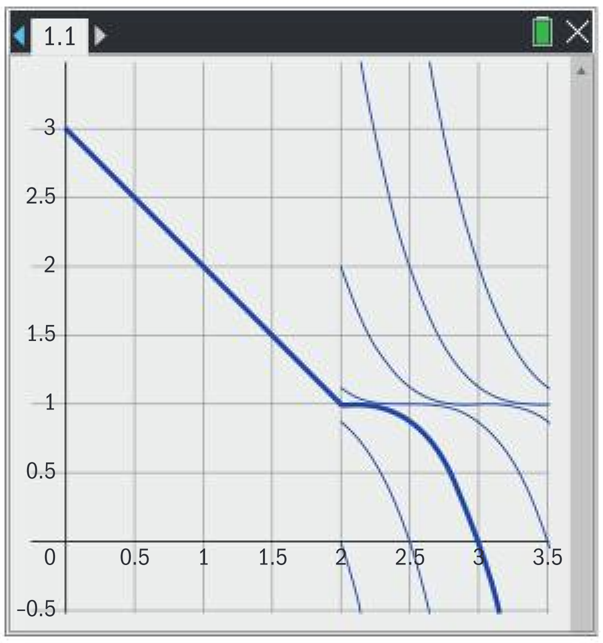
In some cases, you use the domain of one of the pieces to determine the parameters of another in order to "join" the pieces to make a continuous curve.

For example,

$$f(x) = \begin{cases} 3 - x & 0 \le x \le 2 \\ 1 - (x - a)^3 & x > 2 \end{cases}$$

is a continuous function if the two pieces meet at one point. You find this point by solving $f(2) = 3 - 2 = 1 = 1 - (2 - a)^3$ for the parameter $a$. The solution to this equation is $a = 2$.

This value of the parameter forces the two pieces of $f$ to meet, as shown in diagram.




The bold line is

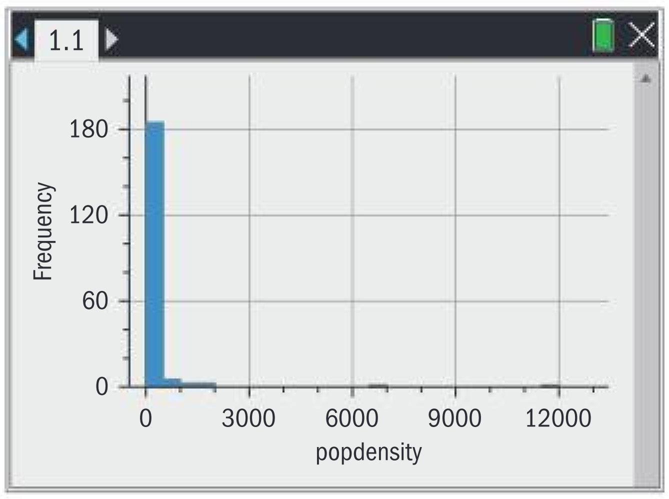
$$f(x) = \begin{cases} 3 - x & 0 \le x \le 2 \\ 1 - (x - 2)^3 & x > 2 \end{cases}$$

and the other curves are those for different values of the parameter.

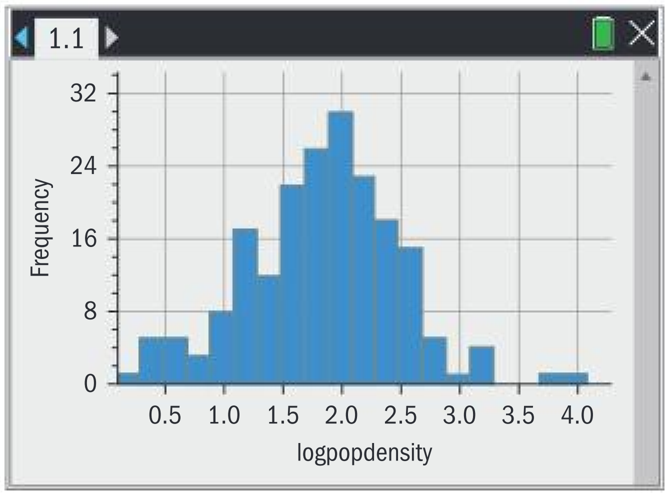
## Scaling numbers and linearizing data

The application of logarithms helps you represent and explore data in useful and revealing perspectives, aiding the modelling process.

For example, a data set of population density for 197 countries measured in the number of people per square km has maximum value 11 929 (Monaco) and minimum value 1.93 (Western Sahara). A histogram of the data shows a very high frequency for population densities of less than 1000 people per square km. The wide range of values on both axes limit the usefulness of this diagram.




Creating a new data set in which each population density $p$ is replaced by $\log_{10}p$ is an example of applying logarithmic scaling to the data. This histogram of the scaled data set reveals more about the data.




For example, Armenia has a population density of approximately 100, which when scaled is represented by $\log_{10} 100 = 2$.

This places Armenia in the modal class of this data set.

The logarithmic scale can then be used to compare population densities. For example, there are 8 countries with population densities approximately one tenth of Armenia's. This is because there are 8 countries in the class with midpoint 1.0, meaning that their population densities are approximately $10^1 = 10$. There is only one country (Monaco) with population density approximately 100 times that of Armenia. This is because Monaco is in the class with midpoint 4, meaning that its population density is approximately $10^4 = 10\,000$.

On a logarithmic scale base $a$, with each increase of one unit on the logarithmic scale, the value of the unscaled variable increases by a factor of $a$. This means that you can use logarithmic scales to represent and explore the size of a change in a variable in proportion to itself.

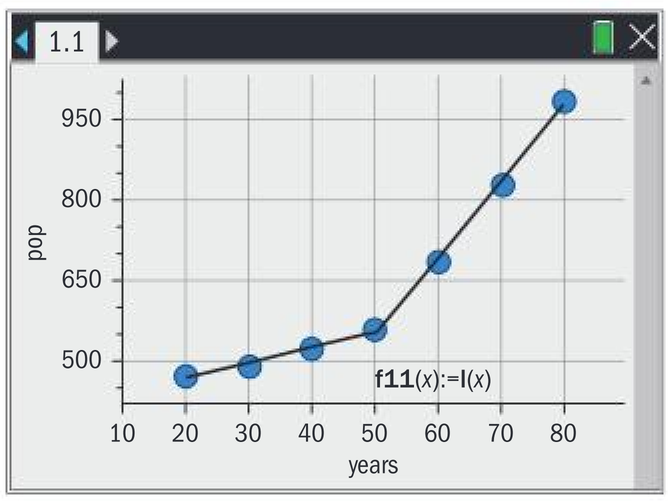
In chemistry, the acidity of a solution (pH) is measured on a logarithmic scale base 10. This means that a change in pH from 3.7 to 4.7 represents an decrease in acidity by a factor of 10.

You can model a data set by application of graph transforms, algebra to determine parameters, regression tools, or by a combination of these methods. Linearization gives you another view of a data set which is valuable because it applies logarithmic scaling to make patterns in the data stand out, as well as giving you another method by which to determine parameters.

For example, consider the task of modelling China's population over the years 1920 to 1980:

| Year | 1920 | 1930 | 1940 | 1950 | 1960 | 1970 | 1980 |
|---|---|---|---|---|---|---|---|
| Population in millions | 472 | 489 | 520.8 | 556.6 | 682.0 | 825.8 | 981.2 |

The graph of the data, where years are the number of years after 1900 is shown.




The data can be modelled by a piecewise function

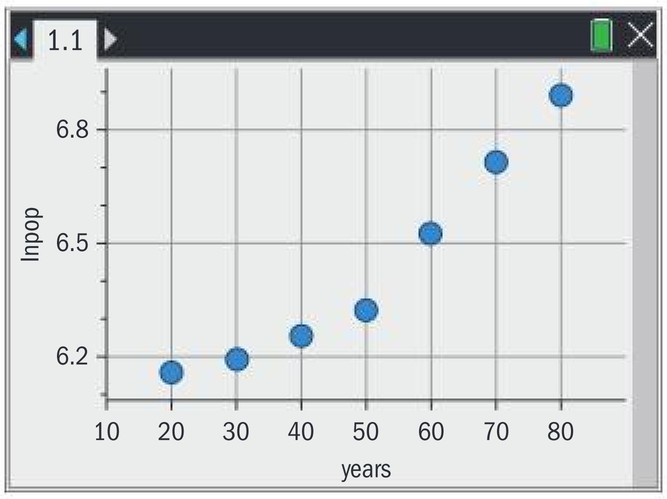
$$l(x) = \begin{cases} 2.856x + 409.6 & 20 \le x < 50 \\ 14.18x - 160.0 & 50 \le x \le 80 \end{cases}$$

The $r^2$ values for each piece are 0.977 and 0.998 respectively, showing an extremely close fit to the data.

However, population growth is most commonly modelled by an exponential function. Linearization can be applied to explore the data further.

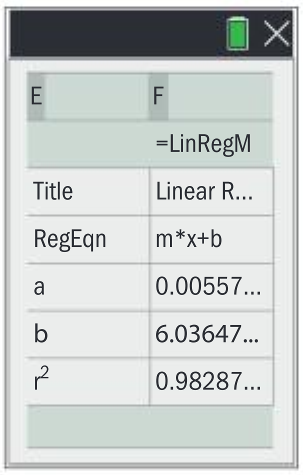
## Linearization for exponential relationships

The steps of the linearization process to determine the parameters of an exponential relationship are described and applied as follows:




(1) Apply logarithmic scaling to the dependent variable $y$ and graph it against the independent variable $x$. (This is an example of a semi-log graph)

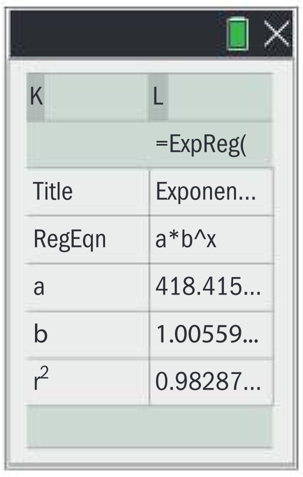
The graph looks very similar to the one on the previous page! However, this is evidence that the data very closely fit an exponential model, as you will see.




(2) Apply linear regression to find the best-fit line(s) in the form $\ln y = mx + c$

The model is
$\ln y = (0.00577..)x + (6.036..)$
with $r^2$ value 0.98287...

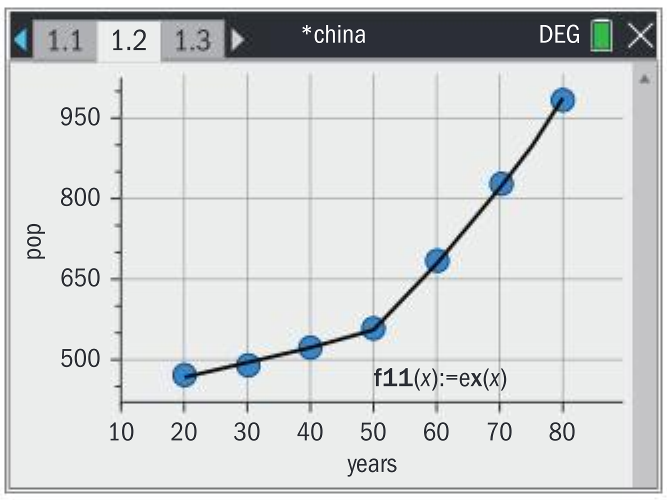
Use your GDC memory to store the values to the full accuracy given by your GDC.




(3) Apply algebra to find the exponential model
$y = e^{mx+c} = e^c e^{mx}$

(4) You can now write the exponential model
$y = e^{(6.036..)} e^{(0.00577..)x}$ which simplifies to
$y = 418.415(1.00559)^x$

> This is exactly how your GDC works out exponential regression. For this reason, $r^2$ value for the exponential and linear regressions are equal.




(5) By repeating these steps, the piecewise exponential model e($x$) can be completed:

$$e(x) = \begin{cases} 418(1.006^x) & 20 \le x < 50 \\ 217.7(1.019^x) & 50 \le x \le 80 \end{cases}$$

has $r^2$ values for each piece of 0.9829 and 0.999 respectively, showing a better fit than the linear piecewise model.

This is an important advantage given the size of the values of the dependent variable.

Reflection: The linearization process showed that there is an exponential model which fits the data points more closely than a linear model since each $r^2$ value is greater for the exponential model. The exponential model is also more appropriate to the context of population growth.

Note that the model e($x$) shows two population **growth rates**: 0.6% and 1.9% per year. The model $l(x)$ has two **population increases**: 2.856 million and 14.18 million per year. e($x$) expresses proportional change of population, whereas $l(x)$ expresses the size of change of population. With linearization you can explore contexts where modelling the rate of growth is the more appropriate than modelling the size of change.

## Linearization for power relationships

The steps of the linearization process for determining the parameters of a power relationship are as follows.

(1) Graph the logarithm of the dependent variable $y$ against the logarithm of the independent variable $x$. (This is a log-log graph.)

(2) Apply linear regression to find the best-fit line(s) in the form $\ln y = m \ln x + c$

(3) Apply algebra to find the power model $y = e^{m \ln(x) + c} = e^c e^{m \ln(x)} = e^c x^m$.

### Assessment tip

If it is not clear which type of linearization to apply, find the best fit lines for the semi-log and the log-log graphs and choose the best fit.

### Example 2.4.7

Jonathan is interested in a career as a sound engineer. He uses his phone to emit a sound and measures its loudness at the following distances from his phone.

| Distance (cm) ($x$) | 20 | 40 | 80 | 100 | 160 | 240 | 320 | 400 | 500 | 600 |
|---|---|---|---|---|---|---|---|---|---|---|
| Loudness (Db) ($y$) | 60 | 27 | 11 | 6 | 5 | 4 | 3 | 2.7 | 1 | 0.6 |

(a) Use linearization to determine if the data is best modelled by an exponential or a power function.

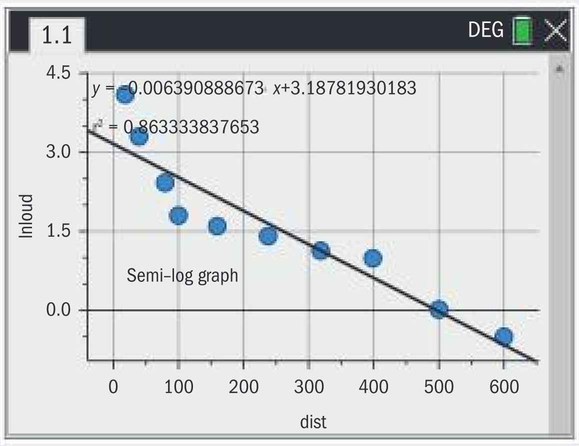
(b) Hence determine the parameters of your model, correct to five significant figures.

(c) Jonathan reads that loudness follows the inverse square law: it is inversely proportional to the square of distance. Describe if his model is consistent with this claim.

**Solution**

(a) Linearizing the data shows that the log-log graph has the best fit because the $r^2$ is greatest: 0.949 > 0.863

Hence the data is best modelled by a power function.

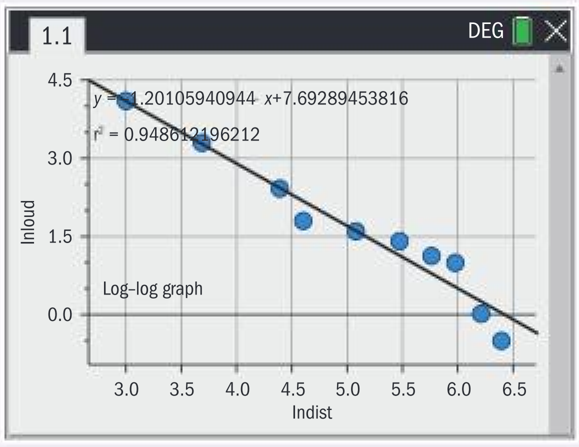


(b) $\ln y = (-1.201\,05..)\ \ln x + (7.692\ldots)$ so

$$y = e^{(7.692\ldots)}\ x^{(-1.201\,05..)}$$

Hence

$$y = 2192.7x^{(-1.2011)}$$




(c) Jonathan's data is best modelled by a power relationship, and the inverse square law is also a power law.

However, Jonathan's power is far from the theoretical value of –2, so he may need to reflect on his data collection methodology.

## Assessment tip

In an examination question that examines your knowledge and understanding of linearization, show all steps in your working to achieve maximum marks instead of using your GDC.

# SL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

**1.** The point $A$ has coordinates (–2, 1) and the point $B$ has coordinates (6, 5).

**a.** Write down the coordinates of $M$, the midpoint of the line segment $AB$.

The point $C$ has coordinates (0, 8)

**b.** Find the gradient of the line $CM$.

**c.** Find the equation of the line $CM$. Write your answer in the form $ax+by+d=0$ where $a$, $b$ and $d$ are integers.

**2.** In an experiment, the intensity $I$ of light on a surface varies inversely as the square of the distance $d$ of the light source from the surface. The intensity of light is measured in candelas (cd) and the distance from the surface in centimetres. You are given that the measurements $I = 97$ cd when $d = 32$ cm were made.

**a.** Find an equation relating $I$ and $d$.

**b.** Hence find the value of $I$ when $d = 6.7$ cm

**3.** Consider the function $h(x)=\frac{5x-10}{6-2x}$

**a.** Show that the coordinates of the $x$- and $y$-intercepts are (2, 0) and $\left(0, -\frac{5}{3}\right)$ respectively.

**b.** Determine the equation of the horizontal asymptote.

**c.** Explain why the largest possible domain of $h$ is not the set of real numbers.

**d.** Sketch the graph of $h$.

**4.** The equation of line $L_1$ is $y-2=\frac{1}{5}(x+10)$

**a.** Write down the gradient of $L_1$

Point $A$ lies on $L_1$ and has $y$-coordinate 9

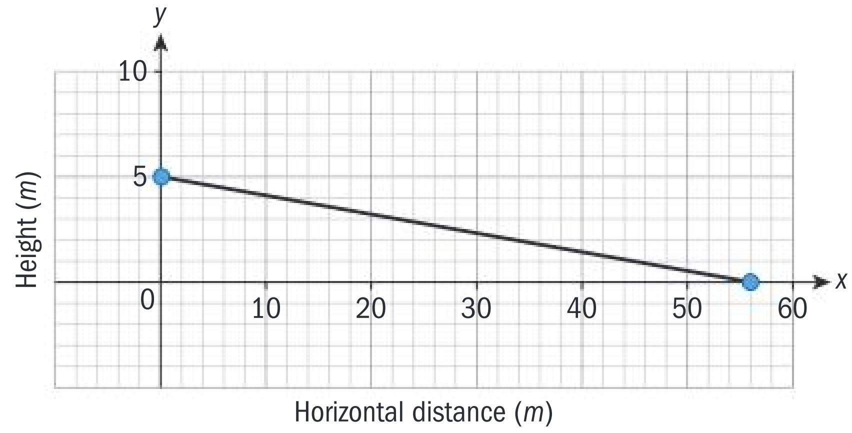
**b.** Find the $x$-coordinate of $A$.

Line $L_2$ is perpendicular to $L_1$ and passes through the $y$-intercept of $L_1$.

**c.** Determine the equation of $L_2$. Write your answer in the form $y = mx + c$.

**5.** City planners are exploring two designs, $A$ and $B$, for a car park. Safety regulations state that the gradient of the car park must not exceed 0.09

Design $A$ has a decent of 5 m, as shown in the diagram.




Design $B$ is modelled by the equation $4x+50y+350=0$

Determine if either design will pass the safety regulations.

**6.** A steel girder is heated. Its length $L$ in metres can be modelled by the linear function $L = aT + b$, where $T$ is the temperature in degrees Celsius °C. At 231°C, the length of the steel girder is 4.53 metres.

**a.** Write down an equation that shows this information.

At 301 °C, the length of the steel girder is 4.71 metres.

**a.** Write down an equation that shows this second piece of information.

**b.** Hence find the length of the girder at 276 °C

## GROUP 2

**7.** The number of people $T$ viewing a post by a celebrity on social media is modelled by the function $M(t) = R\times 2^t$, $0\le t\le 10$, where $R$ is a constant and $t$ is the time in hours since the post was made. Five hours after the post was made, the number of people who had viewed the post was 99 328.

**a.** Find the value of $R$.

**b.** Interpret what $R$ represents in this context.

**c.** Determine if the number of people viewing the post will exceed 1500$R$ in the domain given.

**d.** State a reason why it would not be reasonable for the domain of $M$ to include all real numbers.

8. The depreciation of David's car in Euros (€) can be modelled by the function

$C(n) = 13300 \times 0.86^n + 550$, $n \geq 0$ where $n$ is the number of years after David bought his car.

a. Find the value of David's car after 6 years.

When the value of David's car falls below €2000, he can exchange it for a €3500 discount towards the value of an electric car.

b. Calculate after how many complete years David will be able to exchange his car for the discount.

David predicts that the value of his car will never be less than €$L$.

c. Write down the greatest possible value of $L$.

9. Let $f(x) = \frac{2x^2 - 8}{x^2 + 5} + 3$

a. Sketch the graph of $f$, indicating the coordinates of all intercepts and the position and equation of the horizontal asymptote.

b. Let $P(t) = f(t)$, $t \geq 0$. This is used to model the population $P$ of an island in thousands, $t$ years after January 1, 2020.

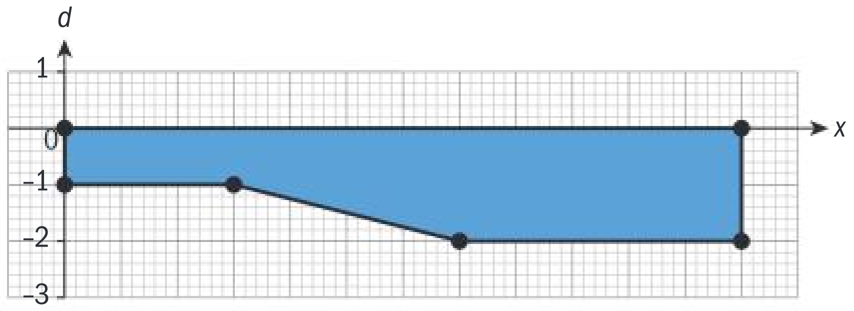
i. Predict the island's population on January 1, 2023, to the nearest 100.

ii. Predict the long-term trend of the population.

10. The depth (in metres) of a swimming pool when it is full is modelled by the function $d$:

$$d(x) = \begin{cases} -1 & 0 \leq x < 3 \\ \frac{-1}{4}x - \frac{1}{4} & 3 \leq x < 7 \\ -2 & 7 \leq x \leq 12 \end{cases}$$

where $x$ is the distance in metres from the left end of the swimming pool, as shown in the diagram:




a. Apply the function to complete the table:

| $x$ | 1 | 3 | 5.5 | 12 |
|---|---|---|---|---|
| Depth | | | | |

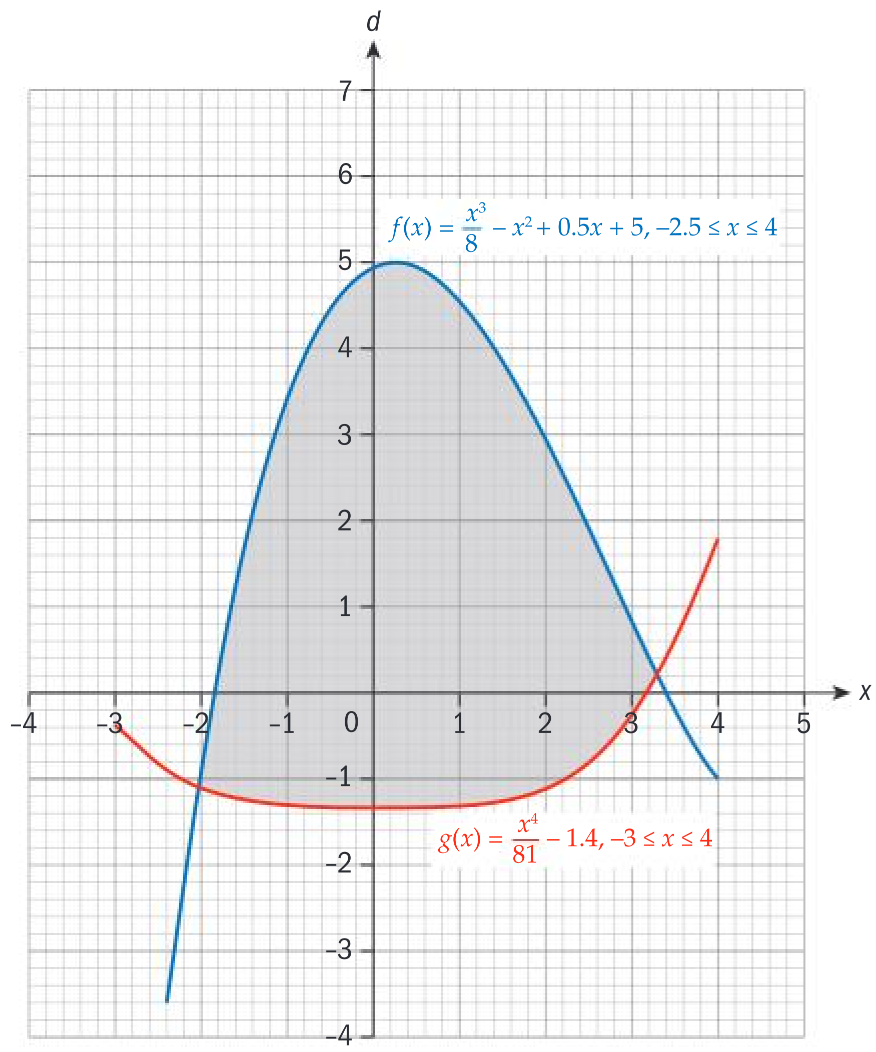
b. Given that the swimming pool is 3 m wide, calculate the volume of the water it holds when full.

11. The perimeter of a swimming pool is modelled by two functions:

$f(x) = \frac{x^3}{8} - x^2 + 0.5x + 5$, with domain $-2.5 \leq x \leq 4$

and $g(x) = \frac{x^4}{81} - 1.4$, with domain $-3 \leq x \leq 4$

A top view of the pool is shown in the diagram as a shaded region, and all distances are measured in metres.




a. Find the range of $f$.

b. Find the greatest possible difference between the $x$-coordinates of any two points in the pool.

c. By considering the graph of $d(x) = f(x) - g(x)$, find the greatest possible difference between the $y$-coordinates of any two points in the pool.

## GROUP 3

**12.** 

The volume of an icosahedron varies directly with the cube of the length of its edge $x$.

An icosahedral die with edge 1.5 cm has a volume of 7.36 $cm^3$

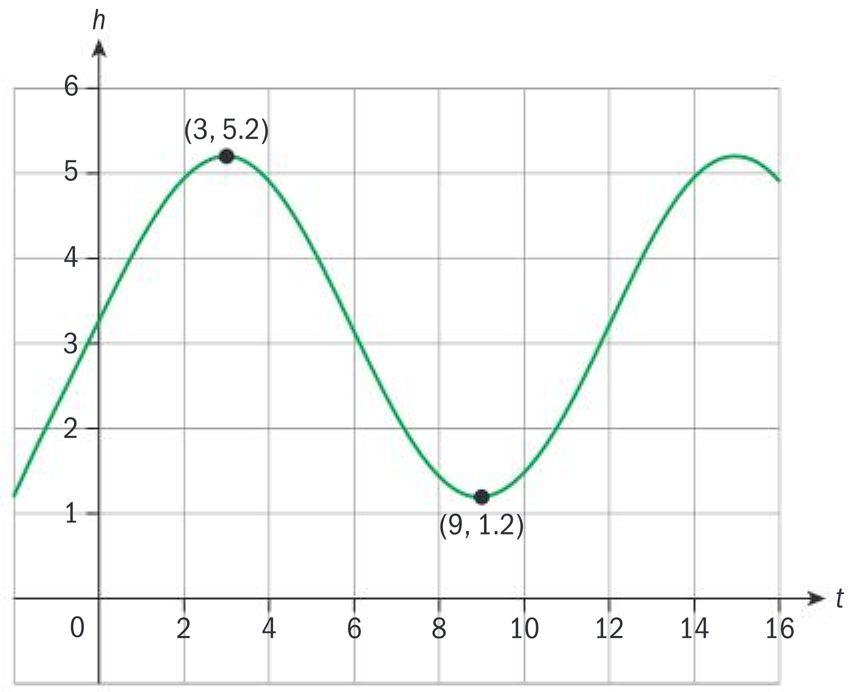
**a.** Find the equation for the volume $V$ of an icosahedron in terms of its edge $x$.

**b.** An icosahedral greenhouse with edge 5 m is planned for a gardening show.
Find the volume of the greenhouse in $m^3$.

**c.** A jewel in the shape of an icosahedron has volume $0.5\,cm^3$, correct to 1 decimal place.
Calculate the upper and lower bounds of the edge length of the jewel.

**13.** The depth of water in a harbour is modelled by the function $h(t) = r + p\sin(st°)$, where $h$ is measured in metres and $t$ represents the number of hours after midnight.

The following diagram shows the graph of $h$.



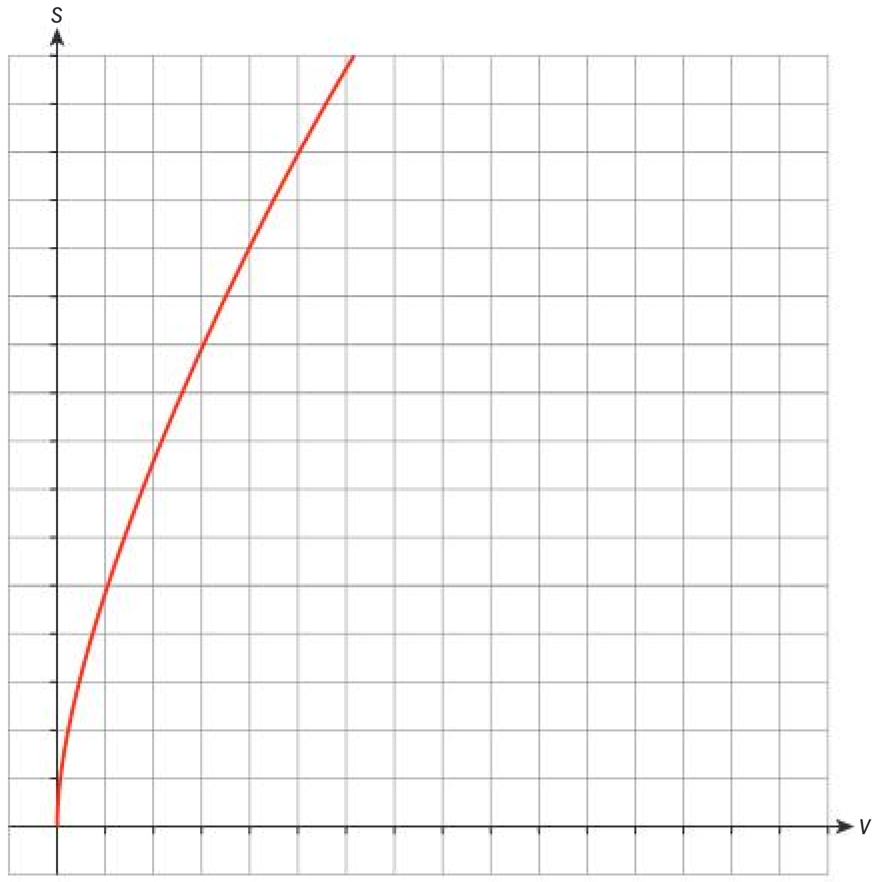

The graph has a maximum at (3, 5.2) and minimum at (9, 1.2)

**a.** Find the values of $r$, $p$ and $s$.

**b.** Interpret what $r$ represents in this context.

**c.** George's boat can set off from the harbour if the depth of water is at least 4 m, and he wants to sail in the afternoon. Determine the earliest time at which he can depart the harbour in his boat.

**14.** The surface area $S$ of a given cube can be represented by the function $S(V) = 6\left(\sqrt[3]{V}\right)^2$, $V \geq 0$ where $V$ is the volume of the cube.
The graph of $S$ is shown for $0 \leq V \leq 8$.




**a.** Write down the value of $S(0)$

**b.** Write down the value of $S(8)$ and interpret its meaning in the context of the question.

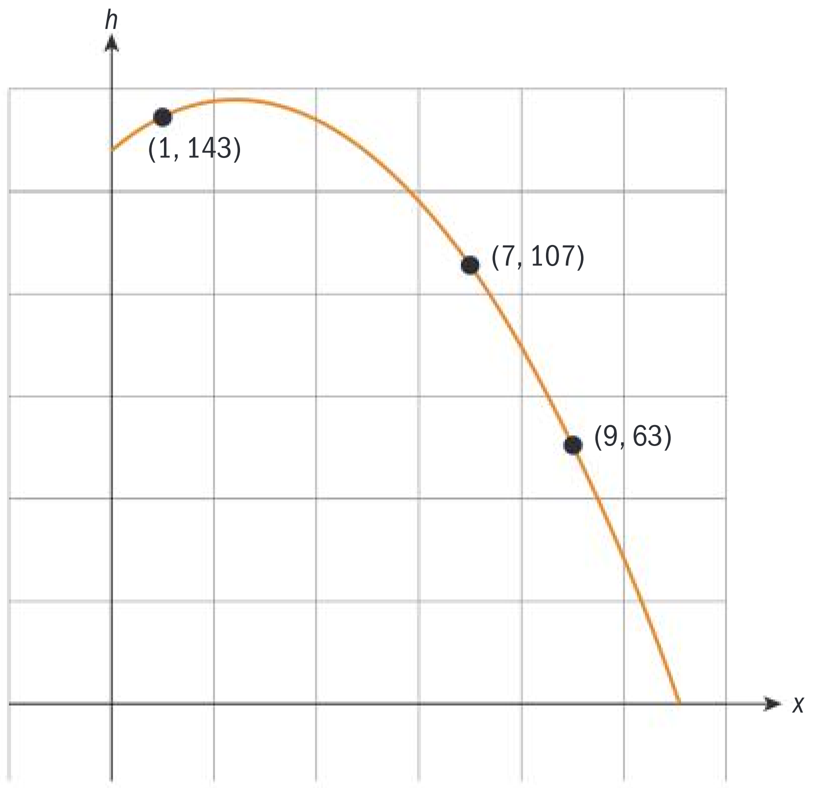
The range of $S(V)$ is $p \leq S(V) \leq q$

**c.** Write down the values of $p$ and of $q$.

**d.** Draw the graph of the inverse function $S^{-1}(V)$ on the diagram given.

**e.** In the context of the question, explain the meaning of $S^{-1}(6) = 1$.

**15.** In an experiment, a football is kicked from the top of a tower. Its vertical height $H$ is a function of its horizontal distance $x$ from the base of the tower. Both $H$ and $x$ are measured in metres. The experiment is filmed and the positions (1, 143), (7, 107) and (9, 63) are recorded. $H$ is modelled by the function $H(x) = px^2 + qx + r$.



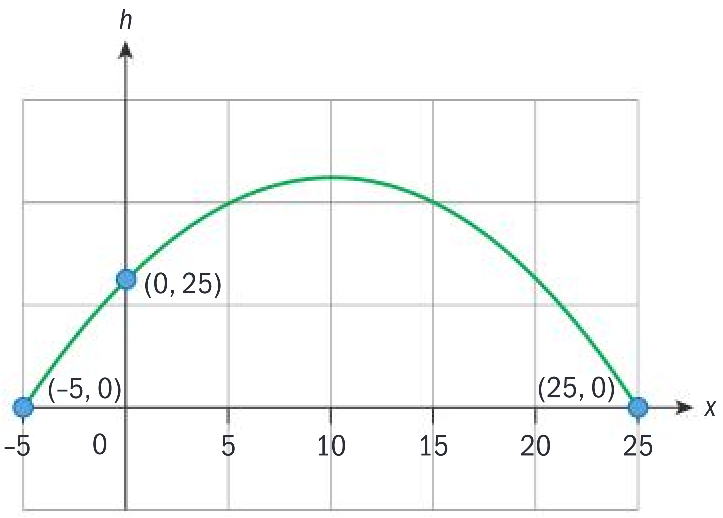

**a.** Write down a system of equations in $p$, $q$ and $r$. Hence, determine the parameters of $H$.

**b.** Hence, write down the height of the tower.

**c.** Determine the coordinates of the point at which the football is at its maximum height.

**d.** Determine the coordinates of the point where the football reaches the ground.

**16.** A roof design of a greenhouse can be modelled by the function $h(x) = ax^2 + 4x + c$. The graph of $h$ is shown for $-5 \le x \le 25$. Both $h$ and $x$ are measured in metres.




**a.** Write down the value of $c$.

**b.** Find the value of $a$.

**c.** Hence find the height of the roof.

The roof is required to fit inside a building in the shape of a cuboid with rectangular cross-sectional area.

**d.** Use your answer to **c** to determine if the dome can fit in a rectangular area of 1400 m$^2$ with length 32 m.

**17.** A new suburban area of a city is planned with a network of roads. A health centre is planned at $H(16, 2)$, representing a position 16 km east and 2 km north of the central plaza $P(0, 0)$. The fire station is located at $F(2, 20)$. A road named Middle Way is planned as a line $L$, the perpendicular bisector of $HF$.

**a.** Find the equation of $L$.

Two roads pass through $P$: North Avenue is the $y$-axis and East Way is the $x$-axis.

**b.** Find the points at which $L$ crosses North Avenue and East Way.

**c.** Find the points at which $HF$ crosses North Avenue and East Way.

**d.** Hence find the shortest distance from the central plaza to the fire station using a selection of the roads Middle Way, North Avenue, East Way and $HF$.

**18.** You are given the 24-hour average temperatures for Beijing[1], where $m = 0$ represents January, $m = 1$ represents February and so on.

| Month ($m$) | Temperature °C ($T$) |
|---|---|
| 0 | −4.6 |
| 1 | −1.8 |
| 2 | 4.7 |
| 3 | 13.6 |
| 4 | 20.0 |
| 5 | 24.4 |
| 6 | 26.0 |
| 7 | 24.7 |
| 8 | 19.8 |
| 9 | 12.6 |
| 10 | 3.9 |
| 11 | −2.6 |

**a.** Plot the data using technology. $m$ is the independent variable and $T$ the dependent variable.

**b.** Assuming this data set repeats every year, model the data by determining the parameters in the model $T = k\cos(pm°) + w$, where $k < 0$.

**c.** Add the graph of $T$ to the data plot and comment on how well your model fits.

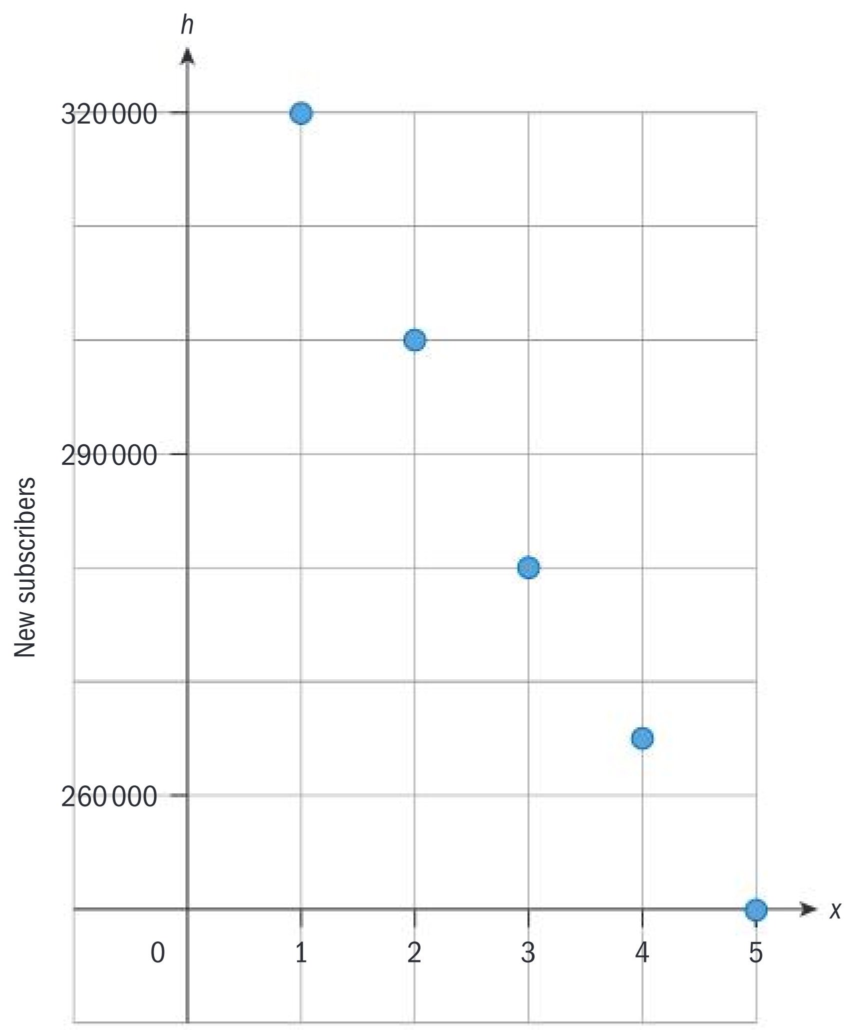
1 http://www.worldclimate.com/cgi-bin/data.pl?ref=N39E116+1102+54511W

## PAPER 2

**1.** The management of the social media site Chumnet collect data on the number of new subscribers to their site every month, $w$ weeks after start-up.

| $w$ | New subscribers $S$ |
|---|---|
| 1 | 320 000 |
| 2 | 300 000 |
| 3 | 280 000 |
| 4 | 265 000 |
| 5 | 250 000 |

Chumnet management graph the data and then model the data with linear and exponential models as below.




Linear model: $S = mw + c$, $r^2 = 0.994$

Exponential model: $S = AB^w$, $r^2 = 0.998$

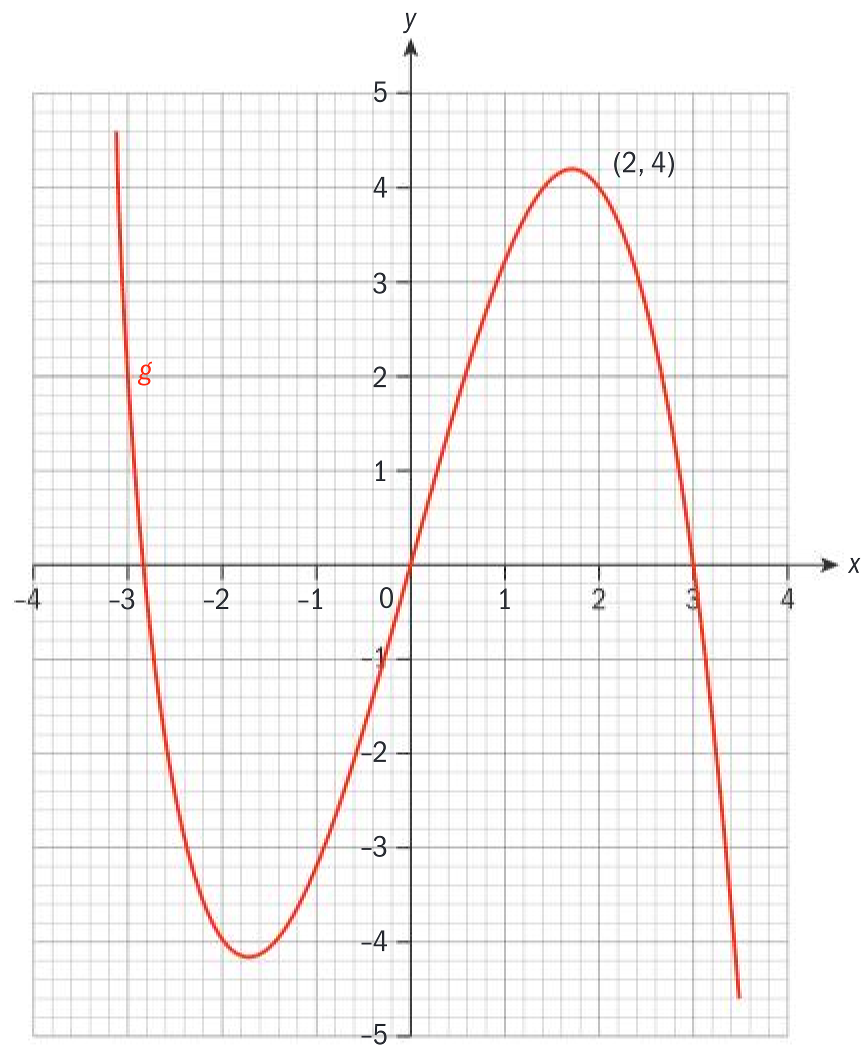
**a.** State, with reasons, why an exponential model is more appropriate for the number of new subscribers.

**b.** Use exponential regression to determine the values of $A$ and of $B$, correct to six significant figures.

**c.** Predict the number of new subscribers in week 10, to the nearest 1000.

**d.** Predict the number of weeks when the number of new subscribers first falls below 50 000.

**2.** The graph shows part of a function $g$. The graph of $y = g(x)$ passes through the point (2, 4)




**a.** Write down the domain of $g$.

**b.** Write down the value of $g(2)$.

**c.** Draw the graph of $y = 1 - 0.5x$ on the same axes.

**d.** Hence, find the solutions to the equation $g(x) = 1 - 0.5x$ correct to the nearest 0.2

**e.** For how many values of $k$ would the equation $g(x) = k$ have exactly two solutions? Explain your answer by adding a sketch to the graph.

# HL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

1. a. Sketch the function $f(x)=\ln(x+2)-4$

   b. State the largest possible domain of $f$

   c. Find $f^{-1}(x)$

   d. State the domain and range of $f^{-1}$

2. a. Given that $f(x)=3x+2$ and $g(x)=2\cos(x)-1$, find $g\circ f$

   b. Hence write down the period of $f\circ g$

   c. Given that $h(x)=ax+b$, find $a$ and $b$ so that $h\circ g$ has the same amplitude as $s(x)=5\sin(4x-1)+3$ and the same principal axis as $t(x)=2\cos(x)-4$

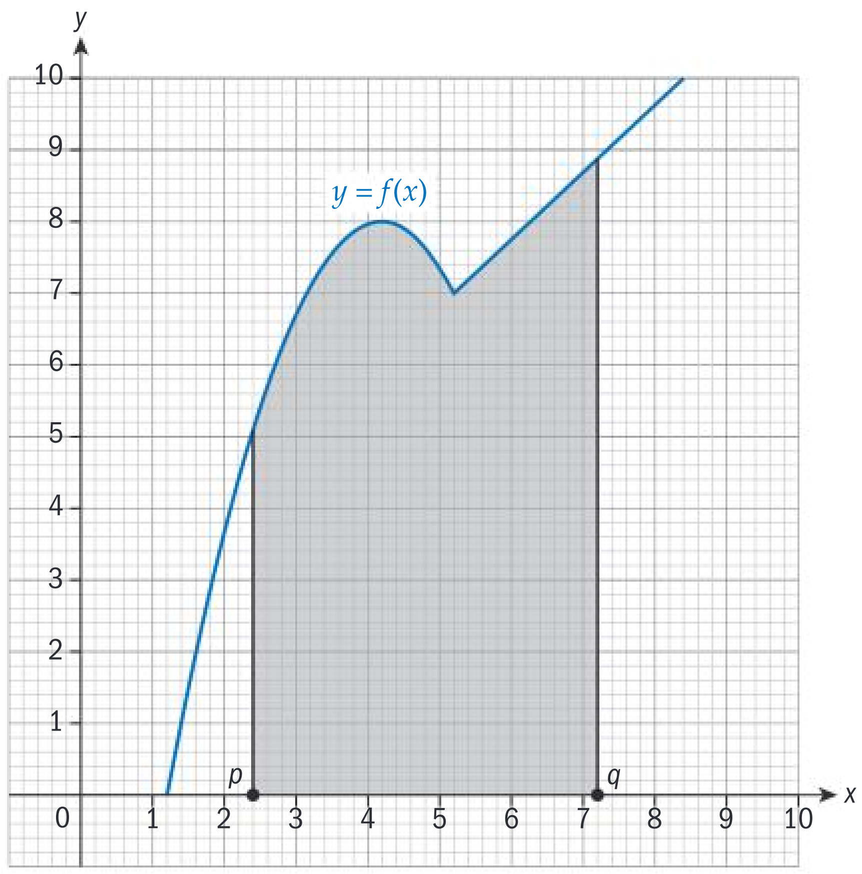
   d. For these values of $a$ and $b$, determine the range of $h\circ t$.

3. Consider the function

$$f(x)=\begin{cases} 8-(x-a)^2 & x<5 \\ x+2 & x\geq 5 \end{cases}$$

   a. Find the value of $a$ in order to ensure that $f$ is a continuous function.

   The graph of $y=f(x)$ is used to model designs for the roof of a classroom. You are given a sketch of the graph on an interval $[p, q]$ on which the area under the graph is 36.2 m$^2$.




   The architects transform the graph to model different designs. For each transformation, find the area under the graph, giving reasons for each answer.

   b. $f(x)$ is transformed to $f(x-2)$

   c. $f(x)$ is transformed to $f(x)+3$

   d. $f(x)$ is transformed to $2f(0.5x)$

4. Sarah bakes a cake and takes it out of the oven to cool. She measures the internal temperature of the cake over time and collects the following data:

| $t$(minutes) | 0 | 15 | 30 | 45 | 60 | 75 |
|---|---|---|---|---|---|---|
| $C$ (°C) | 98 | 69 | 50 | 28 | 23 | 20 |

   Sarah believes that the temperature of the cake $C$ can be modelled by a function of the form $C(t) = ae^{kt} + 18$ where $t$ is the time in minutes after the cake is taken from the oven to cool.

   a. Show that $\ln(C-18)=kt+\ln(a)$.

   b. Find the equation of the regression line of $\ln(C-18)$ on $t$.

   c. Find the value of $k$ and of $a$.

   d. Sarah wants to decorate the cake when its temperature first falls under 19 °C. Predict how long Sarah will have to wait until she decorates the cake.

5. The depth of water $D$ in a harbour is measured in metres at $t$ hours after midnight.

| $t$(hours) | $D(m)$ |
|---|---|
| 0 | 6.5 |
| 1 | 4.6 |
| 2 | 3.4 |
| 3 | 2.3 |
| 4 | 2.1 |
| 5 | 2.6 |

| $t$(hours) | $D(m)$ |
|---|---|
| 6 | 3.4 |
| 7 | 4.7 |
| 8 | 6.5 |
| 9 | 7.6 |
| 10 | 8.1 |
| 11 | 8 |

   a. Use sinusoidal regression to model the data and write your model in the form $f(t)=a\sin(bt+c)+d$

   The depth of water in a different harbour in the same time zone is modelled by $g(t)=-3\cos(0.5t)+4.7$

   b. Find the time in hours for which the depth of water in **both** harbours is above 3.7 m each morning.

6. Consider the functions $f(x)=0.2x-3$ and $g(x)=x^3+7$

a. Find $g^{-1}(x)$ and $f^{-1}(x)$

b. Find $(f\circ g)(x)$ and $(f\circ g)^{-1}(x)$

c. Hence, show that $(f\circ g)^{-1}(x)=(g^{-1}\circ f^{-1})(x)$

d. Given that $h(x)=\ln(x)$, show that $((f\circ g)\circ h)(x)=(f\circ(g\circ h))(x)$

## GROUP 2

7. You are given the logistic function $f(x)=\dfrac{L}{1+Ce^{-kx}}$ where $L,k,C>0$

a. Explain why $y=L$ is a horizontal asymptote of $y=f(x)$

b. Show that $f$ has no turning point.

c. Show that the $x$-coordinate of the point where the values of $f$ are changing fastest is $x=\dfrac{\ln C}{k}$

d. Hence, find the $y$-coordinate of the point where the values of $f$ are changing fastest.

8. The graph of the function $C(n)=n^2$ is transformed to the graph of $y_2=2n^2+1$

a. Describe the sequence of transformations $T$ that transform the graph of $C$ to the graph of $y_2$

b. Find the image of the point (3, 9) after applying $T$.

The graph of the function $v(t)=\sin(t)$ is transformed to the graph of $y_3=2.1\sin(-3t)$

c. Describe a sequence of transformations $S$ that transforms the graph of $v$ to the graph of $y_3$

d. Find the image of the point $\left(\dfrac{\pi}{3},\dfrac{\sqrt{3}}{2}\right)$ after applying $S$.

9. Consider the function $n(x)=\dfrac{4x}{x^2+1}$, $x\in\mathbb{R}$.

The graph of $n$ gives a shape called Newton's serpentine.

a. Find the coordinates of the turning points of $n$ and the equation of the asymptote.

b. Sketch the graph of $n$.

c. Find the minimum value of $a$ such that $f(x)=n(x)$, $x\geq a$ has an inverse. Justify your answer.

d. Hence, find $n^{-1}(x)$, stating its domain and range.

10. A population of bacteria grows in a Petri dish. The number of bacteria $N_1$ is recorded at different times as follows:

| $t$ (hours) | 0 | 0.5 | 1 | 2 |
|---|---|---|---|---|
| $N_1$ | 30 | 70 | 320 | 1070 |

a. Use logistic regression to find a logistic model $N_1(t)$

b. Hence, predict the maximum possible population for the bacteria.

c. Find the time taken for the population to reach 90% of this maximum value.

A similar experiment takes place in a larger Petri dish at the same time. The number of bacteria is modelled by $N_2(t)=\dfrac{1351}{1+90e^{-2.5t}}$

d. Find the interval in which $N_2$ is more than 10% greater than $N_1$

11. Let $f(x)=\dfrac{1}{x^2+1}$ and $g(x)=1-\ln(x+2)$.

Find the two solutions of the equation $(f\circ g)(x)=(g\circ f)(x)$

12. A metal girder of length 10 metres is being heated in an oven as part of the process needed to harden the girder. The temperature of the oven $E$ is modelled by the function $E(t)=107+0.15t$, where $t$ is time in seconds since the hardening process began. The length $L$ of the girder, in metres, is modelled by the function $L(t)=10+1.2\times10^{-5}t$

Find $L$ as a function of $E$.

## GROUP 3

13. A small business produces and sells $x$ units of a product. The business receives revenue $R$, measured in 1000's of Euros, modelled by the function

$R(x)=200-\dfrac{1600}{x+8}-x$, where $R\geq 0$ and $0\leq x\leq a$

a. Sketch the graph of $y=R(x)$, showing the coordinates of the maximum and both axes intercepts.

b. Hence determine the largest possible value of $a$.

The business explores various models to try to increase their maximum revenue.
One model is $S(x)=1.1R(0.9x)$.

**c.** Describe the transformations that transform the graph of $R$ to that of $S$.

**d.** Add a sketch of $y = S(x)$ on the same set of axes as in part **a**.

**e.** Comment on the effect of the transformations on the maximum revenue and the units sold, giving your answers correct to the nearest whole number.

**14.** Chaya is exploring the time taken for her daily commute to school. First, she cycles 3 km at an average speed of $v$ km h$^{-1}$. She then waits, on average, for 10 minutes for a bus which travels the remaining 8 km to Chaya's school. The bus has an average speed of 20 km h$^{-1}$.

**a.** Show that Chaya's average speed for her commute can be modelled by the function $C(v) = \frac{330v}{17v+90}$

**b.** Find $C^{-1}(v)$

**c.** Hence, find the value of $v$ required for Chaya to have an average speed of 15 km h$^{-1}$ for her commute.

**d.** Chaya's friend claims she can carry out the same journey with an average speed of 20 km h$^{-1}$. Determine if this claim is true, stating your reasons.

**15.** In economics, the quantity of output per month $O$ is related to:

- the number of hours of labour per month $N$ and
- the number of hours of physical capital used in a month $K$

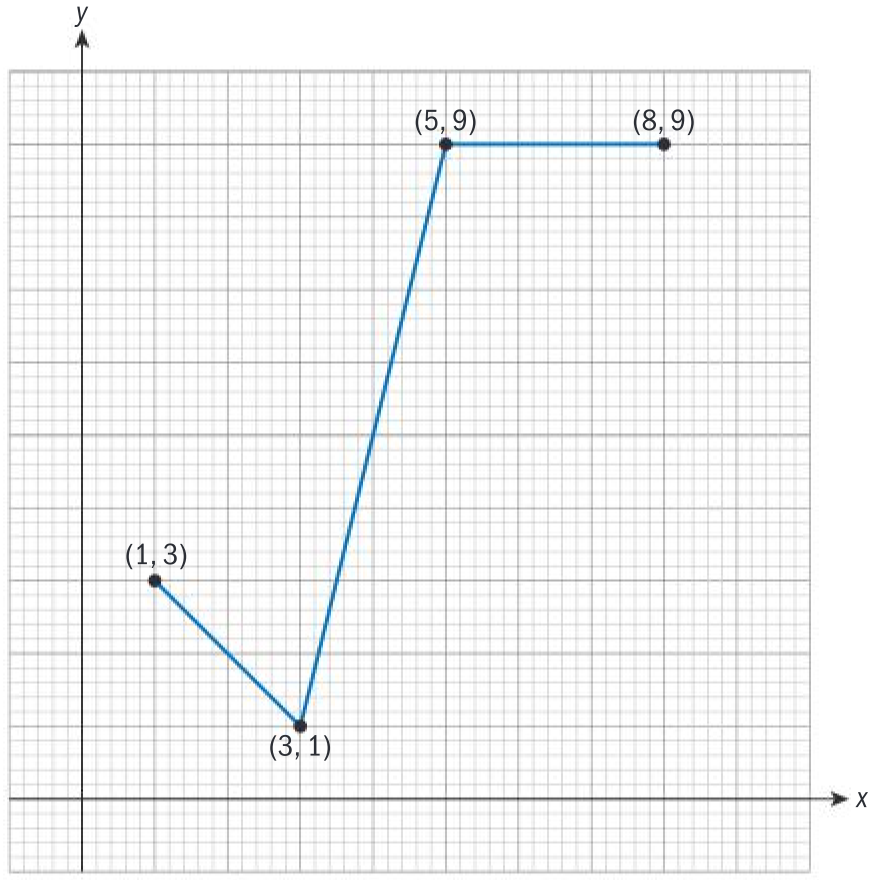
by the equation $O = AN^bK^c$ where $A, b, c \in \mathbb{R}$.

Data is collected from a factory in three separate months.

| Month | $O$ (units/month) | $N$ (hours/month) | $K$ (hours/month) |
|---|---|---|---|
| 1 | 6546 | 75 | 37 |
| 2 | 9216 | 80 | 45 |
| 3 | 3939 | 30 | 41 |

Find $b$, $c$, and $A$.

**16.** The graph of $y = f(x)$ is shown below.




The equation of $f$ is $f(x) = \begin{cases} a - x & 1 \le x < b \\ (x-c)^2 & b \le x < d \\ e & d \le x < 8 \end{cases}$

**a.** Find the values of $a$, $b$, $c$, $d$ and $e$

The graph of $f$ is translated by the vector $\begin{pmatrix} 2.5 \\ 1 \end{pmatrix}$ to give the graph of a function $h$, whose graph is then vertically stretched with scale factor 2 to give the graph of a function $g$.

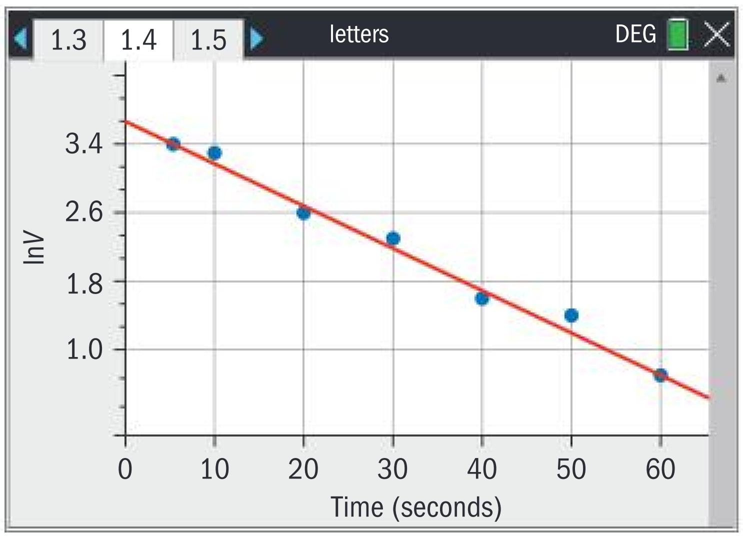
**b.** Find the image of (3, 1) under this sequence of transformations.

**c.** Find the equation of $g$.

**17.** Tony carries out an experiment to investigate the discharge of voltage $V$ through a resistor. He measures the voltage at various times $t$ and writes down the following data:

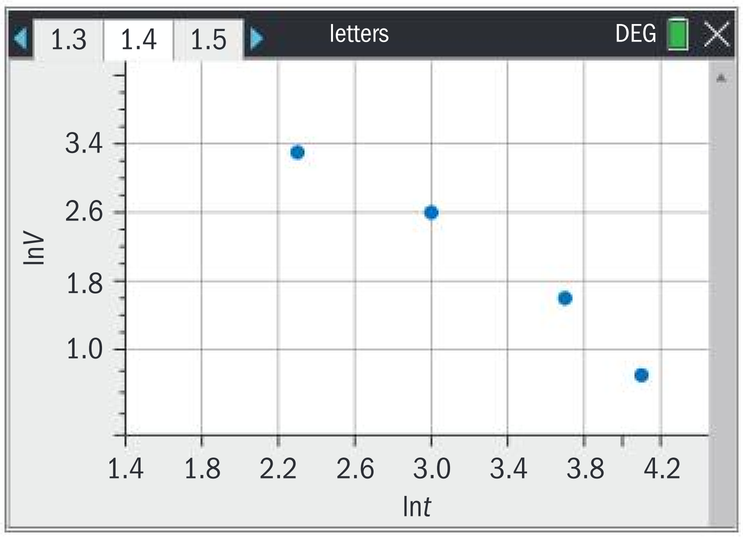
| $t$ (seconds) | 5 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $V$ (Volts) | 30 | 27 | 14 | 10 | 5 | 4 | 2 |

Tony wants to know which of $V = ae^{ct}$ or $V = pt^q$ best models his data. Tony plots the following semi-log graph of $\ln V$ against $t$ and finds the equation of the line of best fit using technology. (See GDC screenshot on the next page.)




**a.** Write down the product-moment correlation coefficient of the line of best fit shown on the semi-log graph.

**b.** Plot a log-log graph of $\ln V$ against $\ln t$ using technology and use it to plot the missing points on this diagram:

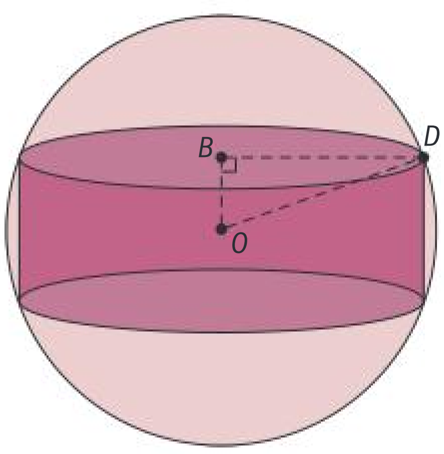



**c.** Write down the product-moment correlation coefficient of the line of best fit of the log-log graph.

**d.** Hence, determine which model best fits Tony's data, justifying your answer.

**e.** Hence, determine the parameters of Tony's model.

**18.** A cylinder is inscribed in a sphere $S$ of radius 6 cm. In the diagram, $O$ is the centre of the sphere, $[BD]$ is the radius of the cylinder.




**a.** Show that the volume of the cylinder $V$ is modelled by the function

$V(x) = 2\pi x(36 - x^2)$, where $OB = x$.

**b.** State the domain of $V$.

Xavier is an artist. He defines a "perfect" cylinder as one that has the same value for its volume as for its surface area.

**c.** Find the equation of the surface area of the cylinder inscribed in $S$.

**d.** Hence find the radius and height of the perfect cylinder that exists in $S$.

## PAPER 2

**1.** Consider the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

**a.** Write down the largest possible domain of $f$.

**b.** Sketch the graph of $f$, showing the coordinates of the maximum point, the equation of the asymptote and the equation of the axis of symmetry.

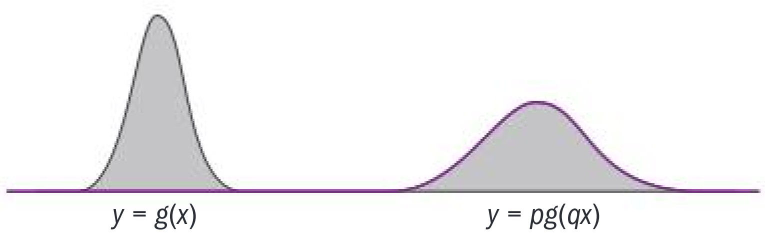
**c.** Use technology to find the coordinates of the point where the values of $f$ are growing fastest.

Consider the function $g(x) = f(x - 15)$

**d.** Find the coordinates of the maximum point of $g$.

**e.** Find the area enclosed by the graph of $g$, the $x$-axis and the lines $x = 10$ and $x = 20$

The sketch shows the graph of $y = g(x)$ and $h(x) = pg(qx)$




**f.** Determine the values of $p$ and of $q$ such that the axis of symmetry of $h$ is $x = 30$ and the $y$-coordinate of the maximum point of $h$ is one half of that of $g$.

**g.** Explain why the area enclosed by the graph of $h$, the $x$-axis and the lines $x = 20$ and $x = 40$ is the same as your answer for part **e**.

**2.** Johannes Kepler (1571–1630) studied planetary motion and is credited with discovering the relationship between the time taken for a planet to orbit the sun and its average distance from the sun.

| Planet | $d$ (Average distance from sun, in millions of km) | $t$ (Time of one orbit, days) |
|---|---|---|
| Mercury | 57.9 | 88 |
| Venus | 108.2 | 225 |
| Earth | 149.6 | 365 |
| Mars | 228.0 | 687 |
| Jupiter | 778.6 | 4333 |
| Saturn | 1427.4 | 10760 |
| Uranus | 2838.2 | 30686 |
| Neptune | 4497.8 | 60191 |
| Pluto | 5913.5 | 90739 |

**a.** Plot $t$ against $d$ using technology, then plot $\ln t$ against $\ln D$ using technology on a separate graph. Hence comment on why scaling both variables in a log-log graph helps represent the data better than an unscaled graph would.

**b.** Use the best-fit line to find the parameters $a$ and $n$ of the function $t = ad^n$

**c.** If a new planet was discovered 8500 million km from the sun, apply your model from part **b** to predict the time it would take to orbit the sun to the nearest day, stating any assumptions that you make.

**d.** Use your model to predict the time for the dwarf planet Ceres to orbit the sun. Ceres' average distance from the sun is 413 million km.

**e.** Calculate the percentage error in your calculation, given that the actual time for one orbit is 1682 days. Comment on your answer.

The following data shows the time required for each of Jupiter's moons to complete one orbit of Jupiter.

| moon | $X$ (Average distance from Jupiter, km) | $Y$ (Time of one orbit, hours) |
|---|---|---|
| Io | 422000 | 42.5 |
| Europa | 671000 | 85.2 |
| Ganymede | 1072000 | 171.7 |
| Callisto | 1883000 | 400.5 |

**f.** Investigate if your model in part **b** can be adapted to model this data set.

## PAPER 3

In a biology project, Rea investigates the rate at which a toxin can kill a population of dangerous bacteria.

Rea defines the variable $t$ as the time in minutes after she administers a dose of toxin to a population of bacteria, and the variable $B_1$ as the number of bacteria remaining after $t$ minutes. The initial population is 100 bacteria. Rea records the following data:

| $t$ | $B_1$ |
|---|---|
| 0 | 100 |
| 500 | 71 |
| 1000 | 25 |
| 1500 | 6 |
| 2000 | 2 |

**a.** Plot the values of $B_1$ against $t$ on your GDC. Use regression tools to model the data with:

**i.** a cubic model

**ii.** a logistic model.

Write down the equations of each model.

**b. i.** Sketch the data points and each model on one set of axes.

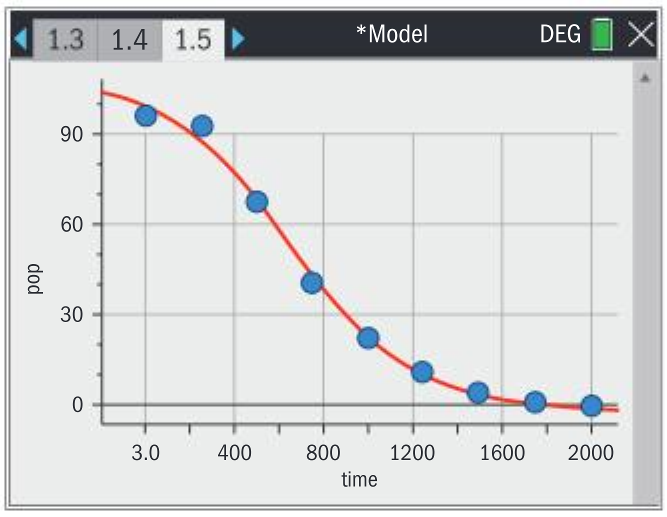
**ii.** Use your graph to comment on how well the equations model the context.

Rea predicted that the toxin would eventually eliminate the population of bacteria.

**iii.** Comment on how well each model fits Rea's prediction.

Rea expands her data set by repeating her experiment and counting the number of bacteria every 250 minutes.

She plots her data and uses regression to fit a logistic model $B_2 = \dfrac{112.41}{1+0.08928e^{0.00368t}}$ as shown on the GDC screenshot on the next page.




| $t$ | $B_2$ |
|---|---|
| 0 | 100 |
| 250 | 97 |
| 500 | 71 |
| 750 | 44 |
| 1000 | 25 |
| 1250 | 13 |
| 1500 | 6 |
| 1750 | 3 |
| 2000 | 2 |

Other researchers tell Rea that in general, the population of bacteria **increases** initially while the toxin is taking effect.

**c.** Suggest an improvement that could be made to the shape of Rea's model $B_2$ for it to better model her expanded data set.

After reading widely, Rea constructs a modified model $B_3 = \frac{600}{4+e^{-0.011t}+e^{0.0029t}}$

**d.** Use technology to plot the data points of Rea's expanded data set and the modified model on the same set of axes. Comment on how the modified model fits with Rea's expectation that the population should initially increase.

In a replication of the experiment, Rea records 105 bacteria after 100 minutes.

**e.** Calculate the percentage error between this value and the value predicted by the modified model.

**f.** Find the derivative of the model $B_3 = \frac{600}{4+e^{-0.011t}+e^{0.0029t}}$

**g.** Hence, show that the time at which the population will reach its maximum is $t = 95.9$

**h.** Hence, demonstrate that the parameters in the general model $y = \frac{p}{q+e^{rt}+e^{st}}$ which determine the time at which the population reaches its maximum are $r$ and $s$.

Rea reads the work of another researcher who carries out an identical experiment, claiming that the maximum rate of increase of $B_3$ is the same size as its maximum rate of decrease.

**i.** Determine if this claim is true for Rea's experiment.

# 3 GEOMETRY AND TRIGONOMETRY

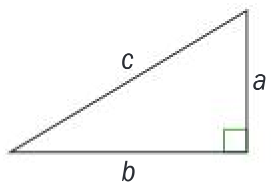
## 3.1 WORKING WITH TRIANGLES

### You must know:

- ✔ the sine rule, cosine rule and the area of a triangle formula
- ✔ the definition of angles of elevation and depression
- ✔ how to use bearings.

### You should be able to:

- ✔ construct labelled diagrams from written statements
- ✔ use Pythagoras' theorem and sine, cosine and tangent ratios to find the sides and angles of right-angled triangles
- ✔ identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles.




### Right-angled triangles

Pythagoras' theorem describes the relationship between the lengths of the sides in a right-angled triangle.

In the triangle shown, $c$ is the longest side, the **hypotenuse.**

$$c^2 = a^2 + b^2$$

**Assessment tip**

The formulae for the trigonometric ratios are not in the formula book, so they need to be memorized.

This can be rearranged so that the lengths of one of the shorter sides can be found.

$$a^2 = c^2 - b^2$$

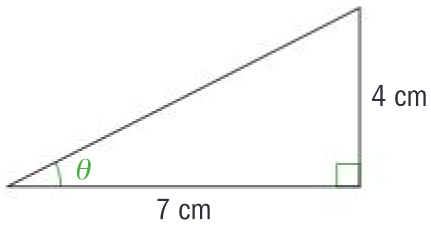
If you are given, or need to find, an angle, then you should use the trigonometric ratios.

$$\sin\theta = \frac{\text{opposite}}{\text{hypoteneuse}},\ \cos\theta = \frac{\text{adjacent}}{\text{hypoteneuse}},\ \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

**Note**

These are often remembered using the mnemonic device SOH-CAH-TOA.

If finding an angle, begin by finding the sine, cosine or tangent of that angle and then take the inverse of that value.

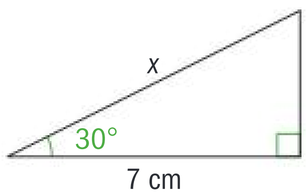



In this example $\tan\theta = \frac{4}{7}$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{4}{7}\right) \approx 29.7°$$

**Assessment tip**

The inverse functions can also be written as $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$.

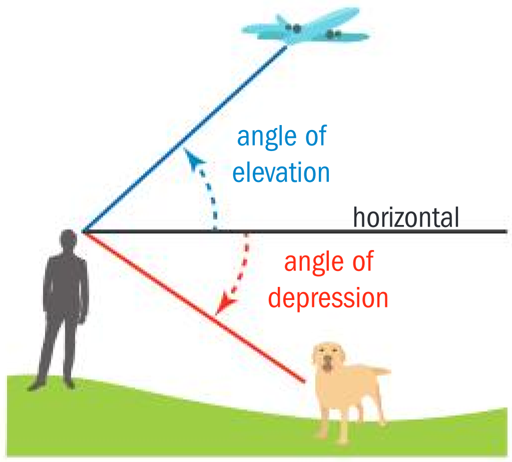



To find a missing side, you need to rearrange the formula.

Here $\cos 30° = \frac{7}{x} \Rightarrow x = \frac{7}{\cos 30°} \approx 8.08\,\text{cm}$

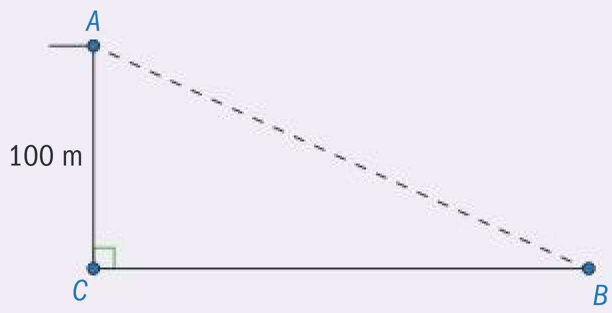
## Angles of elevation and depression

The angle of elevation is the angle you look up, from the horizontal, to see an object. The angle of depression is the angle you look down through, from the horizontal, to see an object.

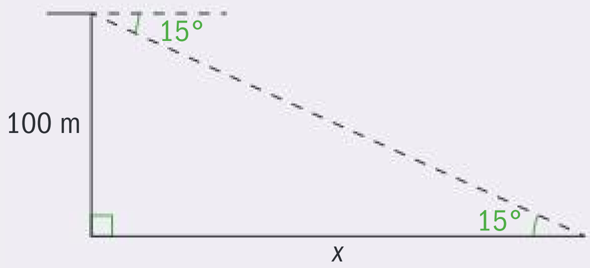



### Example 3.1.1

The angle of depression from the point $A$ on top of a vertical cliff to a boat $B$ on the sea below is 15°. The cliff is 100 m high and the foot of the cliff $C$ is at the same level as the boat. Find, correct to the nearest metre, the horizontal distance $BC$ from the foot of the cliff to the boat.




**Solution**




$$\tan 15° = \frac{100}{x}$$

$$x = \frac{100}{\tan 15°} = 373.205$$

The distance is 373 m, to the nearest metre.

Always begin by drawing a clear diagram and add the given information.

The angle of depression is outside the triangle, but because of the parallel horizontal lines it is equal to the angle of elevation from the boat to the top of the cliff.

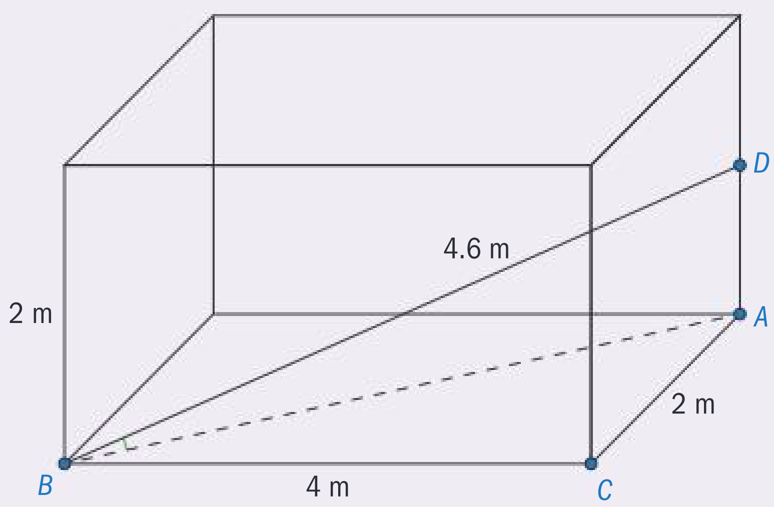
The two sides involved in the question are the opposite and the adjacent (the opposite is given and the adjacent needs to be found), so the tan formula is used.

## Triangles in 3 dimensions

When considering three dimensional shapes it is important to be able to construct right-angled triangles within the shape in order to calculate angles and lengths.

### Example 3.1.2

A storage container is made in the shape of a cuboid. It has width 2 m, length 4 m and height 2 m, as shown in the diagram. Georg wishes to store the mast from his boat in the container. The mast is 4.6 m long. The base of the mast is placed at point $B$ and the top of the mast rests at point $D$.




**Assessment tip**

Use the 5 minute reading time to decide which right-angled triangles need to be used.

Find:

(a) $AD$, the height of the top of the mast above the floor of the storage container

(b) the angle the mast makes with the floor.

**Solution**

(a) $AB^2 = 4^2 + 2^2 = 20$

From the diagram it is clear that to find $AB$ you need to use the right-angled triangle $ABD$.

Only one length is known so another needs to be found, using the right-angled triangle $ABC$.

$AB = \sqrt{20}$

The value for $AB$ is an intermediate answer so it is best kept as an exact number, either by leaving in the square root symbol or by making sure you use the full decimal value given by your GDC.

Using Pythagoras in triangle $ABD$

$AD^2 = 4.6^2 - \left(\sqrt{20}\right)^2 = 1.16$

$AD = \sqrt{1.16} \approx 1.08$ m

(b) Let angle $ABD = \theta$

As all the sides are known you can select any of the trig ratios.

$\cos\theta = \frac{\sqrt{20}}{4.6} \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{20}}{4.6}\right)$

$= 13.540... \approx 13.5°$

## Solving non-right-angled triangles

The sine and cosine rules can be used for any triangle.

The sine and cosine rule are in section SL 3.2 in the formula books.

### The sine rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

To be able to use the sine rule you must know either two sides and one of the opposite angles, or two angles and one of the opposite sides.

### The cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$, which can be rearranged to give $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

The letters in a question are unlikely to be the same as those in the formula so remember that angle $A$ is always opposite the side with length $a$, and similarly for $B$ and $C$.

In the first form of the cosine rule, always place the side opposite the angle being used on the left side of the equation.

## The area of a triangle

The formula for the area of a triangle is in section SL 3.2 of the formula book.

There are two formulae for the area of a triangle.

