SL Paper 1

The vertices A, B, C of an acute angled triangle have position vectors ***a***, ***b***, ***c*** with respect to an origin O.

The mid-point of [BC] is denoted by D. The point E lies on [AD] such that $\mathrm{AE} = 2\mathrm{DE}$.

The perpendiculars from B to [AC] and C to [AB] meet at the point F.

a.i. Show that the position vector of E is [4]

$$\frac{1}{3}\,(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}).$$

a.ii. Explain briefly why this result shows that the three medians of a triangle are concurrent. [1]

b.i. Show that the position vector ***f*** of F satisfies the equations [3]

$$(\boldsymbol{b} - \boldsymbol{f}) \bullet (\boldsymbol{c} - \boldsymbol{a}) = 0$$

$$(\boldsymbol{c} - \boldsymbol{f}) \bullet (\boldsymbol{a} - \boldsymbol{b}) = 0.$$

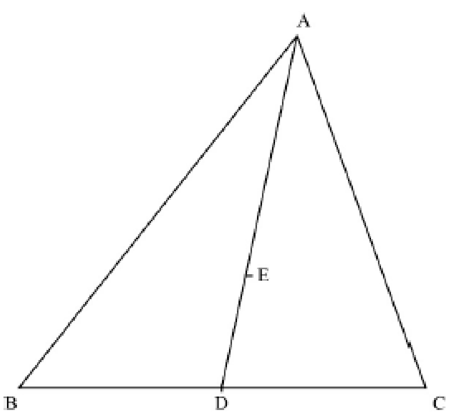
b.ii. Show, by expanding these equations, that [3]

$$(\boldsymbol{a} - \boldsymbol{f}) \bullet (\boldsymbol{c} - \boldsymbol{b}) = 0.$$

b.iii. Explain briefly why this result shows that the three altitudes of a triangle are concurrent. [1]

Markscheme

a.i.



$\boldsymbol{d} = \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$ ***(M1)A1***

$\boldsymbol{e} = \boldsymbol{d} + \frac{1}{3}(\boldsymbol{a} - \boldsymbol{d})$ ***M1***

$= \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) + \frac{1}{3}(\boldsymbol{a} - \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}))$ ***A1***

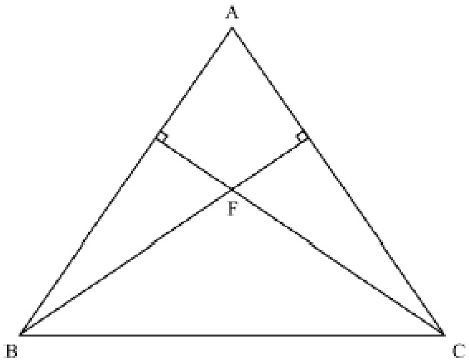
$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

[??? marks]

a.ii.(because of the symmetry of the result), the other two medians also pass through E. ***R1***

[??? marks]

b.i.



$\overrightarrow{\mathrm{BF}}$ = ***f*** – ***b*** and $\overrightarrow{\mathrm{AC}}$ = ***c*** – ***a*** ***A1***

since FB is perpendicular to AC, (***b*** – ***f***) • (***c*** – ***a***) = 0 ***R1AG***

similarly since FC is perpendicular to BA, (***c*** – ***f***) • (***a*** – ***b***) = 0 ***R1AG***

[??? marks]

b.ii.expanding these equations and adding, ***M1***

b • ***c*** – ***b*** • ***a*** – ***f*** • ***c*** + ***f*** • ***a*** + ***c*** • ***a*** – ***c*** • ***b*** – ***f*** • ***a*** + ***f*** • ***b*** = 0 ***A1***

– ***b*** • ***a*** – ***f*** • ***c*** + ***c*** • ***a*** + ***f*** • ***b*** = 0 ***A1***

leading to (***a*** – ***f***) • (***c*** – ***b***) = 0 ***AG***

[??? marks]

b.iii.this result shows that AF is perpendicular to BC so that the three altitudes are concurrent (at F) ***R1***

[??? marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]

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a. The point $\mathrm{T}(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Show that the tangent to the parabola at T has equation $y = \frac{x}{t} + at$. [3]

b. The distinct points $\mathrm{P}(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p, q \neq 0$, also lie on the parabola $y^2 = 4ax$. Given that the line (PQ) passes through the focus, show that [8]

(i) $pq = -1$;

(ii) the tangents to the parabola at P and Q, intersect on the directrix.

Markscheme

a. $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} = \frac{1}{t}$ ***A1***

Note: Accept parametric differentiation.

the equation of the tangent is

$y - 2at = \frac{1}{t}(x - at^2)$ ***A1***

$y = \frac{x}{t} + at$ ***AG***

Note: Accept equivalent based on $y = mx + c$.

[3 marks]

b. (i) the focus F is $(a, 0)$ ***A1***

EITHER

the gradient of (PQ) is $\frac{2a(p-q)}{a(p^2-q^2)} = \frac{2}{p+q}$ ***M1A1***

the equation of (PQ) is $y = \frac{2x}{p+q} + \frac{2apq}{p+q}$ ***A1***

substitute $x = a$, $y = 0$ ***M1***

$pq = -1$ ***AG***

OR

the condition for PFQ to be collinear is

$\frac{2a(p-q)}{a(p^2-q^2)} = \frac{2ap}{ap^2-a}$ ***M1A1***

$\frac{2}{p+q} = \frac{2p}{p^2-1}$ ***A1***

$p^2 - 1 = p^2 + pq$ ***A1***

$pq = -1$ ***AG***

Note: There are alternative approaches.

(ii) the equations of the tangents at P and Q are

$y = \frac{x}{p} + ap$ and $y = \frac{x}{q} + aq$

the tangents meet where

$\frac{x}{p} + ap = \frac{x}{q} + aq$ ***M1***

$x = apq = -a$ ***A1***

the equation of the directrix is $x = -a$ ***R1***

so that the tangents meet on the directrix ***AG***

[8 marks]

Examiners report

a. [N/A]

b. [N/A]

The normal at the point $\mathrm{T}(at^2,\ 2at),\ t \neq 0$, on the parabola $y^2 = 4ax$ meets the parabola again at the point $\mathrm{S}(as^2,\ 2as)$.

a. Show that $t^2 + st + 2 = 0$. [7]

b. Given that $\mathrm{S\hat{O}T}$ is a right-angle, where O is the origin, determine the possible values of t. [5]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ ***(M1)***

$= \frac{2a}{2at} = \frac{1}{t}$ ***A1***

the gradient of the normal $= -t$ ***(A1)***

the equation of the normal at T is $y - 2at = -t(x - at^2)$ ***A1***

substituting the coordinates of S, ***M1***

$2as - 2at = -t(as^2 - at^2)$

$2a(s - t) = -at(s - t)(s + t)$ ***A1***

$2 = -t(s + t) = -st - t^2$ ***A1***

$t^2 + st + 2 = 0$ ***AG***

[7 marks]

b. gradient of $\mathrm{OT} = \frac{2at}{at^2} = \frac{2}{t}$ ***A1***

gradient of $\mathrm{OS} = \frac{2as}{as^2} = \frac{2}{s}$ ***(A1)***

the condition for perpendicularity is $\frac{2}{t} \times \frac{2}{s} = -1$ ***M1***

$t^2 - 4 + 2 = 0$ ***A1***

$t = \pm\sqrt{2}$ ***A1***

[5 marks]

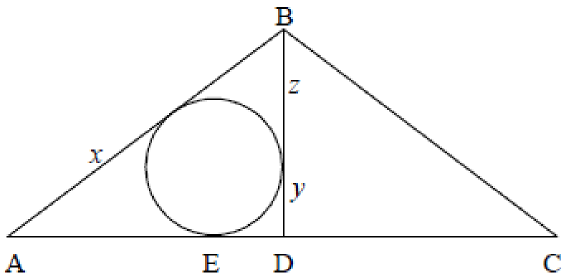
Examiners report

a. [N/A]
b. [N/A]

The triangle ABC is isosceles and AB = BC = 5. D is the midpoint of AC and BD = 4.

Find the lengths of the tangents from A, B and D to the circle inscribed in the triangle ABD.

Markscheme



AD $= 3$ ***(A1)***

Let the lengths of the tangents be as shown.

