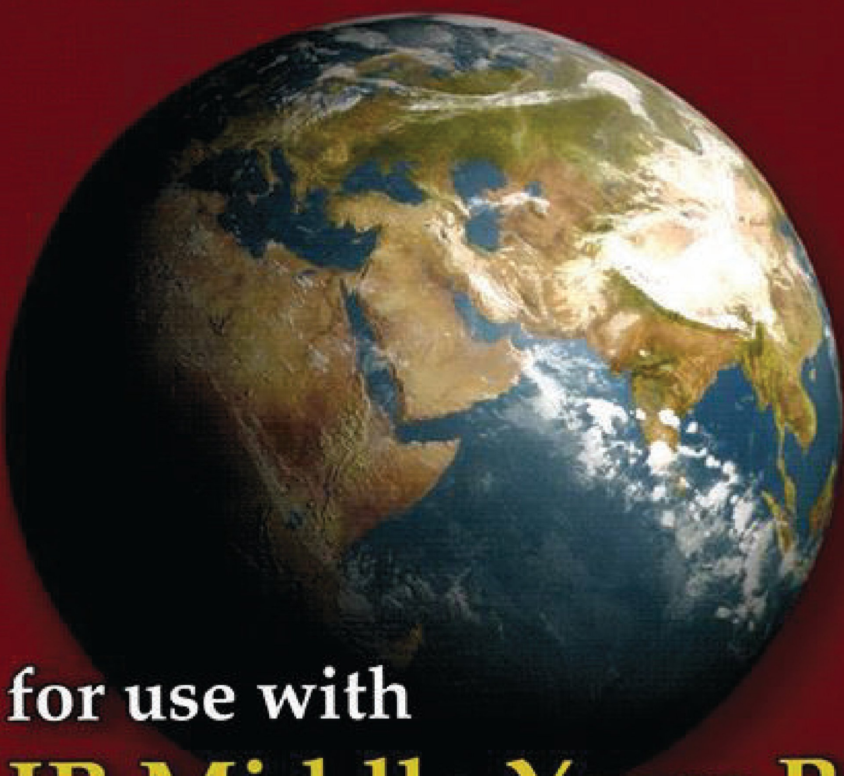
HAESE MATHEMATICS

Mathematics

for the international student

10

MYP 5 (Standard)



Michael Haese

Sandra Haese

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for use with

IB Middle Years Programme

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Specialists in mathematics publishing

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MATHEMATICS FOR THE INTERNATIONAL STUDENT 10
MYP 5 (Standard)

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FOREWORD

MYP 5 (Standard) has been designed and written for the IB Middle Years Program (MYP) Mathematics framework. The textbook covers the Standard content outlined in the framework.

This book may also be used as a general textbook at about 10th Grade level in classes where students are preparing for the study of mathematics at a standard level in their final two years of high school. We have developed this book independently of the International Baccalaureate Organization (IBO) in consultation with experienced teachers of IB Mathematics. The text is not endorsed by the IBO.

Each chapter begins with an Opening Problem, offering an insight into the application of the mathematics that will be studied in the chapter. Important information and key notes are highlighted, while worked examples provide step-by-step instructions with concise and relevant explanations. Discussions, Activities, Investigations, Puzzles, and Research exercises are used throughout the chapters to develop understanding, problem solving, and reasoning, within an interactive environment.

There are two background knowledge chapters available online:

Background knowledge 1: Working with numbers
Background knowledge 2: Algebra

Students should be familiar with the content of these chapters before they enter MYP 5. The background knowledge chapters have been included because some students will need to use them, however students should not spend too much time on them.

The trigonometry material is divided into three chapters to make the volume of work easier to manage. We included a section on negative, complementary, and supplementary angles because we feel it makes understanding the multiples of $30°$ and $45°$ much easier, and because it ties in students' previous work on transformations. Secondly, the supplementary angle formula is necessary to justify the area of a triangle rule and cosine rule in non-right angled triangle trigonometry.

The final two chapters are available online:

Chapter 26: Vectors
Chapter 27: Introduction to calculus

These chapters are included for students wishing to do the diploma Mathematics SL course. We feel that being introduced to vectors and/or calculus in Grade 10 provides a considerable advantage in the following years. If time does not permit during the year, the serious student may consider getting ahead over the summer holidays before their diploma begins.

We understand the emphasis that the IB MYP places on the six Global Contexts, and in response there are online links to ideas for projects and investigations to help busy teachers (see p. 6).

Frequent use of the interactive online features should nurture a much deeper understanding and appreciation of mathematical concepts. The inclusion of our Self Tutor software (see p. 4) is intended to help students who have been absent from classes or who experience difficulty understanding the material.

The book contains many problems to cater for a range of student abilities and interests, and efforts have been made to contextualise problems so that students can see the practical applications of the mathematics they are studying.

We welcome your feedback. Email: info@haesemathematics.com.au

Web: www.haesemathematics.com.au

PMH, SHH, MH, EK, PV

ACKNOWLEDGEMENTS

The authors and publishers would like to thank all those teachers who have read proofs and offered advice and encouragement.

ONLINE FEATURES

There are a range of interactive features which are available online.
With the purchase of a new hard copy textbook, you will gain 15 months subscription to our online product.
This subscription can be renewed annually for a small fee.

COMPATIBILITY

For iPads, tablets, and other mobile devices, the interactive features may not work. However, the electronic version of the textbook and additional chapters can be viewed online using any of these devices.

REGISTERING

You will need to register to access the online features of this textbook.
Visit www.haesemathematics.com.au/register and follow the instructions. Once you have registered, you can:

- activate your electronic textbook
- use your account to purchase additional digital products.

To activate your electronic textbook, contact Haese Mathematics. On providing proof of purchase, your electronic textbook will be activated. **It is important that you keep your receipt as proof of purchase.**

For general queries about registering and licence keys:

- Visit our Snowflake help page: http://snowflake.haesemathematics.com.au/help
- Contact Haese Mathematics: info@haesemathematics.com.au

ONLINE VERSION OF THE TEXTBOOK

The entire text of the book can be viewed online, allowing you to leave your textbook at school.

The online text contains the four additional chapters:

- **Background knowledge 1: Working with numbers**
- **Background knowledge 2: Algebra**
- **Chapter 26: Vectors**
- **Chapter 27: Introduction to calculus**

SELF TUTOR

Self tutor is an exciting feature of this book.

The **Self Tutor** icon on each worked example denotes an active online link.

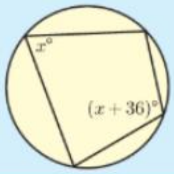
Simply 'click' on the **Self Tutor** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

For example:

Example 7 **Self Tutor**

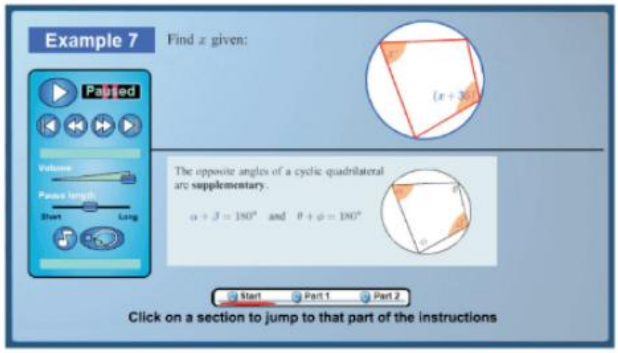
Find x:



Click here

The angles given are opposite angles of a cyclic quadrilateral.

$$\therefore \quad x + (x+36) = 180$$
$$\therefore \quad 2x + 36 = 180$$
$$\therefore \quad 2x = 144$$
$$\therefore \quad x = 72$$



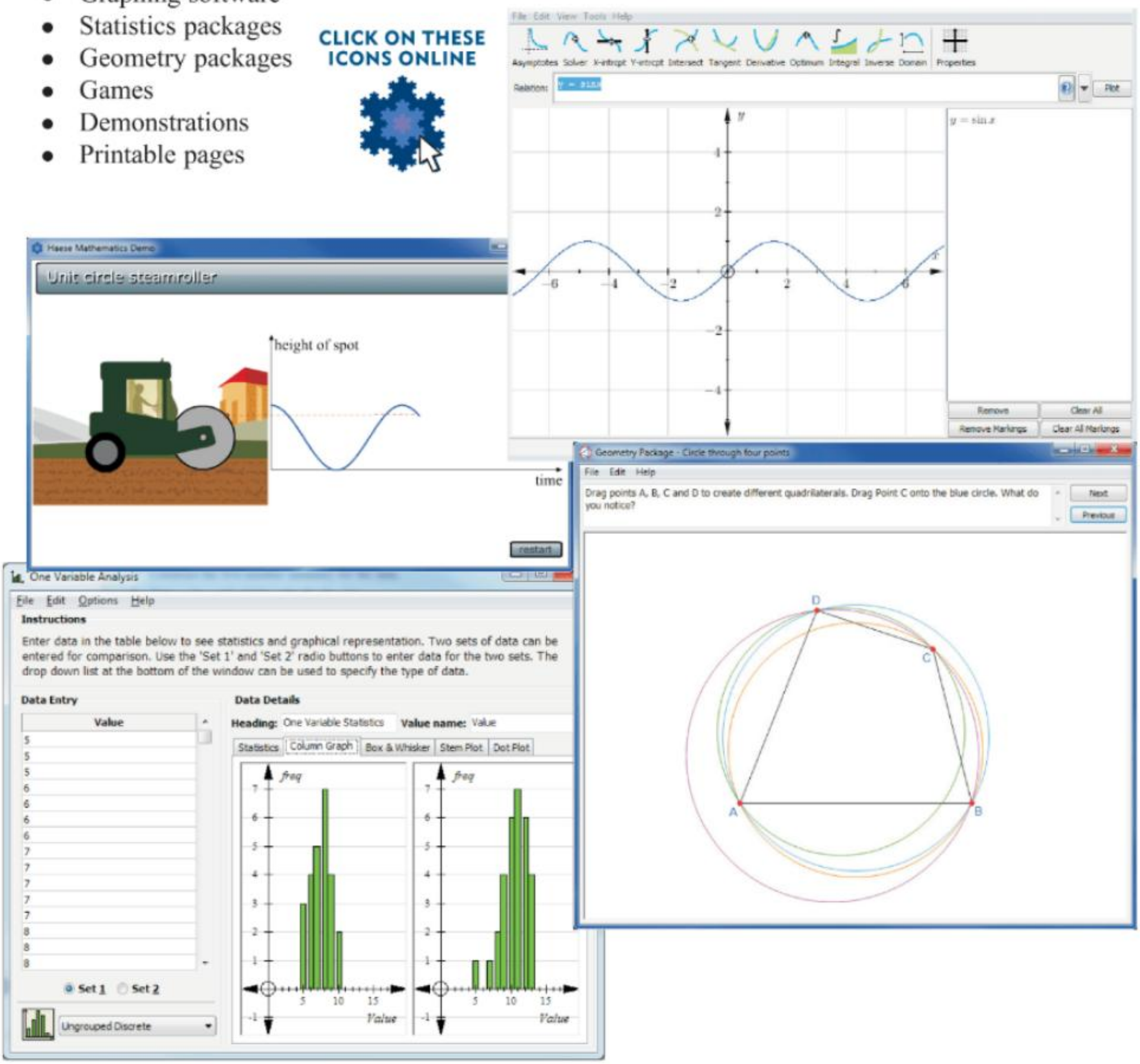
See **Chapter 19**, **Deductive geometry**, p. 362

INTERACTIVE LINKS

Throughout your electronic textbook, you will find interactive links to:

- Graphing software
- Statistics packages
- Geometry packages
- Games
- Demonstrations
- Printable pages

CLICK ON THESE ICONS ONLINE



GLOBAL CONTEXTS

The International Baccalaureate Middle Years Programme focuses teaching and learning through six Global Contexts:

- Identities and relationships
- Orientation in space and time
- Personal and cultural expression
- Scientific and technical innovation
- Globalisation and sustainability
- Fairness and development

Click on the heading to access the online link.

The Global Contexts are intended as a focus for developing connections between different subject areas in the curriculum, and to promote an understanding of the interrelatedness of different branches of knowledge and the coherence of knowledge as a whole.

Global context

click here

How much time do we have?

Statement of inquiry:	Collecting and interpreting data can help us to understand our place in the world.
Global context:	Identities and relationships
Key concept:	Relationships
Related concepts:	Change, Representation
Objectives:	Knowing and understanding, Applying mathematics in real-life contexts
Approaches to learning:	Communication, Self-management

There are six projects in this book, one for each of the Global Contexts:

Chapter 11:	Statistics	p. 225	**AIR PASSENGER NUMBERS** Orientation in space and time
Chapter 14:	Probability	p. 269	**HOW MUCH TIME DO WE HAVE?** Identities and relationships
Chapter 15:	Formulae	p. 300	**NEWTONIAN MECHANICS** Scientific and technical innovation
Chapter 20:	Quadratic functions	p. 390	**ARCHES** Personal and cultural expression
Chapter 22:	Number sequences	p. 429	**SUSTAINABLE FARMING** Globalisation and sustainability
Chapter 24:	Bivariate statistics	p. 473	**WHAT IS A DOLLAR WORTH TO YOU?** Fairness and development

Each project contains a series of questions, divided into:

- Factual questions (in green)
- Conceptual questions (in blue)
- Debatable questions (in red).

These questions should help guide the unit of work.

The projects are also accompanied by the general descriptor and a task-specific descriptor for each of the relevant assessment criteria, to help teachers assess the unit of work.

TABLE OF CONTENTS

Extension questions and exercises are marked in red

Background knowledge 1: Working with numbers

Contents:

CHAPTER

Click on the icon to access this chapter

Background knowledge 2: Algebra

Contents:

CHAPTER

Click on the icon to access this chapter

GRAPHICS CALCULATOR INSTRUCTIONS

Graphics calculator instruction booklets are available for the **Casio fx-9860G Plus**, **Casio fx-CG20**, **TI-84 Plus**, and the **TI-$n$spire**. Click on the relevant icon below.

CASIO fx-9860G Plus | CASIO fx-CG20 | TI-84 Plus | TI-$n$spire

When additional calculator help may be needed, specific instructions are available from icons within the text.

GRAPHICS CALCULATOR INSTRUCTIONS

Chapter 1

Indices

Contents:

OPENING PROBLEM

Approximately 58 400 000 vehicles cross the Sydney Harbour Bridge each year.

Things to think about:

- **a** Can you write this number in the form $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer?
- **b** Can you use the number in this form to estimate how many vehicles cross the bridge:
  - **i** each day
  - **ii** over 10 years?

A INDEX NOTATION

We often deal with numbers that are repeatedly multiplied together, such as $5 \times 5 \times 5$. We can use **indices** or **exponents** to conveniently represent such expressions.

Using **index notation**, we represent $5 \times 5 \times 5$ as 5^3, which reads "5 to the power 3". We say that 5 is the **base**, and 3 is the **index** or **power** or **exponent**.

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times .... \times a}_{n \text{ factors}}$$

Example 1 — Self Tutor

By dividing continuously by the prime numbers 2, 3, 5, 7, ..., write as a product of prime factors in index form:

a 144 **b** 4312

a

$$\begin{array}{r|r} 2 & 144 \\ \hline 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

$\therefore\ 144 = 2^4 \times 3^2$

b

$$\begin{array}{r|r} 2 & 4312 \\ \hline 2 & 2156 \\ \hline 2 & 1078 \\ \hline 7 & 539 \\ \hline 7 & 77 \\ \hline 11 & 11 \\ \hline & 1 \end{array}$$

$\therefore\ 4312 = 2^3 \times 7^2 \times 11$

EXERCISE 1A.1

1 Find the integer equal to:

a 5^3 **b** 3^3 **c** 2^5 **d** 7^3

e $2^2 \times 3^3 \times 5$ **f** $2^3 \times 3 \times 7^2$ **g** $3^2 \times 5^2 \times 13$ **h** $2^4 \times 5^2 \times 11$

2 By dividing continuously by the prime numbers 2, 3, 5, 7, ..., write as a product of prime factors in index form:

a 50 **b** 98 **c** 108 **d** 375
e 1128 **f** 784 **g** 952 **h** 6500

3 The following numbers can be written as 2^n. Find n.

a 32 **b** 256 **c** 4096

4 The following numbers can be written as 3^n. Find n.

a 27 **b** 729 **c** 59 049

5 By considering $3^1, 3^2, 3^3, 3^4, 3^5, ...$ and looking for a pattern, find the last digit of 3^{33}.

6 Find the last digit of 7^{77}.

NEGATIVE BASES

So far we have only considered **positive** bases raised to a power.

However, the base can also be negative. To indicate this we need to use brackets.

Notice that $(-3)^2 = -3 \times -3 = 9$ whereas $-3^2 = -(3^2) = -(3 \times 3) = -9$

Consider the statements below:

$$\begin{aligned} (-1)^1 &= -1 \\ (-1)^2 &= -1 \times -1 = 1 \\ (-1)^3 &= -1 \times -1 \times -1 = -1 \\ (-1)^4 &= -1 \times -1 \times -1 \times -1 = 1 \end{aligned} \qquad \begin{aligned} (-2)^1 &= -2 \\ (-2)^2 &= -2 \times -2 = 4 \\ (-2)^3 &= -2 \times -2 \times -2 = -8 \\ (-2)^4 &= -2 \times -2 \times -2 \times -2 = 16 \end{aligned}$$

From the pattern above it can be seen that:

- a **negative** base raised to an **odd** power is **negative**
- a **negative** base raised to an **even** power is **positive**.

Example 2 Self Tutor

Evaluate:

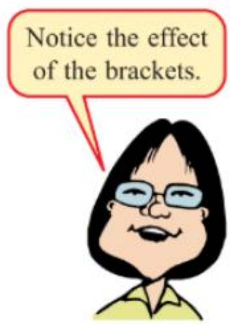
a $(-2)^4$ **b** -2^4 **c** $(-2)^5$ **d** $-(-2)^5$

a $(-2)^4$
$= 16$

b -2^4
$= -1 \times 2^4$
$= -16$

c $(-2)^5$
$= -32$

d $-(-2)^5$
$= -1 \times (-2)^5$
$= -1 \times -32$
$= 32$



EXERCISE 1A.2

1 Evaluate:

a $(-1)^5$ b $(-1)^6$ c $(-1)^9$ d $(-1)^{16}$ e $-(-1)^8$

2 Evaluate:

a $(-3)^2$ b $(-3)^3$ c -3^3 d $-(-3)^3$

3 Evaluate:

a $(-5)^2$ b -5^2 c $(-5)^3$ d $-(-5)^3$

CALCULATOR USE

You can evaluate powers on your calculator. Instructions can be found by clicking on the icon.

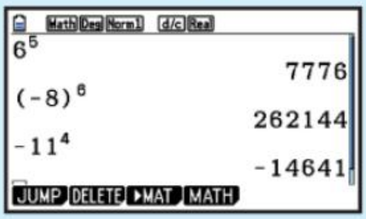
GRAPHICS CALCULATOR INSTRUCTIONS

Example 3 Self Tutor

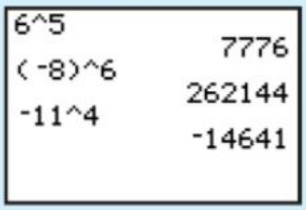
Find, using your calculator:

a 6^5 b $(-8)^6$ c -11^4

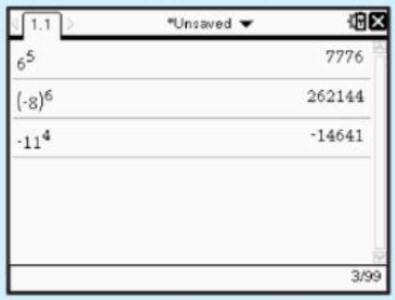
Casio fx-CG20



TI-84 Plus



TI-*n*spire



a $6^5 = 7776$ b $(-8)^6 = 262\,144$ c $-11^4 = -14\,641$

EXERCISE 1A.3

1 Use your calculator to find the value of the following, recording the entire display:

a 2^8 b $(-5)^4$ c -3^5 d 7^4 e 8^3

f $(-7)^6$ g -7^6 h 1.05^{12} i -0.623^{11} j $(-2.11)^{17}$

2 a Use your calculator to find the value of:

i 9^{-1} ii $\frac{1}{9^1}$ iii 4^{-2} iv $\frac{1}{4^2}$

v 3^{-4} vi $\frac{1}{3^4}$ vii 15^0 viii 97^0

b Discuss what happens when a number is raised:

i to a negative power ii to the power zero.

B INDEX LAWS

In previous years we have seen the following **index laws**:

If the bases a and b are both positive, and the indices m and n are integers, then:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\dfrac{a^m}{a^n} = a^{m-n}$	To **divide** numbers with the same base, keep the base and **subtract** the indices.
$(a^m)^n = a^{mn}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	The power of a quotient is the quotient of the powers.
$a^0 = 1, \quad a \neq 0$	Any non-zero number raised to the power of zero is 1.
$a^{-n} = \dfrac{1}{a^n}$	and in particular, $a^{-1} = \dfrac{1}{a}$.

Example 4 Self Tutor

Simplify using the index laws:

a $x^5 \times x^3$ **b** $\dfrac{p^6}{p^2}$ **c** $(y^2)^3$

a $x^5 \times x^3$
$= x^{5+3}$
$= x^8$

b $\dfrac{p^6}{p^2}$
$= p^{6-2}$
$= p^4$

c $(y^2)^3$
$= y^{2\times 3}$
$= y^6$

Example 5 Self Tutor

Express in simplest form with a prime number base:

a 9^4 **b** 4×2^p **c** $\dfrac{3^x}{9^y}$ **d** 25^{x-1}

a 9^4
$= (3^2)^4$
$= 3^{2\times 4}$
$= 3^8$

b 4×2^p
$= 2^2 \times 2^p$
$= 2^{2+p}$

c $\dfrac{3^x}{9^y}$
$= \dfrac{3^x}{(3^2)^y}$
$= \dfrac{3^x}{3^{2y}}$
$= 3^{x-2y}$

d 25^{x-1}
$= (5^2)^{x-1}$
$= 5^{2(x-1)}$
$= 5^{2x-2}$

EXERCISE 1B

1 Simplify using the index laws:

a $3^2 \times 3^5$ **b** $x^6 \times x^3$ **c** $x^5 \times x^n$ **d** $t^3 \times t^4 \times t^5$

e $\frac{7^9}{7^5}$ **f** $\frac{x^7}{x^3}$ **g** $\frac{t^6}{t^x}$ **h** $t^{3m} \div t$

i $(5^3)^2$ **j** $(t^4)^3$ **k** $(y^3)^m$ **l** $(a^{3m})^4$

2 Express in simplest form with a prime number base:

a 121 **b** 32 **c** 81 **d** 4^2

e 25^2 **f** $7^t \times 49$ **g** $3^a \div 9$ **h** $8^p \div 4$

i $\frac{7^n}{7^{n-2}}$ **j** $\frac{9}{3^x}$ **k** $(25^t)^2$ **l** $16^{k-3} \times 2^{-k}$

m $\frac{4^a}{2^b}$ **n** $\frac{8^x}{16^y}$ **o** $\frac{125^{x+1}}{5^{x-1}}$ **p** $\frac{27^{a+2}}{3^a \times 9^a}$

Example 6 **Self Tutor**

Remove the brackets of:

a $(2x)^3$ **b** $\left(\frac{3c}{b}\right)^4$

a $(2x)^3$
$= 2^3 \times x^3$
$= 8x^3$

b $\left(\frac{3c}{b}\right)^4$
$= \frac{3^4 \times c^4}{b^4}$
$= \frac{81c^4}{b^4}$

Each factor within the brackets is raised to the power outside them.

3 Remove the brackets of:

a $(xy)^2$ **b** $(ab)^3$ **c** $(xyz)^2$ **d** $(3b)^3$

e $(5a)^4$ **f** $(10xy)^5$ **g** $\left(\frac{p}{q}\right)^2$ **h** $\left(\frac{m}{n}\right)^3$

i $\left(\frac{x}{3}\right)^4$ **j** $\left(\frac{5}{z}\right)^3$ **k** $\left(\frac{2a}{b}\right)^4$ **l** $\left(\frac{3x}{4y}\right)^3$

4 Simplify the following expressions using one or more of the index laws:

a $4b^2 \times 2b^3$ **b** $\frac{a^6b^3}{a^4b}$ **c** $3ab^2 \times 2a^3$ **d** $\frac{5x^3y^2}{15xy}$

e $\left(\frac{a^2}{5b}\right)^3$ **f** $\frac{24t^6r^4}{15t^6r^2}$ **g** $\frac{(4c^3d^2)^2}{c^2d}$ **h** $\frac{10k^7}{(2k)^5}$

Example 7 **Self Tutor**

Simplify, giving your answers in simplest rational form:

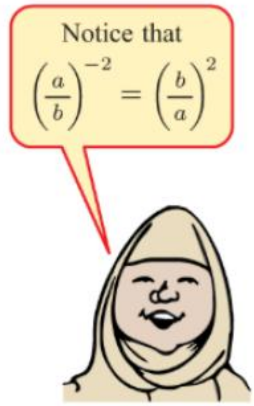
a 7^0 **b** 3^{-2} **c** $3^0 - 3^{-1}$ **d** $\left(\frac{5}{3}\right)^{-2}$

a $7^0 = 1$

b $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

c $3^0 - 3^{-1} = 1 - \frac{1}{3} = \frac{2}{3}$

d $\left(\frac{5}{3}\right)^{-2} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$



5 Simplify, giving your answers in simplest rational form:

a 3^0 **b** 6^{-1} **c** 4^{-1} **d** 5^0
e 4^2 **f** 4^{-2} **g** 5^3 **h** 5^{-3}
i 7^2 **j** 7^{-2} **k** 10^3 **l** 10^{-3}

6 Write as a power of 10:

a 1000 **b** 1 000 000 **c** 0.001 **d** 0.000 000 01

7 Write as a power of 2, 3, or 5:

a 8 **b** $\frac{1}{8}$ **c** 27 **d** $\frac{1}{27}$
e 125 **f** $\frac{1}{125}$ **g** 32 **h** $\frac{1}{32}$
i 81 **j** $\frac{1}{81}$ **k** $\frac{1}{25}$ **l** 1

8 Simplify, giving your answers in simplest rational form:

a $\left(\frac{1}{2}\right)^0$ **b** $\dfrac{5^4}{5^4}$ **c** $2t^0$ **d** $(2t)^0$
e 3×4^0 **f** $\dfrac{5^3}{5^5}$ **g** $\dfrac{x^4}{x^9}$ **h** $\left(\frac{3}{8}\right)^{-1}$
i $\left(\frac{2}{3}\right)^{-1}$ **j** $2^0 + 2^1$ **k** $5^0 - 5^{-1}$ **l** $3^0 + 3^1 - 3^{-1}$
m $\left(\frac{1}{3}\right)^{-2}$ **n** $\left(\frac{2}{3}\right)^{-3}$ **o** $\left(1\frac{1}{2}\right)^{-3}$ **p** $\left(2\frac{1}{2}\right)^{-2}$

9 Write the following without brackets or negative indices:

a $(3b)^{-1}$ **b** $3b^{-1}$ **c** $7a^{-1}$ **d** $(7a)^{-1}$
e $\left(\dfrac{1}{t}\right)^{-2}$ **f** $\left(\dfrac{3x}{y}\right)^{-1}$ **g** $(5t)^{-2}$ **h** $(5t^{-2})^{-1}$
i xy^{-1} **j** $(xy)^{-1}$ **k** xy^{-3} **l** $(xy)^{-3}$
m $(3pq)^{-1}$ **n** $3(pq)^{-1}$ **o** $3pq^{-1}$ **p** $\dfrac{(xy)^3}{y^{-2}}$
q $(5x^{-2}y^3)^3$ **r** $\left(\dfrac{c}{2d^3}\right)^{-2}$ **s** $\left(\dfrac{3r^{-3}}{t}\right)^{-2}$ **t** $\left(\dfrac{2p}{5q^{-2}}\right)^{-3}$

10 Use the index laws to show that, for positive a and b, and integer n:

a $\dfrac{1}{a^{-n}} = a^n$ **b** $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$

11 The units for speed *kilometres per hour* can be written as km/h, or in *index form* as $\text{km}\,\text{h}^{-1}$. Write these units in index form:

a m/s
b grams per second
c cubic metres/hour
d square centimetres per second
e cubic centimetres per minute
f metres per second per second.

INVESTIGATION — RATIONAL INDICES

This Investigation will help you discover the meaning of numbers raised to rational indices of the form $\frac{p}{q}$ where $p, q \in \mathbb{Z},\ q \neq 0$.

What to do:

1 **a** Copy and complete: $\sqrt{x} \times \sqrt{x} = ...$

b Use the index law $a^m \times a^n = a^{m+n}$ to simplify $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$.

c Use the index law $(a^m)^n = a^{mn}$ to simplify $\left(x^{\frac{1}{2}}\right)^2$.

d What is the relationship between $\sqrt{x}$ and $x^{\frac{1}{2}}$?

2 **a** Copy and complete: $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = ...$

b Use the index law $a^m \times a^n = a^{m+n}$ to simplify $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$.

c Use the index law $(a^m)^n = a^{mn}$ to simplify $\left(x^{\frac{1}{3}}\right)^3$.

d What is the relationship between $\sqrt[3]{x}$ and $x^{\frac{1}{3}}$?

3 Explain why $\sqrt[n]{x} = x^{\frac{1}{n}}$ for all $n \in \mathbb{Z},\ n \neq 0$.

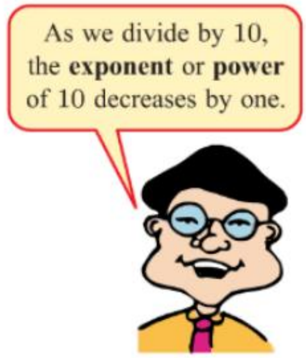
4 **a** Use the index law $(a^m)^n = a^{mn}$ to simplify: **i** $(x^p)^{\frac{1}{q}}$ **ii** $\left(x^{\frac{1}{q}}\right)^p$

b Hence explain why $x^{\frac{p}{q}} = (\sqrt[q]{x})^p = \sqrt[q]{x^p}$.

C SCIENTIFIC NOTATION (STANDARD FORM)

Observe the pattern:

$$\begin{array}{rrcll}
 & 10\,000 & = & 10^4 & \\
\div 10 & 1000 & = & 10^3 & -1 \\
\div 10 & 100 & = & 10^2 & -1 \\
\div 10 & 10 & = & 10^1 & -1 \\
\div 10 & 1 & = & 10^0 & -1 \\
\div 10 & \frac{1}{10} & = & 10^{-1} & -1 \\
\div 10 & \frac{1}{100} & = & 10^{-2} & -1 \\
\div 10 & \frac{1}{1000} & = & 10^{-3} & -1
\end{array}$$



We can use this pattern to simplify the writing of very large and very small numbers.

For example, $5\,000\,000 = 5 \times 1\,000\,000 = 5 \times 10^6$ and $0.000\,003 = \dfrac{3}{1\,000\,000} = 3 \times \dfrac{1}{1\,000\,000} = 3 \times 10^{-6}$

Scientific notation involves writing any given number as *a number between* 1 *inclusive and* 10, multiplied by a *power of* 10. The result has the form

$a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer.

Example 8 Self Tutor

Write in scientific notation:

a $23\,600\,000$ **b** $0.000\,023\,6$

a $23\,600\,000$
$= 2.36 \times 10^7$

b $0.000\,023\,6$
$= 2.36 \times 10^{-5}$

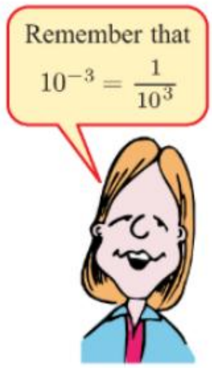
Example 9 Self Tutor

Write as an ordinary decimal number:

a 2.57×10^4 **b** 7.853×10^{-3}

a 2.57×10^4
$= 2.5700 \times 10\,000$
$= 25\,700$

b 7.853×10^{-3}
$= 0007.853 \div 10^3$
$= 0.007\,853$



EXERCISE 1C

1 Write using scientific notation:

a 4816 **b** 4 816 000 **c** 4.816 **d** 0.048 16

2 Write as an ordinary decimal number:

a 3.12×10^2 **b** 3.12×10^{-3} **c** 3.12×10^5 **d** 3.12×10^0

3 Write using scientific notation:

a 230 **b** 53 900 **c** 0.0361 **d** 0.006 80
e 3.26 **f** 0.5821 **g** 361 000 000 **h** 0.000 001 674

4 Write as an ordinary decimal number:

a 2.3×10^3 **b** 2.3×10^{-2} **c** 5.64×10^5 **d** 7.931×10^{-4}
e 9.97×10^0 **f** 6.04×10^7 **g** 4.215×10^{-1} **h** 3.621×10^{-8}

5 Express the following quantities using scientific notation:

- **a** There are approximately 4 million red blood cells in a drop of blood.
- **b** The thickness of a coin is about 0.0008 m.
- **c** Earth's radius is about 6.38 million metres.
- **d** A Rubik's Cube has approximately 43 252 000 000 000 000 000 possible arrangements.

6 Express the following quantities as ordinary decimal numbers:

- **a** The Amazon River is approximately 6.99×10^6 m long.
- **b** A piece of paper is about 1.8×10^{-2} cm thick.
- **c** A test tube holds 3.2×10^7 bacteria.
- **d** A mushroom weighs 8.2×10^{-6} tonnes.

Example 10 **Self Tutor**

Simplify, writing your answer in scientific notation:

a $(3 \times 10^4) \times (8 \times 10^3)$ **b** $\dfrac{2 \times 10^{-3}}{5 \times 10^{-8}}$

a
$$\begin{aligned} &(3 \times 10^4) \times (8 \times 10^3) \\ &= 24 \times 10^{4+3} \\ &= (2.4 \times 10^1) \times 10^7 \\ &= 2.4 \times 10^8 \end{aligned}$$

b
$$\begin{aligned} &\frac{2 \times 10^{-3}}{5 \times 10^{-8}} \\ &= \tfrac{2}{5} \times 10^{-3-(-8)} \\ &= 0.4 \times 10^5 \\ &= (4 \times 10^{-1}) \times 10^5 \\ &= 4 \times 10^4 \end{aligned}$$

7 Simplify, writing your answers in scientific notation:

- **a** $(3 \times 10^3) \times (2 \times 10^7)$
- **b** $(4 \times 10^3) \times (7 \times 10^5)$
- **c** $(8 \times 10^{-4}) \times (7 \times 10^{-5})$
- **d** $(9 \times 10^{-5}) \times (6 \times 10^{-2})$
- **e** $(3 \times 10^5)^2$
- **f** $(4 \times 10^7)^2$
- **g** $(2 \times 10^{-3})^4$
- **h** $(5 \times 10^{-3})^3$

8 Simplify, writing your answers in scientific notation:

- **a** $\dfrac{8 \times 10^6}{4 \times 10^3}$
- **b** $\dfrac{9 \times 10^{-3}}{3 \times 10^{-1}}$
- **c** $\dfrac{4 \times 10^6}{2 \times 10^{-2}}$

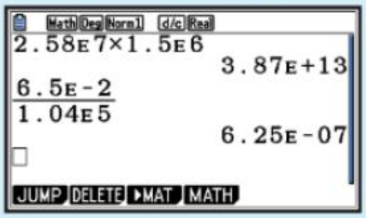
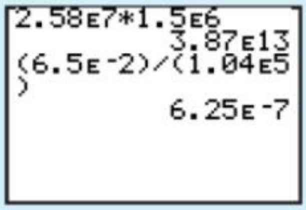
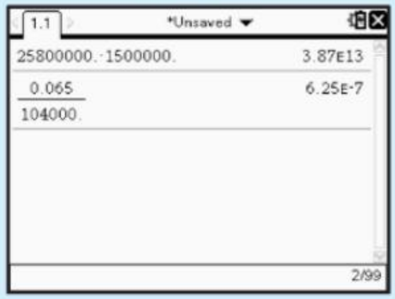
9
- **a** How many times larger is 3×10^{11} than 3×10^8?
- **b**
  - **i** Which is smaller, 5×10^{-16} or 5×10^{-21}?
  - **ii** By how many times is it smaller than the other number?
- **c** How many times larger is 4×10^6 than 8×10^{-5}?

Example 11 **Self Tutor**

Use your calculator to find:

a $(2.58 \times 10^7) \times (1.5 \times 10^6)$ **b** $\dfrac{6.5 \times 10^{-2}}{1.04 \times 10^5}$

GRAPHICS CALCULATOR INSTRUCTIONS

Casio fx-CG20	TI-84 Plus	TI-*n*spire
		

a $(2.58 \times 10^7) \times (1.5 \times 10^6) = 3.87 \times 10^{13}$

b $\dfrac{6.5 \times 10^{-2}}{1.04 \times 10^5} = 6.25 \times 10^{-7}$

10 Calculate the following, giving each answer in scientific notation. The decimal part should be rounded to 3 significant figures.

a $(4.7 \times 10^5) \times (8.53 \times 10^7)$ **b** $(2.7 \times 10^{-3}) \times (9.6 \times 10^{14})$

c $\dfrac{3.4 \times 10^7}{4.8 \times 10^{15}}$ **d** $\dfrac{7.3 \times 10^{-7}}{1.5 \times 10^4}$

e $(2.83 \times 10^3)^2$ **f** $(5.96 \times 10^{-5})^2$

g $\dfrac{(3.56 \times 10^4)^2}{8.05 \times 10^{-5}}$ **h** $\dfrac{2.9 \times 10^2}{(7.62 \times 10^7)^3}$

11 Answer the **Opening Problem** on page **14**.

12 Use your calculator to answer the following:

a A rocket travels in space at 4×10^5 km h^{-1}. Assuming 1 year $\approx$ 365.25 days, how far will it travel in:

i 30 days **ii** 20 years?

b A bullet travelling at an average speed of 2×10^3 km h^{-1} hits a target 500 m away. Find the time of the bullet's flight, in seconds.

Remember that speed $= \dfrac{\text{distance}}{\text{time}}$.

c Mars has volume 1.31×10^{21} m^3.
Pluto has volume 4.93×10^{19} m^3.
How many times bigger is Mars than Pluto?

d Microbe C has mass 2.63×10^{-5} grams.
Microbe D has mass 8×10^{-7} grams.

i Which microbe is heavier?

ii How many times heavier is it, than the other microbe?

13 The table alongside shows the land areas of the Canadian provinces (shaded yellow) and territories (shaded blue).

- **a** Find the total land area of Canada.
- **b** Place the *provinces* in order from largest to smallest.
- **c** How many times larger is:
  - **i** Quebec than Manitoba
  - **ii** Nunavut than Prince Edward Island?
- **d** What percentage of the land area of Canada, is included in Nova Scotia?

	Land area (km^2)
Ontario	9.2×10^5
Quebec	1.4×10^6
Nova Scotia	5.3×10^4
New Brunswick	7.1×10^4
Manitoba	5.5×10^5
British Columbia	9.3×10^5
Prince Edward Island	5.7×10^3
Saskatchewan	5.9×10^5
Alberta	6.4×10^5
Newfoundland and Labrador	3.7×10^5
Northwest Territories	1.2×10^6
Yukon	4.7×10^5
Nunavut	1.9×10^6

HISTORICAL NOTE

The ancient Indians explored the concept of expressing very large and very small numbers. In the *Lalitavistara Sutra*, a Sanskrit text dating from around the 4th century, it is written that the Buddha gave a description of the size of an atom.

In terms of the length of a finger bone, the Buddha stated that:

".... each was the length of

seven grains of barley, each of which was the length of
seven mustard seeds, each of which was the length of
seven poppy seeds, each of which was the length of
seven particles of dust stirred up by a cow, each of which was the length of
seven specks of dust disturbed by a ram, each of which was the length of
seven specks of dust stirred up by a hare, each of which was the length of
seven specks of dust carried away by the wind, each of which was the length of
seven tiny specks of dust, each of which was the length of
seven minute specks of dust, each of which was the length of
seven particles of the first atoms."

What to do:

1. Assuming a finger bone is 4 cm long, use the Buddha's description to estimate the length of an atom, in metres. Write your answer in scientific notation.
2. Research the size of a carbon atom. How accurate is the estimate in **1**?

REVIEW SET 1A

1 By dividing continuously by the prime numbers 2, 3, 5, 7, ..., write as a product of prime factors in index form:

a 96 **b** 180 **c** 154 **d** 2125

2 The following numbers can be written in the form 7^n. Find n.

a 343 **b** $\frac{1}{49}$ **c** 16 807 **d** 1

3 Simplify using the index laws:

a $k^5 \times k^3$ **b** $\dfrac{p^6}{p}$ **c** $(m^6)^8$

4 Remove the brackets of:

a $(3w)^2$ **b** $(2x^2y)^3$ **c** $\left(\frac{a}{b}\right)^6$ **d** $\left(\frac{1}{5n}\right)^3$

5 Simplify, giving answers in simplest rational form:

a 7^{-1} **b** $\left(\frac{4}{3}\right)^{-1}$ **c** $11^0 - 11^{-1}$ **d** $\left(1\frac{3}{4}\right)^{-2}$

6 Write using scientific notation:

a 59 000 **b** 0.009 **c** 6 085 000 **d** 0.000 007 71

7 Write as an ordinary decimal number:

a 6.23×10^5 **b** 3.008×10^{-4} **c** 4.597×10^0

8 Write without brackets or negative indices:

a $(mn)^{-2}$ **b** mn^{-2} **c** $\left(\dfrac{x}{5y^2}\right)^{-3}$

9 Simplify, writing your answers in scientific notation:

a $(6 \times 10^5) \times (3 \times 10^6)$ **b** $(8 \times 10^9) \times (5 \times 10^{-4})$ **c** $\dfrac{9 \times 10^{-5}}{6 \times 10^3}$

10 Write the answers to the following in scientific notation:

a The speed of light in a vacuum is about 2.998×10^8 m s^{-1}.
Assuming 1 year $\approx$ 365.25 days, determine how far light travels in:

i 1 hour **ii** 1 day **iii** 1 year.

b How long does it take for light to travel:

i 1 m **ii** 1 cm **iii** 1 mm?

c In air, light travels at 2.989×10^8 m s^{-1} and sound travels at 343.2 m s^{-1}.
How many times faster is light than sound?

REVIEW SET 1B

1 Evaluate:

a -7^3 **b** $(-7)^3$ **c** $-(-7)^3$ **d** $-(7^3)^{-1}$

2 Simplify, giving answers in simplest rational form:

a $\left(\frac{3}{5}\right)^2$ **b** $\left(\frac{3}{5}\right)^{-2}$ **c** $5^0 - 5^{-1}$ **d** $\left(1\frac{1}{3}\right)^3$

3 Simplify using the index laws:

a $m^2 \times m^3$ **b** $m^2 \times m^{-3}$ **c** $\dfrac{m^2}{m^3}$ **d** $\dfrac{m^2}{m^{-3}}$

4 Express in simplest form with a prime number base:

a 8^2 **b** $\dfrac{25^x}{125}$ **c** $\dfrac{49^{k+3}}{7^{k-1}}$

5 Simplify using the index laws:

a $\left(\dfrac{3}{a}\right)^2$ **b** $(2pq)^2$ **c** $\left(\dfrac{x}{y}\right)^{-1}$ **d** $\dfrac{1}{x^{-1}}$

6 Simplify using one or more of the index laws:

a $5c^3 \times 3c^4$ **b** $\dfrac{14x^5y^2}{2x^2y}$ **c** $\left(\dfrac{3p}{q^{-3}}\right)^2$

7 Write without brackets or negative indices:

a $\left(\dfrac{a}{b}\right)^{-2}$ **b** $\left(\dfrac{3a^{-1}}{2b^2}\right)^{-3}$

8 Simplify, writing your answers in scientific notation:

a $(7 \times 10^5) \times (3 \times 10^9)$ **b** $\dfrac{8 \times 10^7}{2 \times 10^{-3}}$ **c** $\dfrac{2.7 \times 10^{-4}}{4.5 \times 10^7}$

9 How many times larger is 3.5×10^{11} than 5×10^9?

10 The table alongside shows the diameters of the planets in the solar system.

a Find the diameter of Saturn in:

i kilometres **ii** centimetres.

b Find the radius of Venus.

c Write the planets in order of size, from smallest to largest.

d How many times greater is the diameter of:

i Uranus than Mercury

ii Jupiter than Mars?

Planet	*Diameter*
Mercury	4.88×10^6 m
Venus	1.21×10^7 m
Earth	1.27×10^7 m
Mars	6.79×10^6 m
Jupiter	1.40×10^8 m
Saturn	1.21×10^8 m
Uranus	5.11×10^7 m
Neptune	4.95×10^7 m

Chapter 2

Sets and Venn diagrams

Contents:

OPENING PROBLEM

A city has two newspapers, The Sun and The Advertiser. 56% of the people read The Sun and 71% of the people read The Advertiser. 18% read neither newspaper.

Things to think about:

a How can we represent this information on a diagram?

b What percentage of the people read:

i both of the newspapers

ii at least one of the newspapers

iii The Sun, but not The Advertiser

iv exactly one of the two newspapers?

A NUMBER SETS

A **set** is a collection of numbers or objects.

Each number or object is called an **element** or **member** of the set.

SET NOTATION

We often use a capital letter to represent a set.

For example:

- if V is the set of all vowels, then $V = \{\text{vowels}\} = \{a, e, i, o, u\}$
- if E is the set of all even numbers, then $E = \{\text{even numbers}\} = \{2, 4, 6, 8, 10, 12, ...\}$.

We use the symbol $\in$ to mean *is an element of* and $\notin$ to mean *is not an element of*.

So, for the set $E = \{2, 4, 6, 8, 10, 12, ...\}$, we can say $6 \in E$ but $11 \notin E$.

COUNTING ELEMENTS OF SETS

The number of elements in set S is written $n(S)$.

A set which contains a finite number of elements is called a **finite set**.

A set which contains an infinite number of elements is called an **infinite set**.

For example:

- the set of vowels V has 5 elements. V is a finite set, and $n(V) = 5$
- the set of even numbers E is an infinite set.

THE EMPTY SET

The set $\{\ \}$ or $\varnothing$ is called the **empty set** or **null set**, and contains no elements.

EXERCISE 2A

1 Write using set notation:

- **a** 8 is an element of set P.
- **b** k is not an element of set S.
- **c** 14 is not an element of the set of all odd numbers.
- **d** There are 9 elements in set Y.

2 For each of the following sets:

- **i** List the elements of the set.
- **ii** Determine whether the set is finite or infinite.
- **iii** If the set is finite, find the number of elements in the set.