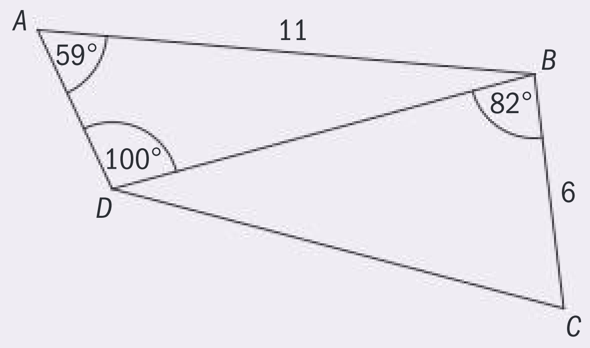
The first is: Area $= \frac{1}{2} \times \text{base} \times \text{height}$

When the height of the triangle is not known the following formula should be used: Area $= \frac{1}{2}ab\sin C$

Notice that this is the product of two sides and the sine of the angle between them.

### Example 3.1.3

The following diagram shows quadrilateral $ABCD$.




$AB = 11$ cm, $BC = 6$ cm, $B\hat{A}D = 59°$, $A\hat{D}B = 100°$, and $C\hat{B}D = 82°$

(a) Find $DB$

(b) Find $DC$

(c) Find the area of triangle $DBC$

**Solution**

(a) $\frac{DB}{\sin 59°} = \frac{11}{\sin 100°}$

$DB = \frac{11}{\sin 100°} \times \sin 59°$

$9.574... \approx 9.57$ cm

As an angle and the length of the opposite side are given the sine rule should be used.

When deciding which form of the sine rule to use, remember it is easier to begin with the unknown quantity on top.

(b) $DC^2 = (9.574...)^2 + 6^2 - 2 \times 9.574... \times 6 \times \cos 82°$

$= 111.677...$

$\Rightarrow DC = \sqrt{111.677...}$

$= 10.5677... \approx 10.6$ cm

To find $DC$ we need to use the cosine rule as there is no known side length opposite the angle given.

(c) Area $= \frac{1}{2} \times 9.57... \times 6 \times \sin 82°$

$= 28.44... \approx 28.4 \text{ cm}^2$

We need to use the two sides adjacent to the known angle.

**Assessment tip**

Note that in the exam $DB$ is read as 'the length of the line from $D$ to $B$' which is what is being asked for here. The name of the line segment from $D$ to $B$ would be written $[DB]$.

**Assessment tip**

In an exam, for your final answer, write down both a longer expression from your calculator as well as the answer rounded to three significant figures.

## Bearings

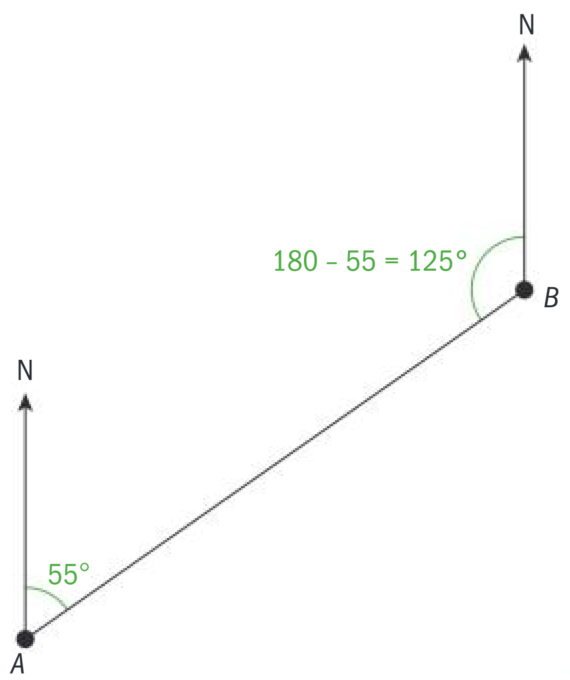
A bearing gives the direction of one point from another. It is measured in degrees clockwise from the north line. In bearings questions, always mark the north line at each point and then use the parallel lines rules to add angles.

For example:

If the bearing of $B$ from $A$ is 55°, find the bearing of $A$ from $B$.

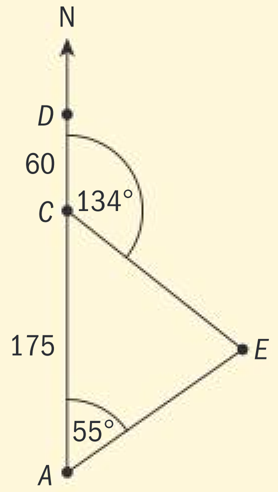
Using the properties of parallelograms, it can be seen that the angle anticlockwise from N at $B$ is: $180 - 55 = 125°$

Hence the bearing of $A$ from $B$ is $360 - 125 = 235°$



**SAMPLE STUDENT ANSWER**

A ship is sailing north from a point $A$ towards point $D$. Point $C$ is 175 km north of $A$. Point $D$ is 60 km north of $C$. There is an island at $E$. The bearing of $E$ from $A$ is 055°. The bearing of $E$ from $C$ is 134°. This is shown in the following diagram.



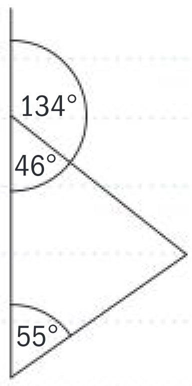

(a) Find the bearing of $A$ from $E$

(b) Find $CE$

(c) Find $DE$

▼ The student's diagram does not include a north line at $E$. Student has correctly calculated angle $CEA$ but the bearing needs to be measured from north. The angle between $CE$ and north is 180 – 134 = 46 so bearing is 360 – 46 – 78 = 236°

(a)




$180 - 134 = 46$

$180 - (99 + 46) = 78$

$A$ from $E = 078°$

▲ Student has correctly identified they need to use sine rule as they have a side and an opposite angle and got the correct answer.

(b) CE

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{175}{\sin 78°} = \frac{CE}{\sin 55°}$$

$$178.9 = \frac{b}{\sin 55°} \Rightarrow b = (\sin 55°)(178.9)$$

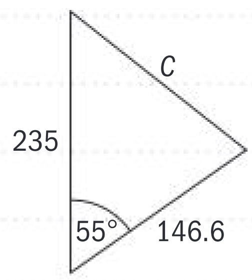
$b = 146.6$

$CE = 146.6$ km

▼ One mark awarded for use of the cosine rule but the side from part b has been put in the wrong place, and the wrong triangle has been used. The correct triangle to use is $CDE$ and the correct answer is 193 km.

▼ The student has lost time by writing out the complete calculation when it would be expected that the value would be calculated directly from the second line of working.

(c) DE




$C^2 = a^2 + b^2 - 2ab\cos C$

$C^2 = (235)^2 + (146.6)^2 - 2(239)(146.6)\cos 55°$

$= 55225 + 21491 - 68 - (470)(146.6)(0.57)$

$= 76716.6 - (68902)(0.57) = 76716.6 - 39520.6$

$C^2 = 37196$

$C = \sqrt{37196}$ Distance can't be negative

$C = 192.9$

$DE = 192.9$ km

# 3.2 SECTORS OF A CIRCLE; VOLUMES AND SURFACE AREAS

## You must know:

- ✔ the formulae for length of an arc and area of a sector in a circle when given an angle in degrees
- ✔ the formulae for the distance between two points in three-dimensional space, and their midpoint
- ✔ the formulae for volume and surface area of three-dimensional solids including a right-pyramid, right-cone, sphere and hemisphere.

## You should be able to:

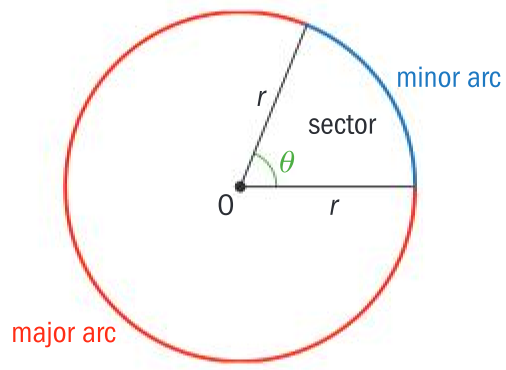
- ✔ identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles
- ✔ find distances between points in three-dimensional space when given their coordinates
- ✔ find the volume and surface area of compound shapes when the sides are formed by polygons.

The formulae for the circumference $C$ and area $A$ of a circle with radius $r$ are:

$C = 2\pi r$ and $A = \pi r^2$

The length $l$ of an arc and the area $A$ of a sector of a circle with an angle $\theta$ at the centre are given by the following formulae:

$l = \frac{\theta}{360} \times 2\pi r$ and $A = \frac{\theta}{360} \times \pi r^2$

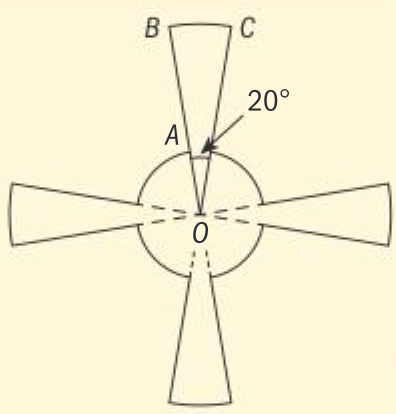



These formulae are in section SL 3.4 in the formula book.

**SAMPLE STUDENT ANSWER**

This diagram shows a metallic pendant made out of four equal sectors of a larger circle of radius $OB = 9$ cm and four equal sectors of a smaller circle of radius $OA = 3$ cm.
Angle $BOC = 20°$

Find the area of the pendant.




Circle $= \pi r^2$
$= \pi\, 9$

Inner Sector $= 9 \times \frac{20}{360} \times \pi r^2$
$= 2\pi$

∴ Circle $= 9\pi - 2\pi$
$= 7\pi$

Sector $= \frac{20}{360} \times \pi \times 4 \times 9^2$
$= 18\pi$

∴ Area $= 7\pi + 18\pi$
$= 25\pi$

or

$78.54 cm^2$

▼ The student is not making his/her method clear. The examiner would struggle to work out the method being used.

▲ The student is using exact answers to ensure the final answer will be correct to 3 sf. If not using exact values, make sure you use the full number from your calculator.

▼ The answers given are correct but the numerical answer should be given to 3 sf.

▲ The method is not clear, but the calculations are clearly set out.

## Volumes and surface areas

You need to be able to find:

the **volumes** of cuboids, cylinders, prisms, right pyramids, right cones and spheres

the **areas** of parallelograms, triangles, trapezoids and circles

the **surface areas** of cylinders, cones and spheres and other three-dimensional shapes whose surface is made up of polygons or circles.

**Assessment tip**

These formulae are in two different sections of the formula book: Prior Learning and SL3.1

### Example 3.2.1

A cylinder of radius $r$ cm and height $h$ cm has a volume of 500 cm$^3$

(a) Write down an equation in terms of $r$ and $h$ for the volume.

The cylinder is to be constructed so it has a minimum surface area.

(b) Use your answer to part (a) to find an expression for the surface area $S$ in terms of $r$.

(c) Find the values of (i) $r$ (ii) $h$ such that the surface area is a minimum.

**Solution**

(a) $500 = \pi r^2 h$

(b) $S = 2\pi rh + 2\pi r^2$

From part (a): $h = \dfrac{500}{\pi r^2}$

Use this value of $h$ in the formula for $S$:

$$S = 2\pi r \times \frac{500}{\pi r^2} + 2\pi r^2$$

$$= \frac{1000}{r} + 2\pi r^2$$

All the formulae needed are in the prior learning section of the formula book.

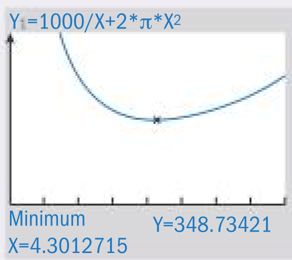
Don't forget to add the area of the two circles at the top and bottom of the cylinder.

(c) (i) From the GDC $S$ is minimized when $r \approx 3.41$ cm

(ii) $h = \dfrac{500}{\pi r^2} = \dfrac{500}{\pi \times 3.41^2}$

$\approx 13.7$ cm

Now the expression has been reduced to one variable $r$, a graph can be drawn.




Though the solution is giving intermediate answers to 3 sf, the calculation of $h$ should be done using the full value of $r$ from the GDC.

**Assessment tip**

Always write out the formula with the values substituted in; there are no method marks for just writing $V = \pi r^2 h$

**Assessment tip**

This question could have been set as part of Topic 5: Calculus in which case you might have been asked to differentiate to find the minimum. Because there is no method specified here you should choose the easiest route, which is to use the GDC.

Using your GDC to find key points on a curve is in section 2.3. Finding maximum and minimum points using calculus is in section 5.1

## Coordinate systems

Sometimes two- or three-dimensional shapes are defined by a set of coordinates. In these cases, lengths can be calculated using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ where $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are the endpoints of the line segment.

Similarly, the coordinates of the midpoint of the line segment with endpoints $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Both formulae are in section SL 3.1 of the formula book.

### Example 3.2.2

A right square-based pyramid is set in a coordinate system with the origin $O$ at one of the corners of the base. The coordinates of the corner $A$, directly opposite $O$, are (15.2, 15.2, 0) where the units of length are metres, and the $z$-coordinate represents the height above the base.

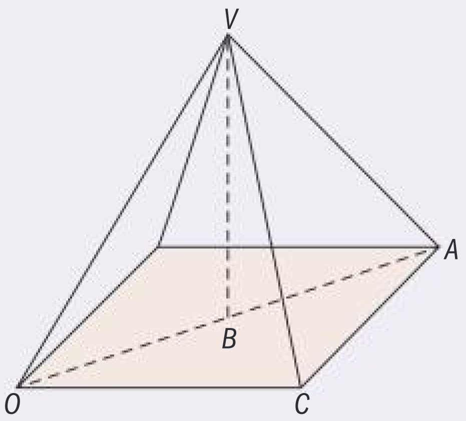
(a) Find the coordinates of $B$, the midpoint of $[OA]$

Let the vertex of the pyramid be $V$.
The pyramid has a height of 10.4 m

(b) Write down the coordinates of $V$.

(c) Find the length of $[OV]$

$C$ is a corner of the base of the pyramid, adjacent to $O$. The coordinates of $C$ are (10.7, 0, 0)




(d) Show that the surface area of the pyramid is 300 m$^2$, to 2 sf.

**Solution**

(a) (7.6, 7.6, 0)

Using $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

(b) (7.6, 7.6, 10.4)

A right pyramid has its vertex directly over the centre of the base.

(c) $\sqrt{(7.6-0)^2+(7.6-0)^2+(10.4-0)^2}$

$= 14.9559... \approx 15.0$ m

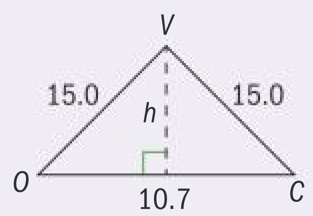
Using $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(d) Let the height of triangle $OCV$ be $h$

Each side is an isosceles triangle with base 10.7

$h^2 = 15.0^2 - \left(\frac{10.7}{2}\right)^2$

$h = 13.966... \approx 14.0$ m




Area $= \frac{1}{2} \times 10.7 \times 14.0$

$= 74.719... \approx 74.7$ m$^2$

Surface area of pyramid = $4 \times 74.7 = 298.87... \approx 300$ m$^2$

An equivalent method is to find the coordinates of the midpoint, $M$, of $[OC]$ and the length of $[VM]$ using the distance formula.

**Assessment tip**

Remember that in exams $[AB]$ should be read as the line segment joining the points $A$ and $B$.

**Assessment tip**

An exam mark scheme would include both the unrounded and rounded answers. Often both need to be seen for you to be awarded the final mark.

# 3.3 VORONOI DIAGRAMS

## You must know:

- ✔ properties of perpendicular bisectors
- ✔ terms used in Voronoi diagrams, such as: sites, vertices, edges, cells
- ✔ the edge of a cell is the perpendicular bisector of the line segment between two sites
- ✔ nearest neighbour interpolation assigns all points within a cell the same value of a quantity (e.g. rainfall) as the value of the site
- ✔ that the solution to the "toxic waste dump" problem lies at the vertex where three cells meet.

## You should be able to:

- ✔ find the equation of a perpendicular bisector when given either two points, or the equation of a line segment and its midpoint
- ✔ create a new cell on a Voronoi diagram when a new site is added
- ✔ find the equation of an edge of a cell, identify the site closest to a given point, or calculate the area of a cell
- ✔ identify when it is appropriate to use the nearest neighbor interpolation and when to apply the toxic waste dump problem.

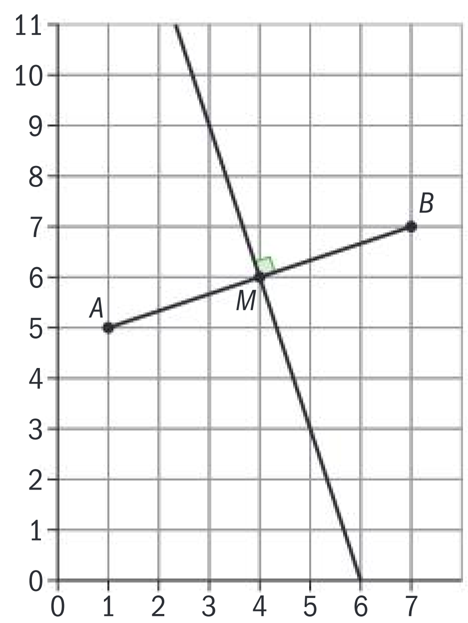
Look back on how to find the gradient and equation of a straight line from section 2.1

### Perpendicular bisectors

The perpendicular bisector of [*AB*] is the line which passes through the midpoint of [*AB*] and is perpendicular to [*AB*].

If the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ are perpendicular then $m_2=-\frac{1}{m_1}$

If asked to show two lines are perpendicular, show that $m_1 \times m_2 = -1$




In this diagram, *M* is the midpoint of [*AB*].

The gradient of [*AB*] is $\frac{1}{3}$ and the perpendicular bisector has gradient −3.

### Example 3.3.1

(a) Find, in the form $y=mx+c$, the equation for the perpendicular bisector of the line joining $A(1, 5)$ and $B(7, 7)$

(b) Verify that the point $C(8, -6)$ lies on this line.

(c) Find the distance *CA*.

**Solution**

(a) The gradient of [*AB*] = $\frac{7-5}{7-1}=\frac{2}{6}=\frac{1}{3}$

Using the formula: gradient = $\frac{y_2-y_1}{x_2-x_1}$

The gradient of perpendicular bisector = −3

The line will pass through the midpoint of [*AB*] = $\left(\frac{1+7}{2}, \frac{5+7}{2}\right)=(4, 6)$

Using the formula $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ (which is in the Prior Learning section in the formula book)

Equation is $y-6=-3(x-4)$

$\Rightarrow y=-3x+18$

Using the point-gradient form of the equation of a straight line.

(b) $-3 \times 8 + 18 = -6$

This is verified by substituting the value for $x$ into the equation.

(c) $CA = \sqrt{(8-1)^2 + (-6-5)^2}$
$= \sqrt{49+121}$
$= \sqrt{170} = 13.038... \approx 13.0$

Using the formula
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
(which is in the Prior Learning section in the formula book)

Every point on the perpendicular bisector of $[AB]$ is an equal distance from both $A$ and $B$.

A Voronoi diagram consists of a set of points called **sites** and the diagram is divided into regions called **cells**, each containing one site. All points within a cell are closer to that site than to any other site.

The boundaries of the cells are called **edges**. The point where the edges meet is called a **vertex** of the cell.

The edges of the cells lie along the perpendicular bisectors of the sites.

In examinations, there will always be three cells meeting at a vertex.

A vertex is always an equal distance from the three adjacent sites.

The **toxic waste dump problem** is finding the point on the Voronoi diagram that is furthest from any of the sites.

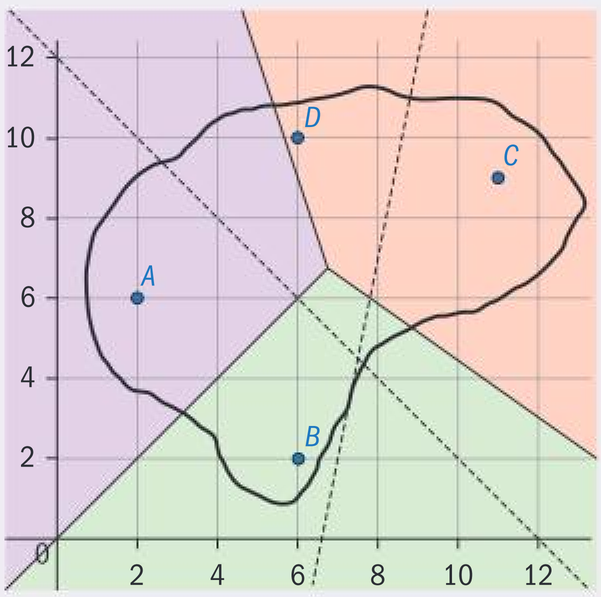
**Nearest neighbour interpolation** assigns the value at the site to all points in the cell. For example, if the sites are weather stations recording temperature for a forecasting website then every point in the cell is assumed to have the same temperature as at the weather station in that cell.

**Assessment tip**

In an exam, the solution of the toxic waste dump problem will always be at a vertex of the Voronoi diagram where three cells meet.

## Example 3.3.2

The diagram shows a map of an island with transmission towers $A(2, 6)$, $B(6, 2)$ and $C(11, 9)$. A Voronoi diagram for $A$, $B$ and $C$ has been constructed to show the area which is closest to each of the three towers. This is shown with the solid lines in the diagram.




A fourth tower $D$ is then added at the point (6, 10). The perpendicular bisectors between $D$ and $A$ and between $D$ and $C$ are shown as dashed lines.

(a) Write down the equation of the perpendicular bisector of $[DB]$.

(b) Complete the Voronoi diagram for the four sites, indicating clearly the added cell.

The strength of signal from a tower at a point is proportional to the distance of the point from the tower.

**Assessment tip**

In an exam you will be given a copy of the Voronoi diagram to complete.

(c) A receiver is at the point (10, 4). From which tower will it receive the strongest signal?

A fifth tower is to be added in order to boost the signal. It is to be placed on the island at the point as far as possible from each of the four existing towers.

It is given that the equation of the perpendicular bisector of [$DC$] is $y = 5x - 33$

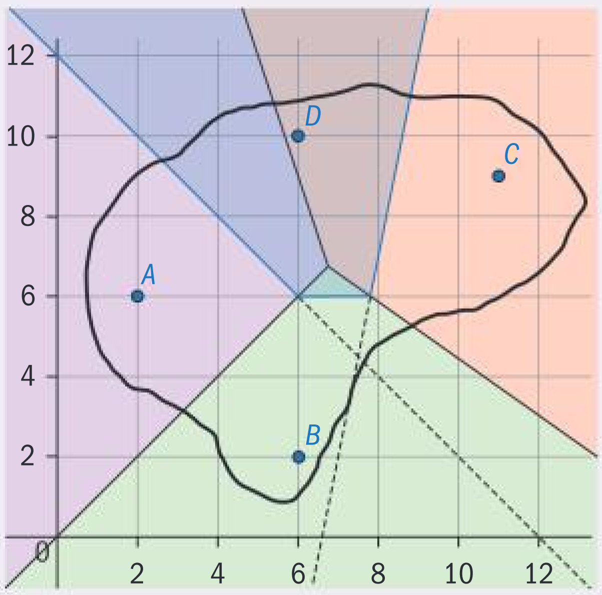
(d) Find the coordinates of the point where the fifth tower will be built.

**Solution**

(a) $y = 6$

The perpendicular bisector will be a horizontal line passing through the midpoint of [$DB$]

(b)




First draw the final perpendicular bisector.

The edges of the new cell can be formed by tracing around the perpendicular bisectors added with the new site, moving to a new edge whenever you reach a vertex where three edges meet.

(c) Transmitter $B$

(10, 4) is in the cell which contains tower $B$.

(d) Of the two vertices, the one furthest from the other vertices is at the point is where the line $y = 5x - 33$ meets $y = 6$

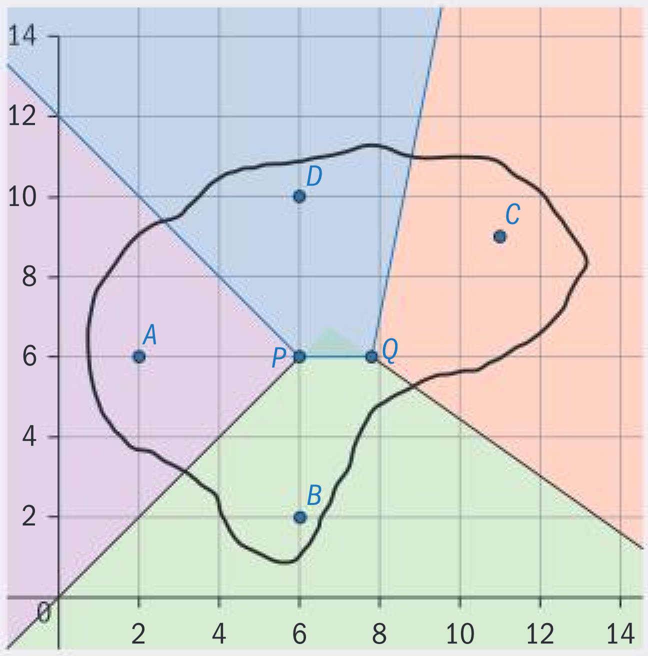
$6 = 5x - 33 \Rightarrow 5x = 39$

$x = \frac{39}{5} = 7.8$

The coordinates are (7.8, 6)

This question is an example of the toxic waste dump problem.

The point furthest from the other sites will be at a vertex where three cells meet. The distance from each of $A$, $B$ and $C$ to $P$ will be 2. The distance from $B$, $C$ or $D$ to $Q$ is clearly more than 2 so this is where the fifth tower should be built.




Sometimes you will be asked to find the areas of the cells. These are likely to be in the shape of triangles or trapezoids – see the formulae in the formula book Prior Learning section.

# 3.4 TRIGONOMETRIC FUNCTIONS (AHL)

## You must know:

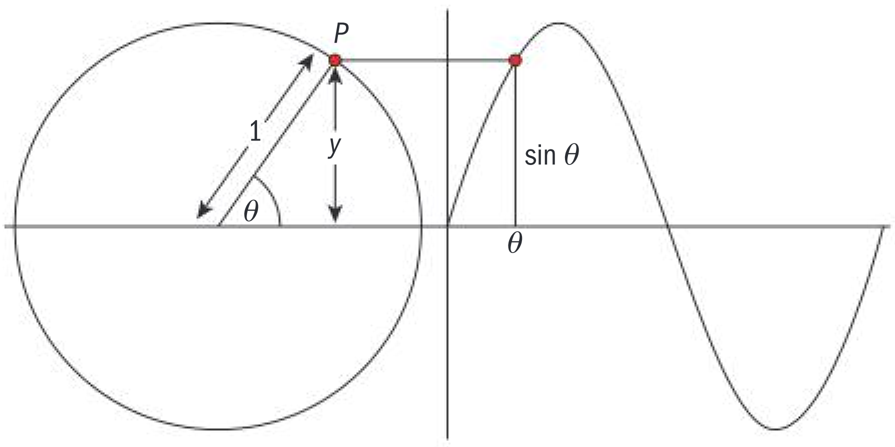
- ✔ the definition of a radian
- ✔ the definitions of $\cos\theta$ and $\sin\theta$ in terms of the unit circle
- ✔ the Pythagorean identity: $\cos^2\theta + \sin^2\theta = 1$
- ✔ the definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$
- ✔ when given one angle and two sides of a triangle there might be two possible triangles that could be drawn.

## You should be able to:

- ✔ convert between degrees and radians and vice versa
- ✔ put your GDC into the correct mode when using trigonometric functions
- ✔ use the correct formula for finding the length of an arc or the area of a sector when the angle is given in radians
- ✔ solve a triangle using the sine rule or the cosine rule in the ambiguous case
- ✔ solve trigonometric equations graphically in a finite interval.

### The unit circle

The sine function can be defined as: $\sin\theta$ is the $y$-coordinate of the point $P$ when it has rotated through an angle $\theta$ around a unit circle centred at the origin.




The link between trigonometric functions and circular motion is revisited in the section 3.6

An instance of the application of the use of these formulae can be seen in Example 3.6.8 in the section on motion with variable velocity.

Similarly, the cosine function can be defined as: $\cos\theta$ is the $x$-coordinate of the point $P$ when it has rotated through an angle $\theta$ around a unit circle centred at the origin.

The following identities can be derived directly from the unit circle definitions of sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1 \text{ and } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

### The ambiguous case of the sine rule

Another identity that can be derived from the unit circle is that $\sin\theta = \sin(180° - \theta)$ or $\sin(\pi - \theta)$ when in radians.

This means there are always two possible solutions between 0° and 180° for $\sin\theta = p$ when $0 \le p < 1$

This means that sometimes there are two possible triangles that can be drawn when given two sides and an angle opposite one of the sides. 'Ambiguous' here means that without more information it is impossible to tell which triangle is intended.

### Example 3.4.1

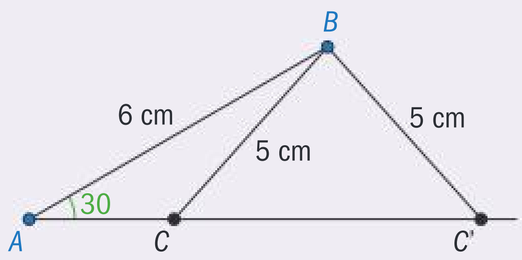
A triangle $ABC$ is to be constructed with $\hat{A} = 30°$, $AB = 6$ cm and $BC = 5$ cm

(a) Show, on the same diagram, two possible triangles that could be formed, labelling the two positions of $C$ as $C$ and $C'$.

(b) Find the area of the smaller triangle.

**Solution**

(a)




The two triangles $ABC$ and $ABC'$ both satisfy the conditions given in the question.

(b) Using the sine rule

$$\frac{\sin(A\hat{C}B)}{6} = \frac{\sin 30}{5}$$

Of the two possible solutions to $\sin(A\hat{C}B) = 0.6$ it is clear from the diagram that the larger angle is required.

$$\Rightarrow \sin(A\hat{C}B) = \frac{\sin 30}{5} \times 6 = 0.6$$

Since $\sin\theta = \sin(180 - \theta)$

$$\Rightarrow A\hat{C}'B = 36.869\ldots \text{ and}$$

$A\hat{C}B = 180 - A\hat{C}'B$

$$A\hat{C}B = 180 - 36.869\ldots = 143.1\ldots$$

$$A\hat{B}C = 180 - 30 - 143.1\ldots = 6.869\ldots$$

To find the area of the smaller of the two triangles it will be necessary to calculate either $AC$ or the angle $ABC$.

$$\text{Area} = \frac{1}{2} \times 6 \times 5 \times \sin(6.869\ldots)$$

$$= 1.79 \text{ cm}^2$$

Make sure you always use the full display on your GDC to avoid errors due to premature rounding.

**Assessment tip**

Have your GDC set to radians, and change to degrees only when the question uses degrees.

## Radian measure

**Radians** are an alternative angle measure to degrees.

$\pi$ radians are equal to 180°

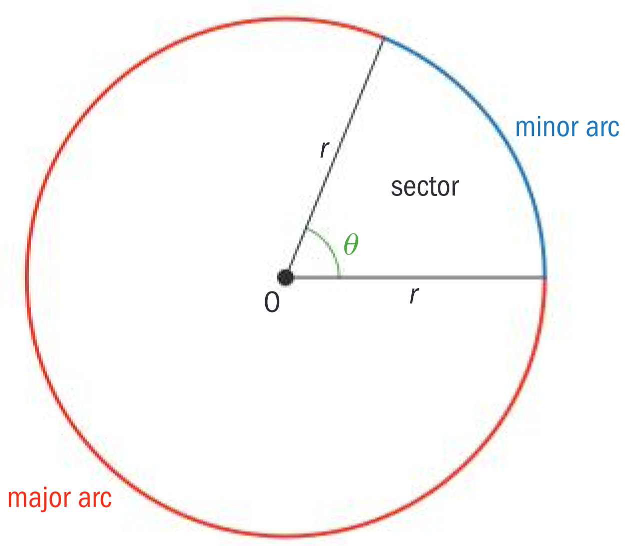
This means that

- to convert from degrees to radians you multiply by $\frac{\pi}{180}$
- to convert from radians to degrees you multiply by $\frac{180}{\pi}$

If measuring the angle of a sector in radians rather than degrees, the formulae for the length of arc ($l$) and area of sector ($A$) are simplified.

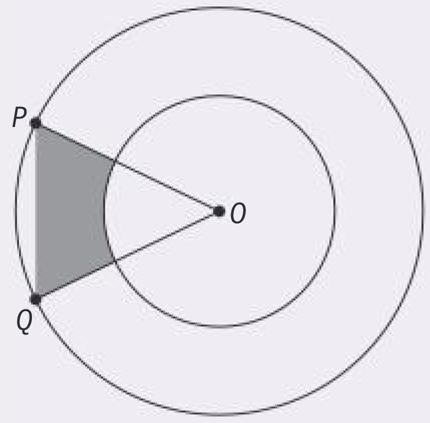
$l = r\theta$ and $A = \frac{1}{2}r^2\theta$

These formulae are given in section AHL 3.7 in the formula book.



## Example 3.4.2

The diagram below shows two concentric circles with centre $O$ and radii 3 cm and 6 cm. The points $P$ and $Q$ lie on the larger circle and $P\hat{O}Q = \theta$, where $0 < \theta < \frac{\pi}{2}$.




(a) Show that the area of the shaded region is $4.5(4\sin\theta - \theta)$

(b) Find the value of $\theta$ that maximizes this area.

**Solution**

(a) Area of triangle $POQ =$ $\frac{1}{2} \times 6 \times 6 \times \sin\theta = 18\sin\theta$

Using area of triangle $= \frac{1}{2}ab\sin C$

Area of sector $= \frac{1}{2} \times 3^2 \times \theta = 4.5\theta$

Working in radians and using the area of sector formula.

Area of shaded region $= 18\sin\theta - 4.5\theta$

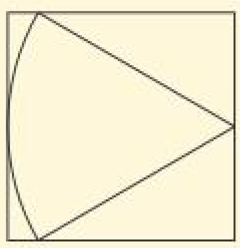
$= 4.5(4\sin\theta - \theta)$

(b) $1.318\,11... \approx 1.32$

From the GDC – you need to ensure your GDC is in radian mode. Note the answer could also be found using differentiation and finding $\theta = \arccos\left(\frac{1}{4}\right)$

**SAMPLE STUDENT ANSWER**

A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is 7 cm$^2$, find the dimensions of the rectangle, giving your answers to the nearest millimetre.



Let the radius of the circle be $r$, so $7 = \frac{1}{2}r^2 \times 1$

$r = \sqrt{14} = 3.741657386$

So, the length of the rectangle is 3.74

Angle is $\frac{180}{\pi} = 57.3°$

$x = \frac{180 - 57.2}{2} = 61.4°$

height $= 2 \times 3.74\cos(61.4) = 3.59$ cm

▲ The student has correctly spotted the length of the rectangle is equal to the radius, which was correctly calculated.

▼ The student has rounded incorrectly and used the incorrect rounding in the rest of the question, leading to the wrong answer. You should always write down a full answer as well as a rounded answer and always use the full answer from the GDC in subsequent working.

▼ It would have been easier to stay in radians and calculate $x$ as $\frac{\pi - 1}{2}$. In this case the calculator would need to be in radian mode for the final calculation.

# 3.5 MATRIX TRANSFORMATIONS (AHL)

## You must know:

- ✔ the order in which to multiply matrices for a compound transformation
- ✔ area of image = $|\det T| \times$ the area of the object
- ✔ if a line makes an angle of $\alpha$ with the $x$-axis then the gradient of the line is $\tan\alpha$.

## You should be able to:

- ✔ find the matrix for a given transformation from the formula book or by considering the image of (1, 0) and (0, 1)
- ✔ find the image of a given point under a transformation
- ✔ find the point that has a given image
- ✔ use iterative techniques to generate a complex shape.

### Linear transformations

A **linear transformation** can be represented by a $2 \times 2$ matrix and as such will always leave the point (0, 0) unchanged.

Examples of linear transformations are **rotations** about the origin, **reflections** in lines passing through the origin, **enlargements** with the origin as a centre and horizontal and vertical one-way **stretches** with the $y$-axis or $x$-axis invariant.

You will need to know how to find the matrix representing a transformation, either by using the form of the general matrix given in the formula book or by considering the effect of the transformation on two points.

The general form of matrices representing each of these transformations are in section AHL 3.9 of the formula book.

**Assessment tip**

Make sure you know how to find the angle ($\alpha$) between a line and the $x$-axis from its gradient ($m$) using $m = \tan\alpha$.

**Example 3.5.1**

(a) Find the matrix that represents a reflection in the line $y = 2x$

(b) Find the matrix that transforms the point (1, 2) to (2, –4) and (4, –3) to (8, 6)

**Solution**

(a) $m = 2 \Rightarrow \tan\theta = 2 \Rightarrow \theta = 63.43...$

$$\begin{pmatrix} \cos(2\times 63.43...) & \sin(2\times 63.43...) \\ \sin(2\times 63.43...) & -\cos(2\times 63.43...) \end{pmatrix}$$

$$= \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$

The matrix for a reflection is given in the formula book but first the value of $\theta$ needs to be found.

(b) Let the transformation matrix be $\mathbf{T}$

$$\mathbf{T}\begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ -4 & 6 \end{pmatrix}$$

$$\Rightarrow \mathbf{T} = \begin{pmatrix} 2 & 8 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix}^{-1}$$

$$\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Because we are given two points a matrix equation can be formed.

Note that we need to post-multiply both sides by the inverse.

The final answer is obtained directly from the GDC.

### Assessment tip

The quickest way to find a matrix that represents a reflection in either axis, the lines $y = \pm x$, or a rotation by a multiple of 90° is to consider the effect on the points (1, 0) and (0, 1) rather than use the formulae.

If (1, 0) goes to $(a, b)$ and (0, 1) goes to $(c, d)$ then the transformation matrix is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. See Example 3.5.2

## Compound transformations

If a point with position vector $\boldsymbol{x}$ undergoes the transformation represented by matrix $\mathbf{R}$, the image will be $\mathbf{R}\boldsymbol{x}$. If this point then undergoes a transformation represented by matrix $\mathbf{S}$, then the image of $\boldsymbol{x}$ after the two transformations is $\mathbf{S}(\mathbf{R}\boldsymbol{x}) = (\mathbf{SR})\,\boldsymbol{x}$
From this we can see that the single matrix that represents the transformation $\mathbf{R}$ followed by $\mathbf{S}$ is $\mathbf{SR}$.

### Assessment tip

Make sure you know the order in which to multiply matrices to represent a compound transformation.

### Example 3.5.2

(a) Find the single matrix that represents a reflection in the line $y = x$ followed by a rotation of 90° clockwise.

(b) Show that performing this transformation twice will return each point to its starting position.

**Solution**

(a) Reflection in $y = x$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Either use the fact that (1, 0) goes to (0, 1) and (0, 1) goes to (1, 0) or use the formula in the formula book and the fact that $y = x$ makes an angle of 45° with the $x$-axis to find the matrix for the reflection.

Rotation of 90° clockwise is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The rotation matrix is found in a similar way.

The single matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note the order of matrix multiplication used to find the matrix for the compound transformation.

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Squaring the matrix gives the matrix for performing the transformation twice.

Because the compound transformation results in the identity matrix performing the transformation twice will return each point to its starting position.

## Affine transformations

These are transformations of the form $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \boldsymbol{b}$ where $\mathbf{A}$ is a $2 \times 2$ matrix and $\boldsymbol{b}$ is a $2 \times 1$ vector and $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$. In an affine transformation $\boldsymbol{x}$ undergoes a linear transformation followed by a translation.

### Example 3.5.3

Let the transformation **T** be defined by $\boldsymbol{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, where $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$ under the transformation **T**.

(a) Find the coordinates of the image of (2, 2) under **T**

(b) Find the coordinates of the point that remains in the same position when transformed by **T**.

**Solution**

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$,

(5, 12)

$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is substituted into the expression.

(b) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 2a + b - 1 \\ a + 3b + 4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$a + b = 1,\ a + 2b = -4$

$a = 6,\ b = -5$

So the coordinates of the invariant point are (6, 5)

Form two equations and solve simultaneously.

> **Assessment tip**
>
> Be sure to note if the question specifically asks for the coordinates of the points. If so, these should be given, not the position vector.

## Fractals and self-similar shapes

Compound shapes can be formed by repeating a transformation first on an object and then successively on the subsequent images. The final image is then made up of transformed images of the original shape.

The transformation can be written in the form $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \boldsymbol{b}$

where **A** is a $2 \times 2$ matrix and $\boldsymbol{b}$ is a $2 \times 1$ vector.

$\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ is the position vector of $\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$ following the transformation.

### Example 3.5.4

A transformation is given by the relation $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Find an expression for $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in the form $\mathbf{A}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \boldsymbol{b}$

**Solution**

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Write down the two expressions for $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ using the interative formula, then substitute the first expression into the second.

$= \begin{pmatrix} -1 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

Simplify the right hand side by multiplying out the matrices and adding the two vectors.

## The area of an image

If an object undergoes a transformation represented by a matrix **T** then:

The area of the image = | det **T** | × the area of the object.

**Assessment tip**

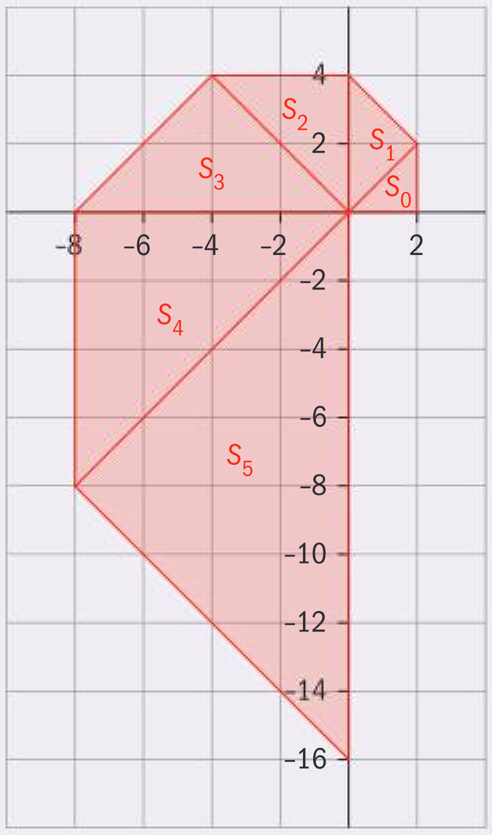
This formula is not given in the formula book so it will need to be memorized.

### Example 3.5.5

It is given that $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$

(a) Write down the matrix that represents a 45° anti-clockwise rotation about (0, 0)

A design is created by rotating a triangle five times through 45° anti-clockwise and enlarging it by a factor of $\sqrt{2}$ each time.
This is shown in the diagram.




(b) Find the single matrix that will take points in triangle $S_0$ to points in triangle $S_1$

(c) Verify your matrix will take the point (2, 2) onto the corresponding point in $S_1$

(d) Find the single matrix that will take points in triangle $S_0$ to points in triangle $S_5$

(e) Write down the area of triangle $S_0$ and hence find the area of triangle $S_5$

**Solution**

(a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

This matrix can be taken from the formula book. It is acceptable to use decimal equivalents, but because the exact forms were given in the question it might be better to use them.

(b) $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

On this occasion the same matrix is obtained irrespective of the order.

(c) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

From the diagram it is clear that the corresponding point is (0, 4)

(d) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^5 = \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix}$

When a transformation is repeated 5 times the single matrix that will represent that transformation is the original matrix raised to the power 5.

(e) Area of $S_0 = \frac{1}{2} \times 2 \times 2 = 2$

$\det \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix} = 16 + 16 = 32$

Area $= 32 \times 2 = 64$

Whenever an area is asked for in a question on matrix transformations it is likely that the formula *The area of the image* = | *det* **T** | × *the area of the object* will be used.

# 3.6 VECTORS

## You should know:

- ✔ unit vectors; base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$
- ✔ components of a vector; column representation
- ✔ position vectors: $\overrightarrow{OA} = \boldsymbol{a}$
- ✔ vector equation of a line in two and three dimensions
- ✔ the relative position of $B$ from $A$ is $\overrightarrow{AB}$
- ✔ how to model linear motion with constant velocity in two and three dimensions
- ✔ how to model motion with variable velocity in two dimensions
- ✔ how to model projectile and circular motion
- ✔ $f(t-a)$ indicates a time shift of $a$
- ✔ the definition and calculation of the scalar product of two vectors
- ✔ the definition and calculation of the vector product of two vectors
- ✔ the component of vector $\boldsymbol{a}$ acting in the direction of vector $\boldsymbol{b}$ is $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} = |\boldsymbol{a}|\cos\theta$ the component of a vector $a$ acting perpendicular to vector $b$, in the plane formed by the two vectors, is $\frac{|a \times b|}{|b|} = |a|\sin\theta$

## You should be able to:

- ✔ use algebraic and geometric approaches to calculate the sum and difference of two vectors, multiply by a scalar, find the magnitude of a vector $|\boldsymbol{v}|$ from components
- ✔ find the resultant as the sum of two or more vectors
- ✔ convert the vector equation of a line to parametric form
- ✔ find the position of an object and the relative position of two objects moving with constant or variable velocity
- ✔ find a velocity vector given a speed and direction
- ✔ find times and distances when two objects are closest to each other
- ✔ calculate the angle between two vectors using $\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}|\cos\theta$, where $\theta$ is the angle between two non-zero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, and ascertain whether the vectors are perpendicular ($\boldsymbol{v}\cdot\boldsymbol{w} = 0$)
- ✔ use $|\boldsymbol{v}\times\boldsymbol{w}|$ to find the area of a parallelogram (and hence a triangle)
- ✔ find the component of a vector acting in a given direction or perpendicular to that direction.

### Assessment tip

Other quantities such as force, or electric field strength may be used in questions but you will not require any knowledge of these quantities beyond what is given in the question.

## Properties of vectors

A **vector** represents a quantity which has both **magnitude** and **direction**. The most common vector quantities you will meet in the course are the displacement or position of an object, its velocity and its acceleration.

Vectors can be written as column vectors or in terms of the base vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is equivalent to $2\boldsymbol{i} - \boldsymbol{j}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is equivalent to $2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

The **magnitude** of a vector $v$ is the length of $v$ and is written as $|v|$.

It is found using the formula $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

This formula is given in section AHL 3.10 of the formula book.

The magnitudes of the two vectors given above are:

$\sqrt{2^2 + (-1)^2} = \sqrt{5}$ and $\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$

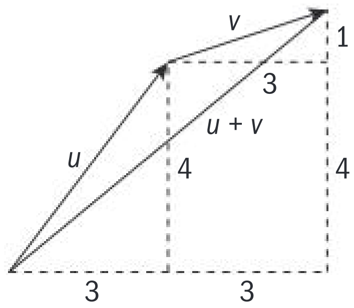
Any vector that is a scalar multiple of another is parallel to it. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is parallel to $\begin{pmatrix} 10 \\ -5 \end{pmatrix}$ because $\begin{pmatrix} 10 \\ -5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Parallel vectors are said to have the same direction.

A **unit** vector has a magnitude of one, hence $\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$ is a unit vector in the same direction as $\boldsymbol{v}$.

$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are unit vectors in the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

The sum of two or more vectors will give the single vector which is equivalent to the combined effect of the **individual** vectors. This vector is called the **resultant** vector. Let the vector $\boldsymbol{u}$ be $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and the vector $\boldsymbol{v}$ be $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Then:

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

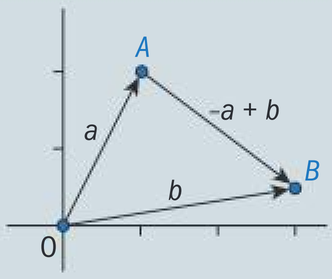



As can be seen in the diagram $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ is equivalent to a displacement of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ followed by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Similarly, if the two vectors represent forces acting on an object then the resultant effect of the two forces would be the same as a single force of $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$.

The **position vector** of a point $A$ is the vector from the origin to $A$. It is written as $\overrightarrow{OA}$ or often as just $\boldsymbol{a}$.

The vector $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b}$ or $\boldsymbol{b} - \boldsymbol{a}$




## Example 3.6.1

In still water, a boat has a speed of 2 ms$^{-1}$ and is being steered in the direction $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

(a) Find the velocity of the boat as a column vector.

While travelling with the same velocity, the boat enters a section of river which is flowing with a velocity of $\begin{pmatrix} 0 \\ 1.5 \end{pmatrix}$ ms$^{-1}$.

(b) Find the velocity of the boat relative to the land, and state whether it is moving upstream or downstream.

Solution

(a) $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ has a magnitude of $\sqrt{3^2+(-4)^2}=5$

A unit vector in the same direction as $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is

therefore $\frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

The velocity of the boat is therefore

$\frac{2}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}=\begin{pmatrix} 1.2 \\ -1.6 \end{pmatrix}$

The velocity is in the same direction as $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ so must be a multiple of it but have a magnitude of 2.

The required vector will be 2 times the unit vector in the direction $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

(b) $\begin{pmatrix} 1.2 \\ -1.6 \end{pmatrix}+\begin{pmatrix} 0 \\ 1.5 \end{pmatrix}=\begin{pmatrix} 1.2 \\ -0.1 \end{pmatrix}$

It is moving upstream.

The velocity relative to the land is the velocity of the boat plus the velocity of the river.

The –0.1 indicates the boat is moving upstream.

**Note**

The speed of an object is equal to the magnitude of its velocity vector.

## The scalar product

These formulae are in section AHL 3.13 in the formula book.

The **scalar** (or **'dot'**) **product** of two vectors $\boldsymbol{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

is defined as $\boldsymbol{v}\cdot\boldsymbol{w}=v_1w_1+v_2w_2+v_3w_3$

Additionally $\boldsymbol{v}\cdot\boldsymbol{w}=|\boldsymbol{v}||\boldsymbol{w}|\cos\theta$ where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

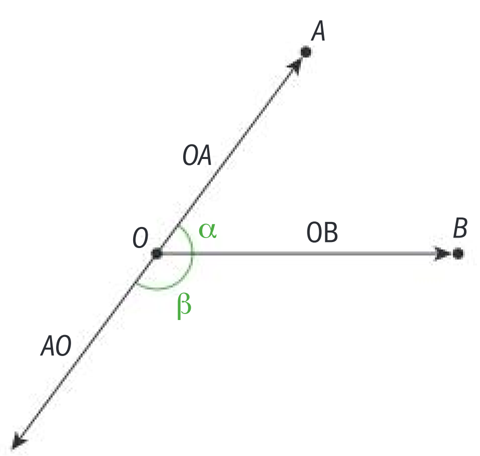
This can be rearranged to give a formula for the angle between two vectors as $\cos\theta=\frac{v_1w_1+v_2w_2+v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$

The main use of the scalar product is to find the angle between two vectors.

If two vectors are perpendicular then their scalar product is equal to 0.

**Note**

The angle $A\hat{O}B$ is the angle between vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ $(\alpha)$ and not between vectors $\overrightarrow{AO}$ and $\overrightarrow{OB}$ $(\beta)$.




### Example 3.6.2

A triangle $OAB$ has vertices at (0, 0, 0), (4, 8, 3) and (8, 6, 3)

(a) Find the lengths $OA$ and $OB$.

(b) Find the angle $A\hat{O}B$.

(c) Hence find the area of the triangle $OAB$.

(d) Show that that angle $O\hat{A}B=90°$.

Solution

(a) $OA=\sqrt{4^2+8^2+3^3}$

$=\sqrt{89}\approx 9.43$

$OB=\sqrt{8^2+6^2+3^2}$

$=\sqrt{109}\approx 10.4$

$OA$ is equal to the magnitude of the vector $\overrightarrow{OA}$, which is the position vector of $A$

(b) $\cos(A\hat{O}B) = \dfrac{4\times 8+8\times 6+3\times 3}{\sqrt{89}\times\sqrt{109}}$

$= 0.9036\ldots$

$A\hat{O}B \approx 25.4°$

This required angle is the one between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

(c) Area $= \dfrac{1}{2}\times\sqrt{89}\times\sqrt{109}$

$\sin(25.363\ldots)$

$= 21.0950\ldots \approx 21.1$

Using the formula for area and the lengths of sides already found.

(d) $\overrightarrow{AO} = \begin{pmatrix} -4 \\ -8 \\ -3 \end{pmatrix}$,

This is the angle between $\overrightarrow{AO}$ and $\overrightarrow{AB}$

$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

$\overrightarrow{AB}$ is calculated using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ or $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Using the key fact above if $\overrightarrow{AO}\times\overrightarrow{AB} = 0$ the vectors are perpendicular

$\overrightarrow{AO}\times\overrightarrow{AB} = -4\times 4 + -8\times -2 + -3\times 0 = 0$

Hence $O\hat{A}B = 90°$

## Vector product

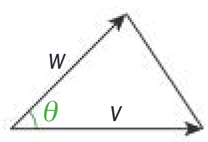
The **vector** (or '**cross**') **product** of two vectors $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

is defined as $v \times w = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

Additionally $|v\times w| = |v||w|\sin\theta$, where $\theta$ is the angle between $v$ and $w$.

Using the formula for the area of a triangle $\frac{1}{2}ab\sin C$ we can see that

the area of the triangle shown is $\frac{1}{2}|v||w|\sin\theta$




Hence the area of the triangle formed by the vectors $v$ and $w$ is $\frac{1}{2}|v\times w|$

The direction of the vector product of ***v*** and ***w*** is perpendicular to both ***v*** and ***w***

**Assessment tip**

It is very easy to make a mistake when calculating a vector product. Make sure you clearly set out each step.

The formula book section 3.13 has the equivalent formula for the area of a parallelogram $A = |\boldsymbol{v}\times\boldsymbol{w}|$ where ***v*** and ***w*** form two adjacent sides of the parallelogram.

### Example 3.6.3

The points $A$ (2.1, 3.5, 4.2), $B$ (3.2, 2.3, 1.4) and $C$ (0.8, 3.9, 1.6) lie at the three corners of a sail.

(a) Find the area of the sail.

A gust of wind strikes the sail perpendicular to its surface with a speed of 12 ms$^{-1}$.

(b) Find the velocity of the wind as a column vector.

**Solution**

(a) $\overrightarrow{AB} = \begin{pmatrix} 1.1 \\ -1.2 \\ -2.8 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1.3 \\ 0.4 \\ -2.6 \end{pmatrix}$

Using the formula $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix} 1.1 \\ -1.2 \\ -2.8 \end{pmatrix} \times \begin{pmatrix} -1.3 \\ 0.4 \\ -2.6 \end{pmatrix} = \begin{pmatrix} 4.24 \\ 6.5 \\ -1.12 \end{pmatrix}$

The area of the sail is the area of the triangle formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and is equal to $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Area $= \frac{1}{2}\sqrt{4.24^2 + 6.5^2 + (-1.12)^2}$

$= \frac{1}{2} \times 7.841... \approx 3.92$

As the wind is perpendicular to the sail it is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and so has the same direction as $\overrightarrow{AB} \times \overrightarrow{AC}$

(b) $\frac{12}{3.92052...}\begin{pmatrix} 4.24 \\ 6.5 \\ -1.12 \end{pmatrix} \approx \begin{pmatrix} 13.0 \\ 19.9 \\ -3.43 \end{pmatrix}$

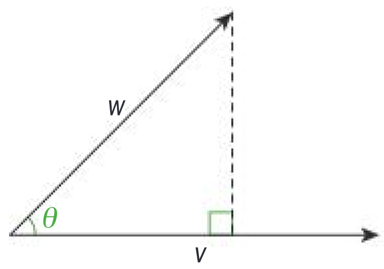
To find the correct magnitude use $\frac{1}{|v|}v$ to find the unit vector and multiply by 12.

## Components of vectors in a given direction

The **components** of a vector tell you how much of the vector is acting along each of the base vectors, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

For example, if a coordinate system is such that $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ represent the directions east, north and vertically up respectively then the velocity vector $\begin{pmatrix} 3 \\ 1 \\ 0.8 \end{pmatrix}$ ms$^{-1}$ or $3\boldsymbol{i} + \boldsymbol{j} + 0.8\boldsymbol{k}$ represents a body moving east at 3 ms$^{-1}$, north at 1 ms$^{-1}$ and is gaining height at 0.8 ms$^{-1}$

It is often useful to know how much of a vector is acting in a given direction.




Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors and let the angle between them be $\theta$.

The component of $\boldsymbol{w}$ acting in the direction of $\boldsymbol{v}$ is $|\boldsymbol{w}|\cos\theta = |\boldsymbol{w}|\frac{\boldsymbol{w} \cdot \boldsymbol{v}}{|\boldsymbol{w}||\boldsymbol{v}|} = \frac{\boldsymbol{w} \cdot \boldsymbol{v}}{|\boldsymbol{v}|}$

The component of $\boldsymbol{w}$ acting perpendicular to the direction of $\boldsymbol{v}$ is

$|\boldsymbol{w}|\sin\theta = |\boldsymbol{w}|\frac{|\boldsymbol{w} \times \boldsymbol{v}|}{|\boldsymbol{w}||\boldsymbol{v}|} = \frac{|\boldsymbol{w} \times \boldsymbol{v}|}{|\boldsymbol{v}|}$

### Example 3.6.4

An object is moving in the direction of the vector $3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$.
It is acted on by a force of $7\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$ Newtons.

Find the component of the force:

(a) in the direction of motion

(b) perpenciular to the motion.

Solution

(a) $\frac{(3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) \cdot (7\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k})}{\sqrt{3^2 + 2^2 + 4^2}}$ using $\frac{w \cdot v}{|v|}$

$= \frac{37}{\sqrt{29}} \approx 6.87$ N

(b) $\frac{|(3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) \times (7\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k})|}{\sqrt{3^2 + 2^2 + 4^2}}$ using $\frac{|w \times v|}{|v|}$

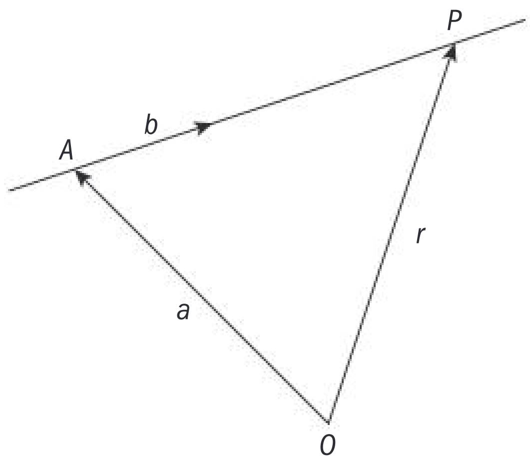
$\frac{|(18\boldsymbol{i} - 13\boldsymbol{j} - 20\boldsymbol{k})|}{\sqrt{29}} = \frac{\sqrt{893}}{\sqrt{29}} \approx 5.55$ N

**Assessment tip**

When answering a vector question, you can use either component form or column form to write the vector.

## The vector equation of a straight line

The vector equation of a straight line can be written as $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{b}$ is a vector in the direction of the line and $\boldsymbol{r}$ is the position vector of the point on the line with parameter $\lambda$.




In three dimensions $\boldsymbol{r}$ represents the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

The vector equation can be written in **parametric** form as three separate equations. For example

$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ can be written as three separate equations:

$x = 2 + 3\lambda,\ y = 1 - 2\lambda,\ z = 4\lambda$

The angle between two lines is the angle between the direction vectors of the lines.

### Example 3.6.5

A chair lift joins an upper station $A$ with coordinates (80, 35, 40) and a lower station $B$ with coordinates (110, 65, 0). The cable joining $A$ and $B$ can be assumed to lie along the straight line joining $A$ and $B$.

(a) Find the vector equation of the straight line joining $A$ and $B$.

A pylon $P$ supports the cable on the chair lift. The top of the pylon makes contact with the line of the cable at a point which is 12 m higher than the position of $B$.

(b) Find the coordinates of $P$.

(c) Find the angle made by the line of the chair lift with the horizontal.

**Solution**

(a) The direction of the line is

$$\overrightarrow{AB} = \begin{pmatrix} 110 \\ 65 \\ 0 \end{pmatrix} - \begin{pmatrix} 80 \\ 35 \\ 40 \end{pmatrix} = \begin{pmatrix} 30 \\ 30 \\ -40 \end{pmatrix}$$

Using the formula $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Equation of line is

$$\boldsymbol{r} = \begin{pmatrix} 80 \\ 35 \\ 40 \end{pmatrix} + \lambda \begin{pmatrix} 30 \\ 30 \\ -40 \end{pmatrix}$$

Any point on the line can be used so

$\boldsymbol{r} = \begin{pmatrix} 110 \\ 65 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 30 \\ 30 \\ -40 \end{pmatrix}$ is also a possible answer.

(b) $z = 12 \Rightarrow 40 - 40\lambda = 12$

$\Rightarrow \lambda = 0.7$

$x = 80 + 0.7 \times 30 = 101$

$y = 35 + 0.7 \times 30 = 56$

Coordinates of $P$ are (101, 56, 12)

The parametric equations can be used to find the value of $\lambda$ and hence the values of $x$ and $y$.

(c) Direction of horizontal line below the chair lift is $\begin{pmatrix} 30 \\ 30 \\ 0 \end{pmatrix}$

Angle between chair lift and horizontal $= \theta$ where

$$\cos\theta = \frac{30 \times 30 + 30 \times 30 + 0 \times -40}{\sqrt{30^2 + 30^2}\sqrt{30^2 + 30^2 + 40^2}}$$
$$= 0.7276...$$
$$\theta \approx 43.3°$$

The angle between the line of the chair lift and the horizontal is the angle between the direction vector and the horizontal.

## Kinematics

The position (displacement) of an object moving with constant velocity will be lie on a straight line of the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$.

In this equation $\boldsymbol{r}$ gives the object's position or **displacement** at time $t$, $\boldsymbol{v}$ is the particle's velocity and $\boldsymbol{a}$ is the position vector of the object when $t = 0$.

If an object departs at time $t = t_1$ rather than $t = 0$ then the equation becomes $\boldsymbol{r} = \boldsymbol{a} + (t - t_1)\boldsymbol{v}$ and $\boldsymbol{a}$ is the object's position at $t = t_1$

**SAMPLE STUDENT ANSWER**

*Distances in this question are in metres.*

Ryan and Jack have model airplanes which take off from level ground. Jack's airplane takes off after Ryan's.

The position of Ryan's airplane $t$ seconds after it takes off is given by $\boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$

(a) Find the speed of Ryan's airplane.

(b) Find the height of Ryan's airplane after two seconds.

The position of Jack's airplane $s$ seconds after it takes off is given by

$\boldsymbol{r} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$

(c) Show that the paths of the airplanes are perpendicular.

The two airplanes collide at the point (–23, 20, 28).

(d) How long after Ryan's airplane takes off does Jack's airplane take off? [5]

(a) Speed $= \sqrt{-4^2 + 2^2 + 4^2} = \sqrt{36} = 6.$

(b) $t = 2$

$r = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 8 \end{pmatrix}.$

(c) $\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} = -16 - 12 + 28 = 0.$

▼ The –4 should be in brackets. Though the final answer was correct it might not be awarded as technically it followed incorrect working!

▼ The student correctly found the displacement of the airplane but did not state that the height is 8 m

▲ The student took the correct direction vectors. The scalar product equaling 0 is sufficient explanation to show they are perpendicular.

(d) If Jack's airplane is a second after Ryan's, then:

$r = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + (t-a)\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$.

time when Ryan's plane gets to (−23, 20, 28) is

$\begin{pmatrix} -23 \\ 20 \\ 28 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \quad -23 = 5 - 4t \rightarrow t = 7$

Check $20 = 6 + 14$

For Jack $\begin{pmatrix} -23 \\ 20 \\ 28 \end{pmatrix} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + (t-a)\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$

$-23 = -39 + 4(t-a)$ but $t = 7$

$-23 = -39 + 28 - 4a \rightarrow a = \frac{-16}{-4} = 4$

▲ The student has correctly used the formula to move the start time.

▲ It is a good idea to check the other coordinates are also satisfied when $t = 7$.

▼ Unfortunately the student did not check their answer here. If they had, the small error in the last line might have been spotted.

If at time $t$ object $A$ has position vector $\boldsymbol{r}_A$ and object $B$ has position vector $\boldsymbol{r}_B$ then The relative position of $B$ from $A$ is the vector $\overrightarrow{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A$.

Their distance apart is $|\overrightarrow{AB}| = |\boldsymbol{r}_B - \boldsymbol{r}_A|$

If two objects collide they must be in the same place at the same time.

### Example 3.6.6

At $t$ hours after 13:00, ship $A$ has position $\boldsymbol{r}_A = \begin{pmatrix} 21 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and ship $B$ has position $\boldsymbol{r}_B = \begin{pmatrix} 7 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ relative to a port, with all distances in kilometres.

(a) Show the two ships would collide if they maintained their velocities and state the time this would occur.

Ship $B$ changes its velocity so it is travelling on a bearing of 045° with the same speed.

(b) Find the shortest distance between the two ships.

**Note**

As a velocity is defined by both speed and direction, it is possible for the velocity to change while the speed remains constant.

**Solution**

(a) The ships will have the same $x$-coordinate when $21 - 2t = 7 + 2t \Rightarrow t = 3.5$

Having found $t$, an alternative method is to solve for $y$ and see if this also gives $t = 3.5$

Substituting into both equations gives

$\boldsymbol{r}_A = \boldsymbol{r}_B = \begin{pmatrix} 14 \\ 13 \end{pmatrix}$, hence collide

Time of collision 16:30

(b) Magnitude of the velocity of $B$ is $\sqrt{2^2 + 4^2} = \sqrt{20}$

The speed of $B$ is the magnitude of its velocity.

A bearing of 045° is in the direction of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

New velocity of B is $\frac{\sqrt{20}}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{10}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3.162... \\ 3.162... \end{pmatrix}$

To find a vector with the correct magnitude use $\frac{1}{|v|}v$ to find the unit vector and multiply by $\sqrt{20}$

$$d = |\overrightarrow{AB}| = |\boldsymbol{r}_B - \boldsymbol{r}_A|$$

$$= \left|\begin{pmatrix} 7 \\ -1 \end{pmatrix} + t\begin{pmatrix} \sqrt{10} \\ \sqrt{10} \end{pmatrix} - \left(\begin{pmatrix} 21 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right)\right|$$

$$= \left|\begin{pmatrix} -14+(2+\sqrt{10})t \\ -7+(-2+\sqrt{10})t \end{pmatrix}\right| \approx \left|\begin{pmatrix} -14+5.16t \\ -7+1.16t \end{pmatrix}\right|$$

$$d = \sqrt{(-14+5.16t)^2 + (-7+1.16t)^2}$$

Hence minimum value of $d \approx 13.8$ km

Simplify $\boldsymbol{r}_B - \boldsymbol{r}_A$ as much as possible before finding an expression for the magnitude.

This last stage is done on the GDC.

Kinematics is also covered in sections 5.4 and 5.5

**Note**

When $x$ is in bold type then it represents a vector; when not bold it represents a single value.

## Motion with variable velocity in two dimensions

If an object's position $\boldsymbol{x}$ is a function of $t$ with $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ then its velocity $\boldsymbol{v} = \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ and its acceleration is $\boldsymbol{a} = \ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$ where the dot notation signifies differentiation with respect to time, so $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

### Example 3.6.7

A heavy ball is thrown from a height of 2 m at an angle of 30° to the horizontal with a speed of 3 ms$^{-1}$.

(a) Find the initial velocity vector, with the $x$ component horizontal and the $y$ component vertical.

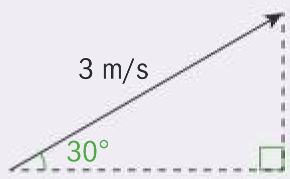
(b) Given that after it has been thrown, the acceleration of the ball is $-9.81\boldsymbol{j}$ ms$^{-1}$, find the horizontal distance from the point of projection that the ball hits the ground.

Solution

(a) $3\cos 30\boldsymbol{i} + 3\sin 30\boldsymbol{j}$

$= 2.598...\boldsymbol{i} + 1.5\boldsymbol{j}$

The components can be found from a diagram of the velocity vector.




(b) $\boldsymbol{v} = \int \boldsymbol{a}\,\mathrm{d}t = c_1\boldsymbol{i} + (-9.81t + c_2)\boldsymbol{j}$

When $t = 0$

$\boldsymbol{v} = 2.598...\boldsymbol{i} + 1.5\boldsymbol{j}$

$\Rightarrow c_1 = 2.598...,\ c_2 = 1.5$

$\boldsymbol{v} \approx 2.60\boldsymbol{i} + (-9.81t + 1.5)\boldsymbol{j}$

When integrating you need to add a constant term to both of the components.

$\boldsymbol{r} = \int \boldsymbol{v}\,\mathrm{d}t = (2.60t + c_3)\boldsymbol{i} + \left(-\frac{9.81}{2}t^2 + 1.5t + c_4\right)\boldsymbol{j}$

When $t = 0$

$\boldsymbol{r} = 2\boldsymbol{j} \Rightarrow c_3 = 0,\ c_4 = 2$

$\boldsymbol{r} = (2.60t)\boldsymbol{i} + \left(-\frac{9.81}{2}t^2 + 1.5t + 2\right)\boldsymbol{j}$

When the object hits the ground, vertical displacement = 0 hence

$-\frac{9.81}{2}t^2 + 1.5t + 2 = 0$

$\Rightarrow t = 0.8095...$

Horizontal displacement is $2.598... \times 0.8095... \approx 2.10$ m

The two components provide two equations which can be used separately.

The negative value of $t$ is disregarded as the equation is only valid for $t \geq 0$

Though this could be solved by the quadratic formula, it is best to solve it directly on the GDC.

The motion of a body which moves only under the action of gravitational force is called **projectile motion**. It is a special case of motion with a variable velocity. Another special case is that of circular motion, as illustrated in the example below.

**Assessment tip**

Even when the question gives vectors in terms of the base vectors, ***i***, ***j*** and ***k*** you can work entirely with column vectors.

### Example 3.6.8

A particle $P$ moves so that its displacement from a point $O$ at time $t$ seconds is given by $\boldsymbol{r} = \begin{pmatrix} 3\cos 2t \\ 3\sin 2t \end{pmatrix}$

(a) Verify that $P$ is always a constant distance from $O$ and state that distance.

(b) Find the time taken for $P$ to complete one full circle.

(c) Find an expression for the velocity of the $P$ at time $t$.

(d) Find the smallest positive value of $t$ at which the velocity is in the direction of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

**Solution**

(a) $|\boldsymbol{r}| = \left| \begin{pmatrix} 3\cos 2t \\ 3\sin 2t \end{pmatrix} \right|$

$= \sqrt{(3\cos 2t)^2 + (3\sin 2t)^2}$

$= \sqrt{9(\cos^2 2t + \sin^2 2t)} = 3$

The distance from $O$ is therefore always constant and equal to 3 units.

The formula $\sin^2\theta + \cos^2\theta = 1$ was given in section 3.4 of this book and is in section AHL 3.8 of the formula book.

The distance being constant means that the particle is moving in a circle.

(b) $P$ will complete a circle when $2t = 2\pi \Rightarrow t \approx 3.14$ seconds

The period of the functions $3\cos 2t$ and $3\sin 2t$ is $\frac{2\pi}{2} = \pi$ (see sections 2.3 and 2.4)

(c) $v = \begin{pmatrix} -6\sin 2t \\ 6\cos 2t \end{pmatrix}$

Each component is differentiated using the chain rule.

(d) This occurs when $6\cos 2t = 0$ and $-6\sin 2t$ is positive

$6\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Hence $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

For $-6\sin 2t$ to be positive $t = \frac{3\pi}{4} \approx 2.36$

If moving in the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then the velocity vector must be a positive multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If it was a negative multiple then it would be moving in the direction of the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$6\cos 2t = 0$ could also be solved directly from a GDC.

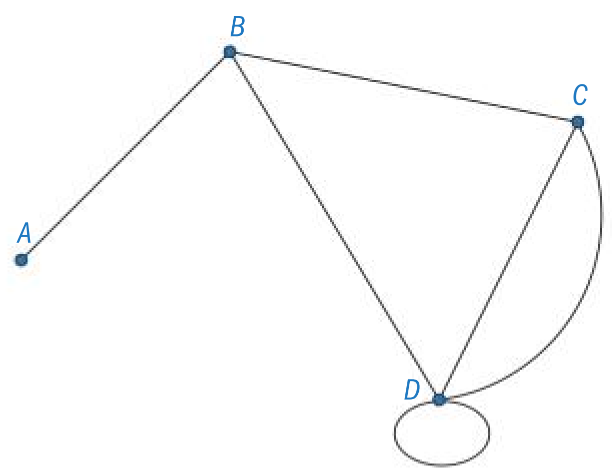
# 3.7 GRAPHS

## You should know:

- ✔ the definitions of: graphs, vertices, edges, adjacent vertices, adjacent edges, degree of a vertex, simple graphs; complete graphs; weighted graphs, directed graphs; in degree and out degree of a directed graph, subgraphs; trees, adjacency matrices, walks, circuits, cycles, connected and strongly connected graphs
- ✔ that given an adjacency matrix **A**, the $(i, j)^{\text{th}}$ entry of $\mathbf{A}^k$ gives the number of $k$ length walks connecting $i$ and $j$
- ✔ that given a transition matrix for a graph the $(i, j)^{\text{th}}$ entry gives the gives the probability of moving from state $j$ to state $i$ in a random walk.
- ✔ Eulerian trails and circuits
- ✔ Hamiltonian paths and cycles
- ✔ Kruskal's and Prim's algorithms for finding minimum spanning trees
- ✔ the Chinese postman problem and algorithm for determining the shortest route around a weighted graph traversing each of the edges at least once
- ✔ the travelling salesman problem to determine the Hamiltonian cycle of least weight in a weighted complete graph

## You should be able to:

- ✔ represent real-world structures (circuits, maps, etc) as graphs (weighted and unweighted)
- ✔ calculate the number of $k$-length walks (or less than $k$-length walks) between two vertices
- ✔ construct the transition matrix for a strongly-connected, undirected or directed graph
- ✔ determine whether an Eulerian trail or circuit exists
- ✔ find a minimum spanning tree using either the matrix or graphical method for Prim's algorithm or the graphical method for Kruskal's
- ✔ apply the Chinese postman algorithm with up to four odd vertices
- ✔ convert a practical travelling salesman problem to a classical problem by completion of a table of least distances.
- ✔ use the nearest neighbour algorithm for determining an upper bound for the travelling salesman problem
- ✔ use the deleted vertex algorithm for determining a lower bound for the travelling salesman problem.



Graph G1

### Graph definitions

Consider the graph G1.

The **vertices** are $A$, $B$, $C$ and $D$ and the **edges** are [$AB$], [$BC$], [$BD$] and [$CD$]

The edges [$AB$] and [$BC$] are **adjacent** because they share a vertex, $B$.

The vertices $A$ and $B$ are **adjacent** because they are connected by an edge.

The edge from $D$ to itself is called a **loop**. Because there is more than one edge from $C$ to $D$ we say there are **multiple edges**.

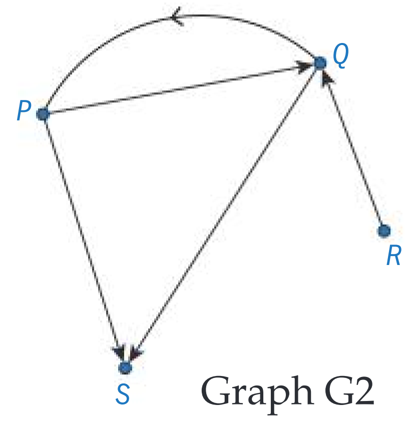
A **simple graph** is one with no loops or multiple edges.

The **degree of a vertex** is the number of edges incident to it, so the degree of $A$ is 1, of $B$ is 3, of $C$ is 3 and of $D$ is 5.

A **directed graph** is one in which it is possible to travel in only one way along the edge. The direction of permissible travel is indicated by an arrow.

For example in Graph G2, it is possible to travel from $P$ to $S$ but not from $S$ to $P$.

On a directed graph, the in-degree of a vertex is the number of edges that lead to the vertex and the out-degree is the number of edges leading from the vertex. Hence the in-degree of vertex $P$ is 1 and the out-degree is 2.



Graph G2

## Adjacency matrices

In an adjacency matrix **A**, the $(i, j)^{\text{th}}$ entry ($a_{ij}$) is the number of edges leading from vertex $i$ to vertex $j$.

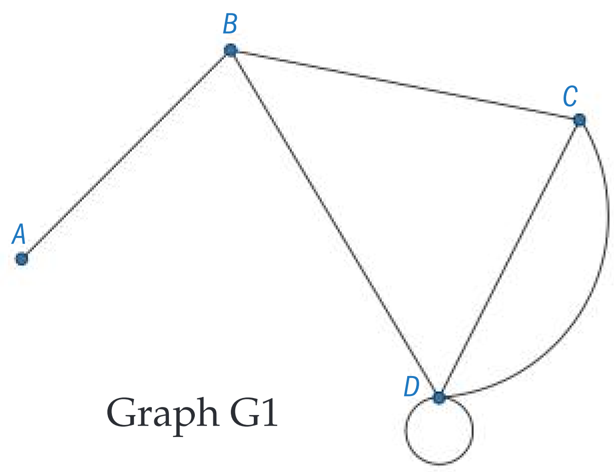
The adjacency matrix for graphs G1 and G2 are shown below.

$$G1 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix} \end{array} \qquad G2 = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

**Note**

This is the opposite to a transition matrix in which the $a_{ij}$ entry gives the probability of moving from vertex $j$ to vertex $i$.

A **walk** is a route from one vertex to another. Its length is the number of edges it traverses. In graph G1 the route $A$ to $B$ to $D$ is a walk of length 2. The walk is usually written as a list of vertices, for example $ABD$.



Graph G1

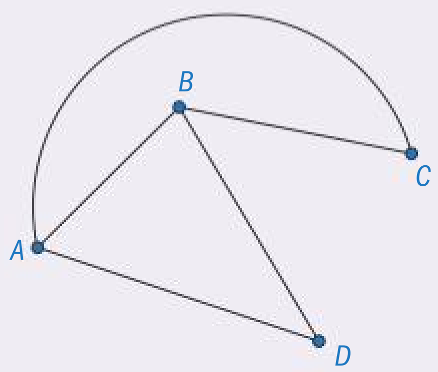
**Note**

The sum of a row or column of an adjacency matrix gives the degree of the vertex represented by that row/column. For a directed graph the sum of a row gives the out-degree and the sum of a column gives the in-degree.

The $(i, j)^{\text{th}}$ entry of the adjacency matrix $A^n$ is the number of walks of length $n$ from $i$ to $j$.

**Example 3.7.1**

(a) Construct an adjacency matrix **M** for this graph.



(b) Find (i) $\mathbf{M}^2$ (ii) $\mathbf{M}^3$

(c) Hence or otherwise find the number of walks of length less than or equal to 3 from $A$ to $D$.

The graph represents airports $A$, $B$, $C$ and $D$ and shows which are connected by direct flights.

Georgina wishes to travel between airports $A$ and $D$ in at most three flights.

(d) Explain in context why the number of possible routes she would consider is fewer than the answer to part (c).

**Solution**

(a) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

The matrix is set up with $A$ in the first row/column, then $B$, $C$ and $D$

(b) (i) $\begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 4 & 5 & 5 \\ 5 & 5 & 2 & 2 \\ 5 & 5 & 2 & 2 \end{pmatrix}$

The powers are found from the GDC

(c) $1+1+5=7$

The number of routes is the number of routes of length 1, 2 and 3, so the entries in the top right corner of the three matrices are added.

(d) Some of the walks involve leaving and returning to $A$ or going to $D$ directly and to $B$ and back to $D$ again.

For example, $ABAD$. The trip from $A$ to $B$ and back again would be unnecessary.