Then,

$x + y = 3$

$y + z = 4$

$x + z = 5$ ***(M1)A1***

Solving,

$x = 2\,, y = 1\,, z = 3$ ***A1A1A1***

[6 marks]

Examiners report

[N/A]

Given that the tangents at the points P and Q on the parabola $y^2 = 4ax$ are perpendicular, find the locus of the midpoint of PQ.

Markscheme

EITHER

attempt to differentiate ***(M1)***

let $y = 2at \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2a$ and $x = at^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 2at$ ***A1***

hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{2a}{2at} = \frac{1}{t}$ ***A1***

let P have coordinates $\left(at_1^2,\ 2at_1\right)$ and Q have coordinates $\left(at_2^2,\ 2at_2\right)$ ***(M1)***

therefore gradient of tangent at P is $\frac{1}{t_1}$ and gradient of tangent at Q is $\frac{1}{t_2}$ ***A1***

since these tangents are perpendicular $\frac{1}{t_1} \times \frac{1}{t_2} = -1 \Rightarrow t_1 t_2 = -1$ ***A1***

mid-point of PQ is $\left(\frac{a\left(t_1^2 + t_2^2\right)}{2},\ a\left(t_1 + t_2\right)\right)$ ***A1***

$y^2 = a^2\left(t_1^2 + 2t_1 t_2 + t_2^2\right)$ ***M1***

$y^2 = a^2\left(\frac{2x}{a} - 2\right)$ $\left(\Rightarrow y^2 = 2ax - 2a^2\right)$ ***A1***

OR

attempt to differentiate ***(M1)***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$

let coordinates of P be $(x_1,\ y_1)$ and the coordinates of Q be $(x_2,\ y_2)$ ***(M1)***

coordinates of midpoint of PQ are $\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$ ***M1***

if the tangents are perpendicular $\frac{2a}{y_1} \times \frac{2a}{y_2} = -1$ ***A1***

$\Rightarrow y_1y_2 = -4a^2$

$y_1^2 + y_2^2 = 4a\left(x_1 + x_2\right)$ ***A1***

$\frac{y_1^2+2y_1y_2+y_2^2}{4} = \frac{4a(x_1+x_2)+2y_1y_2}{4}$ ***M1***

$\left(\frac{y_1+y_2}{2}\right) = 2a\frac{x_1+x_2}{2} - \frac{8a^2}{4}$ ***A1***

hence equation of locus is $y^2 = 2ax - 2a^2$ ***A1***

[9 marks]

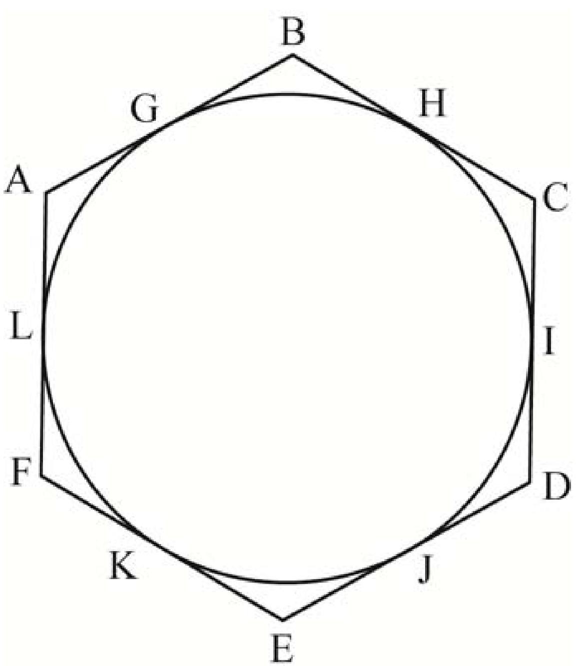
Examiners report

[N/A]

ABCDEF is a hexagon. A circle lies inside the hexagon and touches each of the six sides.

Show that $\mathrm{AB} + \mathrm{CD} + \mathrm{EF} = \mathrm{BC} + \mathrm{DE} + \mathrm{FA}$.

Markscheme



A1

the lengths of the two tangents from a point to a circle are equal ***(R1)***

so that

$\mathrm{AG} = \mathrm{LA}$

$\mathrm{GB} = \mathrm{BH}$

$\mathrm{CI} = \mathrm{HC}$

$\mathrm{ID} = \mathrm{DJ}$

$\mathrm{EK} = \mathrm{JE}$

$\mathrm{KF} = \mathrm{FL}$ ***A1***

adding,

$(\mathrm{AG} + \mathrm{GB}) + (\mathrm{CI} + \mathrm{ID}) + (\mathrm{EK} + \mathrm{KF}) = (\mathrm{BH} + \mathrm{HC}) + (\mathrm{DJ} + \mathrm{JE}) + (\mathrm{FL} + \mathrm{LA})$ ***M1A1***

$\mathrm{AB} + \mathrm{CD} + \mathrm{EF} = \mathrm{BC} + \mathrm{DE} + \mathrm{FA}$ ***AG***

[5 marks]

Examiners report

[N/A]

Consider the curve C given by $y = x^3$.

The tangent at a point P on C meets the curve again at Q. The tangent at Q meets the curve again at R. Denote the x-coordinates of P, Q and R, by $x_1,\ x_2$ and x_3 respectively where $x_1 \neq 0$. Show that, $x_1,\ x_2,\ x_3$ form the first three elements of a divergent geometric sequence.

Markscheme

attempt to find the equation of the tangent at P ***M1***

$y - x_1^3 = 3x_1^2(x - x_1)$ ***A1***

the tangent meets C when

$x^3 - x_1^3 = 3x_1^2(x - x_1)$ ***M1***

attempt to solve the cubic ***M1***

the x-coordinate of Q satisfies

$x^2 + xx_1 - 2x_1^2 = 0$ ***A1***

hence $x_2 = -2x_1$ ***A1***

hence $x_3 = 4x_1$ ***A1***

hence $x_1,\ x_2,\ x_3$ form the first three terms of a geometric sequence with common ratio -2 so the sequence is divergent ***R1AG***

Note: Final ***R1*** is not dependent on final 3 ***A1***s providing they form a geometric sequence.

Total [8 marks]

Examiners report

This question caused a problem for many candidates and only a small number of fully correct answers were seen. Most candidates were able to find a generalised equation of a tangent, but were then unable to see what could be replaced in order to find a quadratic equation that could be solved.

The points P, Q and R, lie on the sides [AB], [AC] and [BC], respectively, of the triangle ABC. The lines (AR), (BQ) and (CP) are concurrent.

Use Ceva's theorem to prove that [PQ] is parallel to [BC] if and only if R is the midpoint of [BC].

Markscheme

suppose R is the midpoint of BC ***M1***

Note: The first mark is for initiating a relevant discussion for "if" or "only if" by Ceva's theorem.

$\frac{\text{AP}}{\text{PB}} \times \frac{\text{BR}}{\text{RC}} \times \frac{\text{CQ}}{\text{QA}} = 1$ ***A1***

$\Rightarrow \frac{\text{AP}}{\text{PB}} = \frac{\text{AQ}}{\text{QC}}$ or equivalent ***A1***

$\Rightarrow \frac{\text{PB}}{\text{AP}} + 1 = \frac{\text{QC}}{\text{AQ}} + 1$ ***(M1)***

$\Rightarrow \frac{\text{AP+PB}}{\text{AP}} = \frac{\text{AQ+QC}}{\text{AQ}}$

$\Rightarrow \frac{\text{AB}}{\text{AP}} = \frac{\text{AC}}{\text{AQ}}$ ***A1***

⇒ triangles APQ and ABC are similar with common base angles ***R1***

so PQ is parallel to BC ***AG***

statement of the converse ***A1***

the argument is reversible ***R1AG***

[8 marks]

Examiners report

This was again a question which a significant number of students were unable to start. For those who did start only a small number understood the significance of "if and only if" meaning that wholly correct answers were not often seen.

The points A, B have coordinates $(-3,\ 0)$, $(5,\ 0)$ respectively. Consider the Apollonius circle C which is the locus of point P where

$$\frac{\text{AP}}{\text{BP}} = k \text{ for } k \neq 1.$$

Given that the centre of C has coordinates $(13,\ 0)$, find

a. (i) the value of k; [11]

(ii) the radius of C;

(iii) the x-intercepts of C.

b. Let M be any point on C and N be the x-intercept of C between A and B. [3]

Prove that $\text{A}\hat{\text{M}}\text{N} = \text{N}\hat{\text{M}}\text{B}$.