- **a** $A = \{\text{factors of } 6\}$
- **b** $B = \{\text{multiples of } 6\}$
- **c** $C = \{\text{factors of } 17\}$
- **d** $D = \{\text{multiples of } 17\}$
- **e** $E = \{\text{prime numbers less than } 20\}$
- **f** $F = \{\text{composite numbers between } 10 \text{ and } 30\}$

3 Let M_3 be the set of all multiples of 3, and F_{60} be the set of factors of 60.

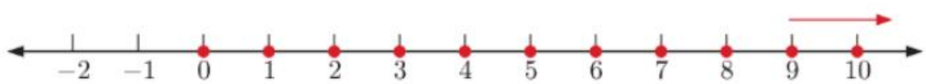
- **a** List the first 8 elements of M_3 in set notation.
- **b** List the elements of F_{60} in set notation.
- **c** What elements are in both M_3 and F_{60}?

4 Find $n(\varnothing)$.

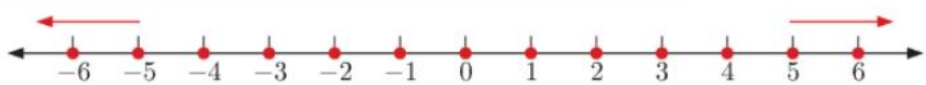
B SPECIAL NUMBER SETS

Following is a list of some special number sets you should be familiar with:

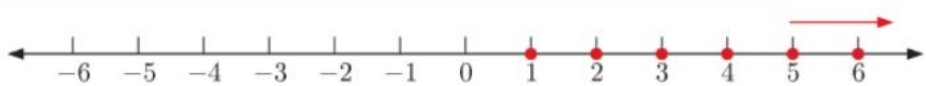
- $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **natural** or **counting numbers**.



- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, ...\}$ is the set of all **integers**.



- $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **positive integers**.



- $\mathbb{Q}$ is the set of all **rational numbers**, or numbers which can be written in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

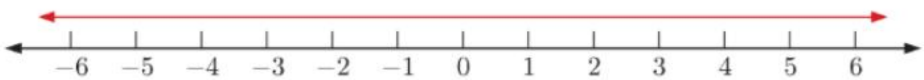
For example: $\frac{15}{4}$, $10\ (= \frac{10}{1})$, $0.5\ (= \frac{1}{2})$, and $-\frac{3}{8}$ are all rational numbers.

We cannot represent all rational numbers on a single number line, because there are infinitely many of them, and in between them are **irrational numbers** which cannot be written in rational form.

For example:

- Radicals or surds such as $\sqrt{2}$ and $\sqrt{7}$ are irrational.
- $\pi \approx 3.141\,592\,65$ is an irrational number.
- Decimal numbers which neither terminate nor recur are irrational.

- $\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.



$\mathbb{R}$ includes all rational and irrational numbers.

$\frac{2}{0}$ and $\sqrt{-2}$ are not real numbers because we cannot write them in decimal form or place them on a number line.

Example 1 — Self Tutor

Show that $0.\overline{36}$, which is $0.363\,636\,36\,....$, is a rational number.

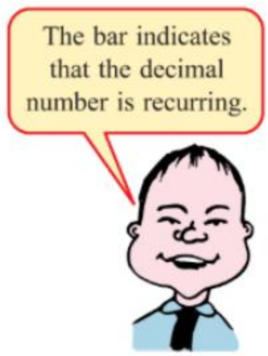
Let $x = 0.\overline{36} = 0.363\,636\,36\,....$

$\therefore\ 100x = 36.363\,636\,.... = 36 + x$

$\therefore\ 99x = 36$

$\therefore\ x = \frac{36}{99} = \frac{4}{11}$

So, $0.\overline{36}$ is actually the rational number $\frac{4}{11}$.



EXERCISE 2B

1 True or false?

a $3 \in \mathbb{Z}^+$ **b** $6 \in \mathbb{Z}$ **c** $\frac{3}{4} \in \mathbb{Q}$ **d** $\sqrt{2} \notin \mathbb{Q}$

e $-\frac{1}{4} \notin \mathbb{Q}$ **f** $2\frac{1}{3} \in \mathbb{Z}$ **g** $0.3684 \in \mathbb{R}$ **h** $\dfrac{1}{0.1} \in \mathbb{Z}$

2 Determine whether each of the following numbers is rational, irrational, or neither:

a 8 **b** -8 **c** $2\frac{1}{3}$ **d** $-3\frac{1}{4}$ **e** $\sqrt{3}$

f $\sqrt{-3}$ **g** $\sqrt{400}$ **h** 9.176 **i** $\frac{1}{0}$ **j** $\pi - \pi$

3 Show that each of the following numbers is rational:

a $0.\overline{7}$ **b** $0.\overline{41}$ **c** $0.\overline{324}$

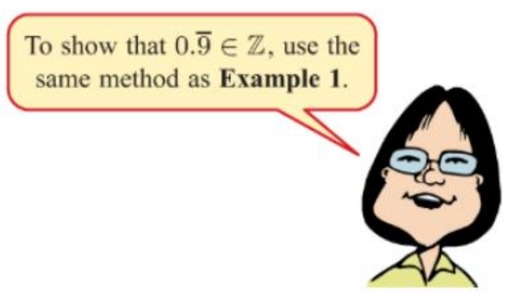
4 Explain why 0.527 is a rational number.

5 Explain why $0.\overline{9} \in \mathbb{Z}$.

6 Give examples to show that these statements are false:

a The sum of two irrationals is irrational.

b The product of two irrationals is irrational.



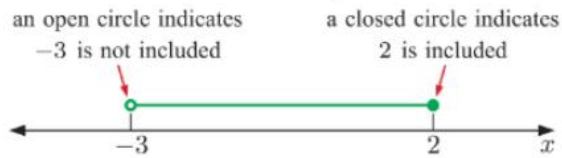
C INTERVAL NOTATION

To avoid having to list all members of a set, we often use a general description of its members. We often describe a set of all values of x with a particular property.

The notation $\{x \mid ......\}$ is used to describe "the set of all x such that ......".

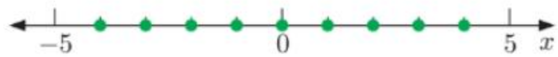
For example:

- $\{x \mid -3 < x \leqslant 2,\ x \in \mathbb{R}\}$
  reads "the set of all real x such that x lies between minus 3 and 2, including 2".
  We can represent the set on a number line as:



  Unless stated otherwise, we *assume* we are dealing with *real* numbers. Thus, the set can also be written as $\{x \mid -3 < x \leqslant 2\}$.

- $\{x \mid -5 < x < 5,\ x \in \mathbb{Z}\}$
  reads "the set of all integers x such that x lies between minus 5 and 5".
  We can represent the set on a number line as:



Example 2 — Self Tutor

Write using interval notation:

a

b

a $\{x \mid 1 \leqslant x \leqslant 5,\ x \in \mathbb{N}\}$
or $\{x \mid 1 \leqslant x \leqslant 5,\ x \in \mathbb{Z}\}$

b $\{x \mid -3 \leqslant x < 6\}$

EXERCISE 2C

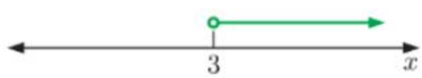
1 Explain the meaning of:

a $\{x \mid x > 4\}$ **b** $\{x \mid x \leqslant 5,\ x \in \mathbb{Z}\}$ **c** $\{y \mid 0 < y < 8\}$

d $\{x \mid 1 \leqslant x \leqslant 4,\ x \in \mathbb{Z}\}$ **e** $\{t \mid 2 < t < 7,\ t \in \mathbb{R}\}$ **f** $\{n \mid n \leqslant 3 \text{ or } n > 6\}$

2 Write using interval notation:

a



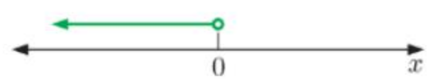
b

c

d

e

f



3 Represent each of the following number sets on a number line:

a $\{x \mid 4 \leqslant x < 8,\ x \in \mathbb{N}\}$
b $\{x \mid -5 < x \leqslant 4,\ x \in \mathbb{Z}\}$
c $\{x \mid -3 < x \leqslant 5,\ x \in \mathbb{R}\}$
d $\{x \mid x > -5,\ x \in \mathbb{Z}\}$
e $\{x \mid x \leqslant 6\}$
f $\{x \mid -5 \leqslant x \leqslant 0\}$

4 Write in interval notation:

a the set of all real numbers greater than 7
b the set of all integers between -8 and 15
c the set of all rational numbers between 4 and 6, including 4.

D SUBSETS AND COMPLEMENT

In this Section we consider some other important terms relating to sets.

SUBSETS

Suppose A and B are two sets. A is a **subset** of B if every element of A is also an element of B. We write $A \subseteq B$.

For example:

- If $E = \{\text{even numbers}\}$ then $E \subseteq \mathbb{Z}$.
- The empty set $\varnothing$ is a subset of every set.

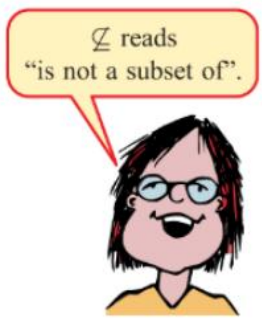
Example 3 **Self Tutor**

Suppose $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 3, 5\}$, and $C = \{3, 5, 8\}$.

Decide whether B or C are subsets of A.

Every element of B is also an element of A, so $B \subseteq A$.

The element 8 of C is not an element of A, so $C \not\subseteq A$.



THE COMPLEMENT OF A SET

If we are given a problem involving sets, the **universal set** U is the set of all elements under consideration.

For example, we might be only interested in integers, or only positive integers.

The **complement** of A, denoted A', is the set of all elements of U which are *not* in A.

$$A' = \{x \mid x \notin A,\ x \in U\}$$

Example 4 **Self Tutor**

Suppose $U = \{x \mid x \leqslant 12,\ x \in \mathbb{Z}^+\}$. Find the complement of:

a $A = \{\text{even numbers in } U\}$ **b** $B = \{\text{prime numbers in } U\}$

a $A' = \{\text{odd numbers in } U\}$
$= \{1, 3, 5, 7, 9, 11\}$

b $B' = \{1, 4, 6, 8, 9, 10, 12\}$

EXERCISE 2D

1 For each of the following sets A and B, decide whether $A \subseteq B$:

a $A = \{2, 5, 6\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
b $A = \{4, 8, 11, 12\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$
c $A = \varnothing$, $B = \{1, 4, 7, 10\}$
d $A = \{5, 10, 15, 20, 25, 30\}$, $B = \{10, 15, 20\}$
e $A = \{6, 7, 8\}$, $B = \mathbb{N}$

2 Are the following statements true or false?

a $\mathbb{Z} \subseteq \mathbb{R}$ **b** $\mathbb{Z}^+ \subseteq \mathbb{Z}$ **c** $\{\frac{1}{3}, \sqrt{2}, 5\} \subseteq \mathbb{Q}$
d $\mathbb{N} \subseteq \mathbb{Q}$ **e** $\mathbb{R} \subseteq \mathbb{Z}$ **f** $\mathbb{Z}^+ \subseteq \mathbb{N}$

3 Let $U = \mathbb{Z}^+$ and $E = \{\text{even integers}\}$. Describe E'.

4 Let $U = \mathbb{Z}^+$ and $P = \{\text{primes}\}$. Describe P'.

5 Suppose $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{3, 6, 9\}$, and $C = \{4, 7\}$. Decide whether B or C are subsets of A.

6 Suppose $P = \{\text{prime numbers less than } 10\}$, $Q = \{\text{multiples of 3 less than } 20\}$, $R = \{3, 5, 7\}$, and $S = \{\text{multiples of 6 less than } 20\}$. Decide whether the following statements are true or false:

a $P \subseteq Q$ **b** $R \subseteq P$ **c** $R \subseteq S$ **d** $S \subseteq Q$

7 Suppose $U = \{x \mid x \leqslant 9,\ x \in \mathbb{Z}^+\}$. Find the complement of:

a $A = \{2, 5, 6\}$ **b** $B = \{\text{prime numbers in } U\}$
c $C = \{\text{odd numbers in } U\}$ **d** $D = \{\text{multiples of 4 in } U\}$
e $E = \varnothing$ **f** $F = \{x \mid x < 3,\ x \in \mathbb{Z}^+\}$.

8 Suppose $U = \{\text{letters of the English alphabet}\}$. Find the complement of:

a $P = \{\text{C, F, J, M, P, U, Y, Z}\}$ **b** $Q = \{\text{consonants}\}$
c $R = \{\text{letters in the word HOSPITAL}\}$ **d** $S = \{\text{letters after J in the alphabet}\}$.

9 Suppose $U = \{x \mid x \leqslant 15,\ x \in \mathbb{Z}^+\}$, $A = \{x \mid 5 \leqslant x < 13,\ x \in \mathbb{Z}^+\}$, and $B = \{x \mid 6 < x < 10,\ x \in \mathbb{Z}^+\}$.

a Write down A' and B'.
b True or false?
i $A \subseteq B$ **ii** $B \subseteq A$ **iii** $A' \subseteq B'$ **iv** $B' \subseteq A'$

E VENN DIAGRAMS

A **Venn diagram** consists of a universal set U represented by a rectangle, and subsets within it that are generally represented by circles or ellipses.

Example 5 Self Tutor

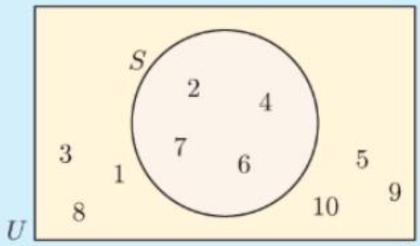
Consider the set $S = \{2, 4, 6, 7\}$ within the universal set $U = \{x \mid x \leqslant 10,\ x \in \mathbb{Z}^+\}$.

a Draw a Venn diagram to show S.

b List the elements of the complement set S'.

c Find: i $n(S)$ ii $n(S')$ iii $n(U)$

a



b $S' = \{1, 3, 5, 8, 9, 10\}$

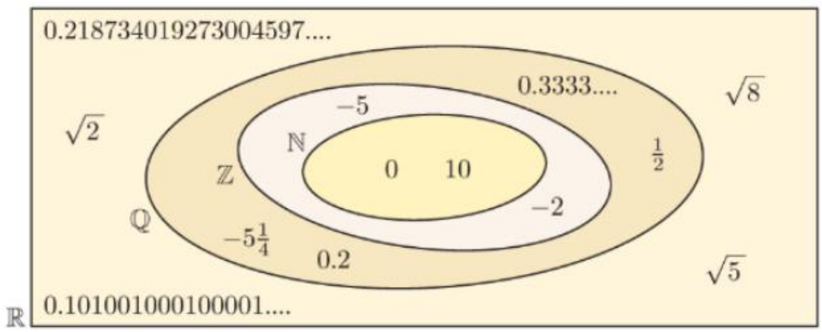
c i $n(S) = 4$

ii $n(S') = 6$

iii $n(U) = 10$

For sets which are subsets of other sets, we can place one circle or ellipse within the other.

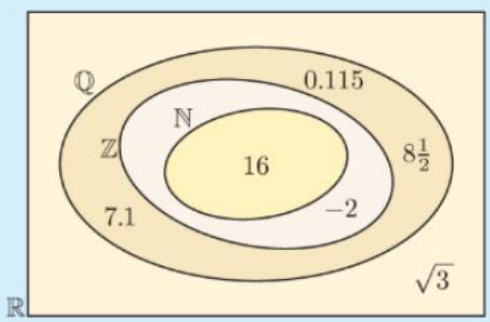
For example, this Venn diagram displays real numbers, rational numbers, integers, and natural numbers:



Example 6 Self Tutor

Illustrate the following numbers on a Venn diagram:

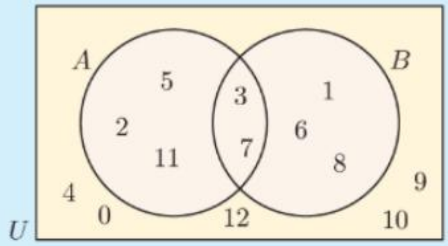
$\sqrt{3}, 8\frac{1}{2}, -2, 7.1, 16, 0.115$



For two sets which have elements in common, we use circles or ellipses which overlap.

Example 7 — Self Tutor

Consider $U = \{x \mid 0 \leqslant x \leqslant 12,\ x \in \mathbb{Z}\}$, $A = \{2, 3, 5, 7, 11\}$, and $B = \{1, 3, 6, 7, 8\}$.
Illustrate A and B on a Venn diagram.



3 and 7 are in both A and B, so the circles representing A and B must overlap.

We place 3 and 7 in the overlap, then fill in the rest of A and the rest of B.

The remaining elements of U are placed outside the two circles.

EXERCISE 2E

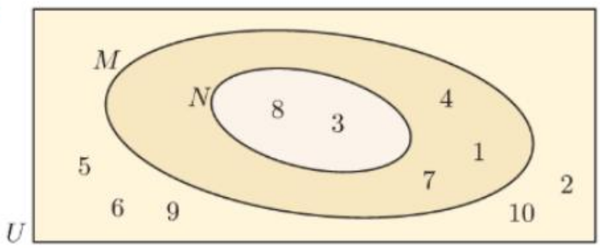
1 Suppose $U = \{x \mid x \leqslant 8,\ x \in \mathbb{Z}^+\}$ and $A = \{\text{prime numbers} \leqslant 8\}$.

- **a** Show set A on a Venn diagram.
- **b** List the set A'.
- **c** Find: **i** $n(A)$ **ii** $n(A')$ **iii** $n(U)$

2 Suppose $U = \{\text{letters of the English alphabet}\}$ and
$V = \{\text{letters of the English alphabet which are vowels}\}$.

- **a** Show these sets on a Venn diagram.
- **b** List the set V'.
- **c** Find: **i** $n(V)$ **ii** $n(V')$ **iii** $n(U)$

3

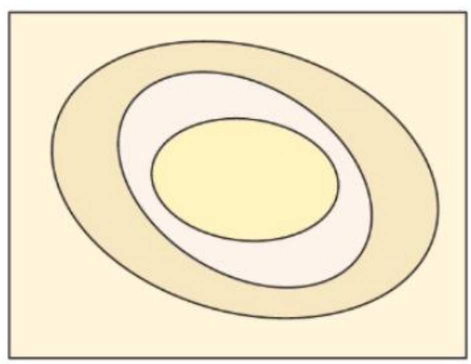


- **a** List the elements of: **i** U **ii** N **iii** M
- **b** Find $n(N)$ and $n(M)$.
- **c** Is $M \subseteq N$?

4 Illustrate A and B on a Venn diagram if:

- **a** $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
- **b** $U = \{4, 5, 6, 7, 8, 9, 10\}$, $A = \{6, 7, 9, 10\}$, $B = \{5, 6, 8, 9\}$
- **c** $U = \{3, 4, 5, 6, 7, 8, 9\}$, $A = \{3, 5, 7, 9\}$, $B = \{4, 6, 8\}$

5 Suppose the universal set is $U = \mathbb{R}$, the set of all real numbers.
$\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ are all subsets of $\mathbb{R}$.



- **a** Copy the given Venn diagram and label the sets U, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$.
- **b** Place these numbers on the Venn diagram:
  $\frac{2}{3}$, $\sqrt{7}$, $0.\overline{4}$, -1, $-8\frac{1}{3}$, 0, 4, and
  $\alpha = 0.564\,105\,923\,6\ldots.$ which does not terminate or recur.
- **c** Shade the region representing the set of irrationals $\mathbb{Q}'$.

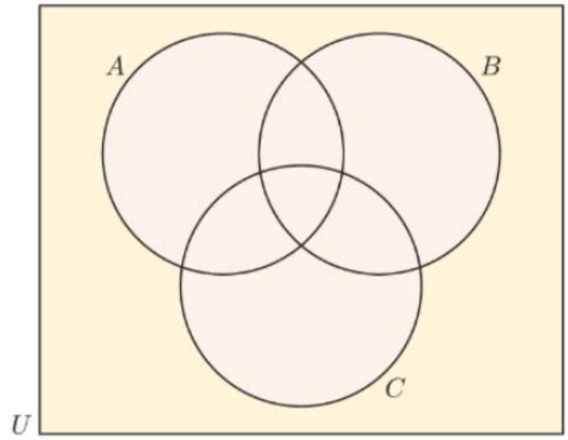
6 Show the following information on a Venn diagram:

a $U = \{\text{triangles}\}$, $E = \{\text{equilateral triangles}\}$, $I = \{\text{isosceles triangles}\}$

b $U = \{\text{quadrilaterals}\}$, $P = \{\text{parallelograms}\}$, $R = \{\text{rectangles}\}$

7 Suppose $U = \{x \mid x \leqslant 30,\ x \in \mathbb{Z}^+\}$,
$A = \{\text{prime numbers} \leqslant 30\}$,
$B = \{\text{multiples of } 5 \leqslant 30\}$,
and $C = \{\text{odd numbers} \leqslant 30\}$.

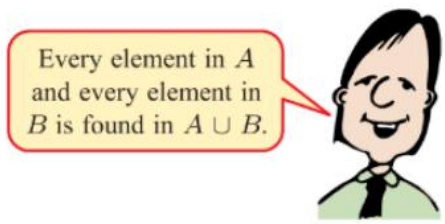
Use the Venn diagram shown to display the elements of the sets.

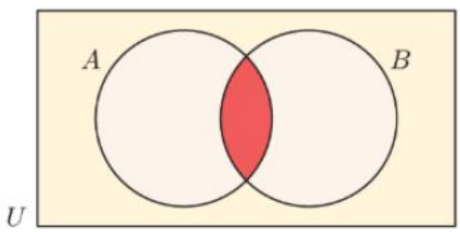


F UNION AND INTERSECTION

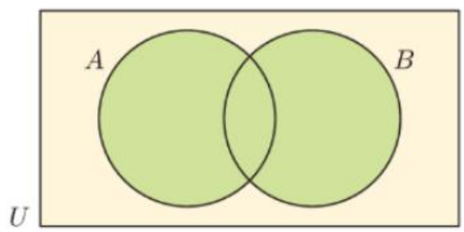
If A and B are two sets, then:

- $A \cap B$ is the **intersection** of A and B, and consists of all elements which are in **both** A **and** B
- $A \cup B$ is the **union** of A and B, and consists of all elements which are in A **or** B (or both).





$A \cap B$ is shaded red.

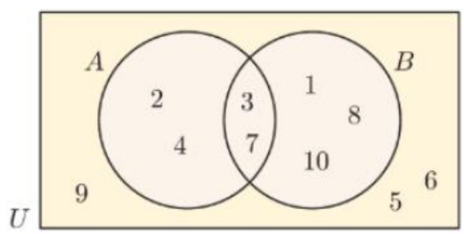


$A \cup B$ is shaded green.

For example, the Venn diagram alongside shows

$A = \{2, 3, 4, 7\}$ and $B = \{1, 3, 7, 8, 10\}$.

We can see that $A \cap B = \{3, 7\}$
and $A \cup B = \{1, 2, 3, 4, 7, 8, 10\}$.



Example 8 **Self Tutor**

Suppose $U = \{\text{positive integers} \leqslant 12\}$, $A = \{\text{primes} \leqslant 12\}$, and $B = \{\text{factors of } 12\}$.

a List the elements of the sets A and B.

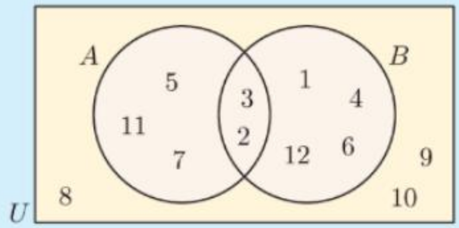
b Show the sets A, B, and U on a Venn diagram.

c List the elements in: **i** A' **ii** $A \cap B$ **iii** $A \cup B$

d Find: **i** $n(A \cap B)$ **ii** $n(A \cup B)$ **iii** $n(B')$

a $A = \{2, 3, 5, 7, 11\}$ and $B = \{1, 2, 3, 4, 6, 12\}$

b



c **i** $A' = \{1, 4, 6, 8, 9, 10, 12\}$ **ii** $A \cap B = \{2, 3\}$

iii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 11, 12\}$

d **i** $n(A \cap B) = 2$ **ii** $n(A \cup B) = 9$

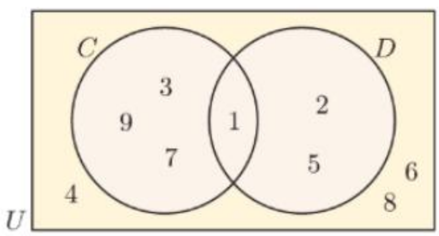
iii $B' = \{5, 7, 8, 9, 10, 11\}$, so $n(B') = 6$

Two sets are **disjoint** or **mutually exclusive** if they have no elements in common.

If A and B are disjoint then $A \cap B = \varnothing$.

EXERCISE 2F.1

1



a List the elements of set:

i C **ii** D **iii** U

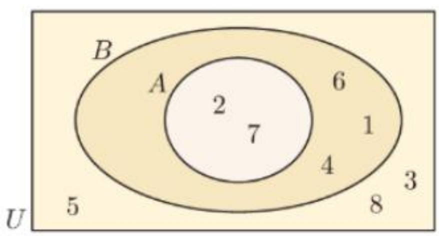
iv $C \cap D$ **v** $C \cup D$

b Find:

i $n(C)$ **ii** $n(D)$ **iii** $n(U)$

iv $n(C \cap D)$ **v** $n(C \cup D)$

2



a List the elements of set:

i A **ii** B **iii** U

iv $A \cap B$ **v** $A \cup B$

b Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(U)$

iv $n(A \cap B)$ **v** $n(A \cup B)$

3 Consider $U = \{x \mid x \leqslant 12,\ x \in \mathbb{Z}^+\}$, $A = \{2, 7, 9, 10, 11\}$, and $B = \{1, 2, 9, 11, 12\}$.

a Show these sets on a Venn diagram.

b List the elements of: **i** $A \cap B$ **ii** $A \cup B$ **iii** B'

c Find: **i** $n(A)$ **ii** $n(B')$ **iii** $n(A \cap B)$ **iv** $n(A \cup B)$

4 If A is the set of all factors of 36 and B is the set of all factors of 63, find:

  a $A \cap B$ **b** $A \cup B$

5 If $X = \{\text{A, B, D, M, N, P, R, T, Z}\}$ and $Y = \{\text{B, C, M, T, W, Z}\}$, find:

  a $X \cap Y$ **b** $X \cup Y$

6 Suppose $U = \{x \mid x \leqslant 30,\ x \in \mathbb{Z}^+\}$, $A = \{\text{factors of } 30\}$, and $B = \{\text{prime numbers} \leqslant 30\}$.

  a Find: **i** $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$ **iv** $n(A \cup B)$

  b Show that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

7 Consider $X = \{1, 3, 5, 7\}$ and $Y = \{2, 4, 6, 8\}$.

  a Simplify $X \cap Y$.

  b What can you say about sets X and Y?

8 Simplify:

  a $A \cup A'$ for any set $A \in U$. **b** $A \cap A'$ for any set $A \in U$.

USING VENN DIAGRAMS TO ILLUSTRATE REGIONS

We can use a Venn diagram to help illustrate the union or intersection of regions.

Shaded regions of a Venn diagram can be used to verify **set identities**. These are equations involving sets which are true for *all* sets.

Examples of set identities include:

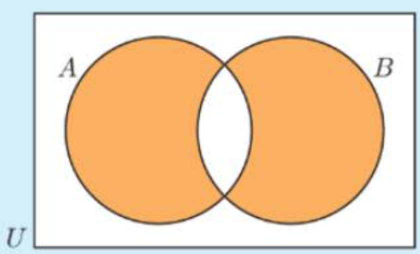
$$A \cup A' = U \qquad A \cap A' = \varnothing$$
$$(A \cup B)' = A' \cap B' \qquad (A \cap B)' = A' \cup B'$$

Example 9 **Self Tutor**

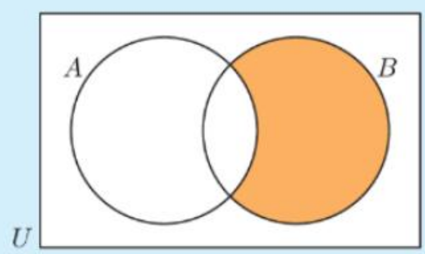
On separate Venn diagrams, shade the region representing:

  a in A or in B but not in both **b** $A' \cap B$

a

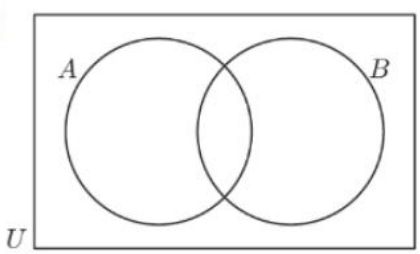


b



EXERCISE 2F.2

1



On separate Venn diagrams, shade regions for:

  a $A \cap B$ **b** $A \cap B'$

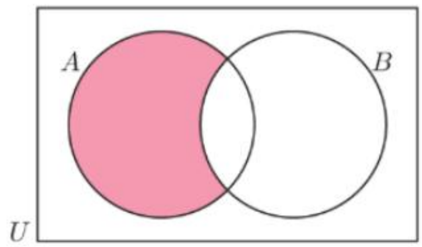
  c $A' \cup B$ **d** $A \cup B'$

  e $A' \cap B$ **f** $(A' \cap B)'$

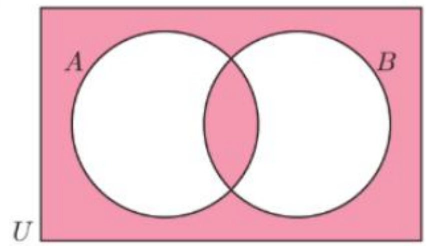
PRINTABLE VENN DIAGRAMS (OVERLAPPING)

2 Describe in words, the shaded region of:

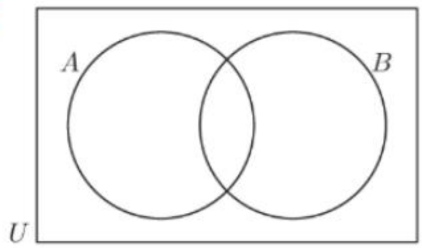
a



b



3



a On separate Venn diagrams, shade regions for:

i $A \cup B$ **ii** $(A \cup B)'$ **iii** $A' \cap B'$
iv $(A \cap B)'$ **v** $A' \cup B'$ **vi** $(A' \cup B')'$

b Hence verify that:

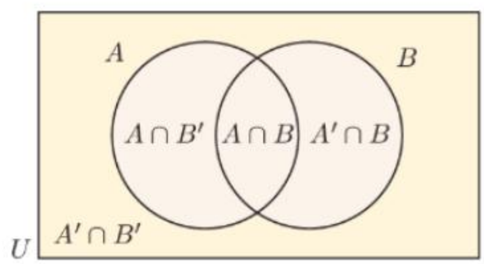
i $(A \cap B)' = A' \cup B'$ **ii** $(A \cup B)' = A' \cap B'$

Click on the icon to practise shading regions representing various subsets.

VENN DIAGRAMS

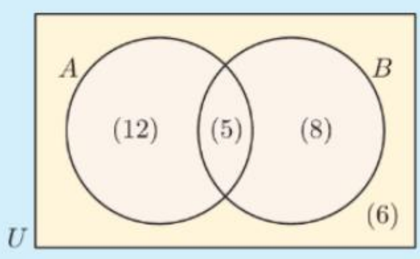
G NUMBERS IN REGIONS

We have seen that a Venn diagram with two overlapping sets A and B contains four regions.



There are many situations where we are only interested in the **number of elements** of U that are in each region. We do not need to show all the elements on the diagram, so instead we write the number of elements in each region in brackets.

Example 10 Self Tutor



In the Venn diagram given, (5) means that there are 5 elements in the set $A \cap B$.

How many elements are there in:

a A **b** B' **c** $A \cup B$
d A, but not B **e** B, but not A **f** neither A nor B?

a $n(A) = 12 + 5 = 17$

b $n(B') = 12 + 6 = 18$

c $n(A \cup B) = 12 + 5 + 8 = 25$

d $n(A, \text{ but not } B) = 12$

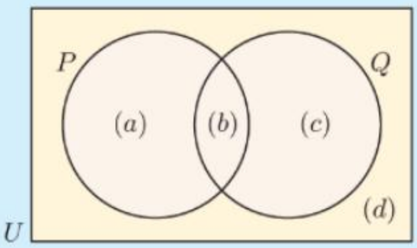
e $n(B, \text{ but not } A) = 8$

f $n(\text{neither } A \text{ nor } B) = 6$

Example 11 **Self Tutor**

Given $n(U) = 25$, $n(P) = 10$, $n(Q) = 12$, and $n(P \cap Q) = 3$, find:

a $n(P \cup Q)$ **b** $n(P$, but not $Q)$



We see that $b = 3$ $\{$as $n(P \cap Q) = 3\}$

$a + b = 10$ $\{$as $n(P) = 10\}$

$b + c = 12$ $\{$as $n(Q) = 12\}$

$a + b + c + d = 25$ $\{$as $n(U) = 25\}$

$\therefore\ b = 3,\ a = 7,$ and $c = 9$

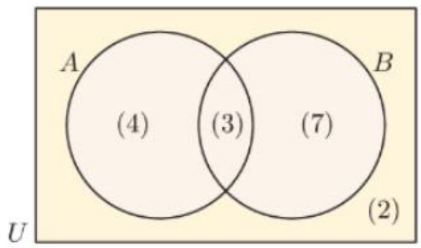
$\therefore\ 7 + 3 + 9 + d = 25$

$\therefore\ d = 6$

a $n(P \cup Q) = a + b + c = 19$ **b** $n(P$, but not $Q) = a = 7$

EXERCISE 2G

1



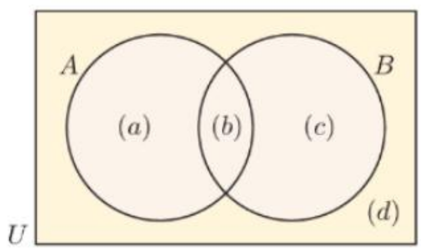
How many elements are there in:

a B **b** A'

c $A \cup B$ **d** A, but not B

e B, but not A **f** neither A nor B?

2 In the Venn diagram below, (a) means that there are a elements in that region.



a Write an expression for:

i $n(A)$ **ii** $n(B)$

iii $n(A \cap B)$ **iv** $n(A \cup B)$

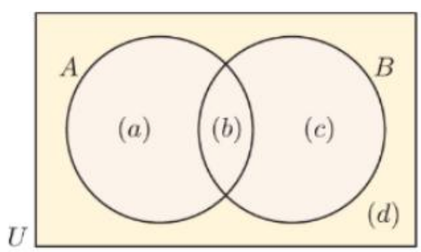
b Show that:

i $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

ii $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

iii if A and B are disjoint, then $n(A \cup B) = n(A) + n(B)$.

3



Use the Venn diagram to show that:

a $n(A \cap B') = n(A) - n(A \cap B)$

b $n(A \cup B') = n(U) - n(A' \cap B)$

4 Given $n(U) = 20$, $n(A) = 12$, $n(B) = 13$, and $n(A \cap B) = 8$, find:

a $n(A \cup B)$ **b** $n(B \cap A')$

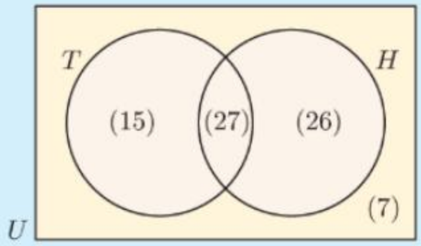
5 Given $n(U) = 28$, $n(M) = 14$, $n(M \cap N) = 3$, and $n(M \cup N) = 18$, find:

a $n(N)$ **b** $n((M \cup N)')$

H PROBLEM SOLVING WITH VENN DIAGRAMS

Example 12 — **Self Tutor**

The Venn diagram alongside illustrates the number of people in a sporting club who play tennis (T) and hockey (H).



Determine the number of people:

a in the club
b who play hockey
c who play both sports
d who play neither sport
e who play at least one sport.

a $n(U) = 15 + 27 + 26 + 7 = 75$
There are 75 people in the club.

b $n(H) = 27 + 26 = 53$
53 people play hockey.

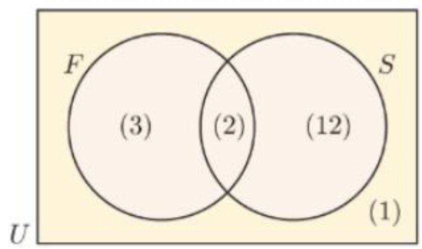
c $n(T \cap H) = 27$
27 people play both sports.

d $n(T' \cap H') = 7$
7 people play neither sport.

e $n(T \cup H) = 15 + 27 + 26 = 68$
68 people play at least one sport.

EXERCISE 2H

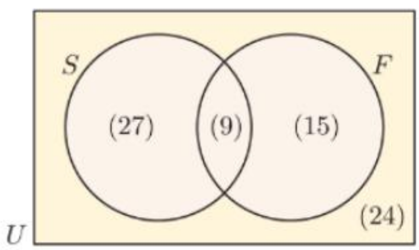
1 The Venn diagram alongside illustrates the number of students in a particular class who study French (F) and Spanish (S).



Determine the number of students:

a in the class
b who study both subjects
c who study at least one of the subjects
d who only study Spanish.

2 In a survey at a resort, people were asked whether they went sailing (S) or fishing (F) during their stay.



Use the Venn diagram to determine the number of people:

a in the survey
b who did both activities
c who did neither activity
d who did exactly one of the activities.

3 In a class of 30 students, 19 study Physics, 17 study Chemistry, and 15 study both subjects.

a Illustrate the information on a Venn diagram.

b Determine the number of students who study:

i at least one of the subjects
ii Physics, but not Chemistry
iii exactly one of the subjects
iv neither subject.

4 In a class of 40 students, 19 play tennis, 20 play netball, and 8 play neither sport. Determine the number of students in the class who:

a do not play netball
b play at least one of the sports
c play exactly one of the sports
d play netball, but not tennis.

5 In a class of 25 students, 15 play hockey, 16 play basketball, and 4 play neither sport. Determine the number of students who play:

a both sports
b hockey but not basketball.

6 In a class of 40 students, 34 like bananas, 22 like pineapples, and 2 dislike both fruits. Find the number of students who:

a like both fruits
b like at least one fruit.

7 In a class of 40 students, 23 have dark hair, 18 have brown eyes, and 26 have dark hair, brown eyes or both. How many students have:

a dark hair and brown eyes
b neither dark hair nor brown eyes
c dark hair but not brown eyes?

8 Answer the **Opening Problem** on page **28**.

9 In a circle of music lovers, 14 people play the piano or violin or both, 8 people are piano players, and 5 people play both instruments. Find the number of violin players.

10 64% of students at a school study a language, and 79% study Mathematics. Every student studies at least one of these subjects. What percentage of students study both a language and Mathematics?

11 Our team scored well in the interschool athletics carnival. Each person was allowed to participate in one running and one jumping event. We gained 8 places in running events. 5 of us gained a place in both running and jumping events, and 14 of us gained exactly one place. In total, how many places were gained by the team?

12 At a certain school there are 90 students studying for their IB diploma. They all study at least one of the subjects Physics, French, or History. 50 are studying Physics, 60 are studying French, and 55 are studying History. 30 students are studying both Physics and French, while 10 students are studying both French and History but not Physics. 20 students are studying all three subjects.

a Construct a Venn diagram to illustrate this information.

b How many students are studying both Physics and History, but not French?

c How many students are studying at least two of the three subjects?

I THE ALGEBRA OF SETS

For the set of real numbers $\mathbb{R}$, we can write laws for the operations $+$ and $\times$:

For any real numbers a, b, and c:

- **commutative** $a+b=b+a$ and $ab=ba$
- **identity** Identity elements 0 and 1 exist such that $a+0=0+a=a$ and $a\times 1=1\times a=a$.
- **associative** $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$
- **distributive** $a(b+c)=ab+ac$

The following are the **laws for the algebra of sets** under the operations $\cup$ and $\cap$:

For any subsets A, B, and C of the universal set U:

- **commutative** $A\cap B=B\cap A$ and $A\cup B=B\cup A$
- **associative** $A\cap(B\cap C)=(A\cap B)\cap C$ and $A\cup(B\cup C)=(A\cup B)\cup C$
- **distributive** $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ and $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
- **identity** $A\cup\varnothing=A$ and $A\cap U=A$
- **complement** $A\cup A'=U$ and $A\cap A'=\varnothing$
- $A\cup U=U$ and $A\cap\varnothing=\varnothing$
- $A\cap A=A$ and $A\cup A=A$
- $(A\cap B)'=A'\cup B'$ and $(A\cup B)'=A'\cap B'$
- $(A')'=A$

EXERCISE 2I

1 With the aid of Venn diagrams, explain why the following laws are valid:

a the *commutative* laws $A\cap B=B\cap A$ and $A\cup B=B\cup A$

b the laws $A\cap A=A$ and $A\cup A=A$

c the *associative* laws $A\cap(B\cap C)=(A\cap B)\cap C$ and $A\cup(B\cup C)=(A\cup B)\cup C$

d the *distributive* laws $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ and $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

e the law $(A')'=A$.

2 Use the laws for the algebra of sets to show that:

a $A\cup(B\cup A')=U$

b $A\cap(B\cap A')=\varnothing$

c $A\cup(B\cap A')=A\cup B$

d $(A'\cup B')'=A\cap B$

e $(A\cup B)\cap(A'\cap B')=\varnothing$

f $(A\cup B)\cap(C\cup D)=(A\cap C)\cup(A\cap D)\cup(B\cap C)\cup(B\cap D)$.

REVIEW SET 2A

1 Explain why 1.3 is a rational number.

2 Is $\sqrt{4000} \in \mathbb{Q}$?

3 Let P be the set of all prime numbers between 20 and 40.

a Is $37 \in P$? **b** Find $n(P)$.

4 Write a statement describing the meaning of $S = \{t \mid -1 \leqslant t < 3\}$.

5 Write using interval notation:

6 For each of the following sets P and Q, decide whether $P \subseteq Q$:

a $P = \{5, 6, 7, 8\}$, $Q = \{1, 2, 3, 4, 5, 6, 7\}$

b $P = \{\text{multiples of 4 between 10 and 30}\}$, $Q = \{\text{even numbers between 0 and 40}\}$

7 Suppose $U = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}^+\}$. Find the complement of:

a $A = \{3, 7, 9\}$ **b** $B = \{\text{composite numbers in } U\}$.

8 Suppose $U = \{x \mid x \leqslant 12, \ x \in \mathbb{Z}^+\}$ and $A = \{\text{multiples of } 3 \leqslant 12\}$.

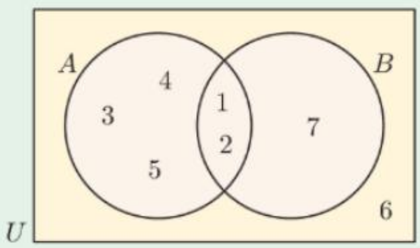
a Show A on a Venn diagram. **b** List the set A'.

c Find $n(A')$.

9 True or false?

a $\mathbb{N} \subseteq \mathbb{Z}^+$ **b** $\mathbb{Q} \subseteq \mathbb{Z}$

10



a List the elements of set:

i A **ii** B **iii** U

iv $A \cup B$ **v** $A \cap B$

b Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(A \cup B)$

11 Consider $U = \{x \mid x \leqslant 10, \ x \in \mathbb{Z}^+\}$, $P = \{2, 3, 5, 7\}$, and $Q = \{2, 4, 6, 8\}$.

a Show these sets on a Venn diagram.

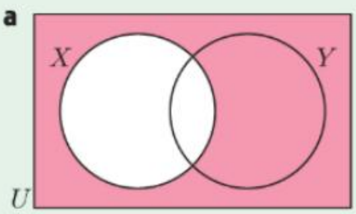
b List the elements of: **i** $P \cap Q$ **ii** $P \cup Q$ **iii** Q'

c Find: **i** $n(P')$ **ii** $n(P \cap Q)$ **iii** $n(P \cup Q)$

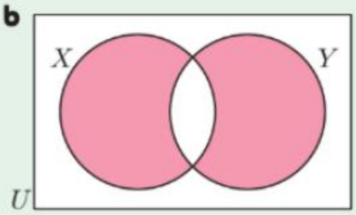
d Is $(P \cap Q) \subseteq P$?

12 Describe in words the shaded region:

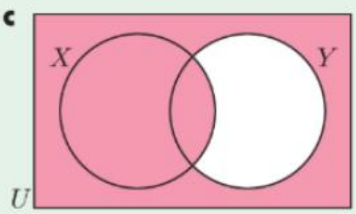
a



b



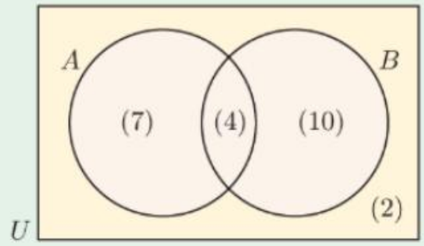
c



13 How many elements are there in:

a A **b** B

c $A \cup B$ **d** neither A nor B?



14 400 families were surveyed. It was found that 90% had a TV set, and 60% had a computer. Every family had at least one of these items. How many families had both a TV set and a computer?

REVIEW SET 2B

1 Is $-2 \in \mathbb{Z}^+$?

2 Show that $0.\overline{51}$ is a rational number.

3 Sketch the number set $\{x \mid x \leqslant 3 \quad \text{or} \quad x > 7,\ x \in \mathbb{R}\}$.

4 For each of the following sets:

i List the elements of the set.
ii Determine whether the set is finite or infinite.
iii If the set is finite, find the number of elements in the set.

a $A = \{\text{factors of } 15\}$ **b** $B = \{\text{multiples of } 8\}$

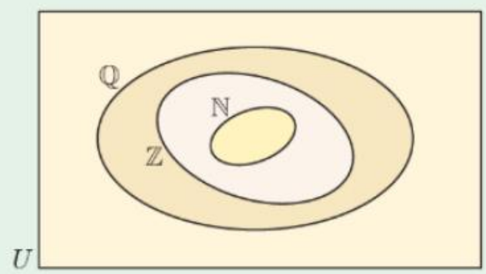
c $C = \{\text{odd numbers between } 30 \text{ and } 50\}$ **d** $D = \{\text{prime numbers less than } 30\}$

5 Suppose $P = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $Q = \{4, 9, 10\}$, and $R = \{5, 6, 12\}$. Decide whether Q and R are subsets of P.