Link to chapter 1 section 1.5 Matrices, and chapter 4 section 4.10 Transition matrices and Markov chains. It is best to review these two sections before reviewing this section.

**Note**

This is the opposite to an adjacency matrix.

**Assessment tip**

In exams, all transition matrices will be for strongly connected graphs.

**Note**

When the graph represents connections between webpages, the steady state values can be used to rank the webpages.

## Transition matrices

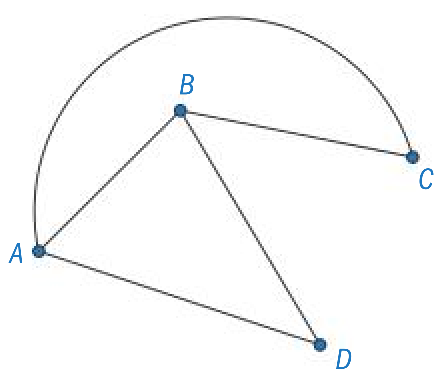
A transition matrix for a graph gives the probability of reaching any given vertex from a starting vertex when all edges are equally likely to be taken (a random walk).

The $(i, j)^{\text{th}}$ entry is the number of edges leading from vertex $j$ to vertex $i$.

The transition matrix for graph in Example 3.7.1 is
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix} \end{array}$$

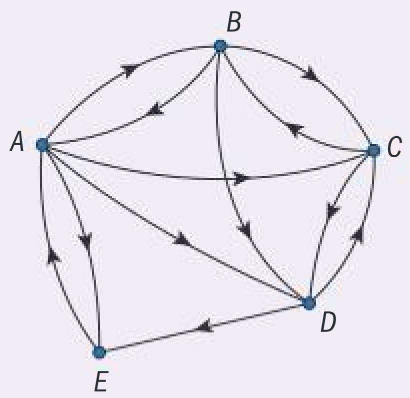
The **steady state vector** for the transition matrix gives the long-term proportion of time spent at each vertex on a random walk.

The steady state will always exist when the graph is **connected** or the directed graph is **strongly connected** (which means you can get from any vertex to any other vertex).



### Example 3.7.2

The directed graph here shows the links between webpages. A search engine performs a random walk through the graph in which it is equally likely to leave a site by any of the possible links.




(a) Find the transition matrix for this random walk.

(b) Find the probability of the search engine being at site $E$ after following three links, given it begins it search at site $A$.

(c) Find the long-term probabilities of being at each of the sites.

**Solution**

(a)

$$\mathbf{T} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 0.333 & 0 & 0 & 1 \\ 0.25 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0.333 & 0 & 0.5 & 0 \\ 0.25 & 0.333 & 0.5 & 0 & 0 \\ 0.25 & 0 & 0 & 0.5 & 0 \end{pmatrix} \end{array}$$

Note: because this is a directed graph the matrix is not symmetrical.

In the first column there are 4 equal probabilities as there are four edges from $A$.

(b) From $\mathbf{T}^3$ the probability is 0.1875

This is the entry in the fifth row of the first column of $\mathbf{T}^3$

(c) Each column in the steady state vector is

$$\begin{pmatrix} 0.222... \\ 0.166... \\ 0.222... \\ 0.222... \\ 0.166... \end{pmatrix}$$

The probability of being at $A$, $C$ and $D$ is 0.222, and of being at $B$ and $E$ is 0.167

Finding the values from a high power of $\mathbf{T}$. The answers have been given to three significant figures but could have been given exactly as fractions.

### Assessment tip

There is no need to write out the whole matrix. You should state that you are finding $\mathbf{T}^3$, so if you entered the values incorrectly or read the wrong value you would get method or follow through marks.

## Graph algorithms

A **weighted graph** is one in which the edges are labelled with **weights**. The weights might represent distances between the vertices or time of travel or costs etc.

A **cycle** is a route that starts and ends at the same vertex without repeating any vertex.

A **circuit** is a route that starts and ends at the same vertex without repeating any edge.

A **tree** is a graph with no cycles.

A **minimum spanning tree** is a tree that connects all the vertices in the graph with the minimum possible total edge weight.

There are two ways to find the minimum spanning tree:

**Kruskal's algorithm:** At each step, add the edge of least weight that is not already included and which does not form a cycle. Repeat until all the edges are connected.

**Prim's algorithm:** Choose a starting vertex. At each stage, add the edge of least weight to the tree already formed and which connects to a new vertex.

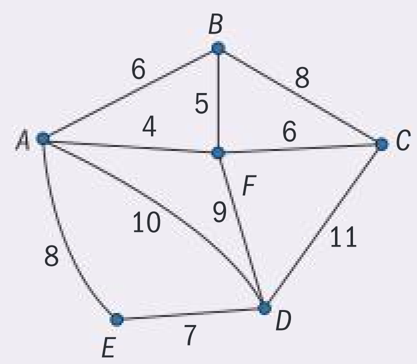
### Example 3.7.3

Find the minimum spanning tree for the graph shown using:

(a) Kruskal's algorithm

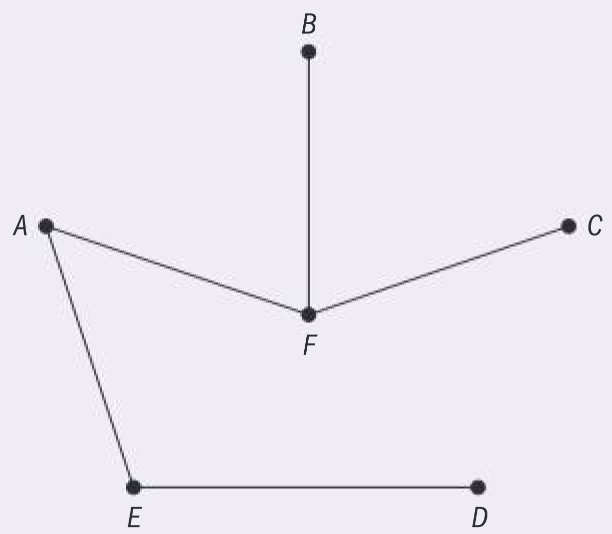
(b) Prim's algorithm

Write down the order in which the edges are selected.




Solution

(a) The order in which the edges are selected is $AF$, $FB$, $FC$, $ED$, $AE$




Note that after $AF$ and $FB$ had been selected there was a choice of two edges with weight 6. $AB$ could not be selected because it would have formed a cycle.

Note that when $ED$ was selected it was not connected to the rest of the tree. In Kruskal's algorithm the edges do not have to be connected until the end, which happens when $AE$ is added.

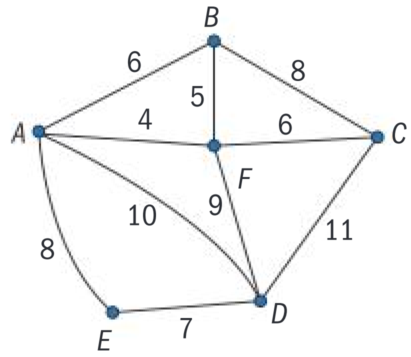
(b) The spanning tree is the same as for part (a) but, beginning with vertex $A$ the order is $AF$, $FB$, $FC$, $AE$, $ED$.

In Prim's algorithm, because you are always adding a new vertex to the tree you are forming, $ED$ could not be added until either $E$ or $D$ was in the tree.

**Assessment tip**

The question will often ask you to write down the order in which the edges are selected – this is to ensure you have used the correct algorithm, so you might not be awarded **any** marks if you don't do this.

The graph in Example 3.7.3 could have been given in a weighted adjacency table (or matrix) as follows:




| | *A* | *B* | *C* | *D* | *E* | *F* |
|---|---|---|---|---|---|---|
| *A* | | 6 | | 10 | 8 | 4 |
| *B* | 6 | | 8 | | | 5 |
| *C* | | 8 | | 11 | | 6 |
| *D* | 10 | | 11 | | 7 | 9 |
| *E* | 8 | | | 7 | | |
| *F* | 4 | 5 | 6 | 9 | | |

Prim's algorithm can be performed on a weighted adjacency table without the need to draw out the graph.

| | 1 | 3 | 4 | 6 | 5 | 2 |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* |
| ~~*A*~~ | | ~~6~~ | | ~~10~~ | ~~8~~ | ~~4~~ |
| ~~*B*~~ | ~~6~~ | | ~~8~~ | | | 5 |
| ~~*C*~~ | | ~~8~~ | | ~~11~~ | | 6 |
| ~~*D*~~ | ~~10~~ | | ~~11~~ | | 7 | ~~9~~ |
| ~~*E*~~ | 8 | | | ~~7~~ | | |
| ~~*F*~~ | 4 | ~~5~~ | ~~6~~ | ~~9~~ | | |

*AF*, *FB*, *FC*, *AE*, *ED*

Cross off row *A* and write a 1 above column *A*.

Choose the smallest entry in that column (4) and circle it.

Cross off that row (*F*) and write a 2 above the corresponding column.

Choose the smallest entry from any of the numbered columns which has not been crossed out (5), circle it and cross out that row (*B*) and number the column 3.

Repeat until all the rows have been selected.

## The Chinese postman problem

A **trail** is a walk that does not repeat any edge.

A **circuit** is a trail that starts and finishes at the same vertex.

An **Eulerian trail** is a trail that traverses all the edges in a graph.

An **Eulerian circuit** is an Eulerian trail which starts and ends at the same vertex.

A graph for which every vertex has an even degree will contain an Eulerian circuit.

A graph which has exactly two vertices of odd degree will have an Eulerian trail which begins at one of the vertices of odd degree and ends at the other.

The Chinese postman problem is to find the walk of minimum weight that travels along every edge of a graph and returns to its starting point. The idea behind the name is that a postman delivering letters would have to walk down every street.

If all the vertices have even degree then the shortest walk will be an Eulerian circuit and no edges need to be repeated.

If there are two vertices of odd degree then the route between these two will need to be repeated and this is done using the route of least weight.

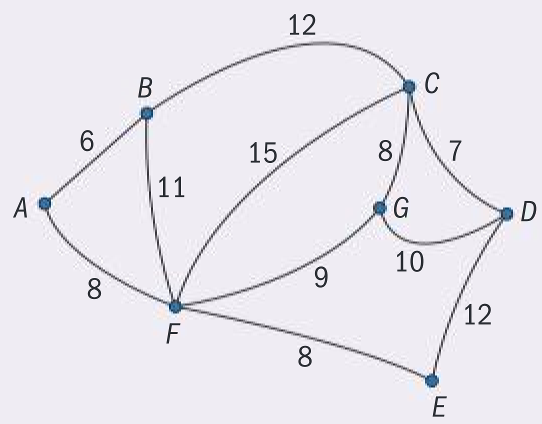
If there are four vertices of odd degree then these need to be connected in pairs using the routes with least total weight.

**Assessment tip**

In an exam there will be at most four vertices with odd degree.

**Example 3.7.4**

The graph shown represents paths in a national park. The vertices are mountain huts and the weights are the distances along each of the paths in kilometres. A walker would like to trek along all of the paths. The total distance of all the paths is 106 km.




(a) Find the least distance he would have to walk if he is starting and finishing at the same mountain hut, and list the paths that he would need to walk along twice.

The walker receives an offer to be dropped and picked up at any of the mountain huts.

(b) Write down which huts he should choose to minimize the distance he has to walk.

**Solution**

(a) The odd vertices are

$B$, $F$, $D$ and $G$

$BF$ and $DG$ $11 + 10 = 21$

$BD$ and $GF$ $19 + 9 = 28$

$BG$ and $DF$ $20 + 19 = 39$

The least distance is $106 + 21 = 127$ km, repeating $BF$ and $DG$

Whenever there are four vertices of odd degree there are three combinations to consider. The weights for all of them must be shown.

(b) The shortest distance is $GF$ so he should choose to be dropped at $B$ and picked up at $D$ or vice versa

This would mean he only has to walk an extra 8 km.

## The travelling salesman problem

A **path** is a walk that does not pass through any vertex more than once.

A **Hamiltonian path** is a path that includes all the vertices in a graph.

A **Hamiltonian cycle** is a Hamiltonian path that starts and ends at the same vertex.

A **complete graph** is one in which all the vertices are adjacent to all other vertices.

The **travelling salesman problem** is to find the Hamiltonian cycle of least weight on a complete weighted graph. This is often referred to as the **classical** travelling salesman problem. The idea behind the name is that the vertices might represent towns and a salesman is looking for the shortest way to travel to all of the towns and then back home.

In many real life or ‘practical’ problems the graph connecting the vertices is not complete. In this case the distances between the unconnected towns is taken as the shortest distance between them. These can be added to a **table of least distances** or shown on a graph so that every vertex is connected and the solution of the practical problem is the same as the solution of the classical problem.

**Assessment tip**

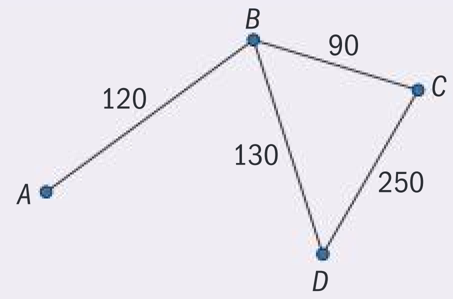
If using a table of least distances, the solution to the practical problem is likely to involve passing through some vertices more than once.

There is no algorithm for finding the solution to the travelling salesman problem. Instead, algorithms are used to find an upper bound and a lower bound for the solution. The closer together these are the better the solution.

The weight of any Hamiltonian cycle is an upper bound to the travelling salesman problem.

### Example 3.7.5

The graph shows times in minutes to travel between towns $A$, $B$, $C$ and $D$. A salesman wishes to visit each of the towns, starting and finishing at $A$.




(a) Complete a table showing the shortest travel times between the towns.

(b) By considering the cycle $ABCDA$, find an upper bound for the time taken by the salesman, stating in order the towns he passes through on this journey.

**Solution**

(a)

| | A | B | C | D |
|---|---|---|---|---|
| A | 0 | 120 | 210 | 250 |
| B | 120 | 0 | 90 | 130 |
| C | 210 | 90 | 0 | 220 |
| D | 250 | 130 | 220 | 0 |

Notice that the shortest time between $C$ and $D$ is not the direct route. This might be because that is a minor road compared with the roads $CB$ and $BD$.

(b) From the table $120 + 90 + 220 + 250 = 680$ minutes

Passing through $A, B, C, B, D, B, A$

Refer back to the graph to see the actual route taken.

The **nearest neighbour algorithm** is often used to find an upper bound to the travelling salesman problem. In the nearest neighbour algorithm you select a starting vertex and then move to the nearest adjacent vertex. This is repeated until all vertices have been reached and the direct route back to the starting vertex is then taken.

The **deleted vertex algorithm** is often used to find a lower bound to the travelling salesman problem.

In the deleted vertex algorithm, one vertex is deleted from the graph along with all the incident edges. The minimum spanning tree of the remaining subgraph is then found.

The weight of the minimum spanning tree plus the weight of the two edges of least weight that connect the deleted vertex to the tree give a lower bound.

**Note**

Both the nearest neighbour algorithm and the deleted vertex algorithm should be performed on a complete graph or on a complete table of least distances.

SAMPLE STUDENT ANSWER

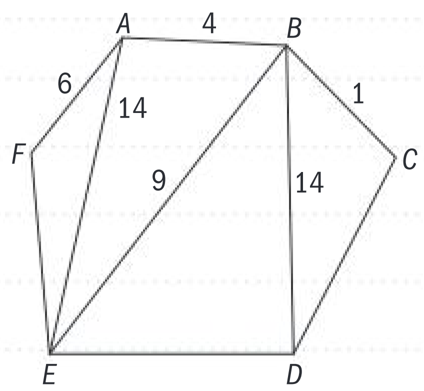
The weights of the edges in the complete graph G are given in the following table.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | – | 4 | 9 | 8 | 14 | 6 |
| B | 4 | – | 1 | 14 | 9 | 3 |
| C | 9 | 1 | – | 5 | 12 | 2 |
| D | 8 | 14 | 5 | – | 11 | 12 |
| E | 14 | 9 | 12 | 11 | – | 7 |
| F | 6 | 3 | 2 | 12 | 7 | – |

(a) Starting at $A$, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G.

(b) By first deleting vertex $A$, use the deleted vertex algorithm together with Kruskal's algorithm to find a lower bound for the travelling salesman problem for G.

▼ The nearest neighbour algorithm can be used directly from the table. It can be helpful to draw the graph, but errors could be made in doing so and time could be lost.




(a) Nearest Neighbor.

AB(4)

BC(1)

CF(2)

FE(7)

EA(14)

▲ The student correctly applied the algorithm to the first few points and showed all the steps, so was awarded some marks.

▼ The student missed out edge $ED$.

Hence, upper bound = 28

(b)

Bc(1), CF(2), CD(5), FE(7)

1 + 2 + 5 + 7 = 15

Lower bound is 15

▲ The student correctly applied Kruskal's and, though not told to do so in the question, wisely listed the order in which the edges were selected, so they were awarded four marks even though the final answer was not correct.

▼ The student forgot to add the two edges of least weight from $A$: 15 + 4 + 6 = 25

# SL PRACTICE QUESTIONS

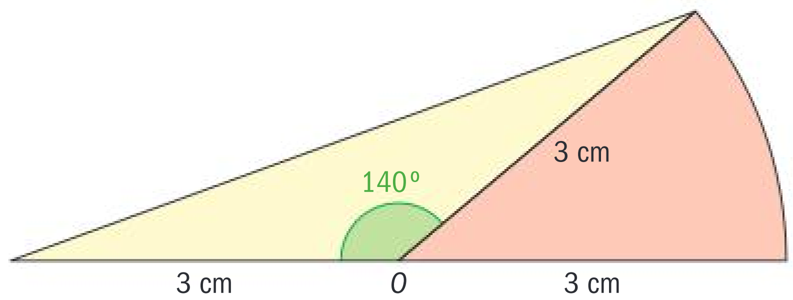
## PAPER 1, GROUP 1

**1.** A logo for a sports team is made up of a sector of a circle and a triangle in two colours, as shown. The sector of the circle has a centre at $O$ and the circle has a radius of 3 cm.

Find the area of:

**a.** the triangular section

**b.** the sector.




**2.** The coordinates of the summits of two adjacent mountains are (102.8, 39.1, 2.9) and (108.3, 42.2, 4.5), where all distances are in kilometres.

Find the straight-line distance between the two summits.

**3.** A cake is in the shape of a cylinder with a radius of 8 cm and a height of 10 cm.

**a.** Find the volume of the cake.

A piece of the cake is removed. The cross-section of the piece is a sector of a circle and forms an angle of 35° at the centre of the cake.

**b.** Find the volume of the piece.

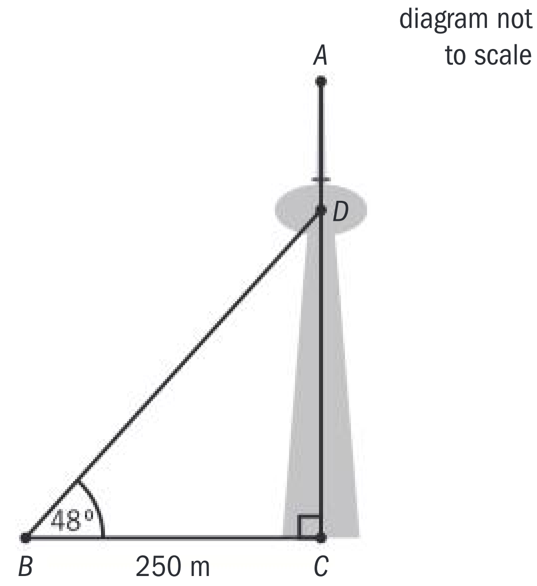
**4.** Let $A$ be the point with coordinates (2, 8) and $B$ be the point with coordinates (–1, 10).

**a.** Find the midpoint of the line segment joining $A$ and $B$.

Let $l$ be the perpendicular bisector of the line segment joining $A$ and $B$.

**b.** Find the equation of $l$.

**5.** $A$ is a point at the top of a vertical communications tower with its base at $C$. The tower has an observation deck $D$, three-quarters of the way to the top of the tower, $A$.

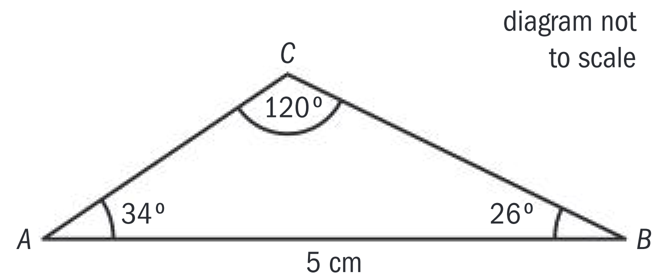



From point $B$, on horizontal ground 250 m from $C$, the angle of elevation to $D$ is 48°

**a.** Calculate $CD$, the height of the observation deck above the ground.

**b.** Calculate the angle of depression from $A$ to $B$.

**6.** A triangular postage stamp, $ABC$, is shown in the diagram below, such that $AB = 5$ cm, $B\hat{A}C = 34°$, $A\hat{B}C = 26°$ and $A\hat{C}B = 120°$

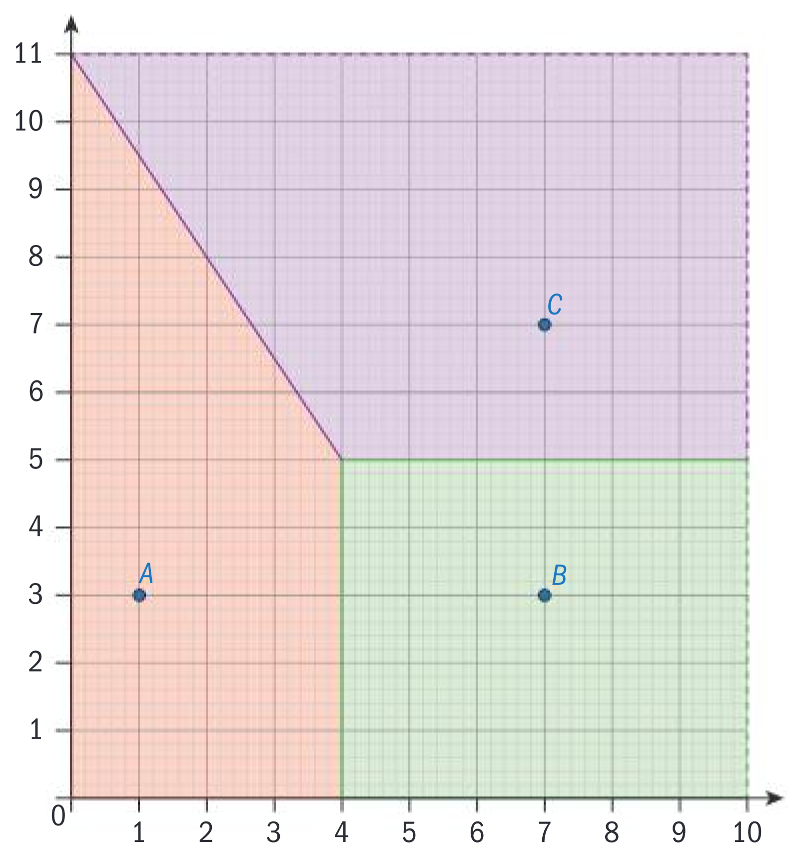



**a.** Find the length of $BC$.

**b.** Find the area of the postage stamp.

## GROUP 2

**7.** The Voronoi diagram below shows three wells, $A$, $B$ and $C$, set in a rectangular area of land. Each unit on the axes represents 1 km.



a. Write the equations of each of the lines forming the three boundaries within the Voronoi diagram.

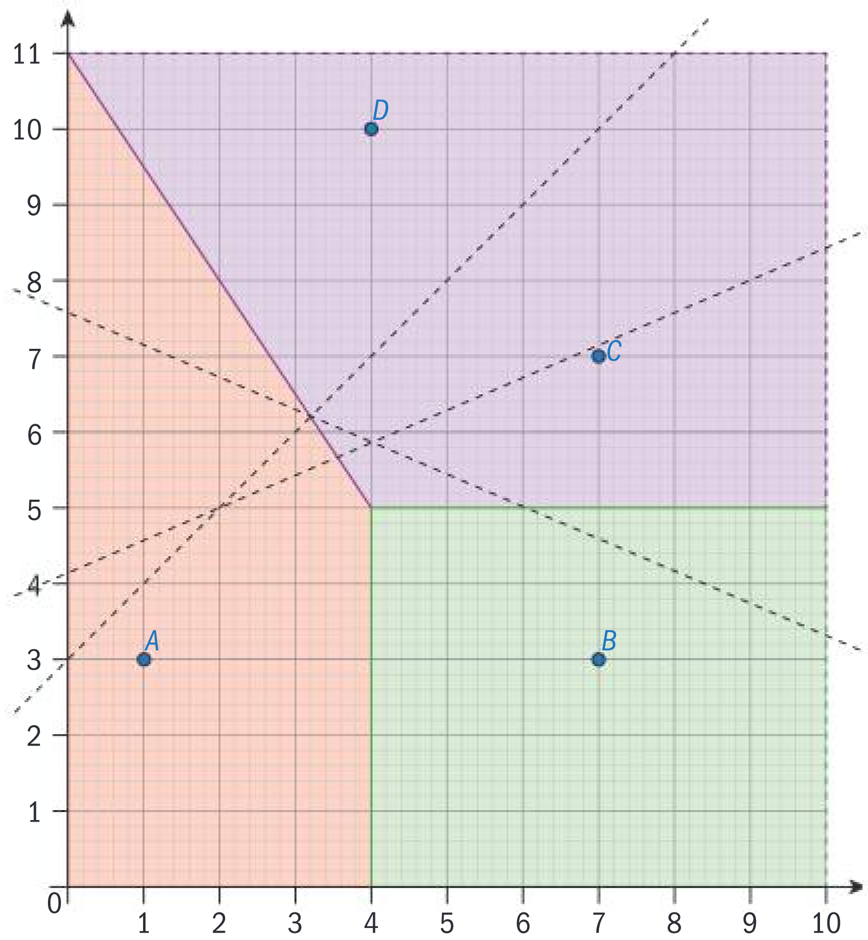
b. Find the area of the region containing:

i. $A$

ii. $B$

iii. $C$

A fourth well is created at the point $D$. The perpendicular bisectors between $D$ and each of $A$, $B$ and $C$ are shown as dashed lines on the diagram.




c. On a copy of the diagram, complete the Voronoi diagram for the four wells.

8. An underground tunnel goes from point $A$ with coordinates (10.1, 3.2, −0.8) to a point $B$ with coordinates (8.5, 1.7, −0.9)

A relief tunnel is to go from a point on the surface at $C$ (9.0, 2.0, 0) to the midpoint of the tunnel joining $A$ and $B$. All positions are given in kilometres.

Find the length of the relief tunnel.

9. A geologist is trying to work out the approximate surface area of a triangular glacier. The coordinates of the three corners of the glacier are $A$(0, 0, 0), $B$(250, 120, 75) and $C$(−120, 230, 140), where distances are in metres.

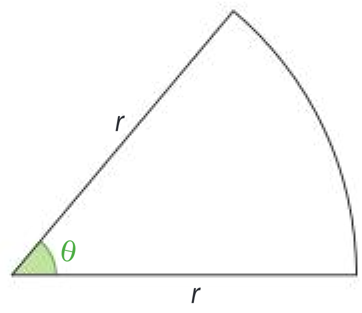
From C, he can see both $A$ and $B$ and measures the angle $A\hat{C}B$ as 47°

Find:

a. the distance from $B$ to $C$

b. the area covered by the glacier.

10. Jorge wishes to bend a 20 cm length of wire into the shape of a circular sector. Let the radius of the sector be $r$ cm and the angle made at the centre by the sector be $\theta$.




a. Find an expression for $\theta$ in terms of $r$.

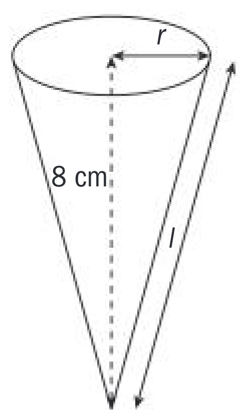
Jorge wants the sector to contain the largest possible area.

b. Show that the area of the sector $A$ can be given by the expression:

$A = 10r - r^2$

c. Hence find the value of $r$ which gives the maximum area.

11. A type of candy is packaged in a right circular cone with a volume of 100 cm$^3$ and vertical height of 8 cm.

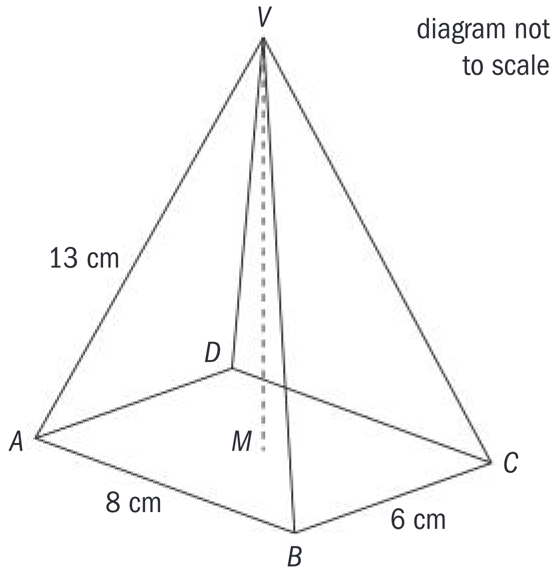



a. Find the radius, $r$, of the circular base of the cone.

b. Find the slant height, $l$, of the cone.

c. Find the curved surface area of the cone.

12. A right pyramid has apex $V$ and rectangular base $ABCD$, with $AB = 8$ cm, $BC = 6$ cm and $VA = 13$ cm. The vertical height of the pyramid is $VM$.

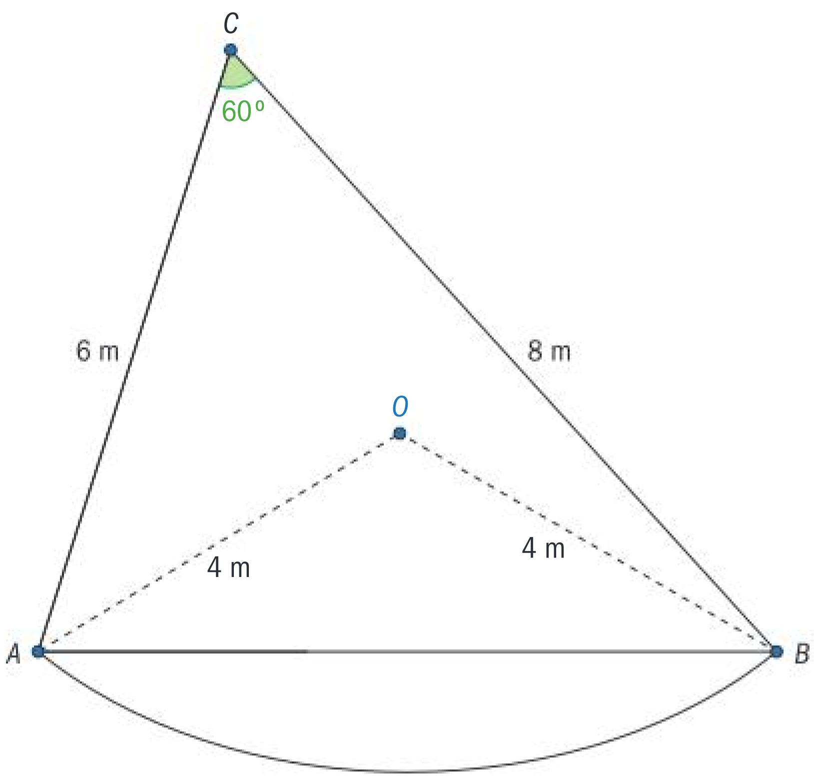



a. Calculate $VM$.

b. Calculate the volume of the pyramid.

## GROUP 3

13. A garden consists of a lawn in the shape of a triangle, shown as $ABC$ in the diagram, and a flower bed formed by the arc of a circle, centre $O$ with a radius of 4 m. The length of $AC$ is 6 m, the length of $BC$ is 8 m and angle $A\hat{C}B = 60°$




The owner of the garden wants to buy some decorative edging to go around the flower bed.

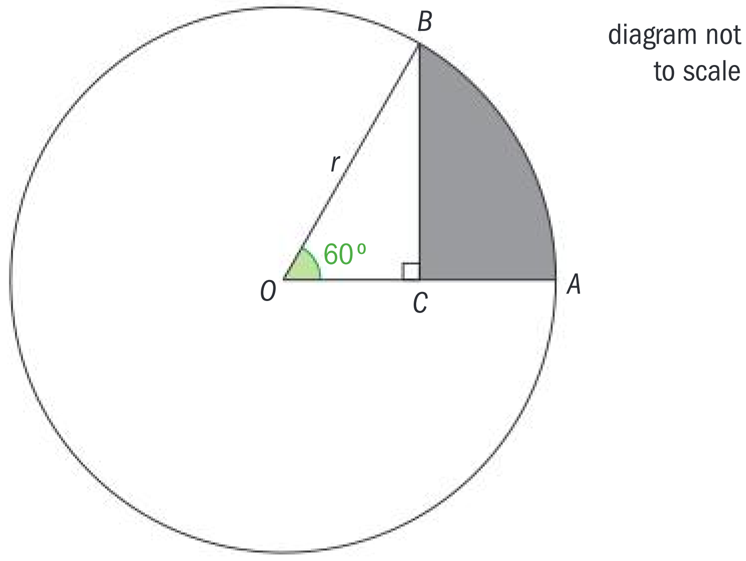
a. Find the length of the side $AB$.

b. Hence, find the length of edging the owner needs to buy.

14. The diagram shows a circle with centre $O$ and radius $r$ cm.

Points $A$ and $B$ lie on the circumference of the circle and $A\hat{O}B = 60°$.

The point $C$ is on $[OA]$ such that $B\hat{C}O = 90°$.




a. Find an expression, in terms of $r$, for:

i. $OC$

ii. the area of the triangle $OBC$.

The area of the shaded region is 50 cm$^3$

b. Find the value of $r$.

15. A large sculpture is in the shape of a right square-based pyramid. Two adjacent corners of the base, $A$ and $B$, have coordinates (0, 0, 0) and (8, 0, 0) respectively. The vertex of the pyramid, $V$, has coordinates ($a$, $a$, 15). All distances are in metres.

a. i. Write down the height of the pyramid.

ii. Find the volume of the pyramid.

b. Write down the value of $a$.

The pyramid has a light positioned half way up the triangular face which has $[AB]$ as its base, and is centred horizontally on that face.

c. Find the coordinates of the light.

The sides of the pyramid are covered in steel panels.

d. Find the area of the panels used.

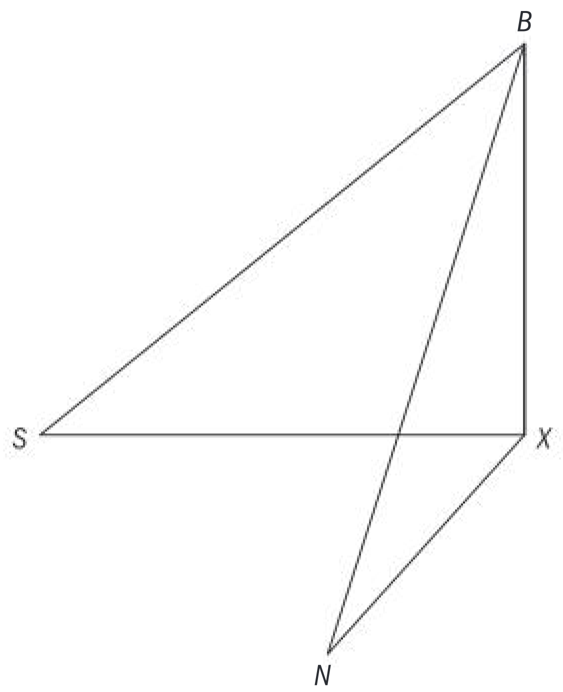
16. Barry is at the top of a cliff, standing 80 m above sea level. He observes two yachts in the sea.

Seaview ($S$) is at an angle of depression of 25°

Nauti Buoy ($N$) is at an angle of depression of 35°

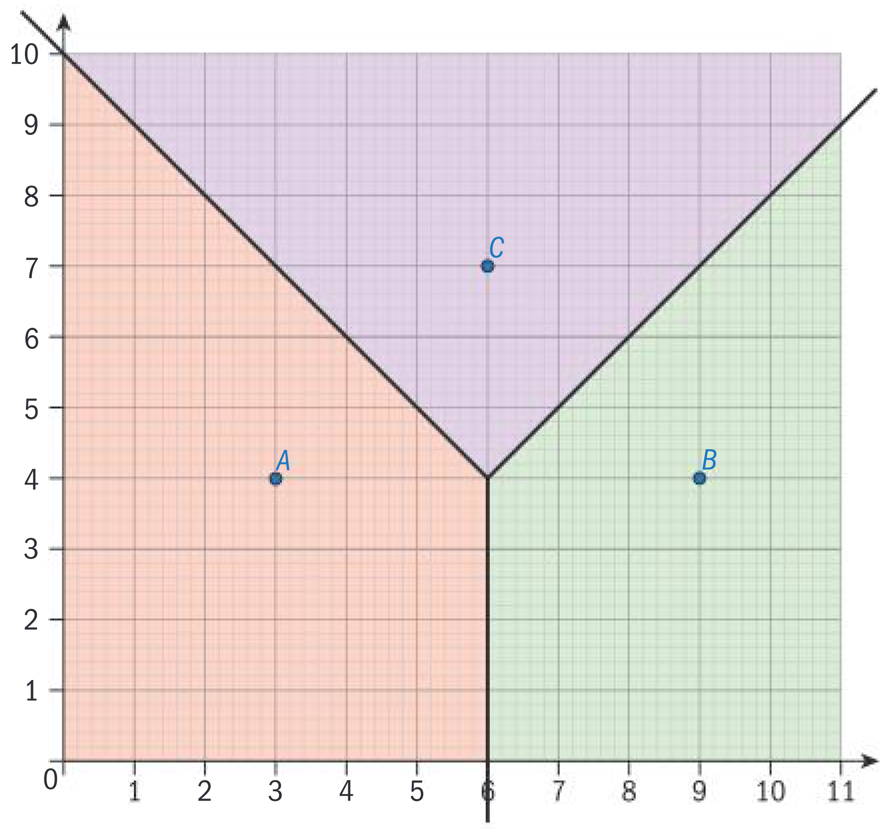
The three-dimensional diagram shows Barry ($B$) and the two yachts at $S$ and $N$.

$X$ lies at the foot of the cliff, and $S\hat{X}N = 70°$




Find the distance between the two yachts, accurate to three significant figures.

**17.** Pollution levels in a city are recorded at three environmental stations $A$, $B$ and $C$. The environmental stations are shown on the Voronoi diagram with coordinates $A(3, 4)$, $B(9, 4)$ and $C(6, 7)$, where distances are given in kilometres.




On a particular day, the levels of pollution according to the Air Quality Health Index are recorded as 6.5 at $A$, 7.2 at $B$ and 3.2 at $C$.

The level of pollution at any point in the city is taken to be the same as the value recorded at the nearest station to that point.

A house is situated at the point $D$ with coordinates (6, 5)

**a.** State the level of pollution that would be assigned to $D$.

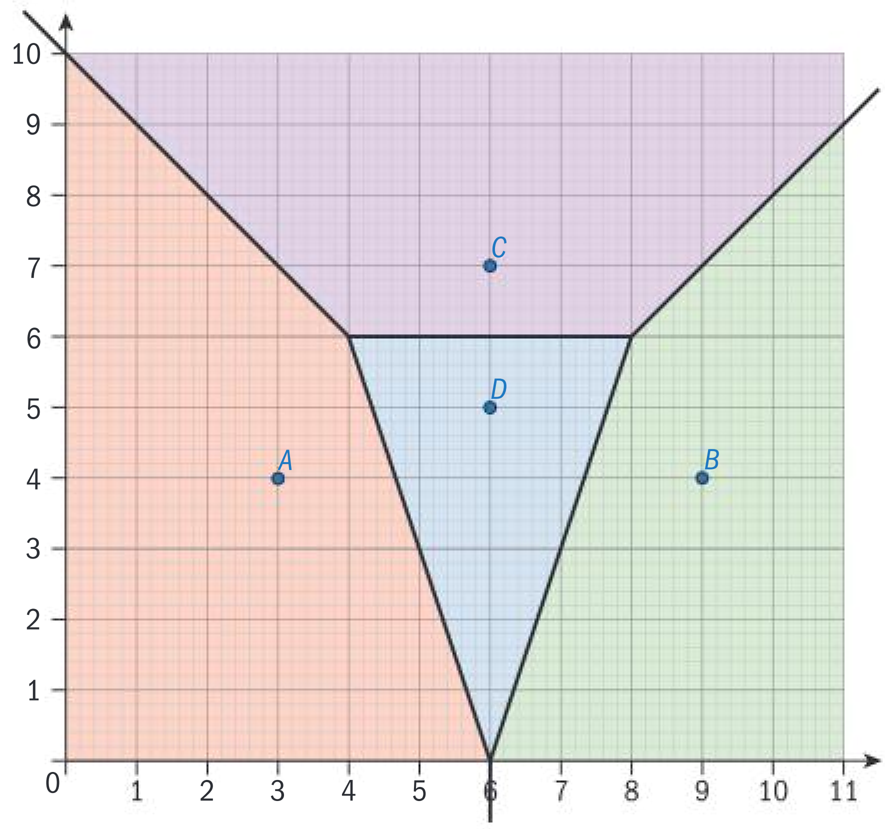
The house owner at $D$ claims that the pollution levels are much worse than the assigned value.

An alternative method is suggested to give a more accurate measure. A new cell for the Voronoi diagram will be created around $D$. The level of pollution, $P_D$, for the point $D$ will then be found using the formula:

$$P_D = \frac{6.5a_A + 7.2a_B + 3.2a_C}{a}$$

where $a$ is the area of the new cell and $a_A$, $a_B$ and $a_C$ are the areas of the regions within the new cell that used to belong to stations $A$, $B$ and $C$ respectively.

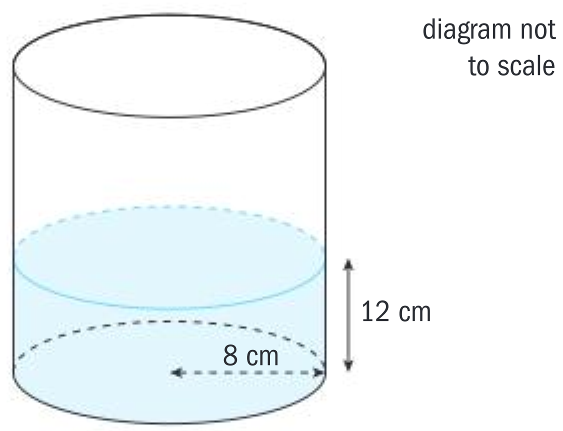
The new Voronoi diagram is shown below with vertices at (4, 6), (8, 6) and (6, 0).




**b.** Find the value of $a$

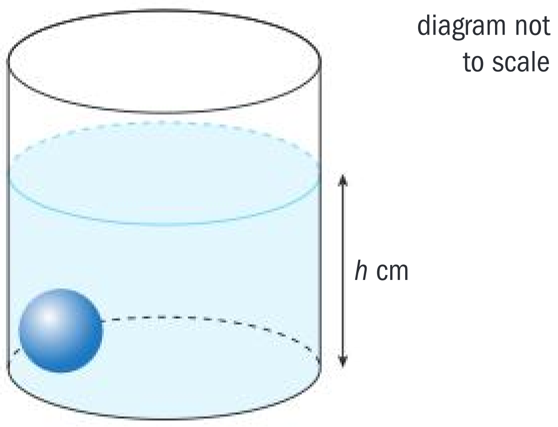
**c.** Find the value of $P_D$

**18.** A cylindrical container with a radius of 8 cm is placed on a flat surface. The container is filled with water to a height of 12 cm, as shown.



**a.** Find the volume of water in the container.

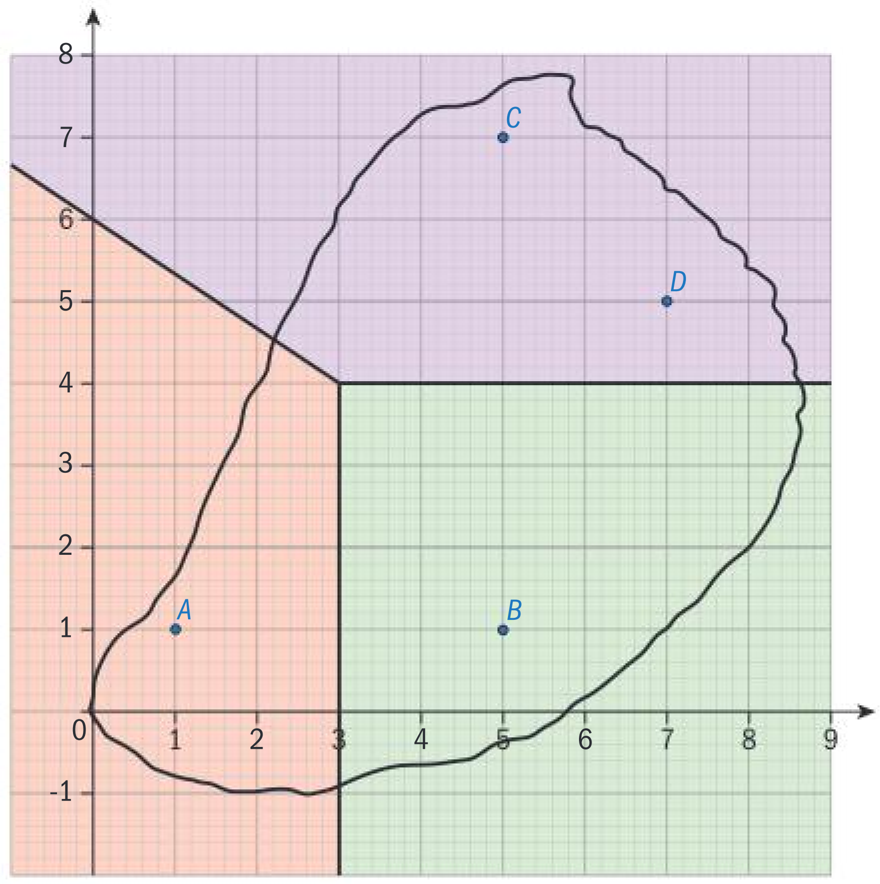
A heavy ball with a radius of 2.9 cm is dropped into the container. As a result, the height of the water increases to $h$ cm, as shown in the diagram.




**b.** Find the value of $h$.

## PAPER 2

**1.** An island has four schools which are shown on a coordinate grid at the points $A(1, 1)$, $B(5, 1)$, $C(5, 7)$ and $D(7, 5)$. Also shown is the Voronoi diagram for the schools $A$, $B$ and $C$.




**a.** Find:

**i.** the midpoint of $[BD]$

**ii.** the gradient of $[BD]$

**b.** Hence show that the equation of the perpendicular bisector of $[BD]$ is $y = -\frac{1}{2}x + 6$

**c.** Find the equation of the perpendicular bisector of $[CD]$

**d.** Complete the Voronoi diagram for the schools $A$, $B$, $C$ and $D$

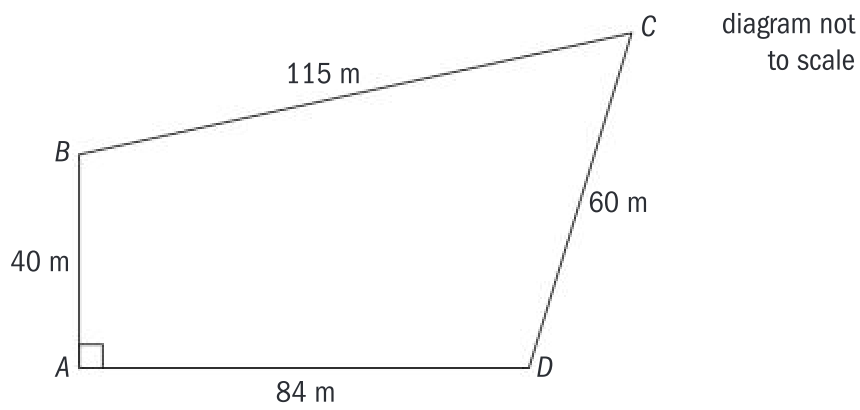
(Note: in an exam, you would have a copy of the grid on which to draw the Voronoi diagram.)

A new school is to be built on the island which needs to be an equal distance from schools $A$, $B$ and $C$.

**e.** Write down the coordinates of the point where the new school should be built.

**f.** Give the straight line distance of the new school from each of the other schools.

**2.** Abdallah owns a plot of land near the River Nile, in the form of a quadrilateral $ABCD$. The lengths of the sides are $AB = 40$ m, $BC = 115$ m, $CD = 60$ m, $AD = 84$ m, and $B\hat{A}D = 90°$. This information is shown on the diagram.




**a.** Show that $BD = 93$ m, correct to the nearest metre.

**b.** Calculate the size of angle $B\hat{C}D$.

**c.** Find the area of $ABCD$.

The formula that the ancient Egyptians used to estimate the area of a quadrilateral $ABCD$ is:

$$\text{Area} = \frac{(AB + CD)(AD + BC)}{4}$$

Abdallah uses this formula to estimate the area of his plot of land.

**d. i.** Calculate Abdallah's estimate for the area.

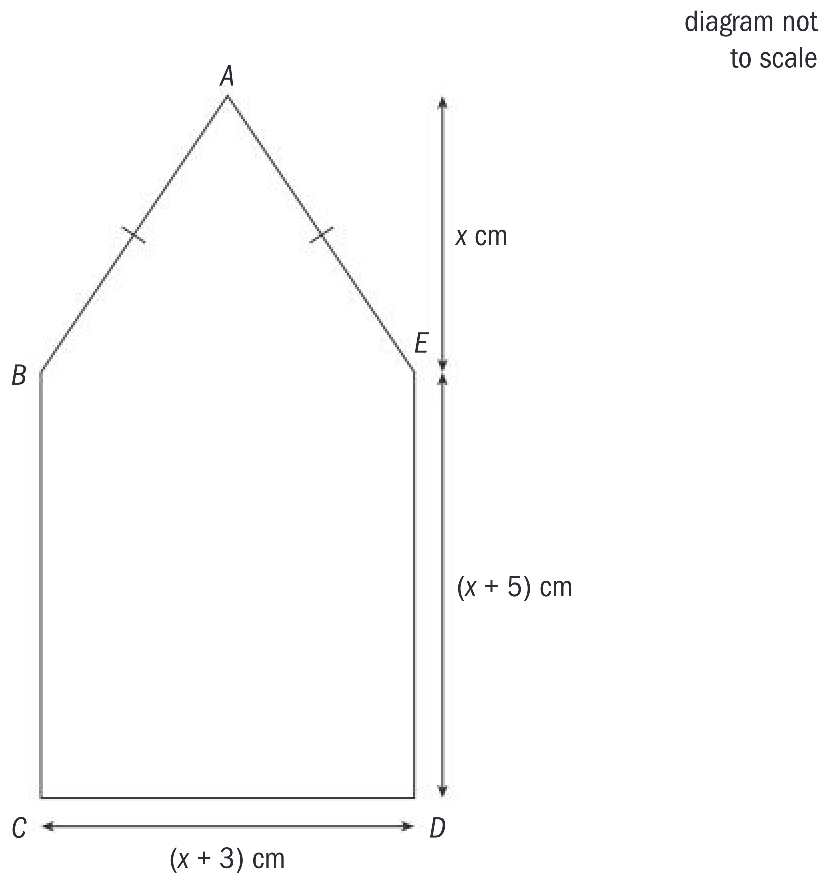
**ii.** Find the percentage error in Abdallah's estimate.

**3.** The base of an electric iron can be modelled as a pentagon $ABCDE$, where:

$BCDE$ is a rectangle with sides of length $(x + 3)$ cm and $(x + 5)$ cm

$ABE$ is an isosceles triangle with $AB = AE$, and a height of $x$ cm

The area of $ABCDE$ is 222 cm$^2$.




**a. i.** Write down an **equation** for the area of *ABCDE*, using the information given.

**ii.** Show that the equation in part **i** simplifies to $3x^2 + 19x - 414 = 0$

**b.** Find the length of *CD*.

**c.** Show that $B\hat{A}E = 67.4°$, correct to 1 decimal place.

Insulation tape is wrapped around the perimeter of the base of the iron *ABCDE*.

**d.** Find the length of the perimeter of *ABCDE*.

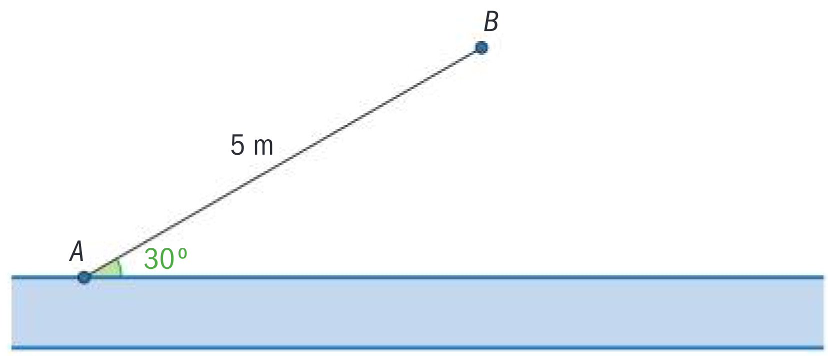
*F* is the point on *AB* such that $BF = 8$ cm. A heating element in the iron runs in a straight line from *C* to *F*.

**e.** Calculate the length of *CF*.

# HL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

**1.** A wall 5 m long makes an angle of 30° with a straight section of river, as shown in the diagram. The owner of the field on the one side of the river wants to make a triangular pen with the wall *AB* as one of the sides and the river as a second side. For the third side, he will use a metal pole 3.5 m long which he does not wish to cut. He will attach one end of the pole to the end of the wall at point *B*.




**a.** Show that there are two possible positions for the other end of the pole.

**b.** Find the area of the smaller of the two pens.

**2.** The transformation matrix $\mathbf{T} = \begin{pmatrix} 2 & 1 \\ a & -1 \end{pmatrix}$ maps the point (4, 3) onto (*b*, 3). Find:

**a.** the value of *a* and the value of *b*

**b.** the point that is mapped onto (0, –7)

**3.** An aircraft is flying such that, at *t* hours after 11 am, its position relative to its destination airport is given by the equation:

$$\boldsymbol{r} = \begin{pmatrix} 105 \\ 226 \\ 12 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The *x* and *y* directions are east and north respectively, and the *z* direction gives the vertical height of the aircraft.

If the aircraft's velocity remains constant, it will land at the airport at 11:30 am.

**a.** Find the values of *a*, *b* and *c*.

**b.** Find the speed of the aircraft.

**4.** Three points in three-dimensional space have coordinates

*A*(1, 0, 5), *B*(2, 4, 3) and *C*(1, –2, 4)

**a.** Find the vector

**i.** $\overrightarrow{AB}$ **ii.** $\overrightarrow{AC}$

**b.** Hence or otherwise, find the area of triangle *ABC*.

**5.** A particle has acceleration $\ddot{x}$ given by $\ddot{x} = \begin{pmatrix} 6t \\ 2 \end{pmatrix}$

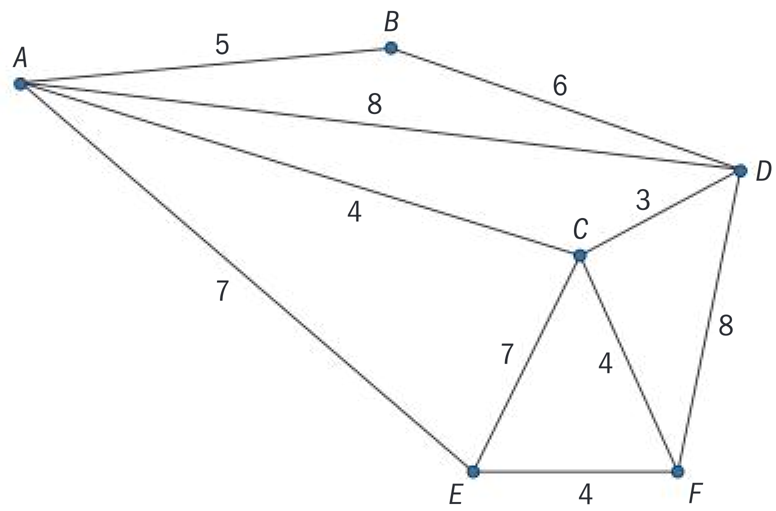
Given at $t = 0$ the particle is at (0, 0) and has velocity $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, find:

**a.** an expression for its velocity at time *t*

**b.** an expression for its displacement from (0, 0) at time *t*

**c.** its distance from (0, 0) at $t = 2$

**6.** Sameer is trying to design a road system to connect six towns, $A$, $B$, $C$, $D$, $E$ and $F$. The possible roads and the costs of building them are shown in the graph. Each vertex represents a town. Each edge represents a road and the weight of each edge is the cost of building that road. Sameer needs to design the lowest cost road system that will connect the six towns.




**a.** Name an algorithm that will allow Sameer to find the lowest cost road system.

**b.** Find the lowest cost road system and state the cost of building it. Clearly show the steps of the algorithm.

## GROUP 2

**7.** In triangle $ABC$, $AB = 5$ cm, $BC = 4$ cm and $B\hat{A}C = \frac{\pi}{7}$

**a.** Use the cosine rule to find the two possible values for $AC$.

**b.** Hence, find the area of the larger of the two possible triangles.

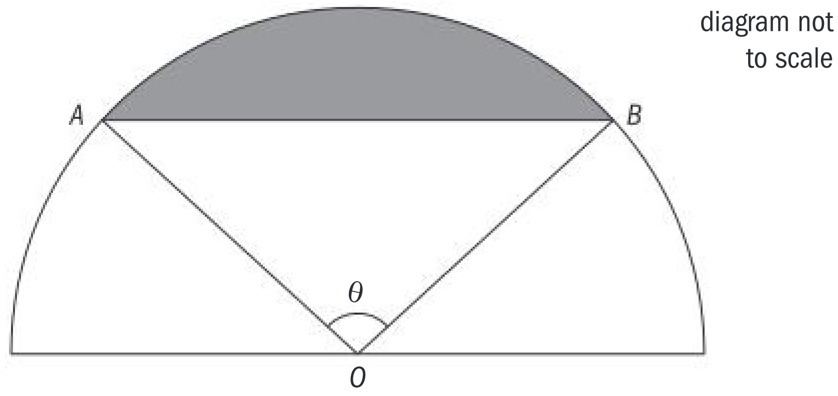
**8.** Triangle $T$ has coordinates (2, 3), (2, 5) and (6, 4).

**a.** Find the area of $T$.

A transformation **M** is represented by the matrix $\begin{pmatrix} 2\sin\theta & -2\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

**b.** Show that the area of the image of $T$ under this transformation is independent of $\theta$.

**9.** The diagram shows a semi-circle of diameter 20 cm, centred at $O$, with two points $A$ and $B$ on the circumference such that $A\hat{O}B = \theta$, where $\theta$ is measured in radians.

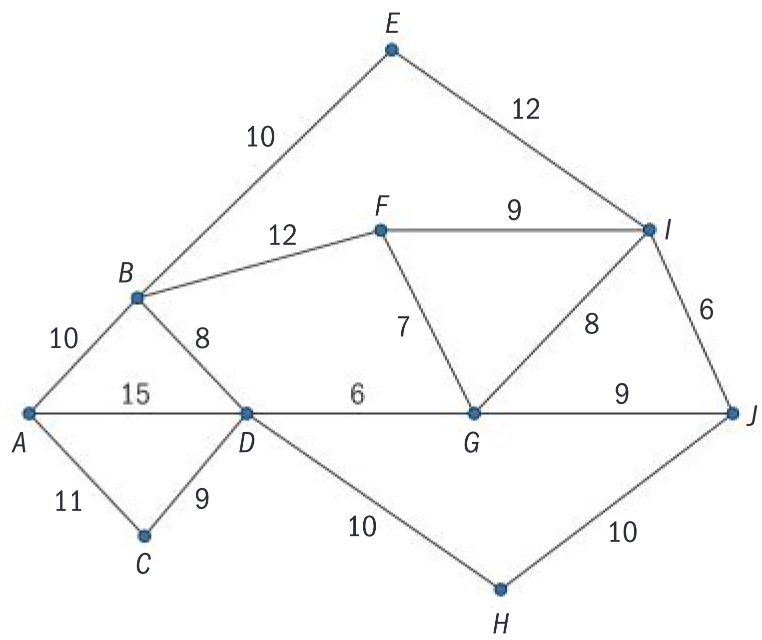



**a.** Show that the shaded area can be expressed as $50\theta - 50\sin\theta$

**b.** Find the value of $\theta$ for which the shaded area is equal to half that of the unshaded area, giving your answer correct to four significant figures.

**10.** Let $\boldsymbol{u} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = m\boldsymbol{j} + n\boldsymbol{k}$. Given that $\boldsymbol{v}$ is a unit vector perpendicular to $\boldsymbol{u}$, find the possible values for $m$ and $n$.

**11.** The graph shows the costs in USD to travel by bus between 10 towns.




The total cost for travelling each of the routes exactly once is 152 USD.

**a.** Explain why the graph has no Euler circuit.

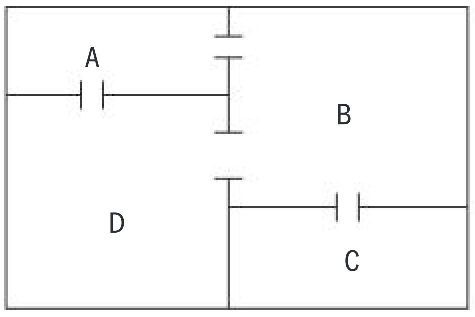
A tourist would like to travel along all the bus routes for the least possible cost, and to start and finish in the same town.

**b.** Use the Chinese postman algorithm to find the least possible cost and state the bus routes which need to be repeated.

The tourist realizes that she can reduce her cost and the number of repeated routes if she starts and finishes at different towns.

**c.** State which towns she should choose to start and finish at, and how much money she would save on bus travel.

12. A security guard patrols four rooms in a museum. The rooms and all the connecting doors are shown in the floor plan below.




While patrolling, the security guard decides he will leave each room by a randomly selected door.

a. Show the floor plan as a graph with the vertices as the rooms and the edges as the connecting doors.

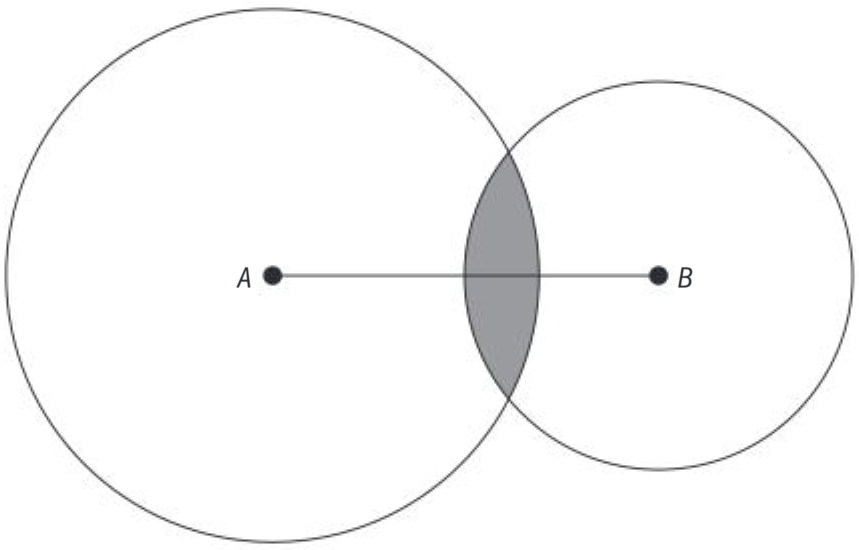
b. Find the transition matrix for the graph.

c. Find the steady state vector for this matrix.

d. Use your answer to part **c** to say which room the security guard spends most time in and which room he spends least time in. State an assumption that you are making.

## GROUP 3

13. Boat $A$ is situated 10 km away from boat $B$, and each boat has a marine radio transmitter on board. The range of the transmitter on boat $A$ is 7 km, and on boat $B$ it is 5 km. The region in which both transmitters can be detected is represented by the shaded region in the diagram. Find the area of this region.




14. The matrix **S** represents an enlargement scale factor 1.1, centred at (0, 0), and the matrix **R** represents a reflection in the line $y = x$.

a. Find:

i. **S**

ii. **R**

Let **T** = **SR**

b. Find an expression for the matrix $\mathbf{T}^n$, when:

i. $n$ is odd

ii. $n$ is even

Let $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{T}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let the distance of $(x_n, y_n)$ from the origin be $d_n$

$N$ is the smallest value of $n$ such that $d_n > 20$

c. Find $(x_N, y_N)$

15. At 13:00, a boat is sailing directly to a port with a velocity of $5.2\boldsymbol{i} + 3.9\boldsymbol{j}$ kmh$^{-1}$ where $\boldsymbol{i}$ is due east and $\boldsymbol{j}$ is due north.

a. Find the speed of the boat.

b. Find the bearing on which the boat is sailing.

The boat is 12 km from the port. Assume that the velocity of the boat remains constant.

c. Find the time, to the nearest minute, when the boat reaches the port.

d. Find an expression for the displacement of the boat from the port in terms of $t$, where $t$ is the number of hours after 13:00

16. Two students are standing 20 m apart and are throwing beanbags towards each other.

Student $A$ throws his beanbag towards student $B$ from a height of 1.5 m with an initial velocity of 5 ms$^{-1}$ and at an angle of 30°

After release, the acceleration of the beanbag is $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ ms$^{-2}$

a. Find the initial velocity of the beanbag as a column vector.

b. Find the displacement from $A$ of the beanbag at time $t$, where $t$ is the number of seconds after it is released.

Student $B$ throws her beanbag towards student $A$ at $t = 0$. The initial speed of the beanbag is 6 ms$^{-1}$ and it is released from a height of 1 m.

c. Given that the beanbags collide, find the angle of release of student $B$'s beanbag.

**17.** Using suitable units, the displacement of a planet from a star centred at (0, 0) can modelled by the equation

$$\boldsymbol{r} = \begin{pmatrix} 2\sin\left(\frac{\pi t}{2}\right) \\ 3\cos\left(\frac{\pi t}{2}\right) \end{pmatrix}$$

**a.** Sketch the position of the planet at times:

**i.** $t = 0$

**ii.** $t = 1$

**iii.** $t = 2$

**iv.** $t = 3$

**b.** Find an expression for the velocity, $v$, of the planet at time $t$.

**c.** Hence, show that the speed of the planet at time $t$ is $|\boldsymbol{v}| = \frac{\pi}{2}\sqrt{4 + 5\sin^2\left(\frac{\pi t}{2}\right)}$

**d.** Hence, write down the first two positive values of $t$ at which the speed is maximum.

**e.** How does your answer to part **d** relate to the distance of the planet from the star?

**18.** A graph $G$ has adjacency matrix **M** given by:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

**a.** Draw the graph $G$.

**b. i.** Find the number of walks of length 6 starting and ending at $A$.

**ii.** Write down how many of these walks will pass through **all** the vertices.

**c.** List all the walks of length 3 starting at $A$ and ending at $D$.

**d.** Find the probability that a random walk of length 3 beginning at $A$ will finish at $D$.

## PAPER 2

**1.** *Note: distances are in metres and time is in seconds in this question.*

Two particles $P_1$ and $P_2$ both leave from a point $A$ along two different straight lines.

$t$ seconds after leaving $A$, the position of $P_1$ is given by $\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

**a.** Write down the coordinates of $A$.

Five seconds after leaving $A$, $P_1$ is at point $B$.

**b.** Find:

**i.** $\overrightarrow{AB}$

**ii.** $|\overrightarrow{AB}|$

$P_2$ leaves $A$ three seconds after $P_1$. Two seconds after it leaves $A$, $P_2$ is at the point $C$, where $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}$

**c.** Write down the velocity vector for $P_2$

**d.** Hence write down an equation for the displacement of $P_2$ in terms of $t$, for $t \geq 3$

**e.** Find $\cos B\hat{A}C$

**f.** Hence or otherwise, find the distance between $P_1$ and $P_2$ when $t = 5$

**2. a.** Use Prim's algorithm, beginning at vertex $A$, to find the length of the minimum spanning tree for the graph represented by the weighted adjacency table below. Write down the order in which the edges are selected.

| | A | B | C | D |
|---|---|---|---|---|
| A | | 12 | 15 | 9 |
| B | 12 | | 8 | 11 |
| C | 15 | 8 | | 18 |
| D | 9 | 11 | 18 | |

The weighted adjacency table shown above is a table of least distances, in kilometres, for the journeys between four hospitals $A$, $B$, $C$ and $D$.

Rainer needs to deliver medicine to these four hospitals and uses the table to help him decide his route.

On one occasion, another hospital *E* is added to the list.

Rainer knows that he can reach *E* only by first going to either *A* or *B* and the distance from *A* to *E* is 5 km and from *B* to *E* is 8 km.

**b.** Copy and complete the table of least distances shown below so it includes hospital *E*.

| | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| *A* | | 12 | 15 | 9 | 5 |
| *B* | 12 | | 8 | 11 | 8 |
| *C* | 15 | 8 | | 18 | |
| *D* | 9 | 11 | 18 | | |
| *E* | 5 | 8 | | | |

Rainer needs to begin at hospital *A*.

**c.** Beginning at *A*, use the nearest neighbour algorithm to find an upper bound for Rainer's journey.

**d.** By first deleting hospital *E*, use the deleted vertex algorithm and your answer to part **a** to find a lower bound for Rainer's journey.

**e.** Find a different lower bound by deleting vertex *B*.

**f.** Hence, write down the smallest interval containing the least length of the route Rainer needs to take.

## PAPER 3

**1.** George is working on an animation that requires him to rotate points about a centre other than (0, 0). He has been told that this is possible using affine transformations, so he is investigating how these can be constructed.

**a.** Describe fully the transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

An affine transformation, **T**, is defined as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ for } n \geq 1$$

Let $(x_0, y_0)$ be (4, 0)

**b.** Find:

**i.** $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

**ii.** $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

**iii.** $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

**c.** Plot these points on a set of coordinate axes.

**d.** From your diagram, show that the centre of rotation is (–0.5, 1.5)

**e.** Verify that this point is invariant under the affine transformation.

George decides he wants to rotate his points 180° about the point (–4, 2) and wishes to find the affine transformation, **A**, that will do this.

**f.** Write down the matrix representing a rotation of 180° about (0, 0)

**g.** By considering the invariant point under **A**, or otherwise, find **A** in the form

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{M}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \boldsymbol{b}$$

In order to do his calculations more quickly, George needs an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of $x_0$ and $y_0$

**h.** Verify that for **A**, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

**i.** Explain geometrically why $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, whenever $n$ is even.

**j.** Find an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of $x_0$ and $y_0$, when $n$ is odd.

**k.** Find an expression in terms of $\theta$, $a$ and $b$, for the affine transformation that rotates a point $\theta$ anticlockwise about the point $(a, b)$.

# 4 STATISTICS AND PROBABILITY

## 4.1 PROBABILITY

### You should know:

- ✔ the meaning of the terms trial, outcome, equally likely outcomes, relative frequency, sample space ($U$) and event
- ✔ that probability of the sample space is exactly 1
- ✔ the meaning of complementary, mutually exclusive and independent events
- ✔ that probabilities enable us to make predictions, for example the expected number of occurrences
- ✔ that compound events can be represented by Venn, sample space or tree diagrams
- ✔ that sampling with or without replacement affects the probability.

### You should be able to:

- ✔ find the probability of an event
- ✔ apply the formula for complementary events to find probabilities
- ✔ construct Venn, sample space or tree diagrams, or a table of outcomes
- ✔ choose an appropriate diagram with which to represent the sample space
- ✔ recognize when the formula for a conditional probability event can be applied
- ✔ recognize with or without replacement problems by understanding the context of a problem.

In mathematics – as well as throughout the natural and the human sciences – the following language is used when representing and quantifying **probabilities**:

> Experiment: a process by which you obtain an observation.
>
> Trials: repeating an experiment a number of times.
>
> Outcome: what may occur in a trial.
>
> Event: an outcome or set of outcomes.
>
> Sample space: the set of all possible outcomes of an experiment.

Probability enters our daily lives in many ways. Imagine your team qualifies for the final of a competition. As you approach the stadium to watch the final, you wonder how likely they are to win; friends, relatives, sport journalists are all debating who will win. They all have their own opinions based on prior knowledge, emotions and how the teams have been playing recently, forming their own **subjective probabilities**. On social media, you read that in the 11 times the teams have played each other, your team has won 8 times. Before you find your seat in the stadium to watch the game, you buy a lottery ticket from a charity fund-raising stall. You read on the ticket that of the 20 000 tickets sold, 550 will win a prize.

Probability formulae can be applied to the information in this imaginary scenario as follows:

**Relative frequency** (also referred to as experimental probability) is found with the formula:

Relative frequency of $A = \frac{\textit{Frequency of occurence of event A in n trials}}{n}$

Hence the relative frequency of the event "your team wins" after 11 trials is $\frac{8}{11} \approx 0.727$.

Having found this relative frequency, you may reason that today your team has a good chance to win. But if some of the 11 games were held many years ago, critical reflection should apply to this evidence because the trials should all take place under identical conditions.

Probability (also referred to as **theoretical probability**) is found with the formula:

Probability of $A = P(A) = \frac{n(A)}{n(U)}$, where $n(A)$ is the number of outcomes that make $A$ happen and $n(U)$ is the number of outcomes in the sample space.

Hence the probability of the event "you win a lottery prize" is:

$P(\textit{you win a lottery prize}) = \frac{n(A)}{n(U)} = \frac{550}{20000} = 0.0275$

This probability is just under three in a hundred and, although you are unlikely to win a prize, you do have exactly the same chance of winning as anyone else buying a ticket, since the 20 000 tickets are all **equally likely outcomes** in the lottery.

Probabilities are represented on a scale from 0 (impossible) to 1 (certain), enabling you to compare the likelihoods of events.

Relative frequency serves as an approximation to theoretical probability. For example, the theoretical probability of choosing an even number at random from the set {1, 2, 3, 10} is 0.5. However, if you carry out this experiment many times, you may find the relative frequency is not exactly 0.5, due to the randomness of the experiment. Increasing the number of trials results in a more accurate estimate.

These photos show the different sides of a biased die. Notice how each side has a different shape.

## Example 4.1.1

Erin rolls a biased die in an experiment of 1115 trials.

(a) Complete Erin's table.

(b) Erin asks two classmates to verify her results by repeating the experiment: 3 was the outcome in 211 of 1200 trials carried out by Maxine whereas Louis reported a relative frequency of 18.1% in 1000 trials. List the three estimates for the probability of rolling a 3 in ascending order of likelihood.

(c) Explain how Erin could best estimate the probability of rolling a 3, and comment on the limitations of this estimate.

| Outcome | Frequency | Relative frequency (exact) | Relative frequency to 3 sf |
|---|---|---|---|
| 1 | 223 | | |
| 2 | 236 | $\frac{236}{1115}$ | 0.212 |
| 3 | 190 | $\frac{190}{1115}$ | 0.170 |
| 4 | 163 | | 0.146 |
| 5 | 140 | $\frac{140}{1115}$ | |
| 6 | 163 | | |
| Totals: | 1115 trials | | |

Solution

(a)

| Outcome | Frequency | Relative frequency (exact) | Relative frequency to 3 sf |
|---|---|---|---|
| 1 | 223 | $\frac{223}{1115}$ | 0.200 |
| 2 | 236 | $\frac{236}{1115}$ | 0.212 |
| 3 | 190 | $\frac{190}{1115}$ | 0.170 |
| 4 | 163 | $\frac{163}{1115}$ | 0.146 |
| 5 | 140 | $\frac{140}{1115}$ | 0.126 |
| 6 | 163 | $\frac{163}{1115}$ | 0.146 |
| Totals: | 1115 trials | 1 | 1.00 |

(b) The estimates of the probability of rolling a 3 in ascending order are:

Erin: 0.170

Maxine: $\frac{211}{1200} \approx 0.176$

Louis: 18.1% = 0.181

(c) Erin can put all the trials together to form the relative frequency:

$$\frac{190 + 211 + 181}{1115 + 1200 + 100} = \frac{194}{1105} \approx 0.176$$

This would likely be the best estimate of the probability of rolling a 3 because more trials have been carried out.

Collaborating on probability experiments or creating probability simulations with technology is a good way to increase your knowledge and understanding about probability, and can lead you to find interesting topics to explore in your internal assessment.

**Assessment tip**

Unless otherwise stated in the question, answers should be given exactly or to three significant figures. Decimal representations of a probability enable you to easily judge its size, however a fraction is exact and should be used in probability calculations.

An event $A$ can be represented by written statements or algebraic symbols. The **complement** of $A$ is the event that $A$ **does not** happen and is denoted $A'$. The table below gives some examples. It is important to be aware that different written statements can describe the same event. For example, if the outcomes are discrete, "$X$ is less than 4" is the same event as "$X$ is 3 or less"

| Written statement of event $A$ | Algebraic equivalent of $A$ | Written statement of $A'$ | Algebraic equivalent of $A'$ |
|---|---|---|---|
| $X$ is prime | $X \in \{2, 3, 5, 7, 11, \ldots\}$ | $X$ is not prime | $X \notin \{2, 3, 5, 7, 11, \ldots\}$ |
| $X$ is odd | $X \in \{1, 3, 5, 7, 9, \ldots\}$ | $X$ is even | $X \notin \{1, 3, 5, 7, 9, \ldots\}$ or $X \in \{2, 4, 6, 8, \ldots\}$ |
| $X$ is less than 4 | $X < 4$ | $X$ is at least 4 | $X \geq 4$ |
| $X$ is no more than 4 | $X \leq 4$ | $X$ is more than 4 | $X > 4$ |

Since $A$ and $A'$ cover all the possibilities in the sample space, it is a rule that $P(A) + P(A') = 1$

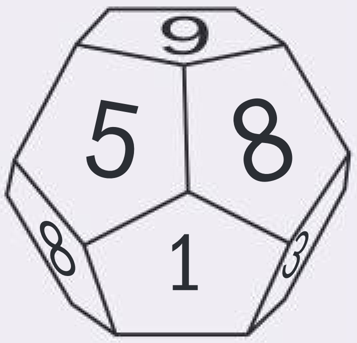
This rule can be applied to solve problems efficiently.

**Assessment tip**

In the five-minute reading time given before you start writing in the exam, take time to read the wording of events carefully to make sure you are clear exactly what the event is asking for.

## Example 4.1.2

Callum numbers the faces on a dodecahedral die with the numbers 1, 2, 3, 5, 8, 13, 4, 8, 16, 4, 5, 6. He rolls the die and records his number obtained as $C$.




Find the probabilities:

(a) $P(C \text{ is prime})$

(b) $P(2 < C \leq 10)$

(c) $P(C \text{ is at least } 4)$

(d) $P(C \text{ is less than } 4)$

(e) $P(C \text{ is not prime})$

(f) $P(C \text{ is } 4 \text{ or more})$

**Solution**

(a) $\text{P}(C \text{ is prime}) = \frac{5}{12}$ — 2, 3, 5 and 13 are prime and 5 appears twice.

(b) $\text{P}(2 < C \leq 10) = \frac{8}{12} = \frac{2}{3}$ — First, carefully list numbers on the die in order: 1,2,3,4,4,5,5,6,8,8,13,16. Then write down the answer.

(c) $\text{P}(C \text{ is at least } 4) = \frac{9}{12} = \frac{3}{4}$ — Your list helps you see there are 9 numbers on the die that are at least 4.

