

Revision answers: Calculus (Topic 6)

Coursebook chapters: 12–15

1. $f'(x) = 3ax^2 + 2bx + 4$

$$f''(x) = 6ax + 2b$$

So,

$$f'(2) = 0 \Rightarrow 12a + 4b + 4 = 0 \Rightarrow 3a + b = -1$$

$$f''(2) = 10 \Rightarrow 12a + 2b = 10 \Rightarrow 6a + b = 5$$

Solving simultaneously gives $a = 2$, $b = -7$

[5 marks]

2. $\int_{-\pi/3}^{\pi/2} \cos\left(\frac{1}{2}x\right) dx = \left[2\sin\left(\frac{1}{2}x\right)\right]_{-\pi/3}^{\pi/2} = 2\left(\frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right)\right) = \sqrt{2} + 1$

[4 marks]

3. (a) $\frac{(x-2)(x-6)}{\sqrt{x}} = \frac{x^2 - 8x + 12}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 12x^{-\frac{1}{2}}$

i.e. $a = \frac{3}{2}$, $b = \frac{1}{2}$, $c = -\frac{1}{2}$

(b) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} - 6x^{-\frac{3}{2}}$

At $x = 4$, $\frac{dy}{dx} = \frac{1}{4}$

\therefore gradient of normal $= -4$

When $x = 4$, $y = -2$.

$$\text{So, } y - (-2) = -4(x - 4) \Rightarrow y = -4x + 14$$

(c) (i) At the intersection with the x -axis, $y = 0$: $0 = -4x + 14 \Rightarrow x = \frac{7}{2}$.

At the intersection with the y -axis, $x = 0$: $y = 14$.

So, $P\left(\frac{7}{2}, 0\right), Q(0, 14)$

(ii) Area $= \frac{1}{2} \times \frac{7}{2} \times 14 = \frac{49}{2}$ [13 marks]

4. From GDC, the intersection point is (0.7628, 0.9137) and the second curve has root 1.0327.

If $a = 0.7628$ and $b = 1.0327$, the area shown is $\int_0^a x^{\frac{1}{3}} dx + \int_a^b (e^{-x^2} - x) dx$.

Area $= 0.5227 + 0.1207 = 0.643$ (3SF) [6 marks]

5. (a) $y' = 12x^3 - 24x^2 + 12x$

At stationary points $y' = 0$

$$\Rightarrow x^3 - 2x^2 + x = 0 \Rightarrow x(x-1)^2 = 0 \Rightarrow x = 0, 1$$

So stationary points are (0, -2) and (1, -1)

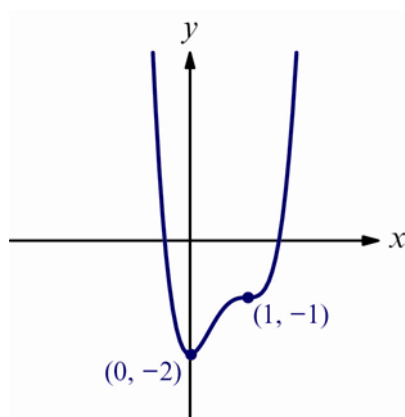
(b) $y'' = 36x^2 - 48x + 12$

At $x = 0, y'' = 12 > 0 \Rightarrow (0, -2)$ local minimum.

At $x = 1, y'' = 0$ so check y' either side of $x = 1$.

When $x = \frac{1}{2}, y' > 0$ and when $x = \frac{3}{2}, y' > 0 \Rightarrow (1, -1)$ (positive) point of inflexion.

(c)



[12 marks]

6. Using the reverse chain rule or the substitution, $u = x^2 - 9$:

$$\int \frac{4x}{x^2 - 9} dx = 2 \ln(x^2 - 9) + c = \ln(x^2 - 9)^2 + c \quad [6 \text{ marks}]$$

7. Using the product rule for (a) and quotient rule for (b):

$$\begin{aligned} \text{(a)} \quad y' &= 2e^{2x} \tan^2 3x + e^{2x} \times 2(\tan 3x) \times 3 \sec^2 3x \\ &= 2e^{2x} \tan^2 3x + 6e^{2x} \tan 3x \sec^2 3x \\ &= 2e^{2x} \tan 3x(\tan 3x + 3 \sec^2 3x) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y' &= \frac{\frac{2x}{1+x^2}(1+x^2) - \ln(1+x^2)2x}{(1+x^2)^2} \\ &= \frac{2x(1 - \ln(1+x^2))}{(1+x^2)^2} \quad [9 \text{ marks}] \end{aligned}$$