6 Suppose $U = \{x \mid x \leqslant 12,\ x \in \mathbb{Z}^+\}$ and $A = \{\text{prime numbers less than } 12\}$. Find:

a A **b** A' **c** $n(A)$ **d** $n(A')$ **e** $n(U)$

7 Illustrate these numbers on a Venn diagram like the one shown:
$-1, \sqrt{2}, 2, 3.1, \pi, 4.\overline{2}$



8 Show this information on a Venn diagram:

a $U = \{10, 11, 12, 13, 14, 15\}$, $A = \{10, 12, 14\}$, $B = \{11, 12, 13\}$

b $U = \{\text{quadrilaterals}\}$, $S = \{\text{squares}\}$, $R = \{\text{rectangles}\}$

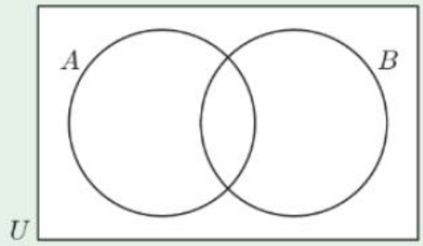
9 If A is the set of all factors of 24 and B is the set of all factors of 18, find:

a $A \cap B$ **b** $A \cup B$

10 On separate Venn diagrams like the one shown, shade the region representing:

a B' **b** in A and in B

c $(A \cup B)'$



11 Given $n(U) = 30$, $n(A) = 14$, $n(B) = 10$, and $n(A \cap B) = 6$, find:

a $n(A \cup B)$ **b** $n(B$, but not $A)$

12 A party of 32 children spent an hour by the river. 23 caught tadpoles, 7 caught fish, and 6 caught neither.

a Illustrate the information on a Venn diagram.

b Find the number of children who caught:

i both tadpoles and fish

ii tadpoles but not fish

iii fish but not tadpoles.

13 In a certain town, three newspapers are published. 20% of the population read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C, and 2% read all 3 newspapers. What percentage of the population read:

a none of the papers **b** at least one of the papers

c exactly one of the papers **d** either A or B

e A only?

14 With the aid of Venn diagrams, explain why the law $(A \cup B)' = A' \cap B'$ is valid.

Chapter 3

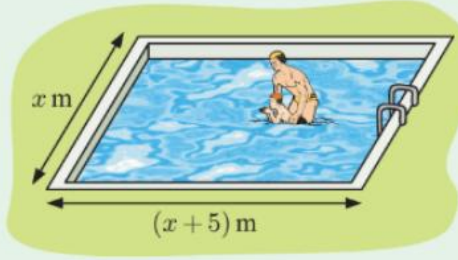
Algebraic expansion and factorisation

Contents:

OPENING PROBLEM

Cezanne wants to install a rectangular swimming pool with length 5 m more than its width.

Let the width be x m, so the length is $(x+5)$ m.



Things to think about:

a Can you write *two* different expressions for the perimeter of the pool? How can we show that these expressions are equal?

b Explain why the surface area of the swimming pool is given by $x(x+5)$ m^2. How can we write this expression without brackets?

The study of **algebra** is vital for many areas of mathematics. We need it to manipulate equations, solve problems for unknown variables, and also to develop higher level mathematical theories.

In this chapter we revise the **expansion** of expressions which involve brackets, and the reverse process which is called **factorisation**.

A REVISION OF EXPANSION LAWS

DISTRIBUTIVE LAW

$$\boldsymbol{a(b+c)=ab+ac}$$

Example 1 Self Tutor

Expand and simplify:

a $2(3x-1)$ **b** $-3x(x+2)$

a
$$\begin{aligned}&2(3x-1)\\&=2\times 3x+2\times(-1)\\&=6x-2\end{aligned}$$

b
$$\begin{aligned}&-3x(x+2)\\&=-3x\times x+-3x\times 2\\&=-3x^2-6x\end{aligned}$$

EXERCISE 3A.1

1 Expand and simplify:

a $5(4+x)$ **b** $2(x-y)$ **c** $3(t-7)$ **d** $7(3+p)$
e $9(b+c)$ **f** $4(x-5)$ **g** $2(6+j)$ **h** $8(q-p)$

2 Expand and simplify:

a $11(4x+y)$ **b** $2(m-7n)$ **c** $6(3g-2h)$ **d** $3(4+3x)$
e $2(3x+z)$ **f** $6(c-3d)$ **g** $5(p+6q)$ **h** $4(3a-bc)$

3 Expand and simplify:

a $2x(5-x)$ **b** $3x(5-x)$ **c** $4a^2(a-3)$ **d** $7n(4+2n)$

e $(3x-2)x$ **f** $(4+2x)x$ **g** $pq(q-p)$ **h** $ab^2(b-1)$

4 Expand and simplify:

a $-3(x+1)$ **b** $-2(x+3)$ **c** $-5(x-2)$ **d** $-6(3-x)$

e $-(a+4)$ **f** $-(x-2)$ **g** $-5(3-x)$ **h** $-2(5-2c)$

5 Expand and simplify:

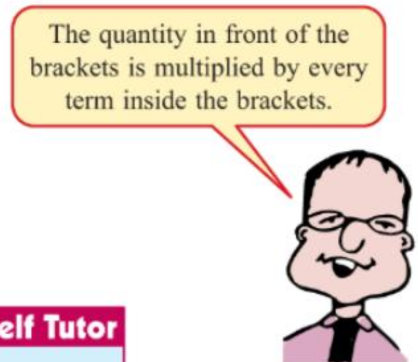
a $-a(2+a)$ **b** $-b(b-4)$ **c** $-c(2+c)$ **d** $-x(-x+7)$

e $-3n(2-n)$ **f** $-4y(y+3)$ **g** $-6a(3-a)$ **h** $-2b(2-5b)$

6 Expand and simplify:

a $3(a^2+3a+1)$ **b** $5(x^2-3x+2)$

c $-4(2c^2-3c-7)$ **d** $2a(3a^2-5a+1)$



Example 2 **Self Tutor**

Expand and simplify:

a $3(x+5)+2(4-x)$ **b** $y(3y-1)-3y(2y-5)$

a
$$\begin{aligned}&3(x+5)+2(4-x)\\&=3\times x+3\times 5+2\times 4+2\times(-x)\\&=3x+15+8-2x\\&=x+23\end{aligned}$$

b
$$\begin{aligned}&y(3y-1)-3y(2y-5)\\&=y\times 3y+y\times(-1)+-3y\times 2y+-3y\times(-5)\\&=3y^2-y-6y^2+15y\\&=14y-3y^2\end{aligned}$$

7 Expand and simplify:

a $4(x+1)+2(x+2)$ **b** $5(x-1)+2(x-3)$

c $3(m+2)-2(m-6)$ **d** $2(x+1)-2(2x+3)$

e $9(x-2)+3(7-4x)$ **f** $9(2-5x)-2(3x+2)$

g $-4(2n-3)-3(3n-5)$ **h** $7(3y-4)+5(1-2y)$

8 Expand and simplify:

a $4x-(1+2x)$ **b** $5-4(2x+1)$ **c** $9-6(2x-4)$

d $9x-(6-2x)$ **e** $5-3(2-5x)$ **f** $12-(3+4x)$

9 Expand and simplify:

a $x(x^2+2x)-x^2(2-x)$ **b** $a(a+b)-b(a-b)$ **c** $x^2(6-x)+3x(x-4)$

THE PRODUCT $(a+b)(c+d)$

$$(a+b)(c+d) = ac + ad + bc + bd$$

Example 3 **Self Tutor**

Expand and simplify:

a $(x+4)(x-3)$ **b** $(2x-5)(-x+3)$

a $(x+4)(x-3)$
$= x \times x + x \times (-3) + 4 \times x + 4 \times (-3)$
$= x^2 - 3x + 4x - 12$
$= x^2 + x - 12$

b $(2x-5)(-x+3)$
$= 2x \times (-x) + 2x \times 3 - 5 \times (-x) - 5 \times 3$
$= -2x^2 + 6x + 5x - 15$
$= -2x^2 + 11x - 15$

EXERCISE 3A.2

1 Expand and simplify:

a $(x+2)(x+5)$ **b** $(x-3)(x+4)$ **c** $(x+5)(x-3)$
d $(x-2)(x-10)$ **e** $(2x+1)(x-3)$ **f** $(3x-4)(2x-5)$
g $(2x+y)(x-y)$ **h** $(x+3)(-2x-1)$ **i** $(x+2y)(-x-1)$

2 Expand and simplify:

a $(a-2)(a-5)$ **b** $(w+x)(y+z)$ **c** $(p+q)(a+b)$
d $(x-1)(3x+2)$ **e** $(1-x)(2x+3)$ **f** $(2x+5)(x-3)$
g $(3x-2)(x+4)$ **h** $(4x-3)(3x-5)$ **i** $(x-1)(x^2+5)$

3 Expand and simplify:

a $(x+3)(x-1)+3(x-5)$ **b** $(x+7)(x-5)+(x+1)(x+4)$
c $(2x+3)(x-2)-(x+1)(x+6)$ **d** $(4t-3)(t+1)-(2t-1)(2t+5)$
e $(4x-1)(3-x)+(2x-3)(3x-2)$ **f** $5(3x-4)(x+2)-(7-x)(8-5x)$

DIFFERENCE OF TWO SQUARES

$$(a+b)(a-b) = a^2 - b^2$$

Example 4 Self Tutor

Expand and simplify:

a $(x+4)(x-4)$

b $(3x-2)(3x+2)$

a $(x+4)(x-4)$
$= x^2 - 4^2$
$= x^2 - 16$

b $(3x-2)(3x+2)$
$= (3x)^2 - 2^2$
$= 9x^2 - 4$

EXERCISE 3A.3

1 Expand and simplify:

a $(y+1)(y-1)$
b $(b+2)(b-2)$
c $(a-7)(a+7)$
d $(x-3)(x+3)$
e $(6-b)(6+b)$
f $(5-x)(5+x)$
g $(8+a)(8-a)$
h $(2+3y)(2-3y)$
i $(7-2a)(7+2a)$
j $(3x+1)(3x-1)$
k $(5-3y)(5+3y)$
l $(-x+2)(-x-2)$
m $(2x+1)(2x-1)$
n $(4-3y)(4+3y)$
o $(3x-4z)(4z+3x)$

2 Expand and simplify:

a $(x+3)(x-3)-(x+6)(x-6)$
b $(5p-2)(5p+2)-p(3p-1)$
c $(3y-z)(3y+z)-(2y+3z)(2y-3z)$
d $(10-x^2)(10+x^2)-(10-3x^2)(10+3x^2)$

PERFECT SQUARES EXPANSION

$$(a+b)^2 = a^2 + 2ab + b^2$$

Example 5 Self Tutor

Expand and simplify:

a $(2x+1)^2$

b $(3-4y)^2$

a $(2x+1)^2$
$= (2x)^2 + 2 \times 2x \times 1 + 1^2$
$= 4x^2 + 4x + 1$

b $(3-4y)^2$
$= 3^2 + 2 \times 3 \times (-4y) + (-4y)^2$
$= 9 - 24y + 16y^2$

EXERCISE 3A.4

1 Expand and simplify:

a $(x+1)^2$
b $(x-1)^2$
c $(x+8)^2$
d $(x-8)^2$
e $(4+y)^2$
f $(4-y)^2$
g $(3x+1)^2$
h $(3x-1)^2$
i $(1+2a)^2$
j $(1-2a)^2$
k $(a+b)^2$
l $(a-b)^2$

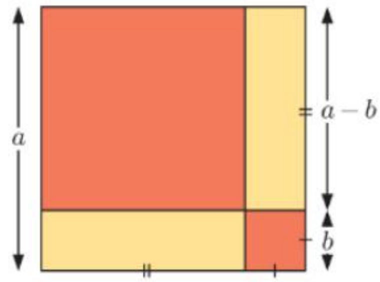
2 Expand and simplify:

a $(x+5)^2$
b $(2x+3)^2$
c $(7+x)^2$
d $(3x+4)^2$
e $(5+x^2)^2$
f $(3x^2+2)^2$
g $(5x+3y)^2$
h $(2x^2+7y)^2$
i $(x^3+8x)^2$

3 Expand and simplify:

- **a** $(x-3)^2$
- **b** $(2-x)^2$
- **c** $(3x-1)^2$
- **d** $(6-5p)^2$
- **e** $(2x-5y)^2$
- **f** $(ab-2)^2$
- **g** $(x^2-5)^2$
- **h** $(4x^2-3y)^2$
- **i** $(-x^2-y^2)^2$

4 Use the diagram alongside to show that $(a-b)^2 = a^2 - 2ab + b^2$.



5 Expand and simplify:

- **a** $(x+9)^2 + (x-2)^2$
- **b** $(3x+1)^2 - (2x-3)^2$
- **c** $(x+8)^2 - (x+2)(x-5)$
- **d** $(5-p)^2 + (p^2-4)^2$
- **e** $(3x^2-1)^2 - 4(1-x)^2$
- **f** $(5x+y^2)^2 - x(x^2-y)^2$

B FURTHER EXPANSION

When expressions containing more than two terms are multiplied together, we can still use the distributive law to expand the brackets. Each term in the first set of brackets is multiplied by each term in the second set of brackets.

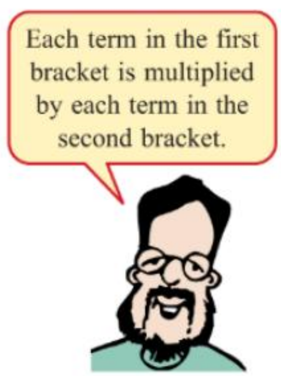
If there are 2 terms in the first brackets and 3 terms in the second brackets, there will be $2 \times 3 = 6$ terms in the expansion. However, when we simplify by collecting like terms, the final answer may contain fewer terms.

Example 6 **Self Tutor**

Expand and simplify: $(x+3)(x^2+2x+4)$

$$(x+3)(x^2+2x+4)$$

$$\begin{aligned} &= x^3 + 2x^2 + 4x && \{x \times \text{each term in 2nd bracket}\} \\ &\quad\; + 3x^2 + 6x + 12 && \{3 \times \text{each term in 2nd bracket}\} \\ &= x^3 + 5x^2 + 10x + 12 && \{\text{collecting like terms}\} \end{aligned}$$



EXERCISE 3B

1 Expand and simplify:

- **a** $(x+2)(x^2+x+4)$
- **b** $(x+3)(x^2+2x-3)$
- **c** $(x+3)(x^2+2x+1)$
- **d** $(x+1)(2x^2-x-5)$
- **e** $(2x+3)(x^2+2x+1)$
- **f** $(2x-5)(x^2-2x-3)$
- **g** $(x+5)(3x^2-x+4)$
- **h** $(4x-1)(2x^2-3x+1)$

Example 7 **Self Tutor**

Expand and simplify: $(x+1)(x-3)(x+2)$

$(x+1)(x-3)(x+2)$
$= (x^2-3x+x-3)(x+2)$ {expanding first two factors}
$= (x^2-2x-3)(x+2)$ {collecting like terms}
$= x^3+2x^2-2x^2-4x-3x-6$ {expanding remaining factors}
$= x^3-7x-6$ {collecting like terms}

2 Expand and simplify:

a $(x+4)(x+3)(x+2)$
b $(x-3)(x-2)(x+4)$
c $(x-3)(x-2)(x-5)$
d $(2x-3)(x+3)(x-1)$
e $(4x+1)(3x-1)(x+1)$
f $(2-x)(3x+1)(x-7)$
g $(x-2)(4-x)(3x+2)$
h $(x+3)(x-1)^2$
i $(x+3)^3$
j $(x-2)^3$

3 State how many terms you would obtain by expanding:

a $(a+b)(c+d)$
b $(a+b+c)(d+e)$
c $(a+b)(c+d+e)$
d $(a+b+c)(d+e+f)$
e $(a+b)(c+d)(e+f)$
f $(a+b+c)(d+e)(f+g)$

4 Expand and simplify:

a $(x^2+3x+1)(x^2-x+3)$
b $(2x^2+x-1)(x^2+3x-2)$
c $(3x^2+x-4)(2x^2-3x+1)$
d $(x^2-3x+2)(x+5)(x-3)$

INVESTIGATION 1 — THE BINOMIAL EXPANSION

Consider $(a+b)^n$ where $n \in \mathbb{Z}^+$.

$\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.

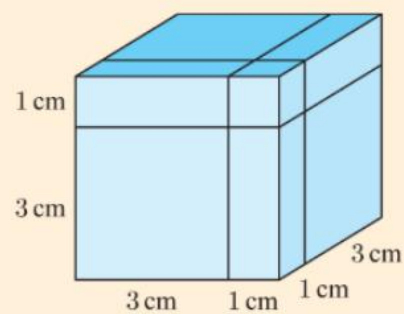
The **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

Part 1: The binomial expansion of $(a+b)^3$

What to do:

1 Find a large potato and cut it to obtain a 4 cm by 4 cm by 4 cm cube.

2 By making 3 cuts parallel to the cube's surfaces, divide the cube into 8 rectangular prisms as shown.



3 How many prisms are:

a 3 by 3 by 3
b 3 by 3 by 1
c 3 by 1 by 1
d 1 by 1 by 1?

4 Now instead of the $4 \text{ cm} \times 4 \text{ cm} \times 4 \text{ cm}$ potato cube, suppose you had a cube with edge length $(a+b)$ cm.

- **a** Explain why the volume of the cube is given by $(a+b)^3$.
- **b** Suppose you made cuts so each edge was divided into a cm and b cm. How many prisms would be:
  - **i** a by a by a
  - **ii** a by a by b
  - **iii** a by b by b
  - **iv** b by b by b?

DEMO

- **c** By adding the volumes of the 8 rectangular prisms, find an expression for the total volume. Hence write down the binomial expansion of $(a+b)^3$.

Part 2: Algebraic expansions

Another method of finding the binomial expansion of $(a+b)^3$ is to expand the brackets:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

What to do:

1 By expanding brackets, show that $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

2 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

- **a** $(x+1)^3$
- **b** $(x+3)^3$
- **c** $(x-2)^3$
- **d** $(x-y)^3$
- **e** $(2+y)^3$
- **f** $(2x-1)^3$
- **g** $(1+3x)^3$
- **h** $(2y-3x)^3$

3 By expanding and simplifying $(a+b)^3(a+b)$, show that $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

4 Use the binomial expansion $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ to expand and simplify:

- **a** $(x+y)^4$
- **b** $(x+2)^4$
- **c** $(x-1)^4$
- **d** $(2x-1)^4$

5 Consider:

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

The expressions on the right hand side of each identity contain the coefficients:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1\end{array}$$

This triangle of numbers is called **Pascal's triangle**.

- **a** Predict the next two rows of Pascal's triangle, and explain how you found them.
- **b** Hence, write down the binomial expansion for:
  - **i** $(a+b)^5$
  - **ii** $(a+b)^6$

C REVISION OF FACTORISATION

Factorisation is the process of writing an expression as a **product** of its **factors**.

Factorisation is the reverse process of expansion, so we use the expansion laws in reverse.

expansion

$$(x-1)(x+7) = x^2+6x-7$$

factorisation

FACTORISING WITH COMMON FACTORS

To factorise an algebraic expression involving a number of terms, we find the HCF of the terms and write it down in front of a set of brackets. We then find the contents of the brackets. This is the reverse of the distributive law.

Example 8 Self Tutor

Fully factorise:

a $6x^2+4x$ **b** $-2x^2-4x$

a
$$\begin{aligned} & 6x^2+4x \\ &= 2\times 3\times x\times x+2\times 2\times x \\ &= 2x(3x+2) \quad \{\text{HCF} = 2x\} \end{aligned}$$

b
$$\begin{aligned} & -2x^2-4x \\ &= -2\times x\times x+-2\times 2\times x \\ &= -2x(x+2) \quad \{\text{HCF} = -2x\} \end{aligned}$$

EXERCISE 3C.1

1 Fully factorise:

a $3a+3b$ **b** $8x-16$ **c** $3p+18$ **d** $7x-14$
e $12+6x$ **f** $ac+bc$ **g** $5a+ab$ **h** $bc-6cd$
i $7x-xy$ **j** $a+ab$ **k** $xy-yz$ **l** $cd-c$

2 Fully factorise:

a $-6a-6b$ **b** $-4-8x$ **c** $-3y-6z$ **d** $-9c-cd$
e $-x-xy$ **f** $-5x^2-20x$ **g** $-9a+9b$ **h** $-3+6b$
i $-8a+4b$ **j** $-7c+cd$ **k** $-a+ab$ **l** $-5x+15x^2$

3 Fully factorise:

a x^2+2x **b** $5x-2x^2$ **c** $4x^2+8x$ **d** $14x-7x^2$
e $6x^2+12x$ **f** x^3+9x^2 **g** x^2y+xy^2 **h** $4x^3-6x^2$
i $9x^3-18xy$ **j** $-12y-3y^2$ **k** $-6x^2+12x$ **l** $-18a^2-9ab$

4 Fully factorise:

a a^3+a^2+a **b** $2a^2+4a+8$ **c** $3a^3-6a^2+9a$

Example 9 Self Tutor

Fully factorise: $-4(a+1)+(a+2)(a+1)$

$$\begin{aligned} & -4(a+1)+(a+2)(a+1) \\ &= (a+1)[-4+(a+2)] \\ &= (a+1)(a-2) \end{aligned}$$

5 Fully factorise:

a $3(x+5)+x(x+5)$
b $a(c-d)+b(c-d)$
c $ab(x-1)+c(x-1)$
d $-5(b+3)+a(b+3)$
e $x(x+2)+(x+2)(x+5)$
f $x(x+4)+x+4$
g $a(x+2)-x-2$
h $y(2+y)-y-2$
i $(x-3)^2+x-3$
j $(x+5)^2+3x+15$
k $2(x-2)^2+4x-8$
l $-(a+b)(a+1)+5(a+b)$
m $(x+4)^2+3(x+4)(x-1)$
n $x(x-1)-6(x-1)(x-5)$

DIFFERENCE OF TWO SQUARES FACTORISATION

$$\boldsymbol{a^2-b^2=(a+b)(a-b)}$$

Example 10 Self Tutor

Fully factorise:

a $4-9y^2$
b $9a-16a^3$

a
$$\begin{aligned} & 4-9y^2 \\ &= 2^2-(3y)^2 \\ &= (2+3y)(2-3y) \end{aligned}$$

b
$$\begin{aligned} & 9a-16a^3 \\ &= a(9-16a^2) \\ &= a(3^2-(4a)^2) \\ &= a(3+4a)(3-4a) \end{aligned}$$

EXERCISE 3C.2

1 Fully factorise:

a x^2-y^2
b p^2-q^2
c x^2-25
d x^2-81
e $4x^2-1$
f $9y^2-16$
g $64-x^2$
h $16-9a^2$
i $1-t^2$

2 Fully factorise:

a $49-4x^2$
b y^2-4x^2
c $4a^2-25b^2$
d $81x^2-16y^2$
e $4x^4-y^2$
f $9a^2b^2-16$
g $(x+3)^2-4$
h $(3x-2)^2-16$
i $(2x-5)^2-(x-4)^2$

3 Fully factorise:

a $2x^2-8$ **b** $3y^2-27$ **c** $2-18x^2$ **d** $4x-9x^3$
e a^3b-ab^3 **f** $50-2x^2y^2$ **g** $9b^3-4b$ **h** x^3-xy^2

PERFECT SQUARES FACTORISATION

$$\boldsymbol{a^2+2ab+b^2=(a+b)^2}$$
$$\boldsymbol{a^2-2ab+b^2=(a-b)^2}$$

Example 11 **Self Tutor**

Factorise:

a $4x^2+4x+1$ **b** $8x^2-24x+18$

a
$$\begin{aligned}&4x^2+4x+1\\&=(2x)^2+2\times 2x\times 1+1^2\\&=(2x+1)^2\end{aligned}$$

b
$$\begin{aligned}&8x^2-24x+18\\&=2(4x^2-12x+9)\\&=2((2x)^2-2\times 2x\times 3+3^2)\\&=2(2x-3)^2\end{aligned}$$

EXERCISE 3C.3

1 Factorise:

a x^2+2x+1 **b** x^2-4x+4 **c** x^2+6x+9
d x^2+4x+4 **e** $x^2-10x+25$ **f** $x^2-20x+100$
g $x^2-8x+16$ **h** $4x^2+28x+49$ **i** $9x^2+30x+25$

2 Factorise:

a $9x^2-6x+1$ **b** $-18x^2+12x-2$ **c** $3x^2+18x+27$
d $-3x^2-18x-27$ **e** $2x^2-20x+50$ **f** $2x^2-16x+32$

D FACTORISING EXPRESSIONS WITH FOUR TERMS

Some expressions with four terms do not have an overall common factor, but can be factorised by pairing the four terms.

For example,
$$\begin{aligned}&\underbrace{ab+ac}+\underbrace{bd+cd}\\&=a(b+c)+d(b+c) &&\{\text{factorising each pair separately}\}\\&=(b+c)(a+d) &&\{\text{removing common factor } (b+c)\}\end{aligned}$$

Example 12 **Self Tutor**

Factorise: $3ab+d+3ad+b$

$$\begin{aligned}&3ab+d+3ad+b\\&=\underbrace{3ab+b}+\underbrace{3ad+d} &&\{\text{putting terms containing } b \text{ together}\}\\&=b(3a+1)+d(3a+1) &&\{\text{factorising each pair}\}\\&=(3a+1)(b+d) &&\{(3a+1) \text{ is a common factor}\}\end{aligned}$$

EXERCISE 3D

1 Factorise:

a $2a+2+ab+b$ **b** $4d+ac+ad+4c$ **c** $ab+6+2b+3a$

d $mn+3p+np+3m$ **e** $2xy-5+10y-x$ **f** $6a-bc-2ac+3b$

Example 13 Self Tutor

Factorise:

a $x^2+2x+5x+10$ **b** $x^2+3x-4x-12$

a $\underbrace{x^2+2x}+\underbrace{5x+10}$

$=x(x+2)+5(x+2)$ {factorising each pair}

$=(x+2)(x+5)$ {$(x+2)$ is a common factor}

b $\underbrace{x^2+3x}\underbrace{-4x-12}$

$=x(x+3)-4(x+3)$ {factorising each pair}

$=(x+3)(x-4)$ {$(x+3)$ is a common factor}

2 Factorise:

a $x^2+2x+4x+8$ **b** $x^2+3x+7x+21$ **c** $x^2+5x+4x+20$

d $2x^2+x+6x+3$ **e** $3x^2+2x+12x+8$ **f** $20x^2+12x+5x+3$

3 Factorise:

a $x^2-4x+5x-20$ **b** $x^2-7x+2x-14$ **c** $x^2-3x-2x+6$

d $x^2-5x-3x+15$ **e** $x^2+7x-8x-56$ **f** $2x^2+x-6x-3$

g $3x^2+2x-12x-8$ **h** $4x^2-3x-8x+6$ **i** $9x^2+2x-9x-2$

E FACTORISING QUADRATIC TRINOMIALS

A **quadratic trinomial** is an algebraic expression of the form ax^2+bx+c where x is a variable and a, b, c are constants, $a \neq 0$.

Consider the expansion of the product $(x+2)(x+5)$:

$$\begin{aligned}(x+2)(x+5) &= x^2+5x+2x+2\times 5 \qquad \text{\{using FOIL\}}\\ &= x^2+[5+2]x+[2\times 5]\\ &= x^2+[\textbf{sum} \text{ of 2 and 5}]x+[\textbf{product} \text{ of 2 and 5}]\\ &= x^2+7x+10\end{aligned}$$

$$\boldsymbol{x^2+px+q=(x+a)(x+b)}$$

where a and b are two numbers whose sum is p, and whose product is q.

So, if we want to factorise the quadratic trinomial $x^2+7x+10$ into $(x+...)(x+...)$ we must find two numbers to fill the vacant places which have a *sum* of 7 and a *product* of 10. The numbers are 2 and 5, so $x^2+7x+10=(x+2)(x+5)$.

Example 14 — Self Tutor

Factorise:

a $x^2 - 7x + 12$ **b** $x^2 - 2x - 15$

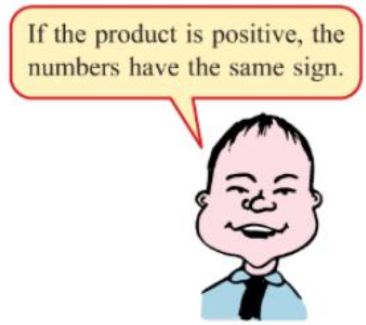
a We need two numbers with sum -7 and product 12.
The numbers are -3 and -4.
$\therefore \quad x^2 - 7x + 12 = (x-3)(x-4)$

b We need two numbers with sum -2 and product -15.
The numbers are -5 and 3.
$\therefore \quad x^2 - 2x - 15 = (x-5)(x+3)$

EXERCISE 3E

1 Factorise:

a $x^2 + 5x + 4$ **b** $x^2 + 7x + 10$ **c** $x^2 + 10x + 21$
d $x^2 + 15x + 54$ **e** $x^2 + 12x + 20$ **f** $x^2 + 9x + 18$
g $x^2 + 14x + 24$ **h** $x^2 + 15x + 36$ **i** $x^2 + 19x + 48$

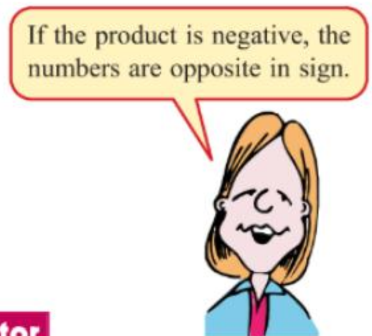


2 Factorise:

a $x^2 - 5x + 4$ **b** $x^2 - 4x + 3$ **c** $x^2 - 5x + 6$
d $x^2 - 13x + 22$ **e** $x^2 - 15x + 56$ **f** $x^2 - 16x + 48$
g $x^2 - 16x + 28$ **h** $x^2 - 25x + 24$ **i** $x^2 - 15x + 36$

3 Factorise:

a $x^2 - 8x - 9$ **b** $x^2 + 4x - 21$ **c** $x^2 - x - 6$
d $x^2 - 3x - 18$ **e** $x^2 + 5x - 24$ **f** $x^2 - 11x - 12$
g $x^2 + 3x - 54$ **h** $x^2 + x - 56$ **i** $x^2 - 3x - 28$
j $x^2 - x - 20$ **k** $x^2 - 2x - 63$ **l** $x^2 + 7x - 60$



Example 15 — Self Tutor

Fully factorise by first removing a common factor:

a $3x^2 + 6x - 72$ **b** $77 + 4x - x^2$

a
$$\begin{aligned} & 3x^2 + 6x - 72 \\ &= 3(x^2 + 2x - 24) && \{3 \text{ is a common factor}\} \\ &= 3(x+6)(x-4) && \{\text{sum} = 2, \quad \text{product} = -24 \\ & && \therefore \text{ the numbers are } 6 \text{ and } -4\} \end{aligned}$$

b
$$\begin{aligned} & 77 + 4x - x^2 \\ &= -x^2 + 4x + 77 && \{\text{writing in descending powers of } x\} \\ &= -1(x^2 - 4x - 77) && \{-1 \text{ is a common factor}\} \\ &= -(x-11)(x+7) && \{\text{sum} = -4, \text{ product} = -77 \\ & && \therefore \text{ the numbers are } -11 \text{ and } 7\} \end{aligned}$$

4 Fully factorise by first removing a common factor:

a $2x^2 + 10x + 8$ **b** $3x^2 - 21x + 18$ **c** $2x^2 + 14x + 24$
d $5x^2 - 30x - 80$ **e** $4x^2 - 8x - 12$ **f** $3x^2 - 42x + 99$
g $2x^2 - 2x - 180$ **h** $3x^2 - 6x - 24$ **i** $2x^2 + 18x + 40$

5 Fully factorise:

a $-x^2 - 3x + 54$ **b** $-x^2 - 7x - 10$ **c** $-x^2 - 10x - 21$
d $-x^2 + 2x + 48$ **e** $6x - x^2 - 9$ **f** $30x - 3x^2 - 63$

F FACTORISATION OF $ax^2 + bx + c,\ a \neq 1$

In this Section we will learn how to factorise quadratic trinomials where the coefficient of x^2 is not 1, and we cannot remove a common factor.

Consider the quadratic trinomial $4x^2 + 11x + 6$.

Using the FOIL rule, we observe that
$$\begin{aligned} &(4x+3)(x+2) \\ &= 4x^2 + 8x + 3x + 6 \\ &= 4x^2 + 11x + 6 \end{aligned}$$

We will now *reverse* the process to factorise $4x^2 + 11x + 6$:

$$\begin{aligned} &4x^2 + 11x + 6 \\ &= 4x^2 + 8x + 3x + 6 && \{\text{'splitting' the middle term}\} \\ &= (4x^2 + 8x) + (3x + 6) && \{\text{grouping in pairs}\} \\ &= 4x(x+2) + 3(x+2) && \{\text{factorising each pair separately}\} \\ &= (4x+3)(x+2) && \{\text{completing the factorisation}\} \end{aligned}$$

But how do we know how to correctly 'split' the middle term? How do we know that $11x$ should be written as $8x + 3x$ rather than $6x + 5x$ or $10x + x$?

INVESTIGATION 2 'SPLITTING' THE MIDDLE TERM

Consider the general quadratic trinomial $ax^2 + bx + c$.

Suppose we 'split' the middle term into $px + qx$, so $ax^2 + bx + c = ax^2 + px + qx + c$.

What to do:

1 Explain why $p + q = b$.

2 Show that $ax^2 + bx + c = x(ax + p) + (qx + c)$.

3 We can only factorise this expression further if the two terms have a common factor. This means that $ax + p = k(qx + c)$ for some k.

  a By equating coefficients, show that $kq = a$ and $kc = p$.
  b Hence, show that $pq = ac$.

This tells us that factorisation by 'splitting' the middle term only works if we can choose p and q such that $p + q = b$ and $pq = ac$.

The following procedure is used to factorise $ax^2 + bx + c$ by 'splitting' the middle term:

Step 1: Find two numbers p and q whose sum is b and whose product is ac.

Step 2: Replace bx by $px + qx$.

Step 3: Complete the factorisation.

Example 16 **Self Tutor**

Factorise:

a $3x^2 + 17x + 10$ **b** $6x^2 - 11x - 10$

a For $3x^2 + 17x + 10$, $ac = 3 \times 10 = 30$ and $b = 17$.
We need two numbers with sum 17 and product 30. These are 2 and 15.

$$\begin{aligned} \therefore\ 3x^2 + 17x + 10 &= 3x^2 + 2x + 15x + 10 \\ &= x(3x+2) + 5(3x+2) \\ &= (3x+2)(x+5) \end{aligned}$$

b For $6x^2 - 11x - 10$, $ac = 6 \times -10 = -60$ and $b = -11$.
We need two numbers with sum -11 and product -60. These are -15 and 4.

$$\begin{aligned} \therefore\ 6x^2 - 11x - 10 &= 6x^2 - 15x + 4x - 10 \\ &= 3x(2x-5) + 2(2x-5) \\ &= (2x-5)(3x+2) \end{aligned}$$

EXERCISE 3F

1 Consider the quadratic trinomial $3x^2 + 7x + 2$.

a Factorise the expression by 'splitting' the middle term into:

i $+6x + x$ **ii** $+x + 6x$

b Are your factorisations in **a** equivalent?

2 Fully factorise:

a $2x^2 + 5x + 3$ **b** $2x^2 + 13x + 18$ **c** $7x^2 + 9x + 2$ **d** $3x^2 + 13x + 4$
e $3x^2 + 8x + 4$ **f** $3x^2 + 16x + 21$ **g** $8x^2 + 14x + 3$ **h** $21x^2 + 17x + 2$
i $6x^2 + 5x + 1$ **j** $6x^2 + 19x + 3$ **k** $10x^2 + 17x + 3$ **l** $14x^2 + 37x + 5$

3 Consider the quadratic trinomial $4x^2 + 4x - 3$.

a Factorise the expression by 'splitting' the middle term into:

i $+6x - 2x$ **ii** $-2x + 6x$

b Are your factorisations in **a** equivalent?

4 Fully factorise:

a $2x^2 - 9x - 5$ **b** $3x^2 + 5x - 2$ **c** $3x^2 - 5x - 2$ **d** $2x^2 + 3x - 2$
e $2x^2 + 3x - 5$ **f** $5x^2 - 8x + 3$ **g** $11x^2 - 9x - 2$ **h** $2x^2 - 3x - 9$
i $3x^2 - 17x + 10$ **j** $5x^2 - 13x - 6$ **k** $3x^2 + 10x - 8$ **l** $2x^2 + 17x - 9$
m $2x^2 + 9x - 18$ **n** $15x^2 + x - 2$ **o** $21x^2 - 62x - 3$

Example 17 **Self Tutor**

Fully factorise: $-5x^2 - 7x + 6$

We remove -1 as a common factor first.

$$\begin{aligned}
&-5x^2 - 7x + 6 \\
&= -1[5x^2 + 7x - 6] \\
&= -[5x^2 + 10x - 3x - 6] \\
&= -[5x(x+2) - 3(x+2)] \\
&= -[(x+2)(5x-3)] \\
&= -(x+2)(5x-3)
\end{aligned}$$

For $5x^2 + 7x - 6$, $ac = -30$ and $b = 7$. The two numbers with product -30 and sum 7 are 10 and -3.

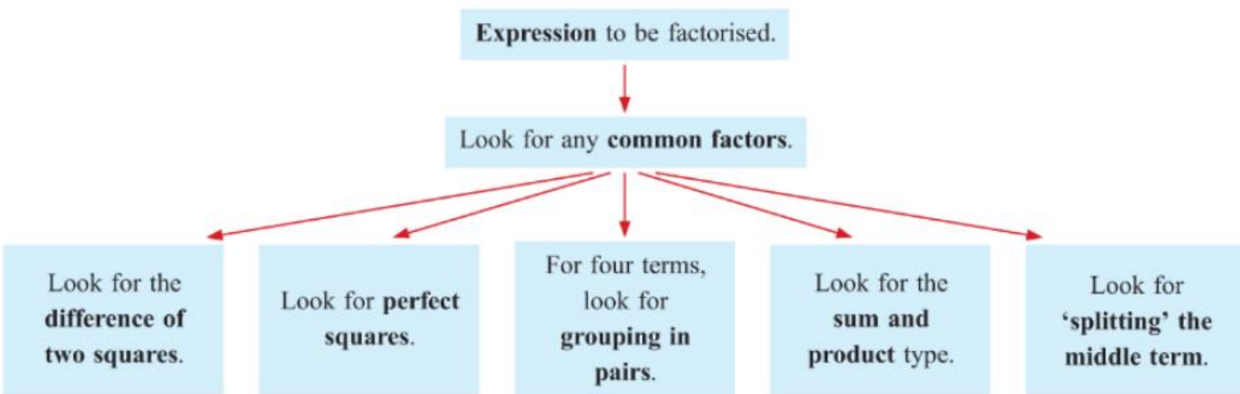
5 Fully factorise by first removing -1 as a common factor:

- **a** $-3x^2 - x + 14$
- **b** $-5x^2 + 11x - 2$
- **c** $-4x^2 - 9x + 9$
- **d** $-9x^2 + 12x - 4$
- **e** $-8x^2 - 14x - 3$
- **f** $-12x^2 + 16x + 3$

G MISCELLANEOUS FACTORISATION

In the following **Exercise** you will need to determine which factorisation method to use.

This flowchart may prove useful:



EXERCISE 3G

1 Fully factorise:

- **a** $3x^2 + 2x$
- **b** $x^2 - 81$
- **c** $2p^2 + 8$
- **d** $3b^2 - 75$
- **e** $2x^2 - 32$
- **f** $n^4 - 4n^2$
- **g** $x^2 - 8x - 9$
- **h** $d^2 + 6d - 7$
- **i** $x^2 + 8x - 9$
- **j** $4t + 8t^2$
- **k** $4x^2 + 12x + 5$
- **l** $2g^2 - 12g - 110$
- **m** $4a^2 - 9d^2$
- **n** $5a^2 - 5a - 10$
- **o** $2c^2 - 8c + 6$
- **p** $2x^2 + 17x + 21$

2 Fully factorise:

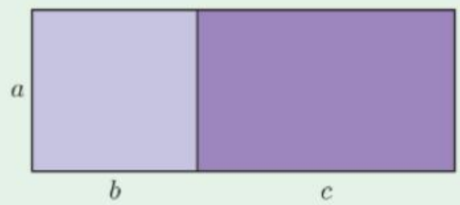
- **a** $7x - 35y$
- **b** $2g^2 - 8$
- **c** $-5x^2 - 10x$
- **d** $m^2 + 3mp$
- **e** $a^2 + 8a + 15$
- **f** $m^2 - 6m + 9$
- **g** $5x^2 + 5xy - 5x^2y$
- **h** $xy + 2x + 2y + 4$
- **i** $y^2 + 5y - 9y - 45$

3 Fully factorise:

a $2x^2 + 10x + x + 5$ b $3y^2 - 147$ c $6x^2 - 29x - 5$
d $4c^2 - 1$ e $3x^2 + 3x - 36$ f $2bx - 6b + 10x - 30$
g $12x^2 + 13x + 3$ h $-2x^2 - 6 + 8x$ i $16x^2 + 8x + 1$
j $4x^2 - 2x^3 - 2x$ k $(a+b)^2 - 9$ l $12x^2 - 38x + 6$

REVIEW SET 3A

1 Use the diagram alongside to show that $a(b+c) = ab + ac$.



2 Expand and simplify:

a $(x+5)(x-6)$ b $(2x+5)(3x-1)$
c $(x+3)(x+2) - (2x-1)(x-6)$

3 Fully factorise:

a $7x^2 - 4x$ b $x^3 + 5x^2 - 6x$ c $x(x-8) + 5(x-8)$

4 Expand and simplify:

a $(x+5)(x-2)(x+1)$ b $(2x-3)(x^2+4x+2)$

5 Fully factorise:

a $16 - 9m^2$ b $x^3 - 81x$ c $(x+7)^2 - 25$

6 Expand and simplify:

a $(t+7)(t-7)$ b $(2y+5)(2y-5)$ c $(2m-5n)^2$

7 Fully factorise:

a $2x^2 + 20x + 50$ b $2b - dc + 2d - bc$

8 Fully factorise:

a $x^2 + 7x - 18$ b $3x^2 - 9x - 30$ c $64 - 2x^2 + 8x$

9 Fully factorise:

a $8x^2 + 10x + 3$ b $5x^2 - 13x + 6$ c $-9x^2 + 3x + 2$

10 a Show that $(2x+9)^2 - (x-3)^2 = 3x^2 + 42x + 72$ by expanding the LHS.
b Factorise $3x^2 + 42x + 72$ by first taking out a common factor.
c Factorise $(2x+9)^2 - (x-3)^2$ using the difference of two squares.

REVIEW SET 3B

1 Expand and simplify:

a $5(4x-5)$ b $-4x(x-3)$ c $2(x+6) + x(3x-7)$

2 Expand and simplify:

a $x(x^2-3)+5(x-4)$ **b** $(a+b)(a-b)-(a+2b)(a-2b)$

3 Fully factorise:

a $2x^2-98$ **b** $(3x+1)^2-(x-4)^2$

4 Answer the **Opening Problem** on page **48**.

5 Fully factorise:

a $x^2+3x-54$ **b** $3x^2+24x+48$

6 How many terms would you obtain by expanding $(a+b+c+d)(e+f)(g+h)$?

7 Fully factorise:

a $x^2-5x-66$ **b** $2x^2+20x-78$ **c** $4x^2-8x-21$

8 Expand and simplify:

a $(3x^2-5)^2$ **b** $(x^2-x+4)(x^2+2x+3)$

9 Fully factorise:

a $-x^2+x+12$ **b** $-6x^2-5x+50$

10 Consider factorising the expression $6x^2+17x+12$.

a Explain why the middle term $17x$ should be 'split' into $9x$ and $8x$.

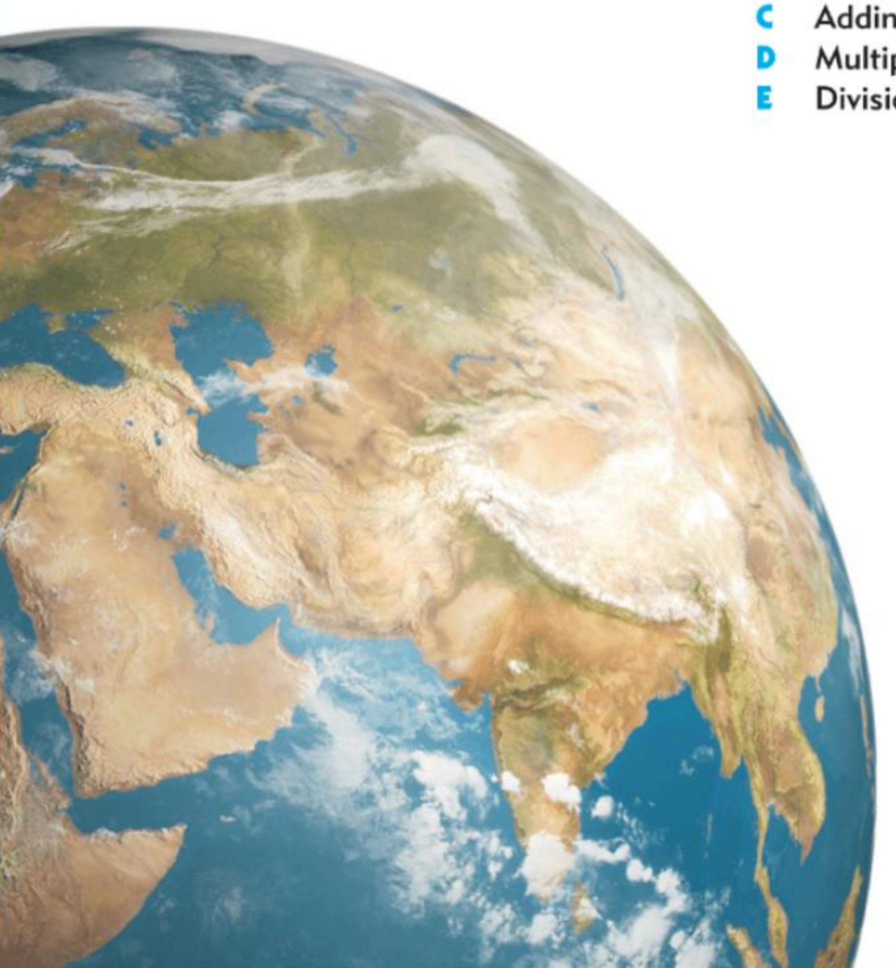
b Factorise $6x^2+17x+12$ by writing $17x$ as $9x+8x$.

c Now factorise $6x^2+17x+12$ by writing $17x$ as $8x+9x$. Check that you get the same answer as in **b**.