Markscheme

a. (i) let $(x,\ y)$ be a point on C

then $(x+3)^2+y^2=k^2\left((x-5)^2+y^2\right)$ ***M1A1A1***

Note: Award ***M1*** for form of an Apollonius circle, ***A1*** for each side.

rearrange, for example,

$(k^2-1)x^2-(10k^2+6)x+(k^2-1)y^2+25k^2-9=0$ ***A1***

equate the x-coordinate of the centre as given by this equation to 13:

$\frac{5k^2+3}{k^2-1}=13$ ***M1A1***

obtain $k^2=2 \Rightarrow k=\sqrt{2}$ ***A1***

(ii) **METHOD 1**

with this value of k, the equation can be reduced to the form

$(x-13)^2+y^2=128$ ***M1A1***

obtain the radius $\sqrt{128}\ \left(=8\sqrt{2}\right)$ ***A1***

METHOD 2

assuming N is the x-intercept of C between A and B

$\frac{\mathrm{AB}}{\mathrm{BN}}=\frac{16-r}{r-8}=\sqrt{2}$ ***M1A1***

$\Rightarrow r=8\sqrt{2}$ ***A1***

Note: Accept answers given in terms of k, if no value of k found in (a)(i).

(iii) x-intercepts are $13\pm 8\sqrt{2}$ ***A1***

[11 marks]

b. because N lies on the circle it satisfies the Apollonius property

hence $\mathrm{AN}=\sqrt{2}\mathrm{NB}$ ***R1***

but as $\mathrm{AM}=\sqrt{2}\mathrm{MB}$ ***R1***

by the converse to the angle-bisector theorem ***R1***

$\mathrm{A\hat{M}N}=\mathrm{N\hat{M}B}$ ***AG***

[3 marks]

Examiners report

a. This question was the first one on the paper to cause a significant problem for the majority of candidates. Many were unable to start and a small number were unable to successfully deal with the algebraic manipulation required from the method they had embarked upon. For those who were successful at part (a), part (b) was often not fully correct, again due to the degree of formality required from the command term “prove”.

b. This question was the first one on the paper to cause a significant problem for the majority of candidates. Many were unable to start and a small number were unable to successfully deal with the algebraic manipulation required from the method they had embarked upon. For those who were successful at part (a), part (b) was often not fully correct, again due to the degree of formality required from the command term “prove”.

The point $\mathrm{P}(x, y)$ moves in such a way that its distance from the point $(1\ ,\ 0)$ is three times its distance from the point $(-1\ ,\ 0)$.

a. Find the equation of the locus of P. [4]

b. Show that this equation represents a circle and state its radius and the coordinates of its centre. [4]

Markscheme

a. We are given that

$\sqrt{(x-1)^2+y^2} = 3\sqrt{(x+1)^2+y^2}$ ***M1A1***

$x^2-2x+1+y^2 = 9(x^2+2x+1+y^2)$ ***A1***

$8x^2+8y^2+20x+8=0$ ***A1***

[4 marks]

b. Rewrite the equation in the form

$\left(x+\frac{5}{4}\right)^2+y^2 = -1+\frac{25}{16} = \frac{9}{16}$ ***M1A1***

This represents a circle with radius $= \frac{3}{4}$; centre $\left(-\frac{5}{4}, 0\right)$ ***A1A1***

Note: Allow ***FT*** from the line above.

[4 marks]

Examiners report

a. [N/A]
b. [N/A]

The parabola P has equation $y^2 = 4ax$. The distinct points U $\left(au^2,\ 2au\right)$ and V $\left(av^2,\ 2av\right)$ lie on P, where $u,\ v \neq 0$. Given that $\mathrm{U\hat{O}V}$ is a right angle, where O denotes the origin,

(a) show that $v = -\frac{4}{\mu}$;

(b) find expressions for the coordinates of W, the midpoint of [UV], in terms of a and u;

(c) show that the locus of W, as u varies, is the parabola P' with equation $y^2 = 2ax - 8a^2$;

(d) determine the coordinates of the vertex of P'.

Markscheme

(a) gradient of OU $= \frac{2au}{au^2} = \frac{2}{u}$ ***A1***

gradient of OV $= \frac{2av}{av^2} = \frac{2}{v}$ ***A1***

since the lines are perpendicular,

$\frac{2}{u} \times \frac{2}{v} = -1$ ***M1***

so $v = -\frac{4}{u}$ ***AG***

[3 marks]

(b) coordinates of W are $\left(\frac{a(u^2+v^2)}{2}, \frac{2a(u+v)}{2}\right)$ ***M1***

$= \left(\frac{a}{2}\left(u^2 + \frac{16}{u^2}\right), a\left(u - \frac{4}{u}\right)\right)$ ***A1***

[2 marks]

(c) putting

$x = \frac{a}{2}\left(u^2 + \frac{16}{u^2}\right);\ y = a\left(u - \frac{4}{u}\right)$ ***M1***

it follows that

$y^2 = a^2\left(u^2 + \frac{16}{u^2} - 8\right)$ ***A1***

$= 2ax - 8a^2$ ***AG***

Note: Accept verification.

[2 marks]

(d) since $y^2 = 2a(x - 4a)$ ***(M1)***

the vertex is at $(4a,\ 0)$ ***A1***

[2 marks]

Examiners report

[N/A]

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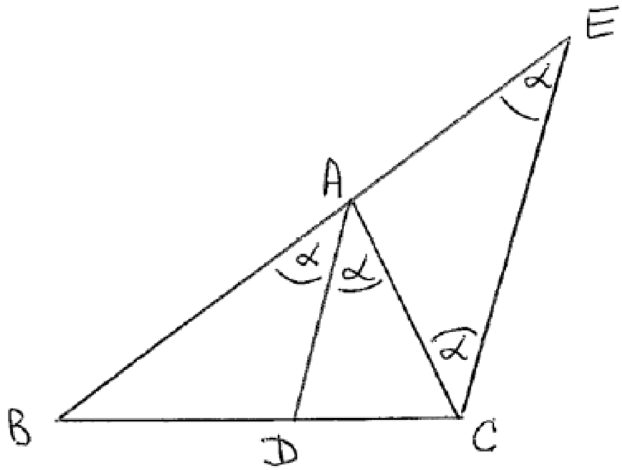
a. Prove the internal angle bisector theorem, namely that the internal bisector of an angle of a triangle divides the side opposite the angle into segments proportional to the sides adjacent to the angle. [6]

b. The bisector of the exterior angle $\widehat{A}$ of the triangle ABC meets (BC) at P. The bisector of the interior angle $\widehat{B}$ meets [AC] at Q. Given that (PQ) meets [AB] at R, use Menelaus' theorem to prove that (CR) bisects the angle $\mathrm{A\widehat{C}B}$. [8]

Markscheme

a. **EITHER**



let [AD] bisect A, draw a line through C parallel to (AD) meeting (AB) at E ***M1***

then $B\widehat{A}D = A\widehat{E}C$ and $D\widehat{A}C = A\widehat{C}E$ ***A1***

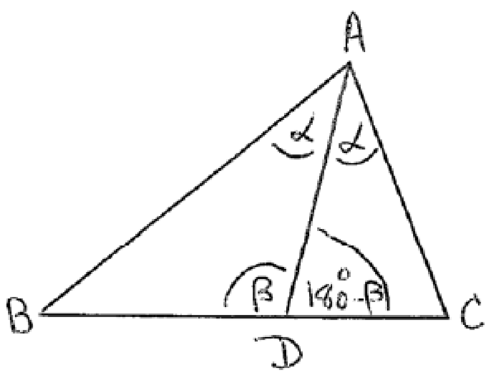
since $B\widehat{A}D = D\widehat{A}C$ it follows that $A\widehat{E}C = A\widehat{C}E$ ***A1***

triangle AEC is therefore isosceles and $AE = AC$ ***A1***

since triangles BAD and BEC are similar

$\frac{BD}{DC} = \frac{AB}{AE} = \frac{AB}{AC}$ ***M1A1***

OR



$\frac{AB}{\sin\beta} = \frac{BD}{\sin\alpha}$ ***M1A1***

$\frac{AC}{\sin(180-\beta)} = \frac{DC}{\sin\alpha}$ ***M1A1***

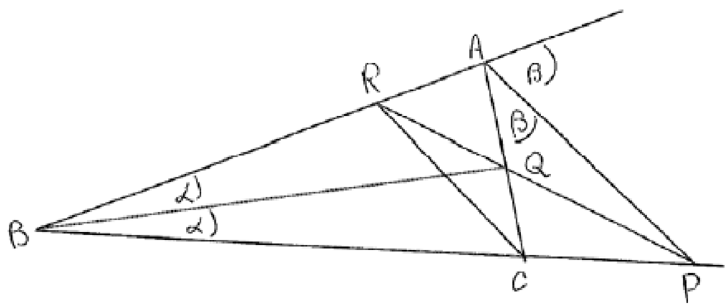
$\sin\beta = \sin(180-\beta)$ ***R1***

$\Rightarrow \frac{AB}{BD} = \frac{AC}{DC}$

$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC}$ ***A1***

[6 marks]

b.