(d) $\text{P}(C \text{ is not prime})$ is $1 - \frac{5}{12} = \frac{7}{12}$ — This is the complement of the event in part (b)

(e) $\text{P}(C \text{ is } 4 \text{ or more}) = \frac{3}{4}$ — P ($C$ is 4 or more) is the same event as P($C$ is at least 4)

### Assessment tip

The expected number of occurrences is similar to the mean value of a data set. Hence it does not need to be rounded to the nearest or lowest integer value. This contrasts with using functions to make a prediction: if $N(5) = 14.7$ predicts the number of elephants in a game reserve 5 years after opening, you would round this up to 15 elephants.

Use of the formula for the expected number of occurrences of an event is one of the many ways that probability allows us to make predictions about random events.

Expected number of occurrences of $A$ in $n$ trials $= n \times P(A)$

## Example 4.1.3

Hannah and Kotryna are exploring dice games. Hannah has a fair octahedral die with sides numbered 1, 2, 3, 4, 5, 6, 7 and 8. She rolls it 95 times and counts the occurrences of the event “roll a factor of 360”.

Kotryna rolls a biased six-sided die 115 times. The possible outcomes and probabilities for Kotryna’s die are given in the following table:

| **Outcome** | 3 | 6 | 8 | 11 | 15 | 21 |
|---|---|---|---|---|---|---|
| **Probability** | 0.13 | 0.15 | 0.12 | 0.15 | 0.25 | 0.2 |

Kotryna counts the occurrences of the event “roll a multiple of 3”

Predict which of Hannah or Kotryna should expect the greatest number of occurrences, justifying your answer.

Solution

The probability of Hannah's event is $\frac{7}{8}$ since all of the numbers on her die are factors of 360 except for 7. Hence the expected number of occurrences for Hannah is $95 \times \frac{7}{8} = 83.125$

Find the probabilities of each event and multiply by the number of trials in each game.

The probability of Kotryna's event is $0.13 + 0.15 + 0.25 + 0.2 = 0.73$

Hence the expected number of occurrences for Kotryna is $115 \times 0.73 = 83.95$

Kotryna should expect the greatest number of occurrences since $83.95 > 83.125$

The expected numbers of occurrences are very close. However, since they predict the likely average number of occurrences, the conclusion is valid.

The formula for expected number of occurrences does not appear in the formula book. However, an equivalent form appears in the formula book in section 4.8; the mean of the binomial distribution. Therefore, you can make a link between these two parts of the course and increase your knowledge and understanding.

## Probability formulae and diagrams

Events can be combined, so that probabilities can be applied in wider contexts. Venn diagrams, sample space diagrams, tree diagrams and tables of outcomes serve as useful representations of the sample space. You use them to solve problems involving combined events.

Once a context has been understood, you choose an appropriate diagram, as shown in the following examples.

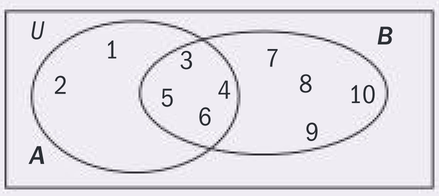
**Context overview for Examples 4.1.4, 4.1.5 and 4.1.6:** In a CAS project to raise money for charity, three dice games $X$, $Y$ and $Z$ are designed. The games each involve rolling a fair cubical die "$A$" numbered 1, 2, 3, 4, 5, 6 and a fair octahedral die "$B$" numbered 3, 4, 5, 6, 7, 8, 9, 10

### Example 4.1.4

In game $X$, the player randomly chooses a number between 1 and 10. Then dice $A$ and $B$ are shown to the player. The player wins if the number chosen appears on exactly one of the dice.

Find the probability of winning game $X$.

Solution




The events relate to membership of a set.

A Venn diagram is an appropriate representation.

$U$ is the sample space.

The probability that the number chosen appears **only** on die $A$ is $\frac{2}{10}$. The probability that the number chosen appears **only** on die $B$ is $\frac{4}{10}$.

Hence the probability of winning game $X$ is $\frac{6}{10}$.

From the Venn diagram, you can solve the problem by writing down the probabilities required.

### Example 4.1.5

In game $Y$, dice $A$ and $B$ are rolled and $T$ is defined as the total of the numbers on the two dice. The player wins if $T$ is a prime number.

Find the probability of winning game $Y$.

**Solution**

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |

Two random quantities are combined by adding them together to give the values of $T$, which represent the sample space.

A sample space diagram is an appropriate representation.

There are 16 outcomes for which $T$ is a prime number in a sample space of 48 equally likely outcomes.

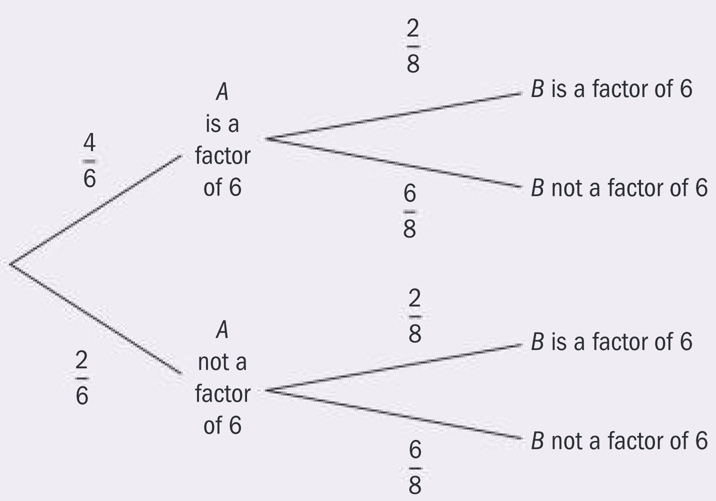
Hence the probability of winning game $Y$ is $\frac{16}{48} = \frac{1}{3}$

Shading or circling the prime numbers helps you count the outcomes accurately.

### Example 4.1.6

In game $Z$, die $A$ is rolled and then die $B$ is rolled. The player wins if exactly one of the two numbers rolled is a factor of 6. Find the probability of winning game $Z$.

**Solution**




One event occurs after another. A tree diagram is an appropriate representation.

Draw the events in order from left to right.

**Assessment tip**

Choosing to construct and apply a probability diagram is often the most efficient way to solve a problem or find a probability. You are encouraged to consider applying a diagram to find a probability before trying to apply a formula.

The probability of winning game $Z$ is

$$\frac{4}{6}\times\frac{6}{8}+\frac{2}{6}\times\frac{2}{8}=\frac{7}{12}$$

Game $Z$ is won if the number rolled on die $A$ is a factor of 6 and the number rolled on die $B$ is not, with probability $\frac{4}{6}\times\frac{6}{8}$ or if the number rolled on die $A$ is not a factor of 6 and the number rolled on die $B$ is a factor of 6, with probability $\frac{2}{6}\times\frac{2}{8}$. The final answer is found by adding these probabilities.

Formulae give another way to represent combined events and find probabilities. You must understand the symbols used in the formulae in order to apply them correctly.

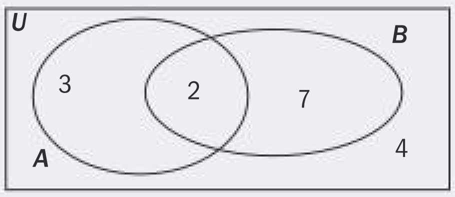
The event $A \cup B$ means $A$ or $B$ occurs. This use of "or" includes the possibility that **both** $A$ and $B$ occur. You say "$A$ or $B$" or "$A$ union $B$" for $A \cup B$. You say "$A$ and $B$" or "$A$ intersection $B$" for $A \cap B$.

The event $A \mid B$ means $A$ occurs given that $B$ has occurred. You say, "$A$ given $B$" or "given that $B$ has happened, $A$ happens".

These symbols are applied in Example 4.1.7.

### Example 4.1.7

In Jan's class of 16 students, the numbers of students who study Art or Biology, or both, are represented in the Venn diagram.




A student from Jan's class is chosen at random. $A$ is the event "the student studies Art" and $B$ is the event "the student studies Biology".

(a) Find the following probabilities:

(i) $P(A)$ (ii) $P(B)$ (iii) $P(A \cap B)$

(iv) $P(A \cup B)$ (v) $P(A \mid B)$

(b) Hence verify that:

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(ii) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

**Solution**

(a)

(i) $P(A)=\frac{3+2}{3+2+7+4}=\frac{5}{16}$ There is a total of 5 Art students in the class of 16.

(ii) $P(B)=\frac{2+7}{3+2+7+4}=\frac{9}{16}$

(iii) $P(A \cap B)=\frac{2}{16}$ 2 students study both Art and Biology.

(iv) $P(A \cup B)=\frac{3+2+7}{3+2+7+4}=\frac{12}{16}$ 12 study Art or Biology. This includes the 2 students who study both.

(v) $P(A \mid B)=\frac{2}{2+7}=\frac{2}{9}$ Given that a Biology student is chosen, the sample space is 9, of whom 2 study Art.

(b) (i) $P(A) + P(B) - P(A \cap B) = \frac{5}{16} + \frac{9}{16} - \frac{2}{16} = \frac{12}{16} = P(A \cup B)$

The formula subtracts the students who study both Art and Biology so they are not counted twice in the total.

(ii) $\frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{16}}{\frac{9}{16}} = \frac{2}{16} \times \frac{16}{9} = \frac{2}{9}$

16 cancels out. The sample space is 9, of whom 2 study Art.

The following formulae are true in general:

> The formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ finds the probability that $A$ or $B$ will occur.

> The formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ finds the probability that $A$ occurs given than $B$ has occurred. Another way to write the formula is $P(A|B)P(B) = P(A \cap B)$. This is known as conditional probability.

How two events relate to each other can easily be confused. Two events $A$ and $B$ are **mutually exclusive** if they cannot both occur at the same time. This means that $P(A \cap B) = 0$. Two events are **independent** if the occurrence of each event does not affect in any way the occurrence of the other. This means that $P(A \mid B) = P(A)$, and therefore $P(A)P(B) = P(A \cap B)$.

Sometimes students think the terms **mutually exclusive** and **independent** mean the same thing. Example 4.1.8 illustrates the difference.

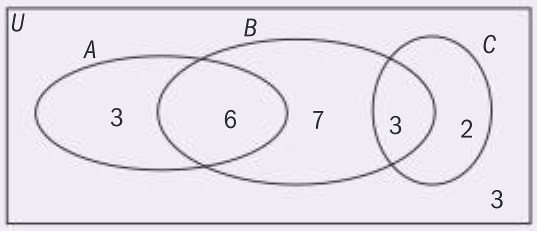
### Example 4.1.8

A survey of 24 students is made to find out how many students study Art, Biology or Chemistry, represented by $A$, $B$ and $C$ respectively. The results are presented in a Venn diagram showing the numbers of students in each set. For example, the number of students who study both Art and Biology is 6.

A student is selected at random from the class.

Determine which pairs of events are (i) mutually exclusive (ii) independent, justifying your answers.

(a) $A$ and $B$ (b) $B$ and $C$ (c) $A$ and $C$




**Solution**

(a) (i) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{6}{24}}{\frac{16}{24}} = \frac{3}{8}$

$P(A) = \frac{3+6}{24} = \frac{3}{8}$

Since $P(A|B) = P(A)$, $A$ and $B$ are independent.

This can also be found quickly without the formula by simplifying $\frac{6}{6+7+3}$ from set $B$ in the Venn diagram.

Equivalently, independence of $A$ and $B$ can be demonstrated by showing $P(A)P(B) = P(A \cap B)$.

(ii) Since $n(A \cap B) = 6 \neq 0$, $A$ and $B$ are not mutually exclusive.

(b) (i) $P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{3}{24}}{\frac{5}{24}} = \frac{3}{5}$

$P(B) = \frac{6+7+3}{24} = \frac{2}{3}$

Since $P(B \mid C) \neq P(B)$, $B$ and $C$ are not independent.

(ii) Since $P(B \cap C) = 3 \neq 0$, $A$ and $B$ are not mutually exclusive.

Write reasons clearly, showing how you replace probabilities in the formulae in order to demonstrate your knowledge and understanding.

(c) (i) $P(A|C) = \frac{0}{\frac{5}{24}} = 0$

$P(A) = \frac{3+6}{24} = \frac{3}{8}$

Since $P(A|C) \neq P(A)$, $A$ and $C$ are not independent.

(ii) Since $P(A \cap C) = 0$, $A$ and $C$ are mutually exclusive.

Reflect on what this shows: Since $A$ and $C$ are mutually exclusive, they cannot be independent because if one occurs, the other cannot.

Note that in a different experiment in which two students are chosen, the probability that both are Art students is $\left(\frac{9}{24}\right)^2 = \frac{9}{64}$ if the same student can be chosen twice (with replacement), or $\frac{9}{24} \times \frac{8}{23} = \frac{3}{23}$ if the same student cannot be chosen twice (without replacement).

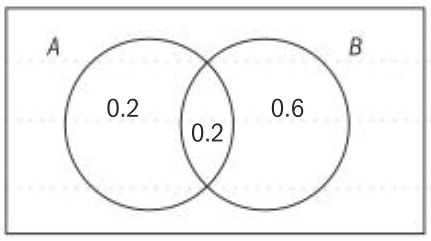
You must note whether a sampling context involves replacement or not before calculation of probabilities.

**SAMPLE STUDENT ANSWER**

Events $A$ and $B$ are independent with $P(A \cap B) = 0.2$ and $P(A' \cap B) = 0.6$

(a) Find $P(B)$

(b) Find $P(A \cup B)$

(a) 


$P(B) = 0.6$

(b) Applying $P(A \cap B) = P(A) \times P(B)$ gives $P(A) = \frac{1}{3}$

Applying $P(A \cup B) = P(A) + P(B)$ gives $P(A \cup B) = \frac{14}{15}$

▲ The student recognizes that a Venn diagram is appropriate.

▼ The student fails to find the total probability for $B$, which is 0.8

▲ The student has applied the correct formula for independent events, gaining a follow through mark.

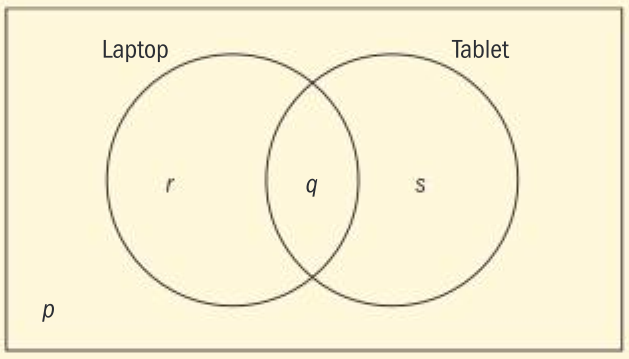
▼ The formula for mutually exclusive events was then applied, which is incorrect. The correct answer for (b) is 0.85

SAMPLE STUDENT ANSWER

In a class of 21 students, 12 own a laptop, 10 own a tablet and 3 own neither.

The following Venn diagram shows the events “own a laptop” and “own a tablet”.

The values $p$, $q$, $r$ and $s$ represent numbers of students.




(a) (i) Write down the value of $p$.

(ii) Find the value of $q$.

(iii) Write down the value of $r$ and of $s$.

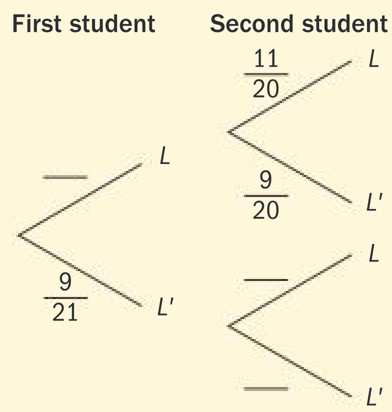
(b) A student is selected at random from the class.

(i) Write down the probability that this student owns a laptop.

(ii) Find the probability that this student owns a laptop or a tablet but not both.

(c) Two students are randomly selected from the class. Let $L$ be the event a “student owns a laptop”.

(i) Copy and complete the following tree diagram.




(ii) Write down the probability that the second student owns a laptop given that the first owns a laptop.

## Assessment tip

The command term “Write down” requires you to show little or no working, whereas “Find” requires you to show your working clearly in order to access full marks.

(a) (i) $p = 3$

(ii) $12 + 10 = 22$

$21 - 3 = 18$, so 4 must be counted twice.

Therefore $q = 4$

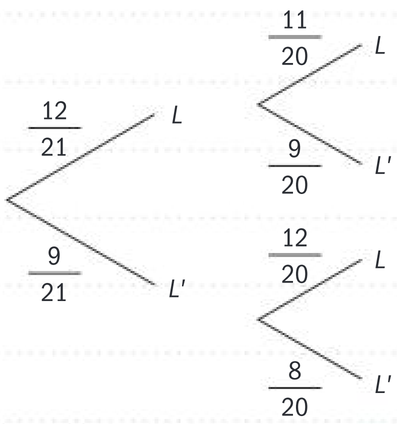
▲ The calculations are clear, precise and complete.

(iii) $r = 12 - 4 = 8$

$s = 10 - 4 = 6$

(b) (i) $\frac{12}{21} = \frac{4}{7} = 0.571$

(c) (i)




(ii) $P(\text{first student owning laptop}) = \frac{12}{21} = P(x)$

$P(2^{nd}\text{ student owning a laptop}) = P(y)$

$\therefore P(y|X) = \frac{P(y \cap x)}{P(x)}$

$= \frac{P(\text{first AND second owning})}{P(\text{Only first owning})}$

$= \frac{\frac{12}{21} \times \frac{11}{20}}{\frac{12}{21}}$

$\frac{\cancel{12} \times 11}{\cancel{21} \times 20} \times \frac{\cancel{21}}{\cancel{12}}$

$= \frac{11}{20}$

▲ The student correctly realizes that the context is "without replacement" and completes the tree diagram correctly.

▼ Although the student found the correct answer, considerable time has been used applying the formula when the answer could have been found immediately from the tree diagram. This is what should be realized from the command term "Write down".

# 4.2 PROBABILITY DISTRIBUTIONS

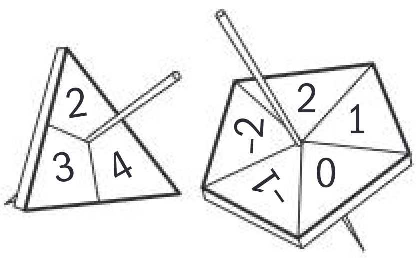
## You should know:

- ✔ the meaning of a discrete random variable
- ✔ probability distributions are generalizations of random processes
- ✔ probability distributions can be represented by a function or a table
- ✔ the sum of all probabilities of a probability distribution is 1 and each is at least zero
- ✔ the expected value of a random variable predicts its average value after many trials
- ✔ the definition of a fair game
- ✔ the context of a finite number of trials and a fixed probability of success can be modelled by the binomial distribution
- ✔ the normal distribution models many naturally occurring continuous data sets
- ✔ the properties of the normal distribution and its curve.

## You should be able to:

- ✔ interpret and write events with the notation of discrete random variables
- ✔ apply a representation of a discrete random variable to calculate the probability of an event
- ✔ apply probability distributions to problems including those involving fair games
- ✔ determine from the context if the binomial distribution is appropriate
- ✔ calculate binomial probabilities with technology
- ✔ calculate normal probabilities with technology
- ✔ interpret, write with, and understand the notation of the binomial distribution and the normal distribution
- ✔ represent normal distribution events using a sketch of the normal curve.

Just as patterns in numbers can be generalized, then expressed as sequences or functions, you can find patterns in probability events that can be generalized. These generalizations give you convenient ways to find probabilities efficiently.




For example, consider an experiment in which a fair three-sided spinner with sides numbered 2, 3 and 4 is spun, along with a fair five-sided spinner with sides numbered –2, –1, 0, 1 and 2.

The numbers are added to give a total $T$. Each outcome is shown in this sample space diagram.

| | **−2** | **−1** | **0** | **1** | **2** |
|---|---|---|---|---|---|
| **2** | 0 | 1 | 2 | 3 | 4 |
| **3** | 1 | 2 | 3 | 4 | 5 |
| **4** | 2 | 3 | 4 | 5 | 6 |

$T$ is a **discrete random variable** because it takes only discrete integer values and, as the spinners are fair, the value of $T$ varies randomly.

The probabilities of each value of $T$ can be represented in a table:

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $P(T=t)$ | $\frac{1}{15}$ | $\frac{2}{15}$ | $\frac{3}{15}$ | $\frac{3}{15}$ | $\frac{3}{15}$ | $\frac{2}{15}$ | $\frac{1}{15}$ |

It can be seen from the table that the probabilities all add to a total of 1. The table shows exactly how this total probability is distributed to each possible outcome.

The table can be generalized and represented as a function $f(t) = P(T = t)$, where

$$f(t) = \begin{cases} \dfrac{t+1}{15} & 0 \le t \le 2 \\ \dfrac{1}{5} & t = 3 \\ \dfrac{7-t}{15} & 4 \le t \le 6 \end{cases} \qquad t \in \mathbb{N}$$

**Note**

You would not be expected to derive this function $f(t)$ without extra information being given. It is inserted here to give you an example of how a probability distribution function can be found from a probability distribution like the one in the table on the previous page.

This **probability distribution function** assigns to each value of the discrete random variable $T$ the corresponding probability of its occurrence and is often referred to as a "pdf".

The **cumulative distribution function** assigns to each value of the discrete random variable $T$ the corresponding probability of $T$ being less than or equal to that value. It is often referred to as the "cdf".

### Example 4.2.1

Two branches of a café chain are investigating their customers' purchasing habits.

Alphacafé finds in a survey that the number of items purchased by a customer in one visit to the café can be modelled by a discrete random variable $A$ with the following distribution:

| $a$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P(A = a)$ | 0.63 | 0.18 | 0.09 | 0.07 | 0.03 |

Bytecafé models the number of items purchased in one visit with a discrete random variable $B$ with probability distribution function $P(B = b) = f(b)$, where $f(b) = \frac{7-b}{20}$, $1 \le b \le 5$, $b \in \mathbb{N}$

(a) Determine in which café that a randomly selected customer is more likely to make at least three purchases in one visit.

(b) Find $P(3 \le B < 5 \mid B > 1)$ and interpret the result.

**Solution**

(a) $P(A \ge 3) = 0.09 + 0.07 + 0.03$
$= 0.19$

Interpret "at least three" and write the event that expresses the probability required.

$P(B \ge 3) = f(3) + f(4) + f(5)$

$= \frac{4}{20} + \frac{3}{20} + \frac{2}{20} = \frac{9}{20} = 0.45$

Apply the table and then the function to find the probabilities.

A randomly selected customer is more likely to make at least three purchases in one visit at Bytecafé.

(b) $P(3 \le B < 5 \mid B > 1) = \frac{P(3 \le B < 5)}{P(B > 1)}$

Apply the formula for conditional probability.

$\frac{f(3) + f(4)}{1 - f(1)} = 0.5$

Given that a customer in Bytecafé buys more than one item, it is just as likely that 3 or 4 items are purchased than not.

Interpret the meaning of the event in context.

Each value of a random variable will occur with a frequency proportional to its probability.

For example, a prediction for the number of items purchased by 100 customers at Alphacafé is found by finding the expected number of occurrences of each event:

| $a$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Expected frequency of of $a$ items purchased | $0.63 \times 100 = 63$ | $0.18 \times 100 = 18$ | $0.09 \times 100 = 9$ | $0.07 \times 100 = 7$ | $0.03 \times 10 = 3$ |

Hence 63 customers are expected to buy only one item, 18 are expected to buy 2 items, and so on.

Finding the mean of this frequency table gives a mean of

$$\frac{63 \times 1 + 18 \times 2 + 9 \times 3 + 7 \times 4 + 3 \times 5}{63 + 18 + 9 + 7 + 3} = \frac{169}{100} = 1.69$$

This process gives the expected value of the discrete random variable $A$. It predicts the likely average number of items **one** customer will purchase. This process is generalized by the following formula:

The expected value of a discrete random variable $X$ is $\mathrm{E}(X) = \sum xP(X = x)$. This is a measure of the central tendency (the mean) of the outcomes of $X$ in a number of trials.

This formula when applied to the probability distribution table of Alphacafé confirms this answer and gives an efficient way to find the expected value of the random variable:

$1 \times 0.63 + 2 \times 0.18 + 3 \times 0.09 + 4 \times 0.07 + 5 \times 0.03 = 1.69$

**Assessment tip**

You can use your GDC to efficiently manage probability distributions by using the GDC memory and sigma notation as shown.

Here, the answer to Example 4.2.1 (b) is found, as well as the expected value of $B$.


The concept of the expected value of a random variable has applications in games of chance in which a player can make a gain or a loss. If the expected gain of a player is zero, the game is called fair.

**Example 4.2.2**

Some students design a game of chance for a CAS project in order to raise money for charity. Players pay $\$k$ to play. The students program a spreadsheet to generate a random number $Z$ and award a cash prize of $\$Z$ according to the probability distribution shown in the table below.

| $z$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| $P(Z = z)$ | 0.4 | 0.3 | 0.2 | $a$ | $b$ |

(a) Determine the values of $a$ and of $b$ if the students decide that the probabilities of winning \$8 and \$16 should be equal.

(b) Hence find the price the students should charge to play the game for the game to be fair.

The students estimate that they can persuade 350 people in their school community to play the game if they charge \$5 to play the game.

(c) Find the money that the students would expect to raise for charity by charging 350 people \$5 to play the game.

Solution

(a) $0.4 + 0.3 + 0.2 + a + b = 1$

$\Rightarrow 2a + 0.9 = 1 \Rightarrow a = 0.05$ and $b = 0.05$

The probabilities must add to 1.

(b) The expected winnings are found by

$1 \times 0.4 + 2 \times 0.3 + 4 \times 0.2 + 8 \times 0.05 + 16 \times 0.05$
$= \$3$

The ticket should cost \$3 for the game to be fair.

If the cost to play is the same as the expected winnings, then the expected gain is zero and the game is fair.

(c) The students would expect that they would raise $350 \times 5 - 350 \times 3 = \$700$ for charity.

## The binomial distribution

Different number patterns generalize to different sequences. Similarly, different random processes are modelled by different distributions. One example of a discrete probability distribution is the **binomial distribution**.

**Context of the binomial distribution:** In a binomial trial, the outcome is either success or failure and you count the number of successes. When an experiment involves a fixed number of trials, each of which has the same probability of success, and the trials are independent of each other, the binomial distribution enables you to find probabilities. The discrete random variable is the number of successes.

For example, you roll a cubical die 5 times and count the number of times a 4 is rolled. The number of trials is 5 and they are independent of each other. The random variable is the number of 4s rolled in 5 trials, and the probability of success is $\frac{1}{6}$ because this is the probability of rolling a 4. Hence, 5 and $\frac{1}{6}$ are the parameters of this binomial distribution.

You write "Let $F$ be the number of 4s rolled in 5 trials. Then $F \sim B\left(5, \frac{1}{6}\right)$"

The latter sentence is short for "$F$ is distributed binomially with parameters 5 and $\frac{1}{6}$."

**Assessment tip**

You are expected to be able to read a context and determine if the binomial distribution is appropriate. You could do this in the five-minute reading time before the examination begins.

**Assessment tip**

Define your random variable and write its distribution to show your knowledge and understanding. This is part of your method and you will be awarded method marks for correct writing.

### Example 4.2.3

For these three contexts, determine if any can be modelled by the binomial distribution, stating your reasons and giving a full description of any that do.

(a) Alex rolls three dice. The probability of rolling three identical numbers is $\frac{1}{36}$. Alex rolls the three dice until he first rolls three identical numbers then stops. The random variable $A$ is the number of rolls made.

(b) Barbara has a box of 11 identical batteries, five of which don't work. She needs working batteries for her torch. She picks a battery at random and tests it. She does not replace it in the box. She carries out this experiment 5 times. The random variable $B$ is the number of batteries that work.

(c) Cards are labelled $A$, $B$, $C$ or $D$ in a pack of 100 cards which is arranged in random order. A card is drawn randomly from the pack and Carolina tries to guess the letter on the card, without looking. After having recorded whether or not she was correct, the card is put back and the pack is randomly rearranged. The number of cards with each label is 25. The random variable $X$ is the number of cards guessed correctly and the experiment is repeated 7 times.

**Solution**

(a) Since the number of trials is not fixed, $A$ does not follow the binomial distribution.

Alex could roll three identical numbers on the first attempt or may fail to do so even after 100 attempts.

(b) The number of trials is fixed, but the probability of choosing a working battery is not constant since batteries are not replaced. $B$ does not follow the binomial distribution.

Recognizing that an experiment is without replacement is important information about the context.

(c) $X$ does follow the binomial distribution since the number of trials is fixed at 7 and because the trials are independent of each other due to the cards being replaced and re-randomized.

$X \sim B\left(7, \frac{1}{4}\right)$

This experiment has been carefully designed to ensure independent trials.

The probability distribution function of $X \sim B(n, p)$ is

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

This formula can be explored in the contexts of Pascal's triangle and counting methods, but is not needed for exams.

When $X \sim B(n, p)$, binomial probabilities are found with technology. The domain of the binomial probability distribution function is $0 \leq x \leq n, x \in \mathbb{N}$.

You are not required to know the formula for the binomial pdf in the examination. You should always find binomial probabilities with technology as shown in the next example.

### Example 4.2.4

Cards are labelled $A$, $B$, $C$ or $D$ in a pack of 100 cards which is arranged in random order. A card is drawn randomly from the pack and Carolina tries to guess the letter on the card, without looking. After having recorded whether or not she was correct, the card is put back and the pack is randomly rearranged. The number of cards with each label is 25. The random variable $X$ is the number of cards guessed correctly and the experiment is repeated 7 times.

(a) Find $P(X = 0)$

(b) Find $P(X \geq 6)$

(c) Find the probability that Carolina guesses fewer than five cards correctly.

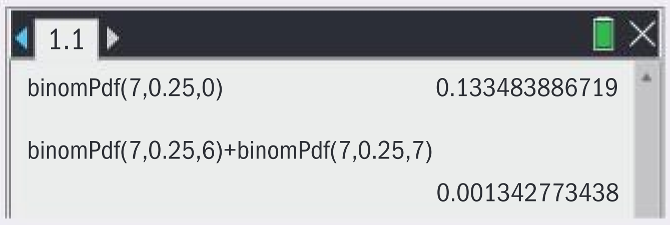
(d) Find the probability that Carolina guesses more than one card but no more than 4 cards correctly.

**Solution**

$X \sim \mathrm{B}\left(7, \frac{1}{4}\right)$ was established in Example 4.2.3

Use technology and write your answers correct to three significant figures.

(a) $P(X = 0) = 0.133$



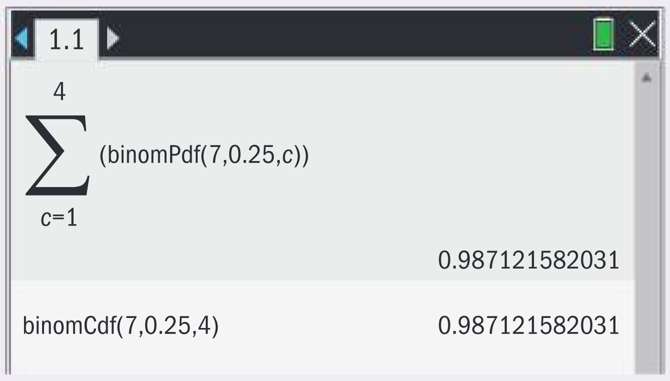

(b) $P(X \geq 6) = 0.001\,34$

$P(X \geq 6) = 1 - P(X \leq 5)$ (since $X$ is discrete) is an equivalent way to find the probability.

(c) The event is $P(X \leq 4)$

$P(X \leq 4) = 0.987$

$P(X < 5) = P(X \leq 4)$ since $X$ is discrete. The cumulative probability function on your GDC has the effect of adding up all the probabilities in this event.




(d) The event is

$P(2 \leq X \leq 4) = 0.542$

**Assessment tip**

Make sure you know how to calculate binomial probabilities with your GDC both with the pdf and the cdf. You first write the distribution, then the event before using your GDC to find the probability.

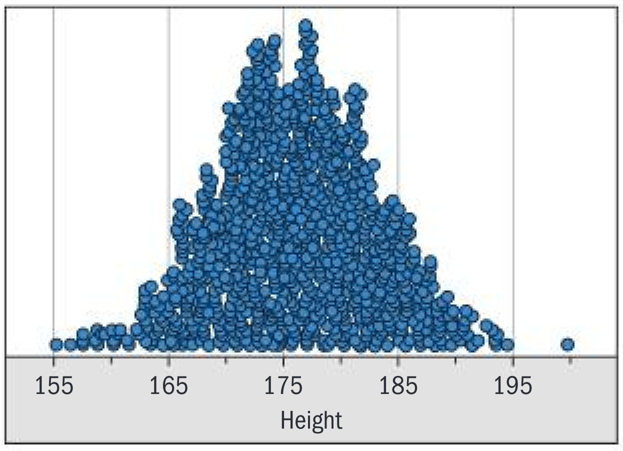
If $X \sim \mathrm{B}(n, p)$ then $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1 - p)$

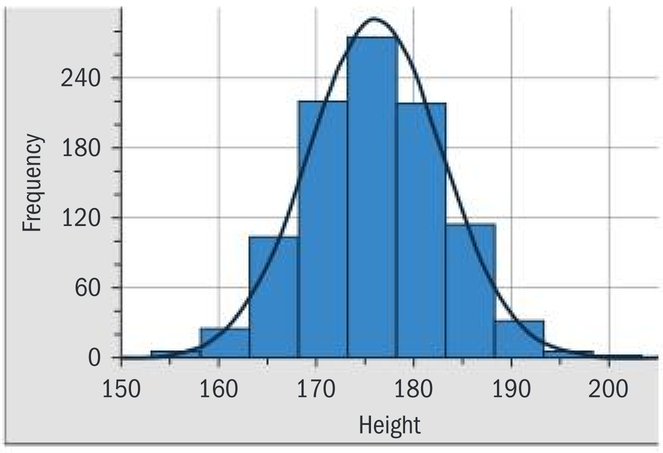
## The normal distribution

There are many examples of random variables that are not discrete because the data are not counted but measured. One example of this type of continuous probability distribution is the normal distribution.

**Context of the normal distribution:** Measured data, which is distributed symmetrically, with most data points found near the mean. Frequencies of data points further from the mean decrease in both directions.

For example, the heights of 1000 males aged 17 is shown in a dot plot on the following page. The same data is then shown in a histogram.




The data has the symmetric "bell" shape characteristic of the normal distribution.

Most data points lie close to the mean, and data points far from the mean are rare. You say that the data are distributed normally or that the data follows a normal distribution.

The data is conveniently represented by a histogram. The shape of the histogram is modelled by the normal curve. The $y$-values of the curve are found from the normal pdf. Many naturally occurring data sets are modelled by the normal curve in this way.

When we randomly select a 17-year-old student from this population and write down his height $H$, this is an example of a **continuous** random variable: the value we select could be the height of a tall student, or of one who is shorter than average and so on. The parameters of this distribution are mean 176 cm and variance 49. You write $H \sim \mathrm{N}(176, 49)$, meaning "$H$ follows the normal distribution with mean 176 and variance 49". These parameters determine the equation of the normal curve. There is a total area of 1 between the normal curve and the $x$-axis.

The probability density function (pdf) of $X \sim N(\mu, \sigma^2)$ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2}$. This formula can be explored in the contexts of graph transformations and integration, but is not needed for exams.

When $X \sim \mathrm{N}(\mu, \sigma^2)$, normal probabilities are found with technology. The domain of the normal probability density function is $x \in \mathbb{R}$

You are not required to know the formula for the normal pdf in the examination nor the concept of probability density. You should always find normal probabilities with technology.

You find probabilities of events by using the normal cumulative distribution function (normal cdf). This finds the area under the normal curve equal to the probability you require.

Because normally distributed data is continuous, $P(X < a) = P(X \le a)$ is always true. This contrasts with discrete probability distributions where "less than" and "less than or equal" always apply to different events.

## Note

The normal distribution notation gives you the variance, but your GDC always requires you to enter the standard deviation.

### Example 4.2.5

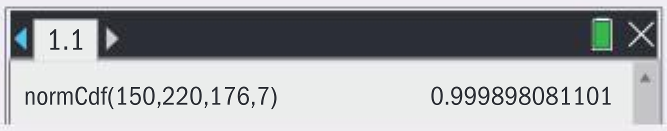
Given that the height $H$ of a randomly selected 17-year-old male follows the normal distribution with mean 176 and variance 49, find the following probabilities and represent them on a sketch.

(a) $P(150 \le H \le 200)$ (b) $P(H < 176)$

(c) $P(169 \le H \le 183)$ (d) $P(162 < H \le 190)$

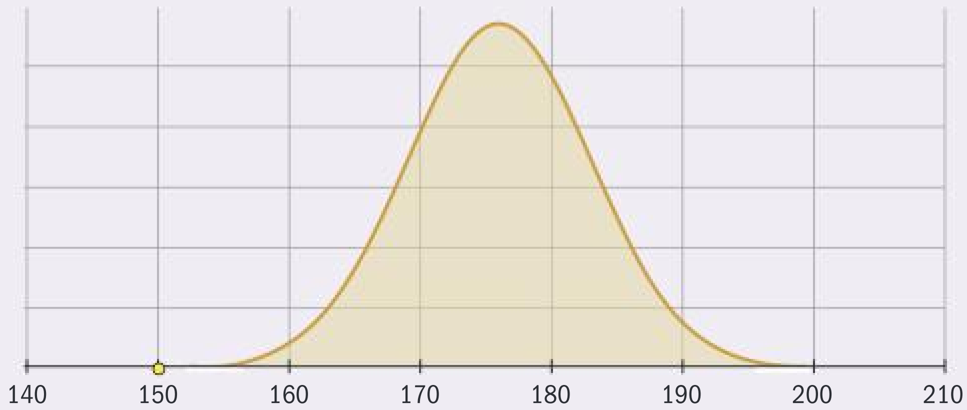
Solution

(a)

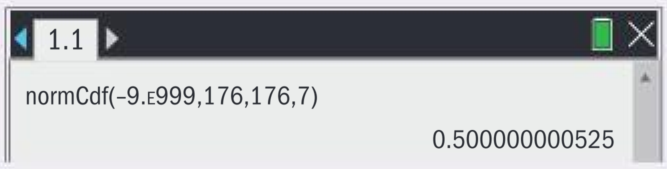



$P(150 \leq H \leq 200) = 1$

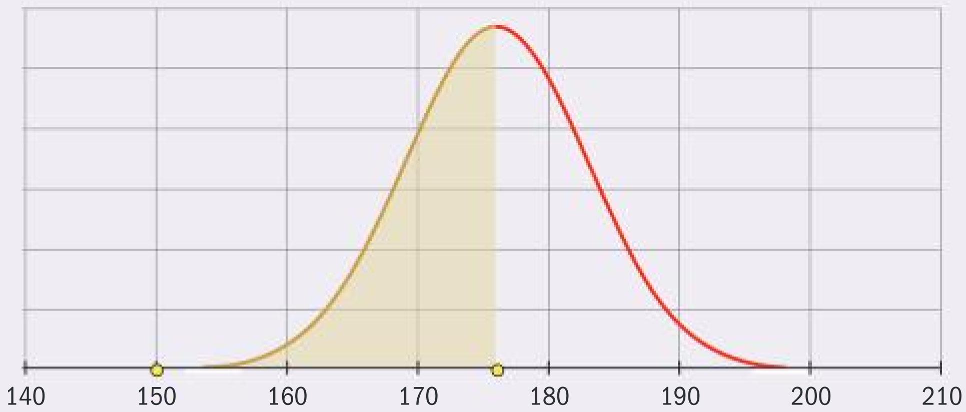
This shows virtually all the total probability of 1 under the normal curve.



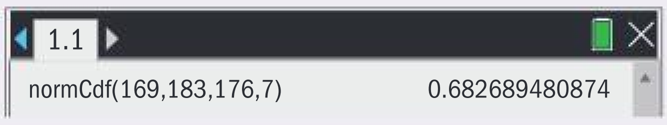

(b)



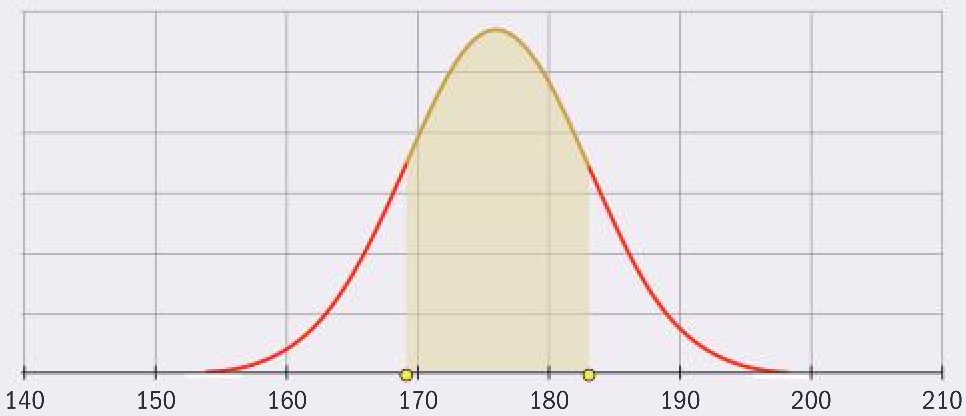

$P(H < 176) = 0.5$




(c)




$P(169 \leq H \leq 183) = 0.683$




**Note**

Though the question gives the variance of the distribution, most calculators require you to input the standard deviation. So in this example, 7 was entered into the GDC, not 49.

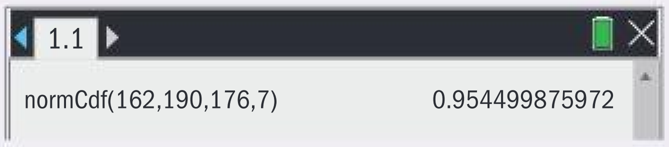
The $x$-axis is an asymptote to the curve.

**Note**

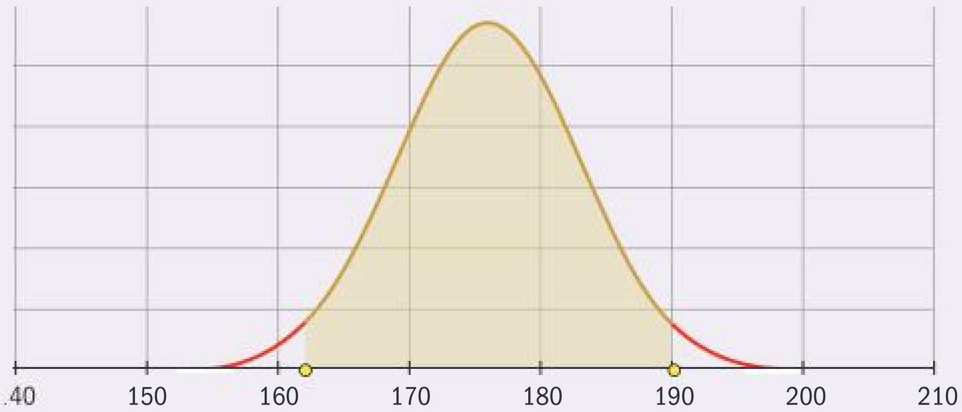
The lower bound of $-9E999$ stands for $-\infty$

The symmetric nature of the normal curve means that the probability the random variable is less than the mean is always exactly 0.5 and the mean is located at the axis of symmetry of the curve. For every normal distribution, the mean, median and mode are all equal.

It is always true that approximately 68% of a normally distributed data set lies within one standard deviation of the mean.

(d) 


$P(162 < H \leq 190) = 0.954$




It is always true that approximately 95% of a normally distributed data set lies within two standard deviations of the mean. This reflects the fact that outcomes far from the mean are relatively rare. Similarly, approximately 99.7% of data always lies within three standard deviations of the mean.

**Assessment tip**

Be sure that you know how to use the normal cdf on your GDC, including how to enter parameters and events.

Note that the values 68%, 95% and 99.7% shown in Example 4.2.5 give you a valuable perspective on how probabilities are distributed within 1, 2 and 3 standard deviations of the mean for every normal distribution. However, these are only approximate values and you should always use your GDC to calculate normal probabilities since using these approximate values would introduce unnecessary rounding errors.

In problem solving applications, be prepared for questions that require applications of more than one probability distribution.

### Example 4.2.6

The time taken for David's morning commute $C$ is distributed normally with mean 67 minutes and standard deviation 5 minutes.

(a) Find the probability that David's morning commute takes between 70 and 80 minutes.

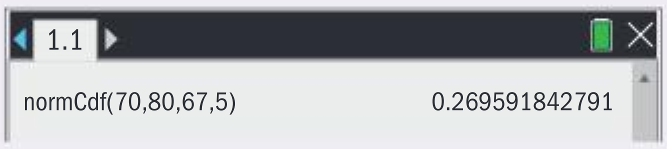
(b) David works five days per week. Find the probability that David's commute takes between 70 and 80 minutes on at least three days, stating any assumptions that you make.

**Solution**

(a) $C \sim \mathrm{N}(67, 5^2)$

$P(70 \leq C \leq 80) = 0 .270$

Write down the distribution and the event. Then find the probability using technology.

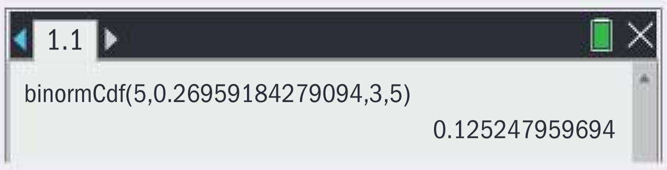



(b) Assuming each of the five days represent independent trials – ie road conditions or adverse weather conditions do not affect the probability, the number of days $D$ on which David's commute takes between 70 and 80 minutes follows a binomial distribution.

$D \sim \mathrm{B}(5, 0.270)$

$P(3 \leq C \leq 5) = 0.125$

Write down the distribution and the event then find the probability using technology.



**SAMPLE STUDENT ANSWER**

Timmy owns a shop. His daily income from selling his goods can be modelled as a normal distribution, with a mean daily income of \$820, and a standard deviation of \$230. To make a profit, Timmy's daily income needs to be greater than \$1000.

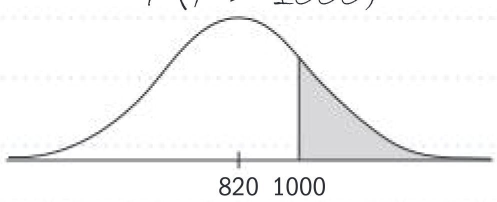
(a) Calculate the probability that, on a randomly selected day, Timmy makes a profit.

The shop is open for 24 days every month.

(b) Calculate the probability that, in a randomly selected month, Timmy makes a profit on between 5 and 10 days (inclusive).

(a) $T \sim N(820, 230^2)$

$P(T > 1000) = 0.217$




(b) $X \sim B(24, 0.217)$

$P(5 \le x \le 10) = P(x \le 10) - P(x \le 5)$

binomcdf(24,0.217,10) − binomcdf(24,0.217,5)

$= 0.9927 - 0.5756$

$= 0.417$

▲ The student correctly interprets the context of the problem as an application of the binomial distribution, and gains a method mark for writing the distribution and its parameters correctly.

▼ The student develops the inequality incorrectly, and writes GDC notation which is not appropriate in the examination. $P(5 \le x \le 10) = P(x \le 10) - P(x \le 4)$, which gives $P(5 \le x \le 10) = 0.613$.

The normal cdf finds probabilities for given values of the random variable. The inverse normal function allows you to find values of the random variable given their probabilities, using your GDC.

## Example 4.2.7

A factory fills bags of rice. The mass of a bag of rice $R$ follows the normal distribution with mean 500 g and standard deviation 4.7 g.

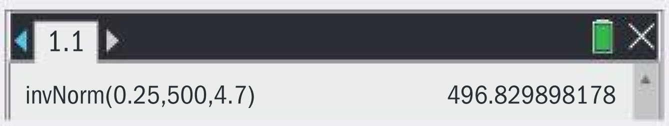
(a) Find the interquartile range of $R$.

(b) Find the mass exceeded by 90% of the bags of rice.

**Solution**

(a) $R \sim N(500, 4.7^2)$ — Write down the distribution.

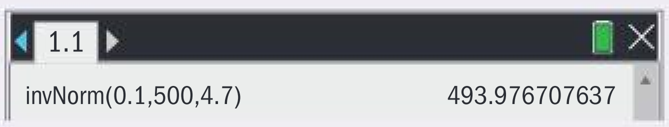
Let the value of the lower quartile be $q$. Then $P(R \le q) = 0.25$ — Introduce a variable to help solve the problem. Write the event.




Find the answer with the inverse normal cdf on your GDC.

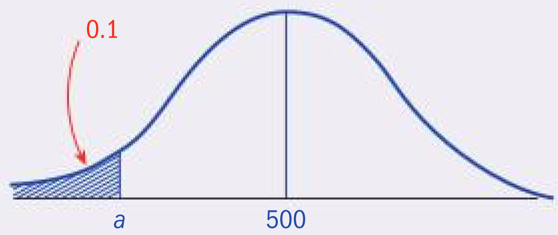
Since $q = 496.82989..$, the inter-quartile range is $2 \times (500 - 496.82989..) = 6.34$ g — Use the full decimal answer in your working to give the IQR correct to three significant figures.

(b) Let the value required be $a$. Then $P(R \le a) = 0.1$




$a = 494$ g

A sketch is a useful tool to show your working and to check the viability of the answer:




The answer of 494 is consistent with the position of $a$ on the sketch.

SAMPLE STUDENT ANSWER

The weights $W$ of newborn babies in Australia are normally distributed with a mean 3.41 kg and standard deviation 0.57 kg. A newborn baby is considered to have a low birth weight if it weighs less than $w$ kg.

(a) Given that 5.3% of newborn babies have a low birth weight, find $w$.

(b) A newborn baby has a low birth weight.

Find the probability that the baby weighs at least 2.15 kg.

▲ The student constructs a well-labelled sketch to visualize the context.

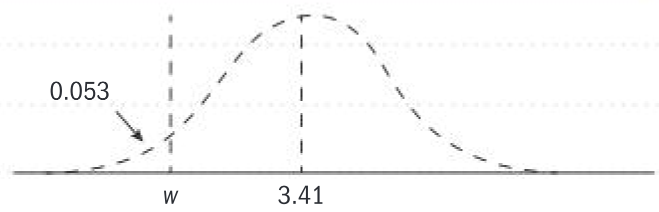
▲ The student recognizes conditional probability in the context of the problem.

▲ The student uses full accuracy in the calculation to avoid accumulation of error.

(a) $P = 0.053$

$W = 2.48863127$

$W = 2.49$




(b) $P(2.15 < X < 2.49)$

$P(2.15 < X < 2.49) = 0.03946567$

$= 0.0395$

$P = \dfrac{0.03946567}{0.053}$

$P = 0.744635283$

$P = 0.745$

# 4.3 TRANSFORMATIONS AND COMBINATIONS OF RANDOM VARIABLES (AHL)

## You should know:

- ✔ the difference between a linear transformation of a random variable and a linear combination of at least two independent random variables
- ✔ the concept of an unbiased estimate of a population parameter
- ✔ a linear combination of $n$ independent normal variables is normally distributed
- ✔ a useful case of this result is that

$$X \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

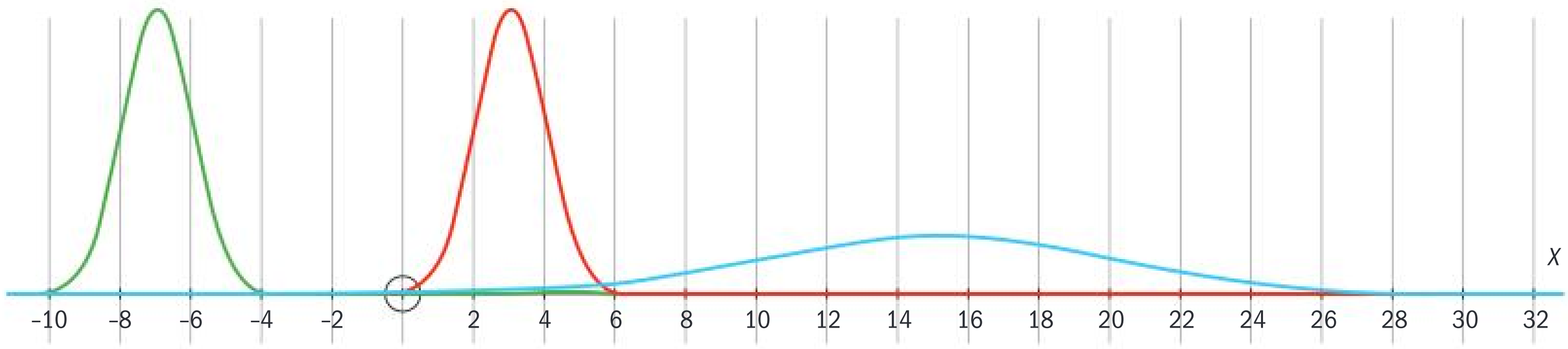
- ✔ the statement of the central limit theorem.

## You should be able to:

- ✔ interpret the context of a problem and apply linear transformations or combinations to the random variable(s) as appropriate
- ✔ find the parameters of a linearly transformed random variable and of a linear combination of independent random variables and apply these to problem solving
- ✔ calculate unbiased estimates of population parameters with technology and with formulae
- ✔ apply the central limit theorem to find probabilities involving a linear combination of independent random variables whose distributions are not known.

You have transformed points and functions in the geometry, trigonometry and functions topics respectively. You can also transform random variables. Transforming a random variable $X$ to $aX + b$ gives a new random variable with different features and parameters.

For example, in the image below the probability distribution represented by the red curve has been transformed in two different ways: it is translated to the left by 10 units to define a distribution represented by the green curve, whereas a horizontal stretch of factor 5 defines the distribution represented by the blue curve. In each distribution, the area under the graph is equal to 1.




| Random variable | Expectation | Variance | Comment |
|---|---|---|---|
| $X+b$ | $\mathrm{E}(X+b)=\mathrm{E}(X)+b$ | $\mathrm{Var}(X+b)=\mathrm{Var}(X)$ | The distribution is translated by $b$ units horizontally, hence the mean increases or decreases by this value. The spread remains the same because a translation does not change the shape of the distribution. |
| $aX$ | $\mathrm{E}(aX)=a\mathrm{E}(X)$ | $\mathrm{Var}(aX)=a^2\,\mathrm{Var}(X)$ | All $x$-values are multiplied by $a$, so the mean is also. Just as the range is $a$ times larger, so too is the standard deviation. |
| $aX+b$ | $\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$ | $\mathrm{Var}(aX+b)=a^2\,\mathrm{Var}(X)$ | Apply the sequence of transforms $X\rightarrow aX\rightarrow ax+b$ to find the effect of the general transformation of $X$ to $aX+b$. |

Similarly, linear combinations of two random variables have the following expectation and variance:

$\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$ and $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

For the latter result, the random variables must be independent. You are not expected to be able to prove these results – what is important is that you can apply them.

These results can be applied repeatedly to the results for linear combinations of $n$ random variables:

$\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$ can be applied again to a linear combination of three random variables, giving

$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm a_3X_3) = \mathrm{E}(a_1X_1) \pm \mathrm{E}(a_2X_2) \pm \mathrm{E}(a_3X_3)$ and so on, hence:

$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = \mathrm{E}(a_1X_1) \pm \mathrm{E}(a_2X_2) \pm \cdots \pm \mathrm{E}(a_nX_n)$

Similarly, $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ can be applied again, giving

$\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm a_3X_3) = a_1^2\mathrm{Var}(a_1X_1) + a_2^2\mathrm{Var}(a_2X_2) + a_3^2\mathrm{Var}(a_3X_3)$ and so on, hence:

$\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = a_1^2\mathrm{Var}(a_1X_1) + a_2^2\mathrm{Var}(a_2X_2) + \cdots + a_n^2\mathrm{Var}(a_nX_n)$.

Again, the latter result is only true when all the random variables are independent of each other.

**Application:**

Estimating population parameters with sample statistics: if the parameters of a population are modelled by a random variable $X$ with $\mathrm{E}(X) = \mu$ and , and $\mathrm{Var}(X) = \sigma^2$, you calculate the sample mean $\bar{x} = \frac{X_1 + X_2 + \cdots + X_n}{n}$ by taking $n$ independent samples from the population.

**Assessment tip**

It is important to correctly distinguish between a transformed random variable and a linear combination.

Then

$$E(\bar{X}) = E\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n\right) = \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \cdots + \frac{1}{n}E(X_n) = \frac{1}{n}(n\mu) = \mu$$

Hence $E(\bar{X}) = \mu$. This means that $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased estimator of $\mu$. It can be shown that the unbiased estimator of $\sigma^2$ is $s_{n-1}^2 = \frac{n}{n-1}s_n^2$ where $s_n^2$ is the variance of the sample. Since $\frac{n}{n-1} > 1$, the unbiased estimate of the population variance is always slightly larger than the sample variance.

**Example 4.3.1**

$Y_i$ are independent random variables with $E(Y_i) = 10$ and $Var(Y_i) = 2$, where $i \in \{1, 2, 3, 4, 5, 6\}$. Find the expected value and variance of each of the following random variables:

(a) $P = 6Y_1$

(b) $Q = Y_1 + 2Y_2 + 3Y_3$

(c) $R = 3Y_1 - 2Y_2 - Y_3$

(d) $S = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6$

**Solution**

(a) $E(6Y_1) = 6E(Y_1) = 60$. $Var(6Y_1) = 36Var(Y_1) = 72$

(b) $E(Y_1 + 2Y_2 + 3Y_3) = E(Y_1) + 2E(Y_2) + 3E(Y_3) = 60$

$Var(Y_1 + 2Y_2 + 3Y_3) = Var(Y_1) + 4Var(Y_2) + 9Var(Y_3) = 28$

(c) $E(3Y_1 - 2Y_2 - Y_3) = 3E(Y_1) - 2E(Y_2) - E(Y_3) = 0$

$Var(3Y_1 - 2Y_2 - Y_3) = 9Var(Y_1) + 4Var(Y_2) + Var(Y_3) = 28$

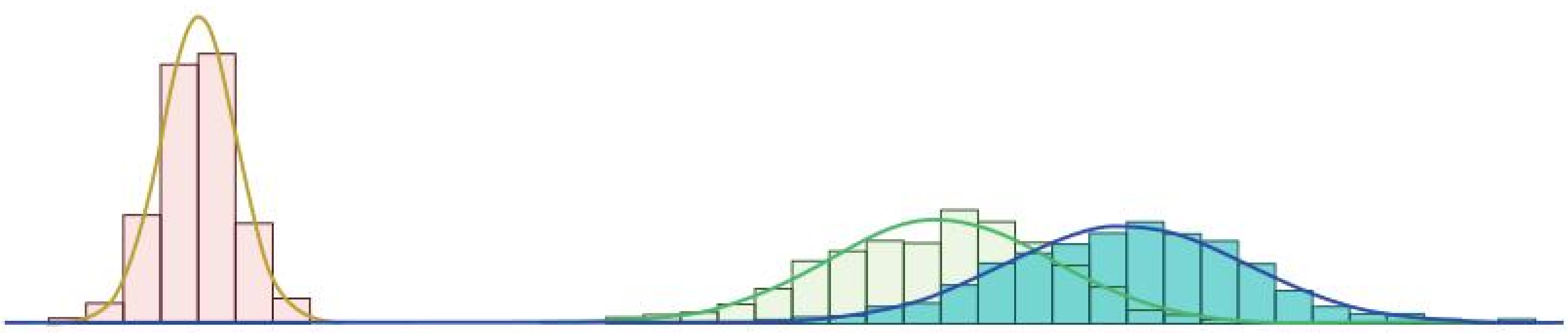
(d) $E(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6) = E(Y_1) + E(Y_2) + E(Y_3) + E(Y_4) + E(Y_5) + E(Y_6) = 60$

$Var(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6) =$

$Var(Y_1) + Var(Y_2) + Var(Y_3) + Var(Y_4) + Var(Y_5) + Var(Y_6) = 12$

The example illustrates that whenever $X, X_1, X_2, \ldots X_n$ are independent random variables each with expected value $\mu$ and variance $\sigma^2$, $Var\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2$
This must not be confused with $Var(nX) = n^2\sigma^2$.

Now the mean and variance of a combination of independent random variables is known, it would be useful to know its distribution. For example, in the diagram below, two independent normal distributions are shown: the time taken by a commuter to walk to her bicycle ($W$) and the time taken riding to her home ($B$). The total time taken for the commute, $T = W + B$, is also distributed normally. The diagram shows that $T$ is modelled by the normal distribution.



$W \sim N(5, 1^2)$ $B \sim N(25, 3^2)$ $T \sim N(30, 1^2 + 3^2)$

More generally, the distribution of a linear combination of $n$ independent normal distributions is normal.

If $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2), i \in \{1, 2, \ldots, n\}$,
then $a_1X_1 + a_2X_2 + \cdots a_nX_n \sim \mathrm{N}(a_1\mu_1 + a_2\mu_2 + \cdots a_n\mu_n, a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots a_n^2\sigma_n^2)$

This can be applied to solve problems.

## Example 4.3.2

Lisa and Miguel are psychologists. The time $L$ taken by Lisa to write a report on a client, can be modelled by a normal distribution with mean 5.7 hours and standard deviation 2 hours. The time taken for Miguel to write a report on one of his clients is modelled by $M \sim \mathrm{N}(4.5, 3.2)$

(a) On a given day, Lisa and Miguel independently write one report each. Find the probability that Lisa's report takes more than twice as long as Miguel's to write.

(b) In one week, Lisa completes two reports and Miguel completes three. Find the probability that the total time taken is more than 30 hours.

**Solution**

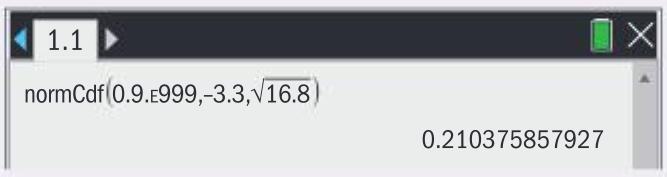
(a) $P(L > 2M) = P(L - 2M > 0)$ — Interpret the context and write down the event.

$\mathrm{E}(L - 2M) = 5.7 - 2 \times 4.5 = -3.3$ — Rearrange the event to a linear combination.

$\mathrm{Var}(L - 2M) = 2^2 + 4 \times 3.2 = 16.8$ — Determine the parameters and use technology to find the probability.

$L - 2M \sim \mathrm{N}(-3.3, 16.8)$

$P(L - 2M > 0) = 0.210$




(b) Let $T = L_1 + L_2 + M_1 + M_2 + M_3$ — Write down the random variable. Note that $2L + 3M$ is not the same as the distribution required.

$\mathrm{E}(T) = 5.7 + 5.7 + 4.5 + 4.5 + 4.5 = 24.9$

$\mathrm{Var}(T) = 4 + 4 + 3.2 + 3.2 + 3.2 = 17.6$

$T \sim \mathrm{N}(24.9, 17.6)$

$P(T > 30) = 0.112$

Consider a **sample** of size $n$ from a population that is modelled by a normal random variable $X_i \sim \mathrm{N}(\mu, \sigma^2)$. Each of the $n$ data points in the sample is an independent random variable $X_i \sim \mathrm{N}(\mu, \sigma^2)$ and $\bar{X} = \frac{X_1 + X_2 + \cdots X_n}{n}$ is a linear combination of these random variables known as the **sample mean.**

A consequence of the previous key point is that you can write down the distribution of the sample mean $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$, and hence make predictions, as shown in the next example.

### Example 4.3.3

Lengths of bamboo canes are distributed normally with a mean of 173 cm and a standard deviation of 3 cm. A garden centre plans to sell the canes in packages of $n$ canes and wants to label the packages with "average length 173 cm".

Find the smallest value of $n$ required so that the probability that the mean length of the $n$ canes is within 1 cm of 173 cm is at least 0.9

**Solution**

$B$ = the length of a randomly chosen cane.

A sample of size $n$ is taken.

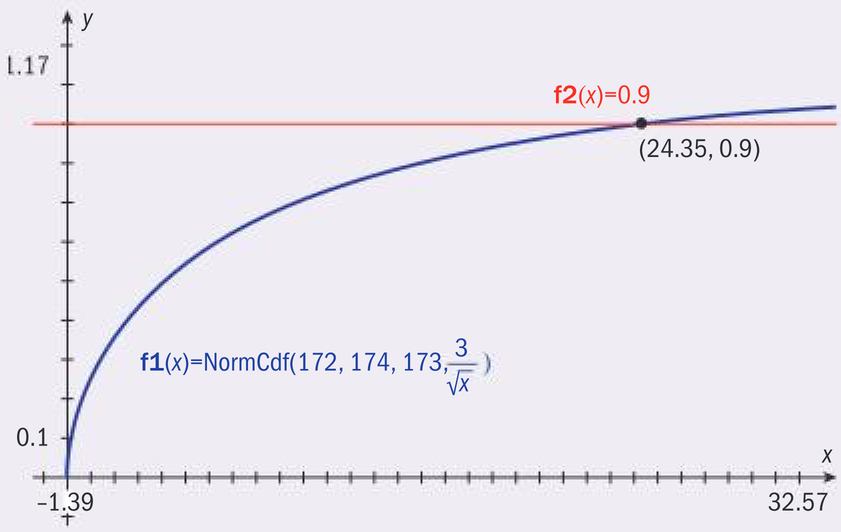
$B \sim \mathrm{N}(173, 3^2) \Rightarrow \overline{B} \sim \mathrm{N}\left(173, \frac{3^2}{n}\right)$

$P(172 < \overline{B} < 174) \geq 0.9$

The smallest value of $n$ is 25 canes.

Write down the distributions and the event.

Solve the problem with technology and provide a sketch of the GDC input to show your method. The range of the functions is [0, 1]. The domain can be found from looking at the table of values of the function.



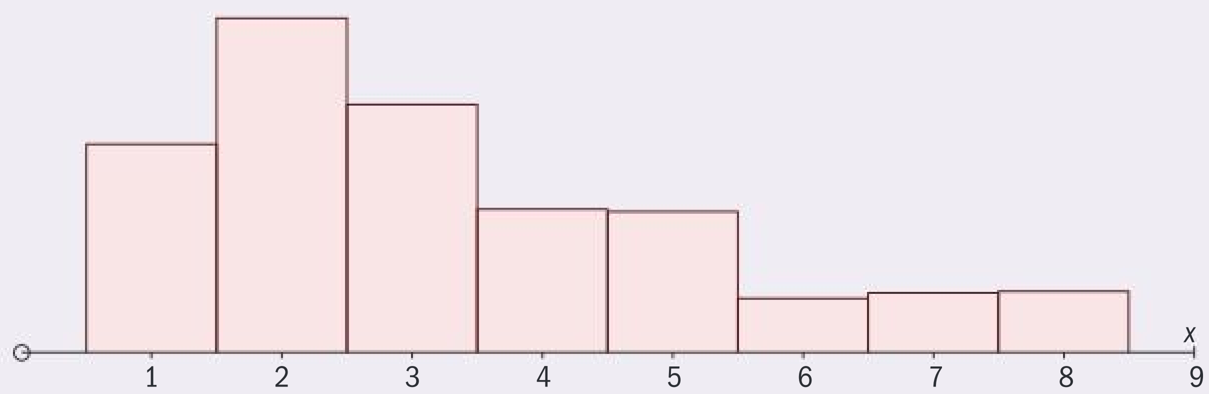

In fact, you can make predictions involving the sample mean in many more contexts.

**Central limit theorem**: When sampling from *any* population of discrete or continuous data with mean $\mu$ and variance $\sigma^2$, the distribution of the sample mean is approximately $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$, if $n$ is large enough.

In examinations, $n > 30$ will be considered sufficient. Therefore, you can make predictions about the sample mean even if you do not know which distribution models the population.

### Example 4.3.4

All residential homes in a city are surveyed in a census. The number of bedrooms has mean 3.30 and standard deviation 1.94 and the distribution is shown below:




A sample of 50 homes is taken from this population. Find the probability that the average number of bedrooms in this sample is between 3.5 and 5.5

**Solution**

Let $\bar{B}$ be the average number of bedrooms.

Then $\bar{B} \sim \mathrm{N}\left(3.30, \frac{1.94^2}{50}\right)$

$P(3.5 < \bar{B} < 5.5) = 0.233$

Write down the distribution by applying the central limit theorem.

1.1

$\text{normCdf}\left(3.5, 5.5, 3.3, \frac{1.94}{\sqrt{50}}\right)$

0.233008078109

**SAMPLE STUDENT ANSWER**

Adam does the crossword in the local newspaper every day. The time taken by Adam, $X$ minutes, to complete it is modelled by the normal distribution $\mathrm{N}(22, 5^2)$.

(a) Given that, on a randomly chosen day, the probability that he completes the crossword in less than $a$ minutes is equal to 0.8, find the value of $a$.

(b) Find the probability that the total time taken for him to complete five randomly chosen crosswords exceeds 120 minutes.

Beatrice also does the crossword in the local newspaper every day. The time taken by Beatrice, $Y$ minutes, to complete the crossword is modelled by the normal distribution $\mathrm{N}(40, 6^2)$.

(c) Find the probability that, on a randomly chosen day, the time taken by Beatrice to complete the crossword is more than twice the time taken by Adam to complete the crossword. Assume that these two times are independent.

Let $x$ be the time for Adam to solve puzzle $x \sim N(22, 5^2)$

(a) $p(x < a) = 0.8$

$\Rightarrow a = 26.2$

(b) let $k$ be time taken to solve 5 puzzles

$k = x_1 + x_2 + x_3 + x_4 + x_5$

$E(k) = 5E(x)$

$var(k) = 5var(x)$

$k \sim N(110, 125)$

$p(k > 120) = 0.185547$

$= 0.186$

(c) ~~$y \sim NE$~~ let $y$ be time taken for Beatrice to solve puzzle

$y \sim N(40, 6^2)$

$p(y > 2x) = p(y - 2x > 0)$

let $m = y - 2x$

$E(m) = E(y) - 2E(x) = 40 - 2(22) = -4$

$var(m) = var(y) + 2^2 var(x) = (6^2) + 4(5^2) = 136$

$m \sim N(-4, 136)$

$p(m > 0) = 0.3658$

$= 0.366$

▲ The student provides clear and complete working, including a clear description of the random variable to aid the interpretation of the context, scoring full marks for part (a).

▲ Carefully setting out the random variable and the event helps the student solve the problem and score full marks.

▼ Although the student gained full marks for this question, it is worth pointing out that in mathematics random variables are written with capital letters as a rule because this helps distinguish them from other variables.

# 4.4 POISSON DISTRIBUTION (AHL)

## You should know:

- ✔ what defines the contexts in which the Poisson distribution is an appropriate model
- ✔ the mean of the Poisson distribution equals the variance
- ✔ the sum of two independent Poisson distributions has a Poisson distribution.

## You should be able to:

- ✔ calculate Poisson probabilities with technology.
- ✔ select between the Poisson, binomial or normal distribution as appropriate for a given context.

Another example of a discrete probability distribution is the Poisson distribution.

**Context of the Poisson distribution**: The Poisson distribution models situations in which events are independent and occur at a uniform average rate in an interval of time or space. The random variable is the number of occurrences.

| Number of buses arriving in 15 minutes ($x$) | Frequency |
|---|---|
| 0 | 3 |
| 1 | 15 |
| 2 | 20 |
| 3 | 25 |
| 4 | 17 |
| 5 | 11 |
| 6 | 6 |
| 7 | 2 |
| 8 | 1 |

For example, a bus company advertises that its buses run with the frequency of one bus per five minutes. Therefore, David reasons that at his bus stop there should be about 3 buses every 15 minutes. He collects data on one hundred 15-minute intervals from 4 pm to 4:15 pm every Monday and presents his findings in a frequency table.

Note that David could not know the maximum number of buses arriving in his data sample until his sample was complete. Also, although David finds the average rate from this sample of size one hundred is 3.11, the number of buses arriving varies from 0 to 8.

The event is a bus arriving at the bus stop in a 15-minute period. The random variable $X$ is the number of buses that arrive at the bus stop in the 15-minute period. The uniform average rate is 3 buses per 15-minute interval. The assumption that the buses arrive independently depends on a combination of factors such as the random nature of traffic congestion, the amount of people embarking at each stop and how long it takes them to buy tickets from the driver.

On the assumption that the Poisson distribution is an appropriate model, you write "$X$ follows a Poisson distribution with parameter 3" or $X \sim \text{Po}(3)$.

> If occurrences of an event are independent and occur at a uniform average rate $m$ in an interval of time or space, the number of occurrences $X$ in the interval follows a Poisson distribution with parameter $m$. You write $X \sim \text{Po}(m)$. Poisson probabilities are found with technology and the domain of the Poisson probability distribution function is $x \in \mathbb{N}$.

For $X \sim \text{Po}(m)$, $E(X) = m$. This reflects the fact that if the event occurs on average $m$ times in the interval, then that is the expected value of $X$. The mean for longer or shorter intervals can be found with direct proportion, as shown in Example 4.4.1

You are not required to know the formula for the Poisson pdf in the examination. You should always find Poisson probabilities with technology as shown in the following example.

### Example 4.4.1

The number of calls received in an hour $C$ by an IT helpline follows a Poisson distribution with mean 7.3. Find the probabilities of the following events:

(a) Exactly 6 calls are received in one hour.

(b) More than 4 but no more than 8 calls are received in one hour.

(c) At least 15 calls are made in a three hour interval, stating any assumptions you make.

**Solution**

(a) $C \sim \text{Po}(7.3)$

$P(C = 6) = 0.142$

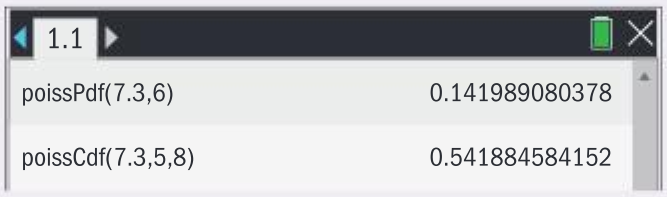
Write down the distribution and the events.

(b) $P(4 < C \leq 8)$

$= P(5 \leq C \leq 8)$

$= 0.542$

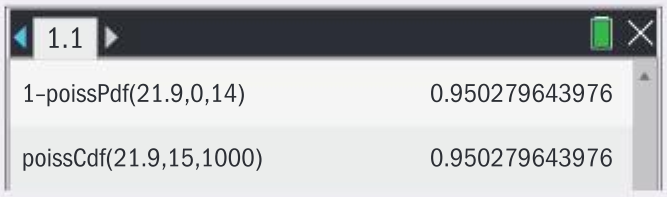
Find the probabilities using technology.




(c) Assuming the rate continues for the three hours, and writing $D$ as the number of calls in a three-hour period, $D \sim \text{Po}(21.9)$

$P(D \geq 15) = 0.950$

Find the rate by direct proportion. There are two equivalent ways to find the answer with technology.




The probability distribution function of $X \sim \text{Po}(m)$ is:

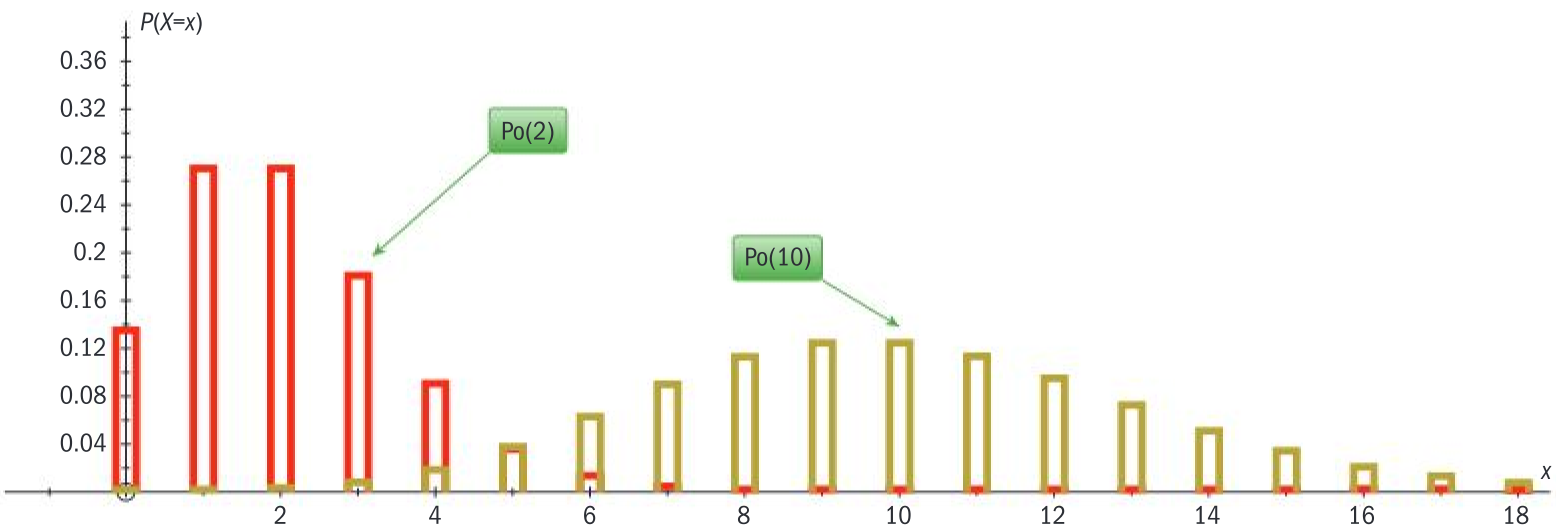
$$P(X = x) = \frac{m^x e^{-m}}{x!}$$

This formula can be explored in the context of functions, but is not needed for exams.

**Assessment tip**

The formulae for the mean and variance of the Poisson distribution are in the formula book, section AHL 4.17

As the value of $m$ increases, so does the spread of the distribution $X \sim \text{Po}(m)$. For example, bar charts of Poisson distributions with means 2 and 10 are shown below, showing a greater spread for mean 10 than for mean 2. This is because for $X \sim \text{Po}(m)$, $\text{Var}(X) = m$.



The expected value and variance of a sum of independent random variables is covered generally in section 4.3 of this book.

If $S \sim \mathrm{Po}(\lambda)$ and $T \sim \mathrm{Po}(\mu)$ are independent Poisson distributions then $S + T \sim \mathrm{Po}(\lambda + \mu)$. This makes sense only if the units for the uniform average rates are the same. It would not make sense to add the rates 1 bus per hour and 5 buses per day to find 6 buses per hour! If the rates are not in the same unit they should be changed to the same unit using direct proportion, as shown in the next example.

**Example 4.4.2**

Medical staff in a hospital work daily shifts of eight hours. The number of admissions $E$ to the emergency room follows a Poisson distribution with parameter 1.7 admissions per hour. The number of admissions $M$ to the maternity ward follows a Poisson distribution with mean 6.1 admissions per shift.

(a) For a shift, find the distribution of $T$, the total number of admissions to the emergency room and to the maternity ward during one shift, stating any assumptions you make.

(b) Hence find $P(T > 24)$.

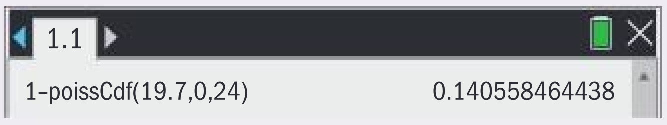
(c) The hospital administrator works a five-day week. Find the probability that on exactly three of these days the total admissions is at least 25, stating any assumptions you make.

Solution

(a) Assuming that the hourly rate of 1.7 continues for the duration of a shift, the parameter of $T$ is $1.7 \times 8 + 6.1 = 19.7$ hence $T \sim \mathrm{Po}(19.7)$

Convert the hourly rate to the rate for one shift using direct proportion.