Chapter 4

Radicals and surds

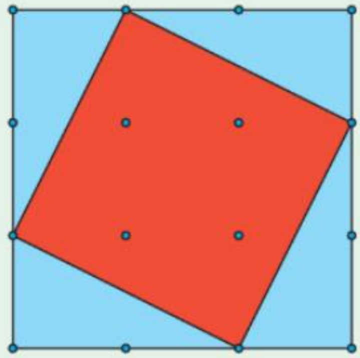
Contents:



OPENING PROBLEM

Pamela's students claim that since an irrational number cannot be written as a fraction, it cannot be placed on a number line.

To demonstrate that an irrational length can be represented, and therefore placed on a number line, Pamela draws this figure.



Things to think about:

a Find:
  i the area of the large square
  ii the total area of the blue triangles.

b Using your answers to **a**, find the area of the red square.

c Hence explain why the side length of the red square is $\sqrt{5}$ units.

d What is the perimeter of the red square?

A RADICALS

In previous years, we have encountered values such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt[3]{8}$. These numbers are known as *radicals*.

A **radical** is a number that is written using the radical sign $\sqrt{}$.

Radicals occur frequently in mathematics, often as solutions to equations involving squared terms. We will see a typical example of this in **Chapter 6** when we study Pythagoras' theorem.

SQUARE ROOTS

The **square root of** $\boldsymbol{a}$, written $\sqrt{a}$, is the *positive* solution of the equation $x^2 = a$.
$\sqrt{a}$ is the *positive* number which obeys the rule $\sqrt{a} \times \sqrt{a} = a$.

For example, $\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{3} \times \sqrt{3} = 3$, $\sqrt{4} \times \sqrt{4} = 4$, and so on.

We know that the square of any real number is non-negative. This means that:

$\sqrt{a}$ is real only if $a \geqslant 0$.

HIGHER ROOTS

In this course we will concentrate mainly on square roots, but it is important to understand that other radicals exist. For example:

The **cube root of** a, written $\sqrt[3]{a}$, satisfies the rule $\left(\sqrt[3]{a}\right)^3 = a$.
If $a > 0$ then $\sqrt[3]{a} > 0$.
If $a < 0$ then $\sqrt[3]{a} < 0$.

We can define higher roots in a similar way.

RATIONAL AND IRRATIONAL RADICALS

In **Chapter 2** we saw that the set of **real numbers** $\mathbb{R}$ can be divided into the set of **rational numbers** $\mathbb{Q}$, and the set of **irrational numbers** $\mathbb{Q}'$.

Remember that:

An **irrational number** is a real number which cannot be written in the form $\frac{p}{q}$, where p and q are integers, $q \neq 0$.

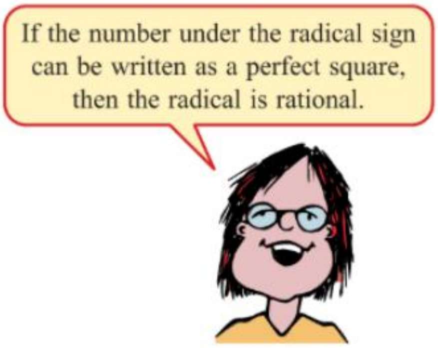
Radical numbers may be rational or irrational. An irrational radical is called a **surd**.

Examples of rational radicals include:

$$\sqrt{4} = 2 = \tfrac{2}{1}$$

$$\sqrt{\tfrac{9}{16}} = \sqrt{\left(\tfrac{3}{4}\right)^2} = \tfrac{3}{4}$$

Two examples of surds are $\sqrt{2} \approx 1.414\,214$ and $\sqrt{3} \approx 1.732\,051$.



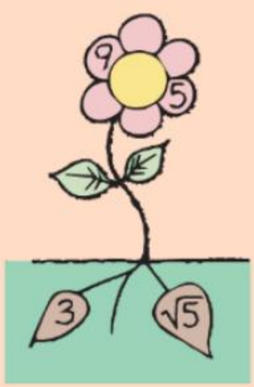
HISTORICAL NOTE

When the Golden Age of the Greeks was past, the writings of the Greeks were preserved, translated into Arabic and extended by Arabic mathematicians in the regions currently known as Iraq and Iran and also in Moslem Spain. The word *surd* came about because of an error in translation.

The Greek word *alogos* meaning 'irrational', or 'without reason', was translated as the Arabic word *asamm* which means 'irrational', but also means 'deaf'. Thus rational and irrational numbers were called 'audible' and 'inaudible' numbers respectively by the Arabic mathematician **Al-Khwarizmi**, from Persia, around 825 AD.

This later led to the Arabic word *asamm* meaning 'deaf' or 'dumb' for irrational numbers being translated into Latin as *surdus* meaning 'deaf' or 'mute'. The European mathematician, Gherardo of Cremona (c. 1150), adopted the word surd.

The origin of the root symbol $\sqrt{}$ is not clear. Some sources suggest that the symbol was first used by Arabic mathematicians. It is believed that the modern square root symbol developed from the letter r which is the first letter in the Latin word radix, meaning 'root', from which we get the word *radical*.



SIMPLIFYING RADICALS

In previous years we have established that:

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ for $a \geqslant 0,\ b \geqslant 0$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0,\ b > 0$

Example 1 Self Tutor

Simplify:

a $\left(\sqrt{5}\right)^2$

b $\left(\frac{1}{\sqrt{5}}\right)^2$

a $\left(\sqrt{5}\right)^2$
$= \sqrt{5} \times \sqrt{5}$
$= 5$

b $\left(\frac{1}{\sqrt{5}}\right)^2$
$= \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}}$
$= \frac{1}{5}$

Example 2 Self Tutor

Simplify:

a $\left(2\sqrt{5}\right)^3$

b $-2\sqrt{5} \times 3\sqrt{5}$

a $\left(2\sqrt{5}\right)^3$
$= 2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$
$= 2 \times 2 \times 2 \times \sqrt{5} \times \sqrt{5} \times \sqrt{5}$
$= 8 \times 5 \times \sqrt{5}$
$= 40\sqrt{5}$

b $-2\sqrt{5} \times 3\sqrt{5}$
$= -2 \times 3 \times \sqrt{5} \times \sqrt{5}$
$= -6 \times 5$
$= -30$

EXERCISE 4A

1 Simplify:

a $\left(\sqrt{7}\right)^2$ **b** $\left(\sqrt{13}\right)^2$ **c** $\left(\sqrt{15}\right)^2$ **d** $\left(\sqrt{24}\right)^2$

e $\left(\sqrt{2}\right)^3$ **f** $\left(\sqrt{2}\right)^4$ **g** $\left(\sqrt{2}\right)^5$ **h** $\left(\frac{1}{\sqrt{2}}\right)^2$

i $\left(\frac{1}{\sqrt{3}}\right)^2$ **j** $\left(\frac{2}{\sqrt{11}}\right)^2$ **k** $\left(\frac{\sqrt{3}}{\sqrt{17}}\right)^2$ **l** $\left(\sqrt{\frac{2}{23}}\right)^2$

2 Simplify:

a $\left(\sqrt[3]{2}\right)^3$ **b** $\left(\sqrt[3]{-5}\right)^3$ **c** $\left(\frac{1}{\sqrt[3]{5}}\right)^3$

3 Simplify:

a $4\sqrt{3} \times \sqrt{3}$ **b** $3\sqrt{2} \times \sqrt{2}$ **c** $\sqrt{5} \times 6\sqrt{5}$

d $3\sqrt{2} \times 4\sqrt{2}$ **e** $-2\sqrt{3} \times 5\sqrt{3}$ **f** $3\sqrt{5} \times \left(-2\sqrt{5}\right)$

g $-2\sqrt{2} \times \left(-3\sqrt{2}\right)$ **h** $\left(3\sqrt{2}\right)^2$ **i** $\left(3\sqrt{2}\right)^3$

j $\left(4\sqrt{3}\right)^2$ **k** $\left(4\sqrt{3}\right)^3$ **l** $\left(2\sqrt{2}\right)^4$

m $\sqrt{5} \times \left(3\sqrt{5}\right)^2$ **n** $-2\sqrt{3} \times \left(5\sqrt{3}\right)^2$ **o** $\left(2\sqrt{2}\right)^3 \times \left(-7\sqrt{2}\right)$

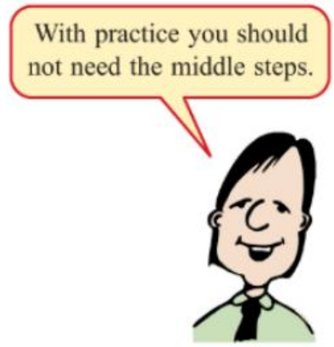
Example 3 — Self Tutor

Write in simplest form:

a $\sqrt{3}\times\sqrt{2}$ **b** $2\sqrt{5}\times 3\sqrt{2}$

a $\sqrt{3}\times\sqrt{2}$
$=\sqrt{3\times 2}$
$=\sqrt{6}$

b $2\sqrt{5}\times 3\sqrt{2}$
$=2\times 3\times\sqrt{5}\times\sqrt{2}$
$=6\times\sqrt{5\times 2}$
$=6\sqrt{10}$



4 Simplify:

a $\sqrt{2}\times\sqrt{5}$ **b** $\sqrt{3}\times\sqrt{7}$ **c** $\sqrt{3}\times\sqrt{11}$
d $\sqrt{7}\times\sqrt{7}$ **e** $\sqrt{3}\times 2\sqrt{3}$ **f** $2\sqrt{2}\times\left(-\sqrt{5}\right)$
g $3\sqrt{3}\times 2\sqrt{2}$ **h** $2\sqrt{3}\times 3\sqrt{5}$ **i** $\sqrt{2}\times\sqrt{3}\times\sqrt{5}$
j $\sqrt{3}\times\sqrt{2}\times 2\sqrt{2}$ **k** $-3\sqrt{2}\times\left(\sqrt{2}\right)^3$ **l** $\left(3\sqrt{2}\right)^3\times\left(\sqrt{3}\right)^3$

Example 4 — Self Tutor

Simplify: **a** $\dfrac{\sqrt{32}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{12}}{2\sqrt{3}}$

a $\dfrac{\sqrt{32}}{\sqrt{2}}$
$=\sqrt{\frac{32}{2}}$
$=\sqrt{16}$
$=4$

b $\dfrac{\sqrt{12}}{2\sqrt{3}}$
$=\frac{1}{2}\sqrt{\frac{12}{3}}$ {using $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$}
$=\frac{1}{2}\sqrt{4}$
$=\frac{1}{2}\times 2$
$=1$

5 Simplify:

a $\dfrac{\sqrt{8}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{\sqrt{8}}$ **c** $\dfrac{\sqrt{18}}{\sqrt{2}}$ **d** $\dfrac{\sqrt{2}}{\sqrt{18}}$
e $\dfrac{\sqrt{20}}{\sqrt{5}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{20}}$ **g** $\dfrac{\sqrt{27}}{\sqrt{3}}$ **h** $\dfrac{\sqrt{18}}{\sqrt{3}}$
i $\dfrac{\sqrt{3}}{\sqrt{30}}$ **j** $\dfrac{\sqrt{50}}{\sqrt{2}}$ **k** $\dfrac{2\sqrt{6}}{\sqrt{24}}$ **l** $\dfrac{5\sqrt{75}}{\sqrt{3}}$

6 Simplify:

a $\sqrt{\frac{1}{25}}$ **b** $\sqrt{\frac{16}{9}}$ **c** $\sqrt{20\frac{1}{4}}$ **d** $\sqrt{3\frac{1}{16}}$

7 **a** Is $\sqrt{9}+\sqrt{16}=\sqrt{9+16}$? Is $\sqrt{25}-\sqrt{16}=\sqrt{25-16}$?
b Are $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$ and $\sqrt{a}-\sqrt{b}=\sqrt{a-b}$ possible laws for radicals?

8 **a** Prove that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for all positive numbers a and b.

Hint: Consider $\left(\sqrt{a}\sqrt{b}\right)^2$ and $\left(\sqrt{ab}\right)^2$.

b Prove that $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0$ and $b > 0$.

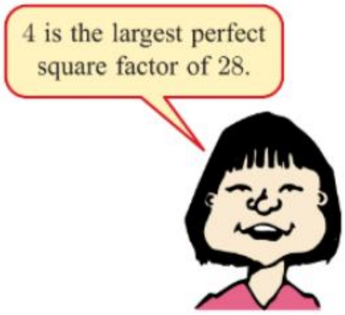
B SIMPLEST RADICAL FORM

A radical is in **simplest form** when the number under the radical sign is the smallest possible integer.

Example 5 Self Tutor

Write $\sqrt{28}$ in simplest radical form.

$$\begin{aligned} &\sqrt{28} \\ &= \sqrt{4 \times 7} \\ &= \sqrt{4} \times \sqrt{7} \\ &= 2\sqrt{7} \end{aligned}$$



EXERCISE 4B

1 Write in simplest radical form:

a $\sqrt{24}$ **b** $\sqrt{50}$ **c** $\sqrt{54}$ **d** $\sqrt{40}$

e $\sqrt{56}$ **f** $\sqrt{63}$ **g** $\sqrt{52}$ **h** $\sqrt{44}$

i $\sqrt{60}$ **j** $\sqrt{90}$ **k** $\sqrt{96}$ **l** $\sqrt{68}$

m $\sqrt{700}$ **n** $\sqrt{175}$ **o** $\sqrt{128}$ **p** $\sqrt{162}$

2 Write in simplest radical form:

a $\sqrt{\frac{5}{9}}$ **b** $\sqrt{\frac{18}{4}}$ **c** $\sqrt{\frac{12}{16}}$ **d** $\sqrt{\frac{75}{36}}$

Example 6 Self Tutor

Write $\dfrac{9-\sqrt{75}}{6}$ in simplest radical form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}$, $n \in \mathbb{Z}$.

$$\begin{aligned} \frac{9-\sqrt{75}}{6} &= \frac{9}{6} - \frac{\sqrt{25 \times 3}}{6} \\ &= \tfrac{3}{2} - \tfrac{5}{6}\sqrt{3} \end{aligned}$$

3 Write in simplest radical form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}$, $n \in \mathbb{Z}$:

a $\dfrac{4+\sqrt{8}}{2}$ **b** $\dfrac{6-\sqrt{12}}{2}$ **c** $\dfrac{4+\sqrt{20}}{4}$ **d** $\dfrac{8-\sqrt{32}}{4}$

e $\dfrac{12+\sqrt{72}}{6}$ **f** $\dfrac{18+\sqrt{27}}{6}$ **g** $\dfrac{14-\sqrt{60}}{8}$ **h** $\dfrac{5-\sqrt{200}}{10}$

C ADDING AND SUBTRACTING RADICALS

We can add and subtract 'like' radicals in the same way as we do 'like' terms in algebra.

For example:

- just as $3a + 2a = 5a$, $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$
- just as $6b - 4b = 2b$, $6\sqrt{3} - 4\sqrt{3} = 2\sqrt{3}$.

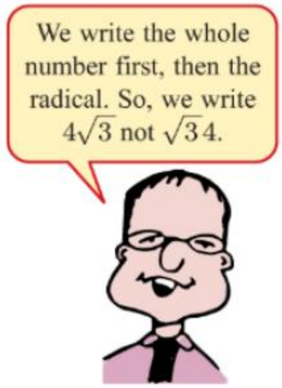
Example 7 Self Tutor

Simplify:

a $3\sqrt{2} - 4\sqrt{2}$ **b** $5\sqrt{3} + 2\sqrt{5} - \sqrt{3} + 7\sqrt{5}$

a
$$\begin{aligned} &3\sqrt{2} - 4\sqrt{2} \\ &= -1\sqrt{2} \\ &= -\sqrt{2} \end{aligned}$$

b
'like' radicals
$$5\sqrt{3} + 2\sqrt{5} - \sqrt{3} + 7\sqrt{5}$$
'like' radicals
$$= 4\sqrt{3} + 9\sqrt{5}$$



EXERCISE 4C

1 Simplify:

a $3\sqrt{2} + 7\sqrt{2}$ **b** $11\sqrt{3} - 8\sqrt{3}$ **c** $6\sqrt{5} - 7\sqrt{5}$

d $-\sqrt{10} + 2\sqrt{10}$ **e** $\sqrt{6} + 7\sqrt{6} - 3\sqrt{6}$ **f** $9\sqrt{15} - 4\sqrt{15} - 11\sqrt{15}$

2 Simplify:

a $5\sqrt{2} - \sqrt{3} + \sqrt{2} - \sqrt{3}$ **b** $4\sqrt{7} + 5\sqrt{6} + 3\sqrt{7} - 2\sqrt{6}$

c $9\sqrt{10} - 5\sqrt{5} + 8\sqrt{5} - \sqrt{10}$ **d** $11\sqrt{3} - 8\sqrt{13} + \sqrt{13} - 13\sqrt{3}$

e $\sqrt{7} + 5\sqrt{11} + 9\sqrt{7} - 4\sqrt{7}$ **f** $-6\sqrt{14} - 3\sqrt{14} - 2\sqrt{6} + 10\sqrt{14}$

g $14 + 6\sqrt{17} - 8\sqrt{17} - 3$ **h** $-8 - \sqrt{15} + 7\sqrt{15} - 13$

Example 8 Self Tutor

Simplify: $\dfrac{\sqrt{7}}{2} + \dfrac{\sqrt{7}}{3}$

$$\begin{aligned} &\frac{\sqrt{7}}{2} + \frac{\sqrt{7}}{3} \qquad \{\text{LCD} = 6\} \\ &= \frac{\sqrt{7} \times 3}{2 \times 3} + \frac{\sqrt{7} \times 2}{3 \times 2} \\ &= \frac{3\sqrt{7}}{6} + \frac{2\sqrt{7}}{6} \\ &= \frac{3\sqrt{7} + 2\sqrt{7}}{6} \\ &= \frac{5\sqrt{7}}{6} \end{aligned}$$

3 Simplify:

a $\frac{\sqrt{5}}{3}+\frac{\sqrt{5}}{4}$ **b** $\frac{\sqrt{6}}{2}-\frac{\sqrt{6}}{7}$ **c** $\frac{5\sqrt{3}}{6}+\frac{\sqrt{3}}{8}$

d $\frac{2\sqrt{11}}{9}-\frac{8\sqrt{11}}{15}$ **e** $\frac{\sqrt{10}}{2}+\frac{\sqrt{10}}{3}+\frac{\sqrt{10}}{4}$ **f** $\sqrt{2}+\frac{5\sqrt{2}}{14}-\frac{7\sqrt{2}}{4}$

4 Show that:

a $\sqrt{20}+\sqrt{5}=\sqrt{45}$ **b** $\sqrt{147}-\sqrt{75}=\sqrt{12}$

5 Answer the **Opening Problem** on page **66**.

D MULTIPLICATIONS INVOLVING RADICALS

The rules for expanding expressions containing radicals are identical to those for ordinary algebra.

$$\begin{aligned} a(b+c)&=ab+ac\\ (a+b)(c+d)&=ac+ad+bc+bd\\ (a+b)^2&=a^2+2ab+b^2\\ (a+b)(a-b)&=a^2-b^2\end{aligned}$$

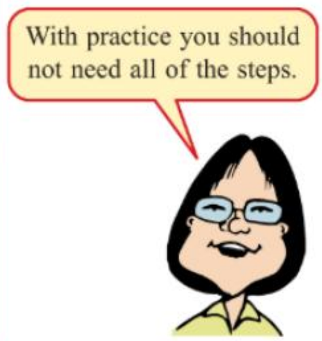
Example 9 Self Tutor

Expand and simplify:

a $\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)$ **b** $\sqrt{3}\left(6-2\sqrt{3}\right)$

a
$$\begin{aligned}&\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)\\ &=\sqrt{2}\times\sqrt{2}+\sqrt{2}\times\sqrt{3}\\ &=2+\sqrt{6}\end{aligned}$$

b
$$\begin{aligned}&\sqrt{3}\left(6-2\sqrt{3}\right)\\ &=\sqrt{3}\times 6+\sqrt{3}\times(-2\sqrt{3})\\ &=6\sqrt{3}-6\end{aligned}$$



EXERCISE 4D

1 Expand and simplify:

a $\sqrt{2}\left(\sqrt{5}+\sqrt{2}\right)$ **b** $\sqrt{2}\left(3-\sqrt{2}\right)$ **c** $\sqrt{3}\left(\sqrt{3}+1\right)$ **d** $\sqrt{3}\left(1-\sqrt{3}\right)$

e $\sqrt{7}\left(7-\sqrt{7}\right)$ **f** $\sqrt{5}\left(2-\sqrt{5}\right)$ **g** $\sqrt{11}\left(2\sqrt{11}-1\right)$ **h** $\sqrt{6}\left(1-2\sqrt{6}\right)$

i $\sqrt{3}\left(\sqrt{3}+\sqrt{2}-1\right)$ **j** $2\sqrt{3}\left(\sqrt{3}-\sqrt{5}\right)$ **k** $2\sqrt{5}\left(3-\sqrt{5}\right)$ **l** $3\sqrt{5}\left(2\sqrt{5}+\sqrt{2}\right)$

Example 10 Self Tutor

Expand and simplify:

a $-\sqrt{2}\left(\sqrt{2}+3\right)$ **b** $-\sqrt{3}\left(7-2\sqrt{3}\right)$

a
$$\begin{aligned}&-\sqrt{2}\left(\sqrt{2}+3\right)\\ &=-\sqrt{2}\times\sqrt{2}+-\sqrt{2}\times 3\\ &=-2-3\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&-\sqrt{3}\left(7-2\sqrt{3}\right)\\ &=-\sqrt{3}\times 7+-\sqrt{3}\times\left(-2\sqrt{3}\right)\\ &=-7\sqrt{3}+6\end{aligned}$$

2 Expand and simplify:

a $-\sqrt{2}\left(3-\sqrt{2}\right)$
b $-\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)$
c $-\sqrt{2}\left(4-\sqrt{2}\right)$
d $-\sqrt{3}\left(1+\sqrt{3}\right)$
e $-\sqrt{3}\left(\sqrt{3}+2\right)$
f $-\sqrt{5}\left(2+\sqrt{5}\right)$
g $-\left(\sqrt{2}+3\right)$
h $-\sqrt{5}\left(\sqrt{5}-4\right)$
i $-\left(3-\sqrt{7}\right)$
j $-\sqrt{11}\left(2-\sqrt{11}\right)$
k $-\left(\sqrt{3}-\sqrt{7}\right)$
l $-2\sqrt{2}\left(1-\sqrt{2}\right)$
m $-3\sqrt{3}\left(5-\sqrt{3}\right)$
n $-7\sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)$
o $\left(-\sqrt{2}\right)^3\left(3-\sqrt{2}\right)$

Example 11 **Self Tutor**

Expand and simplify: $\left(3-\sqrt{2}\right)\left(4+2\sqrt{2}\right)$

$$\begin{aligned}
&\left(3-\sqrt{2}\right)\left(4+2\sqrt{2}\right)\\
&= 12+3\times 2\sqrt{2}+\left(-\sqrt{2}\right)\times 4+\left(-\sqrt{2}\right)\times 2\sqrt{2} \quad \{\text{FOIL rule}\}\\
&= 12+6\sqrt{2}-4\sqrt{2}-4\\
&= 8+2\sqrt{2}
\end{aligned}$$

3 Expand and simplify:

a $\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)$
b $\left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)$
c $\left(\sqrt{3}+2\right)\left(\sqrt{3}-1\right)$
d $\left(4-\sqrt{2}\right)\left(3+\sqrt{2}\right)$
e $\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)$
f $\left(5+\sqrt{7}\right)\left(2-\sqrt{7}\right)$
g $\left(\sqrt{5}+2\right)\left(\sqrt{5}-3\right)$
h $\left(2\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$
i $\left(4-\sqrt{2}\right)\left(3-\sqrt{2}\right)$

Example 12 **Self Tutor**

Expand and simplify:

a $\left(\sqrt{3}+2\right)^2$
b $\left(\sqrt{3}-\sqrt{7}\right)^2$

a
$$\begin{aligned}
&\left(\sqrt{3}+2\right)^2\\
&= \left(\sqrt{3}\right)^2 + 2\times\sqrt{3}\times 2 + 2^2\\
&= 3+4\sqrt{3}+4\\
&= 7+4\sqrt{3}
\end{aligned}$$

b
$$\begin{aligned}
&\left(\sqrt{3}-\sqrt{7}\right)^2\\
&= \left(\sqrt{3}\right)^2 + 2\times\sqrt{3}\times\left(-\sqrt{7}\right) + \left(-\sqrt{7}\right)^2\\
&= 3-2\sqrt{21}+7\\
&= 10-2\sqrt{21}
\end{aligned}$$

4 Expand and simplify:

a $\left(1+\sqrt{2}\right)^2$
b $\left(2-\sqrt{3}\right)^2$
c $\left(\sqrt{3}+2\right)^2$
d $\left(1+\sqrt{5}\right)^2$
e $\left(\sqrt{2}-\sqrt{3}\right)^2$
f $\left(5-\sqrt{2}\right)^2$
g $\left(\sqrt{2}+\sqrt{7}\right)^2$
h $\left(4-\sqrt{6}\right)^2$
i $\left(\sqrt{6}-\sqrt{2}\right)^2$
j $\left(\sqrt{5}+2\sqrt{2}\right)^2$
k $\left(\sqrt{5}-2\sqrt{2}\right)^2$
l $\left(6+\sqrt{8}\right)^2$
m $\left(5\sqrt{2}-1\right)^2$
n $\left(3-2\sqrt{2}\right)^2$
o $\left(1+3\sqrt{2}\right)^2$

Example 13 **Self Tutor**

Expand and simplify:

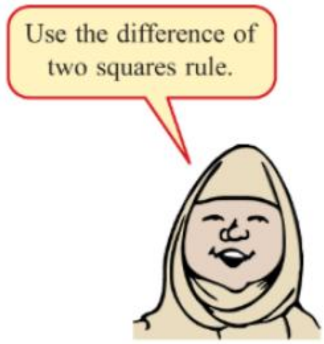
a $(3+\sqrt{2})(3-\sqrt{2})$ **b** $(2\sqrt{3}-5)(2\sqrt{3}+5)$

a $(3+\sqrt{2})(3-\sqrt{2})$
$= 3^2 - (\sqrt{2})^2$
$= 9 - 2$
$= 7$

b $(2\sqrt{3}-5)(2\sqrt{3}+5)$
$= (2\sqrt{3})^2 - 5^2$
$= (4 \times 3) - 25$
$= 12 - 25$
$= -13$

5 Expand and simplify:

a $(4+\sqrt{3})(4-\sqrt{3})$
b $(5-\sqrt{2})(5+\sqrt{2})$
c $(\sqrt{5}-2)(\sqrt{5}+2)$
d $(\sqrt{7}+4)(\sqrt{7}-4)$
e $(3\sqrt{2}+2)(3\sqrt{2}-2)$
f $(2\sqrt{5}-1)(2\sqrt{5}+1)$
g $(5-3\sqrt{3})(5+3\sqrt{3})$
h $(2-4\sqrt{2})(2+4\sqrt{2})$
i $(1+5\sqrt{7})(1-5\sqrt{7})$



6 Expand and simplify:

a $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$
b $(\sqrt{7}+\sqrt{11})(\sqrt{7}-\sqrt{11})$
c $(\sqrt{x}-\sqrt{y})(\sqrt{y}+\sqrt{x})$
d $(3\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})$
e $(3\sqrt{3}+\sqrt{7})(3\sqrt{3}-\sqrt{7})$
f $(2\sqrt{5}-3\sqrt{2})(3\sqrt{2}+2\sqrt{5})$

E DIVISION BY RADICALS

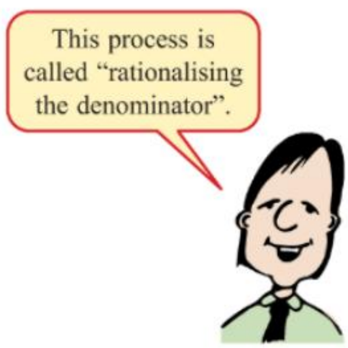
When an expression involves division by a radical, we can write the expression with an **integer denominator** which does **not** contain radicals.

If the denominator contains a simple radical such as $\sqrt{a}$ then we use the rule $\sqrt{a} \times \sqrt{a} = a$.

For example, consider $\dfrac{6}{\sqrt{3}}$.

Since $\dfrac{\sqrt{3}}{\sqrt{3}} = 1$, we can multiply $\dfrac{6}{\sqrt{3}}$ by $\dfrac{\sqrt{3}}{\sqrt{3}}$ without changing its value.

$$\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$
$$= \frac{6\sqrt{3}}{3} \quad \{\text{since } \sqrt{a} \times \sqrt{a} = a\}$$
$$= 2\sqrt{3}$$



Example 14 **Self Tutor**

Express with integer denominator:

a $\dfrac{7}{\sqrt{3}}$ **b** $\dfrac{10}{\sqrt{5}}$ **c** $\dfrac{10}{2\sqrt{2}}$

a $\dfrac{7}{\sqrt{3}}$
$= \dfrac{7}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{7\sqrt{3}}{3}$

b $\dfrac{10}{\sqrt{5}}$
$= \dfrac{10}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{10\sqrt{5}}{5}$
$= 2\sqrt{5}$

c $\dfrac{10}{2\sqrt{2}}$
$= \dfrac{10}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$
$= \dfrac{10\sqrt{2}}{4}$
$= \dfrac{5\sqrt{2}}{2}$

Now suppose the denominator has the form $a + b\sqrt{c}$.

The **radical conjugate** of $a + b\sqrt{c}$ is $a - b\sqrt{c}$.

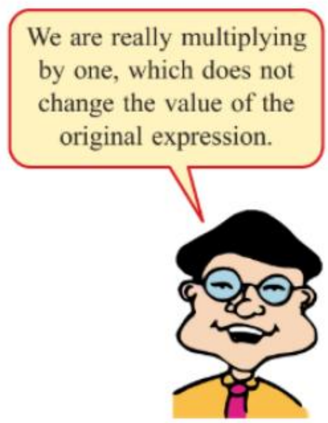
The product of the radical conjugates is
$$\begin{aligned}(a + b\sqrt{c})(a - b\sqrt{c}) &= a^2 - (b\sqrt{c})^2 \\ &= a^2 - b^2c\end{aligned}$$
which does not involve a radical.

So, given a denominator of the form $a + b\sqrt{c}$, we multiply the fraction by $\dfrac{a - b\sqrt{c}}{a - b\sqrt{c}}$.

Example 15 **Self Tutor**

Express $\dfrac{1}{3 + 5\sqrt{2}}$ with integer denominator.

$$\begin{aligned}\frac{1}{3+5\sqrt{2}} &= \left(\frac{1}{3+5\sqrt{2}}\right)\left(\frac{3-5\sqrt{2}}{3-5\sqrt{2}}\right) \\ &= \frac{3-5\sqrt{2}}{3^2 - \left(5\sqrt{2}\right)^2} \qquad \{\text{using } (a+b)(a-b) = a^2 - b^2\} \\ &= \frac{3-5\sqrt{2}}{9-50} \\ &= \frac{5\sqrt{2}-3}{41}\end{aligned}$$



EXERCISE 4E

1 Express with integer denominator:

a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{2}{\sqrt{2}}$ **c** $\dfrac{4}{\sqrt{2}}$ **d** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **e** $\dfrac{\sqrt{7}}{3\sqrt{2}}$

f $\dfrac{1}{\sqrt{3}}$ **g** $\dfrac{3}{\sqrt{3}}$ **h** $\dfrac{4}{\sqrt{3}}$ **i** $\dfrac{\sqrt{7}}{\sqrt{3}}$ **j** $\dfrac{\sqrt{11}}{4\sqrt{3}}$

2 Express with integer denominator:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{3}{\sqrt{5}}$ **c** $\frac{15}{\sqrt{5}}$ **d** $\frac{\sqrt{3}}{\sqrt{5}}$ **e** $\frac{125}{2\sqrt{5}}$

f $\frac{\sqrt{10}}{\sqrt{2}}$ **g** $\frac{1}{2\sqrt{3}}$ **h** $\frac{2\sqrt{2}}{\sqrt{3}}$ **i** $\frac{15}{2\sqrt{5}}$ **j** $\frac{1}{\left(\sqrt{2}\right)^3}$

3 Express with integer denominator:

a $\frac{1}{3-\sqrt{5}}$ **b** $\frac{1}{2+\sqrt{3}}$ **c** $\frac{1}{4-\sqrt{11}}$ **d** $\frac{\sqrt{2}}{5+\sqrt{2}}$

e $\frac{\sqrt{3}}{3+\sqrt{3}}$ **f** $\frac{5}{2-3\sqrt{2}}$ **g** $\frac{-\sqrt{5}}{3+2\sqrt{5}}$ **h** $\frac{3-2\sqrt{7}}{2+3\sqrt{7}}$

4 Write in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\frac{4}{2-\sqrt{2}}$ **b** $\frac{-5}{1+\sqrt{2}}$ **c** $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ **d** $\frac{\sqrt{2}-2}{3-\sqrt{2}}$

e $\frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **f** $\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **g** $\frac{1}{1-\frac{\sqrt{2}}{3}}$ **h** $\frac{\frac{\sqrt{2}}{2}+1}{1-\frac{\sqrt{2}}{4}}$

Example 16 **Self Tutor**

Write $\frac{\sqrt{3}}{1-\sqrt{3}} - \frac{1-2\sqrt{3}}{1+\sqrt{3}}$ in simplest form.

$$\begin{aligned}
\frac{\sqrt{3}}{1-\sqrt{3}} - \frac{1-2\sqrt{3}}{1+\sqrt{3}} &= \left(\frac{\sqrt{3}}{1-\sqrt{3}}\right)\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) - \left(\frac{1-2\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) \\
&= \frac{(\sqrt{3}+3)}{1-3} - \frac{(1-\sqrt{3}-2\sqrt{3}+6)}{1-3} \\
&= \frac{(\sqrt{3}+3)}{-2} - \frac{(7-3\sqrt{3})}{-2} \\
&= \frac{\sqrt{3}+3-7+3\sqrt{3}}{-2} \\
&= \frac{-4+4\sqrt{3}}{-2} \\
&= \frac{-4}{-2} + \frac{4\sqrt{3}}{-2} \\
&= 2-2\sqrt{3}
\end{aligned}$$

5 Write in simplest form:

a $\frac{1+\sqrt{2}}{1-\sqrt{2}} + \frac{1-\sqrt{2}}{1+\sqrt{2}}$ **b** $\frac{2+\sqrt{5}}{2-\sqrt{5}} - \frac{\sqrt{5}}{2+\sqrt{5}}$ **c** $\frac{4-\sqrt{3}}{3-2\sqrt{2}} - \frac{2\sqrt{3}}{3+2\sqrt{2}}$

6 **a** Suppose a and b are positive integers. Show that $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$ is also an integer.

b Write with an integer denominator:

i $\frac{4}{\sqrt{7}+\sqrt{2}}$ **ii** $\frac{2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ **iii** $\frac{\sqrt{3}+2\sqrt{13}}{\sqrt{3}-\sqrt{13}}$

REVIEW SET 4A

1 Simplify:

a $\left(3\sqrt{2}\right)^2$ **b** $-2\sqrt{3} \times 4\sqrt{3}$ **c** $3\sqrt{2} - \sqrt{8}$

d $\left(\sqrt[3]{-27}\right)^3$ **e** $\sqrt{\frac{81}{256}}$ **f** $\left(5\sqrt{3}\right)^3 \times \left(\sqrt{2}\right)^4$

2 Write in simplest radical form:

a $\sqrt{48}$ **b** $\sqrt{864}$

3 Simplify:

a $2\sqrt{3} + 6\sqrt{5} - 3\sqrt{3} - 4\sqrt{5}$ **b** $\dfrac{\sqrt{6}}{3} - \dfrac{\sqrt{6}}{4} + \dfrac{2\sqrt{6}}{5}$

4 Expand and simplify:

a $2\sqrt{3}\left(4 - \sqrt{3}\right)$ **b** $\left(3 - \sqrt{7}\right)^2$ **c** $\left(2 - \sqrt{3}\right)\left(2 + \sqrt{3}\right)$

d $\left(3 + 2\sqrt{5}\right)\left(2 - \sqrt{5}\right)$ **e** $\left(4 - \sqrt{2}\right)\left(3 + 2\sqrt{2}\right)$ **f** $\left(\sqrt{7} + 3\sqrt{8}\right)^2$

5 Express with integer denominator:

a $\dfrac{8}{\sqrt{2}}$ **b** $\dfrac{15}{\sqrt{3}}$ **c** $\dfrac{\sqrt{3}}{4 + \sqrt{2}}$ **d** $\dfrac{5}{6 - 2\sqrt{3}}$

6 Write $\sqrt{\frac{1}{7}}$ in the form $k\sqrt{7}$.

7 Write in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{\frac{\sqrt{3}}{2} + 1}{1 - \frac{\sqrt{3}}{2}}$ **b** $\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}} - \dfrac{2\sqrt{3}}{2 + \sqrt{3}}$

8 Write with integer denominator: $\dfrac{2\sqrt{5} - \sqrt{7}}{\sqrt{5} + 2\sqrt{7}}$

REVIEW SET 4B

1 Simplify:

a $2\sqrt{3} \times 3\sqrt{5}$ **b** $\left(2\sqrt{5}\right)^2$ **c** $5\sqrt{2} - 7\sqrt{2}$

d $-\sqrt{2}\left(2 - \sqrt{2}\right)$ **e** $\left(\sqrt{3}\right)^4$ **f** $\sqrt{3} \times \sqrt{5} \times \sqrt{15}$

2 Write in simplest radical form:

a $\sqrt{75}$ **b** $\sqrt{126}$

3 Write in simplest radical form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}$, $n \in \mathbb{Z}$:

a $\dfrac{3 + \sqrt{24}}{2}$ **b** $\dfrac{8 - \sqrt{72}}{4}$

4 Expand and simplify:

a $\left(5-\sqrt{3}\right)\left(5+\sqrt{3}\right)$ **b** $-\left(2-\sqrt{5}\right)^2$

c $2\sqrt{3}\left(\sqrt{3}-1\right)-2\sqrt{3}$ **d** $\left(2\sqrt{2}-5\right)\left(1-\sqrt{2}\right)$

5 Express with integer denominator:

a $\dfrac{14}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{1-\sqrt{3}}$ **c** $\dfrac{\sqrt{2}}{1+3\sqrt{2}}$ **d** $\dfrac{-5}{5-2\sqrt{3}}$

6 Write in the form $a+b\sqrt{5}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{1-\frac{1}{\sqrt{5}}}{2\sqrt{5}+\frac{1}{\sqrt{5}}}$ **b** $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}-\dfrac{4}{3-\sqrt{5}}$

7 Write in simplest form:

a $\dfrac{2-\sqrt{2}}{1+\sqrt{2}}-\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$ **b** $\dfrac{2+\sqrt{3}}{1+2\sqrt{3}}+\dfrac{-2\sqrt{3}}{1-2\sqrt{3}}$

Chapter 5

Linear equations

Contents:

OPENING PROBLEM

Consider the following statements:

A The sum of two consecutive integers is 17.

B The product of two consecutive integers is 132.

C When 6 is decreased by a number, the result is twice the number.

D The square of an integer is 49.

Things to think about:

a Which statement corresponds to the equation:

i $6 - x = 2x$ **ii** $x(x+1) = 132$ **iii** $x^2 = 49$ **iv** $x + (x+1) = 17$?

b Which of the statements correspond to a *linear* equation?

c What methods can we use to solve linear equations?

We can often convert a worded problem into an **algebraic equation**, then follow a formal procedure to **solve** the equation and hence the problem.

An **algebraic equation** is a mathematical sentence which indicates that two expressions have the same value. It will always contain the symbol $=$.

The **left hand side** (LHS) of an equation is on the left of the $=$ sign.

The **right hand side** (RHS) of an equation is on the right of the $=$ sign.

For example, $\underbrace{3x+7}_{\text{LHS}} = \underbrace{x+11}_{\text{RHS}}$

Linear equations are equations in which the variable is raised only to the power 1.

All linear equations can be written in the form $ax + b = 0$ where a and b are constants, $a \neq 0$, and x is the variable (or unknown).

For example:

- $3x + 4 = 2$, $\frac{2}{3}x + 1 = 6$, and $\dfrac{x-1}{4} = 8$ are linear equations.
- $x^2 + 5x = 7$, $\dfrac{3}{x} = x^3$, and $\sqrt{x} = 8$ are not linear equations.

The **solutions** of an equation are the values of the variable which make the equation true.

Consider the equation $3x + 7 = x + 11$.

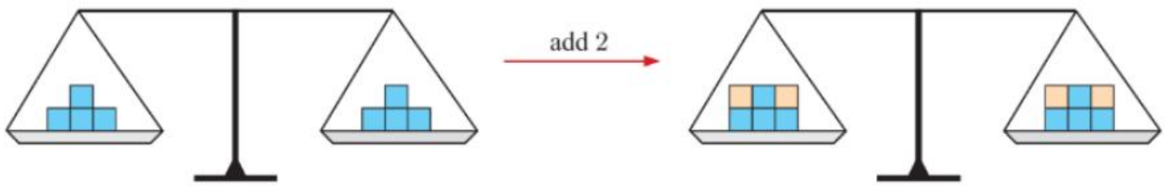
When $x = 2$, LHS $= 3(2) + 7$ and RHS $= 2 + 11$
$= 6 + 7$ $= 13$ also.
$= 13$

So, $x = 2$ is a solution of the equation.

A MAINTAINING BALANCE

For any equation, the LHS must always equal the RHS. We can therefore think of an equation as a set of scales that must always be in **balance**.

The balance of an equation is maintained provided we perform the same operation on **both sides** of the equals sign.



The balance of an equation will be maintained if we:

- add the same amount to both sides
- subtract the same amount from both sides
- multiply both sides by the same amount
- divide both sides by the same amount.

To maintain the balance, whatever operation we perform on one side of the equation, we must also perform on the other.

Example 1 Self Tutor

Write down the equation which results when:

a 5 is added to both sides of $2x - 5 = 7$

b 3 is taken from both sides of $5x + 3 = 18$

c both sides of $\frac{3x-2}{4} = -1$ are multiplied by 4

d both sides of $5x = -15$ are divided by 5.

a
$$\begin{aligned} 2x - 5 &= 7 \\ \therefore\ 2x - 5 + 5 &= 7 + 5 \\ \therefore\ 2x &= 12 \end{aligned}$$

b
$$\begin{aligned} 5x + 3 &= 18 \\ \therefore\ 5x + 3 - 3 &= 18 - 3 \\ \therefore\ 5x &= 15 \end{aligned}$$

c
$$\begin{aligned} \frac{3x-2}{4} &= -1 \\ \therefore\ 4 \times \frac{(3x-2)}{4} &= 4 \times -1 \\ \therefore\ 3x - 2 &= -4 \end{aligned}$$

d
$$\begin{aligned} 5x &= -15 \\ \therefore\ \frac{5x}{5} &= \frac{-15}{5} \\ \therefore\ x &= -3 \end{aligned}$$

EXERCISE 5A

1 Write down the equation which results when we add:

a 5 to both sides of $x - 5 = 2$

b 4 to both sides of $x - 4 = 4$

c 9 to both sides of $2x - 9 = 1$

d 11 to both sides of $3x - 11 = -2$.

2 Write down the equation which results when we subtract:

- **a** 6 from both sides of $x+6=9$
- **b** 7 from both sides of $x+7=-3$
- **c** 4 from both sides of $2x+4=0$
- **d** 8 from both sides of $3x+8=-1$.

3 Write down the equation which results when we multiply both sides of:

- **a** $\frac{x}{3}=4$ by 3
- **b** $\frac{x}{9}=-1$ by 9
- **c** $\frac{x}{-8}=-2$ by -8
- **d** $\frac{x}{-5}=4$ by -5.

4 Write down the equation which results when we divide both sides of:

- **a** $6x=12$ by 6
- **b** $-3x=30$ by -3
- **c** $3(x+2)=-12$ by 3
- **d** $-5(x-1)=-25$ by -5.

B SOLVING LINEAR EQUATIONS

We use the following steps to solve linear equations algebraically:

Step 1: Determine how the expression containing the unknown has been '**built up**'.

Step 2: Perform **inverse operations** on **both sides** of the equation to 'undo' how the expression was 'built up'. In this way we **isolate** the unknown.

Step 3: Check your solution by substitution.