using the angle bisector theorem, ***M1***

$\frac{AQ}{QC} = \frac{AB}{BC}$ and $\frac{BP}{PC} = \frac{AB}{BC}$ ***A1***

using Menelaus' theorem with (PR) as transversal to triangle ABC ***M1***

$\frac{BR}{AR} \times \frac{AQ}{QC} \times \frac{PC}{BP} = (-)1$ ***A1***

substituting the above results, ***M1***

$\frac{BR}{AR} \times \frac{AB}{BC} \times \frac{AC}{AB} = (-)1$ ***A1***

giving

$\frac{BR}{AR} = \frac{BC}{AC}$ ***A1***

[CR] therefore bisects angle C by (the converse to) the angle bisector theorem ***R1AG***

[8 marks]

Examiners report

a. [N/A]
b. [N/A]

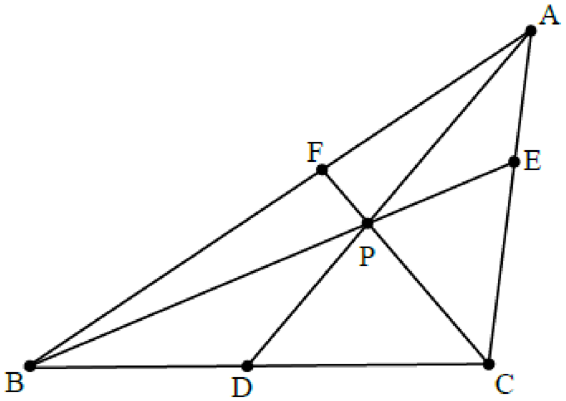
Triangle ABC has points D, E and F on sides [BC], [CA] and [AB] respectively; [AD], [BE] and [CF] intersect at the point P. If 3BD = 2DC and CE = 4EA , calculate the ratios

a. AF : FB . [4]

b. AP : PD [4]

Markscheme

a.



using Ceva's theorem,

$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ ***M1A1***

$\frac{2}{3} \times \frac{4}{1} \times \frac{AF}{FB} = 1$ ***A1***

$\frac{AF}{FB} = \frac{3}{8}$ or AF : FB $= 3 : 8$ ***A1***

[4 marks]

b. using Menelaus' theorem in triangle ACD with BPE as transversal

$\frac{AE}{EC} \times \frac{CB}{BD} \times \frac{DP}{PA} = -1$ ***M1A1***

$\frac{1}{4} \times -\frac{5}{2} \times \frac{DP}{PA} = -1$ ***A1***

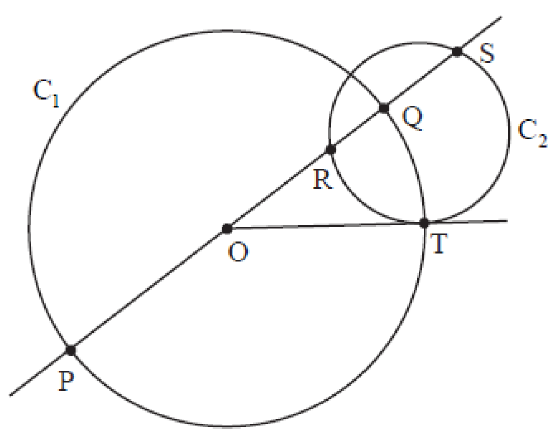
$\frac{DP}{PA} = \frac{8}{5}$ or AP : PD = 5 : 8 ***A1***

[4 marks]

Examiners report

a. This proved difficult for many candidates and often the ratios and negative signs were "blurred".

b. This proved difficult for many candidates and often the ratios and negative signs were "blurred".



The figure shows a circle C_1 with centre O and diameter [PQ] and a circle C_2 which intersects (PQ) at the points R and S. T is one point of intersection of the two circles and (OT) is a tangent to C_2 .

a. Show that $\frac{\mathrm{OR}}{\mathrm{OT}} = \frac{\mathrm{OT}}{\mathrm{OS}}$. [2]

b. (i) Show that $\mathrm{PR} - \mathrm{RQ} = 2\mathrm{OR}$. [6]

(ii) Show that $\frac{\mathrm{PR-RQ}}{\mathrm{PR+RQ}} = \frac{\mathrm{PS-SQ}}{\mathrm{PS+SQ}}$.

Markscheme

a. by the tangent – secant theorem, ***M1***

$\mathrm{OT}^2 = \mathrm{OR} \bullet \mathrm{OS}$ ***A1***

so that $\frac{\mathrm{OR}}{\mathrm{OT}} = \frac{\mathrm{OT}}{\mathrm{OS}}$ ***AG***

[2 marks]

b. (i) $\mathrm{PR} - \mathrm{RQ} = \mathrm{PO} + \mathrm{OR} - (\mathrm{OQ} - \mathrm{OR})$ ***A1***

$= 2\mathrm{OR}$ ***AG***

(ii) attempt to continue the process set up in (b)(i) ***(M1)***

$\mathrm{PR} + \mathrm{RQ} = \mathrm{PO} + \mathrm{OR} + \mathrm{OQ} - \mathrm{OR} = 2\mathrm{OT}$ ***A1***

$\mathrm{PS} - \mathrm{SQ} = \mathrm{PQ} + \mathrm{QS} - \mathrm{SQ} = 2\mathrm{OT}$ ***A1***

$\mathrm{PS} + \mathrm{SQ} = \mathrm{PO} + \mathrm{OS} - \mathrm{OQ} = 2\mathrm{OS}$ ***A1***

it now follows that

$\frac{\mathrm{PR-RQ}}{\mathrm{PR+RQ}} = \frac{\mathrm{OR}}{\mathrm{OT}}$ and $\frac{\mathrm{PS-SQ}}{\mathrm{PS+SQ}} = \frac{\mathrm{OT}}{\mathrm{OS}}$ so using the result in part (a) ***R1***

$\frac{\mathrm{PR-RQ}}{\mathrm{PR+RQ}} = \frac{\mathrm{PS-SQ}}{\mathrm{PS+SQ}}$ ***AG***

[6 marks]

Examiners report

a. Most candidates solved (a) correctly although some used similar triangles instead of the more obvious tangent-secant theorem.

b. Although (b) and then (c) were fairly well signposted, many candidates were unable to cope with the required algebra.

---

a. A triangle T has sides of length 3, 4 and 5. [6]

(i) Find the radius of the circumscribed circle of T .

(ii) Find the radius of the inscribed circle of T .

b. A triangle U has sides of length 4, 5 and 7. [6]

(i) Show that the orthocentre, H, of U lies outside the triangle.

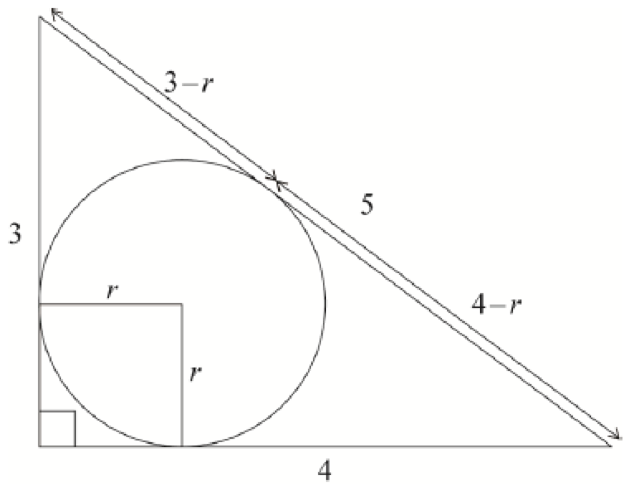
(ii) Show that the foot of the perpendicular from H to the longest side divides it in the ratio $29 : 20$.