(b) $P(T > 24) = 0.141$

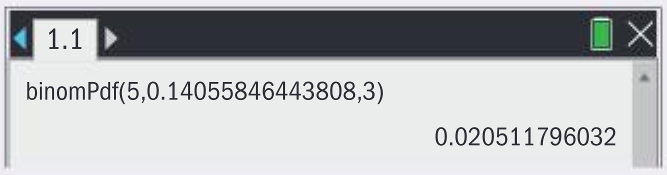



Find the probability with technology.

(c) Let $D$ be the number of days in a five-day week on which a total of at least 25 admissions are made. Then $T \sim \mathrm{B}(5, 0.141)$ assuming that the probability of 0.141 is constant for all five days and that the number of admissions on each day is independent of the admissions on the other days.

$P(D = 3) = 0.0205$

Write down the distribution, the event and find the probability with technology.




**SAMPLE STUDENT ANSWER**

A company produces rectangular sheets of glass of area 5 m$^2$. During manufacturing these glass sheets flaws occur at the rate of 0.5 per 5 m$^2$. It is assumed that the number of flaws per glass sheet follows a Poisson distribution.

(a) Find the probability that a randomly chosen glass sheet contains at least one flaw.

Glass sheets with no flaws earn a profit of \$5. Glass sheets with at least one flaw incur a loss of \$3.

(b) Find the expected profit, $P$ dollars, per glass sheet.

This company also produces larger glass sheets of area 20 m$^2$. The rate of occurrence of flaws remains at 0.5 per 5 m$^2$.

A larger glass sheet is chosen at random.

(c) Find the probability that it contains no flaws.

$X \sim Po(0.5)$ where $x$ is flaws per 5 $m^2$

(a) $p(x \geq 1) = 1 - p(x < 1) = 0.0902$

(b) $\cancel{E(x) = 0.5}$

$\$5 \times 0.9098 - \$3 \times 0.0902 = \$4.28$

(c) $X \sim Po(2)$

$p(x = 0) = 0.135$

▼ The student writes the inequality correctly, but the equivalent form of the event $1 - P(X = 0)$ would set up the use of the GDC better. The correct answer is 0.393

▲ Detailed working enables the award of follow through marks. The correct answer is \$1.85

▲ Direct proportion has been applied correctly, the correct probability has been found with the GDC.

# 4.5 DESCRIPTIVE STATISTICS

### You should know:

- ✔ the difference between discrete and continuous data
- ✔ an outlier is defined as a data item which is more than 1.5 × interquartile range (IQR) from the nearest quartile
- ✔ the mean, median and mode are all measures of central tendency and the interquartile range and standard deviation are measures of spread
- ✔ when calculating the mean or standard deviation from grouped frequency tables the values are estimated by using the midpoints of the intervals
- ✔ the variance is the square of the standard deviation
- ✔ the effect on the mean and standard deviation of a constant change acting on the original data.

### You should be able to:

- ✔ find median, quartiles, percentiles, range and interquartile range (IQR) from a cumulative frequency curve
- ✔ produce and interpret box and whisker plots
- ✔ calculate the mean from a grouped frequency table without technology
- ✔ calculate the mean, standard deviation and quartiles using technology.

The world contains a lot of information in the form of data, but to make sense of this data it needs to be sorted and summarized into a useful form. This section of the course looks at three ways this is done: by finding an 'average' value for the data, a measure of how spread out it is and how closely it fits with either a straight line or a continuously increasing or decreasing function.

## Averages

There are three ways to measure 'central tendency' (or three types of 'average'), which you use in different situations. They are the **mean**, **median** and **mode**.

## Mean

To find the mean, all the values are added and then divided by the number of values. It is often written as $\bar{x}$.

For example, the mean of 10, 12, 17, 24 is $\bar{x} = \frac{10+12+17+24}{4} = 15.75$

We can write the mean as: $\bar{x} = \frac{\sum x}{n}$, where $\sum x$ represents the sum or total of the data.

The formula book writes the formula as $\frac{\sum_{i=1}^{k} f_i x_i}{n}$ where $n = \sum_{i=1}^{k} f_i$.

This is equivalent to the above and is used when the data is given in a frequency table. If the data is given individually then all $f_i$ are equal to 1 and the equation above is formed.

**Note**

Rearranging this formula to give the total in terms of the mean can be useful.

Make sure you know how to find the mean ($\bar{x}$) on your GDC.

### Example 4.5.1

Eight students took a test. Their average score was 14.5 points. A ninth student took the same test a day later and scored 19 points. Find the average score for the nine students.

**Solution**

Total score for the eight students is $14.5 \times 8 = 116$

If $\bar{x} = \frac{\sum x}{n}$ then $\sum x = n\bar{x}$

New total $= 116 + 19 = 135$

New mean $= \frac{135}{9} = 15$

## Median

If the data being considered is put in order of size then the median is the value of the middle number.

**Note**

For a large data set of size $n$ use the rule that the median is the value of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ piece of data.

Consider the ordered data

List 1: 2, 2, 3, 5, 6, 6, 7, 9, 9

List 2: 3, 3, 5, 5, 5, 7, 7, 8, 8, 9

In list 1, the median is 6. Using the formula in the Note box, we have 9 pieces of data so the median is the $\left(\frac{9+1}{2}\right) =$ 5th piece of data.

In list 2 there are 10 pieces of data so the median is the $\left(\frac{10+1}{2}\right) =$ 5.5th piece of data.

In this case we take the mean of the 5th and 6th pieces of data, which is equal to 6.

**Assessment tip**

Usually the median will be found using the statistical functions on the GDC.

The median is also referred to as the 50th **percentile** as it lies halfway (or 50% of the way) through an ordered list of the data.

In addition to the median, the **lower quartile** ($Q_1$) is the value that is 25% of the way through the data (the 25th percentile) and the **upper quartile** ($Q_3$) is three-quarters of the way through the data (the 75th percentile)

If the data in list 1 is entered into a GDC it will give the values $Q_1 = 2.5$, $Q_3 = 8$

**Assessment tip**

In the exam, quartiles will always be obtained from the GDC, a cumulative frequency curve or a box and whisker plot.

## Mode

The mode is the number with the highest frequency, or the most common number. In list 2 the mode is 5.

## Frequency tables

### Example 4.5.2

The following marks out of 8 were achieved by a class. Find:

(a) the number of pupils in the class (b) the mean mark.

| Mark | Frequency |
|---|---|
| 4 | 5 |
| 5 | 6 |
| 6 | 8 |
| 7 | 4 |
| 8 | 2 |

Solution

(a) 25

The total number of students in the class is the sum of the frequencies.

(b) $\dfrac{4\times5+5\times6+6\times8+7\times4+8\times2}{25}$

$=\dfrac{142}{25}=5.68$

When adding up the marks in this frequency table it is much easier to do $4\times5$ than $4+4+4+4+4$

The formula for the mean is in section 4.3 of the formula book:
$\bar{x}=\dfrac{\sum_{i=1}^{n} f_i x_i}{n}$ where $n=\sum_{i=1}^{n} f_i$

You met sigma notation in section 1.2.

**Assessment tip**

In an exam you will usually be calculating a mean using the inbuilt function on your GDC. A common error when doing this is forgetting to let the calculator know there is a frequency list!

## Grouped data

When the data is grouped, it is not possible to know the exact values, so when finding the mean it is assumed all data takes the value of the midpoint of their group.

### Example 4.5.3

The table below shows the length of time in minutes for 74 students to complete a task.

| Time ($t$) | Frequency |
|---|---|
| $0\le t<10$ | 4 |
| $10\le t<20$ | 10 |
| $20\le t<30$ | 22 |
| $30\le t<40$ | 17 |
| $40\le t<50$ | 14 |
| $50\le t<60$ | 7 |

(a) Find the mean of this data.

(b) Write down the modal class.

(c) Find the interval in which the median lies.

Solution

(a)

| L1 | L2 |
|---|---|
| 5 | 4 |
| 15 | 10 |
| 25 | 22 |
| 35 | 17 |
| 45 | 14 |
| 55 | 7 |

31.5 minutes

This is found using the inbuilt function in the GDC (1-variable statistics) and entering the values as the midpoints: 5, 15, 25…

(b) $20 \le t < 30$

The modal class is the interval with highest frequency.

(c) $30 \le t < 40$

The median is the time of the $\left(\frac{74+1}{2}\right) = 37.5$th person. The 37.5th person is in the interval $30 \le t < 40$. The GDC gives the median as 35, which indicates the correct interval, but is unlikely to be the actual median, so should not be quoted.

**Assessment tip**

For grouped data you will only be asked for the interval in which the median lies, not for its value.

## Measures of spread

The **range** of a set of data is the maximum value minus the minimum value. It is not usually a good guide to the spread of the data as it can be affected by a single very large or very small value, called an **outlier**.

The interquartile range (IQR) is literally the 'range between the quartiles' and is equal to $Q_3 - Q_1$. This is the range of the middle 50% of the data and is not affected by any outliers.

The most sophisticated measure of spread is the **standard deviation** ($\sigma$), as it uses all the data. When comparing different sets of data, the one with the highest standard deviation is more spread out. Unlike the IQR, it takes into account all the data points.

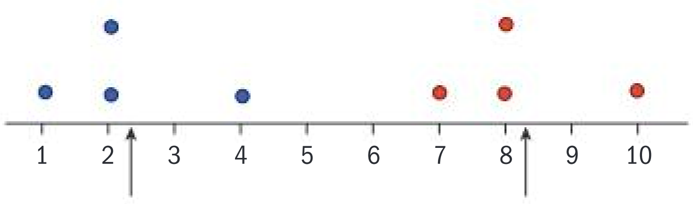
The standard deviation is found directly from the GDC and is usually shown as $\sigma_X$. Using the data from Example 4.5.3, the standard deviation is 13.3 minutes.

**Assessment tip**

The interquartile range will usually be tested in a question on box and whisker plots or cumulative frequency curves.

## Constant changes in the data

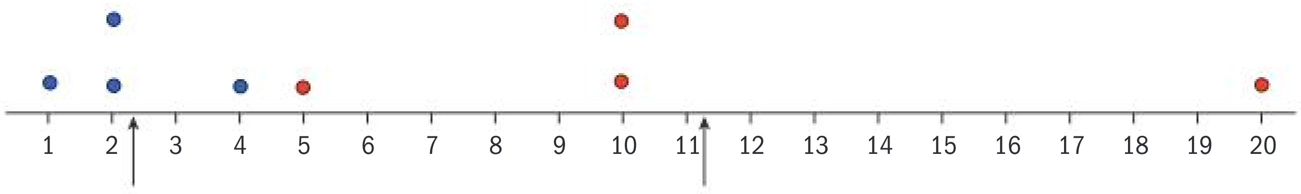
The four values 1, 2, 2, 4 are shown as blue dots in the diagram below. Their mean is 2.25 and their standard deviation is 1.09. The mean is indicated by the arrow on the diagram. If 6 is added to all four pieces of data (shown by the red dots) it can be seen that the mean also increases by 6, but the spread of the data is unchanged and hence the standard deviation is unchanged.




**Note**

Sometimes a question will ask for the variance rather than for the standard deviation. The variance is equal to the square of the standard deviation and has the symbol $\sigma^2$ (see the sample student answer on page 183).

Similarly, if all the data is multiplied by 5, the mean is multiplied by 5. This time though the spread has increased also so the new standard deviation is $5 \times 1.09 = 5.45$




**SAMPLE STUDENT ANSWER**

A data set has $n$ items. The sum of the items is 800 and the mean is 20.

(a) Find $n$.

The standard deviation of this data set is 3. Each value in the set is multiplied by 10.

(b) (i) Write down the value of the new mean.

(ii) Find the value of the new variance.

(a) $\frac{800}{n} = 20 \Rightarrow \frac{800}{20} = n \qquad n = 40$

(b) (i) new mean increases proportionally to increase in data values so $20 \times 10 = 200$ ∴ new mean $= 200$

(ii) $SD = 3$, data spread stays the same; variance $= SD^2$

$3^2 = 9$

variance $= 9$

▲ The student has correctly used the formula for the mean.

▲ The student has calculated the new mean correctly, as when a data set is multiplied by fixed value, so is the mean.

▲ The student has noticed that the question is asking for the variance not the standard deviation.

▼ The standard deviation should also be multiplied by 10, so it becomes 30, and the variance is then 900.

## Presentation of data

A **box and whisker plot** is a way of displaying five summary statistics of a set of data: the minimum, lower quartile, median, upper quartile and the maximum.

Any outliers are shown as a cross and the whiskers extend only to the highest or lowest point which is not an outlier.

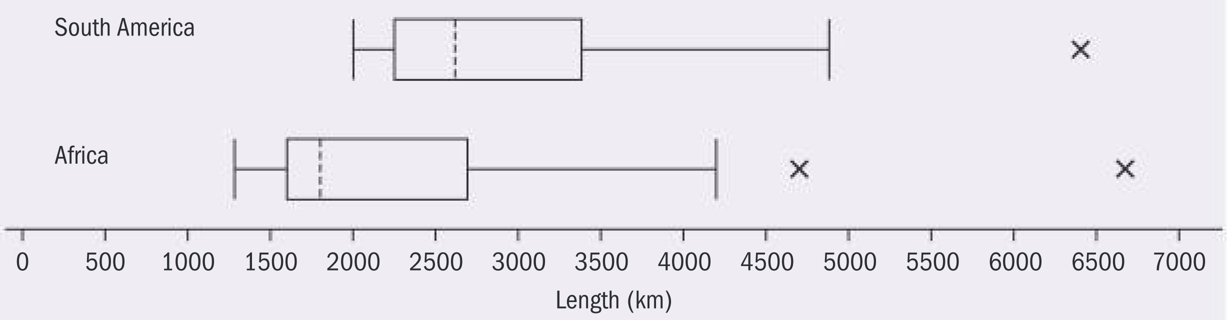
It is a useful way to compare two sets of data.

**Note**

An outlier is defined as any point that is more than $1.5 \times IQR$ above the upper quartile or below the lower quartile.

### Example 4.5.4

Julia is doing a project on the longest rivers in the southern hemisphere. She has recorded the lengths of the 15 longest rivers in Africa and the 15 longest in South America and they are shown on the box and whisker plots below.



(a) Use the plots to determine:

(i) an estimate for the median length of the 15 longest rivers in Africa

(ii) which of Africa or South America has the third longest river in the two continents combined

(iii) the maximum number of rivers in Africa that are longer than the 15th longest river in South America.

Julia then collects information on the 15 longest rivers in Australia and records their lengths in kilometres in the table below.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 680 | 685 | 700 | 752 | 834 | 941 | 969 | 1004 | 1210 | 1300 | 1380 | 1440 | 1472 | 1485 | 2508 |

(b) Construct a box and whisker plot to show this data.

**Solution**

(a) (i) 1750

In an exam, any value from 1700 to 1800 would probably be accepted.

(ii) South America

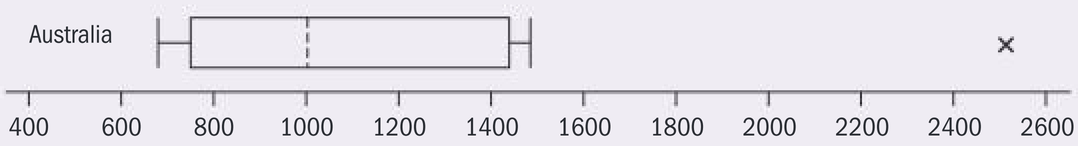
The maximum for South America is longer than the second outlier for Africa.

(iii) Median is the 8th longest river so maximum number is 7

The median for Africa is less than the minimum for South America, so is not included, but the 1st to the 7th longest could all be greater than the minimum for South America.

(b) Median = 1004, $Q_1 = 752$, $Q_3 = 1440$

Outlier will be above $1440 + 1.5(1440 - 752) = 2472$



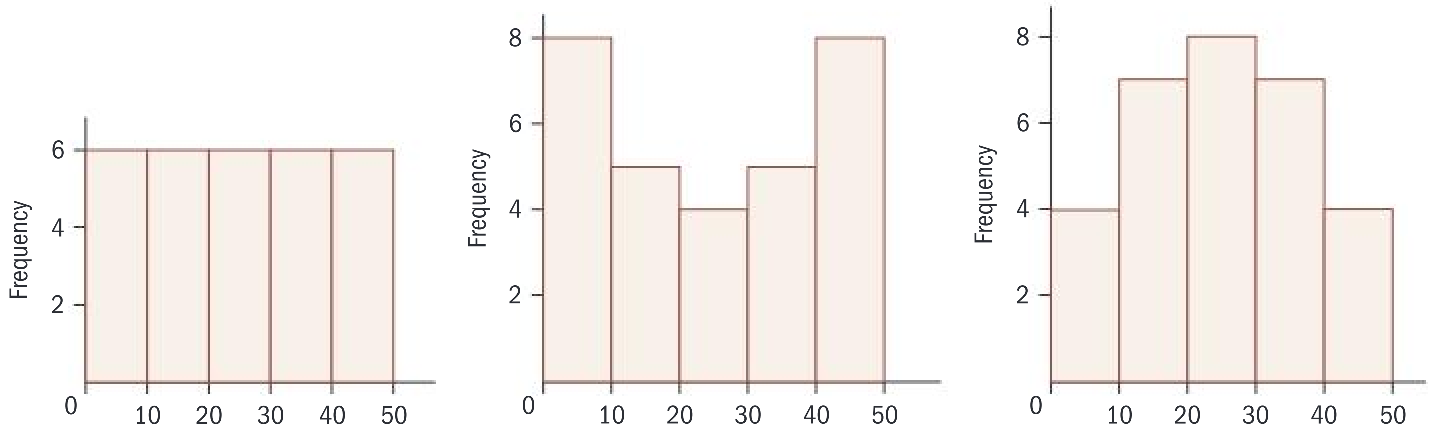

These values can be obtained from the list or from the GDC.

It is important to check for outliers when constructing a box and whisker plot. In this case it is clear from the box and whisker plot that there will be no outliers below the lower quartile.

When an outlier is present, it is important to decide whether or not it should be included in the data. Sometimes it is an important point and should be included. Sometimes it might distort the data and should be dealt with separately, or might even be due to an error in calculation.

## Histograms

A histogram is similar to a box and whisker plot, but for continuous grouped data. The vertical axis is frequency and the horizontal axis represents the range of the measurements as a continuous scale. It is a good way to show the distribution of data. The distributions below all have the same mean and similar standard deviations but very different distributions.




## Cumulative frequency curves

Cumulative frequency curves show the number of values which are less than (or less than or equal to) a given value. They are useful for finding the median and other percentiles for grouped data.

When constructing a cumulative frequency curve, you should always plot the point at the end of the interval, because it indicates how many of the items in the data set are less than a given value.

**Assessment tip**

In an exam you will only ever be asked to find the median for grouped data when using a cumulative frequency curve.

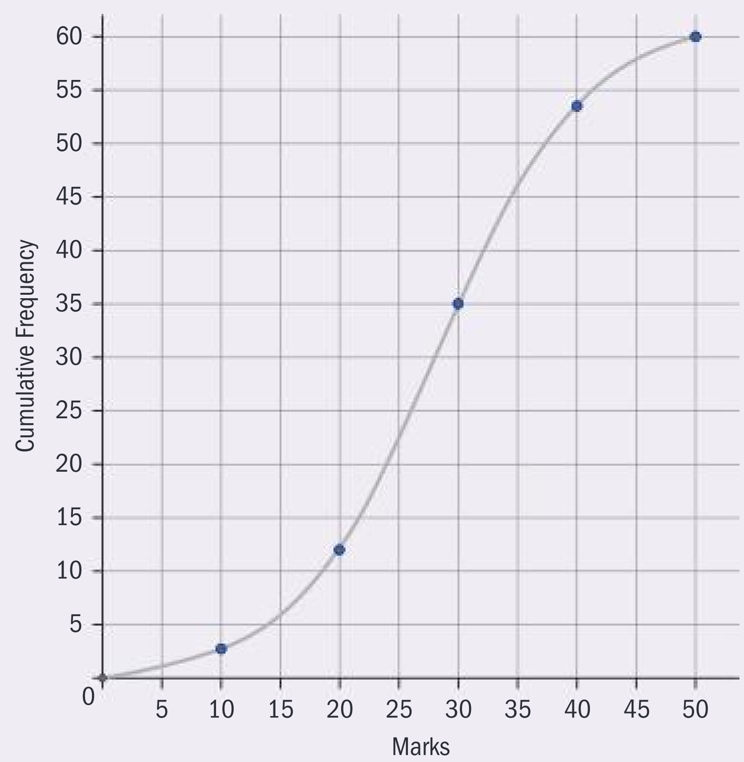
### Example 4.5.5

The scores of 60 students in a test out of 50 marks are given:

| Mark ($x$) | Frequency | Cumulative frequency |
|---|---|---|
| $0 \le x \le 10$ | 3 | 3 |
| $10 < x \le 20$ | $a$ | 12 |
| $20 < x \le 30$ | 23 | $b$ |
| $30 < x \le 40$ | 18 | 53 |
| $40 < x \le 50$ | 7 | 60 |

(a) Find the values of $a$ and $b$

The marks are shown on the cumulative frequency graph here.



(b) From the graph, estimate the value of:

(i) the median score

(ii) the inter-quartile range

(iii) the 80th percentile.

(c) The pass mark for the test was 15. Estimate the number of students who passed the test.

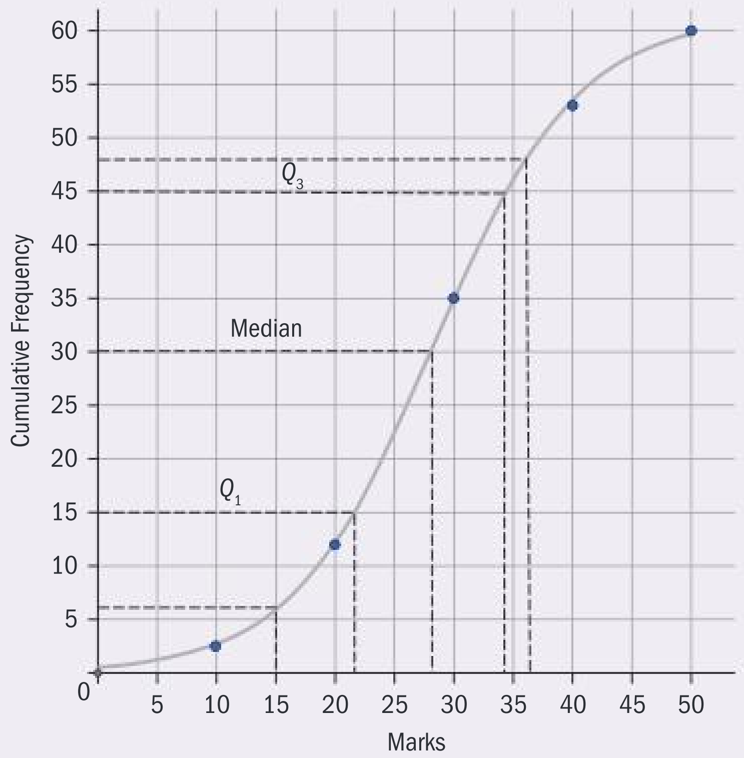
**Solution**

(a) $a = 12 - 3 = 9$

$b = 12 + 23 = 35$

If 12 students scored less than 20 marks there must have been 9 who scored between 10 and 20.

(b)




(i) 28

(ii) $34 - 22 = 12$

(iii) The 48th student scored about 37 out of 50 in the test.

If there are 60 students the median is the score of the 30th student.

If there are 60 students the 80th percentile is the score of the $\frac{80}{100} \times 60 = 48^{th}$ student.

(c) 7 students scored less than 15 and so failed the test. This means about 53 students passed the test

**Assessment tip**

Clearly show the lines you are using on the cumulative frequency diagram.

**Note**

There is no need to add one before dividing by two when finding the median for a cumulative frequency curve, as the scale begins at 0

# 4.6 CORRELATION AND REGRESSION

## You should know:

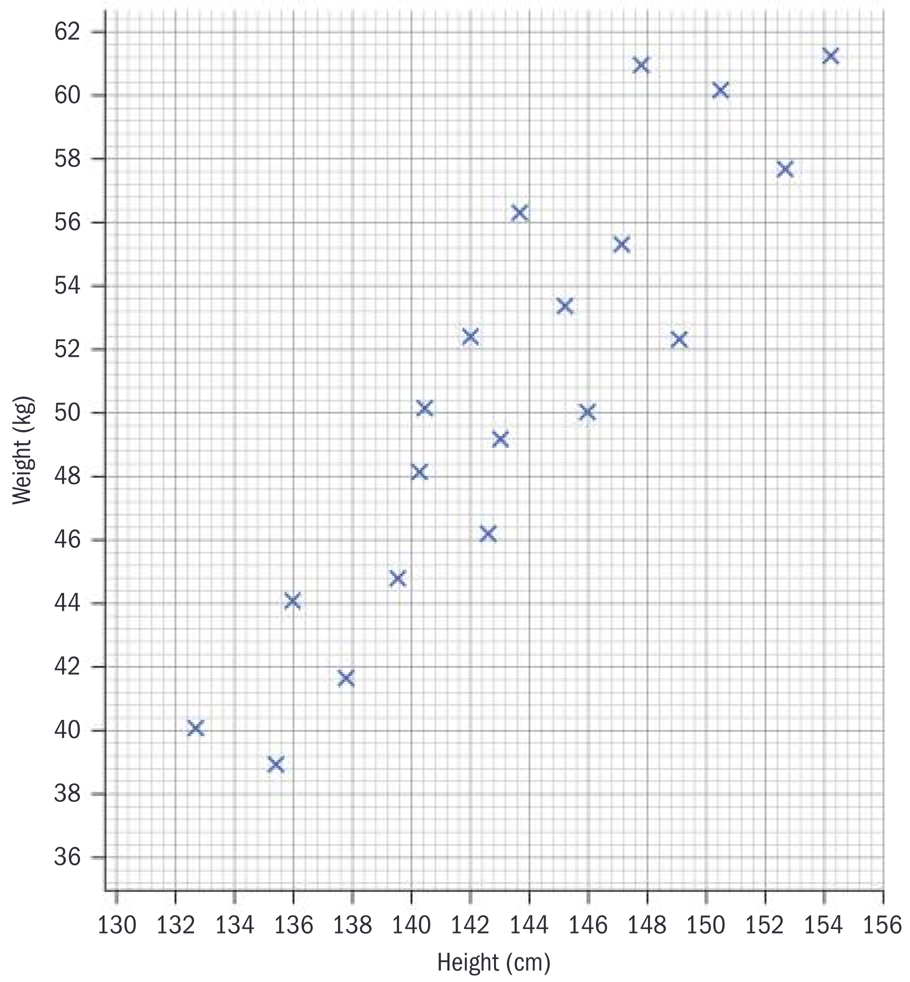
- ✔ the Pearson product moment correlation coefficient ($r$) measures the goodness of fit to a linear model
- ✔ correlation can be strong, moderate or weak depending on how close it is to a straight line; and positive or negative, depending on the sign of the gradient
- ✔ the equation of the regression line of $y$ on $x$ can be used to predict values of $y$ from values of $x$, for values of $x$ within the range of data given
- ✔ Spearman's rank correlation coefficient $r_s$ is used to measure how closely two variables can be modelled by an increasing or a decreasing function
- ✔ if data items are equal when calculating $r_s$, the ranks should be averaged.

## You should be able to:

- ✔ interpret the meaning of the parameters $a$ and $b$ in a linear regression $y = ax + b$
- ✔ calculate $r$ and $r_s$ using technology
- ✔ make the distinction between correlation and causation and know that correlation does not imply causation.

### Scatter diagrams

These help to show whether there is any relationship between two variables. For example the heights and weights of 20 children in a class are recorded and the results shown on the scatter diagram below.



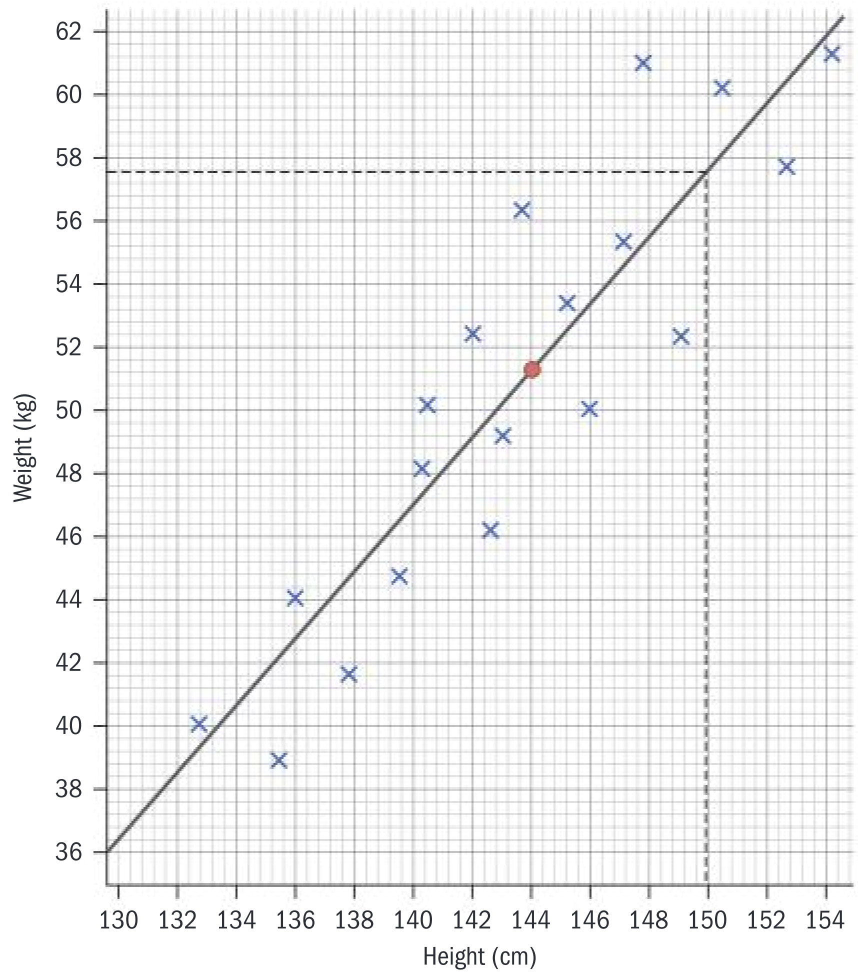

**Note**

The PMCC ($r$) can also be used to assess how well the data fits a linear model.

**Note**

The line of best fit will always pass through the point $(\bar{x}, \bar{y})$ where $\bar{x}$ is the mean of the $x$ values and $\bar{y}$ is the mean of the $y$ values.

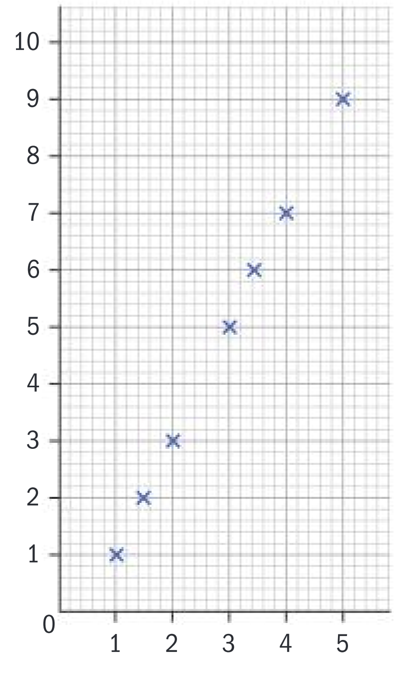
There is an approximately linear relationship here which justifies drawing a **line of best fit** through the data. The line of best fit allows you to find the approximate weight of a student from their height if their height and weight follow a similar relationship.

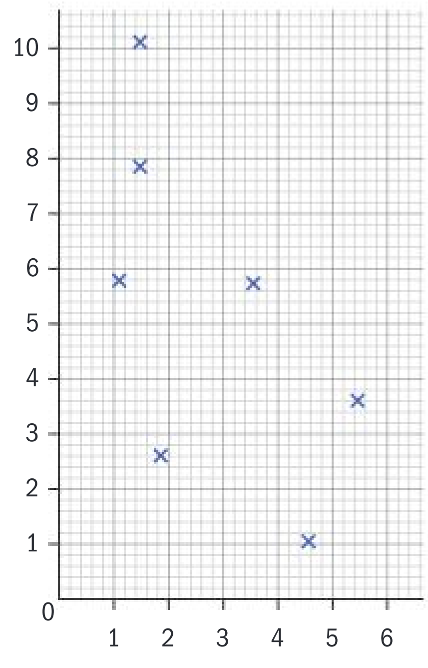


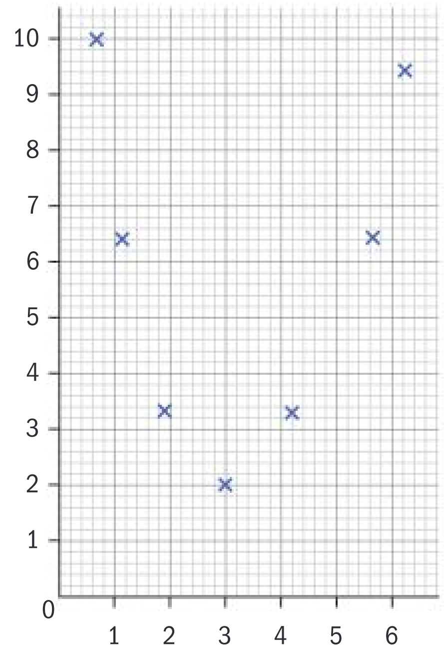

For example, using this model an estimate for the weight of a student with a height of 150 cm would be between 57and 58 kg.

If a set of data points lie close to a line with a positive gradient we say they have a **strong positive correlation**. If they are close to a line with a negative gradient we say they have a **strong negative correlation**. If there is no linear relationship we say there is no correlation.

One measure of how close a set of data points are to a straight line is the (**Pearson**) **product moment correlation coefficient** ($r$) which takes values between –1 and 1.


$r = 1$


$r \approx -0.7$


$r \approx 0$

A high value of the PMCC ($r$) does not necessarily imply causation.
A comparison of ice cream sales in a town on a given day and the number of cases of sunburn on that day will probably show a good correlation, but it is unlikely one is causing the other. Sometimes though a correlation can imply a causation, for example noticing the significant correlation between the number of smokers and the number of cases of lung cancer led doctors to understanding that the increase in one was causing the increase in the other.

## Regression

If the scatter diagram indicates that a linear model connecting the two variables is appropriate, then rather than drawing a line of best fit 'by eye' you can use the one generated by the GDC.

This line is called the **least squares regression line for $y$ on $x$**, or just the **regression line for $y$ on $x$**.

Using this equation, we can estimate values of $y$ for values of $x$ within the range of the data (interpolation).

There are two limitations to using the regression line to estimate values.

1 It would not be wise to assume this line remains the line of best fit outside this range of the data, so it should not be used to estimate values of $y$ beyond the given values of $x$ (extrapolation).

2 The regression line $y$ on $x$ is only the best fit line for estimating values of $y$ from a given value of $x$, so it should not be used for estimating values of $x$ from values of $y$.

**Note**

For the straight line $y = a + bx$ we call $x$ the **independent variable** and $y$ the **dependent variable** because it is given in terms of $x$.

**Note**

To find a value of $x$ from a value of $y$ you would use the regression line for $x$ on $y$, but this is not required for assessment purposes.

**SAMPLE STUDENT ANSWER**

Seven adult men wanted to see if there was a relationship between their Body Mass Index (BMI) and their waist size. Their waist sizes, in cm, were recorded and their BMI calculated.

The following table shows the results.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Waist ($x$ cm) | 58 | 63 | 75 | 82 | 93 | 98 | 105 |
| BMI ($y$) | 19 | 20 | 22 | 23 | 25 | 24 | 26 |

The relationship between $x$ and $y$ can be modelled by the regression equation $y = ax + b$.

(a) (i) Find the value of $a$ and of $b$.

(ii) Find the correlation coefficient.

(b) Use the regression equation to estimate the BMI of an adult man whose waist size is 95 cm.

(a) (i) $a = 0.141$ (ii) $r = 0.978$

$b = 11.1$

(b) $y = 0.141x + 11.1$

$y = 0.141(95) + 11.1$

$y = 24.495$

$y = 24.5$

BMI = 24.5

▲ The student has set out the work clearly and got the correct answers directly from the GDC for part a and by correct substitution for part b.

▼ It is possible on most GDCs to save the regression equation so the value of 95 could be substituted with less work and no possibility of rounding errors.

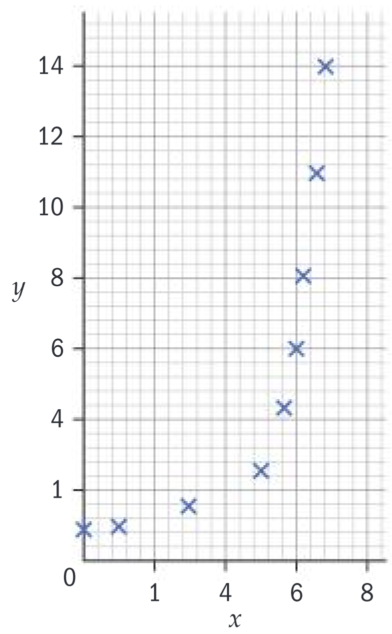
It is important to be able to interpret the value of $a$ and $b$ in the regression equation $y = ax + b$

$a$ is the gradient of the line and gives the increase in $y$ for an increase of 1 unit of $x$. Its units are the number of units of $y$ 'per' unit of $x$.

In the exam question above the value of $a$ gives the increase in BMI (measured in $\text{kg m}^{-2}$) for an increase of one centimetre in the waist measurement, so $0.141\ \text{kg m}^{-2}\,\text{cm}^{-1}$.

$b$ is the value of $y$ when $x$ is 0. Be aware that estimating it will often involve extrapolation and so this value might have no meaning in the context given. In the exam question above the value of $b = 11.1$ would be the BMI for a man with a waist measurement of 0 cm.

## Spearman's rank correlation coefficient




Another measure of correlation is Spearman's rank correlation coefficient. This is a measure of the degree to which the dependent variable increases as the independent variable increases.

For the data in the diagram on the left, the product moment correlation for this data is only moderate (0.78) and misleading as the data does not seem to follow a straight line. But there is a clear relationship in that the value of $y$ increases whenever the value of $x$ does. In this case the Spearman's rank correlation coefficient ($r_s$) will have a value of $r_s = 1$. If the value of $y$ always decreases as the value of $x$ increases then $r_s = -1$.

Spearman's rank correlation coefficient is found by calculating the product moment correlation coefficient for the ranks of the data.

For example, in the table below the value of the data are shown in the top row and their ranks in the second row. Whenever there are equal values each is given the average of the ranks they would otherwise have received. For example, the two 7s would have had ranks 3 and 4 so are each given a rank of 3.5. Similarly, the three 5s, which would have otherwise have received ranks 5, 6 and 7, are all given a rank of 6.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| value | 5 | 7 | 5 | 8 | 12 | 5 | 4 | 7 |
| rank | 6 | 3.5 | 6 | 2 | 1 | 6 | 8 | 3.5 |

### Assessment tip

You may be asked to decide which correlation coefficient is best in which situation.

Remember that the product moment correlation coefficient is used to find how well the data fits to a **straight** line.

An additional point is that the PMCC can be easily distorted by an outlier to give a high correlation. If one exists it should be excluded or Spearman's correlation should be used.

### Example 4.6.1

Kathryn believes that people often pay too much for perfumes so she organizes a test. Students are asked to rank six perfumes without knowing their cost. The overall ranks are given in the table below, along with the costs of the perfume.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Rank | 4 | 6 | 3 | 2 | 5 | 1 |
| Cost ($) | 18 | 18 | 120 | 74 | 68 | 35 |

(a) Explain why it would be inappropriate to use the product moment correlation coefficient.

(b) Find Spearman's rank correlation coefficient and comment on the result.

**Solution**

(a) Because ranks are given in the table, not values.

It would also be justifiable to say that Kathryn is not looking specifically for a linear relationship.

(b)

| A | B | C | D | E | F |
|---|---|---|---|---|---|
| 4 | 6 | 3 | 2 | 5 | 1 |
| 5.5 | 5.5 | 1 | 2 | 3 | 4 |

The cost ranks 5 and 6 are shared by $A$ and $B$.

$r_s = 0.435$

There is only a moderate correlation between the rankings for preference and price.

# 4.7 STATISTICAL TESTS

## You should know:

- ✔ simple random, convenience, systematic, quota and stratified sampling methods
- ✔ the terms: null and alternative hypothesis, significance level, *p*-value and critical value
- ✔ how to find the degrees of freedom for a $\chi^2$ test for independence
- ✔ how to find the expected values for a $\chi^2$ goodness of fit test, for uniform, binomial and normal distributions
- ✔ the degrees of freedom for a $\chi^2$ goodness of fit test is always $n - 1$
- ✔ the difference between a one-tailed and a two-tailed *t*-test
- ✔ the assumptions for a *t*-test, namely that the population is normally distributed and, in exams, the variances are assumed to be equal which means the data is pooled.

## You should be able to:

- ✔ determine whether the data may be normally distributed by consideration of the symmetry of the box and whiskers
- ✔ carry out a $\chi^2$ test using technology and interpret the result
- ✔ carry out a *t*-test for testing the difference between two means, using technology and when all the data is given, and interpret the result.

Up to this point we have used statistics to describe entire populations, for example: the marks of everyone in the class. We have not been using the data to try to find out, or to **infer,** anything about a wider population that had not been directly tested.

However, we often want to find the mean, standard deviation or the distribution of a very large population. In these cases we need to take a **sample** from the population and infer from the sample the **population** statistics.

The first requirement in order to do this successfully is to ensure that the sample is **representative** of the population it comes from.

**Note**

A sample that is unrepresentative is often referred to as **biased**.

## Sampling

A **random** sample is one in which everyone in the population has an equal chance of being selected. It could be found, for example, by giving everyone in the population a number and then using a random number generator to select the sample.

Another method is **systematic sampling.** This saves having to generate many random numbers.

> In systematic sampling, a starting number is selected randomly and a fixed distance is chosen, normally the nearest integer to the size of the population divided by the sample size. The sample begins with the first person chosen and then each subsequent person is the fixed distance down the list from the previous one. If you get to the end of the list you go back to the beginning of the list.

**Note**

Because you start at a random point, everyone has an equal chance of being selected.

For example, if you want to collect a sample of 20 from a school of 242, you would calculate $\frac{242}{20} = 12.1$ and so would choose every 12th person.

Not all samples will be random. If you are doing a survey about cafeteria food it would be easy to ask the first 10 people in the queue, but not everyone would have an equal chance of being asked. Some students may not go to the cafeteria, lunchtimes might be based around year groups so not every year group is asked. This type of sampling has the advantage of being easy to do, and as such is called **convenience sampling**.

Just because a sample is random does not mean that it is representative. Suppose in a school there are equal number of boys and girls. A random sample of 12 could have 10 boys and 2 girls. For some surveys this might not matter but when if does it is best use **stratified sampling**.

**Assessment tip**

If you are asked in an exam to do either a stratified or quota sample, you should assume that the size of the classes should be proportional to those in the overall population, even if this is not explicitly mentioned in the question.

In a stratified sample the population is divided into different groups and, within each group, the sample is selected **randomly**. In a proportional stratified sample the proportion of each group in the sample is the same as the proportion in the population as a whole.

A common alternative to stratified sampling is **quota sampling.**

In quota sampling, a population is divided into different groups and then **convenience sampling** is used to select the number required in each group.

An example of quota sampling would be a survey carried out in a town centre, in which the researcher needs to speak to, for example, 5 men over 30 years old. This sample will not be random, as not everyone is equally likely to be in the town centre, and might be biased because those who choose to speak with the researcher might not be representative of all people in their class.

### Example 4.7.1

A large accounting firm wants to survey how satisfied its clients are with the service it offers. The firm decides to send a questionnaire to 100 randomly selected clients, stratified by the country in which they are based.

(a) Given the firm has 2050 clients in total and of these 65 are based in France, find the number of clients from France that should be in the sample.

Very few of the clients respond to the survey so the firm instructs its employees to ask the next 100 clients that they visit to complete the form.

(b) Write down the type of sampling used in the following cases:

(i) There is no consideration of the country in which the clients are based.

(ii) The numbers surveyed from each country are as calculated in part (a).

Solution

| | |
|---|---|
| (a) $\frac{65}{2050} \times 100 = 3.17 \approx 3$ | The formula is $\frac{\text{number in the class}}{\text{number in the population}} \times \text{sample size}$ |
| (b) (i) convenience<br>(ii) quota | Always give the answer as a whole number |

## Statistical tests

There are many tests that can be done on a sample to learn more about the population. In the Standard Level MAI course you need to know just three; to see if two normally distributed populations have the same mean, to see if two categories are independent of each other and to see if a sample could have come from a population with a given distribution (a goodness of fit test).

In any test, the null hypothesis ($H_0$) is the initial belief that will be kept unless there is strong evidence to reject it. It must be something that can be tested, for example: the population means are the same or the two variables are independent.

The alternative hypothesis ($H_1$) is the one that will be accepted if there is significant evidence to reject the null hypothesis.

Each test will have a **significance level**, normally written as a percentage, for example a 5% significance level. This means that the null hypothesis will be rejected if the probability of obtaining the values in the sample (or more extreme values) when the null hypothesis is true is less than 5% (0.05)

## The $\chi^2$ test

The $\chi^2$ test or chi-squared test (pronounced 'kai' squared) looks at how far a sample is from what would be predicted under the hull hypothesis. If it is too far away then the null hypothesis is rejected.

The values in the sample are referred to as the **observed** values, and the values predicted under the null hypothesis are called the **expected values**.

There are two ways to decide the outcome of a $\chi^2$ test.

The first is to consider the value of $\chi^2_{calc}$ given by your GDC and compare it with a critical value ($\chi^2_{crit}$) for the significance level under consideration. If required the critical value for the test will always be given in the question.

If $\chi^2_{calc} > \chi^2_{crit}$ then the observed values are too far from the expected values so the null hypothesis is rejected.

**Note**

The measure used to judge how far the observed values ($f_o$) are from the expected values ($f_e$) is often written as $\chi^2_{calc} = \sum_{i=1}^{n}\left(\frac{(f_o - f_e)^2}{f_e}\right)$ but this is not required for the exam.

**Note**

The critical value will depend on the **degrees of freedom** of the test. In the SL MAI course this is the number of cells or groups, minus 1.

The most common method though is to consider the $p$-value for the test.

The ***p*-value** for a test is the probability of obtaining the results in the sample (or more extreme ones) when the null hypothesis is true. The null hypothesis is rejected if the $p$-value is less than the significance level of the test.

### Example 4.7.2

A six-sided die is rolled 60 times and the outcomes are recorded in the table below. Test at the 5% significance level whether or not there is evidence that the die is biased.

| Number | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Frequency | 8 | 9 | 7 | 10 | 6 | 20 |

For five degrees of freedom the 5% critical value is 11.07

**Solution**

$H_0$: The die is not biased

$H_1$: The die is biased

The hypotheses must always be stated. The null hypothesis is that the distribution is equal to the one being tested.

Expected values if the die is fair are all $\frac{60}{6} = 10$

The observed and expected values are entered into a GDC and a $\chi^2$ goodness of fit test is performed.

$\chi^2_{calc} = 13 > 11.07$

There is sufficient evidence to reject $H_0$ at the 5% level, so we conclude the die is not fair.

The alternative reasoning would be:

The $p$-value $= 0.0234 < 0.05$

The conclusion is the same.

### Example 4.7.3

(a) Given $X$ follows a normal distribution with a mean of 6.2 and a standard deviation of 2.4, find:

(i) $P(8 < X < 10)$ (ii) $P(X > 10)$

Phil believes that the length ($X$) of pebbles from a beach are normally distributed with a mean of 6.2 cm and a standard deviation of 2.4 cm. To test this hypothesis he measures the length of 150 randomly selected pebbles and records their size in 6 different classes as shown in the table below.

| Length, $x$ (cm) | $x \le 2.0$ | $2 < x \le 4$ | $4 < x \le 6$ | $6 < x \le 8$ | $8 < x \le 10$ | $x > 10$ |
|---|---|---|---|---|---|---|
| Frequency | 13 | 25 | 37 | 43 | 26 | 6 |

In order to perform a $\chi^2$ test, he calculates the expected number of pebbles in each of the classes if their lengths did follow this distribution. His results are shown in the table below.

| Length, $x$ (cm) | $x \le 2.0$ | $2 < x \le 4$ | $4 < x \le 6$ | $6 < x \le 8$ | $8 < x \le 10$ | $x > 10$ |
|---|---|---|---|---|---|---|
| Expected value | 6.0 | 21.0 | 43.1 | 46.0 | $a$ | $b$ |

(b) State the null and alternative hypotheses.

(c) Use your answer to part (a) to find the values of $a$ and $b$.

(d) Perform a $\chi^2$ test at the 5% significance level to see if there is sufficient evidence to reject the null hypothesis. State the degrees of freedom for the test.

**Assessment tip**

Remember the null hypothesis is rejected if $\chi^2_{calc}$ is **greater than** the critical value (the observed values are too far from the expected) or if the $p$-value is **less than** the significance level (the probability of the observed values occurring by chance is even smaller than the significance level).

**Note**

It is 5 degrees of freedom because there are 6 possible outcomes when a die is rolled.

**Assessment tip**

You must always give a reason for your conclusion by comparing $\chi^2_{calc}$ with the critical value or the $p$-value with the significance level.

Solution

(a) (i) $P(8 < X < 10) = 0.1699... \approx 0.170$

(ii) $P(X > 10) = 0.05667... \approx 0.0567$

These are calculated at the start of the question but will be used later.

(b) $H_0$: The lengths of pebbles can be modelled by a $N(6.2, 2.4^2)$ distribution.

$H_1$: They cannot be modelled by this distribution.

(c) $a = 150 \times 0.170 \approx 25.5$

$b = 150 \times 0.0567 \approx 8.50$

The expected value = probability × sample size. Make sure you use the full values for the probability found in part (a)

(d) degrees of freedom $= 6 - 1 = 5$

$p$-value $= 0.0569 > 0.05$

So there is insufficient evidence to reject $H_0$ : The lengths of pebbles can be modelled by a $N(6.2, 2.4^2)$ distribution.

## Assessment tip

In a $\chi^2$ test all the expected frequencies need to be greater than 5. In the Standard Level exam this will always be the case.

## The $\chi^2$ test for independence

A special type of $\chi^2$ test is to test whether or not two variables are independent of each other. For example, you might want to test whether the type of music people like is independent of their age.

This type of test is called the $\chi^2$ test for independence and the observed frequencies are normally given in a **contingency table**.

## Note

Most GDCs will have a different function for the $\chi^2$ test for independence.

SAMPLE STUDENT ANSWER

In a school, students in grades 9 to 12 were asked to select their preferred drink. The choices were milk, juice and water. The data obtained are organized in the following table.

| | Milk | Juice | Water | **Total** |
|---|---|---|---|---|
| Grade 9 | 25 | 34 | 15 | **74** |
| Grade 10 | 31 | $x$ | 13 | **74** |
| Grade 11 | 18 | 35 | 17 | **70** |
| Grade 12 | 9 | 36 | 26 | **71** |
| **Total** | **83** | **135** | **71** | **289** |

A $\chi^2$ test is carried out at the 5% significance level with hypotheses:

$H_0$: the preferred drink is independent of the grade

$H_1$: the preferred drink is not independent of the grade

The $\chi^2$ critical value for this test is 12.6

(a) Write down the value of $x$.

(b) Write down the number of degrees of freedom for this test.

(c) Use your graphic display calculator to find the $\chi^2$ statistic for this test.

(d) State the conclusion for this test. Give a reason for your answer.

## Assessment tip

There are several ways to write your conclusion to the test. The following is an example of acceptable phrasing:

'There is sufficient evidence to reject $H_0$' or 'There is insufficient evidence to reject $H_0$'.

You should avoid 'Accept $H_0$' if the result of the test is not significant.

## Assessment tip

An exam question might ask you for the degrees of freedom of the test. This can normally be read off the GDC but otherwise it can be calculated for an $n \times m$ table as $(m - 1)(n - 1)$.

▲ Correct, it might have been calculated incorrectly first but many GDCs will also state the degrees of freedom so the error can be picked up.

▼ It is actually 18.96 so 19.0 to 3 sf. With no method shown this could lead to a loss of 2 marks. An answer this close though might imply to the examiner that the method was performed correctly.

▲ Conclusion and reason both given.

(a) 30

(b) 6

(c) 18.9

(d) Reject $H_0$ because $18.9 > 12.6$. They are not independent.

## The *t*-test

The *t*-test is used to see if two samples both coming from populations distributed normally and with the same standard deviation could have the same mean. For example, if you want to test whether students in Germany score as highly in Mathematics exams as students in France, you could take a random sample of each and compare the average scores of each sample.

The null hypothesis for a *t*-test is: $H_0$: the means are equal. The alternative might be that the means are not equal (two-tailed test) , or it might be that one is greater than or less than the other (a one-tailed test). Your GDC will ask you to choose.

The *t*-test should be used only if the background population is normally distributed. One way to check this is to look at a box and whisker plot. If it is fairly symmetrical and the inter-quartile range is less than half the range then it is likely to be close to a normal distribution.

### Note

It is not sufficient to say that if the average score of one sample is greater than the other then that must be true for the whole population, but if one is a long way above the other then that could be quite strong evidence for a difference in the population means.

### Example 4.7.4

A teacher wants to see if a revision course he plans to offer is helpful to his students. Half his class volunteer to stay behind after school to attend the course. All the students are then given a test and the scores for the two groups are recorded and listed in the table below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Took revision course | 23 | 45 | 78 | 82 | 56 | 90 | 65 | 72 |
| Did not take revision course | 44 | 32 | 45 | 79 | 48 | 69 | 50 | 35 |

(a) Perform a *t*-test at the 10% significance level to see if there is any evidence that the revision course improved the average scores.

You may assume that the scores come from normal distributions with the same standard deviation.

(b) State why the result of the test might not be valid when deciding if the revision course improves scores.

**Solution**

(a) $H_0$: The revision course does not improve the scores in the test.

$H_1$: The revision course does improve the score

$p$-value $= 0.0893 < 0.1$

There is sufficient evidence at the 10% significance level to reject $H_0$

An alternative general statement which will always be the same for a *t*-test is: The two populations have the same mean.

This is saying that the population mean of those who did the revision course is higher than for those who did not. Hence it is a one-tailed test.

(b) The sample was not randomly selected so there might be other factors to explain why one group did better than the other.

The test has only provided evidence for the two populations not having the same mean, but not what might cause it. If the teacher wanted to be sure it was the revision course he needs to eliminate all other factors.

### Assessment tip

In exams it is always assumed that the standard deviations of both populations are equal. This means you must always choose the **pooled** option on your GDC.

# 4.8 SAMPLING (AHL)

## You should know:

- ✔ the definition of reliability and validity
- ✔ how to obtain reliable and valid data from a sample
- ✔ test-retest and parallel tests for reliability validity tests for criterion validity.

## You should be able to:

- ✔ select relevant variables from many variables
- ✔ choose relevant and appropriate data to analyse
- ✔ use correlation to assess whether a data collection method is reliable/valid.

In section 4.7 we considered the methods for selecting a sample from a population, including convenience, systematic, quota and stratified.

In this section we shall consider how data is obtained from the sample.

In some cases it is clear what is required; for example if a firm wanted information about the length of employees' journey to work then they need just ask that question.

It becomes more difficult if the firm wants to discover how satisfied the employees are with their work.

In this case they might ask employees to rank their satisfaction on a scale, or provide multiple choice questions which do not ask them directly but from which their level of satisfaction could be inferred. Alternatively, the employees in the sample could be interviewed so follow-up questions could also be asked.

Whichever method is used, you need your data collection to be both **valid** and **reliable**.

A **valid test** is one that measures the quality that it claims to be measuring. A valid test can have **content validity**. A Mathematics test on calculus will have content validity if the purpose of the test is to measure how well the students know calculus.

> A **valid test** is one that measures the quality that it claims to be measuring.

If the test result is also going to be used by the teacher to predict the students' final IB Mathematics grade, then the teacher would hope it also has **criterion** validity. In other words it tests for a wider range of qualities than represented just by the content of the test. The calculus test is probably a valid test with regard to predicting the final Mathematics grade a student might achieve, but is unlikely to be a valid way of measuring how well they might do in their History exam.

A **reliable test** will produce similar results each time it is performed on similar samples. A reliable test might not be valid, but all valid tests should be reliable.

> A **reliable** test is one that will produce similar results when used with similar respondents.

You should know two tests for reliability:

**Test-retest.** The test is given twice to the same group. One disadvantage is that the responses in the second test might be influenced by the responses in the first.

**Parallel forms.** Instead of giving the respondents the same test a similar (parallel) one is given. This has the advantage that the respondents have not done the test before but more work is required in making two tests and it is difficult to ensure the two tests are of equal difficulty.

In both cases, if the test is reliable then there should be a high correlation between the two sets of results.

### Example 4.8.1

A school introduces an entrance test for students coming into the school to do the IB diploma. Eight pupils are selected and are ranked on how well they performed in the entrance test, where a rank of 1 means they did best in the test.

These ranks and the students' final points total for the IB diploma are shown in the table.

| Student | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Entrance test | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| IB points total | 28 | 30 | 32 | 30 | 35 | 29 | 36 | 42 |

By finding an appropriate statistic, state whether or not the test is valid for predicting the final points total.

You may assume a correlation coefficient greater than 0.7 will be evidence for a valid test.

**Solution**

Ranks for points total are:

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| 8 | 5.5 | 4 | 5.5 | 3 | 7 | 2 | 1 |

$r_s \approx 0.719$

There is sufficient evidence that the entrance test is a valid predictor of final points total.

The question does not specify whether to use PMCC or Spearman's. Because ranks are given and you are looking for an association rather than a linear relationship Spearman's should be used.

# 4.9 NON-LINEAR REGRESSION (AHL)

## You should know:

✔ the meaning of the terms least squares, regression and residual

✔ the sum of square residuals ($SS_{res}$) and the coefficient of determination ($R^2$) are measures of fit for a model

✔ that many factors affect the validity of a model and the coefficient of determination, by itself, is not a good way to decide between different models.

## You should be able to:

✔ find least squares regression curves using technology for linear, quadratic, cubic, exponential, power and sine regression

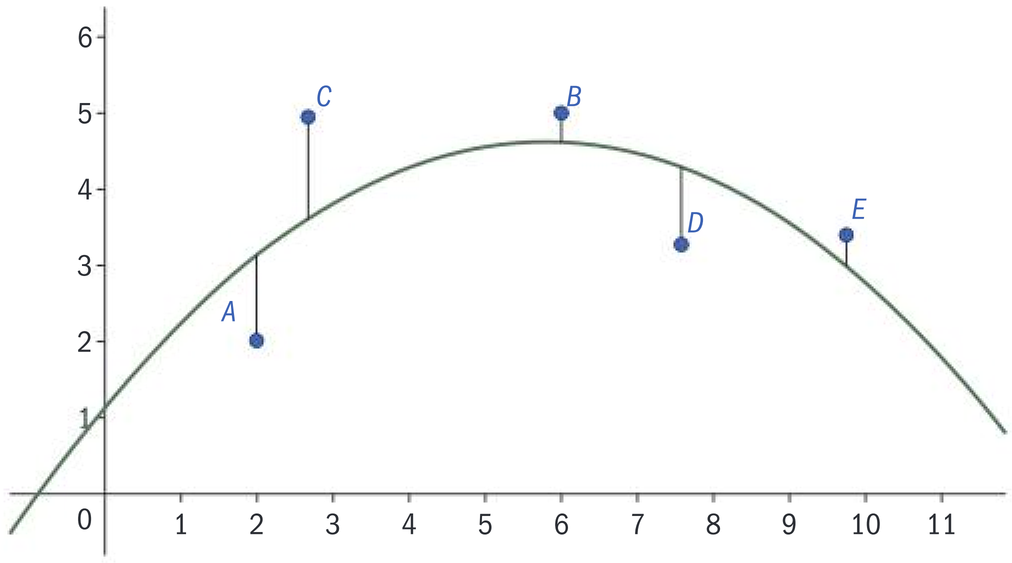
✔ evaluate $R^2$ using technology

✔ interpret the goodness of fit of a curve using the value of $R^2$, as well as the graph and the context for the model.

In section 4.6 you considered how well a straight line fitted a set of data and, when appropriate, how to find the best fit line using least squares linear regression.

In this section we look at finding the best fit curves for a set of data.

When fitting a curve, the GDC considers the residuals which are the distances from the points to the proposed curve. They are shown as the black vertical lines from the points $A$, $B$, $C$, $D$ and $E$ in the diagram below.




The best fit curve is described as the one with the smallest sum of the squares of the residuals, hence the name **least squares regression curve**.

The **coefficient of determination** ($R^2$) is a measure of how well a curve fits a set of points. $R^2 = 1$ implies a perfect fit. $R^2$ can be thought of as giving the proportion of variation in the dependent variable that can be predicted from the independent variable.

When deciding on the form of the appropriate curve you should consider various factors:

1 The context: Is the function periodic, does it have an asymptote, would you expect it to be continuously increasing or decreasing or have maximum or minimum points?

2 The purpose: Are you just trying to fit a curve as closely as possible or hoping to use it to make predictions?

3 If all other factors have been considered, the value of the coefficient of determination can be used to compare how well appropriate curves fit the points.

**Assessment tip**

In exams you will need to be able to fit linear, quadratic, cubic, exponential, power and sine functions.

**Note**

When considering a linear function, $R^2$ is equal to the square of the product moment correlation coefficient.

This links with the section on the modeling process, section 2.3.

## Example 4.9.1

An oven is left to cool and its temperature ($T$ °C) $t$ minutes after it has been turned off are recorded in the table shown.

| $t$ minutes | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $T$ °C | 100 | 78 | 58 | 41 | 30 | 22 |

(a) Find the least squares exponential regression line for the curve in the form $T = ae^{bt}$

(b) Give the value of the coefficient of determination from your GDC for this data and comment on the value.

(c) Give one reason why the exponential model proposed is unlikely to be a valid model in the context given.

### Solution

(a) $T = 104e^{-0.0616t}$

An equivalent form is $T = 104 \times (0.9402...)^t$

One or both of these forms can be obtained directly from the GDC.

To convert to the base e form write as

$T = 104 \times \left(e^{\ln(0.9402...)}\right)^t \approx 104e^{-0.0616t}$

(b) 0.998, this implies a very good fit to the data.

A high value for $R^2$ does not necessarily mean the function is the best model for the data.

(c) Because the asymptote is at $T = 0$ which would imply the temperature inside the oven will eventually reach 0° C.

### Assessment tip

If your GDC only gives one of these forms, make sure you know how to convert to the alternative form in case this form is required in the exam.

Sometimes it may be necessary to decide which of two given curves are the best fit. In this case you would calculate the sum of the square residuals yourself, and the curve with the smallest total is the one that fits the best.

## Example 4.9.2

(a) Find the least squares quadratic regression curve for the following four points:

(1, 5), (2, 7), (4, 7) and (9, 11)

The sum of square residuals for this curve is 1.2

A second model is proposed of the form $y = 3x^{0.5} + 2$

(b) By considering the sum of the square residuals for the new model, determine which curve is the best fit.

### Solution

(a) $y = 0.00961x^2 + 0.585x + 4.92$

This is obtained directly from the GDC.

(b) $(5-5)^2 + (7-6.24...)^2 + (7-8)^2 + (11-11)^2$

$0 + 0.757^2 + 1^2 + 0 = 1.57$

The quadratic curve is the best fit.

The values of $y = 3x^{0.5} + 2$ are calculated for $x = 1$, 2, 4, 9 and subtracted from the $y$-coordinates of the observed points to find the residuals.

# 4.10 CONFIDENCE INTERVALS AND HYPOTHESIS TESTING (AHL)

## You should know:

- ✔ that in a $\chi^2$ test all expected frequencies need to be greater than 5 and you should subtract 1 from the degrees of freedom for every parameter estimated
- ✔ when testing for a population mean or finding a confidence interval the normal distribution is used when $\sigma$ is known and the $t$-distribution when $\sigma$ is unknown
- ✔ a type I error is to reject the null hypothesis when it is true
- ✔ a type II error is to fail to reject the null hypothesis when it is not true.

## You should be able to:

- ✔ find confidence intervals for the mean of a normally distributed population
- ✔ test for a population mean when the population is normally distributed or the sample size large enough for the central limit theorem to apply
- ✔ perform a paired sample $t$-test
- ✔ test for proportion using the binomial distribution
- ✔ test for population mean using the Poisson distribution
- ✔ use technology to test the hypothesis that the population product moment correlation coefficient $\rho$ is 0 for bivariate normal distributions
- ✔ find type I and type II errors for tests involving the following distributions: normal with known variance, Poisson and binomial.

### The $\chi^2$ test revisited

In the Higher Level course there are two refinements for the $\chi^2$ test that you need to be aware of.

In both, the $\chi^2$ goodness of fit test and the $\chi^2$ test for independence the expected values need to be greater than five.

If you have an expected value less than 5 you need to combine two adjacent groups.

**Note**

This is because when working out the $\chi^2$ statistic you divide by the expected values. If these are too small then they distort the distribution.

**Assessment tip**

Your calculator will often not automatically display the expected value matrix, so make sure you always check it.

**Example 4.10.1**

Yenni was conducting a survey to see whether or not a person's favourite genre of film was independent of age. Her results are shown in the contingency table below.

| | Favourite genre | | | |
|---|---|---|---|---|
| Age | Action | Comedy | Horror | Thriller |
| 10–20 | 13 | 7 | 7 | 3 |
| 21–30 | 16 | 8 | 5 | 9 |
| 31–40 | 11 | 12 | 5 | 8 |
| Over 40 | 8 | 7 | 6 | 7 |

Determine whether or not there is sufficient evidence to reject the null hypothesis that favourite genre and age are independent.

**Solution**

The expected number of over 40s who like horror is 4.88 so we must combine the bottom two rows.

| | Favourite genre | | | |
|---|---|---|---|---|
| Age | Action | Comedy | Horror | Independent |
| 10–20 | 13 | 7 | 7 | 3 |
| 21–30 | 16 | 8 | 5 | 9 |
| Over 30 | 19 | 19 | 11 | 15 |

$p$-value $= 0.496$ so insufficient evidence to reject the null hypothesis at either 5 or even 10% that favourite genre of film and age are independent.

Notice that it does not matter that there is a cell in the observed table with an entry less than 5.

You could join adjacent rows or columns. Here it makes sense to join two rows as it is not clear whether Horror and Comedy or Horror and Thriller films would make a more natural pairing!

Note the question does not ask for a particular significance level. In this case it is not significant for any of the standard values, so an appropriate conclusion can be drawn.

If estimating population parameters from the data you subtract 1 from the number of degrees of freedom for each parameter estimated.

For example, if you are testing whether or not your data has come from a normal distribution you will need to know an estimate for the mean and standard deviation so you can work out the expected values. The best choices would be to use the sample mean as an estimate for the population mean and $s_{n-1}$ as an estimate for the population standard deviation. In this case the degrees of freedom would be $n-1-2=n-3$ where $n$ is the number of cells.

Example 4.10.2 illustrates the technique for a binomial distribution. This is the trickiest one as it is easy to be confused by the number of trials and the number of times the trial was repeated!

### Example 4.10.2

The owner of a garden centre suspects he is receiving a variable quality of seeds from a company. Though there is often a good rate of germination, in some batches it seems that fewer than expected germinate. To test this theory the garden centre owner plants four seeds from each of 100 batches and records how many germinate in each row. If the rate of germination is independent of the batch it came from, he would expect the number germinating to follow a binomial distribution. The results of his test are shown in the table.

| Number germinating | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency | 11 | 20 | 25 | 30 | 14 |

(a) Find the mean number of seeds germinating per sample of four.

(b) Find the expected number of seeds that would germinate for a binomial distribution with the same mean as found in part (a).

(c) Perform an appropriate test at a 5% significance level to see if there is evidence to suggest that the rate of germination does not follow a binomial distribution.

**Solution**

(a) $\bar{x} = 2.16$

This is obtained directly from the GDC.

(b) For a binomial distribution $\bar{x} = np$

$\Rightarrow 2.16 = 4p \Rightarrow p = 0.54$

Note: the $n$ here is the number of seeds in each trial.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X=x)$ | 0.0447 | 0.210 | 0.370 | 0.290 | 0.0850 |
| Expected value | 4.48 | 21.0 | 37.0 | 29.0 | 8.50 |

The estimate for $p$ can also be calculated by finding the total number of seeds that germinate and dividing by 400.

The probabilities are calculated using the binomial distribution function on the GDC and then multiplied by 100 to obtain the expected values.

(c) $H_0$: The seeds follow a binomial distribution.

$H_1$: The seeds do not follow a binomial distribution.

In this case we are not testing for a particular value of parameter.

| $x$ | 0 or 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Expected value | 25.5 | 37.0 | 29.0 | 8.5 |

Because one of the expected values is less than 5, two columns need to be combined.

Number of degrees of freedom $= 4 - 1 - 1 = 2$

As the value of $p$ has been estimated then the degrees of freedom reduce by 1.

$p$-value $= 0.0131 < 0.05$ so there is sufficient evidence to reject the claim that the rate of germination follows a binomial distribution.

## The *z*-test and the *t*-test

You have already met the $t$-test in section 4.7; at Higher Level it is looked at in more detail.

We first consider whether a single set of data could have come from a population with a given mean, rather than testing whether two sets of data could have come from populations with the same mean.

Consider a sample of size $n$ taken from a population which follows a normal distribution with a mean of $\mu$ and a standard deviation of $\sigma$, and let $\bar{X}$ be the distribution of the sample mean.

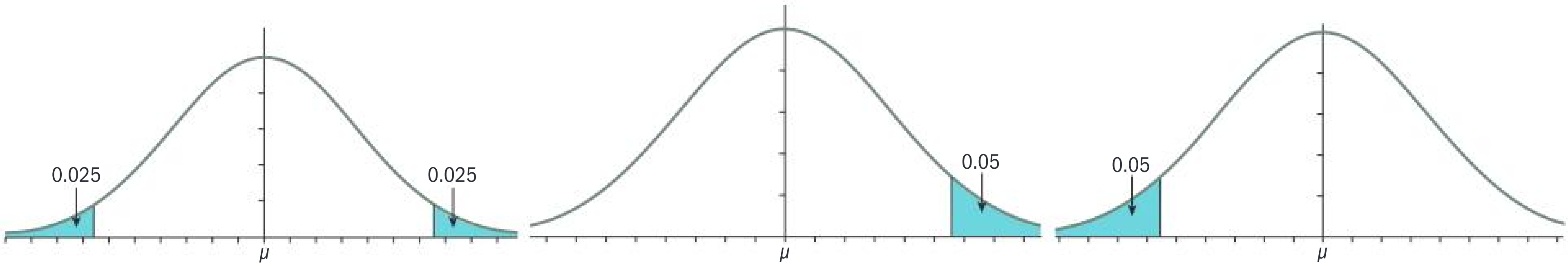
From section 4.3 we know that $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$

Under the null hypothesis that the population mean is equal to $\mu$ we would expect $\bar{X}$ to be close to this value.

We would therefore reject the null hypothesis if it was too far from the assumed value of $\mu$. 'Too far' is chosen to mean that the probability of $\bar{X}$ being this far away by chance is less than the significance level (for example 5%). From the diagram below we would reject the null hypothesis that the population mean is equal to $\mu$ at the 5% level if $\bar{x}$ fell in the **critical regions** shown.

**Note**

Because the variance for $\bar{X}$ is $\frac{\sigma^2}{n}$ the larger the sample the closer we would expect the sample mean to be to $\mu$.




two-tailed test, $H_1$: mean $\neq \mu$ one-tailed test, $H_1$: mean $> \mu$ one-tailed test, $H_1$: mean $< \mu$

If the population is normally distributed and we are given the population standard deviation in the question, then we can assume the sample mean is also normally distributed and we perform a *z*-test.

Even if we do not know that the population is normally distributed, if the sample size is large enough we can still assume that $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ by the central limit theorem, see section 4.3.

**Note**

Here you are given the population standard deviation, so you use the z-test.

### Example 4.10.3

A sample of size 10 is taken from a population which is normally distributed with a standard deviation of 2.

The sample mean will be used to test, at the 5% significance level, the hypotheses:

$H_0$: The population mean $\mu$ is equal to 5
$H_1$: The population mean $\mu$ is greater than 5

(a) Find the critical region for this test.

(b) It is found that $\bar{x} = 6.5$
State whether or not the null hypothesis should be rejected

(c) Find the *p*-value and verify this confirms the decision in part (b).

**Solution**

(a) We require that under $H_0$

$P(\bar{X} > a) = 0.05$

$a = 6.04$

The critical region is $\bar{X} > 6.04$

This is a one-tailed test and we will reject $H_0$ if the value of $\bar{x}$ is too large.

To find $a$ we use the inverse normal function on the GDC with $\mu = 5$ and $\sigma = \frac{2}{\sqrt{10}}$

(b) $\bar{x} = 6.5 > 6.04$ and hence we reject $H_0$: $\mu = 5$

$\bar{x} = 6.5$ lies in the critical region, so $H_0$ is rejected.

(c) $p$-value $= 0.008\,85 < 0.05$ so reject $H_0$

This is done using the *z*-test on the GDC and entering **statistics** rather than **data** (see below)

If the population is normally distributed but the population standard deviation is unknown and needs to be estimated, then the distribution of the sample mean is no longer normal and a *t*-test should be used.

**Note**

$s_{n-1}$ is denoted by $s_x$ on most calculators and is used as the estimator for the population standard deviation, $\sigma$. It should not be confused with the standard deviation of the sample, $s_n$ which is denoted by $\sigma_x$ on most calculators.

There are several options for testing for a mean on the GDC so make sure you can use all of them.

1 *z*-test or *t*-test? Use the *z*-test when the population standard deviation is known and the *t*-test when it has to be estimated from the data (using $s_{n-1}$).

2 Are you entering statistics ($\bar{x}$ and $n$ for the *z*-test and $\bar{x}$, $n$ and $s_{n-1}$ for the *t*-test) or are you entering the full set of data into the calculator?

3 Is it a two-tailed or a one-tailed test? And if one-tailed, are you testing for the mean being smaller than that given in the null hypothesis or larger?

### Example 4.10.4

Anders lives near shop A, where they sell muffins advertised as weighing 100 g. Anders is not convinced by this so he buys six muffins and records their weights in the table:

| Muffin | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Weight (g) | 95 | 98 | 94 | 100 | 99 | 92 |

(a) Perform a test to see whether there is evidence that the mean weight of the muffins ($\mu_A$) is less than 100 g. You may assume the weights of the muffins are normally distributed.

Another shop (B) opens nearby, so Anders buys 10 muffins from them and tests to see whether there is evidence that the mean weight of muffins sold at shop B ($\mu_B$) is greater than $\mu_A$.

The 10 muffins have a mean weight of 98 g and a standard deviation of 3 g.

(b) Perform the test and state whether there is evidence that the mean weight of muffins in shop B is greater than the mean weight in shop A.

You may assume the weights of muffins from shop B are normally distributed and standard deviations of both populations are equal.

**Solution**

(a) $H_0$: $\mu_A = 100$

$H_1$: $\mu_A < 100$

$p$-value $= 0.0177 < 0.05$ sufficient evidence to reject $H_0$: $\mu_A = 100$

Always give the null and alternative hypothesis.

As the standard deviation is estimated from the data this is a $t$-test. It is a one-tailed test as you are testing whether the mean is less than 100.

(b) $H_0$: $\mu_A = \mu_B$

$H_1$: $\mu_A < \mu_B$

$s_{n-1} = \sqrt{\frac{10}{9}} \times 3 = \sqrt{10}$

$s_{n-1}$ needs to be calculated for shop B as only the standard deviation of the sample $s_n$ was given.

For shop A

$\bar{x} = 96.33\ldots$, $s_{n-1} = 3.141\ldots$

$p$-value $= 0.162 > 0.05$

So insufficient evidence to reject $H_0$: $\mu_A = \mu_B$

Most GDCs do not allow a mix of data and statistics so the relevant data for shop A needs to be collected first and then entered into the 2-sample $t$-test option on the GDC.

**Note**

If prompted to include the degrees of freedom for a $t$-test, it is $n-1$.

**Assessment tip**

Often a question will give you the sample standard deviation ($s_n$). Don't forget to convert this to $s_{n-1}$ before entering it into the GDC by using $s_{n-1} = \sqrt{\frac{n}{n-1}}s_n$ (section 4.14 of the formula book).

**Assessment tip**

Tests will normally be done using the inbuilt functions on the GDC and quoting $p$-values. Critical regions are normally only used when finding type II errors.

Two sample $t$-tests were met in section 4.3. In the standard level course the data will always be given in full. In the higher level course you might need to enter the correct statistics instead.

**Assessment tip**

It is easy to make an error in entering data so make sure you write down any working you do so you can pick up method marks, even if your final answer is incorrect.

## Paired sample test

It is easy to miss a 'paired sample test' as two sets of data will be given so you may be tempted to use the two-sample $t$-test, but if the data is paired in an obvious way, you need to instead look at the difference between the paired values.

### Example 4.10.5

Five runners at a club spend two weeks at an overseas training camp. Before going, they record their times taken to run 400 m. After the camp they again time themselves over 400 m. The before and after results are shown below.

| Runner | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Time before, $T_1$ (seconds) | 55.2 | 60.1 | 56.8 | 53.5 | 59.3 |
| Time after $T_2$ (seconds) | 55.0 | 57.8 | 57.0 | 53.4 | 58.8 |

Perform a suitable test at the 5% significance level to see if there is evidence that the camp has improved their times. You may assume that the times are normally distributed.

**Solution**

(a) Let the mean difference in times $(T_1 - T_2)$ be $\mu_D$

$H_0$: $\mu_D = 0$

$H_1$: $\mu_D > 0$

| Difference | 0.2 | 2.3 | −0.2 | 0.1 | 0.5 |
|---|---|---|---|---|---|

$p$-value $= 0.131 > 0.05$ so insufficient evidence to reject the null hypothesis that the camp has not improved the running times.

Note: If $\mu_D$ was the mean difference of $T_2 - T_1$ then $H_1$: $\mu_D < 0$ would indicate an improvement.

The null hypothesis for a paired sample test will always be that the mean difference is equal to a particular value, normally 0.

The differences are entered into the GDC in the same way as for any other $t$-test.

**Assessment tip**

A $p$% confidence interval is often thought of as 'there is a $p$% chance that the mean is in this interval'. Though useful, this is not the correct definition, and so should not be quoted in an exam.

## Confidence intervals

Often instead of testing data against a particular mean it is more useful to give an interval or range within which the mean is likely to lie.

A $p$% confidence interval is such that if a sample was collected many times then the population mean $\mu$ would fall within the confidence interval $p$% of the time.

Confidence intervals are normally obtained directly from the GDC. If the population standard deviation is known then the normal (Z) distribution is used; if not, then the $t$-distribution is used, with $n-1$ degrees of freedom.

## Test for the correlation coefficient

It is possible to test whether or not two categories are independent by using the $\chi^2$ test. An alternative is to test to see if the product moment correlation coefficient between the two variables is equal to zero. This will be done when values, rather than frequencies, are given though it is important to remember that the correlation coefficient is only looking for a linear relationship.

A necessary assumption for the test of the correlation coefficient is that both variables are normally distributed.

The statistic used in this test is the sample correlation coefficient, $r$. In the same way that a sample mean not being equal to 0 does not imply the population mean is not 0, we cannot say that just because $r \neq 0$ the correlation coefficient for the whole population ($\rho$) is not 0. What we can say is that if it is sufficiently far from 0, then there is strong evidence that the $\rho$ is also not equal to 0, and hence the two variables are not independent.

**Assessment tip**

In exams, this test will be performed using the inbuilt function on the GDC and the data will always be given in the question.