Example 2 Self Tutor

Solve for x: $3x+7=22$

$\therefore\ 3x+7=22$

$\therefore\ 3x+7-7=22-7$ {subtracting 7 from both sides}

$\therefore\ 3x=15$

$\therefore\ \frac{3x}{3}=\frac{15}{3}$ {dividing both sides by 3}

$\therefore\ x=5$

Check: LHS $=3\times 5+7=22$ ✓

EXERCISE 5B.1

1 Solve for x:

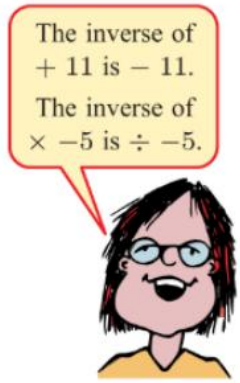
- **a** $x+11=0$
- **b** $4x=-12$
- **c** $5x+35=0$
- **d** $4x-5=-17$
- **e** $5x+3=28$
- **f** $3x-9=18$
- **g** $8x-1=7$
- **h** $3x+5=-10$
- **i** $13+7x=-1$
- **j** $14=3x+5$
- **k** $4x-7=-13$
- **l** $-3=2x+9$

Example 3 **Self Tutor**

Solve for x: $11 - 5x = 26$

$$\begin{aligned} 11 - 5x &= 26 \\ \therefore\ 11 - 5x - 11 &= 26 - 11 \quad \{\text{subtracting } 11 \text{ from both sides}\} \\ \therefore\ -5x &= 15 \\ \therefore\ \frac{-5x}{-5} &= \frac{15}{-5} \quad \{\text{dividing both sides by } -5\} \\ \therefore\ x &= -3 \end{aligned}$$

Check: LHS $= 11 - 5 \times -3 = 11 + 15 = 26$ ✓



2 Solve for x:

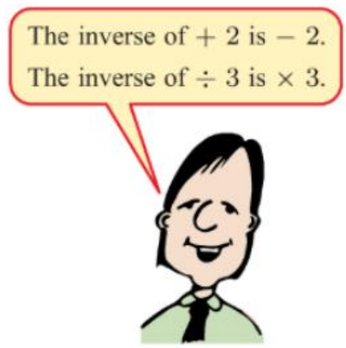
a $8 - x = -3$
b $-4x = 22$
c $3 - 2x = 11$
d $3 - 7x = -4$
e $6 - 4x = -8$
f $17 - 2x = -5$
g $15 = 3 - 2x$
h $24 - 3x = -9$
i $4 = 3 - 2x$
j $13 = -1 - 7x$
k $-21 = 3 - 6x$
l $23 = -4 - 3x$

Example 4 **Self Tutor**

Solve for x: $\frac{x}{3} + 2 = -2$

$$\begin{aligned} \frac{x}{3} + 2 &= -2 \\ \therefore\ \frac{x}{3} + 2 - 2 &= -2 - 2 \quad \{\text{subtracting } 2 \text{ from both sides}\} \\ \therefore\ \frac{x}{3} &= -4 \\ \therefore\ \frac{x}{3} \times 3 &= -4 \times 3 \quad \{\text{multiplying both sides by } 3\} \\ \therefore\ x &= -12 \end{aligned}$$

Check: LHS $= \frac{-12}{3} + 2 = -4 + 2 = -2$ ✓



3 Solve for x:

a $\frac{x}{4} = 7$
b $\frac{2x}{5} = -6$
c $\frac{x}{2} + 3 = -5$
d $\frac{x}{4} - 2 = -5$
e $\frac{x-1}{3} = 6$
f $\frac{x+5}{6} = -1$
g $4 = \frac{2+x}{3}$
h $-1 + \frac{x}{3} = 7$

Example 5 Self Tutor

Solve for x: $\dfrac{4x+3}{5} = -2$

$$\frac{4x+3}{5} = -2$$

$\therefore\ 5 \times \dfrac{(4x+3)}{5} = -2 \times 5$ {multiplying both sides by 5}

$\therefore\ 4x+3 = -10$

$\therefore\ 4x+3-3 = -10-3$ {subtracting 3 from both sides}

$\therefore\ 4x = -13$

$\therefore\ \dfrac{4x}{4} = \dfrac{-13}{4}$ {dividing both sides by 4}

$\therefore\ x = -3\frac{1}{4}$

Check: LHS $= \dfrac{4(-\frac{13}{4})+3}{5} = \dfrac{-13+3}{5} = \dfrac{-10}{5} = -2$ ✓

4 Solve for x:

a $\dfrac{2x+11}{3} = 0$

b $\frac{1}{2}(3x+1) = -4$

c $\dfrac{1+2x}{5} = 7$

d $\dfrac{1-2x}{5} = 3$

e $\frac{1}{4}(1-3x) = -2$

f $\frac{1}{4}(5-2x) = -3$

EQUATIONS WITH A REPEATED UNKNOWN

In situations where the unknown or variable appears more than once, we need to **expand** any brackets, **collect like terms**, and then **solve** the equation.

Example 6 Self Tutor

Solve for x: $5(x+1) - 2x = -7$

$5(x+1) - 2x = -7$

$\therefore\ 5x+5-2x = -7$ {expanding brackets}

$\therefore\ 3x+5 = -7$

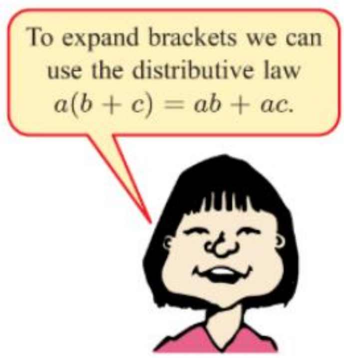
$\therefore\ 3x+5-5 = -7-5$ {subtracting 5 from both sides}

$\therefore\ 3x = -12$

$\therefore\ \dfrac{3x}{3} = \dfrac{-12}{3}$ {dividing both sides by 3}

$\therefore\ x = -4$

Check: LHS $= 5(-4+1) - 2(-4) = 5(-3) + 8 = -7$ ✓



EXERCISE 5B.2

1 Solve for x:

a $3(x-2) - x = 12$

b $4(x+2) - 2x = -16$

c $5(x-3) + 4x = -6$

d $2(3x+2) - x = -6$

e $5(2x-1) - 4x = 11$

f $-2(4x+3) + 2x = 12$

2 Solve for x:

a $3(x+2)+2(x+4)=-1$

b $5(x+1)-3(x+2)=11$

c $4(x-3)-2(x-1)=-6$

d $3(3x+1)-4(x+1)=14$

e $2(3+2x)+3(x-4)=8$

f $4(5x-3)-3(2x-5)=17$

EQUATIONS WHERE THE UNKNOWN APPEARS ON BOTH SIDES

If the unknown appears on **both sides** of the equation, we:

- **expand** any brackets and **collect like terms**
- move the **unknown to one side** of the equation and the remaining terms to the other side
- **solve** the equation.

Example 7 **Self Tutor**

Solve for x: $5x+2=3x-5$

$$5x+2=3x-5$$

$$\therefore\ 5x+2-3x=3x-5-3x \quad \{\text{subtracting } 3x \text{ from both sides}\}$$

$$\therefore\ 2x+2=-5$$

$$\therefore\ 2x+2-2=-5-2 \quad \{\text{subtracting 2 from both sides}\}$$

$$\therefore\ 2x=-7$$

$$\therefore\ \frac{2x}{2}=\frac{-7}{2} \quad \{\text{dividing both sides by 2}\}$$

$$\therefore\ x=-3\tfrac{1}{2}$$

Check: $\text{LHS}=5\left(-\frac{7}{2}\right)+2=-\frac{35}{2}+2=-\frac{31}{2}$

$\text{RHS}=3\left(-\frac{7}{2}\right)-5=-\frac{21}{2}-5=-\frac{31}{2}$ ✓

EXERCISE 5B.3

1 Solve for x:

a $5x+2=3x+14$

b $8x+7=4x-5$

c $7x+3=2x+9$

d $3x-8=5x-2$

e $x-3=5x+11$

f $3+x=15+4x$

g $6+2x=15-x$

h $3x+7=15-x$

i $5+x=11-2x$

j $17-3x=4-x$

k $8-x=x+6$

l $9-2x=3-x$

2 Solve for x:

a $5(2-3x)=-8-6x$

b $4(2x-1)+9=3x$

c $8-(2-x)=2x$

d $5x-4(4-x)=x+12$

e $4(x-1)=1-(3-x)$

f $4(x-6)+7x=5(2x-1)$

g $3(2x-4)=5x-3(12-x)$

Example 8 | **Self Tutor**

Solve for x: $\dfrac{2x+3}{4} = \dfrac{x-2}{3}$

$$\frac{2x+3}{4} = \frac{x-2}{3}$$

$$\therefore \frac{3 \times (2x+3)}{3 \times 4} = \frac{4 \times (x-2)}{4 \times 3} \qquad \{\text{LCD} = 12\}$$

$$\therefore 3(2x+3) = 4(x-2) \qquad \{\text{equating numerators}\}$$

$$\therefore 6x+9 = 4x-8 \qquad \{\text{expanding brackets}\}$$

$$\therefore 6x+9-4x = 4x-8-4x \qquad \{\text{subtracting } 4x \text{ from both sides}\}$$

$$\therefore 2x+9 = -8$$

$$\therefore 2x+9-9 = -8-9 \qquad \{\text{subtracting 9 from both sides}\}$$

$$\therefore 2x = -17$$

$$\therefore \frac{2x}{2} = \frac{-17}{2} \qquad \{\text{dividing both sides by 2}\}$$

$$\therefore x = -8\tfrac{1}{2}$$

Check: $\text{LHS} = \dfrac{2(-\frac{17}{2})+3}{4} = \dfrac{-17+3}{4} = \dfrac{-14}{4} = -\dfrac{7}{2}$

$\text{RHS} = \dfrac{(-\frac{17}{2})-2}{3} = \dfrac{-\frac{21}{2}}{3} = -\dfrac{21}{6} = -\dfrac{7}{2}$ ✓

3 Solve for x:

a $\dfrac{x+6}{2} = \dfrac{x}{3}$ **b** $\dfrac{2x-11}{7} = \dfrac{3x}{5}$ **c** $\dfrac{x+4}{2} = \dfrac{2x-3}{3}$

d $\dfrac{3x+2}{2} = \dfrac{x-1}{4}$ **e** $\dfrac{1-x}{2} = \dfrac{x+2}{3}$ **f** $\dfrac{x+5}{2} = 1-x$

g $\dfrac{2x+7}{3} = x+4$ **h** $\dfrac{3(2x+9)}{2} = x-8$

C PROBLEM SOLVING

Many problems can be translated into **algebraic equations**. To solve problems using algebra, we follow these steps:

Step 1: Decide on the unknown quantity and allocate it a variable such as x.
Step 2: Translate the problem into an equation.
Step 3: Solve the equation by isolating the variable.
Step 4: Check that the solution satisfies the original problem.
Step 5: Write the answer in sentence form, describing how the solution relates to the original problem.

Example 9 Self Tutor

The sum of 3 consecutive even integers is 132. Find the smallest integer.

Let x be the smallest even integer

$\therefore$ the next is $x+2$, and the largest is $x+4$.

$$\begin{aligned} \text{So,} \quad x+(x+2)+(x+4) &= 132 && \{\text{their sum is } 132\} \\ \therefore\ 3x+6 &= 132 \\ \therefore\ 3x+6-6 &= 132-6 && \{\text{subtracting 6 from both sides}\} \\ \therefore\ 3x &= 126 \\ \therefore\ \frac{3x}{3} &= \frac{126}{3} && \{\text{dividing both sides by 3}\} \\ \therefore\ x &= 42 \end{aligned}$$

$\therefore$ the smallest integer is 42.

EXERCISE 5C

1 When a number is trebled and the result is decreased by 5, the answer is 19. Find the number.

2 Two consecutive integers have a sum of 173. Find the numbers.

3 Three consecutive integers add to 108. Find the smallest of them.

4 When a number is decreased by 1 and the resulting number is halved, the answer is 45. Find the number.

Example 10 Self Tutor

If twice a number is subtracted from 11, the result is 4 more than the number. What is the number?

Let x be the number.

$$\begin{aligned} \therefore\ 11-2x &= x+4 \\ \therefore\ 11-2x+2x &= x+4+2x && \{\text{adding } 2x \text{ to both sides}\} \\ \therefore\ 11 &= 3x+4 \\ \therefore\ 11-4 &= 3x+4-4 && \{\text{subtracting 4 from both sides}\} \\ \therefore\ 7 &= 3x \\ \therefore\ \frac{7}{3} &= \frac{3x}{3} && \{\text{dividing both sides by 3}\} \\ \therefore\ x &= 2\tfrac{1}{3} \end{aligned}$$

So, the number is $2\frac{1}{3}$.

5 When a number is subtracted from 35, the result is 11 more than the number. Find the number.

6 When a number is increased by 4 and the result is halved, the answer is equal to the original number. Find the number.

7 Three times a number is equal to 17 minus the number. Find the number.

8 Five times a number is 2 more than the number. Find the number.

9 When one third of a number is subtracted from twice the number, the answer is 15. Find the number.

Example 11 **Self Tutor**

Cans of sardines are sold in two sizes. Small cans cost \$2 each, and large cans cost \$3 each. If 15 cans of sardines were bought for a total of \$38, how many small cans were bought?

Size	*Cost per can*	*Number bought*	*Value*
small	\$2	x	\$$2x$
large	\$3	$15 - x$	\$$3(15 - x)$
		15	\$38

$2x + 3(15 - x) = 38$

$\therefore\ 2x + 45 - 3x = 38$ {expanding brackets}

$\therefore\ 45 - x = 38$

$\therefore\ 45 - x - 45 = 38 - 45$ {subtracting 45 from both sides}

$\therefore\ -x = -7$

$\therefore\ x = 7$

So, 7 small cans were bought.

10 Isaac is going to boarding school. He buys school shirts at €35 each and trousers at €49 each. Altogether he buys 9 items, and their total cost is €357. How many shirts does he buy?

11 I have 36 coins in my pocket, all of which are 5-cent or 10-cent coins. If their total value is \$3.20, how many 5-cent coins do I have?

12 Oranges cost 25 pence each and apples cost 30 pence each. I bought 5 more oranges than apples, and the total cost was £4.55. How many apples did I buy?

13 Cans of soup are sold in two sizes. 250 g cans cost \$2.95 each, and 500 g cans cost \$4.50 each. I bought a total of 12 cans, and they cost me \$43.15. How many 500 g cans did I buy?

REVIEW SET 5A

1 Answer the **Opening Problem** on page **80**.

2 Write down the equation which results when:

a 4 is subtracted from both sides of $2x + 4 = 11$

b both sides of $\frac{x+4}{3} = -1$ are multiplied by 3

c 7 is added to both sides of $3x - 7 = -5$

d both sides of $-4x = 16$ are divided by -4.

3 Solve for x:

a $2x + 5 = 13$ **b** $9 = 5x + 14$ **c** $7 + 4x = -5$

d $3 - 2x = 11$ **e** $-4x - 3 = 13$ **f** $15 = 4 - 11x$

4 Solve for x:

a $\frac{x}{2} = -3$ **b** $\frac{x}{5} = \frac{4}{7}$ **c** $\frac{x+1}{3} = -2$

d $\frac{3-2x}{7} = -5$ **e** $\frac{1}{3}(3-x) = 2$ **f** $\frac{3x}{7} - 5 = -2$

5 Solve for x:

a $2(x+1) - x = 3$ **b** $3x - (x-5) = 9$

c $3(1-x) + 2(x+3) = 7$ **d** $2(x+4) + 3(5-2x) = 8$

6 Solve for x:

a $3x - 1 = 2x + 5$ **b** $2(3x-1) + 4 = 4(x-3)$

c $\frac{4x+5}{3} = \frac{x}{2}$ **d** $\frac{1-3x}{4} = \frac{x-2}{2}$

7 When a number is increased by 11 and the result is doubled, the answer is 48. Find the number.

8 The sum of three consecutive integers is 63. Find the smallest of the integers.

9 When 7 times a certain number is decreased by 11, the result is 31 more than the number. Find the number.

10 I have 25 coins consisting of 5-cent and 50-cent pieces. If the total value is \$7.10, how many 5-cent coins do I have?

REVIEW SET 5B

1 Write down the equation which results when:

a both sides of $\frac{6-x}{2} = -\frac{1}{2}$ are multiplied by 2

b 3 is added to both sides of $5x - 3 = 8$

c both sides of $3(2x-1) = 6$ are divided by 3

d 11 is subtracted from both sides of $9x + 11 = 20$.

2 Solve for x:

a $13 = 2x - 7$ **b** $5x - 4 = 16$ **c** $11 + 3x = -4$

d $-3x - 1 = 8$ **e** $-12 - 5x = 13$ **f** $16 = 2 - 3x$

3 Solve for x:

a $\frac{x}{-4} = 5$ **b** $\frac{x}{2} = \frac{3}{8}$ **c** $\frac{4-x}{3} = -1$

d $\frac{5x+3}{2} = -6$ **e** $\frac{1}{4}(2-3x) = 5$ **f** $2 - \frac{3x}{5} = -1$

4 Solve for x:

a $6x - 2(x-5) = 8$ **b** $3(4-3x) + 6x = 15$

c $2(3x+1) - 3(x-2) = 5$ **d** $5(2x-3) - (3-x) = -14$

5 Solve for x:

a $3 - x = 4x - 7$ **b** $2(3-2x) = 3(4-x) - 5$

c $3(2x-1) + 9 = 2(x+7)$ **d** $\frac{2x+1}{3} = \frac{4-x}{6}$

6 Four times a number is equal to the number plus 15. Find the number.

7 The sum of two consecutive odd integers is 36. Find the larger integer.

8 Five more than a certain number is nine less than three times the number. Find the number.

9 Writing pads cost €1.35 each and pens cost €0.85 each. I bought twice as many pens as pads, and the total cost was €18.30. How many pads did I buy?

10 Sadao likes collecting action figures. In total he has 44 figures belonging to three categories. He has 2 more transformers than chogokin, and he has 1 more anime figure than transformers. How many chogokin does Sadao have?

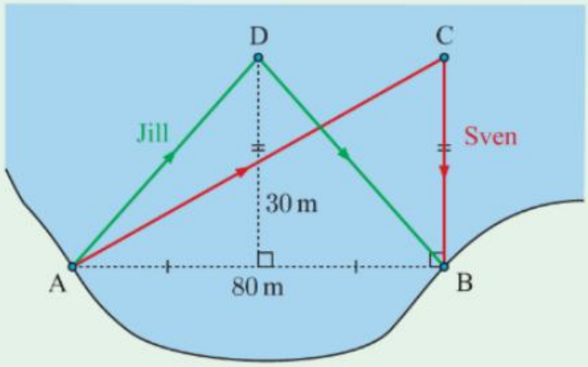
Chapter 6

Pythagoras' theorem

Contents:

OPENING PROBLEM

Sven has challenged Jill to a swimming race in a lake. Both swimmers will swim from A to B, but each needs to touch a buoy along the way. Sven thinks he is a very good swimmer, so he will swim to buoy C on his way to B. Jill will swim to buoy D, as shown.



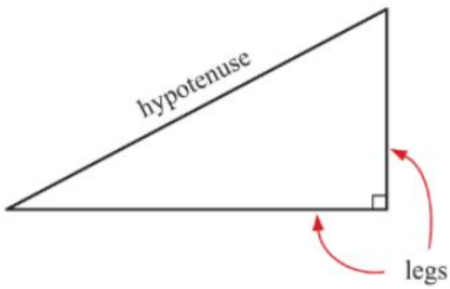
Things to think about:

- **a** How far will Sven swim?
- **b** How far will Jill swim?
- **c** How much further will Sven swim than Jill?

A **right angled triangle** is a triangle which has a right angle as one of its angles.

The side opposite the right angle is called the **hypotenuse**, and is the **longest** side of the triangle.

The other two sides are called the **legs** of the triangle.



Around 500 BC, the Greek mathematician **Pythagoras** discovered a rule which connects the squares of the lengths of the sides of right angled triangles.

The rule often results in equations of the form $x^2 = k$ which we then need to solve.

A SOLVING $x^2 = k$

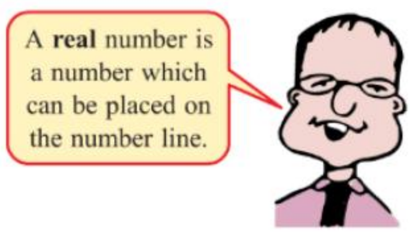
In **Chapter 4** we dealt with radicals like $\sqrt{3}$.

Notice that $\sqrt{3} \times \sqrt{3} = 3$ and $(-\sqrt{3}) \times (-\sqrt{3}) = 3$ {as negative $\times$ negative $=$ positive}.

So, if we were asked to solve the equation $x^2 = 3$, it is clear that x could equal $\sqrt{3}$ or $-\sqrt{3}$. The squares of both of these numbers are 3.

We write the solutions as $x = \pm\sqrt{3}$, which reads "plus or minus the square root of 3".

> Consider $x^2 = k$.
> If $k > 0$, then $x = \pm\sqrt{k}$.
> If $k = 0$, then $x = 0$ is the only solution.
> If $k < 0$, then there are **no real solutions**.



Example 1 Self Tutor

Solve for x:

a $x^2 = 9$ **b** $x^2 = 13$ **c** $x^2 = -2$

a $x^2 = 9$

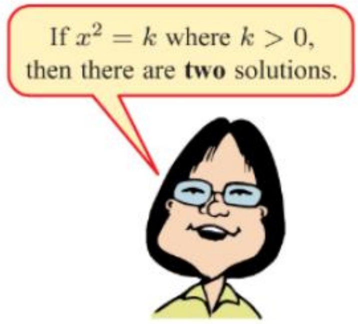
$\therefore x = \pm\sqrt{9}$

$\therefore x = \pm 3$ $\{x = 3$ or $x = -3\}$

b $x^2 = 13$

$\therefore x = \pm\sqrt{13}$ $\{x = \sqrt{13}$ or $x = -\sqrt{13}\}$

c $x^2 = -2$ has no real solutions, since x^2 cannot be negative.



Example 2 Self Tutor

Solve for x:

a $x^2 + 9 = 16$ **b** $21 + x^2 = 25$

a $x^2 + 9 = 16$

$\therefore x^2 = 7$ {subtracting 9 from both sides}

$\therefore x = \pm\sqrt{7}$

b $21 + x^2 = 25$

$\therefore x^2 = 4$ {subtracting 21 from both sides}

$\therefore x = \pm\sqrt{4}$

$\therefore x = \pm 2$

EXERCISE 6A

1 Solve for x:

a $x^2 = 4$ **b** $x^2 = 16$ **c** $x^2 = 64$ **d** $x^2 = 0$

e $x^2 = -1$ **f** $x^2 = 15$ **g** $x^2 = 81$ **h** $x^2 = 20$

i $x^2 = 121$ **j** $x^2 = -3$ **k** $x^2 = 169$ **l** $x^2 = -25$

2 Solve for x:

a $x^2 + 9 = 25$ **b** $x^2 + 20 = 36$ **c** $x^2 + 1 = 10$

d $x^2 + 11 = 47$ **e** $12 + x^2 = 37$ **f** $x^2 + 25 = 169$

g $8 + x^2 = 19$ **h** $x^2 + 11 = 37$ **i** $17 + x^2 = 25$

Example 3 Self Tutor

Solve for x:

a $3x^2 = 72$ **b** $x^2 + (3x)^2 = 100$

a $3x^2 = 72$

$\therefore x^2 = 24$ {dividing both sides by 3}

$\therefore x = \pm\sqrt{24}$

$\therefore x = \pm\sqrt{4}\sqrt{6}$

$\therefore x = \pm 2\sqrt{6}$

b $x^2 + (3x)^2 = 100$

$\therefore x^2 + 9x^2 = 100$ {index law}

$\therefore 10x^2 = 100$

$\therefore x^2 = 10$ {dividing both sides by 10}

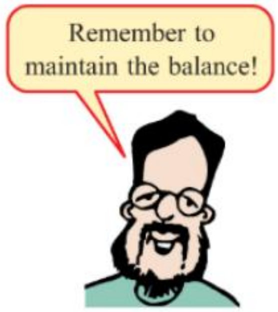
$\therefore x = \pm\sqrt{10}$

3 Solve for x:

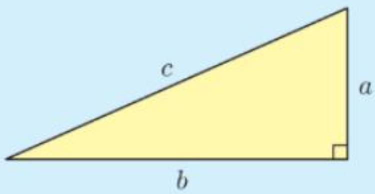
a $2x^2 = 32$
b $3x^2 = 27$
c $4x^2 = 64$
d $x^2 + x^2 = 128$
e $x^2 + 4x^2 = 125$
f $x^2 + 9x^2 = 40$
g $x^2 + (2x)^2 = 45$
h $x^2 + (4x)^2 = 34$
i $x^2 + (3x)^2 = 30$

4 Solve for x:

a $x^2 + 27 = (2x)^2$
b $(3x)^2 = x^2 + 32$
c $(2x)^2 + 80 = (3x)^2$
d $x^2 + 45 = (4x)^2$



B PYTHAGORAS' THEOREM

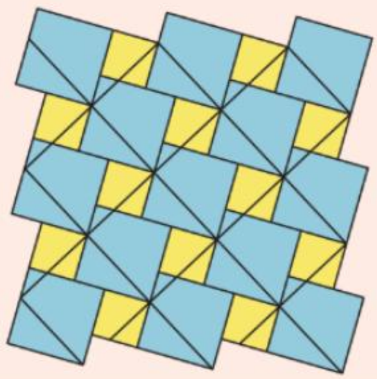


In a right angled triangle with hypotenuse c and legs a and b, $c^2 = a^2 + b^2$.

RESEARCH

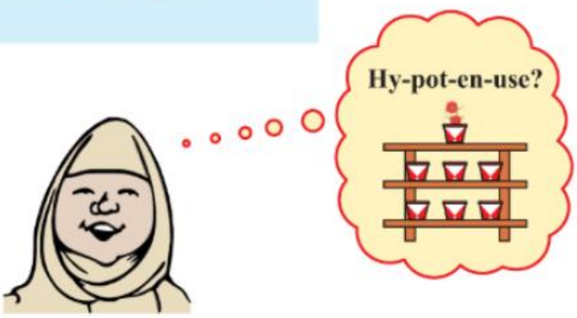
The Persian mathematician **Al-Nayrīzī** (865 - 922 AD) came from Nayriz, Iran. Around 900 AD, he used the tiling arrangement shown to prove Pythagoras' theorem.

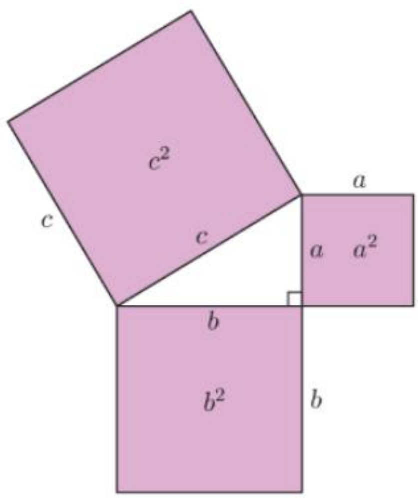
Research the method of **five-piece dissection** he used for his proof.



In geometric form, **Pythagoras' theorem** states that:

In any right angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

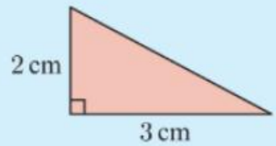




We can use Pythagoras' theorem to find unknown side lengths in right angled triangles.

Example 4 **Self Tutor**

Find the length of the hypotenuse in:



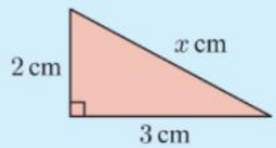
Let the hypotenuse have length x cm.

$\therefore\ x^2 = 3^2 + 2^2$ {Pythagoras}

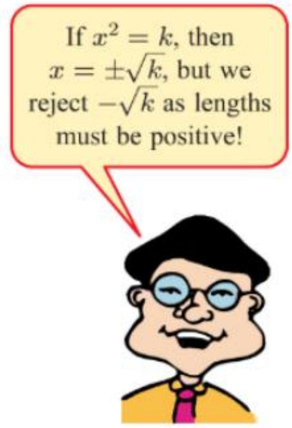
$\therefore\ x^2 = 9 + 4$

$\therefore\ x^2 = 13$

$\therefore\ x = \sqrt{13}$ {as $x > 0$}

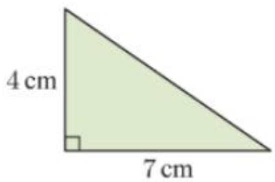


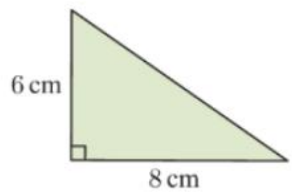
The hypotenuse is $\sqrt{13}$ cm long.

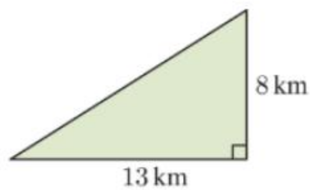


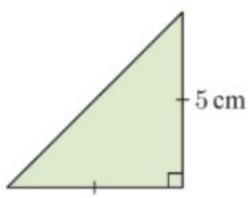
EXERCISE 6B

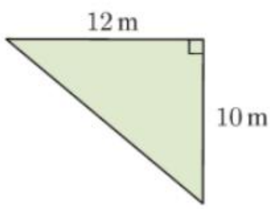
1 Find the length of the hypotenuse of each of the following triangles, leaving your answer in simplest radical form where appropriate:

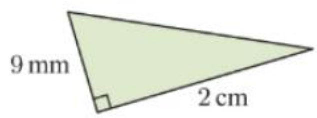
a


b


c


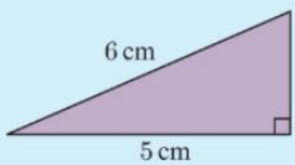
d


e


f


Example 5 **Self Tutor**

Find the length of the third side of the triangle.



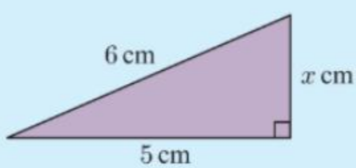
Let the third side have length x cm.

$\therefore\ x^2 + 5^2 = 6^2$ {Pythagoras}

$\therefore\ x^2 + 25 = 36$

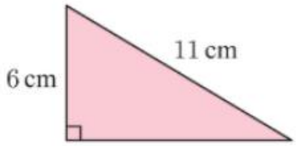
$\therefore\ x^2 = 11$

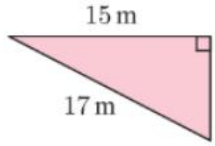
$\therefore\ x = \sqrt{11}$ {as $x > 0$}

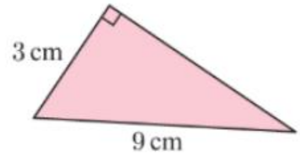


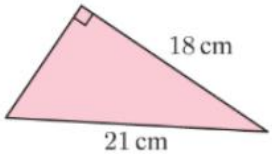
The third side is $\sqrt{11}$ cm long.

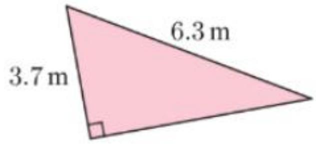
2 Find the length of the third side of each of the following right angled triangles.
Where appropriate, leave your answer in simplest radical form.

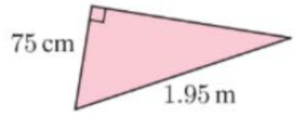
a


b


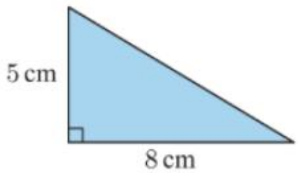
c


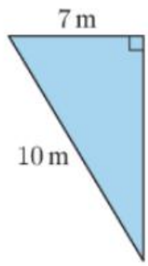
d


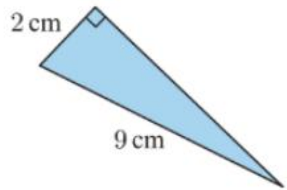
e


f


3 Find the length of the unknown side of each of the following right angled triangles. Give your answer to 1 decimal place where appropriate.

a


b


c


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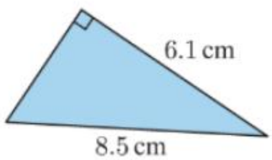

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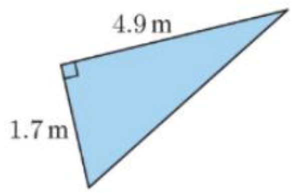

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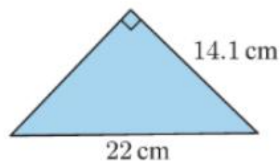
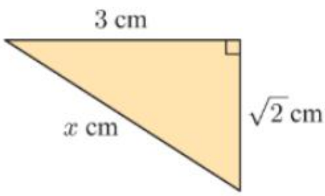
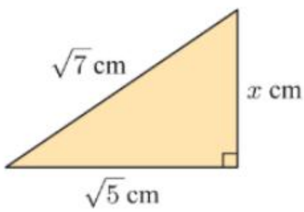
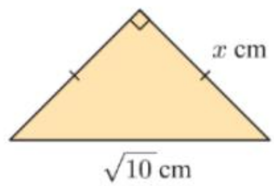
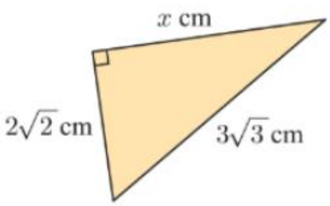


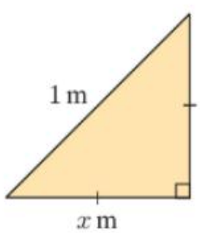
4 Find x in each of the following:

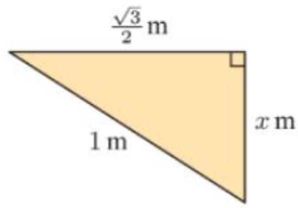
a


b


c


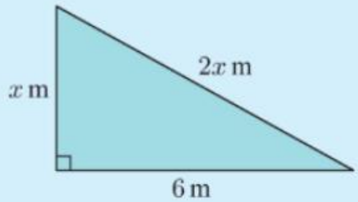
d


e


f


Example 6 — Self Tutor

Find x:



$(2x)^2 = x^2 + 6^2$ {Pythagoras}

$\therefore\ 4x^2 = x^2 + 36$

$\therefore\ 3x^2 = 36$

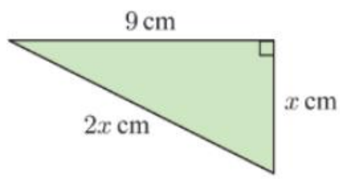
$\therefore\ x^2 = 12$

$\therefore\ x = \sqrt{12}$ {as $x > 0$}

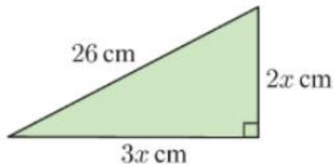
$\therefore\ x = 2\sqrt{3}$

5 Find the value of x:

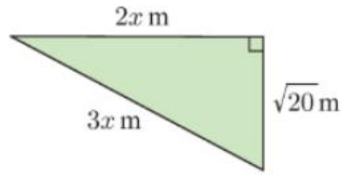
a



b

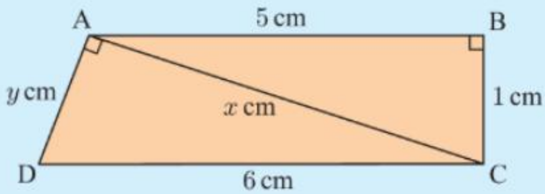


c



Example 7 — Self Tutor

Find the values of the unknowns:



In triangle ABC, the hypotenuse is x cm.

$\therefore\ x^2 = 5^2 + 1^2$ {Pythagoras}

$\therefore\ x^2 = 26$

$\therefore\ x = \sqrt{26}$ {as $x > 0$}

In triangle ACD, the hypotenuse is 6 cm.

$\therefore\ y^2 + (\sqrt{26})^2 = 6^2$ {Pythagoras}

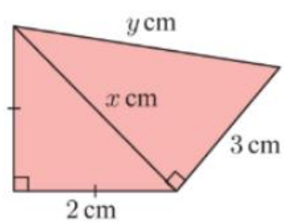
$\therefore\ y^2 + 26 = 36$

$\therefore\ y^2 = 10$

$\therefore\ y = \sqrt{10}$ {as $y > 0$}

6 Find the values of the unknowns:

a



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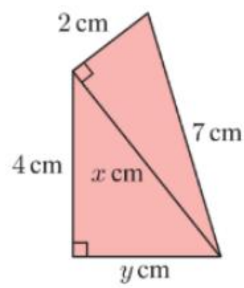
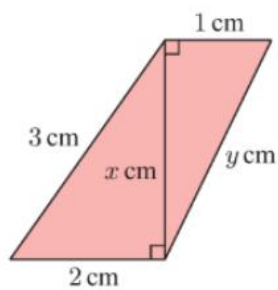
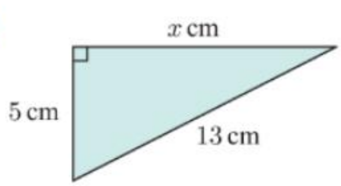


c



7 Find x:

a



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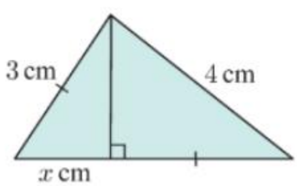

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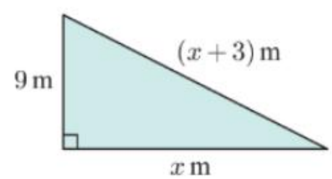
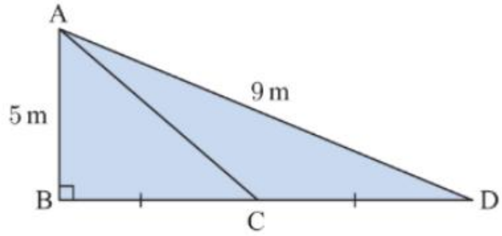
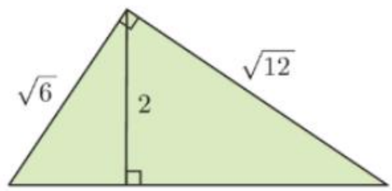


8 Find the length of [AC]:

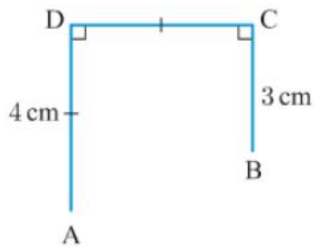


9 Use the figure below to show that $\sqrt{2}+\sqrt{8}=\sqrt{18}$.

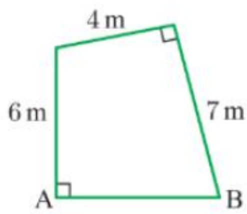


10 Find the distance AB in each of the following figures:

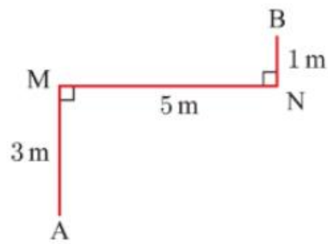
a



b

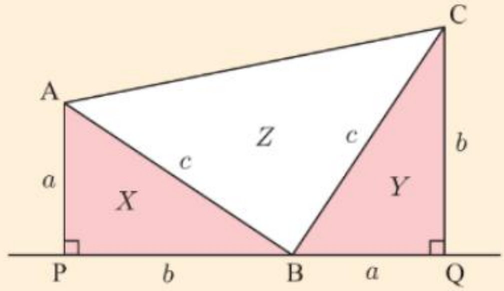


c



INVESTIGATION — PRESIDENT GARFIELD'S PROOF

Prior to being President of the United States, **James Garfield** used the diagram alongside to prove Pythagoras' theorem. When he found this proof he was so pleased he gave cigars out to his many friends.



What to do:

1 Two identical right angled triangles, ABP and BCQ, are placed on a line. What can you deduce about $A\widehat{B}C$? Explain your answer.

2 Find the areas of triangles X, Y, and Z.
Hence, express area X + area Y + area Z in simplest form.

3 The combined regions X, Y, and Z form a trapezium. Find:

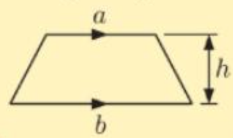
a the average length of the parallel sides

b the distance between the parallel sides

c the area of the trapezium in terms of a and b.

4 Use your results from **2** and **3 c** to find a relationship between a, b, and c.

Area of trapezium
$$=\left(\frac{a+b}{2}\right)\times h$$

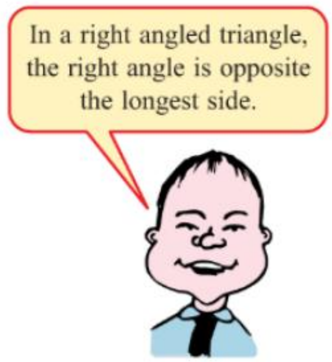


C THE CONVERSE OF PYTHAGORAS' THEOREM

If we know all of the side lengths of a triangle, we can use the **converse of Pythagoras' theorem** to test whether the triangle is right angled.

GEOMETRY PACKAGE

If a triangle has sides of length a, b, and c units where $a^2 + b^2 = c^2$, then the triangle is right angled.



Example 8 — Self Tutor

Is a triangle with side lengths 6 cm, 8 cm, and 5 cm right angled?

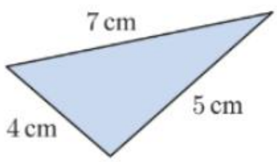
The two shorter sides have lengths 5 cm and 6 cm.

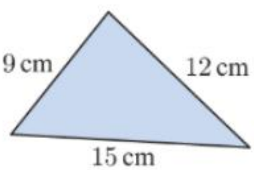
Now $5^2 + 6^2 = 25 + 36 = 61$, but $8^2 = 64$.

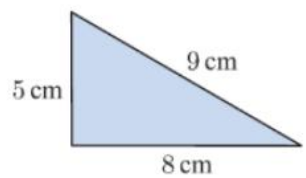
$\therefore$ $5^2 + 6^2 \neq 8^2$, and hence the triangle is not right angled.

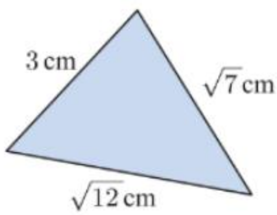
EXERCISE 6C

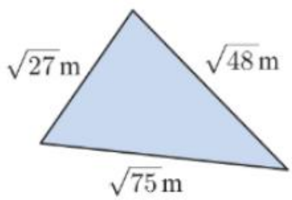
1 The following figures are not drawn to scale. Which of the triangles are right angled?

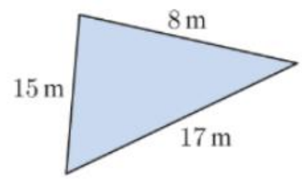
a


b


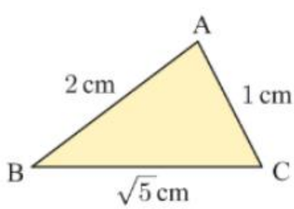
c


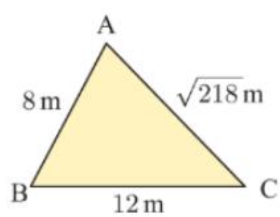
d


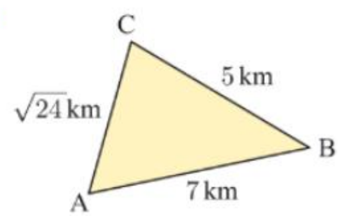
e


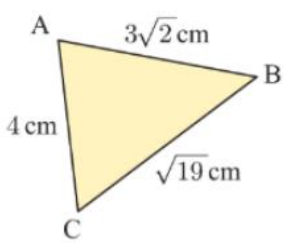
f


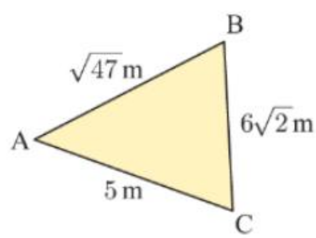
2 The following triangles are not drawn to scale. If any of them is right angled, identify the right angle.

a


b


c


d


e


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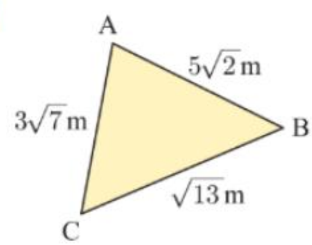
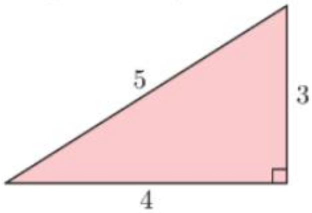


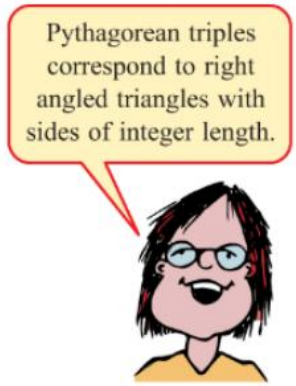
D PYTHAGOREAN TRIPLES

The set of positive integers $\{a, b, c\}$ where $a < b < c$ is a **Pythagorean triple** if it obeys the rule $a^2 + b^2 = c^2$.

For example, $\{3, 4, 5\}$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$.

Other examples of Pythagorean triples include $\{5, 12, 13\}$ and $\{8, 15, 17\}$.





Example 9 — Self Tutor

Determine whether the following sets of numbers are Pythagorean triples:

a $\{5, 8, 9\}$ **b** $\{6, 8, 10\}$ **c** $\{2, 3, \sqrt{13}\}$

a $5^2 + 8^2 = 25 + 64 = 89$
and $9^2 = 81$
Since $5^2 + 8^2 \neq 9^2$, $\{5, 8, 9\}$ is not a Pythagorean triple.

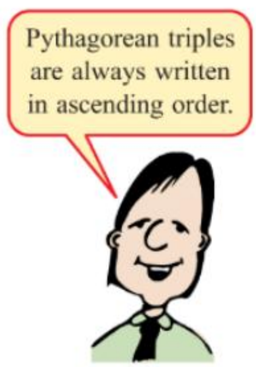
b $6^2 + 8^2 = 36 + 64 = 100$
and $10^2 = 100$
Since $6^2 + 8^2 = 10^2$, $\{6, 8, 10\}$ is a Pythagorean triple.

c $\{2, 3, \sqrt{13}\}$ is not a Pythagorean triple, as these numbers are not all positive integers.

Example 10 — Self Tutor

Find k given that $\{9, k, 15\}$ is a Pythagorean triple.

$$9^2 + k^2 = 15^2$$
$$\therefore\ 81 + k^2 = 225$$
$$\therefore\ k^2 = 144$$
$$\therefore\ k = 12 \qquad \{\text{as } k > 0\}$$



EXERCISE 6D

1 Determine whether the following are Pythagorean triples:

a $\{15, 20, 25\}$ **b** $\{5, 6, 7\}$ **c** $\{14, 48, 50\}$

d $\{1, 6, \sqrt{37}\}$ **e** $\{20, 48, 52\}$ **f** $\{-15, 8, 17\}$

2 Find k given that the following are Pythagorean triples:

- **a** $\{12,\ 16,\ k\}$
- **b** $\{k,\ 24,\ 26\}$
- **c** $\{14,\ k,\ 50\}$
- **d** $\{8,\ k,\ k+2\}$
- **e** $\{20,\ k,\ k+8\}$
- **f** $\{k,\ 60,\ k+50\}$

3 **a** Given that $\{a,\ b,\ c\}$ is a Pythagorean triple and k is a positive integer, show that $\{ka,\ kb,\ kc\}$ is also a Pythagorean triple.

b Use the multiples $k = 2,\ 3,\ 4,$ and 5 to construct new Pythagorean triples from these Pythagorean triples:

- **i** $\{3,\ 4,\ 5\}$
- **ii** $\{5,\ 12,\ 13\}$

4 Show that $\{2n+1,\ 2n^2+2n,\ 2n^2+2n+1\}$ is a Pythagorean triple for:

- **a** $n=1$
- **b** $n=2$
- **c** $n=3$
- **d** $n=4$

5 **a** Given that $\{a,\ b,\ c\}$ and $\{d,\ e,\ f\}$ are Pythagorean triples, show that $\{be-ad,\ bd+ae,\ cf\}$ is also a Pythagorean triple.

b Given that $\{3,\ 4,\ 5\}$ and $\{8,\ 15,\ 17\}$ are Pythagorean triples, use **a** to construct a new Pythagorean triple. Use technology to check your answer.