Markscheme

a. (i) T is a right angled triangle $\Rightarrow$ the hypotenuse is a diameter ***(M1)***

circumradius $= 2.5$ ***A1***

(ii) diagram seen with some sensible unknown(s) given ***(A1)***



need to solve $3 - r + 4 - r = 5$ (or equivalent) ***M1A1***

$r = 1$ ***A1***

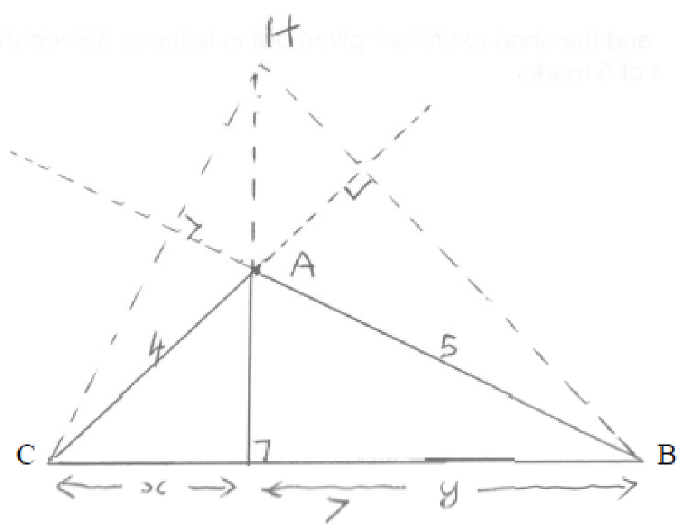
[6 marks]

b. (i) recognition that $7^2 > 4^2 + 5^2$ ***M1*** □

therefore one of the angles is obtuse ***R1***

so the orthocentre, H, of U lies outside of the triangle ***AG***

(ii) foot of perpendicular from H to longest side is the foot of the perpendicular from A to the longest side ***(R1)***



EITHER

attempt to solve $4^2 - x^2 = 5^2 - (7 - x)^2$ or $4^2 - (7 - y)^2 = 5^2 - y^2$ ***M1A1*** obtain $x = \frac{20}{7}$ or $y = \frac{29}{7}$ ***A1***

OR

if $\hat{B}$ is the smallest angle

$\cos \hat{B} = \frac{25+49-16}{2\times5\times7}$ ***M1***

$= \frac{58}{70} = \frac{29}{35}$

$y = 5 \times \frac{29}{35} = \frac{29}{7}$ ***A1***

$x = 7 - \frac{29}{7} = \frac{20}{7}$ ***A1***

THEN

ratio 29 : 20 (accept 20 : 29) ***AG***

Note: Accept the use of Stewart's theorem.

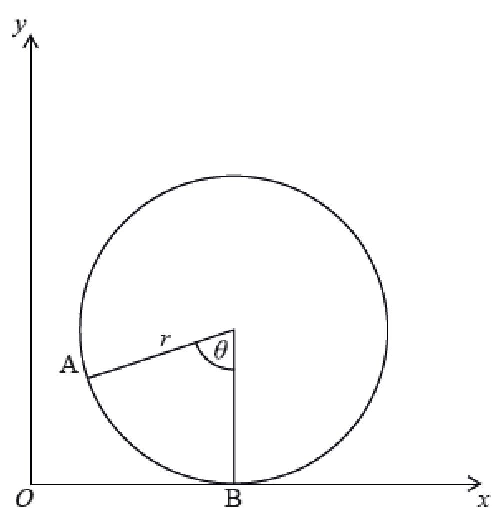
[6 marks]

Examiners report

a. A few fully correct answers were seen to this question, but many candidates were unable to make much progress after part a) (i) and a significant minority made no attempt at all. A few fully correct answers were seen to part a) (ii) and part b) (i). In both part a) (ii) and part b) (ii) a majority of candidates were unable to draw a meaningful diagram to enable them to start the question.

b. A few fully correct answers were seen to this question, but many candidates were unable to make much progress after part a) (i) and a significant minority made no attempt at all. A few fully correct answers were seen to part a) (ii) and part b) (i). In both part a) (ii) and part b) (ii) a majority of candidates were unable to draw a meaningful diagram to enable them to start the question.

A wheel of radius r rolls, without slipping, along a straight path with the plane of the wheel remaining vertical. A point A on the circumference of the wheel is initially at O. When the wheel is rolled, the radius rotates through an angle of θ and the point of contact is now at B, where the length of the arc AB is equal to the distance OB. This is shown in the following diagram.



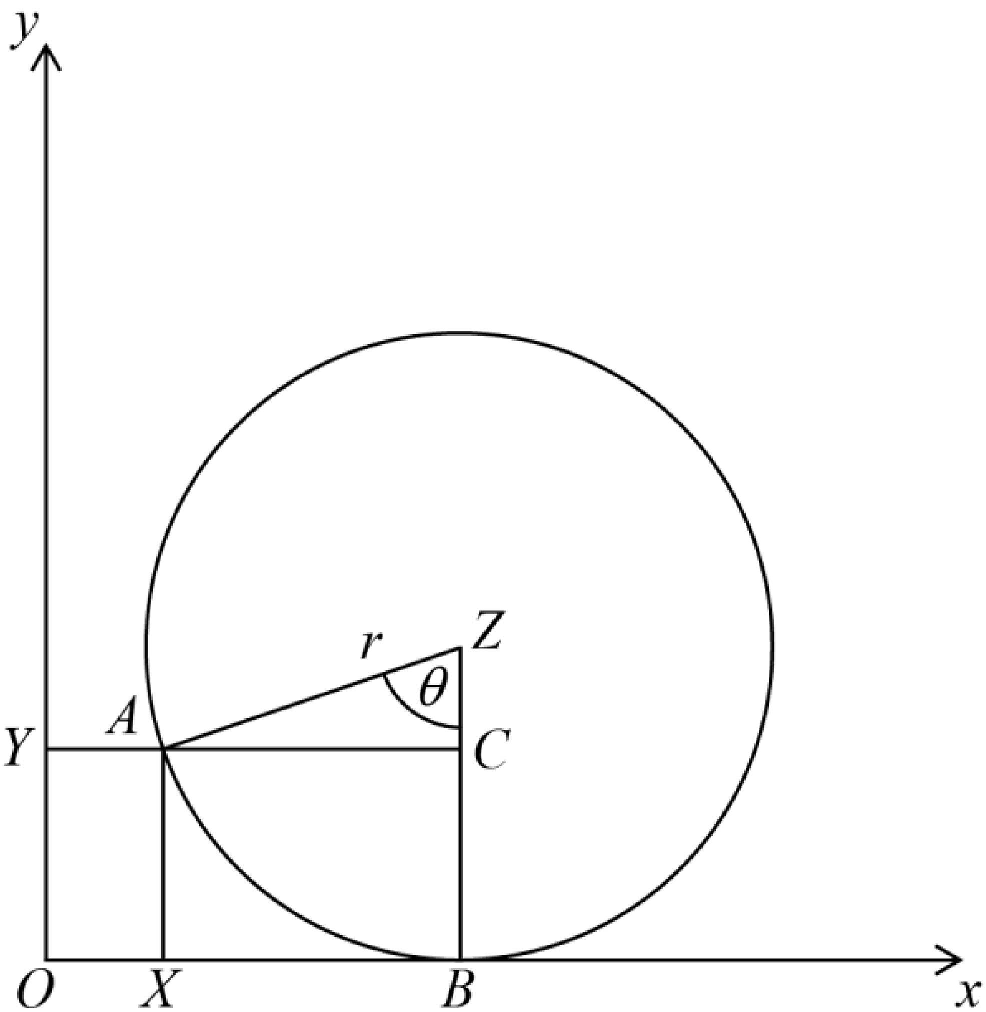
a. Find the coordinates of A in terms of r and θ. [3]

b. As the wheel rolls, the point A traces out a curve. Show that the gradient of this curve is $\cot\left(\frac{1}{2}\theta\right)$. [6]

c. Find the equation of the tangent to the curve when $\theta = \frac{\pi}{3}$. [3]

Markscheme

a.