**Example 4.10.6**

A teacher is convinced that time spent playing games on a computer affects the marks obtained in class assessments. To test this, he selects 8 students and asks them how much time each week they spend gaming and compares this with the scores obtained in the next assessment. The responses are shown in the table.

| Student | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Time on computer (hours) | 12 | 0 | 15 | 8 | 9 | 4 | 0 | 8 |
| Score in assessment | 34 | 61 | 55 | 38 | 78 | 65 | 46 | 82 |

Perform a test at the 10% significance level to see if there is any evidence to reject the hypothesis that there is no correlation between time spent on a computer and score in the assessment.

You may assume that the variables are normally distributed.

**Note**

Make sure you know where to find this test on your GDC. It is likely to be in a menu for regression tests or Linear regression tests.

**Solution**

$H_0$: $\rho = 0$ $H_1$: $\rho \neq 0$

$p$-value $= 0.836 > 0.1$ there is insufficient evidence to reject $H_0$ at the 10% significance level.

Even though the teacher is probably looking for a negative correlation the question is only asking whether or not $\rho = 0$

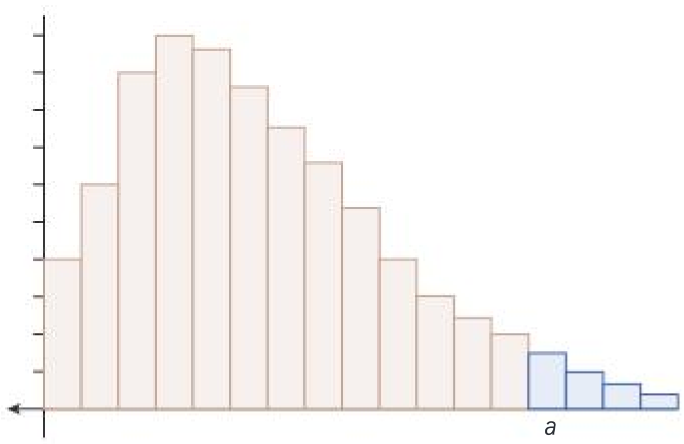
## Test for the mean of a Poisson distribution

It is possible to use the $\chi^2$ test to see if a sample could come from a Poisson distribution with a particular mean. But if a significant result is obtained it is not clear if it is due to the distribution not being Poisson or it being Poisson but with a different mean. However, if it is known the distribution is Poisson then it is easy to construct a test for the population mean ($\mu$).

Assume $X \sim \text{Po}(\mu)$. If we wish to test $H_0$: $\mu = \mu_0$ against $H_1$: $\mu > \mu_0$ then we reject $H_0$ if the value of $X$ obtained in our test is very unlikely if the mean were $\mu_0$.

So for a test with $H_1$: $\mu > \mu_0$ at a 5% significance level, the null hypothesis is rejected if $X \geq a$ where $a$ is the smallest value such that $P(X \geq a) \leq 0.05$ if $\mu = \mu_0$

In the diagram, the critical region is coloured blue. The sum of the probabilities coloured blue will be less than 0.05




There are two approaches to performing the test, either by finding the critical region or by calculating $P(X \geq x)$ where $x$ is the observed value.

**Example 4.10.7**

Over a long period of time it is known that the number of imperfections in printed material follows a Poisson distribution with a mean of 1.2 every 10 m.

After improvements on the machine used to print the material it is felt that the rate at which imperfections occur will be reduced.
To test this belief at the 5% significance level a sample of length 200 m is checked.

(a) Find the critical region for this test.

It is given that 19 imperfections are found in the sample.

(b) State the conclusion of the test.

(c) Find the $p$-value for the test and confirm this gives the same result as in part (b)

**Solution**

(a) Under $H_0$ the distribution is Po(24)

The mean is $1.2 \times 20 = 24$

$H_0$: $\mu = 24$ $H_1$: $\mu < 24$

To find the critical region enter the cumulative Poisson distribution as a function into your GDC and read off from the table the last value which has a probability of less than 0.05

| $X$ | $F1$ |
|---|---|
| 14 | 1.98253328235E⁻2 |
| 15 | 3.44000940596E⁻2 |
| 16 | 5.62622359137E⁻2 |

$P(X \le 15) = 0.0344$, hence the critical region is $X \le 15$

(b) $19 \ge 15$ so there is insufficient evidence to reject $H_0$

(c) $P(X \le 19) = 0.18026... > 0.05$

Hence there is insufficient evidence to reject $H_0$

It is more usual to perform the test by finding the $p$-value rather than finding the critical region.

## Test for a population proportion using the binomial distribution

**Assessment tip**

The tests for binomial and Poisson will always be one-tailed in the exam.

**Example 4.10.8**

A company employed by a political party is gathering evidence about its popularity in a town. They know that nationally the party is supported by 55% of the population but suspect it might be less in this town.

They select a randomly chosen sample of 25 people and find that 11 say they support the party. Is this sufficient evidence at the 5% level to believe support in the town is less than 55%?

**Solution**

$H_0$: $p = 0.55$ $H_1$: $p < 0.55$

Under $H_0$ the distribution would be B(25, 0.55), (assuming those in the sample were randomly chosen and independent.)

Let $X$ be the number of people in the sample who vote for the party.

If the numbers in our sample supporting the party were too small then this would be strong evidence to accept the alternative hypothesis.

$P(X \le 11) = 0.183 > 0.05$

There is insufficient evidence to reject $H_0$ at the 5% significance level.

In fact the probability is greater than 0.05 which means 11 is not in the critical region.

## Type I and type II errors

> The are two possible errors when performing a hypothesis test.
>
> Type I error: Rejecting $H_0$ when $H_0$ is true
>
> Type II error: Failing to reject $H_0$ when $H_0$ is not true.

If we let $C$ be the event of our sample statistic falling in the critical region, and so $H_0$ is rejected, we will write the probability of a type I error as $P(C \mid H_0)$.

The probaility of a type I error is, by definition, the significance level of the test when the distribution is continuous. For a discrete distribution such as the Poisson or binomial distribution they might be slightly different (see Example 4.10.9)

The probability of a type II error is more difficult. It can be calculated only if an alternative value to the parameter is known. We will denote the probability of a type II error as $P(C' \mid H_1)$, with $H_1$ standing for this alternative value.

The process for calculating a type II error is always the same:

1 Find the critical region, given $H_0$ is true.

2 Find the probability of **not** being in the critical region given $H_1$ is true, $P(C' \mid H_1)$, or $1 - P(C \mid H_1)$

### Example 4.10.9

The principal of a large school is trying to decide whether or not to introduce a homework club, and will not do so if fewer than 20% of the students would be willing to attend. She decides to ask a sample of 50 students to find data on the proportion $p$ of students in the school who would attend the club.

She will test the hypothesis $H_0$: $p = 0.2$ against $H_1$: $p < 0.2$, at the 5% significance level.

Assume the sample is random and independent.

(a) Find the critical region for the test.

(b) Write down the probability of a type I error, to 3 sf

(c) If the proportion of students in the school who would attend is 15%, find the probability of a type II error and explain what this would mean in context.

**Solution**

(a) Let $X$ be the number of people in the sample who say they will attend.

Assuming $H_0$ is true $X \sim B(50, 0.2)$

$H_0$ is rejected if $X$ is too small, so the critical region is $X \le a$ where

$P(X \le a) < 0.05$

$a = 5$

Hence the critical region is $X \le 5$

$a$ is the largest value of $X$ for which the probability of $X \le a$ is smaller than 0.05. This can be found by entering the cumulative binomial function into a GDC as a function.

| X | $Y_1$ |
|---|---|
| 0 | 1.4E–5 |
| 1 | 1.9E–4 |
| 2 | 0.0013 |
| 3 | 0.0057 |
| 4 | 0.0185 |
| 5 | 0.048 |
| 6 | 0.1034 |
| 7 | 0.1904 |
| 8 | 0.3073 |
| 9 | 0.4437 |

(b) 0.0480

A type I error is the probability of rejecting $H_0$ when it is true and hence is the $P(X \le 5) = 0.0480$

(c) Given $X \sim B(50, 0.15)$ the probability of being in the critical region is

$P(X \leq 5 \mid p = 0.15) = 0.21935$

so the probability of accepting $H_0$ is $1 - 0.21935 \approx 0.781$

This would be the probabilty of going ahead with the homework club even though there was not 20% support.

Some GDCs will allow the direct calculation of not being in the critical region, $P(X \geq 6 \mid p = 0.15)$

**SAMPLE STUDENT ANSWER**

A smartphone's battery life is defined as the number of hours a fully charged battery can be used before the smartphone stops working. A company claims that the battery life of a model of smartphone is, on average, 9.5 hours. To test this claim, an experiment is conducted on a random sample of 20 smartphones of this model. For each smartphone, the battery life, $b$ hours, is measured and the sample mean $\bar{b}$ is calculated. It can be assumed the battery lives are normally distributed with standard deviation 0.4 hours.

(a) State suitable hypotheses for a two-tailed test.

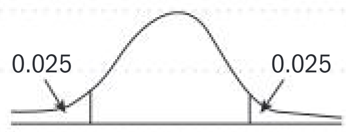
(b) Find the critical region for testing $\bar{b}$ at the 5% significance level.

It is then found that this model of smartphone has an average battery life of 9.8 hours.

(c) Find the probability of making a Type II error.

(a) $H_0: \mu = 9.5$ $\quad H_A: \mu \neq 9.5$

(b) $n = 20$




$X \sim N(b, 0.4^2)$

$X \sim N(9.5, 0.4^2)$

$z_1 = \text{invNorm}(0.975, 9.5, 0.4) = 10.28$

$z_2 = \text{invNorm}(0.025, 9.5, 0.4) = 8.716$

Critical region $= [8.72, 10.3]$

(c) $P(\text{Type II}) = P(\text{Accepting } H_0 / H_1 \text{ is true})$

for $\quad X \sim N(9.8, 0.4^2)$

$P(8.72 < X < 10.3 = 0.89)$

▼ The statistic is $\bar{X}$ not $X$ so, assuming $H_0$ is true the distribution needed is $\bar{X} \sim N\left(9.5, \frac{0.4^2}{20}\right)$.
The correct answer for part (b) is $\bar{X} > 9.68$ and $\bar{X} < 9.32$.

▲ Correct method using the inverse normal function to find the boundaries for the critical region.

▼ The student has given the acceptance region and not the critical region. The correct answer is [9.475, 9.525]

▲ Clearly setting out the correct expression for a type II error.

▼ Once again the distribution for $X$ and not $\bar{X}$ is used. The correct answer for part (c) is 0.0816.

▲ Apart from the wrong distribution the method is correct. The acceptance region is between the two critical values.

# 4.11 TRANSITION MATRICES AND MARKOV CHAINS (AHL)

## You should know:

- ✔ the initial state vector represents the initial state of the system
- ✔ the transition matrix represents the probabilities of transitioning between each state
- ✔ the definition of a regular Markov chain
- ✔ the steady state matrix gives the long-term probabilities of being in each state.

## You should be able to:

- ✔ represent the transitions in a system with transition diagrams or matrices
- ✔ interpret values in transition matrices
- ✔ calculate steady state and long-term probabilities by repeated multiplication of the transition matrix or by solving a system of equations.

You have studied contexts with sequences of independent events, such as the binomial distribution. However, in some contexts the probability of an event depends on previous events. Consider two simulated weather pattern sequences below. They each show how the weather changes state between fair (F) and rain (R) on 19 consecutive days.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | R | R | R | F | F | F | F | F | F | R | F | F | F | R | R | R | R | R | R |
| *B* | F | R | F | R | F | F | R | R | F | R | R | R | F | R | R | F | F | R | F |

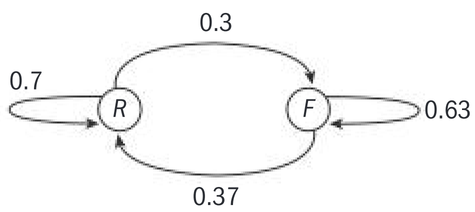
Sequence *A* consists of 19 trials where the probability of F or R tomorrow is dependent on what the weather was today. Sequence *B* consists of 19 independent trials where the probability of F or R are equally likely. This is not an appropriate model for a weather system; we do not predict weather with independent events, because when we predict tomorrow's weather, we use the state of the weather today in our judgement. This is a simple example of a Markov chain.

> A sequence of states in which the probability of the next state occurring depends only on the current state is a Markov chain.

Markov chains are presented in transition diagrams and in transition matrices. Both representations give the probabilities of the states changing to a future state given their current state. Sequence *A* was generated by a computer simulation using the probabilities given in the following transition matrix **T** and its transition diagram:

Current state

$$\mathbf{T} = \text{Future state}\ \begin{array}{c c} & \begin{array}{cc} \mathrm{R} & \mathrm{F} \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{F} \end{array} & \begin{pmatrix} 0.7 & 0.37 \\ 0.3 & 0.63 \end{pmatrix} \end{array}$$



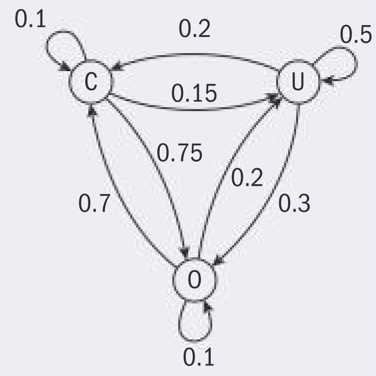

For example, the probability it is fair tomorrow given that it is raining today is 0.3. Another way to say this is that the transition probability from R to F is 0.3.

The probabilities in each column of a transition matrix **T** add to one. Each column of a transition matrix shows the probabilities of transition from its current state to each future state.

With transition matrices, you can make predictions about future states as shown in Example 4.11.1

### Example 4.11.1

Electric cars are rented by ZEcar in a large city for commuter journeys between three zones: Central (C), Urban (U) and Outer (O). In one rental period, customers can pick up a car from a zone before 09:00 and return their car to a zone after 17:00. The management of ZEcar construct this transition diagram to show the probabilities of a car transitioning from one zone to another over one rental period.




(a) Represent the transition probabilities in a transition matrix **T** in which the columns represent the location of an electric car at the start of a day and the rows represent the location of the car at the end of a day.

(b) Find $\mathbf{T}^2$. Interpret the element in the second row and the third column of $\mathbf{T}^2$.

**Solution**

(a)

Take care to represent the rows and columns as the question requires.

$$\mathbf{T} = \text{Future zone}\ \begin{array}{c} \\ \text{C} \\ \text{U} \\ \text{O} \end{array} \begin{array}{c} \text{Current zone} \\ \begin{array}{ccc} \text{C} & \text{U} & \text{O} \end{array} \\ \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.15 & 0.5 & 0.2 \\ 0.75 & 0.3 & 0.1 \end{pmatrix} \end{array}$$

(b) $\mathbf{T}^2 = \begin{pmatrix} 0.565 & 0.33 & 0.18 \\ 0.24 & 0.34 & 0.225 \\ 0.195 & 0.33 & 0.595 \end{pmatrix}$

Find $\mathbf{T}^2$ with technology.

The element 0.225 represents the probability that a car beginning in O at the start of a day is found in U at the end of the next day.

The result can be confirmed by considering the meaning of the product of the row and column shown:

$$\begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.15 & 0.5 & 0.2 \\ 0.75 & 0.3 & 0.1 \end{pmatrix}\begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.15 & 0.5 & 0.2 \\ 0.75 & 0.3 & 0.1 \end{pmatrix}$$

The product of the column and row shown finds the total probability that a car transitions from O to U by summing the probabilities that of the transitions O-C-U, O-U-U and O-O-U.

Just as the transition probability from O to U after two transitions is 0.225, the same process can be applied $n$ times: the elements of $\mathbf{T}^n$ give all the transition probabilities between each state after $n$ transitions. This can be applied to make long term predictions.

Each column of a transition matrix $\mathbf{T}^n$ shows the probabilities of transition from its current state to each future state after $n$ transitions.

For example, ZEcar begin with putting 50, 50 and 100 cars in zones C, U and O respectively. With this information you can write down

$s_0 = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.5 \end{pmatrix}$, the **initial state probability matrix** which gives the initial probabilities of finding a randomly chosen car in each of the zones. Hence you can predict the distribution of the cars after 7 transitions by finding $s_7 = \mathbf{T}^7 s_0$ where

$$\mathbf{T}^7 s_0 = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.15 & 0.5 & 0.2 \\ 0.75 & 0.3 & 0.1 \end{pmatrix}^7 \begin{pmatrix} 0.25 \\ 0.25 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.360 \\ 0.260 \\ 0.379 \end{pmatrix}$$

Therefore, by comparing $s_7$ to $s_0$, the management of ZEcar can identify a trend of migration of their cars from zone O to zone C over one week, helping them to manage and plan their business.

> The elements of $s_n = \mathbf{T}^n s_0$ predict the probabilities of being in each state, $n$ transitions after the initial state probability matrix $s_0$.

Just as values of geometric sequence $u_n = u_1 r^{n-1}$ exhibit convergence or divergence according to the value of $r$, the behavior of the sequence $s_n = \mathbf{T}^n s_0$ depends on $\mathbf{T}$. Markov chains are classified according to the long term behavior of $\mathbf{T}^n$: A regular Markov chain with transition matrix $\mathbf{T}$ is one for which there exists $n \in \mathbb{Z}^+$ for which all the elements in $\mathbf{T}^n$ are greater than zero, whereas other types of Markov chains outside the syllabus include absorbing and periodic, with applications in psychology and biology respectively.

For example, if $\mathbf{T} = \begin{pmatrix} 0.4 & 0.75 \\ 0.6 & 0.25 \end{pmatrix}$ then some members of the sequence

$\mathbf{T}^n$ are $\mathbf{T}^3 = \begin{pmatrix} 0.5365 & 0.579375 \\ 0.4635 & 0.420625 \end{pmatrix}$,

$\mathbf{T}^9 = \begin{pmatrix} 0.55552052638281 & 0.55559934202149 \\ 0.44447947361719 & 0.44440065797851 \end{pmatrix}$ and

$\mathbf{T}^{100} = \begin{pmatrix} 0.555555555556 & 0.555555555556 \\ 0.444444444444 & 0.444444444444 \end{pmatrix}$

The elements of $\mathbf{T}^n$ converge to values found in the **steady state probability matrix** $\mathbf{P} = \begin{pmatrix} \frac{5}{9} & \frac{5}{9} \\ \frac{4}{9} & \frac{4}{9} \end{pmatrix}$.

The consequences of knowing $P$ are:

- $s_n = \mathbf{T}^n s_0$ becomes $s_n = \mathbf{P}s_0$ in the long term. This means that applying $\mathbf{P}$ predicts the **long-term probability matrix** $\mathbf{P}s_0$ of being in each state.
- Since the elements of $\mathbf{T}^n$ converge to those of $\mathbf{P}$, $s_{n+1} = \mathbf{T}(\mathbf{P}s_0) = \mathbf{P}s_0$. This is because $\mathbf{P}s_0$ is the long-term probability matrix, so carrying out another transition does not make a change to the state.
- The solution to $\mathbf{T}\boldsymbol{u} = \boldsymbol{u}$ is independent of the values of $s_0$.

$\mathbf{T}(\mathbf{P}s_0) = \mathbf{P}s_0$ can be written as $\mathbf{T}\boldsymbol{u} = \boldsymbol{u}$ where $\boldsymbol{u}$ is the long-term probability vector. Therefore $\boldsymbol{u}$ is the eigenvector of $\mathbf{T}$ corresponding to the eigenvalue of 1. See also section 1.6

$\mathbf{T}\boldsymbol{u} = \boldsymbol{u}$ can also be solved by solving a system of linear equations. See also section 1.5

These consequences are applied in the following example.

**Example 4.11.2**

Two smartphone companies, Alfa and Better, operate in a rural area. Customers sign up for yearly subscriptions. Each year, 25% of Alfa customers switch to Better, and 15% of Better customers switch to Alpha.

(a) Write down a transition matrix **T** representing the switches made between the two companies in one year, assuming that there are no additional losses or gains of customers.

(b) Find the steady state probability matrix **P** by repeated multiplication of **T**.

(c) Hence find the long-term probability matrix for each initial state:

(i) 80% of the smartphone customers subscribe with Alfa

(ii) 50% of the smartphone customers subscribe with Better

(d) Solve the equation $\mathbf{T}\boldsymbol{u} = \boldsymbol{u}$, where $\boldsymbol{u}$ is the long-term probability matrix, by solving a system of equations.

**Solution**

(a)

Current company

$$\mathbf{T} = \text{Future company} \begin{array}{c c} & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} 0.75 & 0.15 \\ 0.25 & 0.85 \end{pmatrix} \end{array}$$

It is not essential to label your transition matrix, but some labelling may help you plan your work and be consistent with the meaning of the column and row headings.

(b) $$\begin{pmatrix} 0.75 & 0.15 \\ 0.25 & 0.85 \end{pmatrix}^{100} = \begin{pmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{pmatrix} = \mathbf{P}$$

Experiment with high powers of the matrix with your GDC until you can identify a steady state.

(c) (i) $$\begin{pmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{pmatrix}\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

(d) $$\begin{pmatrix} 0.75 & 0.15 \\ 0.25 & 0.85 \end{pmatrix}\begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} q \\ 1-q \end{pmatrix} \Rightarrow$$

$$0.75q + 0.15 - 0.15q = q \Rightarrow$$

$$q = \frac{0.15}{0.4} = 0.375. \text{ Hence } \boldsymbol{u} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

$0.25q + 0.85(1 - q) = 1 - q$ will give the same answer.

# SL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

**1.** Sheldon is comparing the cost of identical soft drinks in Germany and Italy. As he travels through the two countries, he collects the following data in euros.

| Germany | Italy |
|---|---|
| 1.80 | 1.50 |
| 2.00 | 1.70 |
| 1.70 | 3.00 |
| 1.80 | 1.90 |
| 2.00 | 1.60 |
| 2.10 | 1.70 |
| 2.20 | 1.80 |
| 1.60 | 1.50 |
| 1.80 | |

**a.** Find the sample means for the costs in the two countries.

Sheldon decides to use a two-sample pooled $t$-test at the 5% significance level to test the hypothesis that drinks are cheaper on average in Italy than in Germany.

**b.** State whether this is a one-tailed or a two-tailed test

**c. i.** State the null and alternative hypotheses.

**ii.** Find the $p$-value for the test.

**iii.** Write down the conclusion of the test, justifying your answer.

**2.** Student lockers in a corridor of Wiles Academy are numbered with consecutive integers starting at 21 and ending at 53. At the end of term, a locker is chosen at random. Find the probability that the locker number is:

**a.** 49 or higher

**b.** 26 or less

**c.** a multiple of 9

**d.** not a multiple of 9

**e.** a factor of 120

**3.** A survey in a large city is conducted in order to find out more about the lives of the people living there. The number of beds in a randomly chosen home, $T$, is modelled by the discrete probability distribution represented in the table.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $P(T=t)$ | 0.47 | $3a$ | 0.17 | 0.09 | $a$ | 0.02 | 0.01 |

**a.** Find the value of $a$.

**b.** Find the probability that a randomly selected home has exactly 4 beds, given that it has more than 2.

**4.** A restaurant has four choices of main course. Lisa thinks that the choice of main course will be independent of gender, whereas Michelle believes that it is not. They decide to do a test and record which course is chosen by 60 randomly selected customers.

The results are shown in the table.

| | Male | Female | Total |
|---|---|---|---|
| Beef | $a$ | 8 | 20 |
| Fish | 5 | 9 | 14 |
| Vegetarian | 8 | $b$ | 15 |
| Seafood pasta | 3 | 8 | 11 |
| Total | 28 | 32 | 60 |

**a.** Write down the values of $a$ and $b$.

Lisa and Michelle decide to perform a $\chi^2$ test for independence at the 5% significance level.

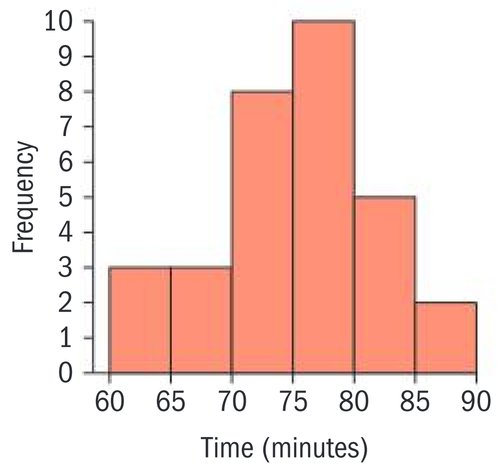
**b. i.** Write down the null and alternative hypotheses for the test.

**ii.** Find the $\chi^2$ statistic for the test.

The critical value for a 5% significance level is 7.81

**iii.** Use your answer to part **ii** to write down the conclusion of the test, clearly justifying your answer.

**5.** A teacher asks the students in a class to record how many minutes they spent working on an extended task, designed to take between one and two hours. The results are shown in the diagram.




**a.** Find the number of students in the class.

**b.** Find estimates for the mean and standard deviation of the time taken by the students.

In addition to the time spent at home on the task, the whole class also spent 30 minutes working on it during their lesson.

**c.** Write down the mean and standard deviation for the total time spent on the task.

**6.** A fair tetrahedral die numbered 2, 3, 5 and 8 is thrown, and a fair cubical die numbered 1, 1, 2, 3, 5 and 8 is thrown. $T$ is the total of the two numbers thrown.

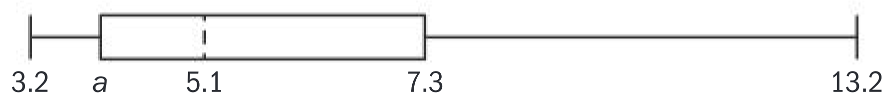
**a.** Find the probability that $T$ is a prime number.

**b.** Find the probability that $T$ is a prime number or a factor of 10.

**c.** Find the probability that $T$ is a prime number or a multiple of 4.

## GROUP 2

**7.** Carole recorded the distances in kilometres that she ran over the course of 6 weeks and showed the results in this box and whisker plot.




The interquartile range for the lengths of the runs is 3.3 km.

**a.** Find the value of $a$.

**b.** Find whether or not Carole should have shown the maximum length as an outlier, fully justifying your answer.

**8.** In a class of 30 students, 10 study Economics, 18 study Art and 3 students study neither Economics nor Art.

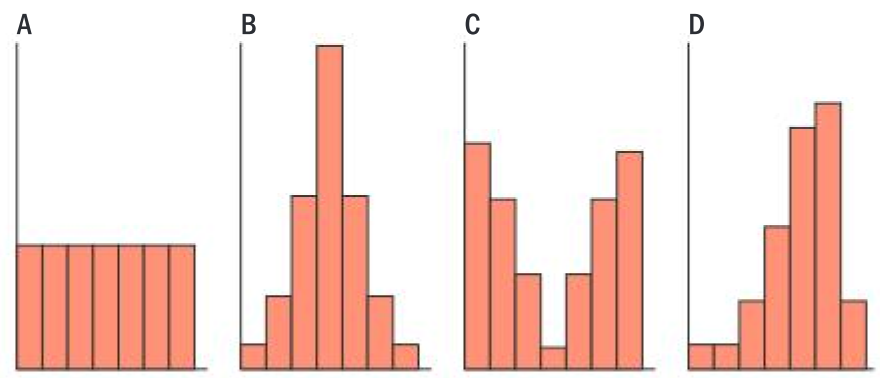
**a.** Find the probability that a randomly selected student from this class studies both Economics and Art.

**b.** Given that a randomly selected student studies Economics, find the probability that the student also studies Art.

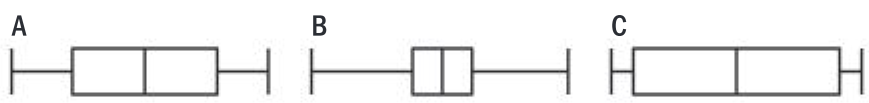
**c.** Determine if the events *studies Economics* and *studies Art* when choosing a student at random are independent. Justify your reasoning.

**9.** The four populations A, B, C and D are the same size and have the same range.

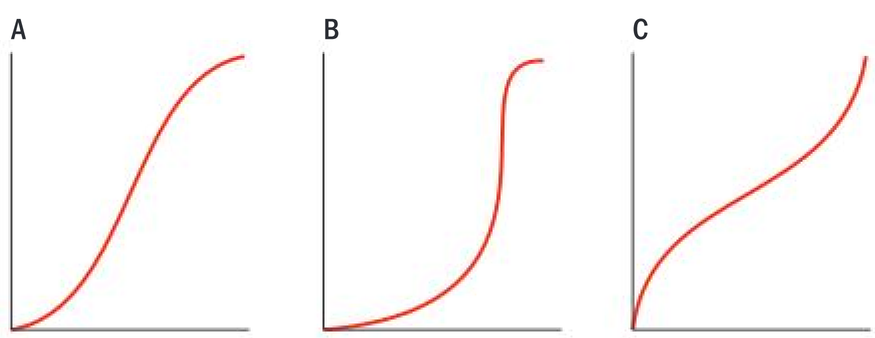
Frequency histograms for the four populations are shown here:




**a.** Each of the three box and whisker plots below corresponds to one of the four populations. Match each box and whisker plot with the correct frequency diagram.




**b.** Each of the three cumulative frequency diagrams below corresponds to one of the four populations. Match one with the correct frequency diagram.




**10.** A machine refills 500 ml washing liquid bottles in order to reduce waste. The volume refilled by the machine is normally distributed with mean 499.3 ml and standard deviation 3.7 ml.

In a quality control procedure, 4% of the bottles are rejected for containing not enough washing liquid. Find the minimum volume, to the nearest ml, that a bottle must contain in order to be accepted.

**11.** Natalia wanted to perform a $t$-test to see if the mean distance from school for students in grade 10 was greater than the mean distance from school for the students in grade 9.

To do so, she decided to collect a sample of 50 students from grades 9 and 10 and to ask them how far they lived from school.

She obtained a list of the students from the principal, arranged alphabetically. Grade 9 had 85 students and grade 10 had 115 students.

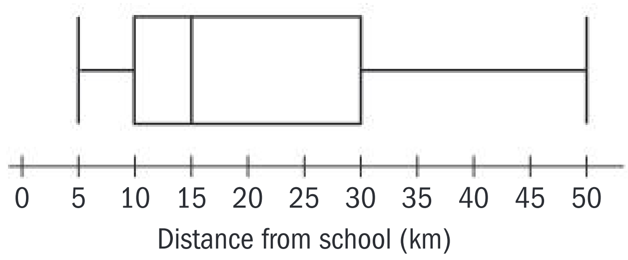
Natalia decided to use stratified sampling to find the number to be selected from each year group.

**a.** Find how many students she should select from:

**i.** grade 9 **ii.** grade 10

**b.** Describe how she could use systematic sampling to select a random sample from grade 9.

Before carrying out her $t$-test, Natalia plotted her values from grade 10 as a box and whisker plot, shown here.




**c.** Explain why the box and whisker plot indicates that the $t$-test might not be valid.

**12.** In Ted's class, the number of vowels in each student's name is shown in this table.

| Number of vowels | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Frequency | 1 | 6 | 4 | 1 | 1 |

Each student's name is written on a card and put in a bag. In an experiment, one card is chosen from the bag.

**a.** The experiment is carried out once. Find the probability that there are at least 3 vowels in the student's name.

**b.** The experiment is repeated 25 times. Each time a card is chosen it is put back in the bag. Find the expected number of times a student with at least 3 vowels in his/her name is chosen.

**c.** The experiment is repeated twice, but without replacing the card after the first selection. Find the probability that exactly one of the two names chosen has at least 3 vowels.

## GROUP 3

**13.** Helen is very keen to improve the mean score in her tests. The tests are out of 50 and she scored 8 out of 50 on her first test. She aims to improve her score by 2 marks in every test until she reaches full marks.

Assuming she succeeds in her aim,

**a.** write down an expression for her score in the $n$th test she takes

**b.** find the number of tests she must take to have mean score over 20.

**14.** The number of points scored in a game $Y$ is modelled by the discrete probability distribution:

| $y$ | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|
| $P(Y = y)$ | 0.2 | $p$ | $q$ | 0.1 | 0.3 |

It is known that $E(Y) = 6.2$

**a.** Find the values of $p$ and of $q$.

**b.** If each point scored gains a cash prize of $y$ USD, find the price of a ticket which would make this a fair game.

**c.** The game is played 10 times. Find the probability that at least 7 USD is won in exactly four of the games.

**15.** Frances grows eight sets of plants in controlled conditions, adding different amounts of nutrients to each set. The data in the table shows the amount of nutrient ($m$) in grams given to each plant and the average height ($h$), in centimetres, of each set of plants after six weeks.

| | Mass ($m$) | Height ($h$) |
|---|---|---|
| A | 0 | 5.2 |
| B | 5 | 9.2 |
| C | 10 | 14.4 |
| D | 15 | 18.5 |
| E | 20 | 21.2 |
| F | 25 | 21.0 |
| G | 30 | 21.0 |
| H | 35 | 21.6 |

**a.** Calculate Spearman's rank correlation coefficient and comment on your result.

Frances decides that it would be better to model the data as a piecewise function.

In this model, the height increases linearly for $0 \le m \le 20$ and is approximately constant for $m > 20$

**b.** Find the line of best fit for $0 \le m \le 20$ in the form $h = am + b$

**c.** Interpret the meaning of $a$.

For $m > 20$, the equation is $h = c$

**d.** Given the function is continuous at $m = 20$, find the value of $c$.

**16.** David is designing computer games involving two experiments on his laptop. The outcomes of experiment A and of experiment B are shown in the tables below, with the probability of each outcome.

| Outcome for A | 1 | 3 | 6 | 10 | 15 |
|---|---|---|---|---|---|
| Probability | 0.1 | 0.1 | 0.15 | 0.5 | 0.15 |

If the outcome of experiment A is a multiple of 3, the player scores a point. All other outcomes score zero.

| Outcome for B | 2 | 3 | 5 | 7 | 11 |
|---|---|---|---|---|---|
| Probability | 0.03 | 0.02 | 0.05 | 0.4 | 0.5 |

If the outcome of experiment B is a factor of 10, the player scores a point. All other outcomes score zero.

$X$ is the number of points scored in 20 trials of A.
$Y$ is the number of points scored in 100 trials of B.

**a.** Write down the distributions of $X$ and of $Y$ stating any assumptions you make.

**b.** Hence show that the expected values of $X$ and of $Y$ are both equal to 8.

**c.** Find the variances of the distributions of $X$ and of $Y$.

**d.** Hence compare and contrast the distribution of $X$ with that of $Y$.

**17.** Brian always turns up to the stadium of his football team exactly 15 minutes before the game kicks off. He knows that the time taken, $T$, for him to queue, gain entry to the stadium and then find his seat is distributed normally with mean 10 minutes and standard deviation 2.5 minutes.

**a.** Find the probability that Brian will find his seat before the game kicks off.

**b.** In a season of 20 games, find the probability that Brian is not in his seat at the start of the game on:

**i.** at least two occasions

**ii.** exactly two occasions.

State any assumptions that you make.

**18.** On her way to and from work, Julia passes through a set of traffic lights.

For a period of 60 days, she records the number of times each day she has to stop at traffic lights ($X$). Her results are shown in the table.

| Number of times stopped ($X$) | 0 | 1 | 2 |
|---|---|---|---|
| Frequency | 12 | 24 | 24 |

She decides to use this data to test the hypothesis that $X$ is distributed binomially, with the probability of being stopped equal to 0.5

**a.** Write down the null and alternative hypotheses for Julia's test.

**b.** Find the expected values for the number of times she has to stop if the null hypothesis is true.

**c.** Perform the test and state why the null hypothesis should be rejected at the 5% level.

**d.** Give one reason why the null hypothesis might not be true.

## PAPER 2 QUESTIONS

**1.** Adam is a beekeeper who collected data about monthly honey production in his beehives. The data for six of his hives is shown in this table.

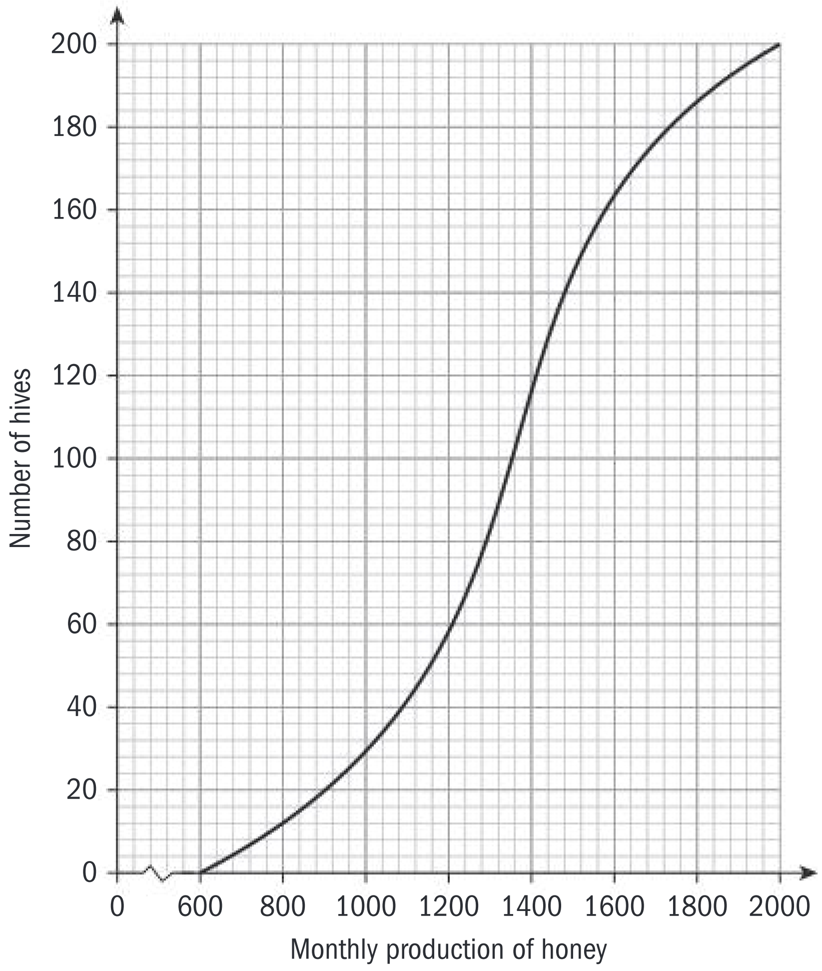
| Number of bees ($N$) | Monthly honey production, in grams ($P$) |
|---|---|
| 190 | 900 |
| 220 | 1100 |
| 250 | 1200 |
| 285 | 1500 |
| 305 | 1700 |
| 320 | 1800 |

The relationship between the variables is modelled by the regression line with equation $P = aN + b$.

**a.** Write down the value of $a$ and of $b$.

**b.** Use this regression line to estimate the monthly honey production from a hive that has 270 bees.

Adam has 200 hives in total. He collects data on the monthly honey production of all the hives. This data is shown in the cumulative frequency graph.




Adam's hives are labelled as low, regular or high production, as defined in this table.

| Type of hive | Monthly honey production, in grams ($P$) |
|---|---|
| low | $P < 1080$ |
| regular | $1080 < P \le k$ |
| high | $P > k$ |

**c.** Write down the number of low production hives.

Adam knows that 128 of his hives have a regular production.

**d.** Find:

**i.** the value of $k$

**ii.** the number of hives that have a high production.

**e.** Adam decides to increase the number of bees in each low production hive. Research suggests that there is a probability of 0.75 that a low production hive becomes a regular production hive. Calculate the probability that 30 low production hives become regular production hives.

**2.** A health survey is carried out in a large city. The heights of 16 to 18-year-old students are measured. The heights of the females are normally distributed with mean 163 cm and standard deviation 10 cm. The heights of the males are normally distributed with a mean of 170 cm and a standard deviation of 11 cm.

**a.** Find the probability that a randomly selected male is taller than 175 cm.

**b.** Given that 15% of the females are shorter than $t$ cm, find $t$.

**c.** Find the inter-quartile range of the heights of the males.

The students surveyed are 51% female and 49% male. If the person is female, the probability they are taller than 175 cm is 0.115.

A student is selected at random.

**d.** Find the probability that the student is taller than 175 cm.

**e.** Given that the student is taller than 175 cm, find the probability that the student is male.

# HL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

**1.** A sample is taken from a population which is normally distributed with a mean equal to $\mu$. The values in the sample are given in the table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 12.1 | 14.2 | 10.7 | 9.8 | 15.6 | 10.6 | 11.8 | 17.2 | 9.4 | 10.5 |

**a.** State the distribution of the sample mean.

**b.** Find the 95% confidence interval for $\mu$.

**2.** **a.** Given that $A \sim \mathrm{Po}(3.1)$ and $B \sim \mathrm{Po}(2.7)$, find $\mathrm{E}(A - 2B)$ and $\mathrm{Var}(A - 2B)$.

**b.** Hence explain why $A - 2B$ does not follow a Poisson distribution.

**3.** For his Science Internal Assessment, Antoine has collected 40 samples and he records how far his results are from the expected results. He wants to test that the difference is due to random error with a mean of 0.

Let $\bar{X}$ be the the sample mean of Antoine's data.

a. State why $\bar{X}$ can be assumed to be normally distributed.

The sample mean for Antoine's data is –0.54 and the standard deviation is 1.2

b. Test, at the 5% signifcance level, whether or not Antoine can assume his errors have a mean of 0.

c. State one implication of your result in the context of the question.

4. Given that $X \sim \text{Po}(4.9)$, $Y \sim \text{B}(7, 0.35)$ and $Z \sim \text{N}(61, 25)$ are independent random variables, find:

a. $\text{E}(2Y + 7)$

b. $\text{Var}(8 - 2X)$

c. $\text{Var}(X + 2Y - Z)$

5. Vedant wishes to test whether or not a coin is fair, or whether it is more likely to show Heads. He decides to flip it 50 times and to record the number of Heads that appear.

Let $p$ be the probability of the coin showing Heads.

a. Write down the null and alternative hypotheses.

b. Find the critical region for a test at the 5% significance level.

c. Given that $p = 0.6$ find the probability of a type II error.

6. *Life drinks* produces one-litre bottles of orange juice. The company wants to determine the amount of vitamin C in milligrams in these bottles.

A random sample of ten bottles is analysed and the results are as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 243 | 251 | 237 | 252 | 257 | 254 | 248 | 250 | 239 | 252 |

Find unbiased estimates of the population mean and variance of the amount of vitamin C in the one-litre bottles.

## GROUP 2

7. Ana believes that the number of goals scored by her favourite football team follows a Poisson distribution. To test her hypothesis, she records the number of goals they score in each game during a 38 game season and the results are shown in the table.

| Number of goals | 0 | 1 | 2 | 3 | 4 | ≥5 |
|---|---|---|---|---|---|---|
| Frequency | 9 | 14 | 12 | 2 | 1 | 0 |

Perform an appropriate test to see if the number of goals scored follows a Poisson distribution.

8. The number of admissions per hour on Saturday evenings to a hospital emergency room follows a Poisson distribution with mean 4.7. Vicky works a shift of 4 hours every Saturday evening.

a. Find the probability that on a given Saturday evening there will be at least 20 admissions to the emergency room during Vicky's shift, stating any assumptions that you make.

Vicky works on 5 consecutive Saturday evening shifts.

b. Find the most likely number of Saturday evening shifts on which there are at least 20 admissions to the emergency room on 5 consecutive Saturday evenings, stating any assumptions that you make.

9. In office A, the number of IT problems that occur in a week follows a Poisson distribution with a mean of 4.2

In office B, the number of IT problems follows a Poisson distribution with a mean of 3.9

a. Write down the distribution of the total number of weekly IT problems in the two offices, stating any assumptions you make.

The firm introduces new software which it is hoped will help reduce the problems.
In the next two weeks there are only 12 IT problems reported.

b. Test, at the 5% significance level, whether this is sufficient evidence that the number of IT problems has been reduced.

**10.** A fabric manufacturer produces high quality silk in sheets of area 10 square metres. During the manufacturing process, flaws in the fabric occur at the rate of 1.8 flaws per 10 square metres. It is assumed that the number of flaws per sheet is modelled by a Poisson distribution.

**a.** Find the probability that a randomly chosen sheet contains at least one flaw.

Silk sheets with no flaws earn a profit of €350. Silk sheets with one flaw are sold at a discount and earn a profit of €150. Any sheets with more than one flaw incur a loss of €100.

**b.** Find the expected profit gained by manufacturing 70 silk sheets, to the nearest €100

**11.** A market research company is conducting a survey to find the proportion of people in a town who would visit a new coffee shop. If the proportion is greater than 0.1 then the company would be likely to build it.

The research company visits the town centre and asks 10 people from each of five age ranges whether or not they would visit the coffee shop and the results are shown in the table below.

| Age ($x$) | Would visit |
|---|---|
| $x < 20$ | 8 |
| $20 \le x < 30$ | 4 |
| $30 \le x < 40$ | 3 |
| $40 \le x < 50$ | 2 |
| $x \ge 50$ | 3 |

**a.** Name this method of sampling a population.

**b.** Find the proportion of the sample who said they would visit the coffee shop.

**c.** Test, at the 1% significance level, the hypothesis that the proportion, $p$, of people in the town who would go to the coffee shop is greater than 0.2

**d.** Give one criticism of the question asked in the survey.

**12.** The effects of two types of fertilizer on the growth of corn are being compared in a study. Corn plants fertilized with type A have a mean height of 253 cm and a standard deviation of 10 cm. Corn plants fertilized with type B have a mean height of 250 cm and a standard deviation of 12 cm.

**a.** Find the probability that a sample of 70 corn plants fertilized with type A has a mean greater than 255 cm.

**b.** Find the probability that a sample of 70 corn plants fertilized with type A has a mean greater than a sample of 80 corn plants fertilized with type B.

## GROUP 3

**13.** A factory produces bags of flour labelled 1 kg. Over a long period time, it is known that the standard deviation of the weight of the bags is 50 g.

The bags are regularly checked to ensure that the mean weight is 1 kg rather than less than 1 kg.

In these checks, 15 bags are taken and the average weight measured. The null hypothesis for the test is that $\mu = 1000$ g and the alternative hypothesis is that $\mu < 1000$ g.

If the mean is less than $a$ grams then the null hypothesis is rejected and the factory is fined.

**a.** Find the value of $a$ if the probability of a type I error is 0.05

The owner of the factory has set the machine to produce bags with an average weight of 980 g.

**b.** Find the probability of a type II error.

**c.** Find the probability that the machine fails at least one of the next three tests.

**14.** Residents in an urban area have a choice of three mobile phone companies: Pi, Mu and Fi. Each year, Pi expects to retain 76% of its customers, losing 10% to Mu and the rest to Fi. Mu expects to retain 70% of its customers, losing 19% to Fi and the rest to Pi. Fi expects to lose 20% of its customers to Pi and 5% to Mu, retaining the rest.

**a.** Construct a transition matrix to show the probabilities of a customer transitioning from each company.

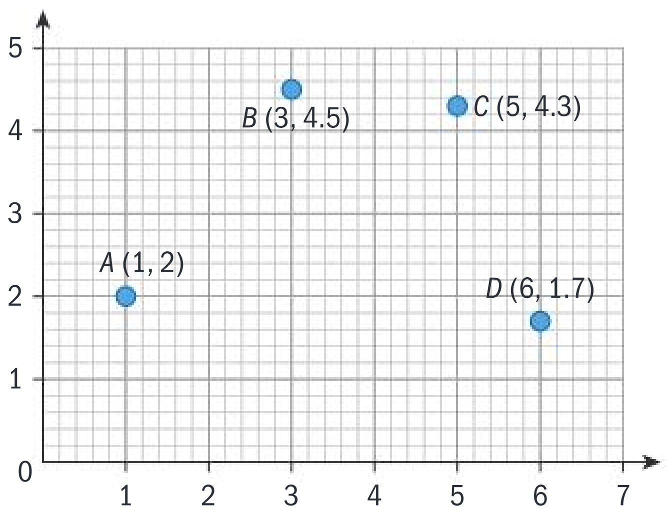
Currently, 32% of the residents are customers of Pi, 40% are customers of Mu and the rest are all with Fi. Fi predict that they will have the largest share of the market after three years.

**b.** Determine if Fi's claim is correct, stating any assumptions you make.

**15.** **a.** Give one reason why you might choose:

**i.** a quadratic curve **ii.** a cubic curve

to model the points shown in the diagram.




Theory suggests that the model will either be

$y = -0.4x^2 + 2.9x$ or $y = -0.155x^3 + 0.944x^2$

The four points have coordinates $A(1, 2)$, $B(3, 4.5)$, $C(5, 4.3)$ and $D(6, 1.7)$

**b.** Find the sum of the square residuals for each curve and state which model you would choose on the basis of this result. Justify your reasoning.

**c.** Find the least squares cubic regression curve for these four points.

**d.** State the $R^2$ value from your calculator and state what this indicates.

**e.** Given that the points represent the height, $y$, at a horizontal distance, $x$, from the initial position of an object projected from ground level, state one reason why either of the first two models might be preferred to the least squares model.

**16.** Callum grows strawberries and raspberries on his farm. The masses of Callum's strawberries are normally distributed with mean 12 g and standard deviation 2.7 g. The masses of Callum's raspberries are normally distributed with mean 4 g and standard deviation 0.5 g.

**a.** Find the probability that the total mass of five randomly chosen strawberries is more than 70 g.

**b.** Find that probability that the mass of a randomly chosen strawberry is more than four times the mass of a randomly chosen raspberry.

Strawberries are packaged in containers of $n$ strawberries. The label on the container states that the average mass of a strawberry in the container is 12 g. If the average mass is above 15 g, the container is rejected as too heavy, and if it is below 11 g it is rejected as being too light.

**c.** If the probability of a package of $n$ being rejected is 0.01, find $n$.

**17.** A group of scientists is assessing posters for a conference. Each poster is either accepted or rejected. Concerns are expressed that each decision is not independent of the previous decisions.

To address these concerns, the group looked at the 250 most recent posters and divided them into 25 groups of 5. The number of posters accepted in each group is shown in the table.

| Number of posters accepted | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency | 9 | 6 | 10 | 7 | 13 | 5 |

**a.** Find the proportion, $p_0$, of posters that are accepted.

**b.** Test to see whether the results above come from a binomial distribution, and state the likely meaning of the result in the context of the question.

**18.** Izzy measured the heights of 291 sunflowers in a botanical garden and found that the heights could be modelled by a normal distribution with mean 189.5 cm and a standard deviation of 15.3 cm.

**a.** The following season, the botanical garden grew 310 sunflowers. Predict the number of sunflowers taller than 195 cm, stating any assumptions that you make.

Izzy notices that the tape she had used to measure the heights was faulty. The scale started at 3 cm, not the zero mark.

**b.** What are the correct values of the mean and variance of the distribution of the heights of the sunflowers?

**c.** Hence find the true answer to part **a** to the nearest whole number.

## PAPER 2

1. A manager is collecting data on how the firm's cafeteria is perceived, and she has five employees fill in a survey. Their mean scores out of 10 are recorded, where a higher score indicates greater satisfaction.

   After assessing the results of the survey, changes are put in place and the same five employees are given the questionnaire again.

   The results are shown in the table.

| Employee | A | B | C | D | E |
|---|---|---|---|---|---|
| First survey | 7.2 | 4.1 | 6.1 | 5.4 | 3.9 |
| Second survey | 7.3 | 5.2 | 6.3 | 6.7 | 4.2 |

   **a.** Find the product moment correlation coefficient between the two sets of data and comment on the value obtained.

   **b.** Perform a test to show that there is significant evidence at the 5% level that there is a correlation between the results of the first survey and the second. You may assume that all necessary conditions for the use of this test are satisfied.

   The manager claims that this result shows her survey is a reliable means of collecting the data.

   **c.** Give the name for this test of reliability.

   The cafeteria manager claims that the data shows the cafeteria has improved between the two surveys.

   **d.** Carry out an appropriate test of the claim of the cafeteria manager.

2. The masses of items of hand luggage carried on to an aircraft by the passengers are normally distributed with mean 9.4 kg and standard deviation 2.8 kg.

   **a.** Find the probability that the mass of a randomly chosen item of hand luggage is between 9 kg and 12 kg.

   The airline sets the hand luggage dimensions such that eight items of hand luggage will fit into each overhead locker. The maximum load each overhead locker can hold is 100 kg.

   **b.** Find the probability that eight items of hand luggage will be too heavy for the overhead locker, stating any assumptions that you make.

   Large cases must be carried in the hold of the aircraft. The masses of the large cases are normally distributed with mean 25 kg and standard deviation 7 kg.

   **c.** Find the probability that the mass of a randomly chosen large case is more than three times that of a randomly chosen item of hand luggage.

   **d.** The Customs Officers at the destination airport select a random sample of three large cases and two items of hand luggage. Find the probability that the total mass of the sample exceeds 105 kg.

## PAPER 3

1. In Euclid city there are only two gym franchises, ClearGym and ResultsNow, who both sell annual membership packages starting on January 1. Each year, ClearGym retains 60% of its members and the rest move to ResultsNow. Each year, ResultsNow retains 70% of its customers and the rest move to ClearGym.

   **a.** Explain why this context can be modelled as a Markov chain.

   **b.** Sketch a transition diagram to represent this context.

   **c.** Hence, write down and label a transition matrix $\mathbf{G}$ to represent the probabilities of customers changing membership between the two companies.

   At the start of 2019, ClearGym had 17 530 customers and ResultsNow had 8956.

   **d.** Predict the number of customers each gym has after one year, to the nearest 100 customers.

   **e.** Find the eigenvalues and the associated eigenvectors of $\mathbf{G}$.

   **f.** Hence, express $\mathbf{G}$ in the form $\mathbf{G} = \mathbf{PDP}^{-1}$

   **g.** Hence, show that

$$\mathbf{G}^n = \frac{1}{7}\begin{bmatrix} 3+4(0.3)^n & 3-3(0.3)^n \\ 4-4(0.3)^n & 4+3(0.3)^n \end{bmatrix}$$

   **h.** Use your result for part **g** to verify your answer to part **d**.

   The management of ClearGym are concerned that in the long term they will lose customers to ResultsNow.

   **i.** Explain if the management of ClearGym are justified in having this concern.

# 5 CALCULUS

## 5.1 DIFFERENTIATION

### You should know:

- ✔ the informal concept of a limit
- ✔ the derivative of a function *f* gives the rate of change of the dependent variable with respect to the independent variable. This is equivalent to the gradient of the curve $y = f(x)$
- ✔ forms of notation for the first derivative, for example: $\frac{dy}{dx}$, $f'(x)$, $\frac{dV}{dt}$
- ✔ $f'(x) > 0$ for increasing functions and $f'(x) < 0$ for decreasing functions
- ✔ the derivative of $f(x) = ax^n$ is $f'(x) = ax^{n-1}$, $n \in \mathbb{Z}$
- ✔ local maximum and minimum points occur where $f'(x) = 0$, but these might not be the greatest or least values in the given domain.

### You should be able to:

- ✔ find derivatives of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$ where all exponents are integers
- ✔ find the equations of tangents and normals at a given point, both analytically and using technology
- ✔ use technology to find values of $f'(x)$ given $f(x)$, and find the solutions of $f'(x) = 0$
- ✔ solve optimization problems in context.

The derivative of a function at a point gives the **gradient** of the graph of the function at that point.

The derivative of a function or the gradient of a curve gives the **rate of change** of the function. The larger the value of the derivative, the greater the rate of change.

A positive value for the derivative means the value of the dependent variable is increasing as the independent variable increases and a negative value means it is decreasing.

The derivative can be written as $\frac{dy}{dx}$ or $f'(x)$

The derivative of a polynomial can be found using the relation:

$y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$ on each of the terms.

**Note**

In the case of an equation like $P = 2t^2 + 3t - 7$ the notation is $\frac{dP}{dt} = 4t + 3$

Note the special case of $y = mx$ and $y = c$ which have derivatives $\frac{dy}{dx} = m$ and $\frac{dy}{dx} = 0$ respectively.

**SAMPLE STUDENT ANSWER**

Consider the function $f(x) = x^3 - 3x^2 + 2x + 2$

(a) Find $f'(x)$

(b) There are two points at which the gradient of the graph of *f* is 11. Find the *x*-coordinates of these points.

(a) $f(x) = x^3 - 3x^2 + 2x^1 + 2$

$f'(x) = 3x^2 - 6x^1 + 2$

$f'(x) = 3x^2 - 6x + 2$

(b) $11 = 3x^2 - 6x + 2$

$3x^2 - 6x + 2 = 11$

$3x^2 - 6x - 9 = 0$

GDC = Poly roots

$\{-1, 3\}$

(1) $f(-1) = (-1)^3 - 3(-1)^2 + 2(-1) + 2$

$= -4$

(2) $f(3) = 3^3 - 3(3)^2 + 2(3) + 2$

$= 8$

Answers:

(a) $f'(x) = 3x^2 - 6x + 2$

(b) $(-1, -4)$, $(3, 8)$

▲ Gradient equal to 11 means $f'(x) = 11$

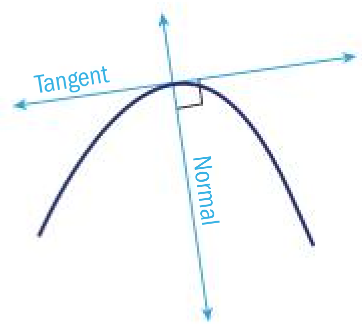
▼ This could be solved by factorizing, but always take the easiest route in an exam

▲ Clearly set out.

▼ Unfortunately the question just asked for $x$-coordinates. The student probably did not lose any marks because they did find the $x$-coordinates, but time was lost unnecessarily in calculating the $y$-coordinates. Always read the question carefully.

## Tangents and normals

The tangent to a curve at a point is the line which just touches the curve at that point and has the same gradient as the curve. The normal to the curve at a point is the line through the point, perpendicular to the tangent.




### Example 5.1.1

Consider the curve $y = \frac{x^2}{4}$

(a) Find $\frac{dy}{dx}$

(b) Find the equation of the tangent to the graph of $y = \frac{x^2}{4}$ at $x = 4$

(c) Find the $x$-coordinate of the point at which the normal to the graph of $y = \frac{x^2}{4}$ has gradient $-\frac{1}{8}$

Remember from section 3.3, that if a line has a gradient of $m$, then the line perpendicular to it has a gradient of $-\frac{1}{m}$

**Solution**

(a) $\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

$\frac{x^2}{4}$ can be written as $\frac{1}{4}x^2$ so the derivative is $\frac{1}{4} \times 2x = \frac{1}{2}x$ or $\frac{x}{2}$

(b) When $x = 4$, $\frac{dy}{dx} = \frac{4}{2} = 2$

Equation is $y = 2x + c$

The equation of the tangent is $y = mx + c$

The first stage is to find the value of $m$, which is the gradient of the tangent and is equal to the gradient of the curve at $x = 4$

When $x = 4,\ y = \frac{16}{4} = 4$

so $4 = 2 \times 4 + c \Rightarrow c = -4$

$y = 2x - 4$

A point on the line is needed to find the value of $c$, so we can use the point where the tangent meets the curve, (4, 4)

An alternative method is to use $y - y_1 = m(x - x_1)$

(c) Gradient of tangent $= 8$

$\frac{x}{2} = 8 \Rightarrow x = 16$

If the gradient of a tangent is equal to $m$ then the gradient of the normal is equal to $-\frac{1}{m}$ and vice versa.

**Note**

All GDCs will be able to find the numerical value of a derivative at a point, and some will also be able to find the equation of a tangent at point.

## Optimization

If $f'(x) > 0$ then the function $f$ is increasing, and if $f'(x) < 0$ the function is decreasing.

When $f'(x) = 0$ the curve will have a local maximum or minimum point. This fact can be used to solve **optimization** problems.

**Note**

Optimization is 'the action of making the best or most effective use of a situation or resource'. In this topic it is finding the maximum or minimum value of a function in a given context.

**Note**

In the HL course, $f'(x) = 0$ can also occur at a horizontal point of inflexion.

**SAMPLE STUDENT ANSWER**

A potter sells $x$ vases per month.

His monthly profit in Australian dollars (AUD) can be modelled by

$P(x) = -\frac{1}{5}x^3 + 7x^2 - 120,\ x \geq 0$

(a) Find the value of $P$ if no vases are sold.

(b) Find $P'(x)$

(c) Hence find the number of vases that will maximize the profit.

(a) $P(0) = -\frac{1}{5}(0)^3 + 7(0)^2 - 120$

$= -120$

(b) $P(x) = -\frac{1}{5}x^3 + 7x^2 - 120$

$= -\frac{3}{5}x^2 + 14x$

(c) $0 = -\frac{3}{5}x^2 + 14x$

$x = \frac{70}{3}$

$= 23.3$

▼ The answer is correct but the candidate is in danger of losing a mark by not beginning the second line with $P'(x) =$

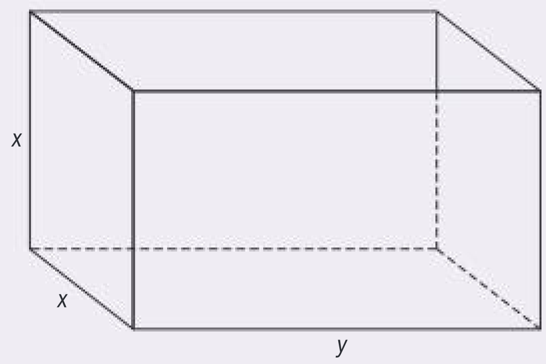
▲ The maximum of the curve occurs when $P'(x) = 0$. It is important to show the full working to indicate to the examiner that the maximum has not been found directly from the GDC.

▼ The number of vases must be a whole number, so the answer needs to be rounded. Always keep the context of the question in mind!

In some optimization questions, the function to be optimized will contain two variables. In this case the question will provide information about a second relationship between the two variables (a constraint) which will allow you to substitute out one of the variables.

## Example 5.1.2

Fred makes an open metal container in the shape of a cuboid, as shown here.




The container has a height $x$ m, width $x$ m and length $y$ m. The volume is 36 m$^3$.

Let $A(x)$ be the outside surface area of the container.

(a) Show that $A(x) = \dfrac{108}{x} + 2x^2$

(b) Find $A'(x)$

(c) Hence find the height of the container which minimizes the surface area.

**Solution**

(a) $A(x) = 2x^2 + 3xy$

$V = 36 = x^2 y \Rightarrow y = \dfrac{36}{x^2}$

There are two square sides and three rectangular sides, as the container is 'open'.

$A(x) = 2x^2 + 3x \times \dfrac{36}{x^2}$

$= 2x^2 + \dfrac{108}{x}$

The equation has two variables so we need to use the condition that the volume is 36 m$^3$ to find an expression for $y$, which can then be substituted into $A(x)$.

(b) $A(x) = 2x^2 + 108x^{-1}$

$A'(x) = 4x - 108x^{-2}$

First write using negative exponents.

(c) $4x - \dfrac{108}{x^2} = 0$

$4x^3 = 108 \Rightarrow x^3 = 27$

$x = 3$

Height is 3 m.

When manipulating an expression it is often easiest to convert a negative exponent to a fraction.

Because the question said 'hence' you need to make it clear that you are using the answer from part (b).

Having written the first line, this equation could then be solved using the GDC, rather than analytically.

If expressions are given as fractions, they should be written with negative exponents before differentiating. For example:

Rewrite $y = \dfrac{2}{x^2}$ as

$y = 2x^{-2} \Rightarrow \dfrac{dy}{dx} = -4x^{-3}$

Remember that if the domain is restricted then the maximum and minimum values might occur at the end points rather than where the derivative is equal to zero. In these cases you should always check by plotting the curve on your GDC.

# 5.2 INTEGRATION

## You should know:

- if $\frac{dy}{dx} = ax^n$ then $y = \frac{a}{n+1}x^{n+1} + c$, for $n \neq -1$, which can be written as
  $\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$
- the area of a region enclosed by a curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$, where $f(x) > 0$, can be found from calculating $\int_a^b f(x)dx$
- the trapezoidal rule.

## You should be able to:

- find the general form of a function when given its derivative or rate of change
- find the value of $c$ using a boundary condition, for example the value of $y$ when $x$ is 0
- use technology to find the area of a region enclosed by a curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$, where $f(x) > 0$
- find an estimate for the value of an area using the trapezoidal rule, with intervals of equal width when given either a table of data or a function.

### Antiderivatives

An antiderivative or integral is useful for deriving an equation from a rate or for finding areas under a curve.

If $\frac{dy}{dx} = ax^n$ then $y = \frac{a}{n+1}x^{n+1} + c$, $n \neq -1$

**Note**

This can be written as

$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$

This formula can be found in section SL 5.5 of the formula book. Notice that it excludes the case where $n = -1$ as the formula in this instance would involve dividing by 0.

**Example 5.2.1**

The rate at which the temperature $T$ is changing $t$ minutes after a heating element is turned on is given by the equation

$\frac{dT}{dt} = 19 - 2t$, $0 \leq t \leq 10$

(a) Find the rate of change of the temperature when $t = 4$

When $t = 0$ the temperature is 5 °C

(b) Find an expression for the temperature at time $t$

(c) Find the maximum value of $T$ for $0 \leq t \leq 10$

**Assessment tip**

In standard level exams, $n$ will always be an integer.

**Solution**

(a) When $t = 4$

$\frac{dT}{dt} = 19 - 8 = 11$

11 °C per minute

The rate of change is another way of asking for the value of the derivative.

Notice that the units of $\frac{dT}{dt}$ are °C per minute.

(b) $T = \int 19 - 2t \, dt$

$= 19t - 2 \times \frac{1}{2}t^2 + c$

$= 19t - t^2 + c$

Note that $\int 19 \, dt = 19t$ because the derivative of $19t$ is 19.

When $t = 0$,
$T = 19 \times 0 - 0^2 + c = 5$

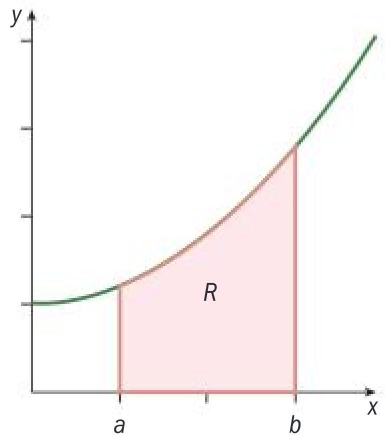
$\Rightarrow c = 5$

$T = 19t - t^2 + 5$

The value of $c$ is found by substituting known values for the variables.

| | |
|---|---|
| (c) The maximum value occurs when $t = 9.5$<br>$T = 95.25\,°\text{C}$ | You should plot the curve on your GDC to ensure the maximum does not occur at $t = 0$ or $t = 10$. Having plotted the curve it is easier to find the maximum directly from the curve rather than solving $\frac{dT}{dt} = 0$ |

## Areas between a curve and the *x*-axis

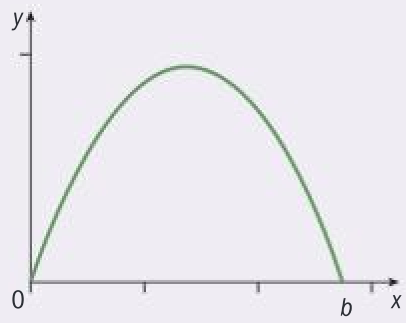



Integrals can be used to find the area between a curve and the $x$-axis.

The notation for the area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ where $b > a$ (shown as $R$ in the diagram) is given by the **definite integral** written as $\int_a^b f(x)\,dx$

### Example 5.2.2

The side wall in a concert hall can be modelled by a curve with equation $y = 2.8x - 0.5x^2$, $0 \le x \le b$ where $x$ is the horizontal distance from the point $O$ and the units are measured in metres.




(a) Find the value of $b$.

(b) Write down an integral which represents the area of the wall.

(c) It is intended to repaint the wall. If one can of paint covers $4.5\,\text{m}^2$ of wall, find the minimum number of cans of paint needed.

**Solution**

| | |
|---|---|
| (a) $b = 5.6$<br>(b) $\int_0^{5.6} 2.8x - 0.5x^2\,dx$ | This can be found by factorizing the equation, or directly from the GDC by entering the equation and finding the zero ($x$-intercept). |
| (c) Area = 14.6 $\text{m}^2$ | Using the integral formula for area. |
| Number of cans required $\frac{14.6}{4.5} \approx 3.24$ | The area is obtained directly from the GDC. |
| 4 cans of paint | To find the minimum number of cans you need to round up. |

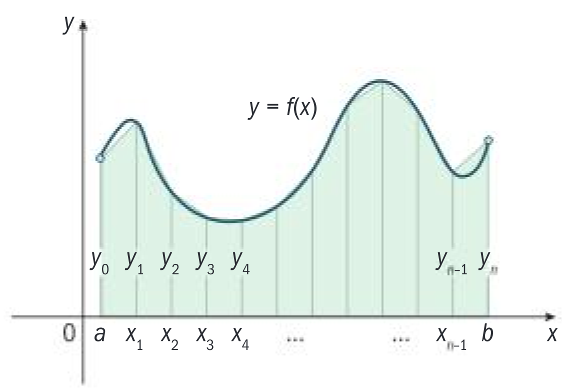
**Assessment tip**

At Higher Level you will need to know how to work out the value of the integral you wrote down in part (b) using integration (anti-derivatives) but for Standard Level you will always use the appropriate function on the calculator. You should though know how to use the notation correctly.

## The trapezoidal rule

Before the arrival of GDCs, an approximation for the area between a curve and the $x$-axis was often found by dividing the area into trapezoids and working out their areas. This is still useful today, particularly when the equation defining the curve is not known.

The area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ with $b > a$, can divided into $n$ trapezoids each with height $h = \frac{b-a}{n}$, as shown.



This formula is in section SL 5.8 of the formula book

The area of the trapezoids can be found using the trapezoidal rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{1}{2}h\left(y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$$

The question may ask you if the trapezoidal rule gives an underestimate or an overestimate of the actual area. This is usually clear from the diagram.

### Example 5.2.3

Use the trapezoidal rule with 4 trapezoids to find an approximate value for the area between the curve $y = 2 + \frac{6}{x}$, the $x$-axis and the lines $x = 1$ and $x = 3$

**Solution**

$h = \frac{3-1}{4} = 0.5$

Using the formula $h = \frac{b-a}{n}$

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
| $y$ | 8 | 6 | 5 | 4.4 | 4 |

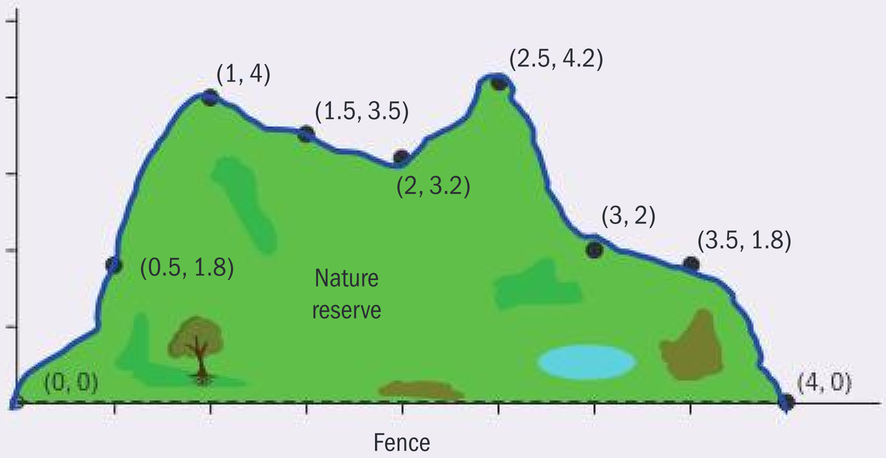
Area $\approx \frac{1}{2} \times 0.5((8+4) + 2(6+5+4.4))$

$= 10.7$

You should enter the function into your GDC and read off the required values. Alternatively they can be worked out individually. For example, when $x = 1$, $y = 2 + 6 = 8$. You should always write down the $y$-values to obtain method marks in case you make an error in your calculations.

### Example 5.2.4

A nature reserve is bounded by a river and a straight fence, as shown in the diagram below. Use the trapezoidal rule and the coordinates of the points shown to find the approximate area of the nature reserve. Each unit represents one kilometre.




**Solution**

Width of trapezoids $= 0.5$

Area $\approx \frac{1}{2} \times 0.5[(0+0) + 2(1.8 + 4 + 3.5 + 3.2 + 4.2 + 2 + 1.8)]$

$= 10.25\ \text{km}^2$

The width can easily be found from the difference between the $x$-values, but could also be calculated using $h = \frac{4-0}{8} = 0.5$

# 5.3 DIFFERENTIATION (AHL)

## You should know

✔ the derivatives of $\sin x$, $\cos x$, $\tan x$, $e^x$, $\ln x$, $x^n$ where $n \in \mathbb{Q}$

✔ if $f''(x) < 0$ the curve is concave down and if $f''(x) > 0$ the curve is concave up

✔ that a point of inflexion is a point at which the concavity changes and the interpretation of this in context.

## You should be able to

✔ use the chain rule, product rule and quotient rules

✔ connect variables with related rates of change

✔ use the second derivative to distinguish between local maximum and local minimum points.

At higher level you need to be aware of the following derivatives:

| $f(x)$ | $\sin x$ | $\cos x$ | $\tan x$ | $e^x$ | $\ln x$ | $x^n, n \in \mathbb{Q}$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $\cos x$ | $-\sin x$ | $\frac{1}{\cos^2 x}$ | $e^x$ | $\frac{1}{x}$ | $nx^{n-1}$ |

These are given in section 5.9 and 5.11 of the formula book.

### The chain rule

The chain rule is used to differentiate composite functions. For example if $u$ is a function of $x$, and $y = g(u)$ then $\frac{dy}{dx} = g'(u) \times u'(x)$ or $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

**Note**

This can be remembered by regarding the terms as fractions and 'cancelling' $du$

The way this is done in practice is to separate the two functions, differentiate both and then multiply them.

So if $y = \sin(x^2)$ then the two functions are $u = x^2$ and $y = \sin u$. These are differentiated and multiplied together:

$$\frac{dy}{dx} = 2x \times \cos u = 2x \cos x^2$$

The chain rule formula is in section 5.9 of the formula book.

Try differentiating the functions in the next example before checking the answers.

**Example 5.3.1**

Differentiate the following functions.

(a) $y = (3x+1)^{\frac{1}{2}}$ (b) $y = e^{\cos x}$ (c) $f(x) = \sin^2 x$

**Solution**

(a) $u = 3x+1,\ y = u^{\frac{1}{2}}$ — First identify the two functions.

$\frac{du}{dx} = 3,\ \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ — Differentiate each one.

$\frac{dy}{dx} = 3 \times \frac{1}{2}u^{-\frac{1}{2}} = \frac{3}{2}(3x+1)^{-\frac{1}{2}}$ — Find the product, changing all variables back to $x$.

(b) $u = \cos x,\ y = e^u$

$\frac{du}{dx} = -\sin x,\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = -\sin x \times e^u = -(\sin x)e^{\cos x}$

$u$ is often the function in brackets, but sometimes the brackets are not shown so you need to consider which is the first function performed.

**Assessment tip**

Make sure your GDC is in radian mode when the question involves calculus and trigonometric functions.

## Assessment tip

It is worth learning these common derivatives rather than using the chain rule on each occasion.

| $f(x)$ | $\sin ax$ | $\cos ax$ | $e^{ax}$ |
|---|---|---|---|
| $f'(x)$ | $a\cos ax$ | $-a\sin ax$ | $ae^{ax}$ |

(c) $f(x) = (\sin x)^2$

$$u = \sin x, f(u) = u^2$$
$$f'(x) = \cos x \times 2u$$
$$= 2\cos x \sin x$$

Rewriting the function makes it easier to see the individual functions that make up the composite function.

After plenty of practice the middle lines are often omitted. Think of it as 'the derivative of the inside function, multiplied by the derivative of the outside function'

## Product and quotient rule

These formulae are in section 5.9 of the formula book.

$$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Again, try differentiating the functions in the next example before checking the answers.

### Example 5.3.2

Differentiate the following functions.

(a) $y = x^2 \ln x$ (b) $f(x) = \dfrac{x}{e^{2x}}$ (c) $s = \dfrac{2t}{(2t+1)^{\frac{1}{2}}}$

**Solution**

(a) $u = x^2$, $v = \ln x$

$$\frac{dy}{dx} = x^2 \times \frac{1}{x} + \ln x \times 2x$$
$$= x + 2x \ln x$$

The product rule is often thought of as: 'keep the first the same and differentiate the second, plus differentiate the first, keep the second the same.'

(b) $u = x$, $v = e^{2x}$

$$\frac{dy}{dx} = \frac{e^{2x} \times 1 - x \times 2e^{2x}}{(e^{2x})^2}$$
$$= \frac{e^{2x}(1-2x)}{(e^{2x})^2} = \frac{1-2x}{e^{2x}}$$

The derivative of $e^{2x}$ is $2e^{2x}$. This can be worked out from the chain rule or simply learned.

If the question just says 'differentiate' there is no need to simplify, though the simplification will make any subsequent calculations easier.

(c) $u = 2t$, $v = (2t+1)^{\frac{1}{2}}$

$$\frac{dv}{dt} = 2 \times \frac{1}{2}(2t+1)^{-\frac{1}{2}} = \frac{1}{(2t+1)^{\frac{1}{2}}}$$

Because the derivative of $v$ is quite complex it should be done first using the chain rule.

$$\frac{dy}{dx} = \frac{(2t+1)^{\frac{1}{2}} \times 2 - 2t \times \frac{1}{(2t+1)^{\frac{1}{2}}}}{\left((2t+1)^{\frac{1}{2}}\right)^2}$$

At this point the derivative has been found but it is likely that some simplification is required.

$$= \frac{2(2t+1)^{\frac{1}{2}} - \frac{2t}{(2t+1)^{\frac{1}{2}}}}{(2t+1)} \times \frac{(2t+1)^{\frac{1}{2}}}{(2t+1)^{\frac{1}{2}}}$$
$$= \frac{2(2t+1) - 2t}{(2t+1)^{\frac{3}{2}}} = \frac{2t+2}{(2t+1)^{\frac{3}{2}}}$$

Whenever you have fractions within fractions remove them by multiplying top and bottom by the denominators within the fraction.

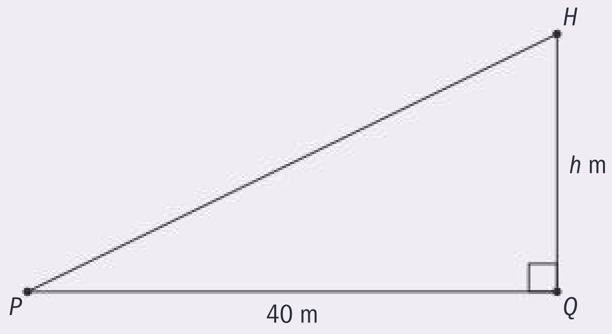
## Related rates

If the rate of change of one quantity is given, the rate of change of a linked quantity can sometimes be calculated using the chain rule.

> **Assessment tip**
>
> Don't forget the rate of change is just another way of saying the derivative.

### Example 5.3.3

A helicopter $H$ is moving vertically upwards with a speed of 10 ms$^{-1}$. The helicopter is $h$ m directly above the point $Q$, which is situated on level ground. The helicopter is observed from point $P$, which is also on ground level, and $PQ = 40$ m.




When $h = 30$, show that the rate of change of $H\hat{P}Q$ is 0.16 radians per second.

**Solution**

$\frac{dh}{dt} = 10$

You are given the rate of change of $h$

Let $H\hat{P}Q = \theta$ hence we need $\frac{d\theta}{dt}$

By the chain rule $\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}$

If you write down what you want and what you are given then the chain rule can be used to find what you need. Here you need $\frac{d\theta}{dh}$ so are looking for an equation linking $\theta$ and $h$.

$h = 40\tan\theta \Rightarrow \frac{dh}{d\theta} = \frac{40}{\cos^2\theta}$

$\frac{d\theta}{dt} = 10 \times \frac{\cos^2\theta}{40} = \frac{\cos^2\theta}{4}$

$\frac{d\theta}{dh} = \frac{1}{\frac{dh}{d\theta}}$, so find whichever is most convenient.