PUZZLE — PYTHAGOREAN TRIPLE SEQUENCES

Consider a sequence of numbers such that any two consecutive numbers are members of a Pythagorean triple.

For example, one such sequence is

$$\{10,\ 24,\ 26\}$$

$$\underbrace{6,\ 10}_{\{6,\ 8,\ 10\}},\ \underbrace{24,\ 18}_{\{18,\ 24,\ 30\}}.$$

What to do:

1. Create a sequence of 6 numbers with this property, which starts with 3 and ends with:
   - **a** 6
   - **b** 50
   - **c** 75.
2. Find the shortest such sequence you can, that starts with 3 and ends with 1000.

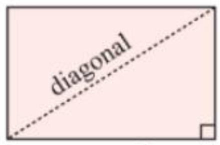
$$\boxed{3}\ ,\ \ldots\ \ ?\ \ \ldots,\ \boxed{1000}$$

E PROBLEM SOLVING USING PYTHAGORAS

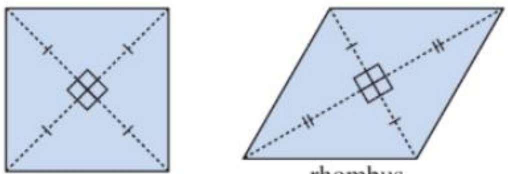
Right angled triangles occur in many practical problems. In these situations we can apply Pythagoras' theorem to help find unknown side lengths. The problem solving method involves the following steps:

Step 1: Draw a neat, clear diagram of the situation.
Step 2: Mark known lengths and right angles on the diagram.
Step 3: Use a symbol such as x to represent the unknown length.
Step 4: Apply Pythagoras' theorem to the right angled triangle.
Step 5: Solve the equation.
Step 6: Where necessary, write your answer in sentence form.

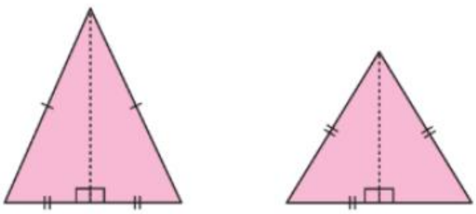
The following special figures contain right angled triangles:



In a **rectangle**, right angles exist between adjacent sides. We can construct a diagonal to form a right angled triangle.



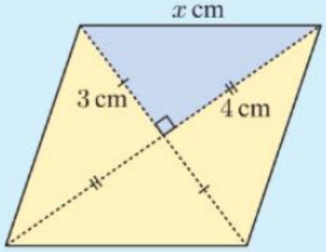
In a **square** and a **rhombus**, the diagonals bisect each other at right angles.



In an **isosceles triangle** and an **equilateral triangle**, the altitude bisects the base at right angles.

Example 11 **Self Tutor**

A rhombus has diagonals of length 6 cm and 8 cm.
Find the length of its sides.



The diagonals of a rhombus bisect at right angles.

Let each side of the rhombus have length x cm.

$\therefore\ x^2 = 3^2 + 4^2$ {Pythagoras}

$\therefore\ x^2 = 25$

$\therefore\ x = 5$ {as $x > 0$}

The sides are 5 cm long.

EXERCISE 6E

1 A rectangle has sides of length 8 cm and 3 cm. Find the length of its diagonals.

2 The longer side of a rectangle is three times the length of the shorter side. The length of the diagonal is 10 cm. Find the dimensions of the rectangle.

3 A rectangle with diagonals of length 20 cm has sides in the ratio $2:1$. Find the:

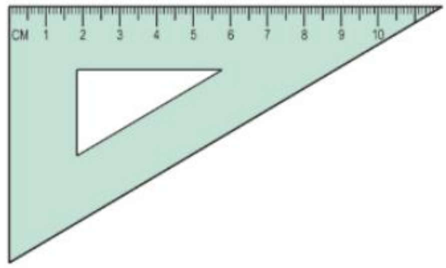
a perimeter **b** area of the rectangle.

4 A rhombus has sides of length 6 cm. One of its diagonals is 10 cm long. Find the length of the other diagonal.

5 A square has diagonals of length 10 cm. Find the length of its sides.

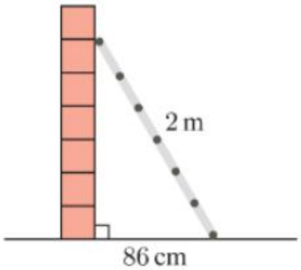
6 A rhombus has diagonals of length 8 cm and 10 cm. Find its perimeter.

7 To check that his set square was right angled, Roger measured its sides. The two shorter sides were 8 cm and 11.55 cm long, and the longest side was 14.05 cm long. Is the set square right angled?



8 An equilateral triangle has sides of length 12 cm. Find the length of one of its altitudes.

9



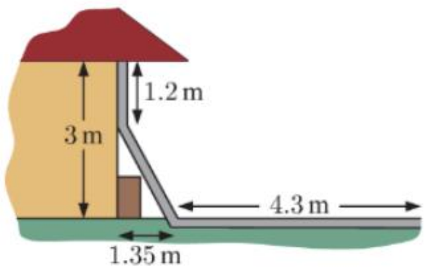
A ladder 2 metres long is leaning against a wall. The base is 86 cm from the wall.

a Write 86 cm in metres.

b How far up the wall does the ladder reach?

10 A 160 m long water pipe runs across the diagonal of a square paddock. What are the lengths of the sides of the paddock?

11

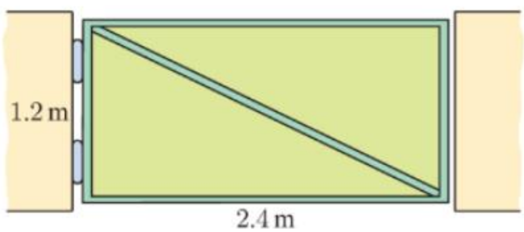


A drain pipe runs down the wall of a house, then out to the road as shown. Find the length of the pipe.

12 A garden gate is 2.4 metres wide and 1.2 metres high. The gate is strengthened by a diagonal strut.

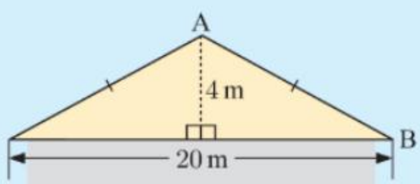
a How long is the strut?

b Calculate the length of steel needed for the frame of the gate, including the strut.

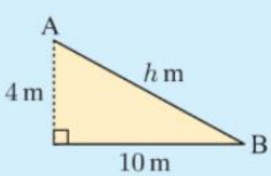


Example 12 **Self Tutor**

Find the length of the truss AB in the given roof structure.

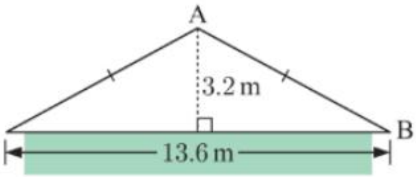


Let the length of the truss be h m.

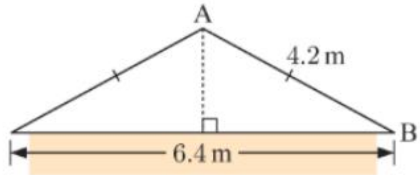


$$h^2 = 4^2 + 10^2 \quad \{\text{Pythagoras}\}$$
$$\therefore\ h^2 = 116$$
$$\therefore\ h = \sqrt{116}$$
$$\therefore\ h \approx 10.7703$$
$$\therefore\ \text{AB} \approx 10.8 \text{ m long}$$

13 Find the length of the truss AB in the roof structure shown:



14 How high is the roof above the walls in the roof structure shown?



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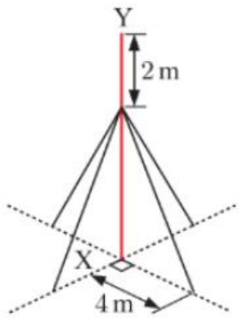
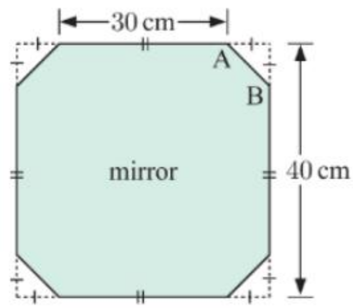


A pole [XY] is 10 m tall above the ground. To stabilise the pole, four wires are connected from a point 2 m below Y, to the ground. Each wire is anchored 4 m from X.
Find the total length of wire.

16 A mirror with the dimensions shown has a timber frame. Find:

a the length of AB

b the total length of timber.

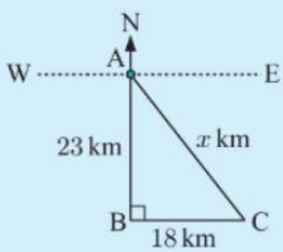


F NAVIGATION PROBLEMS

The problems in this Section involve navigation and compass bearings, including some problems where we test for right angles. Remember that drawing a diagram may be very helpful in solving the problem.

Example 13 Self Tutor

A yacht sails 23 km south and then 18 km east. How far is the yacht from its starting point?



Let the yacht be x km from its starting point.

$x^2 = 23^2 + 18^2$ {Pythagoras}

$\therefore\ x = \sqrt{23^2 + 18^2}$

$\therefore\ x \approx 29.2$

$\therefore$ the yacht is 29.2 km from the start.

EXERCISE 6F

1 A yacht sails 9 km due west and then 7 km due south. How far is it from its starting point?

2 A runner is 9 km west and 6 km south of her destination.

a If she runs in a straight line, how far is it to her destination?

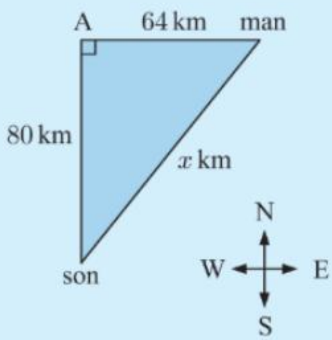
b How long will this run take her if she can run at $10\ \text{km h}^{-1}$?

3 Hayato and Yuki are sailing at sea. From a particular buoy they sail for 240 m in one direction, then turn and sail for 100 m in another. They are now 260 m from the buoy. Was the angle they turned a right angle?

4 Pirate Captain William Hawk left his hat on Treasure Island. He sailed 18 km northeast through the Forbidden Strait, then 11 km southeast to his home before realising it was missing. He sent his parrot to fetch the hat and return it to the boat. How far did the parrot need to fly?

Example 14 — Self Tutor

A man and his son both leave point A at the same time. The man rides his bicycle due east at $16\ \text{km h}^{-1}$. The son rides his bicycle due south at $20\ \text{km h}^{-1}$. How far apart are they after 4 hours?



After 4 hours, the man has travelled $4 \times 16 = 64$ km, and his son has travelled $4 \times 20 = 80$ km.

Let the distance between them be x km.

Thus $x^2 = 64^2 + 80^2$ {Pythagoras}

$\therefore\ x^2 = 10\,496$

$\therefore\ x = \sqrt{10\,496}$ {as $x > 0$}

$\therefore\ x \approx 102$

$\therefore$ they are about 102 km apart after 4 hours.

5 Two ships X and Y leave port P at the same time. X travels due east at a constant speed of $15\ \text{km h}^{-1}$. Y travels due north at a constant speed of $20\ \text{km h}^{-1}$.

a How far have X and Y each travelled after three hours?

b Find the distance between them after three hours.

6 Two runners set off from town A at the same time. One runs due east to town B, and the other runs due south to town C at twice the speed of the first. They both arrive at their destinations two hours later. Given that B and C are 50 km apart, find the average speed of each runner.

7 Answer the **Opening Problem** on page **92**.

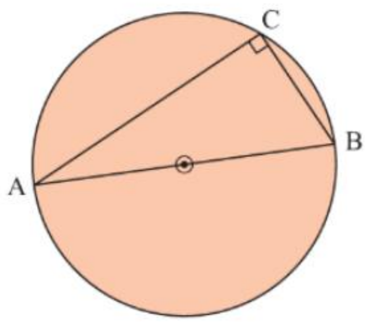
G CIRCLE PROBLEMS

There are also certain properties of circles which involve right angles. They are described in the following theorems:

ANGLE IN A SEMI-CIRCLE

The angle in a semi-circle is a right angle.

No matter where C is placed on the circle, $A\widehat{C}B$ is always a right angle.



DEMO

Example 15 **Self Tutor**

A circle has diameter [XY] of length 13 cm. Z is a point on the circle such that XZ is 5 cm. Find the length YZ.

From the angle in a semi-circle theorem, $X\widehat{Z}Y$ is a right angle.

Let the length YZ be x cm.

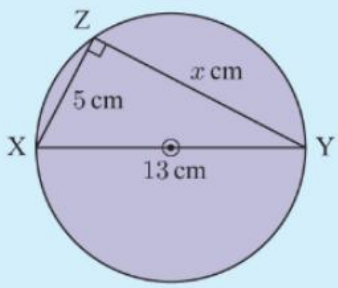
$\therefore\ 5^2 + x^2 = 13^2$ {Pythagoras}

$\therefore\ x^2 = 169 - 25 = 144$

$\therefore\ x = \sqrt{144}$ {as $x > 0$}

$\therefore\ x = 12$

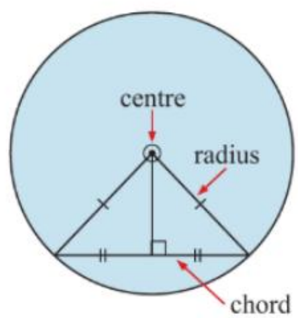
So, YZ has length 12 cm.



A CHORD OF A CIRCLE

The line drawn from the centre of a circle at right angles to a chord, bisects the chord.

The construction of radii from the centre of the circle to the end points of the chord produces an isosceles triangle. The above property then follows from the **isosceles triangle theorem**.



Example 16 Self Tutor

A circle with radius 8 cm has a chord of length 10 cm.
Find the shortest distance from the centre of the circle to the chord.

The shortest distance is the 'perpendicular distance'. The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

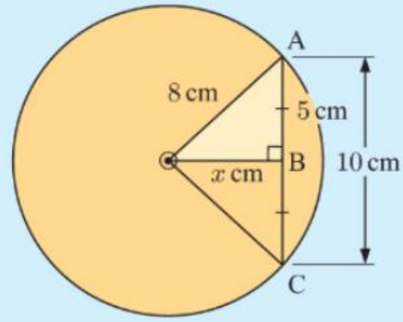
$\therefore$ AB = BC = 5 cm

In $\triangle$AOB, $5^2 + x^2 = 8^2$ {Pythagoras}

$\therefore\ x^2 = 64 - 25 = 39$

$\therefore\ x = \sqrt{39}$ {as $x > 0$}

$\therefore\ x \approx 6.24$

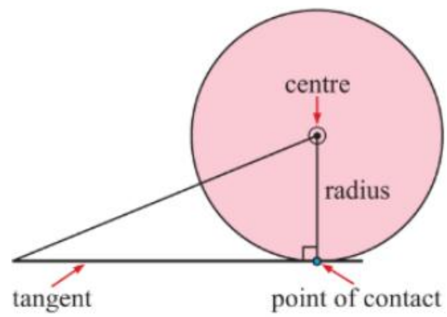


So, the shortest distance is about 6.24 cm.

TANGENT-RADIUS PROPERTY

A tangent to a circle and a radius at the point of contact meet at right angles.

Notice that we can now form a right angled triangle.



Example 17 Self Tutor

A tangent of length 10 cm is drawn to a circle with radius 7 cm. How far is the centre of the circle from the end point of the tangent?

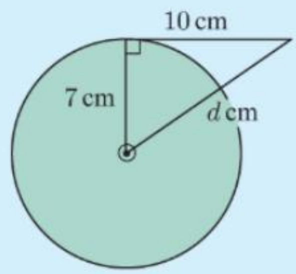
Let the distance be d cm.

$\therefore\ d^2 = 7^2 + 10^2$ {Pythagoras}

$\therefore\ d^2 = 149$

$\therefore\ d = \sqrt{149}$ {as $d > 0$}

$\therefore\ d \approx 12.2$

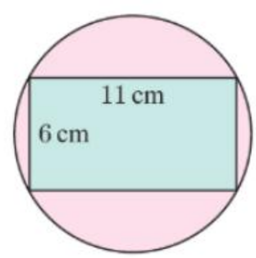


The centre is about 12.2 cm from the end point of the tangent.

EXERCISE 6G

1 A circle has diameter [AB] of length 10 cm. C is a point on the circle such that AC is 8 cm. Find the length BC.

2 A rectangle with side lengths 11 cm and 6 cm is inscribed in a circle. Find the radius of the circle.



3 A circle with radius 4 cm has a chord of length 3 cm. Find the shortest distance from the centre of the circle to the chord.

4 A chord of length 6 cm is 3 cm from the centre of a circle. Find the radius of the circle.

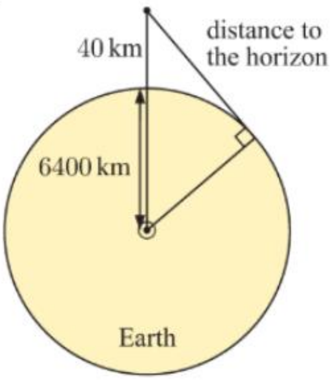
5 A chord is 5 cm from the centre of a circle of radius 8 cm. Find the length of the chord.

6 A circle has radius 3 cm. A tangent is drawn to the circle from point P, which is 9 cm from the circle's centre. How long is the tangent?

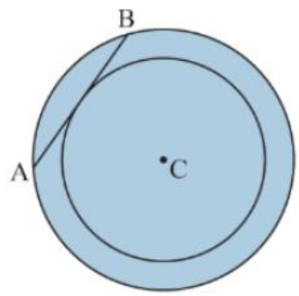
7 A tangent of length 12 cm has end point 16 cm from the circle's centre. Find the radius of the circle.

8 A circular table of diameter 2 m is placed in the corner of a room so that its edges touch two perpendicular walls. Find the shortest distance from the corner of the room to the edge of the table.

9 The radius of the Earth is about 6400 km. Find the distance to the horizon from a rocket which is 40 km above the Earth's surface.

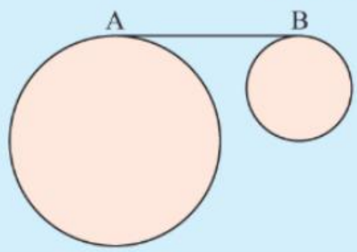


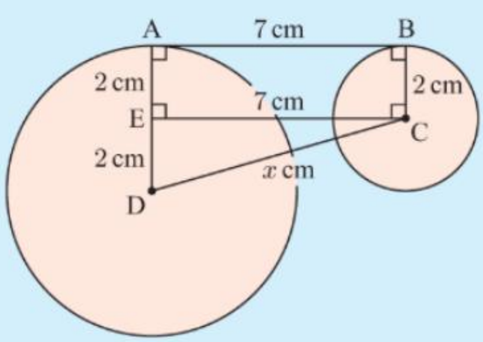
10 C is the centre of two circles with radii 7 cm and 5 cm.
[AB] is a chord of the larger circle, and a tangent of the smaller circle.
Find the length of [AB].



Example 18 **Self Tutor**

Two circles have a common tangent with points of contact A and B which are 7 cm apart. The radii of the circles are 4 cm and 2 cm respectively. Find the distance between the centres.





For centres C and D, we draw [BC], [AD], [CD], and [CE] $\parallel$ [AB].

$\therefore$ ABCE is a rectangle.

$\therefore$ CE $= 7$ cm {as CE = AB}

and DE $= 4 - 2 = 2$ cm

Let the distance between the centres be x cm.

$\therefore\ x^2 = 2^2 + 7^2$ {Pythagoras in $\triangle$DEC}

$\therefore\ x^2 = 53$

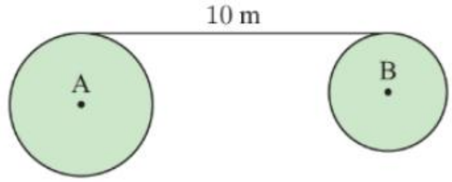
$\therefore\ x = \sqrt{53}$ {as $x > 0$}

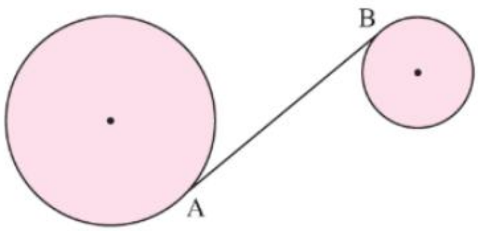
$\therefore\ x \approx 7.28$

The distance between the centres is about 7.28 cm.

11 A and B are the centres of two circles with radii 4 m and 3 m respectively. The illustrated common tangent has length 10 m.

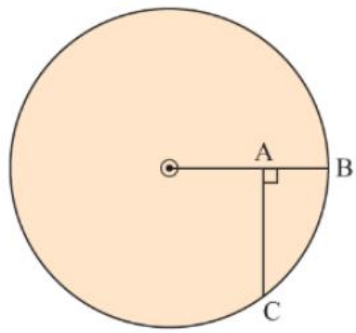
Find the distance between the centres to the nearest millimetre.



12 

The illustration shows two circles of radii 4 cm and 2 cm respectively. The distance between the two centres is 8 cm. Find the length of the common tangent [AB].

13 In the given figure, AB = 1 cm and AC = 3 cm. Find the radius of the circle.

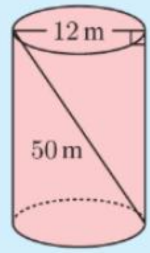


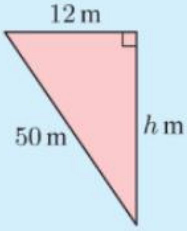
H 3-DIMENSIONAL PROBLEMS

Pythagoras' theorem is often used to find lengths in **3-dimensional** problems.

Example 19 Self Tutor

A 50 m rope is attached inside an empty cylindrical wheat silo of diameter 12 m as shown. How high is the silo?





Let the height be h m.

$\therefore\ h^2 + 12^2 = 50^2$ {Pythagoras}

$\therefore\ h^2 + 144 = 2500$

$\therefore\ h^2 = 2356$

$\therefore\ h \approx 48.5$ {as $h > 0$}

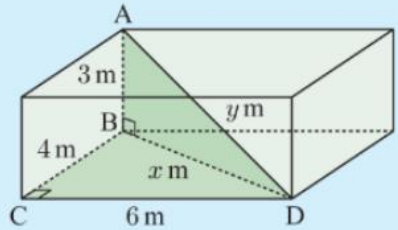
The wheat silo is about 48.5 m high.

Sometimes we need to apply Pythagoras' theorem *twice*.

Example 20 — Self Tutor

A room is 6 m by 4 m, and has a height of 3 m. Find the distance from a corner point on the floor to the opposite corner point on the ceiling.

The required distance is AD. We join [BD].



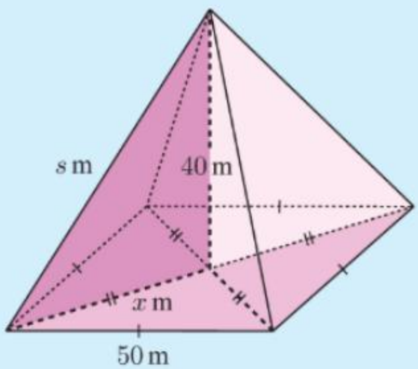
In $\triangle$BCD, $x^2 = 4^2 + 6^2$ {Pythagoras}
In $\triangle$ABD, $y^2 = x^2 + 3^2$ {Pythagoras}

$\therefore\ y^2 = 4^2 + 6^2 + 3^2$
$\therefore\ y^2 = 61$
$\therefore\ y \approx 7.81$ {as $y > 0$}

$\therefore$ the distance is about 7.81 m.

Example 21 — Self Tutor

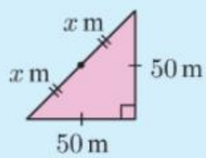
A pyramid of height 40 m has a square base with edges of length 50 m.
Determine the length of the slant edges.



Let a slant edge have length s m.

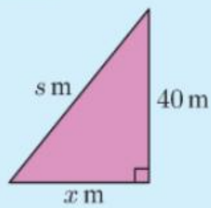
Let half a diagonal have length x m.

Using



$(2x)^2 = 50^2 + 50^2$ {Pythagoras}
$\therefore\ 4x^2 = 5000$
$\therefore\ x^2 = 1250$

Using

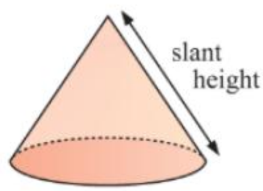


$s^2 = x^2 + 40^2$ {Pythagoras}
$\therefore\ s^2 = 1250 + 1600$
$\therefore\ s^2 = 2850$
$\therefore\ s \approx 53.4$ {as $s > 0$}

Each slant edge is about 53.4 m long.

EXERCISE 6H

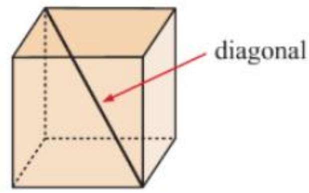
1 A cone has a slant height of 17 cm, and a base radius of 8 cm. How high is the cone?



2 Find the length of the longest nail that could fit entirely within a cylindrical can with radius 3 cm and height 8 cm.

3 A 20 cm nail just fits inside a cylindrical can. Three identical spherical balls need to fit entirely within the can. What is the maximum radius each ball could have?

4 A cube has sides of length 3 cm. Find the length of a diagonal of the cube.

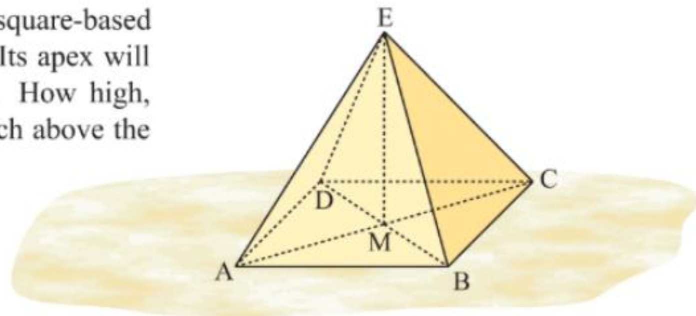


5 A room is 5 m by 3 m, and has a height of 3.5 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

6 A rectangular box has internal dimensions 2 cm by 3 cm by 2 cm. Find the length of the longest toothpick that can be placed within the box.

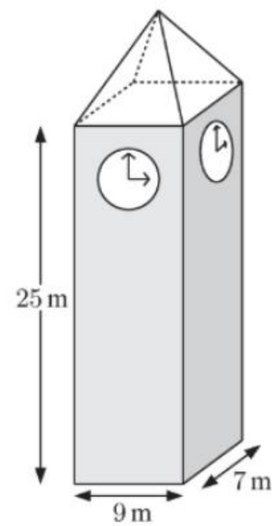
7 Can an 8.5 m long piece of timber be stored in a rectangular shed which is 6 m by 5 m by 2 m high?

8 An Egyptian Pharaoh wishes to build a square-based pyramid with all edges of length 100 m. Its apex will be directly above the centre of its base. How high, to the nearest metre, will the pyramid reach above the desert sands?



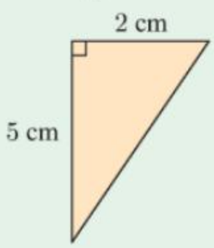
9 A symmetrical square-based pyramid has height 10 cm and slant edges of length 15 cm. Find the dimensions of its square base.

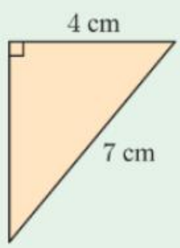
10 A clock tower has the dimensions shown. The slant edges of the pyramid are 7.5 m long. Find the height of the tower, to the nearest cm.

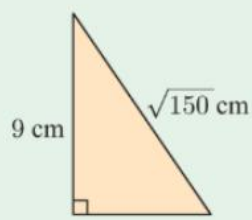


REVIEW SET 6A

1 Find the length of the unknown side in each of the following triangles:

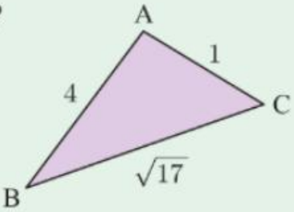
a 

b 

c 

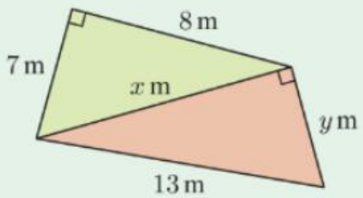
2 Determine whether $\{5, 11, 13\}$ is a Pythagorean triple.

3 Is this triangle right angled?
Explain your answer.

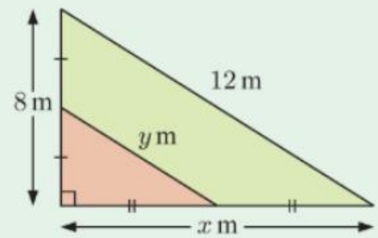


4 Find the values of the unknowns:

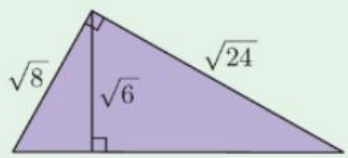
a



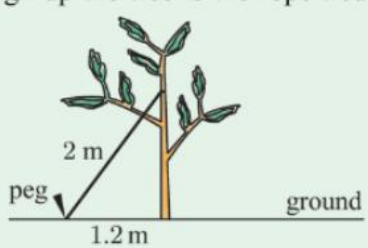
b



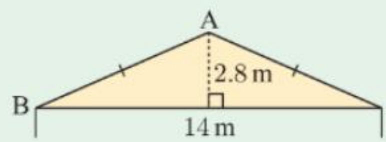
5 Use the figure alongside to show that $\sqrt{2}+\sqrt{18}=\sqrt{32}$.



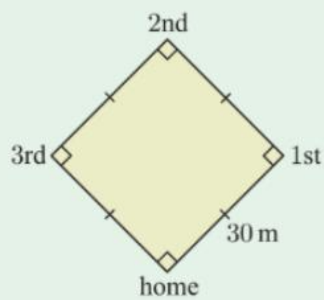
6 A young tree has a 2 m support rope tied to a peg in the ground 1.2 m from its base. How high up the tree is the rope tied?



7 Find the length of the truss AB for the roof structure shown.



8 A softball diamond has sides of length 30 m.
Determine the distance a fielder must throw the ball from second base to reach home base.



9 Mia and Yvette leave home at the same time. Mia walks east at 5 km h^{-1} and Yvette walks at 4 km h^{-1} in another direction. After 90 minutes they are approximately 9.6 km apart.

a How far do they each walk in 90 minutes?

b Show that Yvette travelled at right angles to Mia.

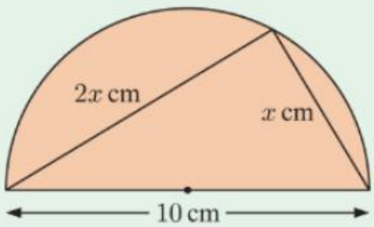
c What directions might Yvette have walked in?

10 A rectangle has diagonals 15 cm long, and one side is 8 cm long. Find the perimeter of the rectangle.

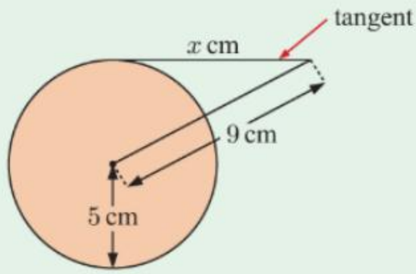
11 A circle has a chord of length 10 cm. The shortest distance from the circle's centre to the chord is 5 cm. Find the radius of the circle.

12 Find x:

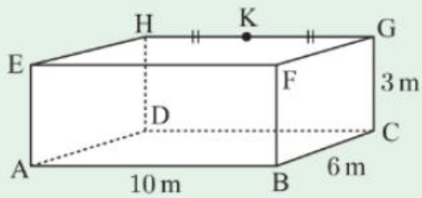
a



b



13



A room is 10 m by 6 m by 3 m. Find the shortest distance from:

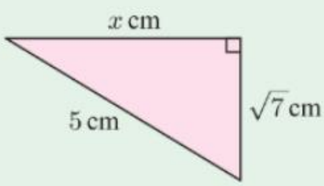
a E to K

b A to K.

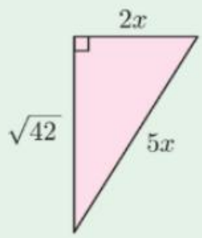
REVIEW SET 6B

1 Find the value of x in the following:

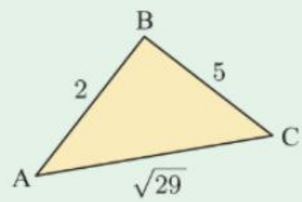
a



b



2 Show that this triangle is right angled, and identify which is the right angle.



3 Find k given that $\{12, k, 37\}$ is a Pythagorean triple.

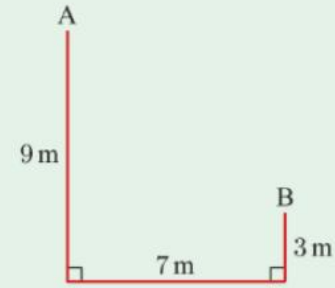
4 The diameter of a circle is 20 cm. Find the shortest distance from a chord of length 16 cm to the centre of the circle.

5 A landscaped garden is 25 metres square. Find the length of a path from one corner to the opposite corner.

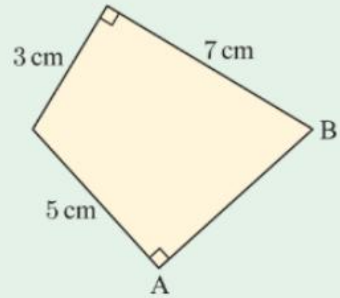
6 A fishing boat is 8 km west and 15 km south of its port. How far must it sail if it returns to port by the shortest distance?

7 Find the distance AB in the following figures:

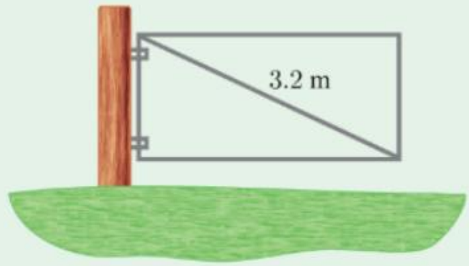
a



b



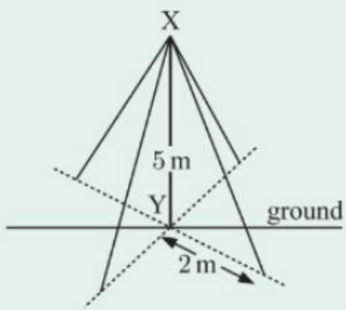
8 A rectangular gate is twice as wide as it is high. It is held in shape by a diagonal strut 3.2 m long. Find the height of the gate to the nearest millimetre.



9 A 15 m ladder reaches three times as far up a vertical wall as the base is out from the wall. How far up the wall does the ladder reach?

10 Can a wooden beam 10.5 m long be placed in a rectangular shed 8 m by 7 m by 3 m?

11



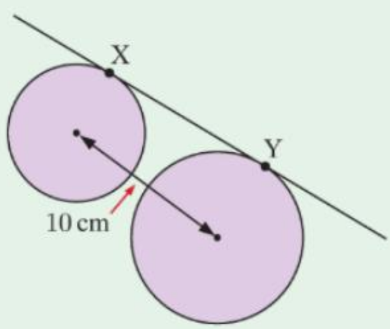
A pole [XY] is 5 metres tall. Four wires from the top of the pole X connect it to the ground.

Each wire is pegged 2 metres from the base of the pole. Find the total length of the four wires.

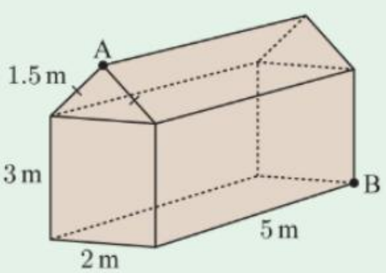
12 Two circles have a common tangent with points of contact X and Y.

The radii of the circles are 4 cm and 5 cm respectively, and the distance between the centres is 10 cm.

Find the length of the common tangent [XY].



13



A barn has the dimensions given.

Find the shortest distance from A to B.

Chapter 7

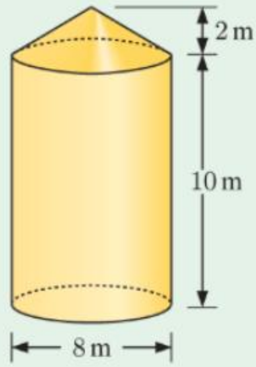
Measurement

Contents:

- A Length and perimeter
- B Area
- C Surface area
- D Volume
- E Capacity

OPENING PROBLEM

A grain silo consists of a concrete cylinder with a conical roof. The entire outside of the silo is to be painted green. The paint costs €70 for each 10 litre can, and each litre covers an area of 12 square metres.



Things to think about:

- **a** What *surface area* of the silo needs to be painted?
- **b** What will be the total cost of the paint?
- **c** What *volume* of grain can the silo contain?
- **d** What is the *capacity* of the silo?

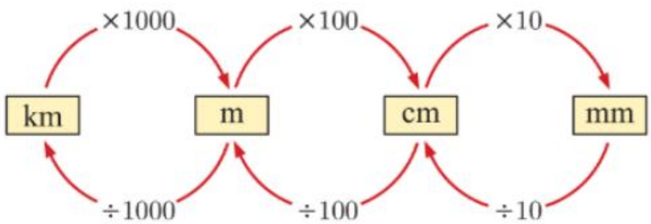
A LENGTH AND PERIMETER

The base unit of length is the **metre** (m). Lengths are also commonly measured in:

- **millimetres** (mm)
- **centimetres** (cm)
- **kilometres** (km)

The units of length are related as follows:

1 kilometre (km) = 1000 metres (m)
1 metre (m) = 100 centimetres (cm)
1 centimetre (cm) = 10 millimetres (mm)

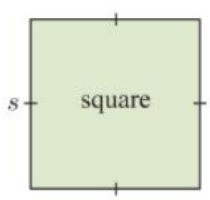


The **perimeter** of a closed figure is the total distance around its **boundary**.

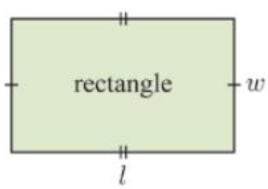
The perimeter of a **polygon** is the sum of the lengths of its sides.

For a **circle**, the perimeter is given the special name **circumference**.

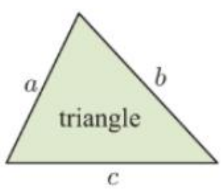
The following are perimeter formulae for commonly occurring shapes:



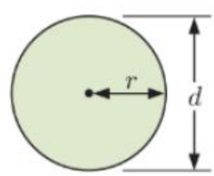
$P = 4s$



$P = 2(l + w)$



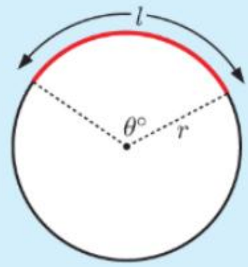
$P = a + b + c$



$C = 2\pi r$ or $C = \pi d$

The **arc length** of part of a circle is a fraction of the circumference of the circle.

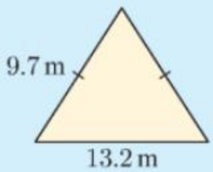
Arc length $l = \dfrac{\theta}{360} \times 2\pi r$



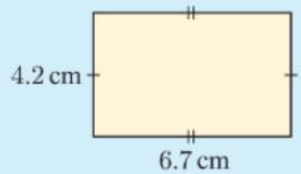
Example 1 — Self Tutor

Find the perimeter of the following figures:

a



b



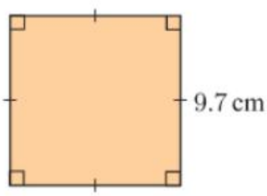
a Perimeter $= (2 \times 9.7 + 13.2)$ m
$= 32.6$ m

b Perimeter $= (2 \times 4.2\ +\ 2 \times 6.7)$ cm
$= (8.4 + 13.4)$ cm
$= 21.8$ cm

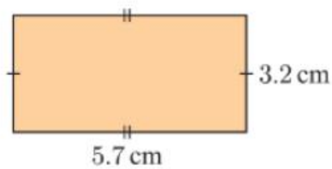
EXERCISE 7A

1 Find the perimeter of the following figures:

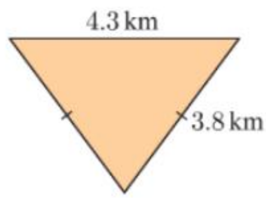
a



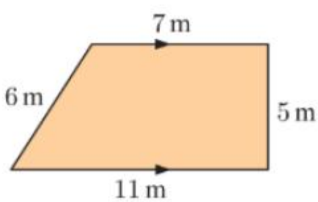
b



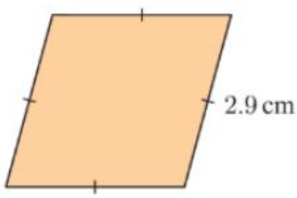
c



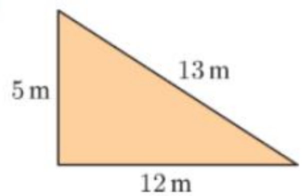
d



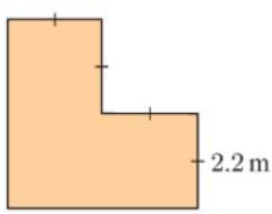
e



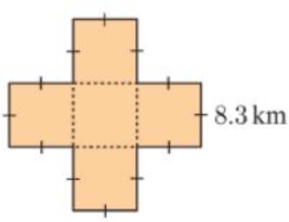
f



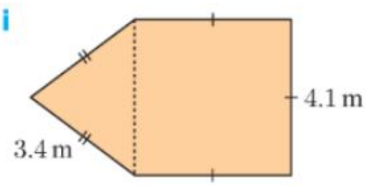
g



h



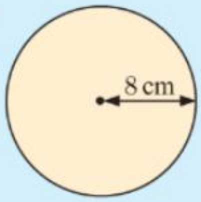
i



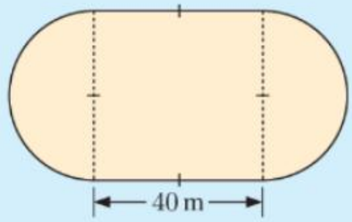
Example 2 **Self Tutor**

Find the perimeter of the following figures:

a

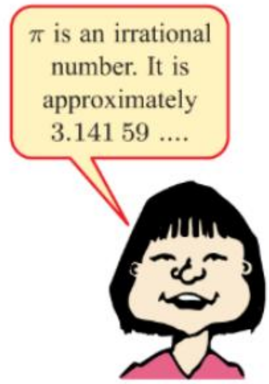


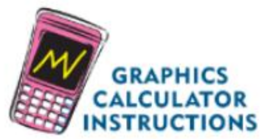
b



a Circumference
$= 2\pi r$
$= 2\pi \times 8$ cm
$= 16\pi$ cm
≈ 50.3 cm

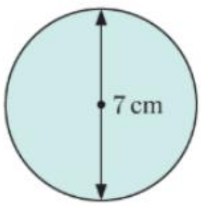
b Perimeter
$=$ circumference of circle
$+\ 2 \times$ length of each straight side
$= ((\pi \times 40) + (2 \times 40))$ m
$= (40\pi + 80)$ m
≈ 206 m



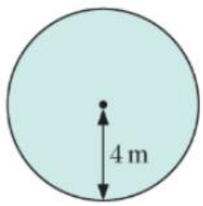


2 Find the perimeter of the following figures:

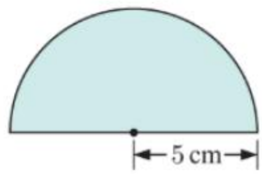
a



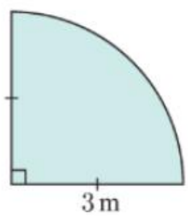
b



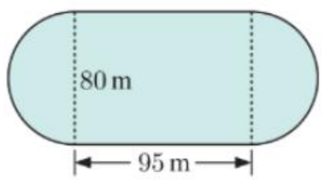
c



d



e



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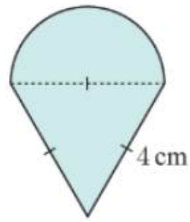

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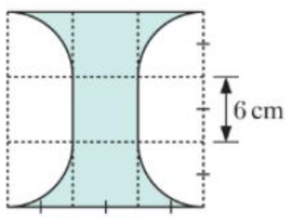

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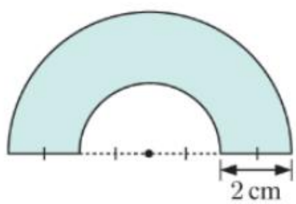
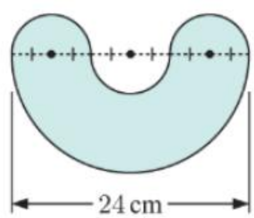
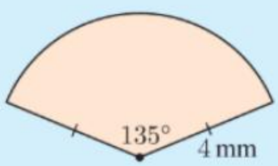


i



Example 3 **Self Tutor**

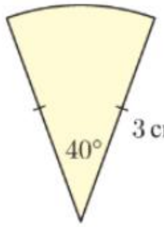
Find the perimeter of the sector:



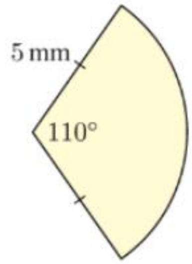
Perimeter
$= (4 + 4)$ mm $+$ length of arc
$= 8$ mm $+ \left(\frac{135}{360}\right) \times 2 \times \pi \times 4$ mm
≈ 17.4 mm

3 Find the perimeter of the following figures:

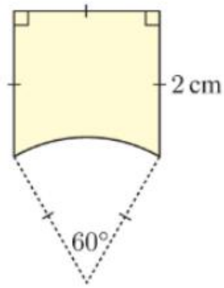
a



b



c



4 A square paddock has sides of length 450 m. It is to be fenced with 3 strands of wire. The wire costs \$0.28 per metre. Find:

a the perimeter of the paddock

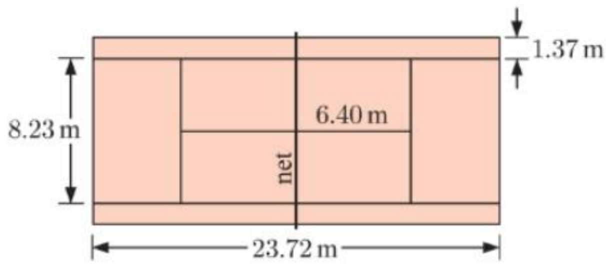
b the total length of wire needed

c the total cost of the wire.