$\mathrm{OX} = \mathrm{OB} - \mathrm{XB} = r\theta - r\sin\theta = x$ ***(M1)A1***

$\mathrm{OY} = \mathrm{ZB} - \mathrm{ZC} = r - r\cos\theta = y$ ***A1***

b. $\frac{\mathrm{d}x}{\mathrm{d}\theta} = r - r\cos\theta$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r\sin\theta}{r - r\cos\theta}$ ***M1***

$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ ***M1A1A1***

$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}$ ***AG***

c. when $\theta = \frac{\pi}{3}$, gradient $= \sqrt{3}$ ***A1***

$x = \frac{\pi}{3}r - r\frac{\sqrt{3}}{2},\ y = r - \frac{r}{2} = \frac{r}{2}$ ***A1***

$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{\pi}{3}r + r\frac{\sqrt{3}}{2}\right)$ or $y = \sqrt{3}x + 2r - \frac{\pi r}{\sqrt{3}}$ ***A1***

Examiners report

a. Many candidates were unable to find the coordinates of the point A which made (b) inaccessible. Many candidates reached the halfway point in (b) but were then unable to use the half angle formulae to obtain the required result. Many of the candidates who failed to solve (b) picked up the A1 in (c) for finding the gradient.

b. Many candidates were unable to find the coordinates of the point A which made (b) inaccessible. Many candidates reached the halfway point in (b) but were then unable to use the half angle formulae to obtain the required result. Many of the candidates who failed to solve (b) picked up the A1 in (c) for finding the gradient.

c. Many candidates were unable to find the coordinates of the point A which made (b) inaccessible. Many candidates reached the halfway point in (b) but were then unable to use the half angle formulae to obtain the required result. Many of the candidates who failed to solve (b) picked up the A1 in (c) for finding the gradient.

(a) The function g is defined by $g(x,\ y) = x^2 + y^2 + dx + ey + f$ and the circle C_1 has equation $g(x, y) = 0$.

(i) Show that the centre of C_1 has coordinates $\left(-\frac{d}{2}, -\frac{e}{2}\right)$ and the radius of C_1 is $\sqrt{\frac{d^2}{4} + \frac{e^2}{4} - f}$.

(ii) The point $\mathrm{P}(a, b)$ lies outside C_1. Show that the length of the tangents from P to C_1 is equal to $\sqrt{g(a,\ b)}$.

(b) The circle C_2 has equation $x^2 + y^2 - 6x - 2y + 6 = 0$.

The line $y = mx$ meets C_2 at the points R and S.

(i) Determine the quadratic equation whose roots are the x-coordinates of R and S.

(ii) **Hence**, given that L denotes the length of the tangents from the origin O to C_2, show that $\mathrm{OR} \times \mathrm{OS} = L^2$.

Markscheme

(a) (i) completing the square,

$\left(x + \frac{d}{2}\right)^2 + \left(y + \frac{e}{2}\right)^2 - \frac{d^2}{4} - \frac{e^2}{4} + f = 0$ ***M1A1***

whence the centre C is the point $\left(-\frac{d}{2}, -\frac{e}{2}\right)$ and the radius is

$\sqrt{\frac{d^2}{4} + \frac{e^2}{4} - f}$ ***AG***

(ii) $\mathrm{CP}^2 = \left(a + \frac{d}{2}\right)^2 + \left(b + \frac{e}{2}\right)^2$ ***(A1)***

let Q denote the point of contact of one of the tangents from P to the circle.

$\mathrm{CQ}^2 = \frac{d^2}{4} + \frac{e^2}{4} - f$ ***(A1)***

using Pythagoras' Theorem in triangle CPQ, ***M1***

$L^2 = \left(a + \frac{d}{2}\right)^2 + \left(b + \frac{e}{2}\right)^2 - \left(\frac{d^2}{4} + \frac{e^2}{4} - f\right)$

$= a^2 + b^2 + da + eb + f = g(a, b)$ ***A1***

therefore $L = \sqrt{g(a,\ b)}$ ***AG***

[6 marks]

(b) (i) the x-coordinates of R, S satisfy

$x^2 + (mx)^2 - 6x - 2mx + 6 = 0$ ***M1***

$(1 + m^2)x^2 - (6 + 2m)x + 6 = 0$ ***A1***

(ii) $L^2 = g(0,\ 0) = 6$ ***A1***

let x_1, x_2 denote the two roots. Then $x_1x_2 = \frac{6}{1+m^2}$ ***A1***

$\text{OR} = \sqrt{x_1^2 + (mx_1)^2} = x_1\sqrt{1+m^2}$ and $\text{OS} = x_2\sqrt{1+m^2}$ ***M1***

therefore

$\text{OR} \times \text{OS} = x_1x_2(1 + m^2) = 6$ ***A1***

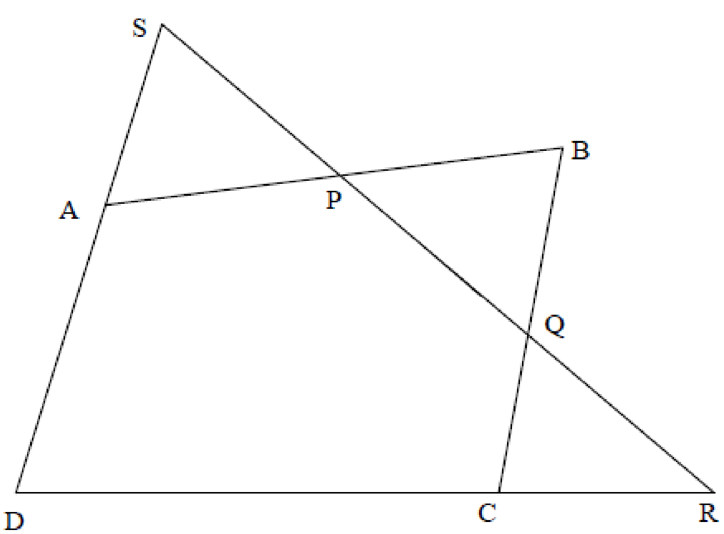
so that $\text{OR} \times \text{OS} = L^2$ ***AG***

[6 marks]

Examiners report

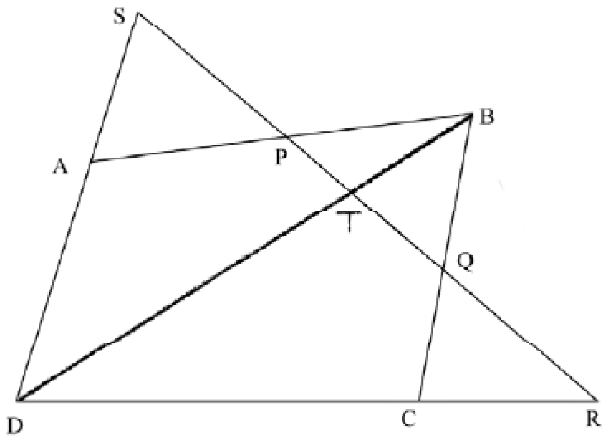
[N/A]

The diagram below shows a quadrilateral ABCD and a straight line which intersects (AB), (BC), (CD), (DA) at the points P, Q, R, S respectively.



Using Menelaus' theorem, show that $\frac{\text{AP}}{\text{PB}} \times \frac{\text{BQ}}{\text{QC}} \times \frac{\text{CR}}{\text{RD}} \times \frac{\text{DS}}{\text{SA}} = 1$.

Markscheme



join BD and let the transversal meet (BD) at T ***(A1)***

apply Menelaus' theorem to triangle ABD with transversal (RS) : ***M1***

$\frac{\text{AP}}{\text{PB}} \times \frac{\text{BT}}{\text{TD}} \times \frac{\text{DS}}{\text{SA}} = (-)1$ ***A1***

apply Menelaus' theorem to triangle CBD with transversal (RS) : ***M1***

$\frac{\text{AP}}{\text{PB}} \times \frac{\text{BQ}}{\text{QC}} \times \frac{\text{CR}}{\text{RD}} \times \frac{\text{DT}}{\text{TB}} = (-)1$ ***A1***

multiplying these two results,

$\frac{\text{AP}}{\text{PB}} \times \frac{\text{BT}}{\text{TD}} \times \frac{\text{DS}}{\text{SA}} \times \frac{\text{BQ}}{\text{QC}} \times \frac{\text{CR}}{\text{RD}} \times \frac{\text{DT}}{\text{TB}} = 1$ ***M1A1***

whence

$\frac{\text{AP}}{\text{PB}} \times \frac{\text{BQ}}{\text{QC}} \times \frac{\text{CR}}{\text{RD}} \times \frac{\text{DS}}{\text{SA}} = 1$ ***AG***

Note: The question can also be solved by joining AC and letting the transversal meet (AC) at T. Menelaus' Theorem then has to be applied to triangles ABC and ACD.