8. (a) (i) $v = \int a \, dt = 3t^2 - 22t + c$

$$v = 35 \text{ when } t = 0 \Rightarrow c = 35$$

$$\therefore v = 3t^2 - 22t + 35$$

$$\text{(ii)} \quad 3t^2 - 22t + 35 = 0 \Rightarrow (3t - 7)(t - 5) = 0 \Rightarrow t = \frac{7}{3}, 5 \text{ seconds}$$

$$\text{(iii)} \quad \text{For local max/min, } v' = 0, \text{ so } 6t - 22 = 0 \Rightarrow t = \frac{11}{3}.$$

$$\text{However, at } t = \frac{11}{3}, v = -\frac{16}{3}, \text{ whereas at } t = 0, v = 35.$$

$$\therefore \text{max speed} = 35 \text{ ms}^{-1}$$

- (b) We know that P has negative velocity for $\frac{7}{3} < t < 5$, so P is moving back towards O here.

So find the displacement for $0 < t < \frac{7}{3}$ and $\frac{7}{3} < t < 5$ separately:

$$s_1 = \int_0^{\frac{7}{3}} v \, dt = \int_0^{\frac{7}{3}} 3t^2 - 22t + 35 \, dt = \left[t^3 - 11t^2 + 35t \right]_0^{\frac{7}{3}} = \frac{931}{27} \text{ m}$$

$$s_2 = \int_{\frac{7}{3}}^5 v \, dt = \int_{\frac{7}{3}}^5 3t^2 - 22t + 35 \, dt = \left[t^3 - 11t^2 + 35t \right]_{\frac{7}{3}}^5 = -\frac{256}{27} \text{ m}$$

$$\therefore s = \frac{931}{27} + \frac{256}{27} = \frac{1197}{27} \approx 44 \text{ m}$$

[11 marks]

9. (a) $\cos 2x = 1 - 2 \sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\therefore \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + c = \frac{1}{4}(2x - \sin 2x) + c$$

(b) $V = \pi \int_0^{\pi/2} y^2 \, dx = \pi \int_0^{\pi/2} x \sin^2 x \, dx$

$$= \pi \left(\left[\frac{x}{4}(2x - \sin 2x) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{4}(2x - \sin 2x) \, dx \right)$$

$$= \pi \left(\left[\frac{x}{4}(2x - \sin 2x) \right]_0^{\pi/2} - \left[\frac{1}{4} \left(x^2 + \frac{1}{2} \cos 2x \right) \right]_0^{\pi/2} \right)$$

$$= \pi \left(\frac{\pi^2 + 4}{16} \right)$$

[12 marks]

10. (a) Height of triangular face $= \sqrt{3}$

Area of triangular face of water of depth h and base b , $A = \frac{1}{2}bh$

By similar triangles: $\frac{h}{b} = \frac{\sqrt{3}}{2} \Rightarrow b = \frac{2h}{\sqrt{3}}$

So, the volume of water at height h ,

$$V = \frac{1}{2}bh \times 6 = \frac{1}{2} \times \frac{2h}{\sqrt{3}} h \times 6 = 2\sqrt{3}h^2.$$

- (b) Capacity of tank $= 6\sqrt{3}$, so when quarter full $V = \frac{3}{2}\sqrt{3}$.

$$\frac{3}{2}\sqrt{3} = 2\sqrt{3}h^2 \Rightarrow h = \frac{\sqrt{3}}{2}$$

Then, $\frac{dV}{dh} = 4\sqrt{3}h$

$$\text{And } \frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{1}{4\sqrt{3}h} \times \frac{600}{10^6} = \frac{3}{20000\sqrt{3}h}$$

$$\text{So, when } h = \frac{\sqrt{3}}{2}, \frac{dh}{dt} = \frac{3}{20000\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{10000} \text{ ms}^{-1} \left(\text{or } \frac{1}{100} \text{ cm}^{-1} \right) \quad [7 \text{ marks}]$$