5 A tennis court has the dimensions shown.

a What is the perimeter of the court?

b Find the total length of all the marked lines, not including the net.



6 An ironman triathlon course includes three stages:

- a 3.8 km swim
- a 180.2 km bicycle ride
- a 42.2 km run.

Find:

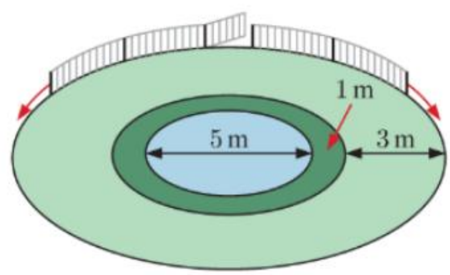
a the total distance around the course

b the average speed of a contestant who took 8 hours 40 minutes to complete the course.

7

Al needs 30 metres of rope. He has a coil in the shed which is 35 cm in diameter and has 29 circles of rope. Will this be enough?

8 At Bushby Park there is a 5 m diameter circular pond which is surrounded by a 1 m wide garden bed, and then a 3 m wide lawn. A safety fence is placed around the lawn with posts every 3 m and a gateway 1.84 m wide. The gate is wrought iron.



a How many metres of safety fence are needed?

b How many posts are needed?

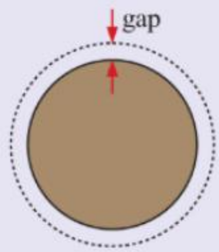
c If the posts cost £15.75 and the safety fence costs £18.35 per metre, calculate the total cost of the fence (excluding the gate).

9 Find the total length of ribbon used to tie a box as illustrated. 25 cm of ribbon is required for the bow.

PUZZLE — ROPE AROUND THE EARTH

Consider these two scenarios:

1 A loop of rope is placed tightly around a circular table. 1 metre of rope is then added to the loop, and the rope is stretched into a circle so there is a gap between the rope and the table.



2

A loop of rope is placed tightly around the Earth. Again, 1 metre of rope is then added to the loop, and the rope is stretched into a circle so there is a gap between the rope and the Earth.

Do you think the gap will be larger in scenario **1** or scenario **2**? Perform some calculations to find out whether you are correct!

B AREA

All around us we see surfaces such as walls, ceilings, paths, and sporting fields. All of these surfaces have boundaries which define their shape.

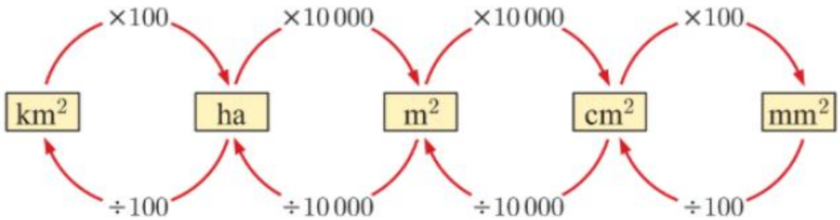
The **area** of a region is the amount of **surface** within its boundaries.

The **area** of surface of a closed figure is measured in terms of the number of square units it encloses.

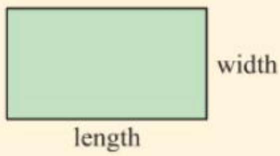
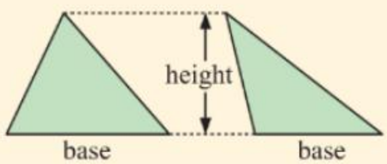
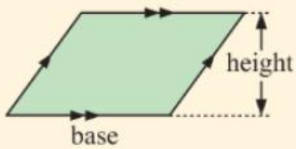
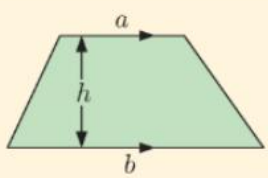
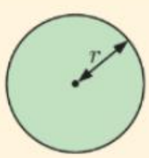
Area can be measured in **square millimetres**, **square centimetres**, **square metres**, and **square kilometres**. There is also another unit called the **hectare** (ha).

$$\begin{aligned}
1\text{ mm}^2 &= 1\text{ mm} \times 1\text{ mm} \\
1\text{ cm}^2 &= 10\text{ mm} \times 10\text{ mm} = 100\text{ mm}^2 \\
1\text{ m}^2 &= 100\text{ cm} \times 100\text{ cm} = 10\,000\text{ cm}^2 \\
1\text{ ha} &= 100\text{ m} \times 100\text{ m} = 10\,000\text{ m}^2 \\
1\text{ km}^2 &= 1000\text{ m} \times 1000\text{ m} = 1\,000\,000\text{ m}^2 \text{ or } 100\text{ ha}
\end{aligned}$$

We can convert units of area using the chart:

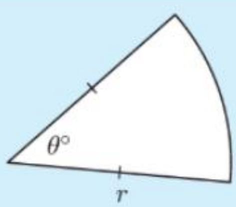


In previous years we have established the following area formulae:

Shape	*Figure*	*Formula*
Rectangle		**Area = length × width**
Triangle		**Area = $\frac{1}{2}$ × base × height**
Parallelogram		**Area = base × height**
Trapezium		**Area =** $\left(\frac{a+b}{2}\right) \times h$
Circle		**Area =** πr^2

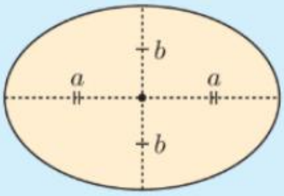
The area of a **sector** is a fraction of the area of the circle it is taken from.

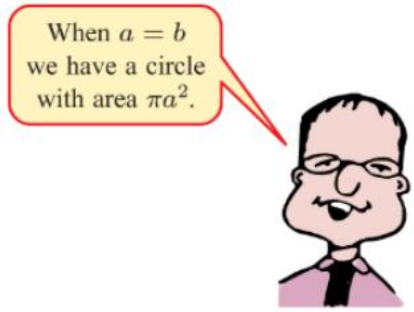
Area of sector = $\left(\frac{\theta}{360}\right) \times \pi r^2$



The area of an **ellipse** with **semi-axes** a and b is given by the formula:

Area $\quad A = \pi ab$

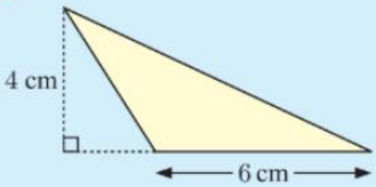




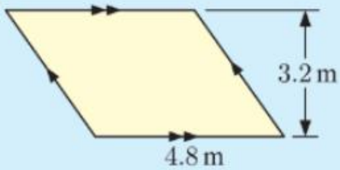
Example 4 **Self Tutor**

Find the area of:

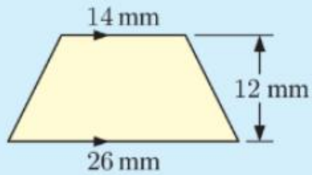
a



b



c



a Area
$= \frac{1}{2}(\text{base} \times \text{height})$
$= \frac{1}{2} \times 6 \times 4 \text{ cm}^2$
$= 12 \text{ cm}^2$

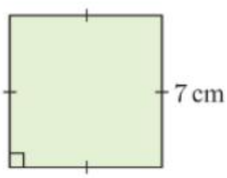
b Area
$= \text{base} \times \text{height}$
$= 4.8 \times 3.2 \text{ m}^2$
$= 15.36 \text{ m}^2$

c Area
$= \left(\frac{a+b}{2}\right) \times h$
$= \left(\frac{14+26}{2}\right) \times 12 \text{ mm}^2$
$= 240 \text{ mm}^2$

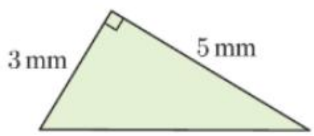
EXERCISE 7B

1 Find the area of the following figures:

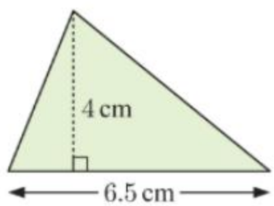
a



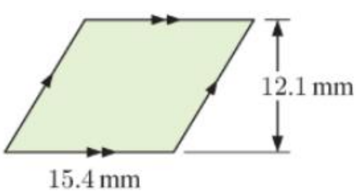
b



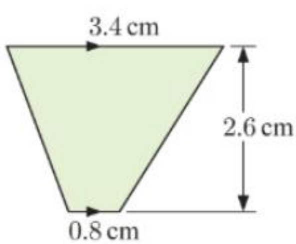
c



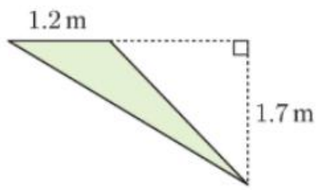
d



e



f



Example 5 **Self Tutor**

An equilateral triangle has sides of length 6 cm. Find its area.

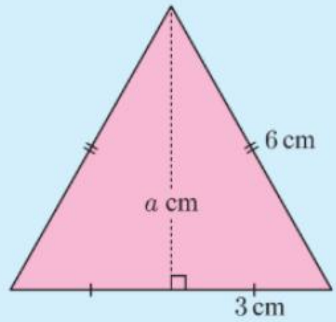
The altitude bisects the base at right angles.

$\therefore\ a^2 + 3^2 = 6^2$ {Pythagoras}

$\therefore\ a^2 + 9 = 36$

$\therefore\ a^2 = 27$

$\therefore\ a = \sqrt{27}$ {as $a > 0$}



$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 6 \times \sqrt{27}$

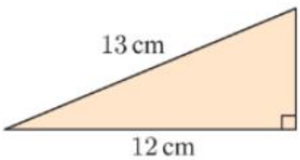
$= 3\sqrt{27}\ \text{cm}^2$

$\approx 15.6\ \text{cm}^2$

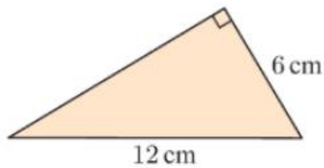
So, the area is about $15.6\ \text{cm}^2$.

2 Find the area of the following figures:

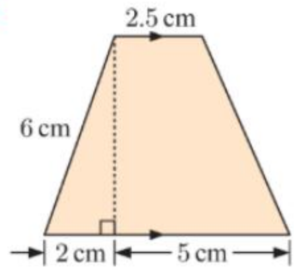
a



b



c

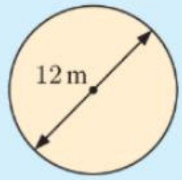


3 An isosceles triangle has equal sides of length 8 cm and a base of length 6 cm. Find the area of the triangle.

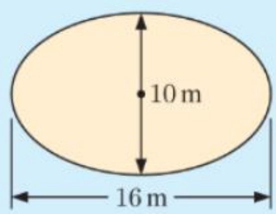
Example 6 **Self Tutor**

Find, to 1 decimal place, the shaded area:

a



b



a $\text{Area} = \pi r^2$

$= \pi \times 6^2$

$= 36\pi$

$\approx 113.1\ \text{m}^2$

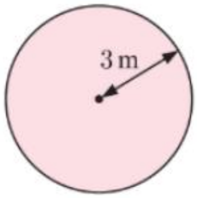
b $\text{Area} = \pi ab$

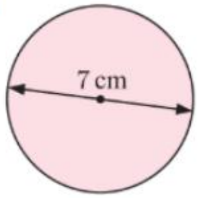
$= \pi \times 8 \times 5$

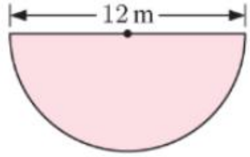
$= 40\pi$

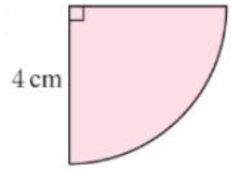
$\approx 125.7\ \text{m}^2$

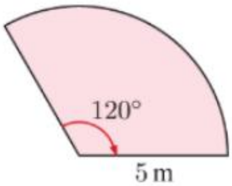
4 Find the shaded area:

a


b


c


d


e


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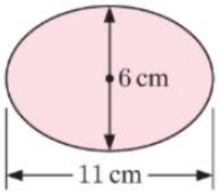
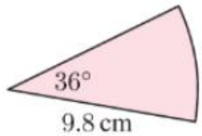
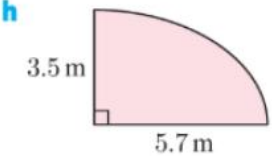


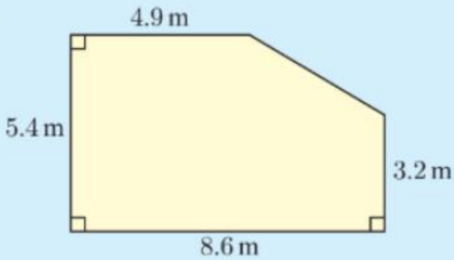
g


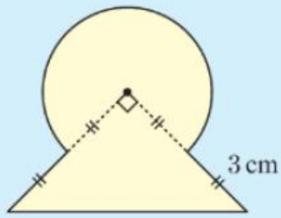
h


quarter ellipse

Example 7 **Self Tutor**

Find the shaded area:

a


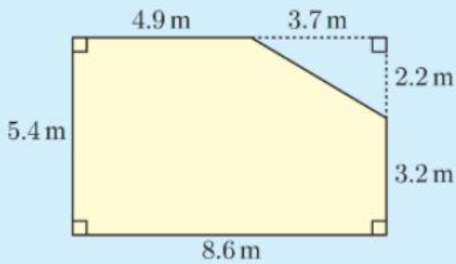
b


a Shaded area

$= $ area of rectangle $-$ area of triangle

$= (8.6 \times 5.4)\ \text{m}^2 - (\frac{1}{2} \times 3.7 \times 2.2)\ \text{m}^2$

$= 42.37\ \text{m}^2$



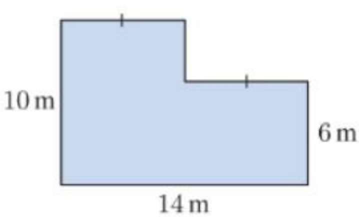
b Shaded area

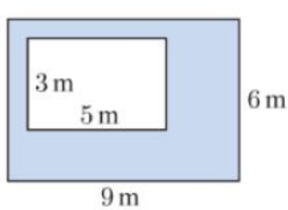
$= $ area of triangle $+$ area of sector

$= (\frac{1}{2} \times 6 \times 6)\ \text{cm}^2 + (\frac{3}{4} \times \pi \times 3^2)\ \text{cm}^2$

$\approx 39.2\ \text{cm}^2$

5 Find the shaded area:

a


b


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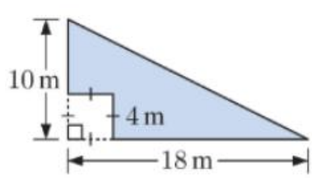

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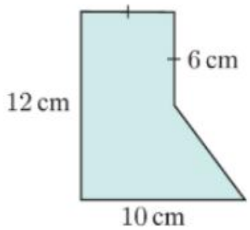

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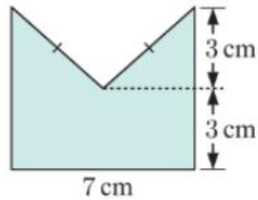

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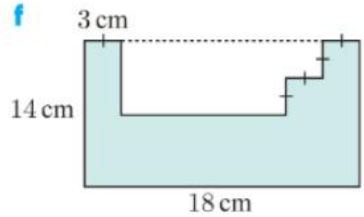
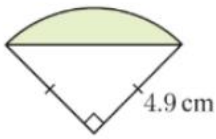


g



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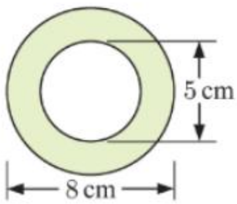
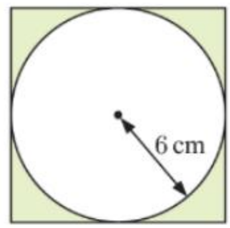
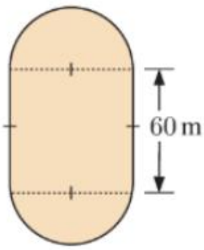


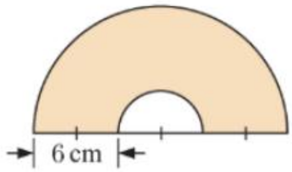
i



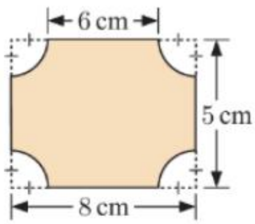
j



k

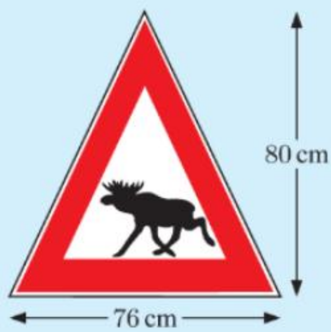


l



Example 8 **Self Tutor**

The Highways Department orders 25 road signs that warn motorists to watch out for moose.



a What area of metal sheeting (in m^2) is required for the 25 signs?

b To allow for wastage when the signs are cut, an extra 20% of the metal is needed. How much metal needs to be purchased?

c The sheet metal costs \$58.40 per m^2. What will its cost be?

a Area of one sign $= \frac{1}{2} \times 0.76 \times 0.8 \text{ m}^2$
$= 0.304 \text{ m}^2$

$\therefore$ the area of 25 signs $= 25 \times 0.304 \text{ m}^2$
$= 7.6 \text{ m}^2$

b Area needed to be purchased $= 7.6 \text{ m}^2 \times 120\%$
$= 9.12 \text{ m}^2$

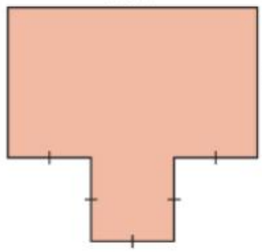
c Cost of metal $= 9.12 \text{ m}^2 \times \$58.40 \text{ per m}^2$
$= \$532.61$

6 

A farmer wishes to fertilise his paddock using 150 kg of superphosphate per hectare. The paddock is 550 m $\times$ 300 m.

What amount of superphosphate will he need to spread?

7 The diagram shows the dimensions of a courtyard. It is to be paved with 60 cm square tiles costing €9.40 each.

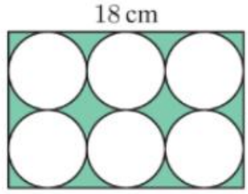


- **a** How many tiles will be needed?
- **b** How much will the tiles cost?

8 A gravel path 1 m wide is placed around a circular garden bed of diameter 2 m. The gravel costs £7.90 per m^2.

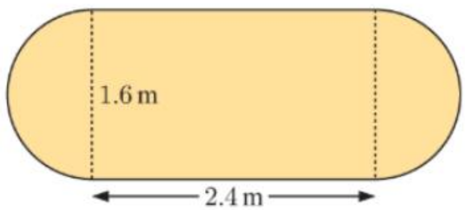
- **a** Find the area of the path.
- **b** Find the cost of the gravel.

9 6 identical metal discs are stamped out of an 18 cm by 12 cm sheet of copper as illustrated. What percentage of the copper is wasted?



10 A 15 cm by 20 cm rectangle has the same perimeter as a square. Which figure has the greater area? Explain your answer.

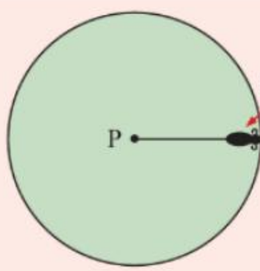
11 The diagram shows the dimensions of a table-top. A protective cloth is cut from a roll 1.6 m wide to exactly fit the table-top. The cloth costs $18.40 per metre of length.



- **a** What length of cloth must be purchased?
- **b** Calculate the cost of the fabric.
- **c** Find the area of the table-top.
- **d** Calculate the percentage of cloth that is wasted.

ACTIVITY 1 SAM THE SHEEP

When Sam the sheep was tethered to a pole P on the back lawn, he was restricted to eating the grass within a circle:

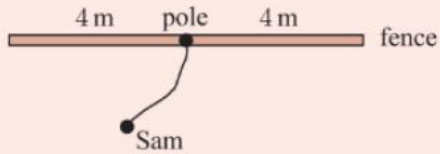


What to do:

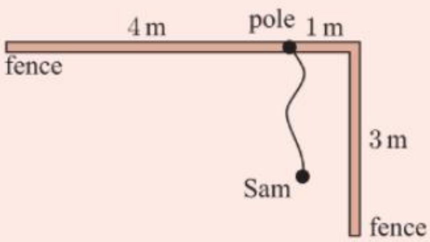
1. For each situation, draw a scale diagram with scale $1 \text{ cm} \equiv 1 \text{ m}$ to show the exact region from which Sam may feed. The rope is 3 m long in each case.

a

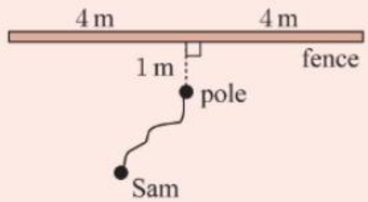
b



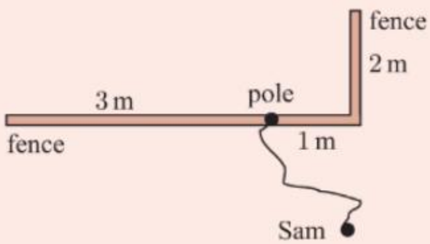
c



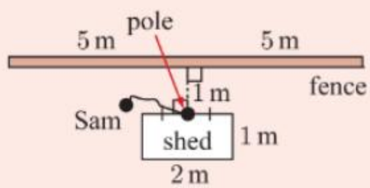
d



e



f



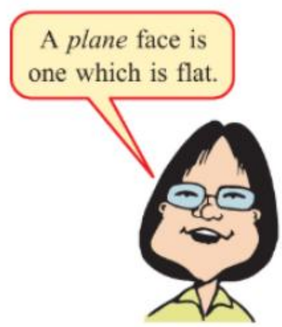
2. For each of the situations above, use your scale diagram and appropriate area formulae to calculate the area of grass that Sam can feed on.

C SURFACE AREA

SOLIDS WITH PLANE FACES

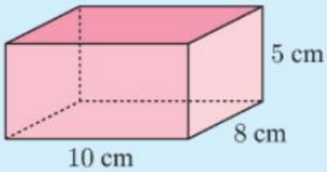
The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

The surface area is therefore the same as the area of the **net** required to make the figure.



Example 9 **Self Tutor**

Find the surface area of the rectangular prism:

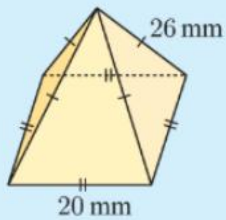


The figure has 2 faces which are $10 \text{ cm} \times 8 \text{ cm}$
2 faces which are $10 \text{ cm} \times 5 \text{ cm}$
and 2 faces which are $8 \text{ cm} \times 5 \text{ cm}$.

Total surface area
$= 2 \times 10 \times 8 \text{ cm}^2 + 2 \times 10 \times 5 \text{ cm}^2 + 2 \times 8 \times 5 \text{ cm}^2$
$= (160 + 100 + 80) \text{ cm}^2$
$= 340 \text{ cm}^2$

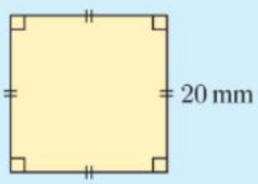
Example 10 **Self Tutor**

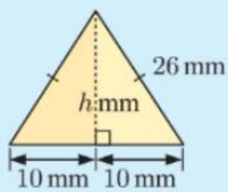
Find the surface area of the square-based pyramid:



The figure has:

- 1 square base
- 4 triangular faces





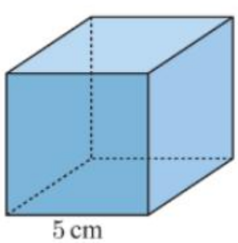
$h^2 + 10^2 = 26^2$ {Pythagoras}
$\therefore \ h^2 + 100 = 676$
$\therefore \ h^2 = 576$
$\therefore \ h = 24$ {as $h > 0$}

Total surface area $= 20 \times 20 \text{ mm}^2 + 4 \times (\frac{1}{2} \times 20 \times 24) \text{ mm}^2$
$= (400 + 960) \text{ mm}^2$
$= 1360 \text{ mm}^2$

EXERCISE 7C.1

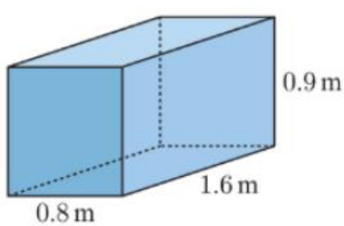
1 Find the surface area of each solid:

a



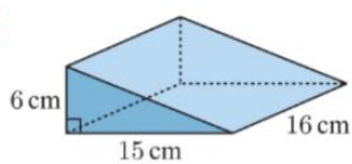
cube

b



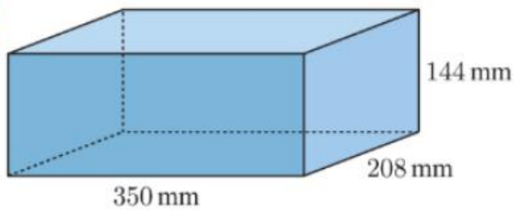
rectangular prism

c



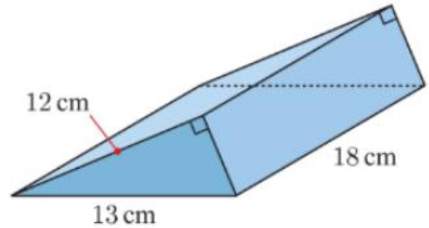
triangular prism

d



rectangular prism

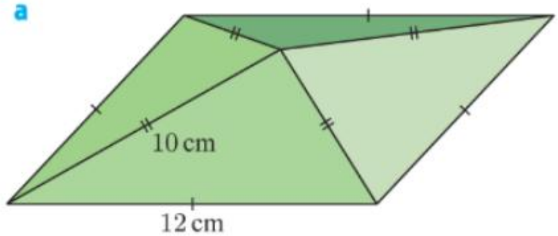
e



triangular prism

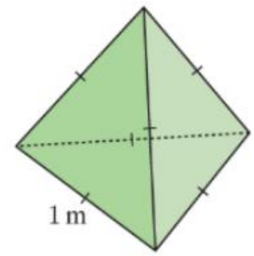
2 Find the surface area of each pyramid:

a



square-based pyramid

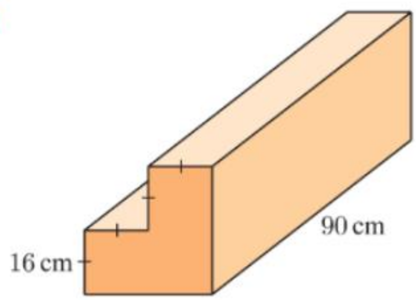
b



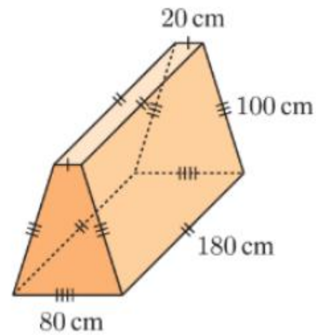
tetrahedron

3 Find the surface area of each solid:

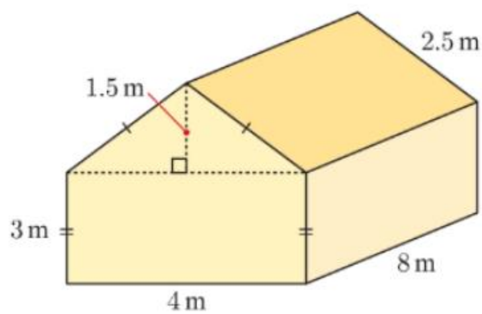
a



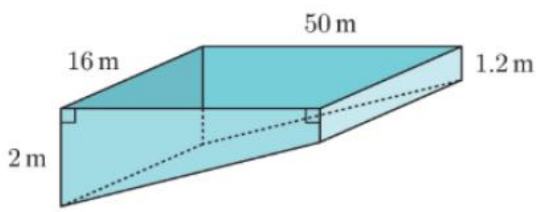
b



4 A marquee with the dimensions as shown is made from canvas. The marquee has no floor. Find the total cost of the canvas if it costs \$31.50 per square metre.



5



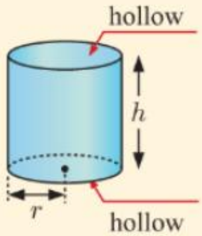
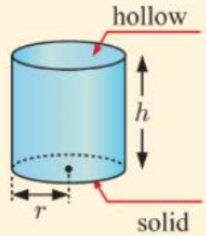
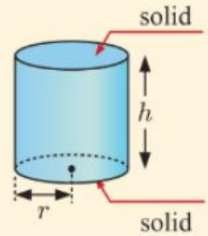
The base and walls of the swimming pool shown are tiled. The tiles cost €25 per m^2.

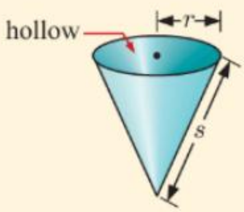
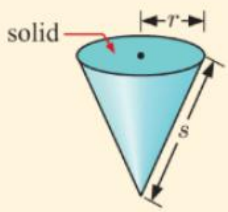
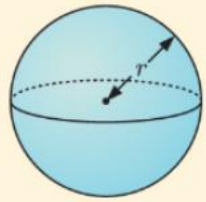
a Find the total area of tiles.

b Find the value of the tiles.

SOLIDS WITH CURVED SURFACES

In previous years you should have seen the following formulae for the outer surface areas of solids with curved surfaces:

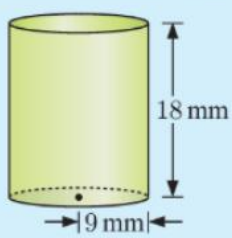
Hollow cylinder (no ends)	**Open can** (one end)	**Solid cylinder** (two ends)
		
$A = 2\pi rh$	$A = 2\pi rh + \pi r^2$	$A = 2\pi rh + 2\pi r^2$

Hollow cone (no end)	**Solid cone** (closed end)	**Sphere**
		
$A = \pi rs$	$A = \pi rs + \pi r^2$	$A = 4\pi r^2$

Example 11 **Self Tutor**

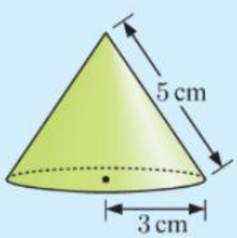
Find the outer surface area of each of these solids:

a



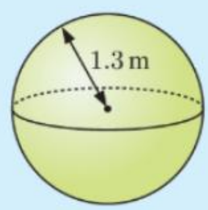
open top

b



solid

c



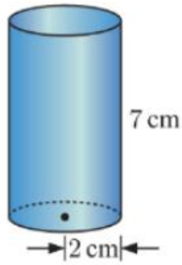
a Surface area $= 2\pi rh + \pi r^2$
$= (2 \times \pi \times 9 \times 18 + \pi \times 9^2)\ \text{mm}^2$
$\approx 1270\ \text{mm}^2$

b Surface area $= \pi rs + \pi r^2$
$= (\pi \times 3 \times 5 + \pi \times 3^2)\ \text{cm}^2$
$\approx 75.4\ \text{cm}^2$

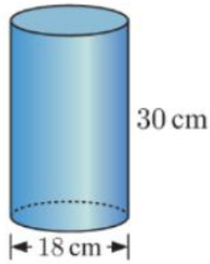
c Surface area $= 4\pi r^2$
$= 4 \times \pi \times 1.3^2\ \text{m}^2$
$\approx 21.2\ \text{m}^2$

EXERCISE 7C.2

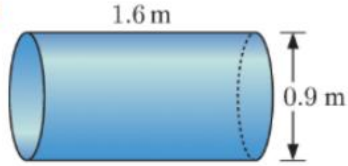
1 Find the outer surface area of each solid:

a


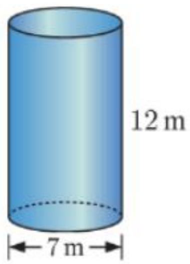
hollow throughout

b


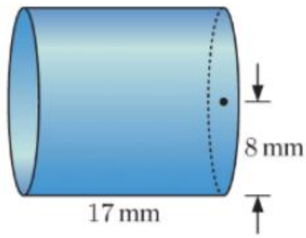
solid

c


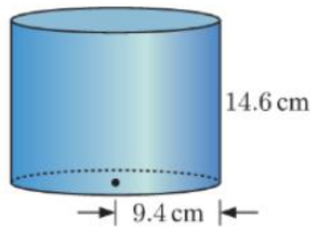
open one end

d


open top

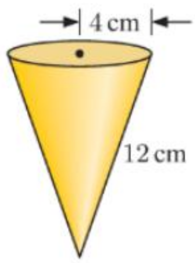
e


hollow throughout

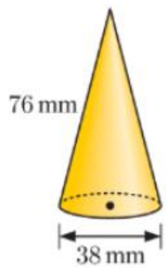
f


solid

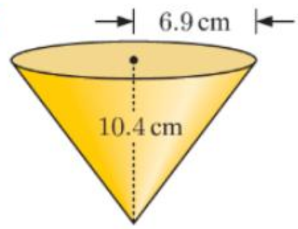
2 Find the outer surface area of each solid:

a


solid

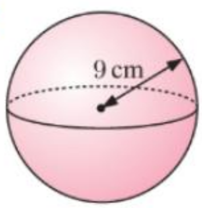
b


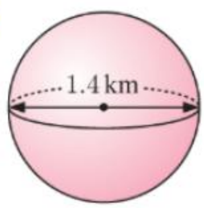
solid

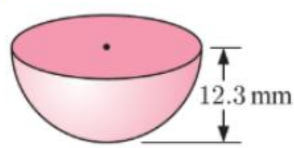
c


open top

3 Find the *total* surface area of each solid:

a


b


c


4 A cylindrical tank is 6 m long and has diameter 8 m. Its outer surface (including both ends) is to be painted bright red. Each litre of paint covers 5 m^2. It is purchased in 5 L cans costing €52.50 each.

a Find the surface area to be painted.

b Find the number of paint cans which must be purchased.

c What is the cost of the paint?

5 We commonly use a sphere to model the Earth, even though it is not a *perfect* sphere. The Earth has a radius of approximately 6400 km.

a Estimate the surface area of the Earth.

b 71% of the Earth's surface is covered by water. Estimate this area.

c China has a land area of 9 706 961 km^2.

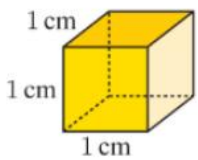
i What percentage of the surface area of the Earth is China?

ii What percentage of the *land* area of the Earth is China?

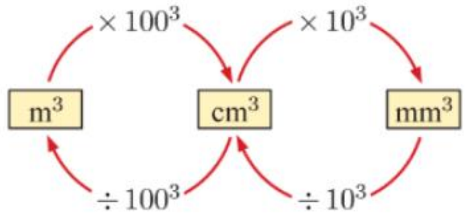
D VOLUME

The **volume** of a solid is the amount of space it occupies.

Volume can be measured in **cubic millimetres**, **cubic centimetres**, or **cubic metres**.

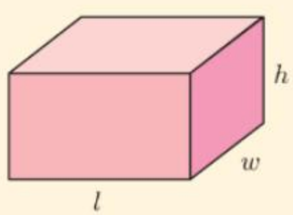
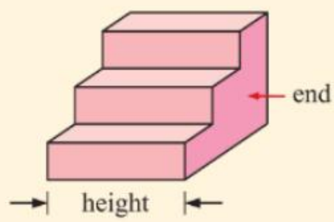
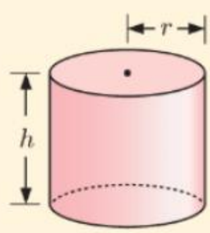


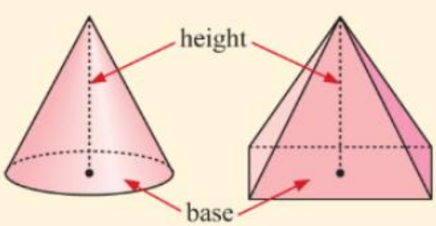
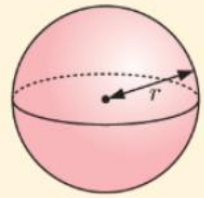
1 cm^3
$= 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm}$
$= 1000 \text{ mm}^3$



Likewise, 1 m^3
$= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm}$
$= 1\,000\,000 \text{ cm}^3$

In previous years we have established the following volume formulae:

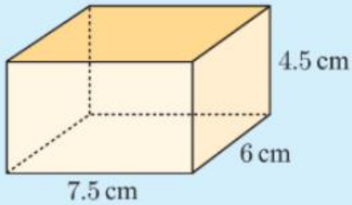
Rectangular prism	Solid of uniform cross-section	Cylinder
		
$V = l \times w \times h$	$V =$ **area of end** $\times$ **height**	$V = \pi r^2 h$

Tapered solid	Sphere
	
$V = \frac{1}{3} \times$ **area of base** $\times$ **height**	$V = \frac{4}{3}\pi r^3$

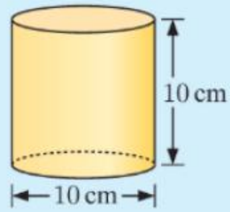
Example 12 — Self Tutor

Find, correct to 3 significant figures, the volume of the following solids:

a



b



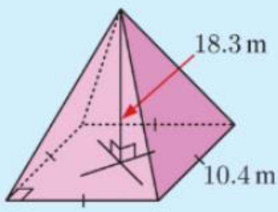
a Volume
$= \text{length} \times \text{width} \times \text{height}$
$= 7.5 \text{ cm} \times 6 \text{ cm} \times 4.5 \text{ cm}$
$\approx 203 \text{ cm}^3$

b Volume
$= \text{area of cross-section} \times \text{height}$
$= \pi r^2 \times h$
$= \pi \times 5^2 \times 10$
$\approx 785 \text{ cm}^3$

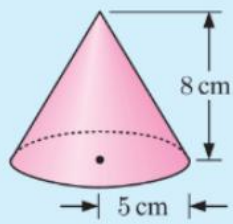
Example 13 — Self Tutor

Find the volume of each solid:

a



b



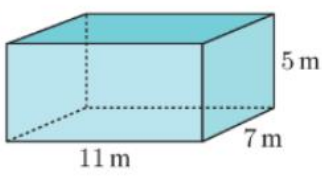
a Volume
$= \frac{1}{3} \times \text{area of base} \times \text{height}$
$= \frac{1}{3} \times 10.4 \times 10.4 \times 18.3 \text{ m}^3$
$\approx 660 \text{ m}^3$

b Volume
$= \frac{1}{3} \times \text{area of base} \times \text{height}$
$= \frac{1}{3} \times \pi \times 5^2 \times 8 \text{ cm}^3$
$\approx 209 \text{ cm}^3$

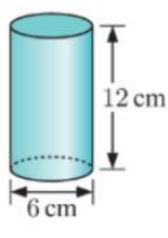
EXERCISE 7D

1 Find the volume of the following:

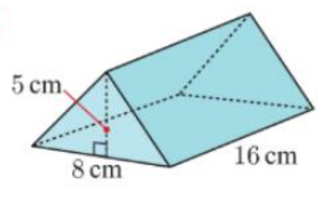
a



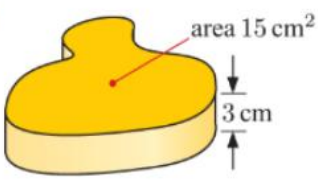
b



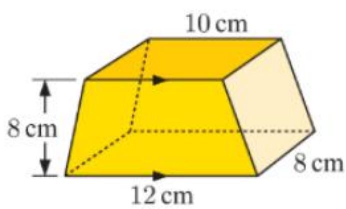
c



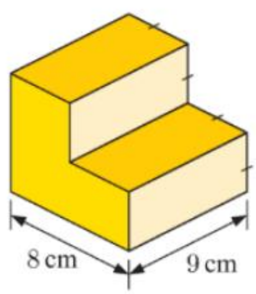
d



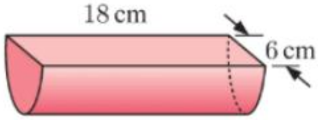
e



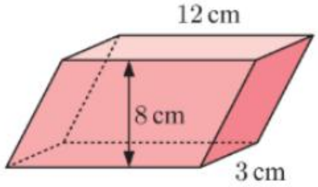
f



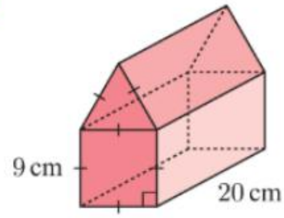
g



h

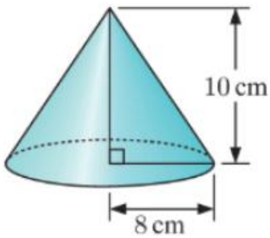


i

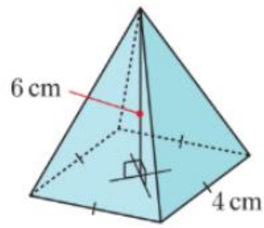


2 Find the volume of the following:

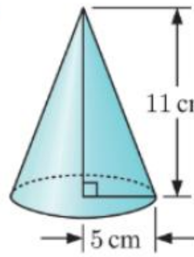
a



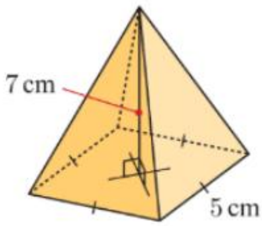
b



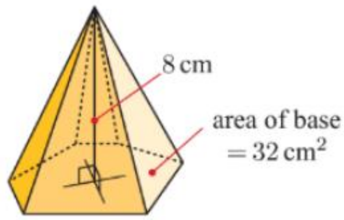
c



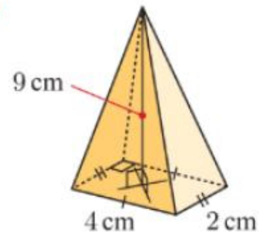
d



e

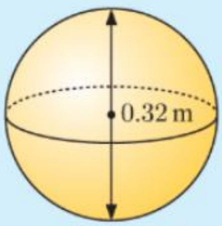


f



Example 14 **Self Tutor**

Find the volume of the sphere in cubic centimetres.



0.32 m = 32 cm

$V = \frac{4}{3}\pi r^3$

$\therefore\ V = \frac{4}{3}\pi \times 16^3$

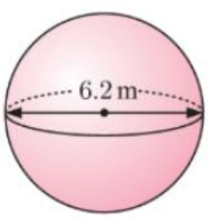
$\therefore\ V \approx 17\,200\ \text{cm}^3$

$\therefore\ V \approx 1.72 \times 10^4\ \text{cm}^3$

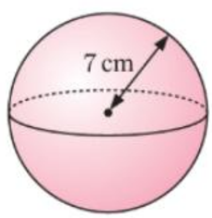
Change the units to centimetres before calculating the volume.

3 Find the volume of the following:

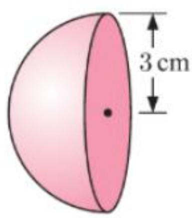
a



b

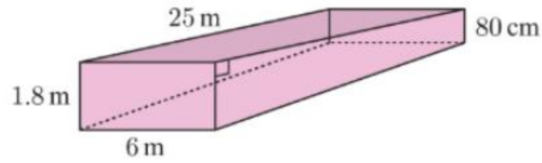


c



4 Find the volume of a rectangular box which measures 24.7 cm by 32.6 cm by 18.8 cm.

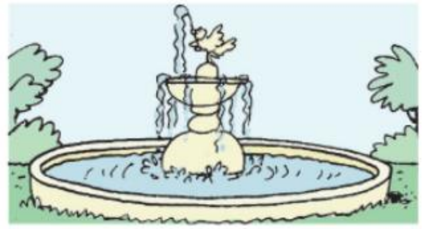
5 A swimming pool has dimensions shown alongside.



a Find the area of a trapezium-shaped side.

b Determine the volume of water required to fill the pool.

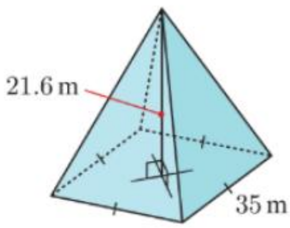
6 In the town square, there is a fountain in the middle of a circular pond. The pond is 6 metres in diameter. A concrete wall 30 cm wide and 60 cm high is built around the edge of the pond.



a Find the area of the top of the wall.

b Find the volume of concrete required for the wall.

7



The *Pyramide du Louvre* in Paris is a square-based pyramid with sides 35 metres long and height 21.6 metres. Calculate the volume of air in this building.

8 A hollow spherical glass bauble contains a winter snowman scene.
The bauble has internal diameter 6.8 cm and external diameter 7.0 cm. What volume of glass was used to make it?

Example 15 **Self Tutor**

Sand from a quarry pours out from a giant hose and forms a conical heap on the ground. The heap has base diameter 25 m and height 8.9 m.

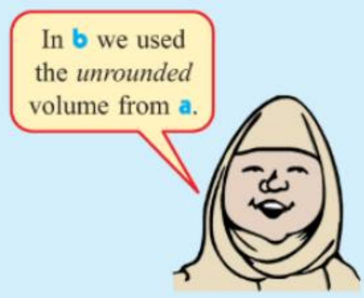
a Find the volume of sand in the heap, to the nearest m^3.

b Find the total mass of the sand given that each cubic metre weighs 2.35 tonnes.

The diameter is 25 m, so $r = 12.5$ m.
The height $h = 8.9$ m.

a Volume
$= \frac{1}{3}\pi r^2 h$
$= \frac{1}{3} \times \pi \times 12.5^2 \times 8.9$
≈ 1456.259
$\approx 1456 \text{ m}^3$

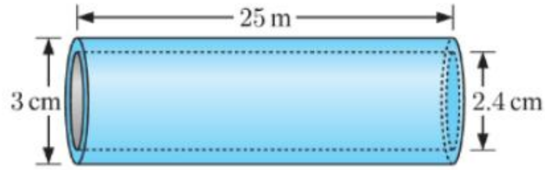
b Total mass
$\approx 1456.259 \times 2.35$
≈ 3422.209
≈ 3420 tonne



9 A conical heap of salt is 3.5 metres in diameter and 4.2 metres high. Each cubic metre weighs 769 kg, and each kg is worth €0.85. Find:

- **a** the volume of the heap of salt
- **b** the total mass of salt
- **c** the total value of the salt.