When $h = 30$, $\cos\theta = 0.8$

$\Rightarrow \frac{d\theta}{dt} = \frac{0.8^2}{4} = 0.16$

Be careful to only substitute in the value for the variable right at the end.

> **Note**
>
> When tackling these questions, write down the rate you want to find, then 'equals' and the rate you are given. Decide then which extra rate you need to ensure the two sides are equal when cancelled, as illustrated in the example.

## Second derivative

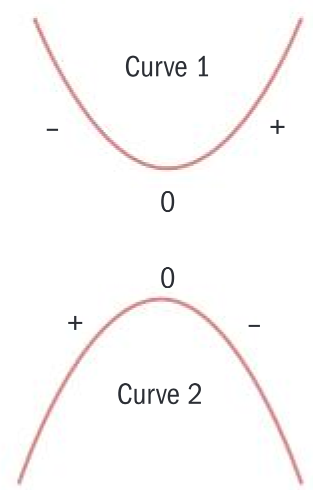
The derivative of a function gives you the rate of change of the dependent variable with respect to the independent variable (the gradient of the graph). The derivative of the derivative (the second derivative), tells you the rate of change of this gradient.

The gradient of curve 1 is increasing and so $f''(x) > 0$.
The curve is described as **concave up**.

The gradient of curve 2 is decreasing and so $f''(x) < 0$.
The curve is described as **concave down**.

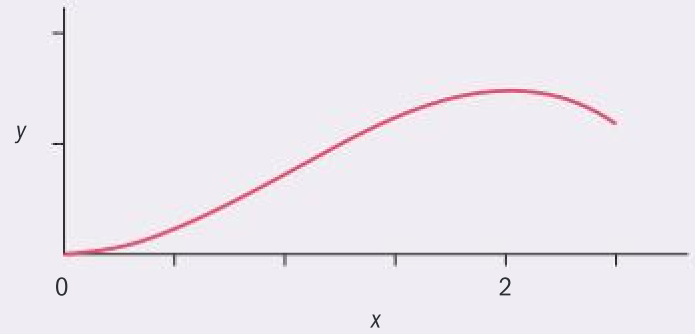
At a **point of inflexion,** a curve changes its concavity from concave down to concave up or vice versa. At a point of inflexion, $f''(x) = 0$.

> **Note**
>
> The second derivative has the notation $\frac{d^2y}{dx^2}$ or $f''(x)$



### Example 5.3.4

Jorge is cycling up a hill which can be modelled by the equation $y = \frac{5}{27}(3x^2 - x^3)$, $0 \le x \le 2.5$, where $x$ and $y$ are the horizontal and vertical distances (in kilometres) from the foot of the hill.




(a) (i) Find $\frac{dy}{dx}$

(ii) Hence show that the summit of the hill occurs when $x = 2$ and find the height of the hill at this point.

(b) Find $\frac{d^2y}{dx^2}$

(c) Use the answer to part (b) to find the values of $x$ and $y$ at which the hill is steepest, for $0 \le x \le 2$

**Solution**

(a) (i) $\frac{dy}{dx} = \frac{5}{27}(6x - 3x^2)$

The brackets could be expanded but there is no need as $\frac{5}{27}$ is a constant term.

(ii) At the summit of the hill $\frac{dy}{dx} = 0$

hence $\frac{5}{27}(6x - 3x^2) = 0$

$\Rightarrow 6x - 3x^2 = 3x(2 - x) = 0$

Hence $x = 2$ or $x = 0$

From the sketch, $x = 2$ is a local maximum.

Because the question says 'hence' this needs to be done using the answer from the previous part. Because the command term is 'show that', a GDC cannot be used and each stage of the working should be shown.

The height is given by $\frac{5}{27}(3 \times 2^2 - 2^3) = \frac{20}{27}$ $\approx 0.741$ km

The height can also be found by entering the equation into your GDC, as the command term is 'find'.

(b) $\frac{d^2y}{dx^2} = \frac{5}{27}(6 - 6x) = \frac{10}{9}(1 - x)$

(c) The hill is steepest when $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 1$

When $x = 1$, $y = \frac{10}{27}$

The hill is steepest when the curve changes from concave up to concave down (the gradient stops increasing). This occurs at the point of inflexion when $\frac{d^2y}{dx^2} = 0$

**Note**

$f'(x) = 0$ and $f''(x) > 0$ at a minimum point

$f'(x) = 0$ and $f''(x) < 0$ at a maximum point

# 5.4 INTEGRATION (AHL)

## You should know:

- ✔ definite and indefinite integrals of $x^n$ where $n \in \mathbb{Q}$, including $n = -1$, $\sin x$, $\cos x$, $\frac{1}{\cos^2 x}$ and $e^x$
- ✔ formulae for volumes of revolution $V = \int_a^b \pi y^2\,dx$ or $V = \int_a^b \pi x^2\,dy$
- ✔ kinematics: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$
- ✔ speed is the magnitude of velocity
- ✔ distance travelled formula, $\int_{t_1}^{t_2} |v(t)|\,dt$

## You should be able to:

- ✔ integrate by inspection, or substitution, integrals of the form $\int_a^b f(g(x))g'(x)dx$
- ✔ find the area of the region enclosed by a curve and the $x$- or $y$-axes in a given interval, including cases where the area is negative
- ✔ use both integration and differentiation to find equations for displacement, velocity or acceleration.

Integration is the reverse of differentiation and so the integrals of the main families of functions can be deduced from the corresponding derivatives.

| $f(x)$ | $x^n, (x \in \mathbb{Q}, x \neq -1)$ | $\frac{1}{x}$ | $\sin x$ | $\cos x$ | $\frac{1}{\cos^2 x}$ | $e^x$ |
|---|---|---|---|---|---|---|
| $\int f(x)dx$ | $\frac{x^{n+1}}{n+1} + c$ | $\ln\lvert x\rvert + c$ | $-\cos x + c$ | $\sin x + c$ | $\tan x + c$ | $e^x + c$ |

Because $f(x) = \ln(x)$ is defined only for $x > 0$ and $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$ the integral of $f(x) = \frac{1}{x}$ is given as $\ln|x| + c$, so it is defined over the same domain.

**Note**

Remember to add the $+\,c$ term when integrating.

These formulae are in sections 5.5 and 5.11 of the formula books.

### Integrals of the form $\int f(u(x))u'(x)dx$

You need to be able to find integrals of the form $\int f(u(x))u'(x)dx$. These consist of a composite function multiplied by a multiple of the derivative of the 'inside' function, for example $\int x\sin(x^2)dx$, where $x$ is a multiple of the derivative of $x^2$. The process is the reverse of differentiating by the chain rule. After a bit of practice the results can be spotted and integrated by inspection, but initially it is worth integrating them using the substitution technique.

**Example 5.4.1**

Find (a) $\int 4xe^{x^2}\,dx$ (b) $\int \sin x \cos^2 x\,dx$ (c) $\int \frac{x}{x^2+3}dx$

**Solution**

(a) Let $u = x^2$

Using the substitution method the 'inside' function of the composite needs to be spotted. As in the chain rule this is usually denoted by $u$.

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

When substituting, all $x$ terms need to be replaced by the equivalent $u$ term. This includes the d$x$. This is substituted by differentiating $u$ and rearranging as if it were a fraction.

$\int 4xe^{x^2}\,dx = \int 4xe^{u}\frac{du}{2x}$

Replace the $x$ term in the composite with $u$ and $dx$ with the expression derived.

$\int 2e^{u}\,du$

$= 2e^{u} + c = 2e^{x^2} + c$

The other term in $x$ will then cancel, leaving an expression just in terms of $u$ which can be easily integrated.

(b) $\int \sin x\cos^2 x\,dx$

$= \int \sin x(\cos x)^2\,dx$

By rewriting the integral it becomes clearer which function is $u$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

Again the substitution for $dx$ is found by differentiating $u$ and rearranging.

$\int \sin x(\cos x)^2\,dx$

$= \int \sin x(u)^2\frac{du}{-\sin x}$

$= -\int u^2\,du = -\frac{1}{3}u^3 + c$

$= -\frac{1}{3}\cos^3 x + c$

(c) $\int \frac{x}{x^2+3}\,dx$

$= \int x(x^2+3)^{-1}\,dx$

It is often easier to set up the substitution by first writing a quotient as a product.

Let $u = x^2 + 3$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

In general, you should always have the modulus signs in the solution, though in this case because $x^2 + 3 > 0$ they could be omitted.

$\int x(x^2+3)^{-1}\,dx$

$= \int xu^{-1}\frac{du}{2x} = \frac{1}{2}\int u^{-1}\,du$

$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+3| + c$

Integrals of the form $f(ax)$ can be found using substitution, but the following are so common the results should be learned and the integrals found directly:

$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$ $\qquad$ $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$ $\qquad$ $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

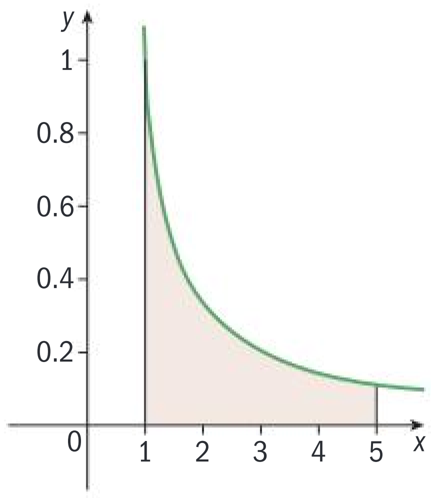
## The definite integral and areas

In section 5.2 it was stated that the area between a curve $y = f(x)$, the $x$-axis and the lines $x = a$ and $x = b$ can be found by calculating $\int_a^b f(x)\,dx$. For the HL course you also need to be able to calculate this value directly using integration.

If $\int f(x)\,dx = F(x) + c$ then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ and when $f(x) > 0$ this is equal to the area between a curve $y = f(x)$, the $x$-axis and the lines $x = a$ and $x = b$

## Example 5.4.2

Find the exact area between the curve $y = \frac{1}{2x-1}$, the $x$-axis and the lines $x = 1$ and $x = 5$




**Solution**

The question is set up in the usual way but because the **exact** value is asked for it cannot be calculated directly on a GDC.

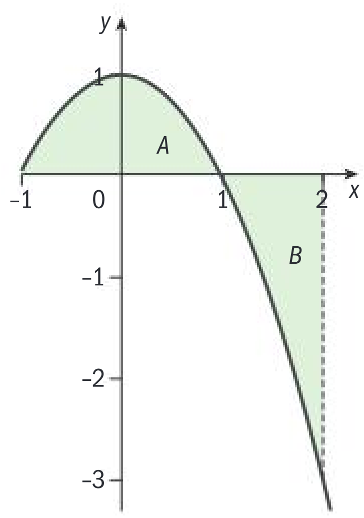
Area $= \int_1^5 \frac{1}{2x-1}\,dx$ — This integral is one of those that should be learned, but it can also be calculated

$= \left[\frac{1}{2}\ln|2x-1|\right]_1^5$ — by writing as $\int_1^5 (2x-1)^{-1}\,dx$ and using the substitution $u = 2x-1$

$= \frac{1}{2}\ln 9 - \frac{1}{2}\ln 1$ — The values are substituted and subtracted.

$= \frac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$ — Unless told otherwise in the question, any of these answers would be acceptable.

If the curve $y = f(x)$ lies entirely below the $x$-axis between the values $x = a$ and $x = b$ then the definite integral $\int_a^b f(x)\,dx$ is negative.




For example, considering the curve $y = 1 - x^2$

$\int_0^1 (1-x^2)\,dx = \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$, hence the area of $A$ is $\frac{2}{3}$,

$\int_1^2 (1-x^2)\,dx = \left[x - \frac{1}{3}x^3\right]_1^2 = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}$, unlike an integral, area is always positive, hence the area of $B$ is $\frac{4}{3}$

The integral between $x = 0$ and $x = 2$ will be equivalent to

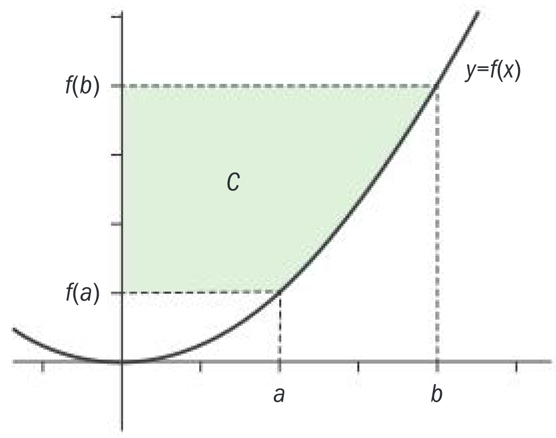
$\int_0^1 (1-x^2)\,dx + \int_1^2 (1-x^2)\,dx$ so $\int_0^2 (1-x^2)\,dx = \frac{2}{3} + \left(-\frac{4}{3}\right) = -\frac{2}{3}$

The area between $y = 1 - x^2$, the $x$-axis and $x = 0$ and $x = 2$ is the area of $A$ plus the area of $B$, hence, $\frac{2}{3} + \left|-\frac{4}{3}\right| = 2$, this can also be calculated directly as $\int_0^2 |1-x^2|\,dx = 2$

**Assessment tip**

When asked to evaluate areas in an exam, use $\int_a^b |f(x)|\,dx$

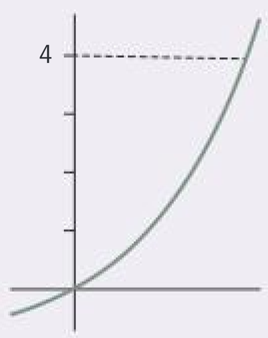
The area between a curve and the $y$-axis (shown as $C$ on the diagram) is given by the formula $\int_{f(a)}^{f(b)} f^{-1}(y)\,dy$




## Example 5.4.3

Find the area between the curve $y = e^{2x} - 1$, the $y$-axis and the line $y = 4$

**Solution**




A sketch is useful to identify the limits of integration. From the sketch we can see the lower limit is 0.

The equation needs to be rearranged to make $x$ the subject.

In section 5.12 of the formula book this is written as $\int_a^b |x|\,dy$, where $a$ and $b$ are values on the $y$-axis.

$y = e^{2x} - 1 \Rightarrow x = \frac{1}{2}\ln(y+1)$ The integral can then be done on the calculator.

$\int_0^4 \left|\frac{1}{2}\ln(y+1)\right| dx \approx 2.02$ There is no need to use modulus signs as the area is entirely on the positive side of the axis, but it is a good habit to always use them when finding areas.

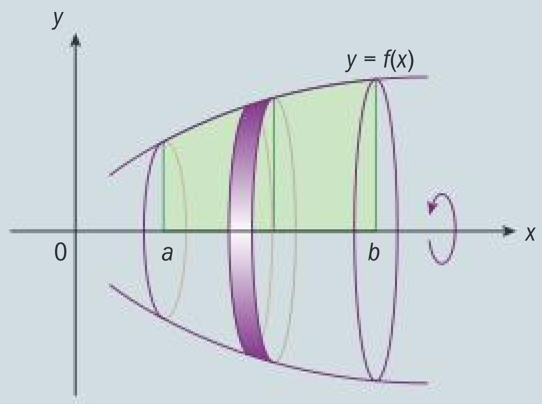
## Volumes of revolution

These formulae are in section 5.12 of the formula book.

The formulae for the volumes of solids of revolution formed by rotating a curve around the axes $x$- or $y$-axis are:

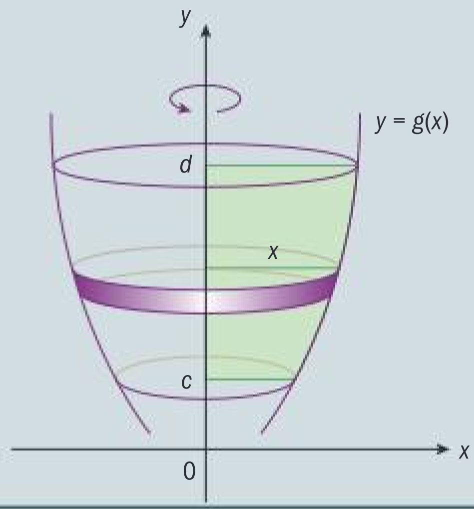
Around the $x$-axis:

Volume $= \int_a^b \pi y^2 dx$



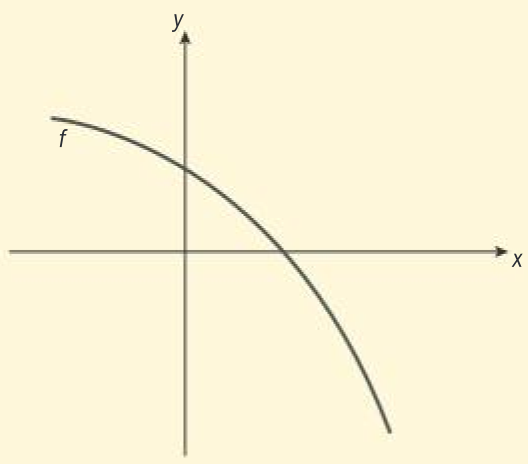

Around the $y$-axis:

Volume $= \int_c^d \pi x^2 dy$




**SAMPLE STUDENT ANSWER**

Let $f(x) = 4 - 2e^x$. The diagram shows part of the graph of $f$.




(a) Find the $x$-intercept of the graph of $f$.

(b) The region enclosed by the graph of $f$, the $x$-axis and the $y$-axis is rotated 360° about the $x$-axis. Find the volume of the solid formed.

(a) $0 = 4 - 2e^x \Rightarrow 2e^x = 4 \Rightarrow e^x = 2 \Rightarrow x = \ln 2 \approx 0.693$

(b) $\pi\int_0^{\ln 2} (4 - 2e^x)^2\, dx = 1.09$

▲ The answer could have been found directly from the GDC, but it is good to work with exact values if easy to do so and if they are needed in later parts.

▲ This could be calculated exactly by expanding the brackets, but the best approach is the one done here, finding it directly from the GDC.

▼ The student has made the common error of forgetting to multiply by π (the correct answer is 3.425) but because the working has been clearly set out there would only be the loss of one mark. If only the answer was written then all the marks in this part would have been lost.

## Kinematics

Velocity ($v$) is the rate of change of displacement ($s$), and acceleration ($a$) is the rate of change of velocity, hence:

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These formulae are in section 5.13 of the formula book.

Remember, displacement is the position of the object relative to a fixed point which is not necessarily equal to the distance travelled by the object. Section 5.13 of the formula book gives the following equations for calculating the displacement of an object and the distance travelled by the object between times $t_1$ and $t_2$

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt \qquad \text{distance travelled} = \int_{t_1}^{t_2} |v(t)|\,dt$$

If boundary conditions are given, displacement can also be calculated by integrating velocity and finding an explicit equation for displacement at time $t$.

**Note**

By the chain rule,

$$a = \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds} = v\frac{dv}{ds}$$

which is useful if acceleration is given as a function of displacement.

### Example 5.4.4

A particle $P$ moves in a straight line. Its acceleration at time $t$ is given by $a = 2t^2 - 7t + 5$ for $t \geq 0$

(a) Find the values of $t$ when $a = 0$

(b) Hence or otherwise, find all possible values of $t$ for which the velocity of $P$ is decreasing.

When $t = 0$, the velocity of $P$ is 4 ms$^{-1}$

(c) Find an expression for the velocity of $P$ at time $t$.

(d) Find the total distance travelled by $P$ when its velocity is decreasing.

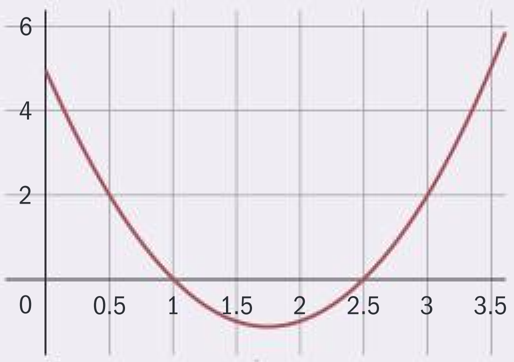
Kinematics in two dimensions is covered in section 3.6

**Solution**

(a) $(2t - 5)(t - 1) = 0$

$t = 1, 2.5$

These values could also be obtained directly from the GDC.




(b) $1 < t < 2.5$

By definition, decreasing velocity occurs where the acceleration is negative.

(c) $v = \int (2t^2 - 7t + 5)\,dt$

$= \frac{2}{3}t^3 - \frac{7}{2}t^2 + 5t + c$

When $t = 0$, $v = 4 \Rightarrow c = 4$

$v = \frac{2}{3}t^3 - \frac{7}{2}t^2 + 5t + 4$

Remember when integrating to find the '$+c$' term. It is not always equal to the initial value.

(d) distance =

$$\int_1^{2.5} \left|\frac{2}{3}t^3 - \frac{7}{2}t^2 + 5t + 4\right| dt \approx 8.41$$

Using the distance formula from the formula book.

**Note**

Decreasing velocity is not always the same as decreasing speed. Think of acceleration as a 'push' (acceleration and force are linked by the equation $F = ma$). When a particle is moving in the negative direction its velocity is negative, so a negative acceleration (a 'push' in the negative direction) will cause the velocity to become more negative which will also result in an increase in speed. Hence if $a$ and $v$ have the same signs the speed is increasing and if they have different signs the speed will decrease.

# 5.5 SOLUTIONS OF DIFFERENTIAL EQUATIONS (AHL)

## You should know:

- ✔ how to construct and use a slope field diagram
- ✔ Euler's method for finding the approximate solution to first order differential equations
- ✔ the solution $x = Ae^{\lambda_1 t}p_1 + Be^{\lambda_2 t}p_2$ for coupled differential equations of the form $\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$ with real, distinct eigenvalues
- ✔ if the eigenvalues are:
  - positive or complex with positive real part, all solutions move away from the origin
  - negative or complex with negative real part, all solutions move towards the origin
  - complex, the solutions form a spiral
  - imaginary, the solutions form a circle or ellipse
  - real with different signs (one positive, one negative) the origin is a saddle point.

## You should be able to:

- ✔ set up a model/differential equation from a context
- ✔ solve a differential equation by separation of variables
- ✔ use Euler's method to solve single variable and coupled differential equations using the inbuilt functions on your GDC
- ✔ use phase diagrams/portraits to qualitatively analyse future paths for distinct, real, complex and imaginary eigenvalues
- ✔ sketch trajectories and identify key features such as equilibrium points, stable populations and saddle points
- ✔ convert second order differential equations of the form $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$ into a system of coupled first order differential equations using the substitution $y = \frac{dx}{dt}$.

A differential equation is one containing a derivative. The simplest can be integrated directly, for example, $\frac{dy}{dx} = e^{2x} \Rightarrow y = \int e^{2x}dx = \frac{1}{2}e^{2x} + c$

If the equation is given in terms of the dependent variable then the solution can sometimes be found by the process of **separating the variables**.

The phrase 'is proportional to' or 'is inversely proportional to' is equivalent to 'varies directly as' or 'varies inversely as'. See Section 2.3, page 62.

### Example 5.5.1

The velocity of an object falling in a viscous liquid is proportional to the square root of the distance $h$ that it has fallen.

It is given that when $h = 1$ cm the velocity of the object is 5 cm s$^{-1}$

(a) Find an equation connecting $v$ and $h$.

(b) Hence find a general equation for the distance fallen by the object at time $t$.

**Solution**

(a) $v = a\sqrt{h}$

when $h = 1$, $v = 5$

hence $a = 5$

$v = 5\sqrt{h}$

The given values are substituted to find the unknown parameter.

(b) $\frac{dh}{dt} = 5h^{\frac{1}{2}} \Rightarrow \int \frac{1}{h^{\frac{1}{2}}} dh = \int 5 dt$

$\int h^{-\frac{1}{2}} dh = \int 5 dt$

$2h^{\frac{1}{2}} = 5t + c$

$h = \left(\frac{5t+c}{2}\right)^2$

Velocity is the derivative of displacement, which here is denoted by $h$.

$\frac{dh}{dt}$ can be treated as a fraction and separated, cross multiplying is then used to put the same variables on the same sides. This gives $\frac{1}{h^{\frac{1}{2}}} dh = 5dt$. The two sides are then integrated.

Only one '$+ c$' term is needed when both sides of an equation are integrated.

A general equation or a **general solution** to a differential equation will still have an unknown constant of integration.

When the rate of growth of a population of size $N$ is proportional to $N$ then the growth is exponential.

$\frac{dN}{dt} = kN \Rightarrow \int \frac{1}{N} dN = \int k dt \Rightarrow \ln N = kt + c$ (the modulus signs are not needed as $N > 0$)

This can be rearranged to give $N = e^{kt+c} = e^c e^{kt} = Ae^{kt}$, where $A = e^c$. This is such a common substitution that normally solutions would go straight from $\ln N = kt + c$ to $N = Ae^{kt}$. As $A$ is the value of $N$ at time $t = 0$ it is often written as $N = N_0 e^{kt}$

The only other set of differential equations that need to be solved without using numerical methods are coupled differential equations of the form:

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

These can be written as the matrix equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

**Note**

These equations could also be written as

$\dot{x} = ax + by$

$\dot{y} = cx + dy$

When the eigenvalues, $\lambda_1$ and $\lambda_2$, of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are real and distinct then the solution is $\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$ where $\boldsymbol{x}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$

This equation is given in section 5.17 of the formula book.

Finding eigenvalues and eigenvectors was covered in section 1.5, from page 37.

Coupled equations of this form are often used to model competing populations.

**Example 5.5.2**

In order to control a population of rabbits on an island, the land owners introduced a population of foxes. The early growth of the two populations can be modelled by the equations below, where $x$ is the population of rabbits (measured in 100s), $y$ is the population of foxes and $t$ is the time in years since the foxes were introduced.

$$\dot{x} = 1.5x - 0.5y$$
$$\dot{y} = x$$

(a) Find general equations for the population of rabbits and foxes, $t$ years after the foxes are introduced.

(b) Given that all parameters in the general solution are positive and the model remains valid for high values of $t$, find the long term ratio of foxes to rabbits.

**Solution**

(a) $\det\begin{pmatrix} 1.5-\lambda & -0.5 \\ 1 & -\lambda \end{pmatrix} = 0$

The equation can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1.5 & -0.5 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$-\lambda(1.5-\lambda)+0.5=0$

$\lambda^2 - 1.5\lambda + 0.5 = 0$

The first step is to find the eigenvalues.

$\lambda = 0.5, 1$

When $\lambda = 0.5$ $\begin{pmatrix} 1 & -0.5 \\ 1 & -0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x - 0.5y = 0 \Rightarrow y = 2x$

To find the eigenvectors solve either

$$\begin{pmatrix} 1.5-\lambda & -0.5 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 1.5 & -0.5 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

When $\lambda = 1$ $\begin{pmatrix} 0.5 & -0.5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x - y = 0 \Rightarrow y = x$

Possible eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The form of the solution can come directly from the formula book (section 5.17). The two equations can also be written separately, giving $x$ and $y$ in terms of $t$:

$$x = Ae^{0.5t} + Be^{t}$$
$$y = 2Ae^{0.5t} + Be^{t}$$

(b) For large values of $t$ the term $Be^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will dominate as $e^{t} > e^{0.5t}$ so the long term ratio of foxes to rabbits will tend to 1 : 100

The ratio of $x$ to $y$ is 1 : 1 but $x$ is the number of rabbits measured in 100s.

## Phase diagrams / portraits

Phase diagrams show the change in the sizes of populations of $x$ and $y$ as $t$ increases. They are useful for indicating how the populations might change for different values of initial populations.

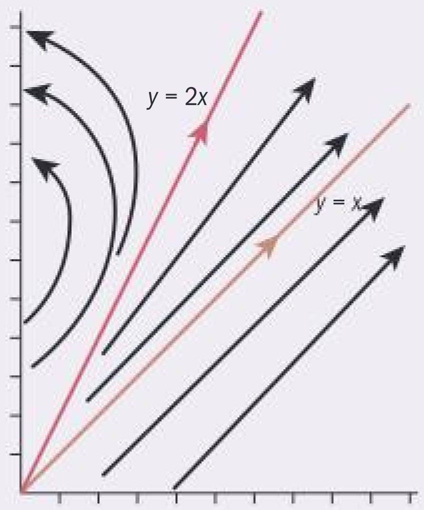
### Example 5.5.3

(a) Draw the phase diagram for the population of foxes and rabbits from example 5.5.2

(b) Use the phase diagram to explain what would happen to the two populations if 5 foxes were introduced when the rabbit population was 200.

**Solution**

(a)



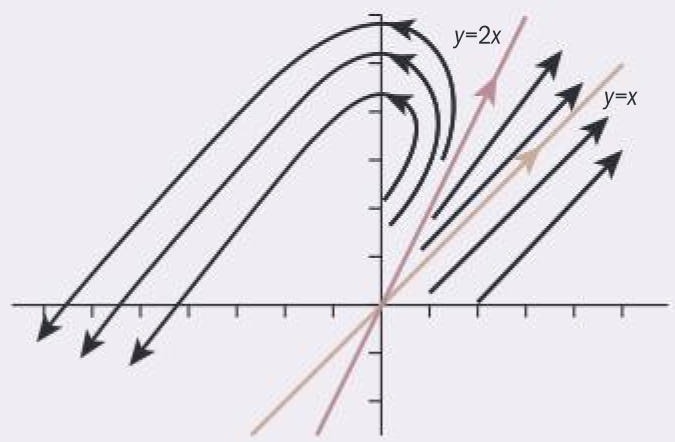

Only the first quadrant is needed as $x, y > 0$

When drawing a phase diagram for a system with real distinct eigenvalues, add the two lines corresponding to the eigenvectors. Because the eigenvalues are positive, the direction along both eigenvectors is away from the origin.

Because $y = x$ is associated with the largest eigenvalue the trajectories will move towards the direction of this line, the eigenvector parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The line of the eigenvectors cannot be crossed so the trajectories in the top part of the diagram need to move towards the direction $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ rather than $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) The trajectory from the point (2, 5) ends at the $y$-axis. This means that the population of rabbits becomes extinct. The model will then be no longer valid, but it is clear that the population of foxes would then also become extinct unless they have an alternative food source to rabbits.




Note: if requested, the particular solution for these initial values could be found by substituting $x = 2, y = 5$ and $t = 0$ into the general solution.

## Imaginary or complex eigenvalues

If the eigenvalues for a coupled system are of the form $a \pm bi$ then, as above, the solution can be written as

$$x = (Ae^{(a+bi)t}p_1 + Be^{(a-bi)t}p_2) = e^{at}(Ae^{bit}p_1 + Be^{-bit}p_2)$$

This time the solution can be split into two parts. As in section 1.4 $e^{bit}$ can be written as $\cos bt + i \sin bt$ and represents circular motion, the period of which is $\frac{2\pi}{b}$

If $a = 0$, then the trajectories form circles or ellipses centred on (0, 0)

If $a > 0$, then as $t$ increases the trajectories spiral away from (0, 0)

If $a < 0$, then as $t$ increases the trajectories spiral towards (0, 0)

**Assessment tip**

In an exam you will not need to find the equation for any system with complex or imaginary eigenvalues but will need to be able to draw and interpret phase portraits.

### Example 5.5.4

Consider the system:

$\dot{x} = x + 3y$

$\dot{y} = -4x - 3y$

Sketch the trajectory for $x$ and $y$ given $x = 2$, $y = 0$ at time $t = 0$

**Solution**

$$\det\begin{pmatrix} 1-\lambda & 3 \\ -4 & -3-\lambda \end{pmatrix} = 0$$

The first step is to find the eigenvalues.

$(1-\lambda)(-3-\lambda) + 12 = 0$

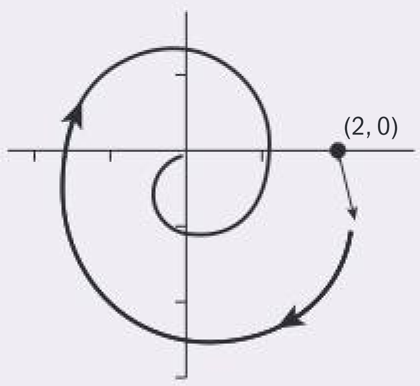
$\lambda^2 + 2\lambda + 9 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-36}}{2} = -1 \pm 2\sqrt{2}\mathrm{i}$

Complex eigenvalues mean the trajectory is a spiral.

The actual equation is not needed, but to draw the trajectory you need to know the sign of the real part (here it is negative so the trajectory spirals inwards). We know the trajectory passes through (2, 0) and spirals inwards. We still need to find whether it spirals clockwise or anticlockwise. This is done by calculating the values of $\dot{x}$ and $\dot{y}$ at (2, 0)

At (2, 0) $\dot{x} = 2$, $\dot{y} = -8$




The fact that $\dot{y}$ is negative means the spiral is clockwise (motion is 'downwards'). The gradient at (2, 0) can be found using the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-8}{2} = -4$

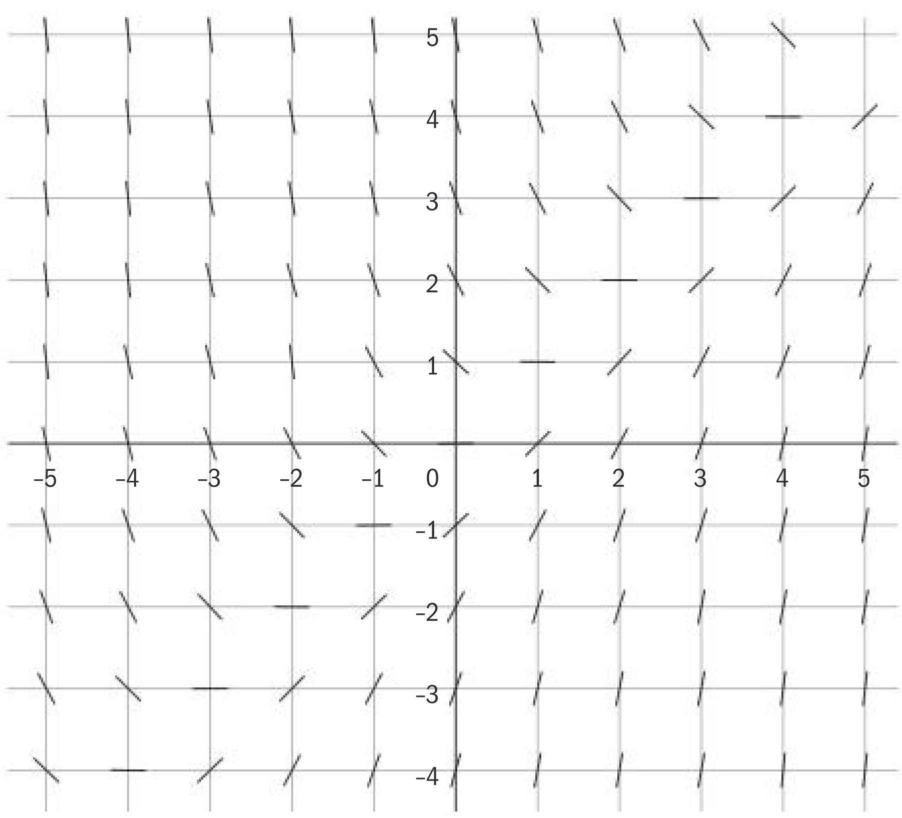
## Numerical solutions

Most differential equations cannot be solved exactly, so approximation methods need to be used to find qualitative or approximate solutions.

### Slope fields

A differential equation will give the gradient of a curve at any value. A **slope field** is a diagram that shows these gradients at many points, indicating them by a small line segment, as in the diagram below, which is the slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$

For example, the point (2, 0) has a gradient of $2 - 0 = 2$



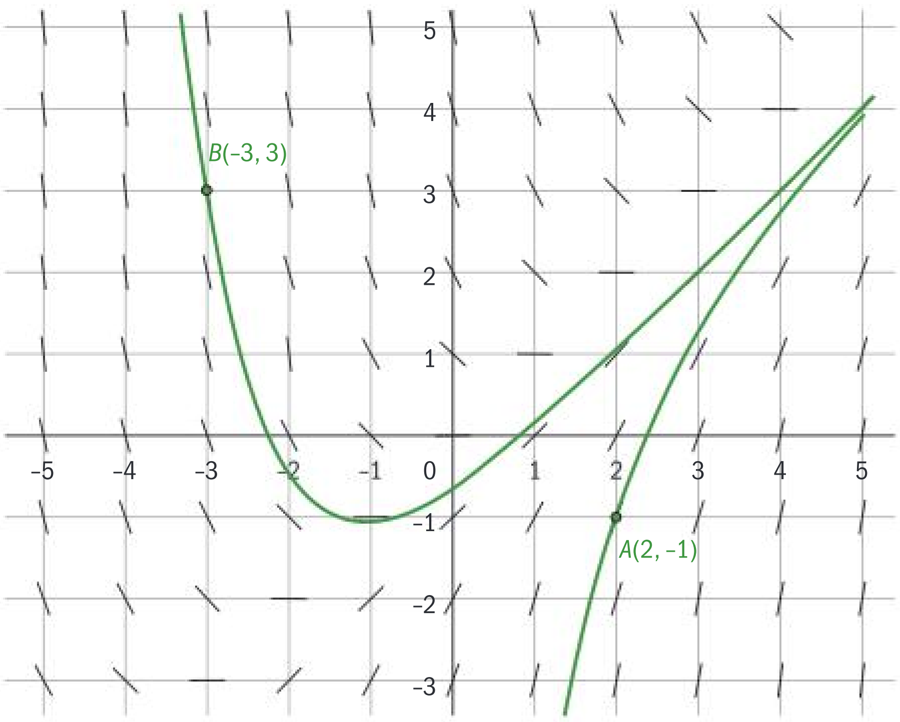

**Note**

From the equation we can see that all maximum or minimum points will occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, which in this case is along the line $y = x$

**Note**

These diagrams are not phase portraits, as phase portraits show trajectories over time. These diagrams simply show solutions to the differential equations.

The diagram indicates the graphs of the solutions for all different starting values. It is possible to use the line segments to construct an approximation for the graph of a particular solution by following the directions of the line segments. The diagram below shows the graphs of the solutions which pass through $A(2, -1)$ and $B(-3, 3)$




It can be seen that the minimum of the graph that passes through $B$ lies on the line $y = x$ as predicted.

Both of the particular solutions indicate there is an asymptote along the line $y = x - 1$ (or $x - y = 1$). This is because the gradient of this line is 1 and on this line the gradient of the curve, $\frac{dy}{dx}$, is also 1 $\left(\frac{dy}{dx} = x - y = 1\right)$

## Euler's method

Euler's method uses the same idea as slope fields. It assumes that the gradient at a point remains constant for a small distance ($h$) beyond the point and uses this to find an approximation for a new point. It then repeats the process.

The formula is $y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$ where $h$ is a constant (step length)

**Note**

The accuracy of the values obtained will depend on the size of $h$ and how far your approximation is from the original point.

**Assessment tip**

Make sure you know how to use Euler's method using the in-built functions on your GDC.

The formula for Euler's method is in section 5.16 of the formula book.

**SAMPLE STUDENT ANSWER**

Consider the differential equation $\frac{dy}{dx} = \sqrt{x + y}$, $(x + y \geq 0)$ satisfying the initial conditions $y = 1$ when $x = 1$. Let $y = c$ when $x = 2$.

Use Euler's method to find an approximation for the value of $c$ using a step length of $h = 0.1$. Give your answer correct to 4 dp.

$y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$

$x_1 = 1.1, \quad y_1 = 1.14142$

$x_{10} = 2, \quad y_{10} = 2.77011$

when $x = 2, \ y = 2.77$

▲ The student has written the formula they are entering into the GDC. If they did no other work this alone would have scored 3 of the 6 marks available for this question.

▲ Time should not be wasted writing out all the intermediate stages, though it is no bad thing to write one or two. By writing this answer here the student achieved two further marks which would not have been given otherwise.

▼ This answer has not been given to the requested degree of accuracy. Without the methods shown above the implied marks for this question would not have been awarded.

These formulae are given in section 5.16 of the formula book.

## Solving coupled differential equations using Euler's method

Most coupled differential equations cannot be solved exactly, so numerical methods need to be used. Consider the coupled system of differential equations $\frac{dx}{dt} = f_1(x, y, t)$, $\frac{dy}{dt} = f_2(x, y, t)$

Euler's method uses the following formulae:

$x_{n+1} = x_n + h \times f_1(x_n, y_n, t)$, $y_{n+1} = y_n + h \times f_2(x_n, y_n, t)$, $t_{n+1} = t_n + h$, where $h$ is a constant (step length)

**Assessment tip**

Make sure you know how to use Euler's method for coupled differential equations using the inbuilt functions on your GDC.

**Note**

An equilibrium point is one at which both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are equal to zero.

### Example 5.5.5

The differential equations used for modelling the population of rabbits and foxes in example 5.5.2 is modified to allow for the fact that as the populations increase the resources to support them are restricted. The new set of differential equations proposed are

$\dot{x} = 1.5x - 0.5y - 0.1x^2$

$\dot{y} = 2x - 0.2xy$

where $t$ is measured in years, $x$ is the number of rabbits (measured in 100s) and $y$ is the number of foxes ($x, y \geq 0$). The populations have two non-zero points of equilibrium.

(a) Find the numbers of rabbits and foxes at each point of equilibrium.

A survey finds that there are 400 rabbits and 6 foxes.

(b) Use Euler's method with a step length of one-tenth of a year to find the number of rabbits and foxes one year later, giving both answers to the nearest whole number.

(c) State the equilibrium point the populations are tending towards.

**Solution**

(a) $\dot{y} = 2x - 0.2xy = 0 \Rightarrow x = 0$ or $y = 10$

If $x = 0$, $\dot{x} = -0.5y = 0$
$\Rightarrow y = 0$

Equilibrium point is (0, 0); no foxes or rabbits.

If $y = 10$:
$\dot{x} = 1.5x - 5 - 0.1x^2 = 0$
$\Rightarrow x = 5, x = 10$

Equilibrium points are (5, 10) and (10, 10)

Hence, population at the equilibrium points are 10 foxes and either 500 or 1000 rabbits.

To find the equilibrium points solve $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ simultaneously.

(b) $x_{n+1} = x_n + 0.1 \times (1.5x_n - 0.5y_n - 0.1(x_n)^2)$

$y_{n+1} = y_n + 0.1 \times (2x_n - 0.2x_n y_n)$

$t_{n+1} = t_n + 0.1$

| T | U1 | U2 |
|---|---|---|
| 0.3 | 4340077248851 | 6.91319756974 |
| 0.4 | 4.52156049834 | 7.18488387399 |
| 0.5 | 4.63670528999 | 7.43945823146 |
| 0.6 | 4.74461344509 | 7.67687705602 |
| 0.7 | 4.84734804162 | 7.89732346311 |
| 0.8 | 4.94461624434 | 8.10117156298 |
| 0.9 | 5.0367578048 | 8.28895112168 |
| 1.0 | 5.12413462759 | 8.46131389752 |
| 1.1 | 5.20712157004 | 8.61900259229 |

After one year the values of $x$ and $y$ are 5.124… and 8.461…

**Assessment tip**

Always write down the formulae for Euler's method so you can be sure of getting some marks, even if you make an error entering them into your GDC.

| | | |
|---|---|---|
| | After one year the number of rabbits is 512 and the number of foxes is 8. | These need to be rounded to give the population to the nearest integer. |
| (c) | 1000 rabbits, 10 foxes | This can be found by continuing the table to find values of $x$ and $y$ as $t$ increases. |

## Second order differential equations

There are many practical processes which can be described using second order differential equations of the form $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$

These can be solved by writing the second order equation as a system of coupled first order equations.

Let $\frac{dx}{dt} = y$, then $\frac{dy}{dt} = f(x, y, t)$

### Example 5.5.6

The motion of parachutist falling under gravity but subject to air resistance can be modelled by the equation:

$$\frac{d^2x}{dt^2} = 9.8 - 0.004\left(\frac{dx}{dt}\right)^2$$

where $x$ is the distance fallen, $t$ seconds after he leaves the plane.

The initial velocity of the parachutist can be taken as equal to zero.

(a) State what $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ represent in the context of the question.

(b) Use the differential equation to find the limit for the velocity of the parachutist.

(c) Use Euler's method with a step length of 0.2 to find (i) the distance fallen (ii) the velocity of the parachutist after 2 seconds.

**Assessment tip**

In an exam, the question will always give the differential equation, so knowledge from outside the syllabus will not be required.

**Note**

This limit is referred to as the **terminal velocity**.

**Solution**

(a) $\frac{d^2x}{dt^2}$ is the acceleration and $\frac{dx}{dt}$ is the velocity of the parachutist

(b) $9.8 - 0.004\left(\frac{dx}{dt}\right)^2 = 0 \Rightarrow \frac{dx}{dt} = \sqrt{\frac{9.8}{0.004}} = 49.5\ \text{m s}^{-1}$

The limit will occur when the acceleration is zero.

(c) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 9.8 - 0.004\left(\frac{dx}{dt}\right)^2 = 9.8 - 0.004y^2$

$x_{n+1} = x_n + 0.2y_n$, $y_{n+1} = y_n + 0.2(9.8 - 0.004y_n^2)$

$t_{n+1} = t_n + 0.2$

It is important to show the formulae that have been used.

When $t = 2$

(i) $x \approx 17.3$ m

(ii) $y = \frac{dx}{dt} \approx 18.8\ \text{m s}^{-1}$

$y$ is the velocity

# SL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

1. Consider the curve $y = 2x^4 - 7x$

   a. Find $\frac{dy}{dx}$

   The point $P(-1, 9)$ lies on the curve.

   b. Find the gradient of the tangent at point $P$.

   c. Find the equation of this tangent in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers.

2. a. Find the value of $\int_1^2 \frac{1}{x} dx$

   b. Use the trapezoidal rule with four trapezoids to find an approximation for $\int_1^2 \frac{1}{x} dx$

   c. Hence, find the percentage error for the approximation found in part **b**

3. A curve $y$ has derivative $\frac{dy}{dx} = 3x^2 - 8x + 1$. Find the equation for $y$, given that it passes through the point (2, 7)

4. A curve $C$ is defined by the equation $y = x^4 - 32x + 8$

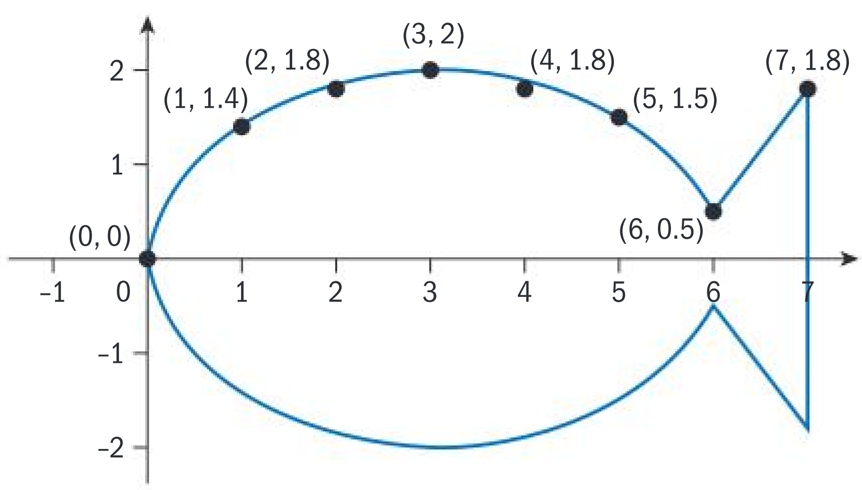
   a. Find $\frac{dy}{dx}$

   Let $P$ be the minimum point on the curve $C$.

   b. Write down an equation satisfied by the $x$-coordinate of $P$.

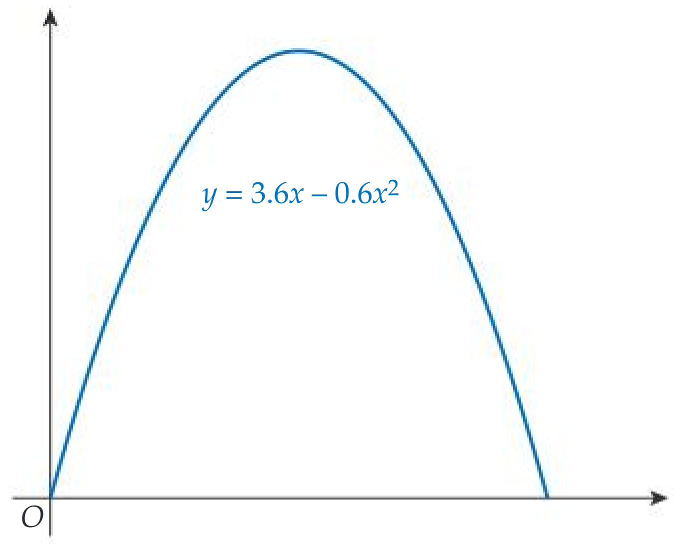
   c. Hence, find the coordinates of $P$.

5. Mika needs to estimate the area of a fish design that she is creating. She places the design on a coordinate grid, marked out in centimetres, and she plots eight points as shown.




   Use the trapezoidal rule to find an approximate value for the area of the fish.

6. A straight tunnel has a uniform cross-section in the shape of a parabola with equation $y = 3.6x - 0.6x^2$, where $y$ is the height of the tunnel in metres at a horizontal distance of $x$ metres from the origin, $O$.




   a. Find the width and the maximum height of the cross-section of the tunnel.

   The tunnel is 50 m long.

   b. Find the volume of earth removed to create the tunnel.

## GROUP 2

7. The equation of a curve is $y = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 6x - 3$

   a. Find $\frac{dy}{dx}$

   The gradient of the curve at a point $P$ is 30.

   b. Find the coordinates of $P$.

8. The average cost $C_A$ of producing $x$ light bulbs changes at a rate of $\frac{dC_A}{dx} = -6x^{-2}$ USD per unit.

   Given the average cost of producing 12 units is 4.5 USD, find:

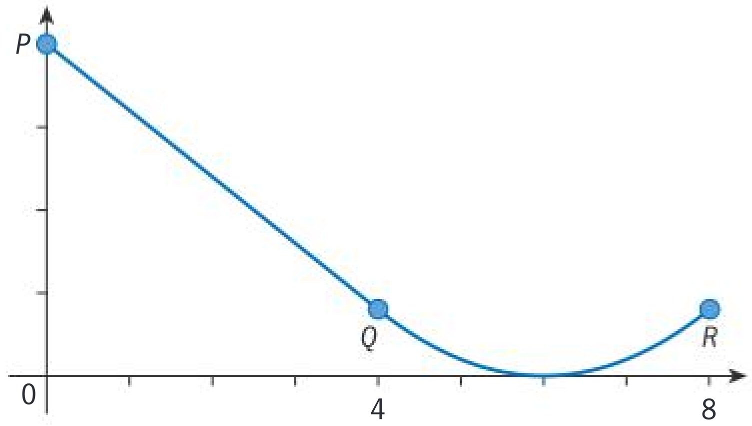
   a. the average cost of producing 15 units

   b. an expression for the total cost $C$ of producing $x$ units.

9. A ramp consists of a straight section $P$ to $Q$, and a curved section $Q$ to $R$.

   The two sections can be modelled by the piecewise function

$$f(x) = \begin{cases} ax + b & 0 \le x < 4 \\ 0.2x^2 - 2.4x + 7.2 & 4 \le x \le 8 \end{cases}$$




The two branches of $f(x)$ need to be continuous and have the same gradient at $x = 4$.

Find the values of $a$ and $b$.

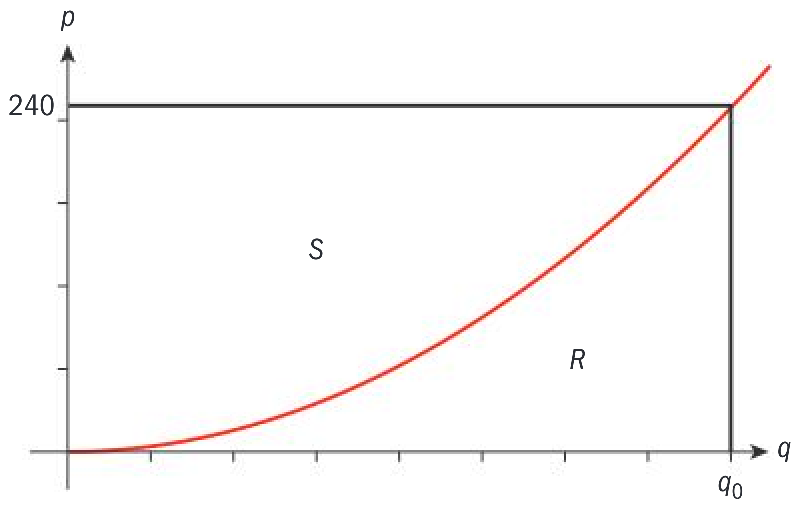
**10.** Let $q$ be the number of units a manufacturer will supply at a price $p$ per unit.

For a particular factory, the supply equation connecting $p$ and $q$ is $p = 0.6q^2$

The factory receives a payment of €240 per unit.

**a.** Find the number of units $(q_0)$ that they will supply for this price.

The graph of the supply function is shown below.




**b.** Find the area of the region $R$, bounded by the curve, the $q$-axis and the line $q = q_0$

Economists call the area of region $S$, bounded by the curve, the $p$-axis and the horizontal line representing price, the **producer surplus**.

**c.** Find the producer surplus when $p = 240$

**11.** Consider the function $f(x) = 2x^3 - 9x^2 + 12x + 1,\ 0 \le x \le 3$

**a.** Find $f'(x)$

**b.** Find the values of $x$ such that $f'(x) = 0$

**c.** Give the range of values of $x$ for which $f'(x) < 0$

**d. i.** Explain why the maximum value of $f$ does not occur when $x$ is equal to either of the values found in part **b**

**ii.** Write down the maximum value of $f$.

**12.** A seller of ice creams uses the following model to estimate the number of sales of ice creams, $n$, given the expected temperature $T$ °C.

$$n = -\frac{1}{50}T^3 + T^2 + 50,\ 0 \le T \le 30$$

**a.** Find the expected number of ice creams sold when the temperature is 20 °C

**b.** Find an expression for the rate of increase in ice cream sales when the temperature is $T$ °C

**c.** Hence, find the temperature at which the rate of increase in ice cream sales is the greatest.

**d.** State the significance of this point on the graph of $n$ against $T$.

## GROUP 3

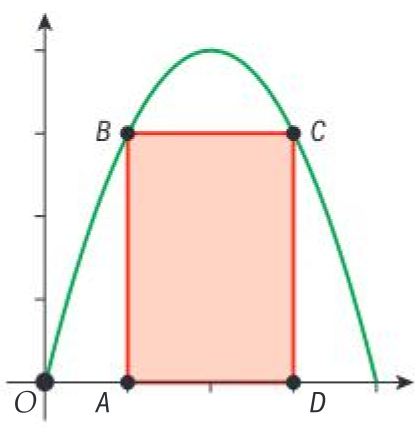
**13.** A function $f$ is given by $f(x) = 2x + \frac{3}{x} - 2,\ x > 0$

**a.** Find the derivative of $f$.

**b.** Show that the $x$-coordinate of the point on the graph of $f$ at which the gradient of the tangent is equal to $-4$ is $\sqrt{\frac{1}{2}}$

**14.** Consider the graph of the parabola given by $f(x) = 4x - x^2$

A rectangle $ABCD$ is drawn with upper vertices $B$ and $C$ on the graph of $f$, and $AD$ on the $x$-axis, as shown in the diagram.




**a.** Let $OA = x$

**i.** Show that $AD = 4 - 2x$

**ii.** Hence write down an expression for the area, $R$, of the rectangle in terms of $x$.

**b.** Find the maximum possible area of the rectangle.

**c.** **i.** Find $\frac{dR}{dx}$

**ii.** Find the rate of change of the area when $x = 1$

**15.** Let $f(x) = ax^3 + bx^2 + cx$, where $a$, $b$ and $c$ are real numbers.
The graph of $f$ passes through the point (2, 2)

**a.** Show that $8a + 4b + 2c = 2$

The graph has a local minimum at (1, –5)

**b.** Find two other equations in $a$, $b$ and $c$.

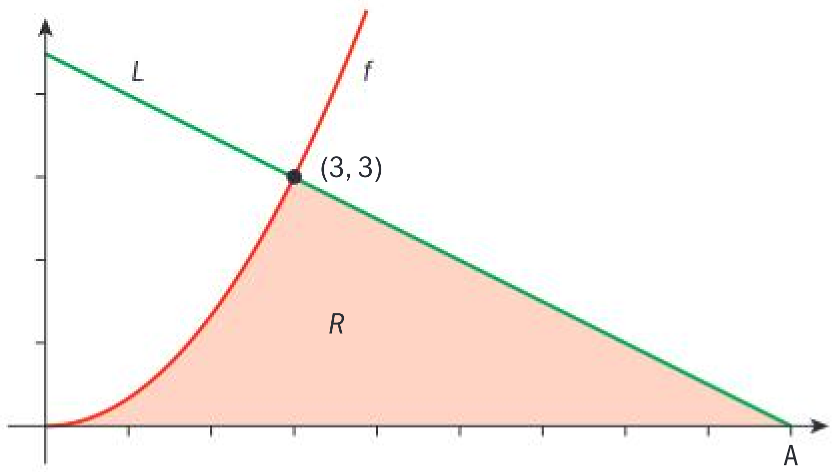
**c.** Find the values of $a$, $b$ and $c$.

**16.** Let $f(x) = \frac{1}{3}x^2$. Line $L$ is the normal to the graph of $f$ at the point (3, 3)

**a.** Show the equation of $L$ is $x + 2y - 9 = 0$

**b.** Point $A$ is the $x$-intercept of $L$. Find the $x$-coordinate of $A$.

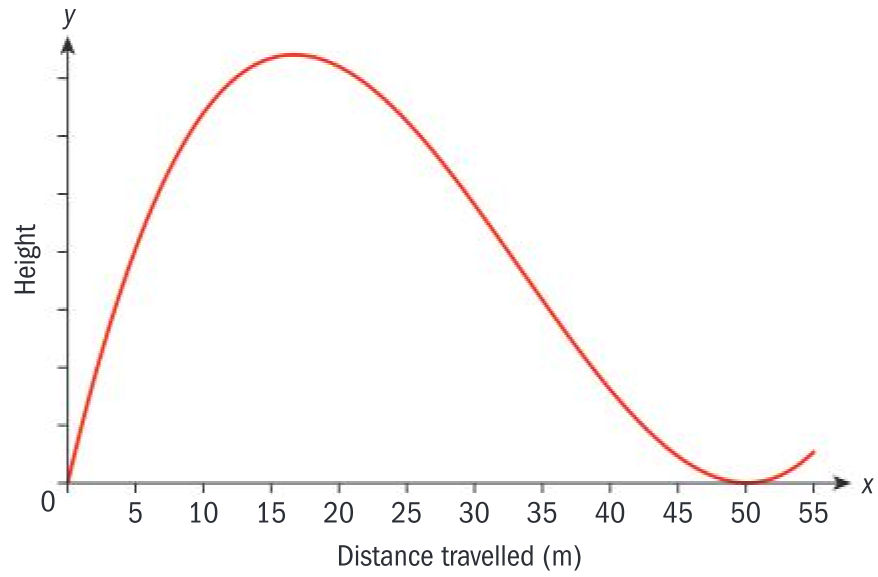
In the diagram, the shaded region $R$ is bounded by the $x$-axis, the graph of $f$ and the line $L$.




**c.** **i.** Write down an expression for the area of $R$.

**ii.** Find the area of $R$.

**17.** The height, $h$, of a car on a roller coaster after travelling a distance $x$ is given by the equation $h(x) = \frac{x^3}{500} - \frac{x^2}{5} + 5x$, $0 \le x \le 55$, where all distances are measured in metres.




Let the point $P(a, b)$ be the coordinates of the local maximum of $h$

**a.** **i.** Find the values of $a$ and $b$.

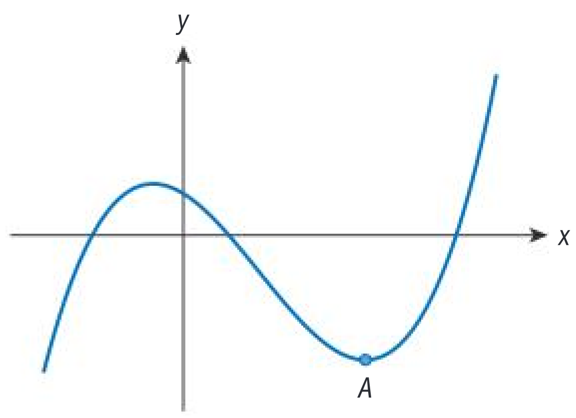
**ii.** State the maximum height of the roller coaster.

**b.** Find $h'(x)$

**c.** Find the steepest gradient of the roller coaster for $a \le x \le 55$

**d.** Find the angle that the roller coaster's track makes with the horizontal at this point.

**18.** The diagram shows the graph of a function $f$. There is a local minimum at the point $A$.




The derivative of $f$ is given by $f'(x) = 3x^2 - 5x - 2$

**a.** Find the $x$-coordinate of $A$.

The coordinates of the $y$-intercept of the graph of $f$ are $(0, 1.5)$

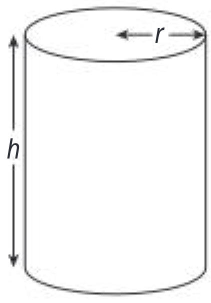
**b.** Find an expression for $f(x)$

The equation $f(x) = k$ has two solutions.

**c.** Find two possible values for $k$.

## PAPER 2

**1.** A closed cylindrical can with radius $r$ centimetres and height $h$ centimetres has a volume of 50 $cm^3$.




**a.** Express $h$ in terms of $r$.

The material for the base and top of the can costs 8¢ per $cm^2$ and the material for the curved sides costs 6¢ per $cm^2$.

**b.** Show that the total cost of the material, $C$, is given by the equation

$$C = 16\pi r^2 + \frac{600}{r}$$

**c.** Find $\frac{dC}{dr}$

**d.** Given that there is a minimum value for $C$, write down an equation it must satisfy.

**e.** Find the value of $r$ for which the cost of the can is minimum.

**f.** Find the value of $h$ for this can.

**2.** The charity committee in a school is planning to sell doughnuts. The committee calculates that the number of doughnuts, $n$, it could sell at price $x$ euros is given by the function $n = \frac{600}{x^2}$, $x \geq 1$

**a.** Write down an expression for the money they would receive from sales when selling the doughnuts for $x$ euros.

They can purchase the doughnuts from a company at a cost of 0.75 euros per doughnut.

**b.** Write down an expression for the cost of purchasing $n$ doughnuts.

The profit, $P$, is found by subtracting the cost of the doughnuts from the money gained from the sales of the doughnuts.

**c.** Find an equation for the profit made when charging $x$ euros per doughnut.

**d.** Find $\frac{dP}{dx}$

**e.** Use your answer to part **d** to find the price at which the charity committee should sell the doughnuts in order to maximize their profit, and find the number of doughnuts they need to buy.

# HL PRACTICE QUESTIONS

## PAPER 1, GROUP 1

**1.** Let $f(x) = \frac{x^2}{e^x}$

**a.** Find $f'(x)$

**b.** Hence or otherwise, find the range of values of $x$ for which $f$ is an increasing function.

**2.** **a.** Sketch the graph of $y = 2(x-1)(x-3)$ for $0 \leq x \leq 4$

**b.** Find $\int_0^4 2(x-1)(x-3)dx$

**c.** Find the total area between the curve, the $x$-axis and the lines $x = 0$ and $x = 4$

**3.** The rate of decay of a radioactive substance $\frac{dM}{dt}$ is directly proportional to the amount of substance, $M$, remaining.

Let $\frac{dM}{dt} = -kM$ where $k > 0$ and $t$ is measured in years. It is given that when $t = 0$, $M = 50$ g

**a.** Find an expression for $M$ in terms of $k$ and $t$.

It is given that after 400 years, 50% of the substance has decayed.

**b.** Find the value of $k$.

**4.** Garlic mustard is an invasive species in large parts of north America. A team of biologists are hoping to control the spread by introducing a type of weevil which is known to restrict its growth.

A simple model is proposed to help calculate the number of weevils that should be introduced in a given area:

$\dot{x} = 2x - 2y$
$\dot{y} = 2x - 3y$

In this model, $x$ is the number of square metres affected and $y$ is the number of weevils.

It is given that the eigenvalues and corresponding eigenvectors for the matrix

$\begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}$ are $-2$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and 1, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

**a.** Sketch a phase portrait for this system with $x, y \geq 0$

**b.** One area has 150 m$^2$ affected. Find the minimum number of weevils that should be introduced to ensure the garlic mustard is removed.

**5.** Given $y = \ln(2x-3)$

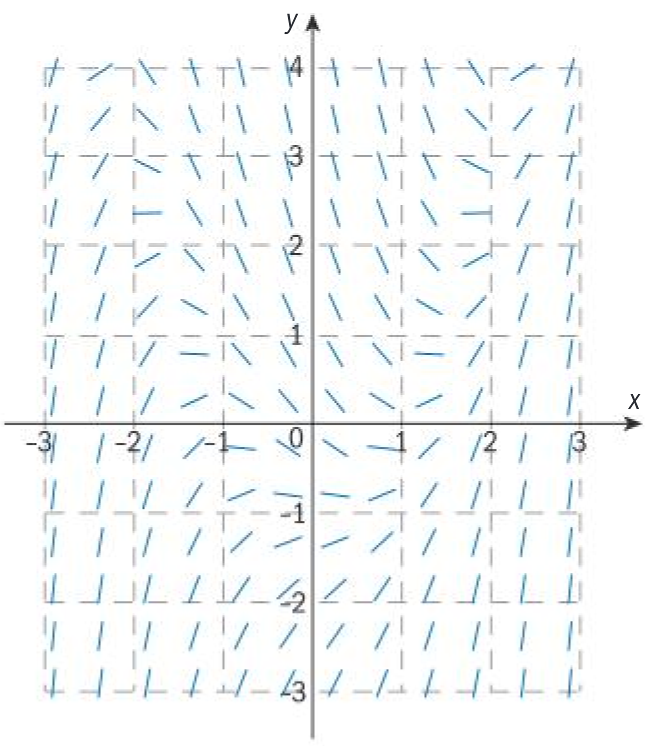
**a.** Find an expression for $\frac{dy}{dx}$

The normal to the curve $y = \ln(2x-3)$ at the point (2, 0) intersects the $y$-axis at $c$.

**b.** Find the value of $c$.

**6.** Consider the differential equation $\frac{dy}{dx} = x^2 - y - 1$.

Part of the slope field for this differential equation is shown.



**a. i.** Find the equation of the curve on which any locally maximum or minimum points must lie.

**ii.** Sketch this curve on the slope field diagram.

Let $C$ be the solution curve that passes through the point (0, 0)

**b.** Use Euler's method with a step length of 0.2 to find the value of $y$ on this curve when $x = 2$

**c.** On the slope field diagram, sketch the solution that passes through (0, 0) for $-3 \le x \le 3$

## GROUP 2

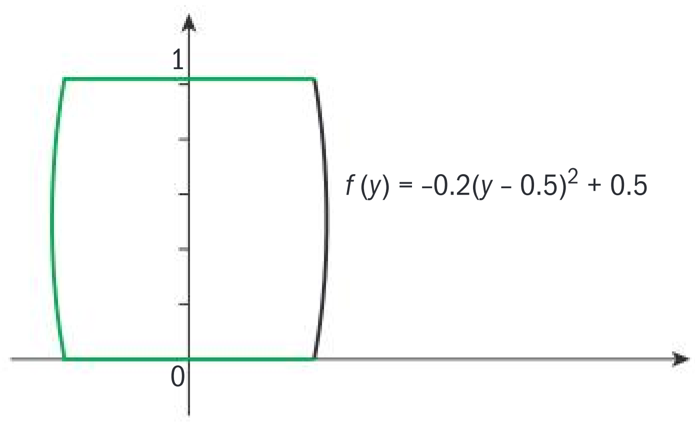
**7.** A particle moves along a straight line so that its velocity, $v$, after $t$ seconds is given by $v(t) = 1.4^t - 2.7$, for $0 \le t \le 5$.

**a.** Find when the particle is at rest.

**b.** Find the acceleration of the particle when $t = 2$

**c.** Find the total distance travelled by the particle in the first 5 seconds.

**8.** Let $f(y) = -0.2(y-0.5)^2 + 0.5$ for $0 \le y \le 1$. Henri uses $f(y)$ as a model to create a barrel. The region enclosed by the graph of $f$, the $y$-axis, the line $y = 0$ and $y = 1$ is rotated 360° about the $y$-axis. This is shown in the diagram where the units are in metres.




**a.** Find the volume of the barrel.

The barrel is filled with water. The volume of the water in the barrel is denoted by $V$ and the rate at which water flows into the barrel can be approximately modelled by $\frac{dV}{dt} = 0.6e^{-t}$ m$^3$ min$^{-1}$, where $t$ is the time in minutes since the barrel began to be filled. It is given that at $t = 0$, $V = 0$

**b.** Find an expression for $V$ in terms of $t$.

**c.** Find the maximum percentage of the barrel that will be filled.

**9.** The size of a population, $N$, can be modelled by the equation $N = te^{-pt}$, where $p > 0$

**a.** Find $\frac{dN}{dt}$

**b.** Hence, find the value of $t$ and corresponding value of $N$ for which $\frac{dN}{dt} = 0$

**c.** Find $\frac{d^2N}{dt^2}$

**d.** Hence, show that the value of $N$ found in part **b** is a maximum for all $p > 0$

**10.** Consider the curve $y = \tan(2x - 4)$

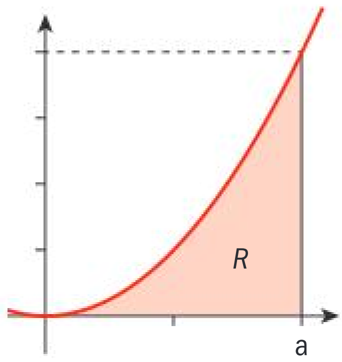
**a.** Find $\frac{dy}{dx}$

**b.** Find $\frac{d^2y}{dx^2}$

Let $a > 0$ be the $x$-coordinate of the point of inflexion whose $x$-coordinate is closest to the origin.

**c.** Find the value of $a$.

**11.** Consider the curve $y = x^2$. The region $R$ is between the curve, the $x$-axis and the line $x = a$.




**a.** Find an expression for the area of $R$.

**b.** Hence or otherwise, find an expression for the area between the curve, the $y$-axis and the line $y = a^2$

**12.** The velocity of a particle is given by the equation $v = 4te^{t^2}$

**a.** Find an expression for the acceleration of the particle at time $t$.

**b.** Find an expression for the displacement of the particle from its initial position at time $t$.

## GROUP 3

**13.** Consider the system of coupled differential equations

$\dot{x} = 3x - 9y$

$\dot{y} = 4x - 3y$

**a.** Use the chain rule to find the value of $\frac{dy}{dx}$ as the trajectories cross:

i the $x$-axis

ii the $y$-axis

**b.** Find the eigenvalues for this system.

**c.** Sketch the trajectory of a particle that begins at the point (2, 0), indicating clearly the direction of motion.

**14.** Consider the curve $f(t) = \frac{350}{1 + 6e^{-t}}$, $t > 0$

**a.** Find $f'(t)$

The function represents the number of goats on an island.

**b.** Write down:

i. the carrying capacity of the island

ii. the number of goats at $t = 0$

**c.** Find the time at which the rate of increase in the population of goats is greatest and state the rate of increase at this time.

**d.** Find the size of the population when the rate of increase is greatest.

**15.** The acceleration of a particle is given by the piecewise function

$$a(t) = \begin{cases} 2t & 0 \le t < 3 \\ 6 & t \ge 3 \end{cases}$$

where $t$ is measured in seconds and $a$ in m s$^{-2}$

It is given that the velocity of the particle, $v$, is continuous at $t = 3$ and when $t = 0, v = -9$

**a.** Find an expression for the velocity of the particle.

**b.** When $t = 5$, find:

i. the total distance travelled by the particle

ii. the particle's distance from its initial position.

**16.** Water is pumped into a tank at a rate of $W = [4\sin(0.25t - 2) + 4]$ litres per hour.

At the same time, water is removed from the tank at a rate of $R = [3\sin(0.25t) + 4]$ litres per hour, where $t$ is measured in hours.

Let $V$ be the volume of water in the tank.

**a.** Find an expression for $\frac{dV}{dt}$

**b.** Given the tank initially contains 100 litres of water, find an expression for the volume of water in the tank at time $t$.

**c.** Find the minimum amount of water in the tank during a 24-hour period.

**d.** Find the difference between the total amount of water flowing into the tank and the total amount of water flowing out of the tank over a 24-hour period.

**17.** Let the concentration of a chemical $A$ be $a$ mg per litre and the concentration of a chemical $B$ be $b$ mg per litre.

The rate of reaction of a chemical $A$ is $\frac{da}{dt}$ and this is proportional to $ab^2$.

The rate of reaction of $A$ is 12 mg per litre per second when the concentration of $A$ is 4 mg per litre and the concentration of $B$ is 2 mg per litre.

**a.** Show that $\frac{da}{dt} = \frac{3}{4}ab^2$

The concentration of $B$ is given by the equation $b = 1.2 - 0.2t$ for $0 \le t \le 6$, where $t$ is the time from the moment the two substances are mixed.

**b.** Find an expression for the concentration of $A$ at time $t$, where $0 \le t \le 6$, given that $a = 0.5$ when $t = 0$

**c.** Hence write down the maximum concentration of $A$ in the interval $0 \le t \le 6$

**18.** At time $t$, a particle has a velocity, $v$, and a displacement, $x$, where $x > 0$, given by the following equation.

$$v = \frac{e^{-0.1x^2}}{x}$$

**a.** Find an expression for the acceleration of the particle in terms of $x$.

**b.** Find an expression for the displacement of the particle in terms of $t$, given that $x = 0$ when $t = 0$.

## PAPER 2

**1.** Consider the following system of coupled differential equations:

$\dot{x} = x + 2y$

$\dot{y} = x$

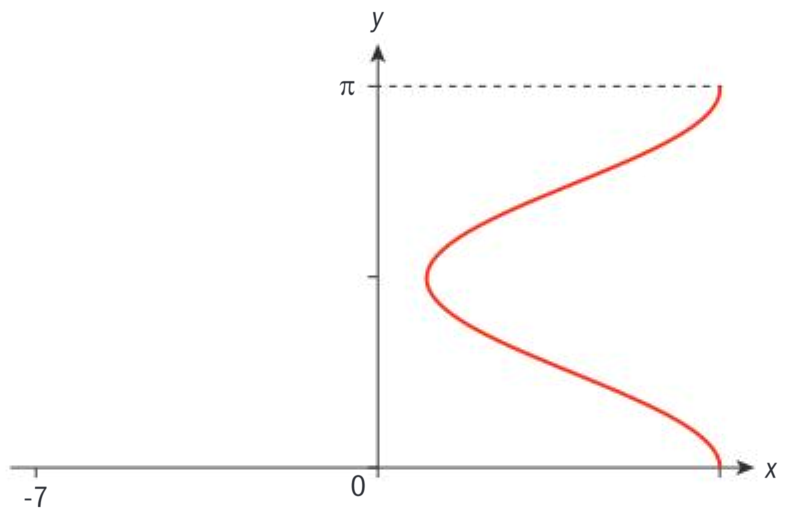
**a.** By first finding the eigenvectors for the system, construct a phase portrait for the system showing clearly the equations of any asymptotes.

Given $x = 3$ and $x = 4$ when $t = 0$:

**b.** find an equation for $x$ in terms of $t$ and an equation for $y$ in terms of $t$

**c.** hence write down the long term ratio of $x : y$

**2.** The graph shows the relation $x = 3\cos 2y + 4,\ 0 \le y \le \pi$




The curve is rotated 360° about the $y$-axis to form a solid of revolution.

**a.** Calculate the volume of the solid formed.

A container with this shape is made with a solid base of diameter 14 cm. The container is filled with water at a rate of 2 cm$^3$ min$^{-1}$. The water depth $t$ minutes after it starts being filled is $h$ cm, $0 \le h \le \pi$, and the volume of water in the container is $V$ cm$^3$

**b. i.** Given that $\frac{dV}{dh} = \pi(3\cos 2h + 4)^2$, find an expression for $\frac{dh}{dt}$

**ii.** Find the value of $\frac{dh}{dt}$ when $h = \frac{\pi}{4}$

**c. i.** Find $\frac{d^2h}{dt^2}$

**ii.** Find the values of $h$ for which $\frac{d^2h}{dt^2} = 0$

**iii.** By making specific reference to the shape of the container, interpret $\frac{dh}{dt}$ at the values of $h$ found in part **ii**

## PAPER 3

**1.** In this question you will use a variety of techniques to model the motion of an object attached to a spring and subject to external forces.

The motion of an object attached to a spring can be modelled by the differential equation $\ddot{x} + 36x = 0$, where $x$ is the displacement of the object from the equilibrium position, $O$.

**a.** Use differentiation to verify that $x = A\sin 6t + B\cos 6t$ is a solution to the differential equation.

It is given that at $t = 0$, $x = 2$ and $\dot{x} = 0$

**b.** Find the values of $A$ and $B$.

**c. i.** Sketch the graph of $x$ against $t$ for $0 \le t \le 2$

ii. Describe the displacement of the object relative to the equilibrium position.

The motion of the object is refined by the installation of a **dampener** which is designed to reduce its motion. The new motion is described by the following differential equation $\ddot{x}+5\dot{x}+36x=0$

d. Write this second-order differential equation as a pair of first-order coupled differential equations.

e. i. Find the eigenvalues for the system in part **d**

ii. Sketch the trajectory of the object on a phase portrait, with $x$ on the horizontal axis and $y$ on the vertical axis, given that when $t=0$, $x=2$ and $y=\dot{x}=0$.

iii. Hence, describe qualitatively the displacement of the object relative to the equilibrium position, $O$.

An additional force now acts on the object, so that the differential equation describing its motion is:

$\ddot{x}+5\dot{x}+36x=2\cos(t)$

f. Write this second-order differential equation as a pair of first-order coupled differential equations.

g. Use Euler's method with a step length of 0.1 to find an approximation for the value of $x$ when $t=1$, given that at $t=0$, $x=2$ and $\dot{x}=0$

h. Sketch the motion of the displacement, $x$, of the object against time for $0 \le t \le 2$

# THE INTERNAL ASSESSMENT: AN EXPLORATION

The internal assessment for SL and HL Mathematics: Applications and Interpretations is the mathematical exploration. This is a report in which you investigate an area of Mathematics. The exploration is a compulsory part of the course which is worth 20% of your final grade. As a guide, the length of the exploration should be between 12 and 20 pages long with double-line spacing and typed in a reasonable font size and style such as Arial 12. The audience for the exploration is your classmates. This means that anyone in your class should be able to follow and understand what you write in your exploration.

Your exploration will be marked using the five criteria named in the table below. The way marks are awarded in criteria A, B, C and D is exactly the same in both SL and HL. Section E, 'use of mathematics', is graded differently between the two courses to reflect the greater sophistication required at HL.

| Criterion | Title | Maximum mark |
|---|---|---|
| A | Presentation | 4 |
| B | Mathematical communication | 4 |
| C | Personal engagement | 3 |
| D | Reflection | 3 |
| E | Use of mathematics | 6 |

**Assessment tip**

You should be familiar with the criteria before starting your exploration, and you should keep a copy of them close to hand throughout the writing process.

Each mark you gain in your exploration adds 1% to your final IB Mathematics total percentage, so doing well in your exploration will have a significant effect on your final grade. You can ask your teacher for informal feedback at any stage, but your teacher can only give written advice and feedback on one draft. It is therefore a good idea to make your draft as complete as you can so that you can get the maximum benefit from this feedback.

Like all internally and externally assessed components in the IB Diploma programme, you must adhere to the IB academic honesty policy throughout your writing, and you must cite sources and provide a bibliography for the sources you have consulted.