The relevant equations are $\frac{\text{AP}}{\text{PB}} \times \frac{\text{BQ}}{\text{QC}} \times \frac{\text{CT}}{\text{TA}} = (-)1$ and $\frac{\text{CT}}{\text{TA}} \times \frac{\text{AS}}{\text{SD}} \times \frac{\text{DR}}{\text{RC}} = (-)1$.

[7 marks]

Examiners report

Questions on pure geometry which require an initial construction to be made are usually either well done or not done at all and this was no exception. The obvious construction was to draw the line BD and use Menelaus' Theorem twice although some candidates took the more difficult route by joining AC which also leads to the solution. However, many candidates did not start the question at all or tried to apply Menelaus' Theorem to existing triangles which was not a successful approach. The examiners saw a number of candidates who produced well set out and well-explained solutions, but there were still a significant number of cases where diagrams were not fully labelled and points were referred to in the working that were not on the diagram. Candidates should realise that to ensure full marks on questions involving geometric proof that there is a certain degree of formality required in the solution.

A circle $x^2 + y^2 + dx + ey + c = 0$ and a straight line $lx + my + n = 0$ intersect. Find the general equation of a circle which passes through the points of intersection, justifying your answer.

Markscheme

METHOD 1

$x^2 + y^2 + dx + ey + c + \lambda(lx + my + n) = 0$ ***M1 A1***

ie $x^2 + y^2 + x(d + \lambda l) + y(e + \lambda m) + c + \lambda n = 0$ ***A1***

since x^2 and y^2 have the same coefficients and there is no xy term, this is a circle ***R1***

we know the pair of points fit the equation. ***R1***

hence this is the required equation.

METHOD 2

Let the general equation be

$x^2 + y^2 + ax + by + q = 0$ ***M1***

The intersection with the given circle satisfies

$(a - d)x + (b - e)y + (q - c) = 0$ ***M1A1***

This must be the same line as $lx + my + n = 0$ ***R1***

Therefore

$a - d = \lambda l \quad \text{giving} \quad a = d + \lambda l$

$b - e = \lambda m \quad \text{giving} \quad b = e + \lambda m$ ***A1***

$q - c = \lambda n \quad \text{giving} \quad q = c + \lambda n$

leading to the required general equation

Note: Award ***M1*** to candidates who only attempt to find the points of intersection of the line and circle

Examiners report

This was the worst answered question on the paper and indeed no complete solution was seen with no candidate having the required insight to write down the solution. Most candidates who attempted the question tried to find the points of intersection of the line and circle which led nowhere.

---

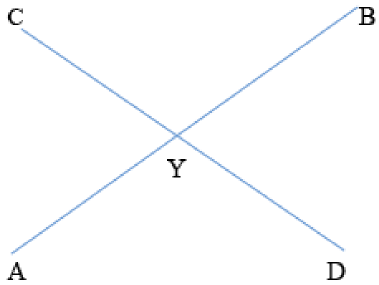
a. Two line segments $[\text{AB}]$ and $[\text{CD}]$ meet internally at the point Y. Given that [6]

$\text{YA} \times \text{YB} = \text{YC} \times \text{YD}$ show that A, B, C and D all lie on the circumference of a circle.

b. Explain why the result also holds if the line segments meet externally at Y. [3]

Markscheme

a. **METHOD 1**



Consider the triangles ACY and DBY ***M1***

Then $\mathrm{YA} \times \mathrm{YB} = \mathrm{YC} \times \mathrm{YD}$

It follows that $\frac{\mathrm{YA}}{\mathrm{YD}} = \frac{\mathrm{YC}}{\mathrm{YB}}$ ***A1***

Also $\mathrm{A\hat{Y}C} = \mathrm{D\hat{Y}B}$ ***A1***

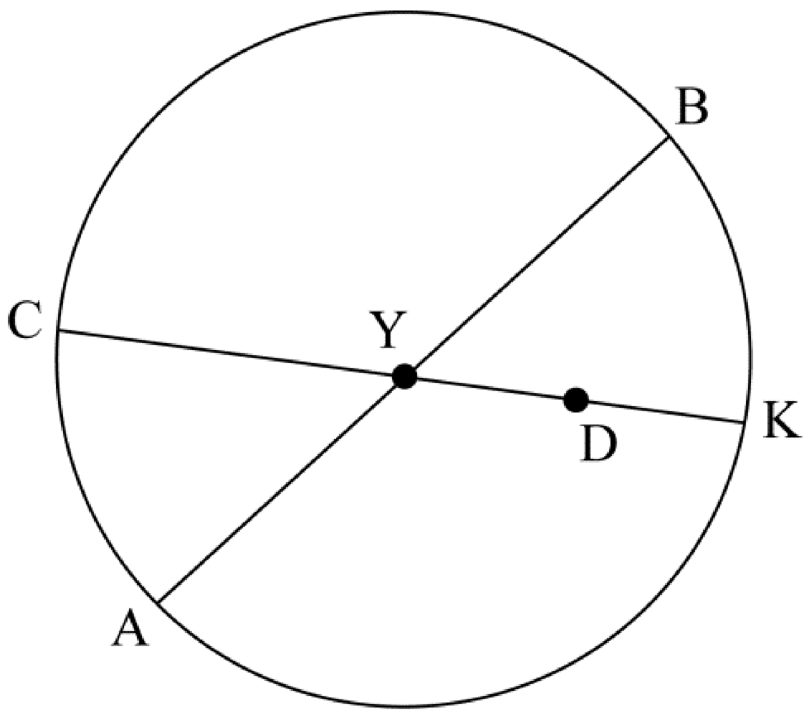
The triangles ACY and DBY are therefore similar ***A1***

So $\mathrm{A\hat{C}Y} = \mathrm{D\hat{B}Y}$ ***A1***

Therefore by the converse to the angles subtended by a chord theorem,

the points $\mathrm{A, B, C, D}$ lie on a circle. ***R1***

METHOD 2



consider the circle passing through ABC ***M1***

the circle then cuts the line (CD) at K ***M1***

Note: May be seen on diagram

since Y lies inside the circle, Y divides the chord CK internally

hence K and D are on the same side of Y ***(R1)***

$\mathrm{YA} \times \mathrm{YB} = \mathrm{YC} \times \mathrm{YK}$ since $\mathrm{A, B, C}$ and K are concyclic ***M1***

$\mathrm{YA} \times \mathrm{YB} = \mathrm{YC} \times \mathrm{YD}$ given

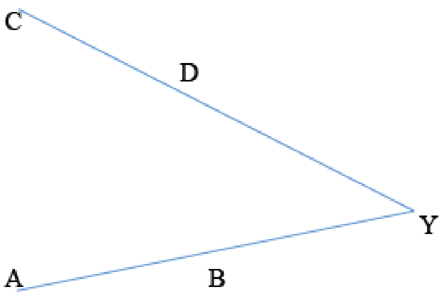
$\Rightarrow \mathrm{YC} \times \mathrm{YK} = \mathrm{YC} \times \mathrm{YD}$ ***A1***

hence K and D are the same point ***R1*** the circle passes through D

Note: Allow an argument based on similar triangles and angles in the segment

Do not allow the use of the converse of the intersecting chords theorem in either (a) or (b)