## Choice of topic

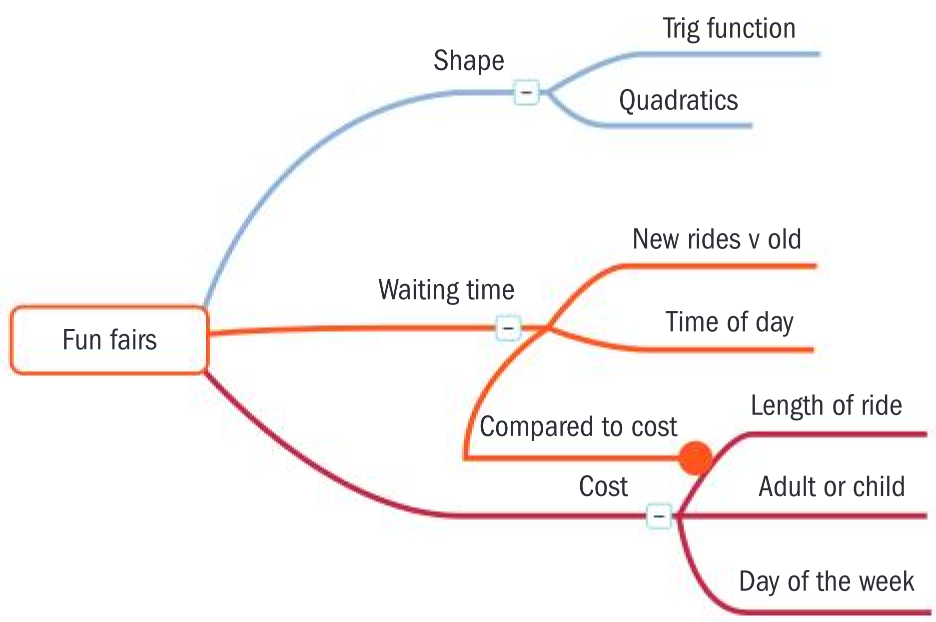
Choosing your topic is an exciting opportunity in which you have a great deal of freedom to explore what you want. However, this choice is one that many students find to be a daunting challenge. It's tempting to rush the process of choosing a topic, but it is important to realize that thinking very carefully about your choice of topic is time well invested. If you choose a topic that really interests you and that you consider important, the exploration will be easier and more enjoyable to write because your motivation will be greater. Choosing the right topic for you increases your chances of scoring more marks.

It is tempting to search online for advice in choosing a topic, but it is better to look inside your own thoughts and consult your teacher than to spend time trying to understand the thoughts of others.

**Assessment tip**

To help you choose a successful topic:

- Your teacher can show you successful topics from previous years. By seeing the huge diversity of topics that have been successful, you will be more aware of what is possible, and your imagination will be engaged.
- Review and reflect on areas of the course that you have enjoyed the most. Could your work in these areas be extended or applied to other contexts that interest you?
- Are there links between mathematics and your hobbies, interests or your plans for future study? In your life outside school or in CAS, there may be a topic "hiding in plain sight" just waiting for you to discover it.
- Draw mind-maps starting with a single area or an idea, and brainstorm how inquiry questions relate to this idea. You could even use an online random word generator to come up with ideas. Set yourself the challenge of coming up with at least three good ideas.




In the above example, you can see areas in a fun fair where mathematics is used and this could develop into a question that you would like to ask, such as "Is there a connection between the length of a ride and the cost of a ticket?" This could be a title for an exploration.

Other examples:

- A student who enjoyed calculus and who wanted to study Aeronautical Engineering at university explored the cross-sectional areas and shapes of aircraft wings, linking forward to his future studies.
- A student who loved skateboarding used video-tracking software to trace the path of the back wheel of her skateboard as she performed tricks and modelled the data, making a link to her hobby.
- A student who was struggling to find a topic but knew he liked cooking arrived at the idea of exploring the rates at which different shapes and sizes of cakes cool by brainstorming the word "cooking".

**Assessment tip**

To help you avoid choosing an unsuccessful topic:

- Avoid topics that are purely descriptive and historical because such topics will limit your grade.
- Avoid topics whose conclusions are widely shared online already because such topics can limit your grade and can lead to academic honesty issues with your work.
- Avoid topics that are much more advanced than the level of your course: this may sound like a strategy to get a high grade, but this approach is not recommended. Applying mathematics that you do not fully understand can limit your grade, as well as use up a disproportionate amount of your time.
- Avoid topics that are too easy: the level of mathematics must be commensurate with the level of your course or else your grade will be limited. Your teacher can advise you if your topic involves mathematics that is too easy.
- Avoid topics that are not practical to carry out: before starting your exploration, you should check that access to any data you will need can be carried out in the time you have, leaving you enough time to write your exploration. Surveys need to be tested out before being given to a population, in order to find any issues with the survey design.

## The assessment criteria unpacked

### Criterion A: Presentation

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | The exploration has some coherence or some organization. |
| 2 | The exploration has some coherence and shows some organization. |
| 3 | The exploration is coherent and well organized. |
| 4 | The exploration is coherent, well organized, and concise. |

This criterion assesses the organization and the coherence of your exploration. Think about documents, articles or books that you have read that you feel are well written, and perhaps others that are not. In a well-written document, you should be able to identify the aim by reading the introduction. The methodology and the conclusion should both be clearly set out. Relevant graphs, tables or diagrams should be in the correct place to help you understand, perhaps by illustrating a point just when you need further explanation. You need to explain what you are doing so that anyone in your class can follow and understand. All these elements of writing add to the organization of your exploration. Larger data sets or material that takes up too much space in the body of the writing, but which may be useful for the reader to refer to, are best put in an appendix.

Coherence refers to how well the elements mentioned above that you have organized fit together. You will have practised presentation skills in other subjects, for example in planning and writing essays. A coherent exploration "flows" and your reader should feel secure when reading your exploration that it is progressing logically and with no "jumps" where he/she is left puzzled as to what is going on, or has to frequently refer back to a previous result.

## Assessment tip

To get a good mark in this criterion you should:

- Write your introduction so that the reader will clearly understand your rationale and aims after reading it.
- Make sure you describe and explain contexts which may be unfamiliar to the reader. For example, if you are exploring games of chance, you must supply enough background information – for example, the rules of a dice game – so that the reader can understand what you are going to explore.
- Integrate your diagrams with your text by positioning them conveniently and referring to features specifically. For example, use labels if appropriate to point out what you are referring to on a graph.
- Swap explorations with a classmate before you submit the draft. You each proof-read and tell each other if you lose touch with the development of what you are presenting: do they get "stuck"? If so, you may need to improve your coherence or your organization.
- Be concise – this means you need to write enough so that the reader understands well, but you do not repeat yourself. For example, copying and pasting content from early in your writing but just changing the parameters would be repetitive and would rule out scoring maximum marks.
- Consider including "signposts" in your writing to help make links. For example, "In the previous section, I showed that …", "In this section I now aim to extend this result to …" are examples of signposts. You may have used them in other subjects.
- Show in the final pages how you have achieved your aim as you write your conclusions.

## Criterion B: Mathematical communication

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | The exploration contains some relevant mathematical communication which is partially appropriate. |
| 2 | The exploration contains some relevant appropriate mathematical communication. |
| 3 | The mathematical communication is relevant, appropriate and is mostly consistent. |
| 4 | The mathematical communication is relevant, appropriate and consistent throughout. |

This criterion assesses how well you communicate Mathematics appropriately, relevantly and consistently. Mathematics can be communicated with notation, symbols and terminology in your writing. However, your writing can be complemented by other types of communication. Imagine reading a trigonometry text that features no triangles, graphs or any type of diagram. It would be very hard to make sense of. In fact, since the first mathematics texts were published by Euclid in 300 BCE, authors have thought hard about how best to communicate Mathematics beyond the power of the written word. You should communicate Mathematics appropriately by being as correct as a professional mathematician should be with notation, symbols, graphs etc, but also you should think about what software can create graphs or diagrams that are specific to your exploration and how they might enhance your communication.

## Assessment tip

To get a good mark in this criterion you should:

- Communicate key terms clearly. For example, if you are exploring an application of mathematics to revenue and cost functions in economics, state the meaning of the independent and dependent variables, their units, domain and range.
- Always avoid word-processing or calculator notation since it is not always the same notation used in Mathematics. Use an equation editor to communicate correctly. For example, 5^x should be written as $5^x$. Also, write $5x$ instead of 5*x. Write $5.67 \times 10^4$ instead of 5.67E4.
- Always label scales and axes in each graph.
- Consistently express your results to an appropriate degree of accuracy.
- Invest time learning how mathematical software can enable you to create powerful ways to communicate your mathematics. For example, three-dimensional diagrams are an appropriate means of communicating the geometry and trigonometry of solids or 3D vector equations of lines. Traces of families of functions are an appropriate way to communicate the effects of changing function parameters. These functionalities are features of dynamic geometry software.

Note that it is not compulsory for you to use multiple forms of mathematical communication. For example, if you are exploring patterns in algebraic expansions, you could communicate your mathematics appropriately just with symbols and you would still be able to gain maximum marks.

## Criterion C: Personal engagement

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | There is evidence of some personal engagement. |
| 2 | There is evidence of significant personal engagement. |
| 3 | There is evidence of outstanding personal engagement. |

This criterion assesses the extent to which you engage with the exploration and make it your own. If you have invested time in choosing a topic that really interests you and that you find important, you have made a good start towards scoring highly in this criterion. This is because your interest has already been engaged, and you will be more able to demonstrate that you take full control and ownership about how the exploration develops. However, what is crucial is that you show written evidence of your engagement within your exploration. This evidence comes in many different forms, some of which you will have encountered in your toolkit lessons.

### Assessment tip

To get a good mark in this criterion you should:

- Think independently and creatively.
- Brainstorm inquiry questions both as part of your planning and as the exploration develops.
- Write in your own voice, reporting on how you progressed with your investigation, what you wondered, what your ideas were, and how and why you chose to engage with an inquiry question.
- Consider which of the following types of personal engagement could be applied to your topic, and apply them if appropriate:
    - creating mathematical models of a real-life context
    - designing and implementing a survey
    - applying technology in a resourceful and effective way
    - taking a risk and showing courage by exploring unfamiliar mathematics or contexts
    - linking to historical, local or global perspectives
    - showing how you address personal interest or satisfy your curiosity
    - creating a simulation or an experiment
    - including hypotheses or conjectures that did not go quite as you expected and explaining why.
- Remember that you are writing a *report* of your investigation. Make sure you report the decisions you made in engaging with its development. Maximum marks depend on you showing evidence of your personal engagement. Only you can provide this evidence by reporting it in your writing.

Note that exploring mathematics that is much more advanced than your mathematics course is not a recommended way to demonstrate personal engagement. The criteria are designed so that you can gain maximum marks by applying mathematics at a level appropriate to the level of your course.

## Criterion D: Reflection

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | There is evidence of limited reflection. |
| 2 | There is evidence of meaningful reflection. |
| 3 | There is substantial evidence of critical reflection. |

This criterion assesses how well you review, analyse and evaluate throughout your exploration. It is easy to neglect this criterion when you are so busy carrying out mathematical processes such as calculations, analysis, modelling or problem solving that your exploration requires. Sometimes it is such a relief when you finally feel you have been successful with the mathematics that you feel your work is done! Or maybe our careers as mathematicians have been conditioned at an early age by a tendency to check with our teacher that our answers are correct, and then just moving on with a happy feeling of success, but with little reflection.

In your exploration, you are expected not simply to state your conclusions, but to consider what they mean, state any limitations they have and reflect critically.

### Assessment tip

To get a good mark in this criterion you should:

- Not leave your reflection until the end of your exploration. Instead, reflect periodically throughout the exploration when appropriate – for example each time you have established a result you can reflect on it.

- Reflect on what your results mean in the real-world context you are exploring. This is a frequent theme in Mathematics: Applications and Interpretations which you may have experienced in investigations, toolkit lessons or exercises.
- Reflect on the appropriate degree of accuracy given the context of your topic.
- Think of reflection and personal engagement working together. You may find that in reflecting on a result, a new line of inquiry suggests itself to you, which you then engage with. If you feel that reflection drives the direction of your exploration, this is evidence of critical reflection and you should report this in your writing.
- Aim to reflect critically: have you considered the validity, implications and reliability of your results and the appropriateness of the mathematics you applied to find them?
- Do not hold back if you think some of your knowledge and understanding of TOK is relevant in writing a critical reflection. After all, TOK is a course about critical thinking.

## Criterion E: Use of mathematics

This criterion assesses to what extent you use relevant mathematics in your exploration. This criterion has the most marks available to you. This is the only criterion for which the requirements for the awarding of marks for SL is not exactly the same as for HL, although there are similarities.

Requirements for SL students:

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | Some relevant mathematics is used. |
| 2 | Some relevant mathematics is used. Limited understanding is demonstrated. |
| 3 | Relevant mathematics commensurate with the level of the course is used. Limited understanding is demonstrated. |
| 4 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated. |
| 5 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is mostly correct. Good knowledge and understanding are demonstrated. |
| 6 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Thorough knowledge and understanding are demonstrated. |

Requirements for HL students:

| Achievement level | Descriptor |
|---|---|
| 0 | The exploration does not reach the standard described by the descriptors below. |
| 1 | Some relevant mathematics is used. Limited understanding is demonstrated. |
| 2 | Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding is demonstrated. |
| 3 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Some knowledge and understanding are demonstrated. |
| 4 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated. |
| 5 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and demonstrates sophistication or rigour. Thorough knowledge and understanding are demonstrated. |
| 6 | Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and demonstrates sophistication and rigour. Thorough knowledge and understanding are demonstrated. |

For both SL and HL, the mathematics you use must be relevant, meaning that it supports the development of your exploration towards meeting its aim. Use of mathematics that is overly complicated (where simple mathematics would be enough to meet the aims of the exploration) is not relevant – and this will limit the marks you are awarded significantly.

For both SL and HL, the aim is for you to demonstrate in your writing that you have thorough knowledge and understanding of the mathematics you use. It is not enough just to have a correct answer to demonstrate thorough knowledge and understanding, but – as is also the case with criteria C and D – you must provide the evidence to *demonstrate* your knowledge and understanding.

Obviously, SL and HL students will all aim for all their mathematics to be correct. The degree to which your mathematics is correct is proportional to the marks you can be awarded, but you must also demonstrate your knowledge and understanding at the same time.

**Assessment tip**

To get a good mark in this criterion, SL and HL students should:

- Check with your teacher when you are planning your exploration that the level of mathematics you propose to use is not too simple nor too advanced, and that it is relevant to your aims.
- Check with your teacher that your plans for using mathematics are achievable in the time and space given. When using mathematics, it is best to focus on quality rather than quantity.
- Don't just use mathematics. Use it and then justify its use by showing why it is appropriate. For example, if you use the formula for independent events, explain why the events are independent in the first place. If you are applying the binomial distribution, explain why the context fits the requirements of the binomial distribution.
- You are encouraged to use technology, but with the same expectations for demonstrating your knowledge and understanding. For example, just finding the value of derivative at a point on a curve with technology does not necessarily demonstrate knowledge and understanding. You should explain what this value means in context.

For HL, there are more expectations placed on your use of mathematics in criterion E than for SL. The degree to which your mathematics is sophisticated, rigorous and precise helps you to access the highest grades in this criterion. You begin to address sophistication at the planning stage of your exploration, with your teacher's guidance, by making sure first of all that your use of mathematics is commensurate with the expectations of HL mathematics, or that it involves a complex use of SL mathematics that goes beyond the expectations of SL.

**Assessment tip**

To get a good mark in this criterion, HL students should, in addition:

- Brainstorm "sophistication" when at the planning stage. Consider looking at your topic from different perspectives, using and understanding challenging concepts, identifying underlying structures to link different areas of mathematics.
- Review how you set out your mathematics. Is it expressed as rigorously as you would find in a mathematics textbook? Are your logic and language clear in your arguments and calculations?
- Consider using logical connectives such as $\therefore$, $\Rightarrow$, $\Leftrightarrow$, $\because$ to help communicate your logic.
- Always use an appropriate level of accuracy.

## Checklist

Here is a checklist to ensure that you have done your best to submit a good exploration.

### Communication and mathematical presentation

- ☐ Have you included a title but no identifying features such as your name, candidate number, school or teacher's name?
- ☐ Did you start with an introduction?
- ☐ Have you stated a clear aim? What is the theme of your exploration? What do you hope it will do?
- ☐ In your conclusion, have you answered or responded to the aim you wrote at the start of your exploration?
- ☐ Do you have a clear rationale? Why did you choose this topic? Why is this topic of interest to you?
- ☐ Can an average student in your class read and follow your exploration? You may want to peer edit with a friend.
- ☐ Does the entire paper focus on the aim and avoid irrelevance?
- ☐ Does the writing flow nicely?
- ☐ Did you include graphs, tables and diagrams with titles at appropriate places and not attach them all at the end?
- ☐ Have you reread and improved your exploration?
- ☐ Did you cite all references in your bibliography and acknowledge direct quotes appropriately?
- ☐ Did you use appropriate mathematical language and representation? (Not computer notation, such as *, ^, etc.)
- ☐ Did you define key terms where necessary?
- ☐ Did you use appropriate technology?
- ☐ Did you think about the degree of accuracy? (For your topic, how many decimal places are relevant?)
- ☐ Did you end with a conclusion and relate it back to your aim and rationale?

### Personal engagement

- ☐ Did you address why you think your topic is interesting or why it appealed to you?
- ☐ Did you use the personal pronoun?
- ☐ Did you ask and answer personal questions ("I wonder if …", "What if …")?

- ☐ Did you present mathematical ideas in your own way (as opposed to copying someone else's theory)?
- ☐ Did you explain any new mathematics or software that you had to employ?
- ☐ Did you try to add "your voice" to the work?
- ☐ Did you relate the results to your own life?

**Reflection**

- ☐ Did you explain what the results of your mathematics means after you have used a technique?
- ☐ Did you ask questions, make conjectures and investigate mathematical ideas?
- ☐ Did you consider the historical and global perspectives of your topic?
- ☐ Did you discuss the implications of your results? (What do they mean? Why are they important?)
- ☐ Did you look for possible limitations and/or extensions of your topic?
- ☐ Did you find any areas where your mathematics may be approximate or slightly inaccurate?
- ☐ Did you make links between your topic and different fields and/or areas of mathematics?

**Use of mathematics**

- ☐ Did you use mathematics from your course?
- ☐ Did you explain unfamiliar mathematics, or apply familiar mathematics to a new situation?
- ☐ Did you create mathematical models for real-world situations, if this applied to your topic?
- ☐ Did you apply problem-solving techniques?
- ☐ Did you look for and explain patterns, if this applied to your topic?

# PRACTICE EXAM PAPERS

## Introduction to the exam-practice papers

At this point you will have re-familiarized yourself with the contents from the topics of the IB Mathematics: Applications and Interpretation syllabus. Additionally you will have picked up some key techniques and skills to refine your exam approach. It is now time to put these skills to the test! In this section you will find practice examination papers: SL Papers 1 and 2, HL Papers 1, 2 and 3, with the same structure as the external assessment that you will complete at the end of the Diploma Programme course. Answers to these papers are available at www.oxfordsecondary.com/ib-prepared-support.

You will require pens, pencil, a ruler and a graphical display calculator for all exams. The use of correction fluid is not allowed. If you do make a mistake, just put a single line through it. If it is a section that is wrong, draw a line at the top and at the bottom and then a single line though the incorrect working.

A clean copy of the *Mathematics: Applications and Interpretation formula booklet* is required for all papers. This will be provided by your school.

There will be 5 minutes of reading time at the start of paper. You may not write during this time but you can prepare your strategy for answering the questions.

All papers will have the instruction “Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.” Make sure that you understand what significant figures are and check that you have followed this instruction at the end of each question.

All questions in all papers are to be attempted. Ensure that you divide your time wisely between the questions. The number of marks allocated to a question is a rough guide to how many minutes it should take to complete. If you become stuck on a question, do not waste time: be prepared to move onto the next question. Put a mark by it so that you can come back to it at the end of the exam if you have time.

Paper 1 consists of short-response questions and Paper 2 extended-response questions. The order in which you do the questions does not matter. However, there is a difficulty gradient going through the papers, with the easier questions at the start.

The Paper 1 questions are to be answered on the examination paper in the specially prepared boxes under each question. If you run out of space then you can complete your answer in an answer booklet. Make sure that you indicate that you have done this in the box and make sure that the question is well labelled in the answer booklet.

The Paper 2 questions are to be answered in the answer booklets provided. There will be an instruction to start each question on a fresh page. Obey this instruction; it will also give you space to go back if you have to complete a part in the question. Make sure that you clearly mark each question number and the subparts. In Paper 2 the question paper will not go to the examiner so if you do any working on the question paper make sure you also transcribe it to the answer booklet. This can happen in particular if there is a diagram in the question that you use as a basis your working. Redraw the diagram in your answer booklet and then add the extra work.

HL Paper 3 consists of two compulsory, extended-response, problem-solving questions. Start each question on a fresh page in the answer booklet.

At the start of each paper there will be the advice “Full marks are not necessarily given for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided that this is shown by written working.

You are therefore advised to show all working. Solutions found from a graphical display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer." It should be no surprise to you when reading this on the exam. This is how you should have been answering all of the questions you have done in the two years leading up to the exam.

## Standard level

### Practice paper 1

Time: 1 hour 30 min

Technology required

Maximum: 80 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**1** [Maximum marks: 4]

Tania is playing a game. The number of points she can win is $X$, where $X$ is a random variable with the following probability distribution.

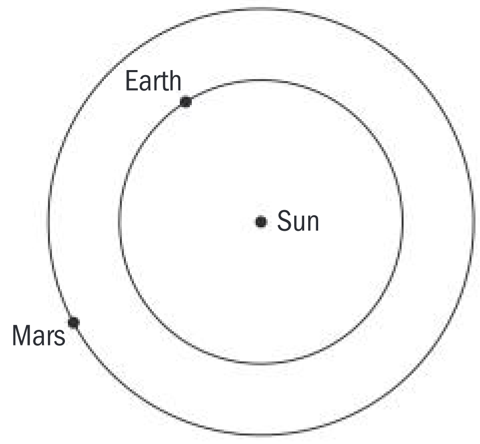
| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | 0.2 | 0.3 | 0.4 | $k$ |

**(a)** Find the value of $k$. [2]

**(b)** Find $E(X)$. [2]

**2** [Maximum marks: 5]

The orbits of the Earth and of Mars around the Sun are shown in the diagram below.




The orbits of Earth and Mars can be considered circular with a radius 150 million kilometres and 228 million kilometres, respectively.

Giving your answers in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$, find:

**(a)** **(i)** the minimum possible distance between the Earth and Mars [2]

**(ii)** the maximum possible distance between the Earth and Mars. [1]

**(b)** Find the distance travelled by the Earth in 100 days (take a year as equal to 365 days). [2]

**3** [Maximum marks: 4]

A sample of 5 holiday resorts were ranked on their popularity. This ranking and the mean number of hours of sunshine per day in the resorts are shown in the table below, where a rank of 1 indicates the most popular resort.

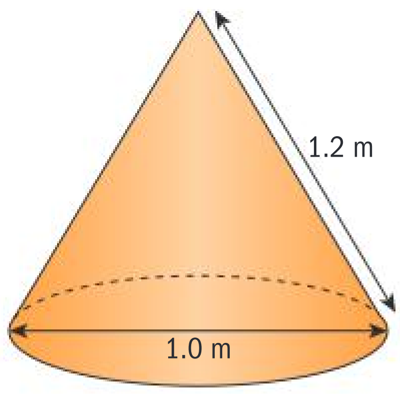
| **Resort** | A | B | C | D | E |
|---|---|---|---|---|---|
| **Popularity rank** | 5 | 4 | 3 | 2 | 1 |
| **Hours of sunshine** | 5.5 | 4.6 | 8.2 | 7.7 | 9.0 |

Calculate Spearman's rank correlation coefficient $r_s$ for this data. [4]

**4** [Maximum marks: 5]

Grain leaks from a silo and forms a cone. To calculate the amount of grain in the cone, a farm worker measures the slant height of the cone as 1.2 m and the diameter of the base of the cone as 1.0 m.

Find the volume of grain in the cone. [5]




**5** [Maximum marks: 8]

The rate of change of the money gained per day (profit), $P$ (measured in 1000 USD) for a factory producing an amount $x$ of chemical $X$ (measured in 1000 kg) is given by the equation

$$\frac{dP}{dx} = 0.6x^2 - 3.6x + 5,\ 0 \le P \le 3$$

**(a)** Given that the factory makes a loss of 500 USD per day when it produces no chemical $X$, find an expression for $P$. [5]

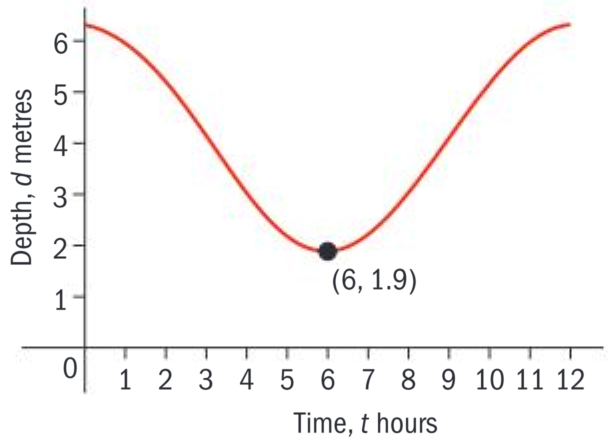
When the factory is close to maximum capacity, the costs of producing more chemical $X$ are greater than the extra profit received.

**(b)** **(i)** Find the amount of chemical $X$ the factory should produce to maximize its profit. [2]

**(ii)** Write down the value of $\frac{dP}{dx}$ at this point. [1]

**6** [Maximum marks: 8]

The graph shows the depth ($d$) of water in a harbour for the 12-hour period between high tides.




At high tide, the depth of water is 6.3 m and at low tide it is 1.9 m. High tide occurs at times $t = 0$ and $t = 12$.

The equation of the curve is $d = a\cos(bt) + c$, $a, b, c \in \mathbb{R}$ where $t$ is the number of hours after a high tide.

**(a)** Find the value of:

**(i)** $a$ [2]

**(ii)** $b$ [2]

**(iii)** $c$ [1]

A boat cannot enter the harbour when the depth of water is less than 4 m.

**(b)** Find the percentage of time during which the boat cannot enter the harbour. [3]

**7** [Maximum marks: 5]

In order to test whether or not loud music affects hearing, a group of 280 people are asked to record how many hours each week they listen to music above a certain volume. The hearing ability of the group is also tested.

The results are shown in the table below.

| | | Level of hearing (1 indicates 'good' hearing) | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Time listening (hours) | $0 \le t < 2$ | 52 | 54 | 22 |
| | $2 \le t < 4$ | 35 | 32 | 17 |
| | $t \ge 4$ | 21 | 28 | 19 |

Use the $\chi^2$ test for independence to see if there is an association between time spent listening to loud music and level of hearing ability.

Carry out the test at the 5% significance level and state clearly the null and alternative hypotheses. [5]

**8** [Maximum marks: 7]

A souvenir manufacturer produces models of a famous statue in different sizes. The volume ($V$) of these models is proportional to the cube of their height ($h$).

The statue that is 10 cm tall has a volume of 225 cm$^3$.

**(a)** Find an expression for $V$ in terms of $h$. [3]

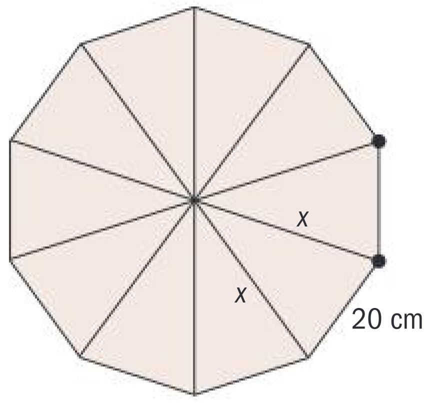
The cost of a statue is proportional to the square root of its volume. The 10 cm statue costs €12.

**(b)** Find the cost of a 20 cm statue. [4]

**9** [Maximum marks: 6]

An umbrella is made up of 10 identical triangles. Two sides of the triangles are formed by the spokes of the umbrella and the third side, at the edge of the umbrella, is of length 20 cm.


A view of the umbrella from above is shown in the diagram.

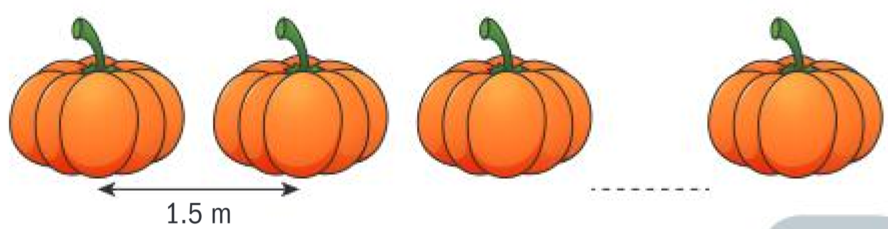



**(a)** Find the value of the length marked $x$ in the diagram. [4]

**(b)** Find the area which would be shielded from the rain by the umbrella. [2]

**10** [Maximum marks: 5]

A farmer is picking pumpkins. Twenty pumpkins are placed in a line, each 1.5 m from the next.



The farmer then parks his truck next to a pumpkin at the end of the line.

Find the distance the farmer needs to walk to collect all the pumpkins, given he walks from the truck to collect each one and then carries it back to the truck. [5]

**11** [Maximum marks: 6]

A total of \$18,000 is invested in three different funds, A, B and C. At the end of the first year, fund A pays out 6% interest on the money invested, fund B pays out 4% interest and fund C pays out 8% interest.

The total interest paid at the end of the first year is \$1210.

The amount invested in fund A is \$4000 more than the amount invested in fund B.

Let the amount of money invested in the three funds be $a$, $b$ and $c$.

Form three equations in $a$, $b$ and $c$ and solve them to find the amount of money invested in each fund. [6]

**12** [Maximum marks: 6]

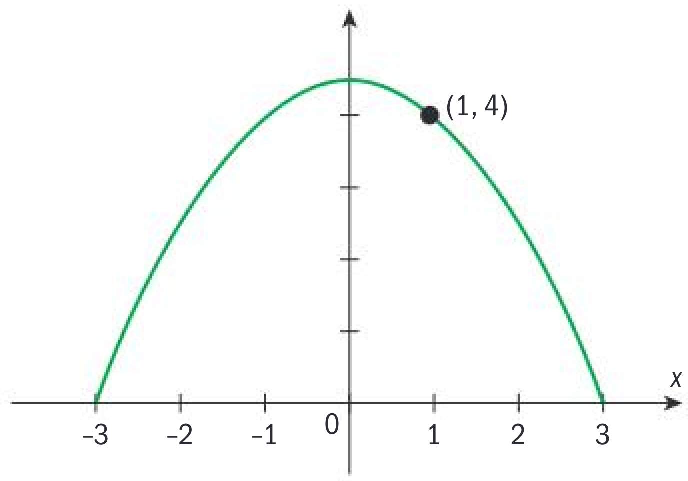
Consider the equation $y = 4.5 - 0.5x^2$

**(a)** **(i)** Find $\frac{dy}{dx}$ [1]

**(ii)** Hence, find the equation of the tangent to the curve at the point (1, 4). [2]

The cross-section of a building can be modelled by the curve $y = 4.5 - 0.5x^2,\ -3 \le x \le 3$

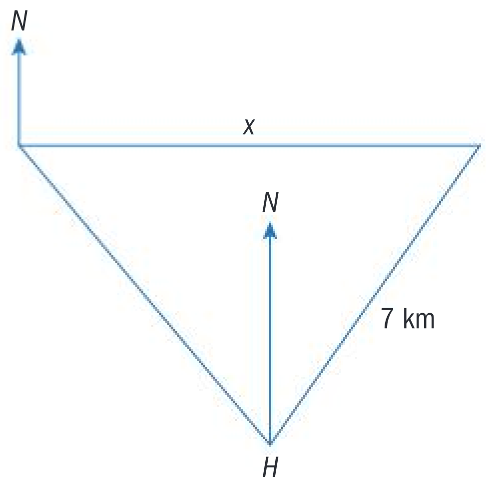
A builder needs to do some work on the roof. In order to reach the highest point, her ladder must reach the point on the roof at (1, 4).




**(b)** Find the minimum length required for her ladder. [3]

**13** [Maximum marks: 5]

A boat sails a distance of 7 km on a bearing of 035° from a harbour $H$. It then sails due west for a distance $x$ kilometres. When the bearing of $H$ from the boat is 140°, the boat changes direction again and sails straight back to $H$.

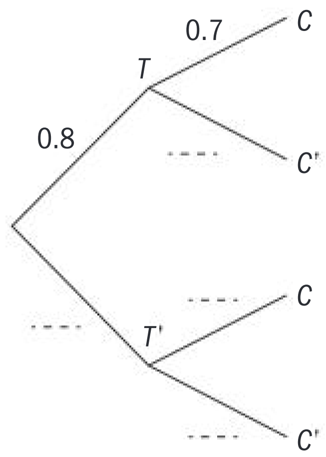



Find the value of $x$. [5]

**14** [Maximum marks: 6]

In order to be allowed to drive, Sabine must take a theory test and then a practical test. She will retake any test that she fails. The probability she passes her theory test ($T$) is 0.8 and the probability she passes the practical test ($C$) is 0.7.

**(a)** Complete the probability tree shown below. [1]




**(b)** Find the probability Sabine will need to retake at least one test. [2]

**(c)** Find the probability she takes exactly three tests before she is allowed to drive. [3]

## Standard level

### Practice paper 2

Time: 1 hour 30 min

Technology required

Maximum: 80 marks

Answer all questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations.

Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**1** [Maximum marks: 16]

In a biology experiment, Pierre is feeding one set of plants with nutrient A and a second set of plants with nutrient B.

The heights, $h$, of 10 plants fed with nutrient A are shown in the table below.

| | | | | | |
|---|---|---|---|---|---|
| Height (cm) | 5.8 | 6.5 | 6.2 | 7.0 | 6.3 |
| | 6.5 | 5.9 | 6.8 | 6.6 | 6.4 |

**(a)** For these data, find:

**(i)** the median [1]

**(ii)** the lower quartile [1]

**(iii)** the upper quartile. [1]

**(b)** **(i)** Find the interquartile range. [1]

A piece of data, $x$, is an outlier if $x < a$ or $x > b$.

**(ii)** Find the values of $a$ and $b$. [3]

**(c)** **(i)** Draw a box plot for this data. [2]

**(ii)** State one reason why the box plot indicates that the sample of plants could come from a normal population. [1]

The heights of the 8 plants fed with nutrient B are shown in the table below.

| | | | | |
|---|---|---|---|---|
| Height (cm) | 6.0 | 7.2 | 6.9 | 7.5 |
| | 7.0 | 7.2 | 6.6 | 7.1 |

Pierre thinks that nutrient B produces taller plants than nutrient A and decides to test this hypothesis at the 1% significance level.

**(d)** **(i)** State the null and alternative hypotheses for Pierre's test. [2]

**(ii)** Find the $p$-value. You may assume that the two populations are normally distributed with equal standard deviations. [3]

**(iii)** Compare the $p$-value with the significance level of the test and state the conclusion of the test. [1]

**2** [Maximum marks: 13]

Sumitra rents an apartment. She pays €400 per month.

**(a)** Assuming no increase in the rent, find the amount she would pay in rent over a 20-year period. [1]

Sumitra has €20,000 in her bank account and it is earning 4% interest per year, compounded monthly at the end of each month.

**(b)** Find the amount of money that would be in Sumitra's account at the end of 20 years if she took no money out of the account. [2]

She is considering buying an apartment. The one she would like to buy costs €130,000

She plans to pay €20,000 as a deposit and the rest in monthly instalments, paid at the end of each month, for 20 years. The bank will charge her 6% interest per year, compounded monthly.

**(c)** Find Sumitra's monthly payment. [3]

At the end of the 20 years Sumitra will move to another city.

**(d)** Find whether it is better for Sumitra to rent or buy an apartment, and the difference in costs between the two routes. You may assume:

- her €20,000 is left in the savings account for all 20 years
- she can sell the flat for €130,000
- there are no other costs associated with buying or renting, no other interest payments and no inflation. [7]

**3** [Maximum marks: 13]

The mass ($X$) of potatoes from a farm are normally distributed with a mean of 190 g and a standard deviation of 35 g.

**(a)** Find the probability that a randomly chosen potato has a mass greater than 220 g. [2]

Potatoes are classed as small, medium and large. A potato is classed as large if its mass is greater than 220 g and as small if its mass is less than $a$ g. 15% of the potatoes are classed as small.

**(b)** Find the value of $a$. [3]

A restaurant collects 800 randomly chosen potatoes from the farm.

**(c)** Find the expected number of potatoes the restaurant will have in each category. [4]

The restaurant's head chef randomly selects 5 potatoes to check their size, replacing each potato before randomly selecting another.

**(d)** Find the probability that the chef selects at least two large potatoes. [4]

**4** [Maximum marks: 13]

In an experiment, mould is being grown in a circular dish. The area of the dish which is covered by the mould is recorded each day.

Past experience has shown that the growth of the mould can be modelled by the equation $A = ae^{bt}$, where $A$ is the area covered and $t$ the time in days that the mould has been growing.

Initially there is 1.2 cm$^2$ covered by the mould and the next day (when $t = 1$) there is 3.4 cm$^2$ covered.

**(a)** **(i)** Write down the value of $a$. [1]

**(ii)** Find the value of $b$, accurate to three significant figures. [2]

**(b)** Using your answers to part **(a)**, find the area of the dish covered by mould when $t = 2$. [1]

The dish is in the shape of a circle with a radius of 5 cm.

The rate of growth slows when 80% of the area of the dish is covered by the mould.

**(c)** Find the value of $t$ at which the rate of growth begins to slow. [3]

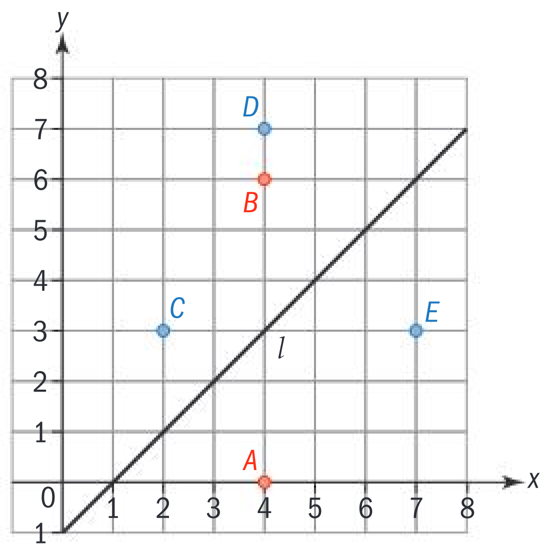
After this time the area of the dish covered by mould can be modelled by the equation $A = c + dt$, up to the point when the dish is completely covered.

**(d)** If the dish is completely covered when $t = 6.0$ find the values of $c$ and $d$. [5]

**(e)** Write down the rate of growth of the mould (in cm$^2$ per day) during the period of linear growth. [1]

**5** [Maximum marks: 14]

The Red team are playing against the Blue team at soccer. Part of the playing area is shown on the diagram below, where all measurements are in metres.




Player $A$ currently has the ball at (4, 0) and wants to pass to player $B$ at (4, 6). Players $C$, $D$ and $E$ are all on the opposing team at the points with coordinates (2, 3), (4, 7) and (7, 3).

The line $l$ is the perpendicular bisector of $B$ and $E$.

**(a)** Write down the equation of $l$. [2]

**(b)** Show that the equation for the perpendicular bisector of $[BC]$ is $2x + 3y = 19.5$ [5]

**(c)** Find the point where $l$ meets $2x + 3y = 19.5$ [2]

Player $A$ needs to pass the ball to a point in the region $R$ which contains those points closer to player $B$ than to any of the opposing players.

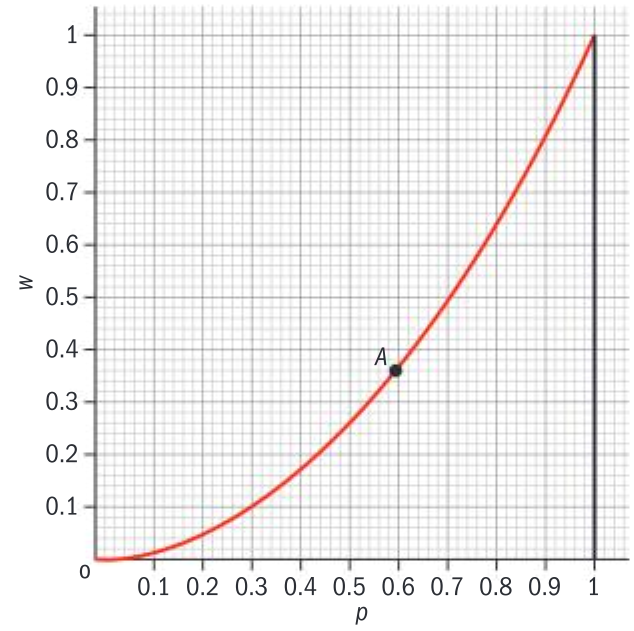
**(d)** **(i)** Draw a sketch of the diagram and on it show the region $R$. [3]

**(ii)** Find the area of $R$. [2]

**6** [Maximum marks: 11]

A Lorenz curve, $w(p)$, shows the percentage of a nation's wealth ($w$) owned by the poorest $p$% of the population.

For example, in the diagram shown, in which the percentages are given as their decimal equivalents, the point $A$ indicates that the poorest 60% in the country own 36% of the wealth of that country.

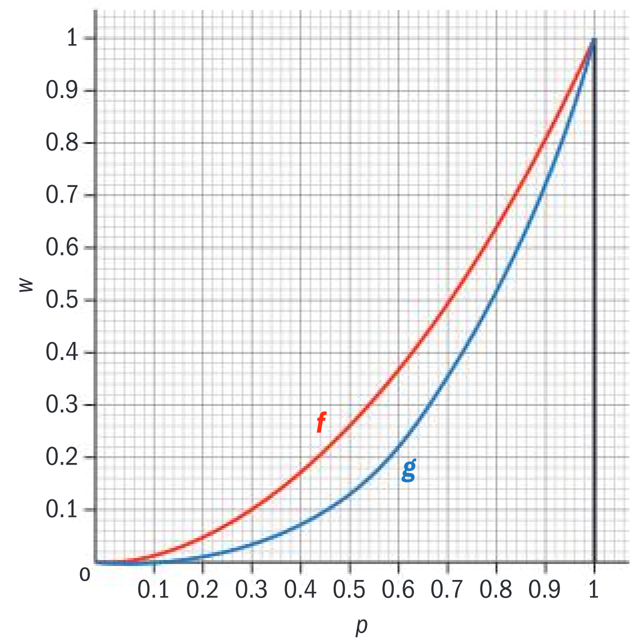



**(a)** Use the diagram above to write down the proportion of wealth owned by:

**(i)** the poorest 40% [1]

**(ii)** the richest 10% of the country. [2]

**(b)** The diagram below shows the wealth distribution in two countries. State which of the two curves ($f$ or $g$) represents a country in which the wealth is distributed more evenly among the population. Justify your answer. [2]



One measure of inequality within a country is the Gini coefficient.

If the area of the region below the Lorenz curve for $0 \le p \le 1$ is $A$, then the Gini coefficient ($G$) is given by $G = 1 - 2A$.

In country $B$, the Lorenz curve has the equation $w = 0.2p^3 + 0.6p^2 - 0.2p$

**(c)** **(i)** Find the proportion of wealth owned by the poorest 50% of the population in country $B$. [2]

**(ii)** Find the Gini coefficient for this country. [4]

# Higher level

## Practice paper 1

Time: 2 hours

Technology required

Maximum: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

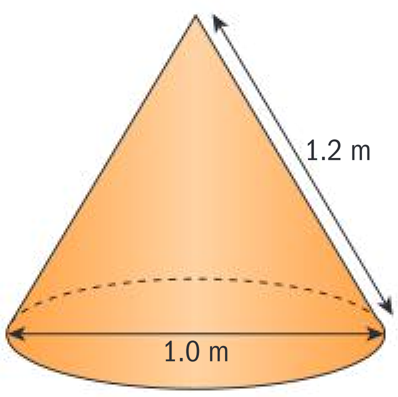
Answers should be supported by working and/or explanations.

Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**1** [Maximum marks: 5]

Grain leaks from a silo and forms a cone. To calculate the amount of grain in the cone, a farm worker measures the slant height of the cone as 1.2 m and the diameter of the base of the cone as 1.0 m.

Find the volume of grain in the cone. [5]




**2** [Maximum marks: 5]

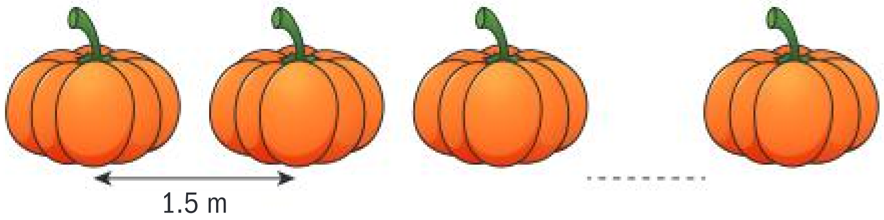
In order to test whether or not loud music affects hearing, a group of 280 people are asked to record how many hours each week they listen to music above a certain volume. The hearing ability of the group is also tested.

The results are shown in the table below.

| | | Level of hearing (1 indicates 'good' hearing) | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Time listening (hours) | $0 \le t < 2$ | 52 | 54 | 22 |
| | $2 \le t < 4$ | 35 | 32 | 17 |
| | $t \ge 4$ | 21 | 28 | 19 |

Use the $\chi^2$ test for independence to see if there is an association between time spent listening to loud music and level of hearing ability. Carry out the test at the 5% significance level and state clearly the null and alternative hypotheses. [5]

**3** [Maximum marks: 5]




A farmer is picking pumpkins. Twenty pumpkins are placed in a line, each 1.5 m from the next. The farmer then parks his truck next to a pumpkin at the end of the line.

Find the distance the farmer needs to walk to collect all the pumpkins, given he walks from the truck to collect each one and then carries it back to the truck. [5]

**4** [Maximum marks: 6]

A total of \$18,000 is invested in three different funds, A, B and C. At the end of the first year, fund A pays out 6% interest on the money invested, fund B pays out 4% interest and fund C pays out 8%.

The total interest paid at the end of the first year is \$1210.

The amount invested in fund A is \$4000 more than the amount invested in fund B.

Let the amount of money invested in the three funds be $a$, $b$ and $c$.

Form three equations in $a$, $b$ and $c$ and solve them to find the amount of money invested in each fund. [6]

**5** [Maximum marks: 6]

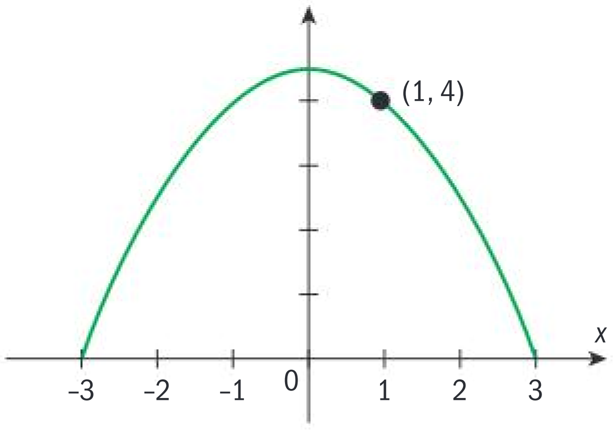
Consider the equation $y = 4.5 - 0.5x^2$

**(a)** **(i)** Find $\frac{dy}{dx}$ [1]

**(ii)** Hence, find the equation of the tangent to the curve at the point (1, 4). [2]

The cross-section of a building can be modelled by the curve $y = 4.5 - 0.5x^2,\ -3 \le x \le 3$

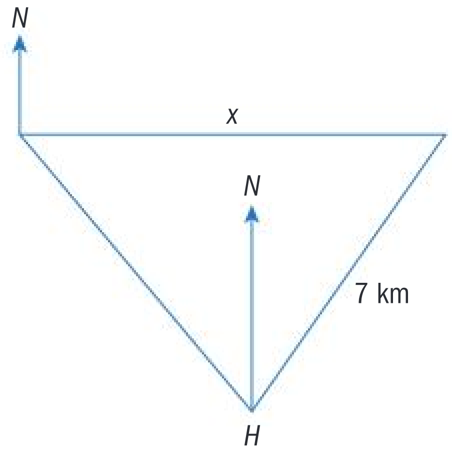
A builder needs to do some work on the roof. In order to reach the highest point, his ladder must reach the point (1, 4).




**(b)** Find the minimum length required for his ladder. [3]

**6** [Maximum marks: 5]

A boat sails a distance of 7 km on a bearing of 035° from a harbour $H$. It then sails due west for a distance $x$ km. When the bearing of $H$ from the boat is 140°, the boat changes direction again and sails straight back to $H$.




Find the value of $x$. [5]

**7** [Maximum marks: 7]

A drone has a velocity $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ km h$^{-1}$ and at 7:00 am it is at the point with coordinates (4, 0, 0).

**(a)** Find the speed of the drone. [2]

**(b)** Write down an expression for the position of the drone $t$ hours after 7:00 am. [1]

**(c)** Find the position of the drone when $t = 2$. [1]

**(d)** Find its distance from (2, 2, 0) at this time. [3]

**8** [Maximum marks: 7]

In order to be allowed to drive, Sabine must take a theory test and then a practical test. If she fails either or both of them, she will retake the ones that she fails. The probability she passes her theory test ($T$) is 0.8 and the probability she passes the practical test ($C$) is 0.7

**(a)** Show this information on a probability tree. [2]

**(b)** Find the probability Sabine will need to retake at least one of the tests. [2]

**(c)** Find the probability she takes exactly three tests before she is allowed to drive. [3]

**9** [Maximum marks: 8]

It is believed that the metabolic rate ($R$) of a mammal is linked to its mass ($m$) by the power law

$$R = cm^d$$

The values of $\ln m$ and $\ln R$ for four mammals are shown in the table below.

| | Mouse | Cat | Sheep | Lion |
|---|---|---|---|---|
| $\ln m$ | −3.69 | 0.916 | 3.76 | 3.91 |
| $\ln R$ | 0.500 | −0.386 | −1.51 | −1.47 |

**(a)** Find the product moment correlation coefficient for these data. [1]

**(b)** Find the least squares regression line for this data in the form $\ln R = a \ln m + b,\ a, b \in \mathbb{R}$ [2]

**(c)** Hence, find estimates for the parameters $c$ and $d$. [3]

**(d)** Use your model to estimate the metabolic rate of an elephant, given its mass is 3850 kg. Give your answer accurate to two significant figures. [1]

The metabolic rate of an elephant is 0.072 ml g$^{-1}$ h$^{-1}$.

**(e)** Find the percentage error in your answer to part **(d)**. [1]

**10** [Maximum marks: 7]

Consider the matrix $\mathbf{A} = \begin{pmatrix} a-3 & -4 \\ 1 & a+2 \end{pmatrix}$

**(a)** **(i)** Find the inverse of the matrix in terms of $a$. [2]

**(ii)** Write down the values of $a$ for which there is no inverse. [2]

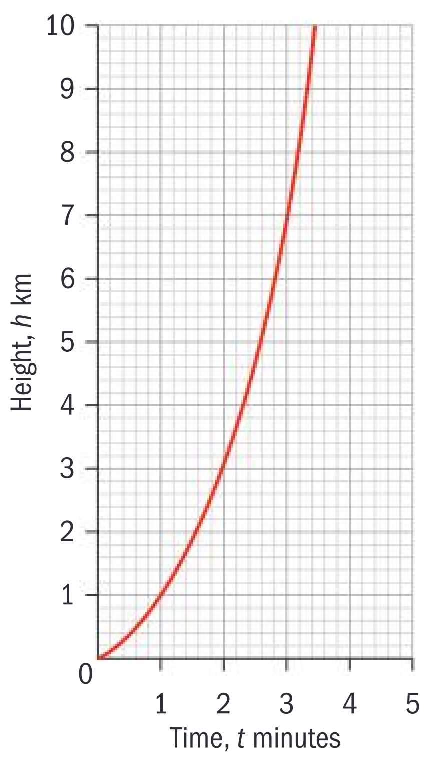
The matrix $\mathbf{A}$ with $a = 1$ is used to encode a sequence of numbers written as a $2 \times 3$ matrix $\mathbf{S}$ by finding the product $\mathbf{AS}$.

The encoded message is $\begin{pmatrix} 2 & -4 & 6 \\ -10 & 8 & -3 \end{pmatrix}$

**(b)** Find $\mathbf{S}$. [3]

**11** [Maximum marks: 7]

A rocket is launched each day at exactly 12:00. Its height in kilometres $t$ minutes after 12:00 is given by the equation $h = 2^t - 1,\ t \geq 0$ and its graph is shown below.




One day, the rocket is given a booster engine that allows it to reach each height in half the usual time.

**(a)** **(i)** Sketch the new graph on the same axes. [2]

**(ii)** Write down the new equation for the rocket's height. [2]

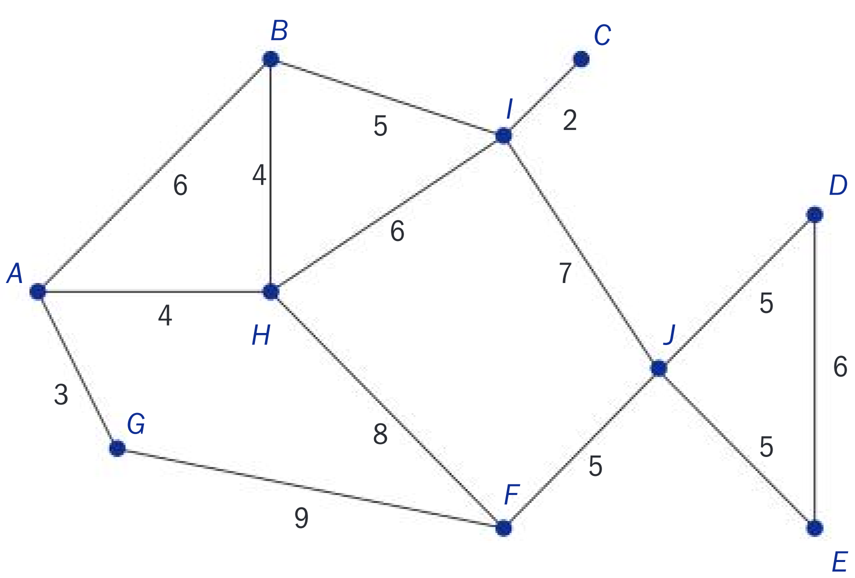
The launch is now delayed by 3 minutes.

**(b)** **(i)** Sketch the new graph on the same axes. [1]

**(ii)** Write down the new equation for the rocket's height, stating the domain. [2]

**12** [Maximum marks: 5]

A graph of a network of paths is shown below. The weights on the graph represent the time in minutes to walk along each of the paths. The sum of all the weights is 75 minutes.




Use the Chinese postman algorithm to find the shortest time required to walk all of the paths, given that the walk can begin and end at any of the vertices. Fully justify your answer. [5]

**13** [Maximum marks: 6]

I am trying to get fit, so I decide I will walk to school more often. If I walk one day, the probability I will drive the next day is 0.6. If I drive one day, the probability I walk the next day is 0.8.

**(a)** Write down the transition matrix for this situation. [2]

**(b)** If I drive on Tuesday, find the probability that I also drive on Friday. [2]

**(c)** On Monday there is a 40% chance I will choose to walk. Find the probability I will walk on Friday. [2]

**14** [Maximum marks: 8]

**(a)** Find the matrix that represents:

**(i)** a reflection in the line $y = \sqrt{3}x$ [3]

**(ii)** a rotation of 60° anticlockwise about (0, 0). [1]

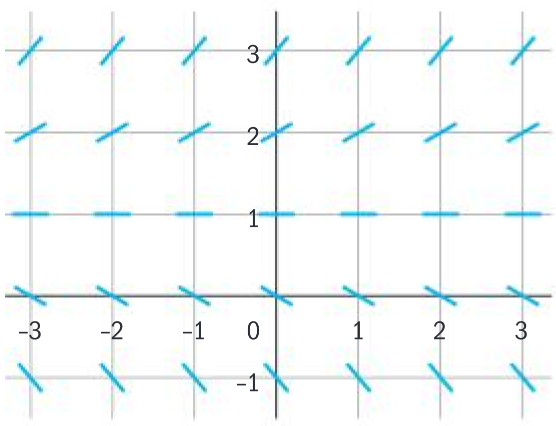
Let **T** be the matrix that represents the compound transformation of a reflection in the line $y = \sqrt{3}x$ followed by a rotation of 60° anticlockwise about (0, 0).

**(b)** Find **T**. [2]

**(c)** Describe the single transformation that has the same effect as the compound transformation represented by **T**. [2]

**15** [Maximum marks: 6]

Part of the slope field for the differential equation $\frac{dy}{dx} = \frac{y-1}{2}$ is shown in the diagram below.




Let $C$ be the curve that satisfies the differential equation and passes through (0, 0).

**(a)** Sketch the graph of $C$ on the diagram. [1]

**(b)** Find the equation for $C$, giving your answer in the form $y = f(x)$. [5]

**16** [Maximum marks: 9]

Julie is playing a game in which she scores three points for a goal and two points for a basket.

The number of goals and baskets she scores follow Poisson distributions with means of 3.2 and 4.5, respectively, and are independent of each other.

**(a)** Find:

**(i)** the expected total number of points she scores in a game [2]

**(ii)** the variance of the total number of points she scores in a game. [3]

A season consists of 40 games.

**(b)** Explain why the mean number of points scored by Julie in a season can be modelled by a normal distribution. [1]

Any player whose mean score in a season is over 20 points is placed on a roll of honour.

**(c)** Find the probability that Julie is placed on the roll of honour. [3]

**17** [Maximum marks: 8]

The population of a unicellular organism is modelled by the logistic function $P = \frac{k}{1+e^{-t}}$, $k > 0$

**(a)** Find $\frac{dP}{dt}$. [2]

**(b)** Show that the point of inflexion on $P$ occurs when $P = \frac{k}{2}$. [6]

# Higher level

## Practice paper 2

Time: 2 hours

Technology required

Maximum: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations.

Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**1** [Maximum marks: 13]

In an experiment, mould is being grown in a circular dish. The area of the dish which is covered by the mould is recorded each day.

Past experience has shown that the growth of the mould can be modelled by the equation $A = ae^{bt}$ where $A$ is the area covered and $t$ the time in days that the mould has been growing.

Initially there is 1.2 cm² covered by the mould and the next day (when $t = 1$) there is 3.4 cm² covered.

**(a)** **(i)** Write down the value of $a$. [1]

**(ii)** Find the value of $b$, accurate to three significant figures. [2]

**(b)** Using your answers to part **(a)**, find the area of the dish covered by mould when $t = 2$. [1]

The dish is in the shape of a circle with a radius of 5 cm.

When 80% of the area of the dish is covered by the mould, the rate of growth slows.

**(c)** Find the value of $t$ at which the rate of growth begins to slow. [3]

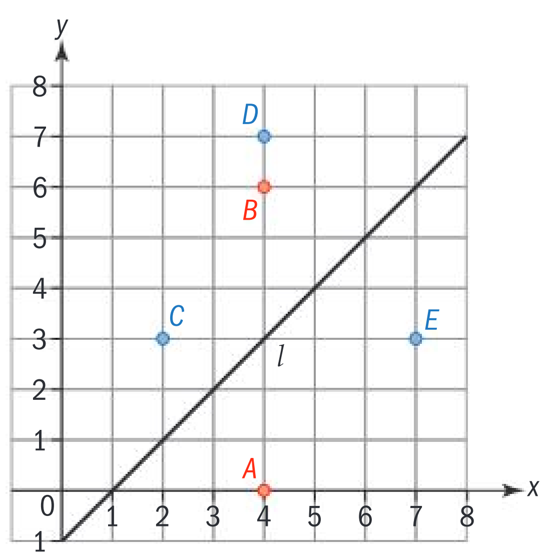
After this time, the area of the dish covered by mould can be modelled by the equation $A = c + dt$, up to the point when the dish is completely covered.

**(d)** If the dish is completely covered when $t = 6.0$, find the values of $c$ and $d$. [5]

**(e)** Write down the rate of growth of the mould in cm² per day during the period of linear growth. [1]

**2** [Maximum marks: 14]

The Red team are playing against the Blue team at soccer. Part of the area being used is shown on the diagram below where all measurements are in metres.




Player $A$ currently has the ball at (4, 0) and wants to pass to player $B$ at (4, 6). Players $C$, $D$ and $E$ are all on the opposing team at the points with coordinates (2, 3), (4, 7) and (7, 3).

The line $l$ is the perpendicular bisector of $B$ and $E$.

**(a)** Write down the equation of $l$. [2]

**(b)** Show that the equation for the perpendicular bisector of $[BC]$ is $2x + 3y = 19.5$ [5]

**(c)** Find the point where $l$ meets $2x + 3y = 19.5$ [2]

Player $A$ needs to pass the ball to a point in the region $R$ which contains those points closer to $B$ than to any of the opposing players.

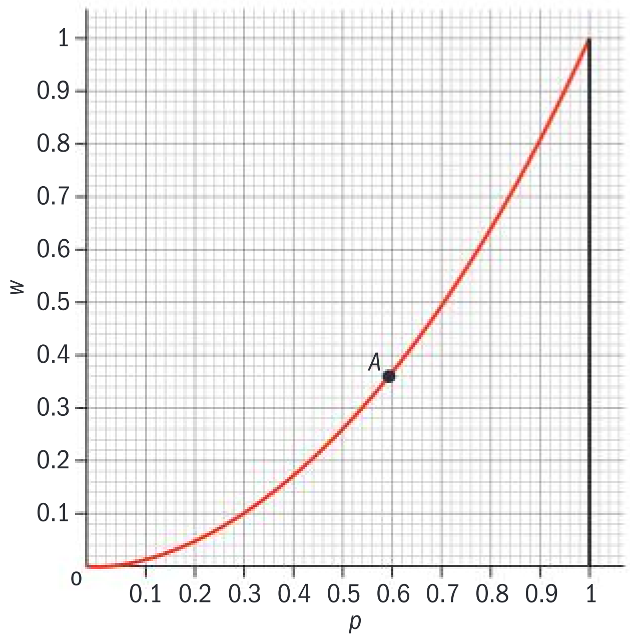
**(d)** **(i)** Draw a sketch of the diagram and on it show the region $R$. [3]

**(ii)** Find the area of $R$. [2]

**3** [Maximum marks: 12]

A Lorenz curve, $w(p)$, shows the percentage of a nation's wealth ($w$) owned by the poorest $p$% of the population.

For example, in the diagram below, in which the percentages are given as their decimal equivalents, the point $A$ indicates that the poorest 60% in the country own 36% of the wealth of that country.

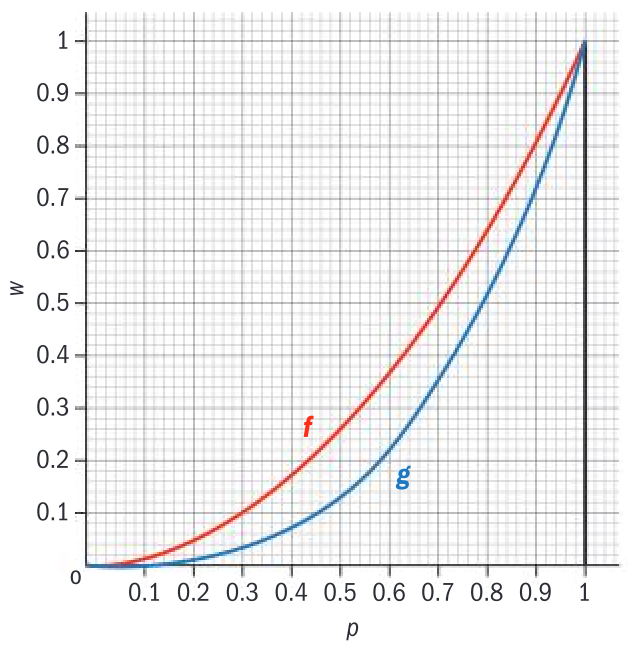



**(a)** Use the diagram above to write down the proportion of wealth owned by:

**(i)** the poorest 40% [1]

**(ii)** the richest 10% of the country. [2]

**(b)** The diagram below shows the wealth distribution in two countries. State which of the two curves ($f$ or $g$) represents a country in which the wealth is distributed more evenly among the population. Justify your answer. [2]




One measure of inequality within a country is the Gini coefficient.

If the area of the region below the Lorenz curve for $0 \le p \le 1$ is $A$, then the Gini coefficient ($G$) is given by $G = 1 - 2A$.

In a particular country, the poorest 50% of the country share 28% of the income and the poorest 80% of the country share 65% of the income.

**(c)** **(i)** Write down the coordinates of four points the curve passes through. [2]

**(ii)** Use cubic regression to find an estimate for the Gini coefficient for this country. [5]

**4** [Maximum marks: 13]

**(a)** Find the eigenvalues and eigenvectors for the matrix $\begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}$ [7]

**(b)** Hence, find the solution to the system of coupled differential equations below, given that at $t = 0$, $x = 2$ and $y = 4$.

$\dot{x} = 2x - 2y \qquad \dot{y} = x + 5y$ [6]

**5** [Maximum marks: 13]

The graph $y = 0.2x^2$, $0 \le x \le 10$, where $x$ is measured in centimetres, is rotated 360° about the $y$–axis to form a bowl.

The bowl is gradually filled with liquid.

**(a)** Find an expression for the volume of liquid in the bowl when it is filled to a height $h$. [6]

The temperature of the surroundings is close to the boiling point of the liquid and so the liquid gradually evaporates from the bowl. The rate of evaporation in $cm^3\,s^{-1}$ is equal to $0.02A$, where $A$ is the area of the liquid exposed to the air.

**(b)** Find an expression for the rate of evaporation of the liquid when it is filled to a height $h$. [3]

The bowl is filled at a constant rate of $5\ cm^3\,s^{-1}$.

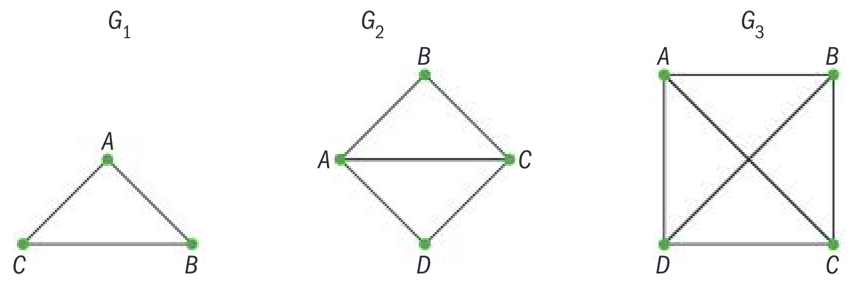
**(c)** **(i)** Find the value of $h$ at which the rate of evaporation equals the rate at which the bowl is being filled. [2]

**(ii)** Find the volume of liquid in the bowl at this point. [2]

**6** [Maximum marks: 11]

An engineer is trying to create an efficient network. To do so, he needs to calculate the number of triangles it contains. A triangle is defined as a set of any three adjacent vertices, irrespective of order.

**(a)** Using the definition given above, write down the number of triangles in each of the three graphs shown. [2]




The engineer creates an adjacency matrix **M** for the graph of his network.

**(b)** **(i)** Write down the adjacency matrix **M** for graph $G_2$. [2]

**(ii)** Hence, find the number of walks of length three from $A$ to $A$ in graph $G_2$. [2]

**(iii)** Explain how the engineer could use an adjacency matrix to find the number of triangles in a simple graph. Fully justify your answer. [4]

**(iv)** Verify your method by using **M** to find the number of triangles in the graph $G_2$. [1]

**7** [Maximum marks: 17]

A homeowner wants to measure the maximum amount of electricity she could hope to generate if she covered her roof in solar panels.

She first calculates the coordinates of the corners of her roof, taking as the origin one of the ground-level corners of the house and got the following measurements: $A(0, 0, 4.8)$, $B(0, 12, 4.8)$, $C(5, 0, 6.6)$ and $D(5, 12, 6.6)$.

**(a)** Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [3]

**(b)** Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]

**(c)** Hence, find the area of the roof, given it is in the shape of a rectangle. [3]

The homeowner takes measurements and finds that the sun is most directly shining on the surface of the roof when it shines in the direction of the vector $\begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix}$.

This occurs at time $T$.

It is given that the intensity of the sun at time $T$ can be written as $I = k\begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix}$

**(d)** Given the magnitude of $I$ is 900 watts per m$^2$, find the value of $k$. [2]

If the solar panels are 100% efficient, the amount of power per m$^2$ (watts per m$^2$) they produce is equal to the component of $I$ that lies along the vector perpendicular to the roof.

**(e)** **(i)** Write down a vector perpendicular to the roof. [1]

**(ii)** Find the component of $I$ which lies along the vector perpendicular to the roof at time $T$. [4]

**(f)** Use your answers to parts **(c)** and **(e)** to calculate the number of watts the homeowner might expect to produce at time $T$ if her solar panels are 20% efficient. [2]

**8** [Maximum marks: 17]

A factory fills bags of flour which are all labelled as containing 1 kg. The machine is set so the weight of flour put into the bags is normally distributed with a mean of 1 kg and a standard deviation of 50 g. A bag which weighs less than 950 g is considered to be underweight. Let the weight of flour in a bag be $X$.

**(a)** Find the probability that a bag of flour chosen at random is underweight, giving your answer to four decimal places. [2]

The weight of flour in the bags is regularly inspected by a weights and measures authority. In the inspection 5 bags are taken and if 2 or more are found to be underweight, the factory is fined €5000.

**(b)** Using your answer (rounded to 4 d.p.) from part **(a)**, find:

**(i)** the probability the factory fails an inspection [4]

**(ii)** the expected loss to the factory over 10 separate inspections. [1]

The method of checking the weight of the bags of flour is changed so that the factory would fail the inspection if the mean weight of the 5 bags, $\overline{X}$, was less than $a$ g.

**(c)** **(i)** Write down the distribution of $\overline{X}$. [3]

**(ii)** Find the value of $a$ given that the probability of the factory failing the inspection has not changed. [2]

The owner of the factory decides to increase the mean weight $\mu$, of flour, added to the bags so that they are less likely to fail the inspection.

Let the cost of flour be €1 per kilogram. Inspections take place after 10 000 bags have been filled. Let the random variable $C$ be the total of: expected payment in fines per inspection, plus the cost of the flour for 10 000 bags.

The owner is keen to minimize his costs.

**(d)** **(i)** Find, to the nearest 5 g, the value of $\mu$ you would recommend the owner uses. In your calculations use the value of $a$ found in **(c)(ii)** rounded to the nearest 10 g. [4]

**(ii)** Explain why you would not recommend he uses a value for $\mu$ which gives a smaller value for $C$. [1]

# Higher level

## Practice paper 3

Time: 1 hour

Technology required

Maximum: 55 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations.

Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**1** [Maximum marks: 25]

This question considers three different methods to discover which of two skiers is the best.

*All three tests should be performed at a 5% significance level and you should clearly state the null and alternative hypothesis, and the conclusion for each of the test.*

George is trying to work out if he is a significantly better skier than his rival Albert. To test this he records their results in their eight most recent races.

| Race | George | | Albert | |
|---|---|---|---|---|
| | Position | Time (seconds) | Position | Time (seconds) |
| 1 | 5 | 90.2 | 7 | 91.3 |
| 2 | 10 | 95.6 | 15 | 97.1 |
| 3 | 15 | 98.5 | 14 | 98.4 |
| 4 | 4 | 90.5 | 8 | 92.1 |
| 5 | 12 | 97.6 | 15 | 99.0 |
| 6 | 13 | 92.5 | 4 | 89.2 |
| 7 | 9 | 97.5 | 8 | 97.4 |
| 8 | 7 | 95.0 | 8 | 95.1 |

**(a)** **(i)** Write down the number of times that George beats Albert. [1]

**(ii)** Use the binomial distribution to test the hypothesis that the probability George beats Albert is equal to 0.5 against the hypothesis that it is greater than 0.5. [6]

George wonders if this is the optimal way to decide who is best as it does not take into account their times in each race.

**(b)** Perform a paired sample test to see if the mean race time for George is less than the mean race time for Albert.

You may assume that the differences in their times are normally distributed. [7]

George discusses his findings with Albert, who suggests that actually the most important information is their position in the race and perhaps another test should be done on whether George's average position is significantly less than his.

**(c)** **(i)** State one reason why George cannot assume the data is normally distributed. [1]

George collects data from 35 of the races in which they have competed over the previous three seasons. He calculates that his mean position was 5.23 with a standard deviation of 2.12, and Albert's mean position was 6.21 with a standard deviation of 4.56. You may assume that both populations have the same variance.

**(ii)** State why the sample mean can now be regarded as following a normal distribution. [1]

**(iii)** Perform an appropriate test to see if this data provides significant evidence that George's average position is less than Albert's. [7]

**(d)** In context state a possible reason for the differences in the conclusions of the tests. [2]

**2** [Maximum marks: 30]

*The aim of the question is to find the equation of a complex sound wave.*

**(a)** **(i)** Expand $(\cos t + \mathrm{i}\sin t)^2$. [2]

Let $z = \cos t + \mathrm{i}\sin t$.

$z^2$ can be written in the form $r(\cos at + \mathrm{i}\sin at),\ a, r \in \mathbb{R}$.

**(ii)** Write down the values of $a$ and $r$. [2]

**(iii)** Use your answer to **(a)(i)** and **(a)(ii)** to show that $\cos 2t = 1 - 2\sin^2 t$ [2]

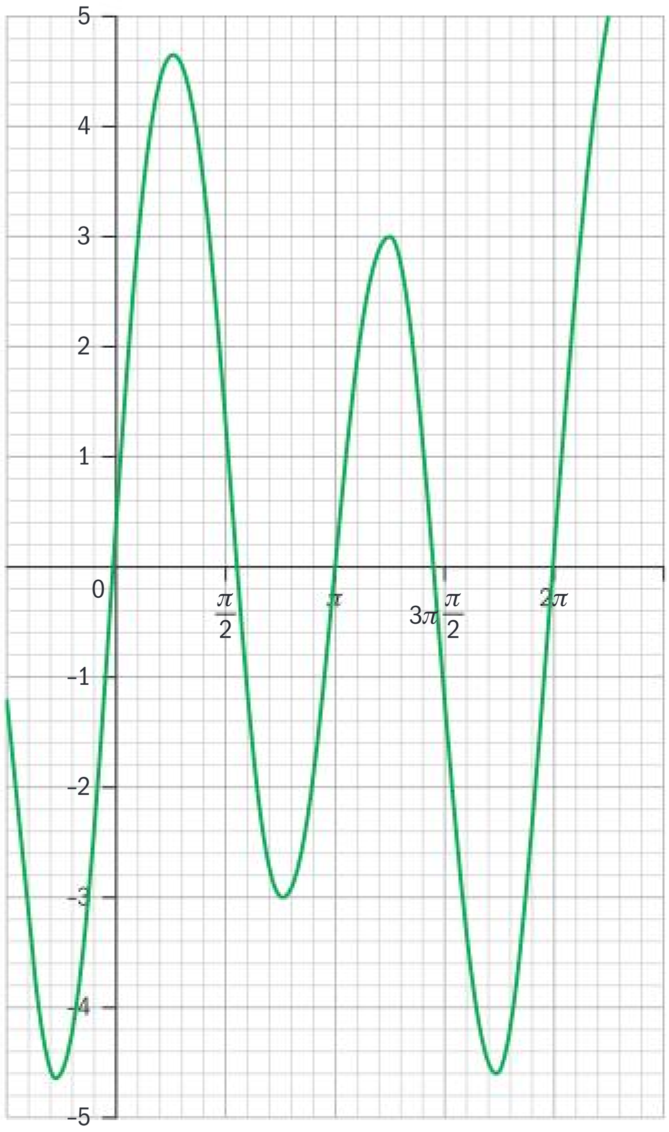
**(b)** Write down $\int_0^{2\pi} \sin 2t \sin t \,\mathrm{d}t$. [1]

**(c)** **(i)** Rearrange the expression from **(a)(iii)** to find an expression for $\sin^2 t$. [2]

**(ii)** Hence find $\int \sin^2 t \,\mathrm{d}t$. [2]

**(iii)** Use your answer to **(c)(ii)** to show that $\int_0^{2\pi} \sin^2 t \,\mathrm{d}t = \pi$. [1]

It is thought that the curve shown can be written in the form $f(t) = a_1 \sin t + a_2 \sin 2t,\ a_1, a_2 \in \mathbb{R}$.




**(d)** Show $\int_0^{2\pi} f(t)\sin t \,\mathrm{d}t = \pi a_1$. [2]

Approximate values of $f(t)$ at intervals of $\frac{\pi}{4}$ are taken from the graph and are given in the table below. The values of $f(t)\sin t$ for these points are then calculated.

It is desired to use all these values to find an estimate for $f(t)$.

| $t$ | $f(t)$ | $f(t)\sin t$ |
|---|---|---|
| 0 | 0 | 0 |
| $\frac{\pi}{4}$ | 4.65 | 3.29 |
| $\frac{\pi}{2}$ | 1.19 | 1.19 |
| $\frac{3\pi}{4}$ | −2.96 | $p$ |
| $\pi$ | 0 | 0 |
| $\frac{5\pi}{4}$ | 2.94 | −2.08 |
| $\frac{3\pi}{2}$ | −1.21 | 1.21 |
| $\frac{7\pi}{4}$ | −4.65 | 3.29 |
| $2\pi$ | 0 | 0 |

**(e)** Find the value of $p$ to three significant figures. [2]

**(f)** **(i)** Use the trapezoidal rule with eight intervals to find an estimate for

$$\int_0^{2\pi} f(t)\sin t \,\mathrm{d}t$$ [3]

**(ii)** Hence write down an estimate for $a_1$. [2]

**(g)** It is given that $\sin^2 2t = \frac{1}{2}(1 - \cos 4t)$.

Use this result to show that $\int_0^{2\pi} f(t)\sin(2t)\,\mathrm{d}t = \pi a_2$. [5]

The table below shows the values of $f(t)\sin 2t$ at intervals of $\frac{\pi}{4}$ for $0 \le t \le 2\pi$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $f(t)\sin 2t$ | 0 | 4.65 | 0 | 2.96 | 0 | 2.94 | 0 | 4.65 | 0 |

**(h)** **(i)** Use the trapezoidal rule with eight intervals to find an estimate for

$$\int_0^{2\pi} f(t)\sin(2t)\,\mathrm{d}t$$ [2]

**(ii)** Hence write down an expression for the function $f(t)$. [2]

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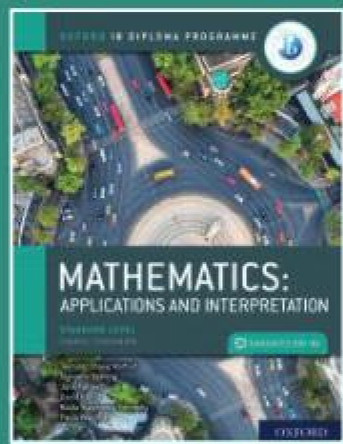


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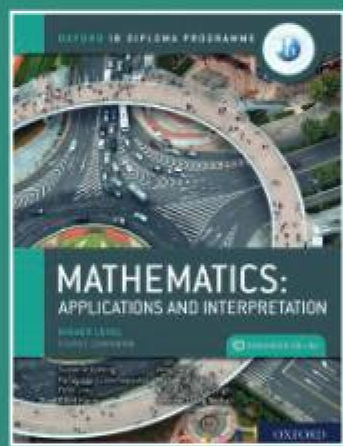


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