b. **METHOD 1**



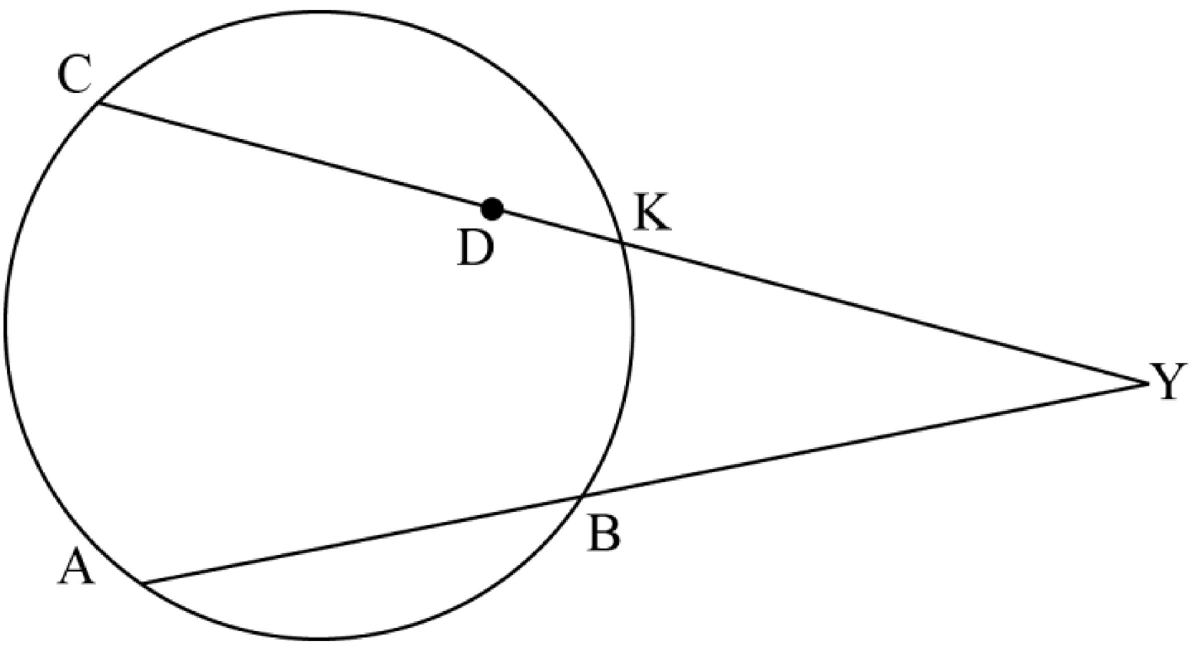
Since the triangles ACY and DBY are still similar $\mathrm{A\hat{C}Y} = \mathrm{D\hat{B}Y}$ ***A1***

Therefore $\mathrm{A\hat{C}Y} + \mathrm{D\hat{B}A} = \mathrm{A\hat{C}Y} + 180^\circ - \mathrm{D\hat{B}Y}$

$= 180^\circ$ ***A1***

ACDB is therefore a cyclic quadrilateral so the points A, B, C, D lie on a circle. ***R1***

METHOD 2



again consider the circle passing through ABC and again let it cut the line CD at K. ***M1***

in this case Y lies outside the circle ABC and therefore Y divides the chord CK externally. ***M1***

by the secant-secant theorem the same working applies as in part (a) ***R1***

and the proof follows identically. ***AG***

Examiners report

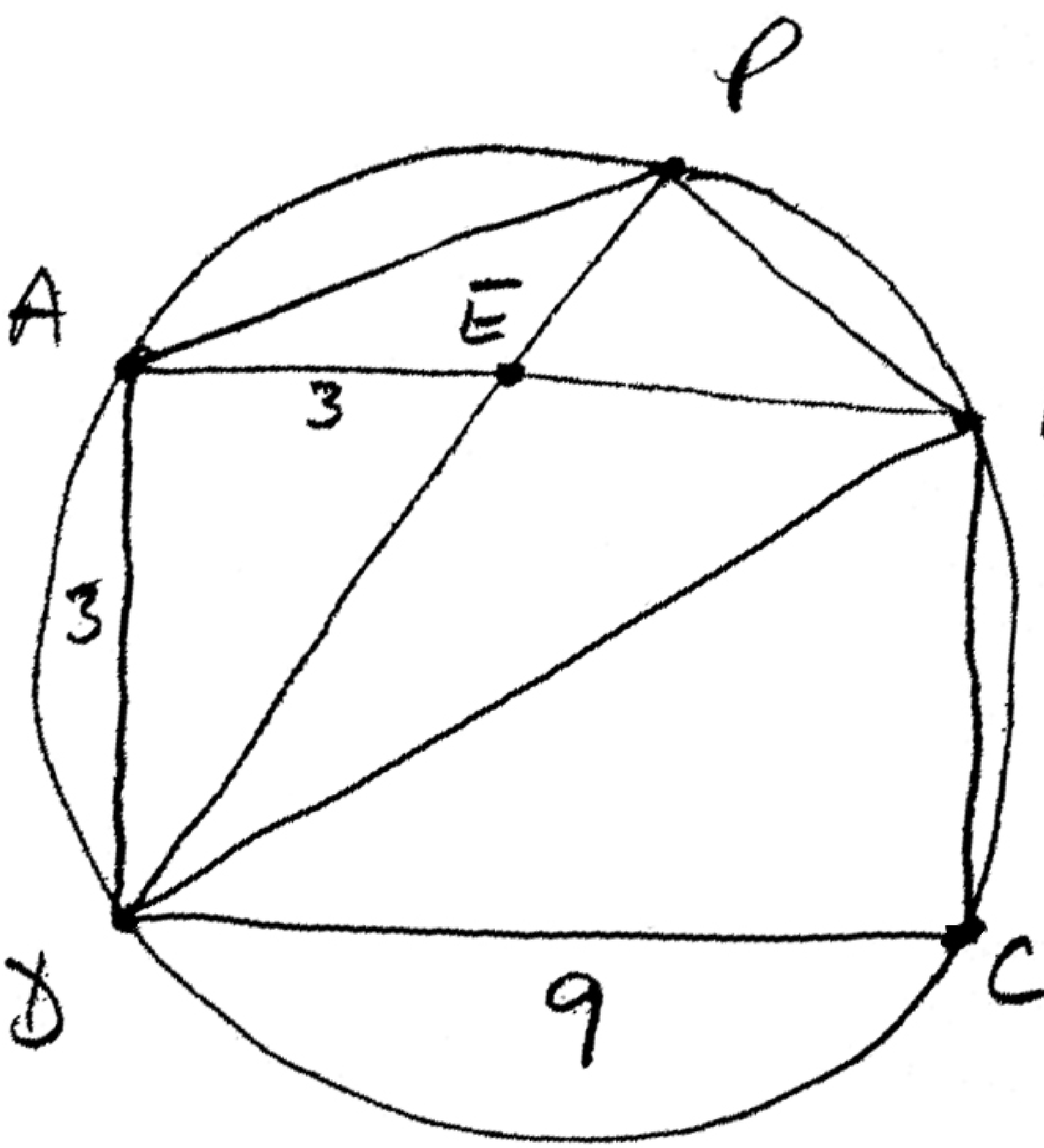
a. Many candidates made no significant attempt at this question. It was expected that solutions would use the intersecting chords theorem but in the event, the majority of candidates who answered the question used similar triangles successfully to prove the required result.

b. Many candidates made no significant attempt at this question. It was expected that solutions would use the intersecting chords theorem but in the event, the majority of candidates who answered the question used similar triangles successfully to prove the required result.

The rectangle ABCD is inscribed in a circle. Sides [AD] and [AB] have lengths 3 cm and (\9\) cm respectively. E is a point on side [AB] such that AE is 3 cm. Side [DE] is produced to meet the circumcircle of ABCD at point P. Use Ptolemy's theorem to calculate the length of chord [AP].

Markscheme

construct diagonal [DB] and the chords [AP] and [PB] ***M1***



since $D\hat{A}B = 90^\circ$, [DB] is the diameter of the circle and $DB = \sqrt{9^2 + 3^2} = 3\sqrt{10}$ ***R1A1*** triangle AED is a right-angled, isosceles triangle so $\mathrm{DE} = 3\sqrt{2}$ ***R1A1***

$\mathrm{A\hat{E}D} = \mathrm{P\hat{E}B} = 45°$ ***M1***

$\Rightarrow \mathrm{PB} = \mathrm{PE} = 6\cos \mathrm{P\hat{E}B} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$ ***M1A1***

using Ptolemy's theorem in quadrilateral APBD

$\mathrm{PB} \times \mathrm{AD} + \mathrm{AP} \times \mathrm{DB} = \mathrm{DP} \times \mathrm{AB}$ ***M1***

$3\sqrt{2} \times 3 + \mathrm{AP} \times 3\sqrt{10} = \left(3\sqrt{2} + 3\sqrt{2}\right) \times 9$ ***A1***

$\mathrm{AP} \times 3\sqrt{10} = 54\sqrt{2} - 9\sqrt{2} = 45\sqrt{2}$ ***A1***

$\mathrm{AP} = \frac{45\sqrt{2}}{3\sqrt{10}} = 3\sqrt{5}$ ***A1***

[12 marks]

Examiners report

The diagrams that some candidates drew were not always helpful to them and sometimes served to confuse what was required and make the problem harder than it was. Candidates were asked to use Ptolemy's theorem but some ignored this request. Lengths of various segments were often written down without any evidence of where they came from.