10 A rubber tube has the dimensions shown.



- **a** Find the cross-sectional area of one end of the tube.
- **b** Hence find the volume of rubber used to make the tube.
- **c** If the rubber weighs 1500 kg per m^3, find the weight of the tube.

11 Josephine has sugar cubes with sides of length 12 mm.

- **a** How many sugar cubes will she need to crush to fill a teaspoon with volume 5 cm^3?
- **b** How much sugar will be left over from the last cube?

12 A block of lead $3.2\text{ cm} \times 2.1\text{ cm} \times 4.6\text{ cm}$ is melted and used to craft solid spheres with radius 6 mm.

- **a** Find the volume of the block of lead.
- **b** Find the volume of each sphere.
- **c** How many spheres can be made?
- **d** What percentage of the lead will be wasted?

ACTIVITY 2 — VOLUME OF CLAY

You will need: Clay, ruler

What to do:

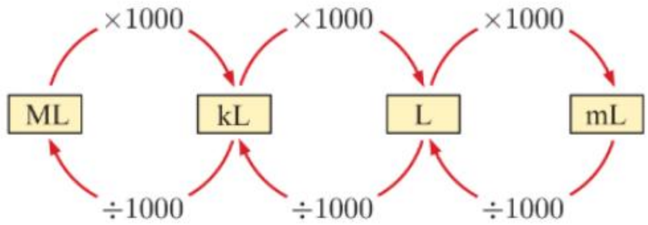
1. Working in groups of four, each group is given a lump of clay.
2. The task of the group is to determine the volume of the lump of clay.
   Mould the clay into each of the following shapes, then take the measurements needed to calculate the volume:
   - sphere
   - rectangular prism
   - cone
   - cylinder
3. Calculate the volume of clay using each shape.
4. Explain why there are differences in the volumes you calculated.
5. Which result do you think is closest to the actual volume of the clay? Explain your answer.

E CAPACITY

The **capacity** of a container is the quantity of fluid or gas used to fill it.

The basic unit of capacity is the **litre** (**L**).

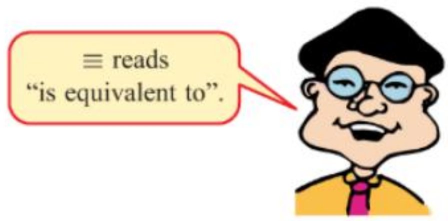
1 litre = 1000 millilitres (mL)
1 kilolitre (kL) = 1000 litres
1 megalitre (ML) = 1000 kilolitres



The units for volume and capacity are connected because **1 millilitre** (**mL**) of fluid fills a container of size 1 cm^3.

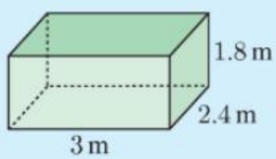
We can therefore construct the following table of equivalent units:

$1 \text{ mL} \equiv 1 \text{ cm}^3$
$1 \text{ L} \equiv 1000 \text{ cm}^3$
$1 \text{ kL} = 1000 \text{ L} \equiv 1 \text{ m}^3$



Example 16 **Self Tutor**

Find the capacity of a 3 m by 2.4 m by 1.8 m tank.

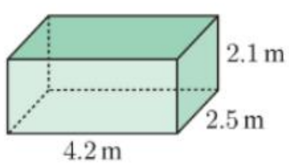


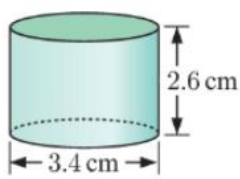
Volume $V = 3 \times 2.4 \times 1.8 \text{ m}^3$
$= 12.96 \text{ m}^3$

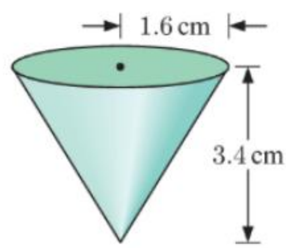
$\therefore$ the capacity of the tank is 12.96 kL.

EXERCISE 7E

1 Find the capacity of each container:

a


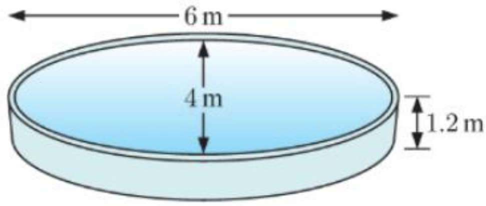
b


c


2 A hemispherical bowl has internal diameter 18 cm. How many litres of water could it contain?

3 How many cylindrical bottles 12 cm high and with 6 cm diameter could be filled from a tank containing 125 L of detergent?

4 How much water is needed to fill this elliptical swimming pool?

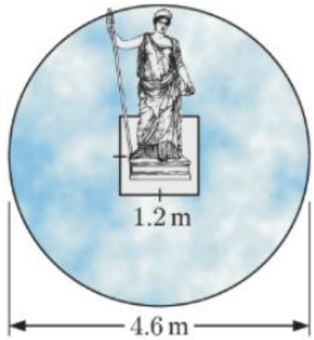


5 2 litres of tomato soup is poured into a rectangular plastic box which has a 20 cm by 12 cm base. To what height does the soup rise?

6 How much helium is needed to fill a spherical balloon with maximum diameter 36 cm?

7 Answer the **Opening Problem** on page **116**.

8 How much water is needed to fill the pond to a depth of 55 cm?



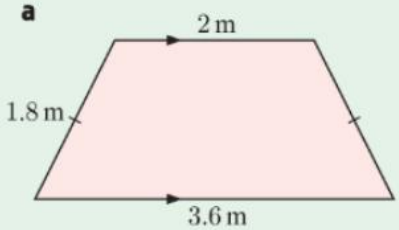
9 When ice melts into liquid water, its volume increases by 10%. Gerard has placed a block of ice $2.4 \text{ cm} \times 3.2 \text{ cm} \times 1.8 \text{ cm}$ on a tile. As the ice melts, the water runs to the edge of the tile and forms drops. Each drop falls when its radius is 3 mm.

  a Find the volume of the block of ice.
  b How many mL of water will the ice melt into?
  c How many mL of water will be in each drop? (Assume each drop is spherical.)
  d How many full-size drops can fall?

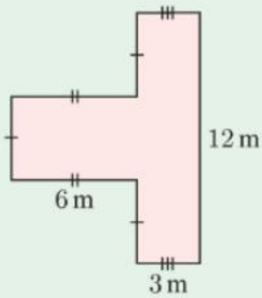
REVIEW SET 7A

1 Find the perimeter and area of each figure:

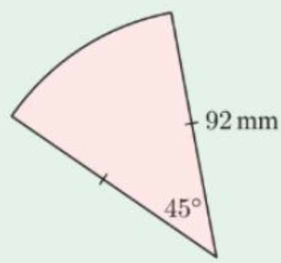
a



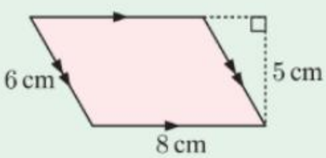
b



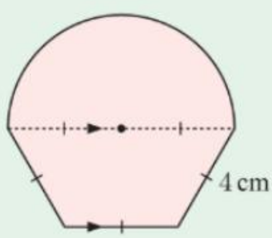
c



d



e



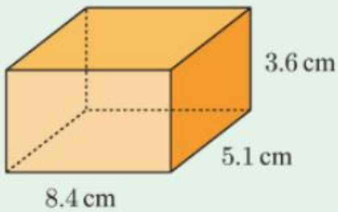
2 A rectangular bathroom measuring 3 m by 2 m is to be decorated on all walls with a single row of patterned tiles. There is a doorway measuring 90 cm wide. Each patterned tile is 15 cm long and costs \$5. Find:

- **a** the total length of patterned tiles required
- **b** the total cost of the tiles.

3 Competitors in a mountain-biking race complete four laps of an 8.5 km circuit. If the winning time is 1 hour 33 minutes, find the average speed of the winner.

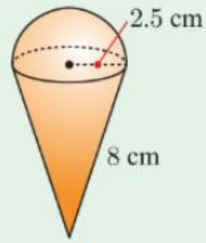
4 Find the outer surface area of:

a



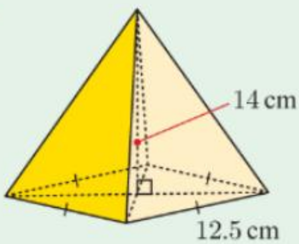
b

c

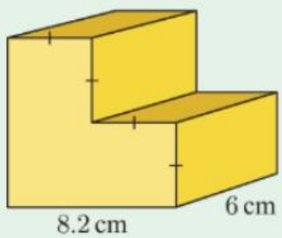


5 Find the volume of the following solids:

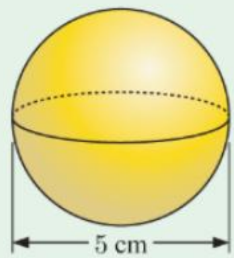
a



b

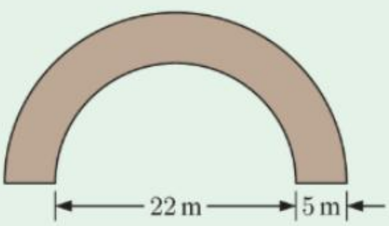


c



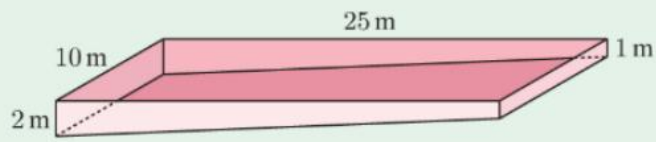
6 After a heavy rain it was found that the water level in a cylindrical tank had risen by 45 cm. If the radius of the tank is 1.2 m, find the volume of water collected.

7 A semi-circular tunnel with the dimensions shown is made of concrete. The tunnel is 220 m long, and the concrete costs €256 per m^3.

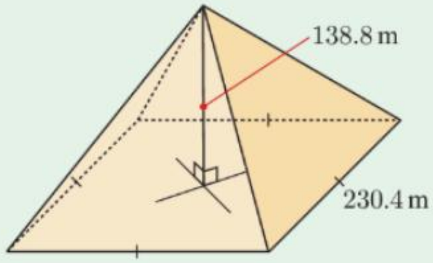


- **a** Find the cross-sectional area of the tunnel.
- **b** Find the volume of concrete used in the tunnel.
- **c** Find the cost of the concrete.

8
- **a** Find the capacity of a swimming pool with the dimensions shown.
- **b** If the pool was filled to a depth 10 cm from the top, how much water would it contain?



9 The Pyramid of Khufu in Egypt has a square base with sides 230.4 m and a height of 138.8 m.



a Find the total volume of the pyramid.

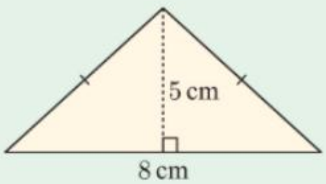
b If each cubic metre of stone weighs 2.67 tonnes, find the total mass of stone used. Give your answer in scientific notation. (Assume the pyramid is solid stone for this part.)

c The King's Chamber in the pyramid is rectangular and measures 10.47 m by 5.23 m by 5.97 m. Find the capacity of the chamber.

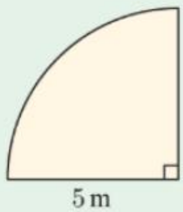
REVIEW SET 7B

1 Find the perimeter and area of each figure:

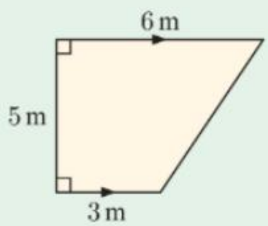
a



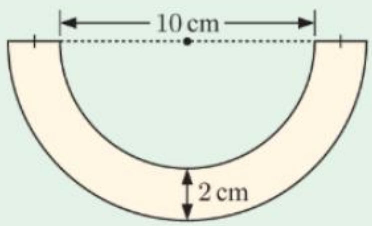
b



c



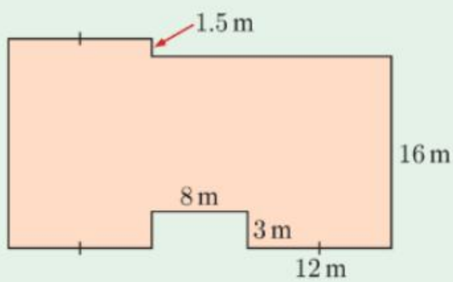
d



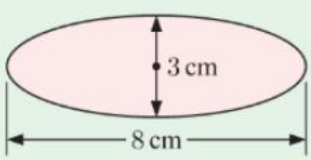
2 New guttering is to be installed around the perimeter of the house with floorplan shown.

a Find the perimeter of the house.

b If the guttering costs \$30 per metre, find the total cost.

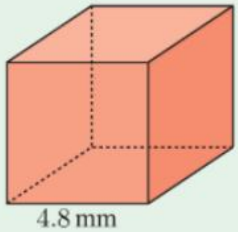


3 Find the area of the ellipse:



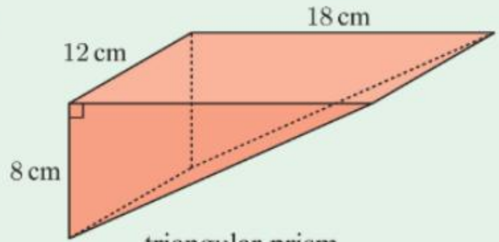
4 Find the outer surface area of:

a



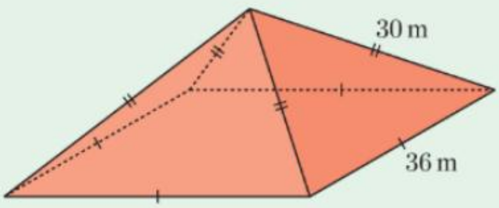
cube

b



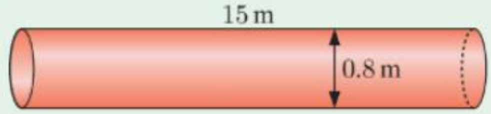
triangular prism

c



square-based pyramid

d



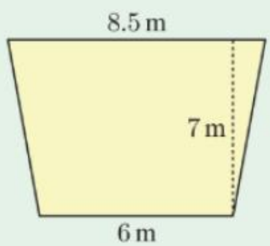
hollow throughout

5 The moon is approximately spherical with radius 1737 km. Estimate:

a the distance around the equator of the moon

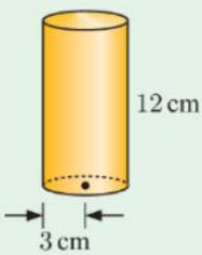
b the surface area of the moon.

6 The diagram shows the cross-section of a railway cutting that needs to be excavated. The cutting will be 56 m long. Find the volume of soil that needs to be excavated.

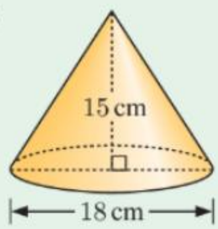


7 Find the volume of the following solids:

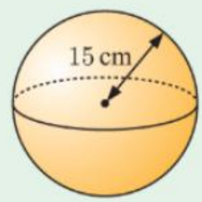
a



b



c

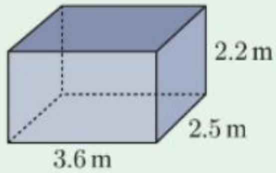


8 A security-conscious man builds himself a round fort of radius 10 m. He surrounds it with a moat 3 m wide. A large dog patrols on the other side of the moat to deter anyone who may attempt to cross it. The dog makes 150 complete circuits of the moat every day. How far does the dog walk every day?

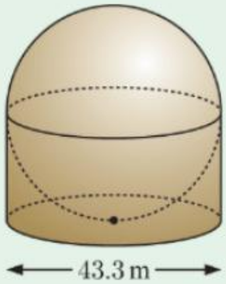
9 Water enters the tank shown at the rate of 60 L per minute.

a Find the capacity of the tank.

b How long will it take to fill the tank?



10 The Pantheon in Rome was built during the reign of emperor Augustus around 27 BC, and rebuilt by the emperor Hadrian around 126 AD. Its centre is designed as a hemisphere on a cylinder so that a sphere with diameter 43.3 m will fit exactly inside it.
Find the capacity of this chamber in kL.



Chapter 8

Algebraic fractions

Contents:

OPENING PROBLEM

To participate in a tennis league, each team must pay \$150 in court fees, and a \$60 team registration fee. The court fees are shared equally between the players in the team, and the registration fee is shared equally between the players and the team coach.

Suppose a team has x players.

Things to think about:

a Can you write the amount that each player must pay as a single fraction involving x?

b How much must each player pay if the team has:

i 4 players **ii** 5 players?

Algebraic fractions are fractions which contain at least one variable or unknown.

The variable may be in the numerator, the denominator, or both the numerator and denominator.

For example, $\frac{x}{7}$, $\frac{-2}{5-y}$, and $\frac{x+2y}{1-y}$ are all algebraic fractions.

A EVALUATING ALGEBRAIC FRACTIONS

To **evaluate** an algebraic fraction, we replace the variables with their known values. We then give our answer in simplest form.

Example 1 Self Tutor

If $a=2$, $b=-3$, and $c=-5$, evaluate:

a $\frac{a-b}{c}$ **b** $\frac{a-c-b}{b-a}$

a
$$\begin{aligned}&\frac{a-b}{c}\\&=\frac{2-(-3)}{(-5)}\\&=\frac{2+3}{-5}\\&=\frac{5}{-5}\\&=-1\end{aligned}$$

b
$$\begin{aligned}&\frac{a-c-b}{b-a}\\&=\frac{2-(-5)-(-3)}{(-3)-2}\\&=\frac{2+5+3}{-3-2}\\&=\frac{10}{-5}\\&=-2\end{aligned}$$

EXERCISE 8A

1 If $a=3$, $b=2$, and $c=6$, evaluate:

a $\frac{c}{2}$ **b** $\frac{c}{a}$ **c** $\frac{a}{c}$ **d** $\frac{c}{b-a}$ **e** $\frac{a+c}{b}$

f $\frac{ab}{c}$ **g** $\frac{a^2}{b}$ **h** $\frac{c^2}{a}$ **i** $\frac{ab^2}{c}$ **j** $\frac{(ab)^2}{c}$

2 If $a = 2$, $b = -3$, and $c = -4$, evaluate:

a $\dfrac{c}{a}$ **b** $\dfrac{a}{c}$ **c** $\dfrac{-1}{b}$ **d** $\dfrac{c^2}{a}$ **e** $\dfrac{c}{a+b}$

f $\dfrac{a-c}{2b}$ **g** $\dfrac{b}{c-a}$ **h** $\dfrac{a-c}{a+c}$ **i** $\dfrac{c-a}{b^2}$ **j** $\dfrac{a^2}{c-b}$

B SIMPLIFYING ALGEBRAIC FRACTIONS

We have observed previously that number fractions can be simplified by cancelling common factors.

For example, $\dfrac{15}{35} = \dfrac{3 \times \cancel{5}^1}{7 \times \cancel{5}_1} = \dfrac{3}{7}$ where the common factor 5 is cancelled.

The same principle can be applied to algebraic fractions.

If the numerator and denominator of an algebraic fraction are both written in factored form and common factors are found, we can simplify by **cancelling the common factors**.

For example,
$$\begin{aligned}\frac{6bc}{3c} &= \frac{{}^2\cancel{6} \times b \times \cancel{c}^1}{{}_1\cancel{3} \times \cancel{c}_1} && \{3 \text{ and } c \text{ are common factors}\}\\ &= \frac{2b}{1} && \{\text{after cancellation}\}\\ &= 2b\end{aligned}$$

Fractions such as $\dfrac{3xy}{7z}$ cannot be simplified since the numerator and denominator do not have any common factors.

To simplify algebraic expressions:

- factorise the numerator and the denominator
- cancel any common factors
- simplify the result.

DISCUSSION

Discuss what is wrong with the cancellation $\dfrac{x + \cancel{4}^2}{\cancel{2}_1} = \dfrac{x+2}{1} = x + 2$.

Example 2 Self Tutor

Simplify: **a** $\dfrac{a^2}{2a}$ **b** $\dfrac{6a^2b}{3b}$ **c** $\dfrac{a+b}{a}$

a
$$\begin{aligned}&\frac{a^2}{2a}\\ &= \frac{a \times \cancel{a}^1}{2 \times \cancel{a}_1}\\ &= \frac{a}{2}\end{aligned}$$

b
$$\begin{aligned}&\frac{6a^2b}{3b}\\ &= \frac{{}^2\cancel{6} \times a \times a \times \cancel{b}^1}{{}_1\cancel{3} \times \cancel{b}_1}\\ &= \frac{2 \times a \times a}{1}\\ &= 2a^2\end{aligned}$$

c $\dfrac{a+b}{a}$ cannot be simplified as $a + b$ is a sum, not a product.

EXERCISE 8B.1

1 Simplify:

a $\dfrac{2a}{4}$ **b** $\dfrac{4m}{2}$ **c** $\dfrac{6a}{a}$ **d** $\dfrac{6a}{2a}$ **e** $\dfrac{2a^2}{a}$

f $\dfrac{2x^3}{2x}$ **g** $\dfrac{2x^3}{x^2}$ **h** $\dfrac{2x^3}{x^3}$ **i** $\dfrac{2a^2}{4a^3}$ **j** $\dfrac{8m^2}{4m}$

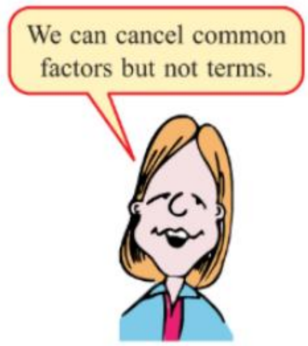
k $\dfrac{4a^2}{a^2}$ **l** $\dfrac{6t}{3t^2}$ **m** $\dfrac{4d^2}{2d}$ **n** $\dfrac{ab^2}{2ab}$ **o** $\dfrac{4ab^2}{6a^2b}$

2 Simplify if possible:

a $\dfrac{2t}{2}$ **b** $\dfrac{2+t}{2}$ **c** $\dfrac{xy}{x}$

d $\dfrac{x+y}{x}$ **e** $\dfrac{ac}{bc}$ **f** $\dfrac{a+c}{b+c}$

g $\dfrac{2a^2}{4a}$ **h** $\dfrac{5a}{9b}$ **i** $\dfrac{14c}{8d}$



Example 3 **Self Tutor**

Simplify: **a** $\dfrac{(-4b)^2}{2b}$ **b** $\dfrac{18}{3(c-1)}$

a
$$\frac{(-4b)^2}{2b} = \frac{(-4b)\times(-4b)}{2\times b} = \frac{{}^{8}\cancel{16}\times b\times \cancel{b}^{1}}{{}_{1}\cancel{2}\times\cancel{b}_{1}} = 8b$$

b
$$\frac{18}{3(c-1)} = \frac{\cancel{18}^{6}}{{}_{1}\cancel{3}(c-1)} = \frac{6}{c-1}$$

3 Simplify:

a $\dfrac{(2a)^2}{a^2}$ **b** $\dfrac{(4n)^2}{8n}$ **c** $\dfrac{(-a)^2}{a}$ **d** $\dfrac{a^2}{(-a)^2}$

e $\dfrac{(-2a)^2}{4}$ **f** $\dfrac{(-3n)^2}{6n}$ **g** $\dfrac{2b}{(2b^2)^2}$ **h** $\dfrac{(3k^2)^2}{18a^3}$

4 Simplify:

a $\dfrac{4(x+5)}{2}$ **b** $\dfrac{2(n+5)}{12}$ **c** $\dfrac{7(b+2)}{14}$ **d** $\dfrac{6(k-2)}{8}$

e $\dfrac{15}{3(t-1)}$ **f** $\dfrac{10}{25(k+4)}$ **g** $\dfrac{4}{12(x-3)}$ **h** $\dfrac{20(p+4)}{12}$

Example 4 **Self Tutor**

Simplify: **a** $\dfrac{(2x+3)(x+4)}{5(2x+3)}$ **b** $\dfrac{12(x+4)^2}{3(x+4)}$

a $\dfrac{{}^1\cancel{(2x+3)}(x+4)}{5\cancel{(2x+3)}_1}$

$= \dfrac{(x+4)}{5}$

$= \dfrac{x+4}{5}$

b $\dfrac{12(x+4)^2}{3(x+4)}$

$= \dfrac{{}^4\cancel{12}(x+4)\cancel{(x+4)}^1}{{}_1\cancel{3}\cancel{(x+4)}_1}$

$= 4(x+4)$

5 Simplify:

a $\dfrac{(x+4)(x+2)}{9(x+4)}$ **b** $\dfrac{12(a-3)}{(a-3)(a+1)}$ **c** $\dfrac{(x+y)(x-y)}{3(x-y)}$ **d** $\dfrac{(x+y)^2}{x+y}$

e $\dfrac{2(x+2)}{(x+2)^2}$ **f** $\dfrac{(a+5)^2}{3(a+5)}$ **g** $\dfrac{2xy(x-y)}{6x(x-y)}$ **h** $\dfrac{5(y+2)(y-3)}{15(y+2)}$

i $\dfrac{x(x+1)(x+2)}{3x(x+2)}$ **j** $\dfrac{3(b-4)}{6(b-4)^2}$ **k** $\dfrac{8(p+q)^2}{12(p+q)}$ **l** $\dfrac{24(r-2)}{15(r-2)^2}$

FACTORISATION AND SIMPLIFICATION

It is often necessary to **factorise** either the numerator or denominator before simplification can take place. To do this we use the rules for factorisation that we have seen previously.

Example 5 **Self Tutor**

Simplify: **a** $\dfrac{3a+9}{3}$ **b** $\dfrac{4a+12}{8}$

a $\dfrac{3a+9}{3}$

$= \dfrac{{}^1\cancel{3}(a+3)}{{}_1\cancel{3}}$

$= a+3$

b $\dfrac{4a+12}{8}$

$= \dfrac{{}^1\cancel{4}(a+3)}{{}_2\cancel{8}}$

$= \dfrac{a+3}{2}$

EXERCISE 8B.2

1 Simplify by factorising:

a $\dfrac{2x+4}{2}$ **b** $\dfrac{3x-6}{3}$ **c** $\dfrac{3x+6}{6}$ **d** $\dfrac{4x-20}{8}$

e $\dfrac{4y+12}{12}$ **f** $\dfrac{6x-30}{4}$ **g** $\dfrac{ax+bx}{x}$ **h** $\dfrac{ax+bx}{cx+dx}$

2 Simplify, if possible:

a $\dfrac{4x+6}{6}$ **b** $\dfrac{4x+6}{5}$ **c** $\dfrac{6a-3}{2}$ **d** $\dfrac{6a-3}{3}$

e $\dfrac{6a+2}{4}$ **f** $\dfrac{3b+9}{2}$ **g** $\dfrac{3b+9}{6}$ **h** $\dfrac{8b-12}{6}$

Example 6 **Self Tutor**

Simplify by factorising:

a $\dfrac{ab+ac}{b+c}$ **b** $\dfrac{6x^2-6xy}{3x-3y}$

a $\dfrac{ab+ac}{b+c}$

$= \dfrac{a(b+c)}{b+c}$ ⟵ HCF is a

$= \dfrac{a\cancel{(b+c)}^1}{\cancel{(b+c)}_1}$

$= a$

b $\dfrac{6x^2-6xy}{3x-3y}$

$= \dfrac{6x(x-y)}{3(x-y)}$ ⟵ HCF is $6x$, ⟵ HCF is 3

$= \dfrac{{}^2\cancel{6}\times x\times \cancel{(x-y)}^1}{{}_1\cancel{3}\times\cancel{(x-y)}_1}$

$= 2x$

3 Simplify by factorising:

a $\dfrac{3x+6}{4x+8}$ **b** $\dfrac{5x-15}{3x-9}$ **c** $\dfrac{ax+bx}{a+b}$ **d** $\dfrac{16x-8}{20x-10}$

e $\dfrac{a+b}{ay+by}$ **f** $\dfrac{ax+bx}{ay+by}$ **g** $\dfrac{4x^2+8x}{x+2}$ **h** $\dfrac{3x^2+9x}{x+3}$

i $\dfrac{5x^2-5xy}{7x-7y}$ **j** $\dfrac{9b^2-9ab}{12b-12a}$ **k** $\dfrac{6x^2-18x}{9x-27}$ **l** $\dfrac{6a+6b}{8a^3+4a^2b}$

Example 7 **Self Tutor**

Simplify: **a** $\dfrac{6a-6b}{b-a}$ **b** $\dfrac{xy^2-xy}{1-y}$

a $\dfrac{6a-6b}{b-a}$

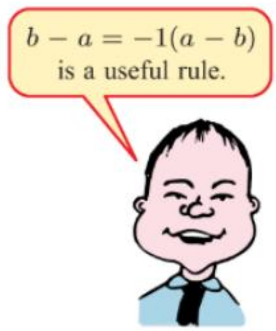
$= \dfrac{6\cancel{(a-b)}^1}{-1\cancel{(a-b)}_1}$

$= -6$

b $\dfrac{xy^2-xy}{1-y}$

$= \dfrac{xy\cancel{(y-1)}^1}{-1\cancel{(y-1)}_1}$

$= -xy$



4 Simplify:

a $\dfrac{2x-2y}{y-x}$ **b** $\dfrac{3x-3y}{2y-2x}$ **c** $\dfrac{m-n}{n-m}$ **d** $\dfrac{r-2s}{4s-2r}$

e $\dfrac{3r-6s}{2s-r}$ **f** $\dfrac{2x-2}{x-x^2}$ **g** $\dfrac{ab^2-ab}{2-2b}$ **h** $\dfrac{4x^2-4x}{2-2x}$

PUZZLE

Click on the icon to load a dynamic puzzle for algebraic fractions.

PUZZLE

C MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS

Variables are used in algebraic fractions to represent unknown numbers. We can treat algebraic fractions in the same way that we treat numerical fractions, since they are in fact *representing* numerical fractions.

The rules for multiplying and dividing algebraic fractions are identical to those used with numerical fractions.

MULTIPLICATION

To **multiply** two or more fractions, we multiply the numerators to form the new numerator, and we multiply the denominators to form the new denominator.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

We can then cancel any common factors, and write our answer in simplest form.

Example 8 Self Tutor

Simplify: **a** $\frac{3}{m} \times \frac{m}{6}$ **b** $\frac{3}{m} \times m^2$

a
$$\frac{3}{m} \times \frac{m}{6}$$
$$= \frac{{}^1\cancel{3} \times \cancel{m}^1}{{}_1\cancel{m} \times \cancel{6}_2}$$
$$= \frac{1}{2}$$

b
$$\frac{3}{m} \times m^2$$
$$= \frac{3}{m} \times \frac{m^2}{1}$$
$$= \frac{3 \times m^2}{m \times 1}$$
$$= \frac{3 \times m \times \cancel{m}^1}{\cancel{m}_1}$$
$$= 3m$$

DIVISION

To **divide** by a fraction, we multiply by its **reciprocal**. The reciprocal is obtained by swapping the numerator and denominator.

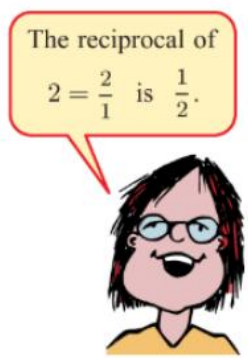
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example 9 Self Tutor

Simplify: **a** $\frac{4}{n} \div \frac{2}{n^2}$ **b** $\frac{3}{a} \div 2$

a
$$\frac{4}{n} \div \frac{2}{n^2}$$
$$= \frac{4}{n} \times \frac{n^2}{2}$$
$$= \frac{4 \times n^2}{n \times 2}$$
$$= \frac{{}^2\cancel{4} \times n \times \cancel{n}^1}{{}_1\cancel{n} \times \cancel{2}_1}$$
$$= 2n$$

b
$$\frac{3}{a} \div 2$$
$$= \frac{3}{a} \times \frac{1}{2}$$
$$= \frac{3 \times 1}{a \times 2}$$
$$= \frac{3}{2a}$$



EXERCISE 8C

1 Simplify:

a $\frac{x}{2} \times \frac{y}{5}$ **b** $\frac{a}{2} \times \frac{3}{a}$ **c** $\frac{a}{2} \times a$ **d** $\frac{a}{4} \times \frac{2}{3a}$

e $\frac{c}{5} \times \frac{1}{c}$ **f** $\frac{c}{5} \times \frac{c}{2}$ **g** $\frac{a}{b} \times \frac{c}{d}$ **h** $\frac{a}{b} \times \frac{b}{a}$

i $\frac{1}{m^2} \times \frac{m}{2}$ **j** $\frac{m}{2} \times \frac{4}{m}$ **k** $\frac{a}{x} \times \frac{x}{b}$ **l** $m \times \frac{4}{m}$

m $\frac{3}{m^2} \times m$ **n** $\left(\frac{a}{b}\right)^2$ **o** $\left(\frac{2}{x}\right)^2$ **p** $\frac{1}{a} \times \frac{a}{b} \times \frac{b}{c}$

2 Simplify:

a $\frac{a}{2} \div \frac{a}{3}$ **b** $\frac{2}{a} \div \frac{2}{3}$ **c** $\frac{3}{4} \div \frac{4}{x}$ **d** $\frac{3}{x} \div \frac{4}{x}$

e $\frac{2}{n} \div \frac{1}{n}$ **f** $\frac{c}{5} \div 5$ **g** $\frac{c}{5} \div c$ **h** $m \div \frac{2}{m}$

i $m \div \frac{m}{2}$ **j** $1 \div \frac{m}{n}$ **k** $\frac{3}{g} \div 4$ **l** $\frac{3}{g} \div \frac{9}{g^2}$

m $\frac{4}{x} \div \frac{x^2}{2}$ **n** $\frac{2}{x} \div \frac{6}{x^3}$ **o** $\frac{a}{b} \div \frac{a^2}{b}$ **p** $\frac{a^2}{5} \div \frac{a}{3}$

Example 10 **Self Tutor**

Simplify:

a $\frac{y^2-y}{y-2} \times \frac{3y-6}{4y-4}$ **b** $\frac{5m-20}{4} \div \frac{2m-8}{3}$

a
$$\frac{y^2-y}{y-2} \times \frac{3y-6}{4y-4}$$
$$= \frac{y\cancel{(y-1)}^1}{\cancel{y-2}_1} \times \frac{3\cancel{(y-2)}^1}{4\cancel{(y-1)}_1}$$
$$= \frac{3y}{4}$$

b
$$\frac{5m-20}{4} \div \frac{2m-8}{3}$$
$$= \frac{5m-20}{4} \times \frac{3}{2m-8}$$
$$= \frac{5\cancel{(m-4)}^1}{4} \times \frac{3}{2\cancel{(m-4)}_1}$$
$$= \frac{15}{8}$$

3 Simplify:

a $\frac{x^2+3x}{x-2} \times \frac{5}{2x+6}$ **b** $\frac{t-5}{t^2+t} \times \frac{4t+4}{3t-15}$ **c** $\frac{4a-28}{a} \div \frac{a-7}{5}$

d $\frac{6k-2}{k+2} \times \frac{2k^2+4k}{9k-3}$ **e** $\frac{x^2-2x}{x+5} \div \frac{8-4x}{2x+10}$ **f** $\frac{m^2-4m}{10m-2} \div \frac{m^2-16}{3m+12}$

D ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

The rules for addition and subtraction of algebraic fractions are identical to those used with numerical fractions.

To **add** two or more fractions, we obtain the *lowest common denominator* and then add the resulting numerators. $\dfrac{a}{c}+\dfrac{b}{c}=\dfrac{a+b}{c}$

To **subtract** two or more fractions, we obtain the *lowest common denominator* and then subtract the resulting numerators. $\dfrac{a}{c}-\dfrac{d}{c}=\dfrac{a-d}{c}$

To find the lowest common denominator of numerical fractions, we look for the **lowest common multiple of the denominators**.

For example:
- when finding $\dfrac{3}{2}+\dfrac{5}{3}$, the lowest common denominator is 6
- when finding $\dfrac{5}{6}+\dfrac{4}{9}$, the lowest common denominator is 18.

The same method is used when there are variables in the denominator.

For example:
- when finding $\dfrac{2}{a}+\dfrac{3}{b}$, the lowest common denominator is ab
- when finding $\dfrac{2}{x}+\dfrac{4}{5x}$, the lowest common denominator is $5x$
- when finding $\dfrac{5}{6x}+\dfrac{4}{9y}$, the lowest common denominator is $18xy$.

Example 11 **Self Tutor**

Simplify: **a** $\dfrac{x}{2}+\dfrac{3x}{4}$ **b** $\dfrac{a}{3}-\dfrac{2a}{5}$

a
$$\begin{aligned}&\frac{x}{2}+\frac{3x}{4} \quad \{\text{LCD}=4\}\\&=\frac{x\times 2}{2\times 2}+\frac{3x}{4}\\&=\frac{2x}{4}+\frac{3x}{4}\\&=\frac{2x+3x}{4}\\&=\frac{5x}{4}\end{aligned}$$

b
$$\begin{aligned}&\frac{a}{3}-\frac{2a}{5} \quad \{\text{LCD}=15\}\\&=\frac{a\times 5}{3\times 5}-\frac{2a\times 3}{5\times 3}\\&=\frac{5a}{15}-\frac{6a}{15}\\&=\frac{5a-6a}{15}\\&=-\frac{a}{15}\end{aligned}$$

EXERCISE 8D

1 Simplify by writing as a single fraction:

a $\dfrac{a}{2}+\dfrac{a}{3}$ **b** $\dfrac{b}{5}-\dfrac{b}{10}$ **c** $\dfrac{c}{4}+\dfrac{3c}{2}$ **d** $\dfrac{d}{2}-\dfrac{3}{5}$

e $\dfrac{5}{8}+\dfrac{x}{12}$ **f** $\dfrac{x}{7}-\dfrac{x}{2}$ **g** $\dfrac{a}{3}+\dfrac{b}{4}$ **h** $\dfrac{t}{3}-\dfrac{5t}{9}$

i $\frac{m}{7}+\frac{2m}{21}$ **j** $\frac{5d}{6}-\frac{d}{3}$ **k** $\frac{3p}{5}-\frac{2p}{7}$ **l** $\frac{7k}{8}-\frac{11k}{18}$

m $\frac{m}{2}+\frac{m}{3}+\frac{m}{6}$ **n** $\frac{a}{2}-\frac{a}{3}+\frac{a}{4}$ **o** $\frac{x}{4}-\frac{x}{3}+\frac{x}{6}$ **p** $2q-\frac{q}{3}+\frac{2q}{7}$

Example 12 Self Tutor

Simplify: **a** $\frac{4}{a}+\frac{3}{b}$ **b** $\frac{5}{x}-\frac{4}{3x}$

a $\frac{4}{a}+\frac{3}{b}$ {LCD $=ab$}

$$=\frac{4\times b}{a\times b}+\frac{3\times a}{b\times a}$$

$$=\frac{4b}{ab}+\frac{3a}{ab}$$

$$=\frac{4b+3a}{ab}$$

b $\frac{5}{x}-\frac{4}{3x}$ {LCD $=3x$}

$$=\frac{5\times 3}{x\times 3}-\frac{4}{3x}$$

$$=\frac{15}{3x}-\frac{4}{3x}$$

$$=\frac{15-4}{3x}$$

$$=\frac{11}{3x}$$

2 Simplify:

a $\frac{7}{a}+\frac{3}{b}$ **b** $\frac{3}{a}+\frac{2}{c}$ **c** $\frac{4}{a}+\frac{5}{d}$ **d** $\frac{2a}{m}-\frac{a}{n}$

e $\frac{a}{x}+\frac{b}{2x}$ **f** $\frac{3}{a}-\frac{1}{2a}$ **g** $\frac{4}{x}-\frac{1}{xy}$ **h** $\frac{5}{x}+\frac{6}{5x}$

i $\frac{11}{3z}-\frac{3}{4z}$ **j** $\frac{a}{b}+\frac{c}{d}$ **k** $\frac{3}{a}+\frac{a}{2}$ **l** $\frac{x}{y}+\frac{2}{3}$

m $\frac{8}{p}-\frac{2}{5}$ **n** $\frac{x}{6y}+\frac{2x}{9y}$ **o** $\frac{1}{8t}-\frac{3}{5t}$ **p** $\frac{5}{2x}+\frac{3}{x^2}$

Example 13 Self Tutor

Simplify: **a** $\frac{b}{3}+1$ **b** $\frac{a}{4}-a$

a $\frac{b}{3}+1$

$$=\frac{b}{3}+\frac{3}{3}$$

$$=\frac{b+3}{3}$$

b $\frac{a}{4}-a$

$$=\frac{a}{4}-\frac{a\times 4}{1\times 4}$$

$$=\frac{a}{4}-\frac{4a}{4}$$

$$=\frac{-3a}{4}$$

$$=-\frac{3a}{4}$$

3 Simplify by writing as a single fraction:

a $\frac{x}{2}+1$ **b** $\frac{y}{3}-1$ **c** $\frac{a}{2}+a$ **d** $\frac{b}{4}-3$

e $\frac{x}{2}-4$ **f** $2+\frac{a}{3}$ **g** $x-\frac{x}{5}$ **h** $2+\frac{1}{x}$

i $5-\frac{2}{x}$ **j** $a+\frac{2}{a}$ **k** $\frac{3}{b}+b$ **l** $\frac{1}{x^2}-2x$

Example 14 Self Tutor

Write as a single fraction: **a** $\frac{x}{6}+\frac{x-2}{3}$ **b** $\frac{x+1}{2}-\frac{x-2}{3}$

a $\frac{x}{6}+\frac{x-2}{3}$ {LCD = 6}

$$=\frac{x}{6}+\frac{2}{2}\left(\frac{x-2}{3}\right)$$
$$=\frac{x}{6}+\frac{2(x-2)}{6}$$
$$=\frac{x+2(x-2)}{6}$$
$$=\frac{x+2x-4}{6}$$
$$=\frac{3x-4}{6}$$

b $\frac{x+1}{2}-\frac{x-2}{3}$ {LCD = 6}

$$=\frac{3}{3}\left(\frac{x+1}{2}\right)-\frac{2}{2}\left(\frac{x-2}{3}\right)$$
$$=\frac{3(x+1)}{6}-\frac{2(x-2)}{6}$$
$$=\frac{3(x+1)-2(x-2)}{6}$$
$$=\frac{3x+3-2x+4}{6}$$
$$=\frac{x+7}{6}$$

4 Write as a single fraction, and hence simplify:

a $\frac{x}{2}+\frac{x+1}{3}$ **b** $\frac{x-1}{4}-\frac{x}{2}$ **c** $\frac{2x}{3}+\frac{x+3}{4}$

d $\frac{x+1}{2}+\frac{x-1}{3}$ **e** $\frac{x-1}{3}+\frac{1-2x}{4}$ **f** $\frac{2x+3}{2}+\frac{2x-3}{3}$

g $\frac{x}{3}+\frac{x+1}{4}$ **h** $\frac{3x+2}{4}+\frac{x}{2}$ **i** $\frac{x}{6}+\frac{3x-1}{5}$

j $\frac{x+1}{5}+\frac{2x-1}{4}$ **k** $\frac{2x-1}{5}-\frac{x}{4}$ **l** $\frac{x}{8}-\frac{1-x}{4}$

m $\frac{x}{5}-\frac{2-x}{10}$ **n** $\frac{x-1}{5}-\frac{2x-7}{3}$ **o** $\frac{1-3x}{4}-\frac{2x+1}{3}$

Example 15 Self Tutor

Write as a single fraction: **a** $\frac{1}{x}+\frac{2}{x-1}$ **b** $\frac{2}{x-1}-\frac{3}{x+1}$

a $\frac{1}{x}+\frac{2}{x-1}$ {LCD = $x(x-1)$}

$$=\frac{1}{x}\left(\frac{x-1}{x-1}\right)+\left(\frac{2}{x-1}\right)\frac{x}{x}$$
$$=\frac{1(x-1)+2x}{x(x-1)}$$
$$=\frac{x-1+2x}{x(x-1)}$$
$$=\frac{3x-1}{x(x-1)}$$

b $\frac{2}{x-1}-\frac{3}{x+1}$ {LCD = $(x-1)(x+1)$}

$$=\left(\frac{2}{x-1}\right)\left(\frac{x+1}{x+1}\right)-\left(\frac{3}{x+1}\right)\left(\frac{x-1}{x-1}\right)$$
$$=\frac{2(x+1)-3(x-1)}{(x-1)(x+1)}$$
$$=\frac{2x+2-3x+3}{(x-1)(x+1)}$$
$$=\frac{-x+5}{(x-1)(x+1)} \text{ or } \frac{5-x}{(x-1)(x+1)}$$

5 Simplify:

a $\frac{3}{x}+\frac{4}{x+1}$ **b** $\frac{5}{x+2}-\frac{3}{x}$ **c** $\frac{4}{x+1}-\frac{3}{x-1}$ **d** $3+\frac{1}{x+2}$

e $\frac{1}{x}+\frac{4}{x-4}$ **f** $\frac{2}{x+3}-4$ **g** $\frac{x+1}{x-1}+\frac{x}{x+1}$ **h** $\frac{5}{x}+\frac{6}{x-2}$

i $\frac{2}{x+2}-\frac{4}{x+1}$ **j** $\frac{x}{x-1}+\frac{4}{2x+1}$ **k** $\frac{5}{x-1}+\frac{x-2}{x+3}$ **l** $\frac{x}{x+5}-\frac{x}{x-3}$

6 Answer the **Opening Problem** on page **144**.

7 Write as a single fraction:

a $\frac{2}{x(x+1)}+\frac{1}{x+1}$ **b** $\frac{2x}{x-3}+\frac{4}{(x+2)(x-3)}$

c $\frac{3}{(x-2)(x+3)}+\frac{x}{x+3}$ **d** $\frac{x+5}{x-2}-\frac{63}{(x-2)(x+7)}$

E EQUATIONS WITH ALGEBRAIC FRACTIONS

To solve equations involving algebraic fractions, we:

- write all fractions with the same **lowest common denominator (LCD)**, and then
- **equate numerators**.

Example 16 Self Tutor

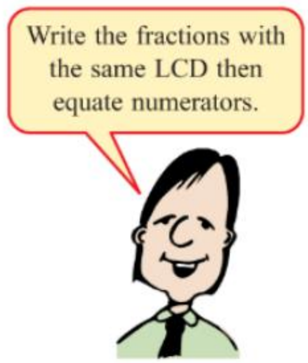
Solve for x: $\frac{6}{x}=\frac{2}{3}$

$\frac{6}{x}=\frac{2}{3}$ {LCD $=3x$}

$\therefore \frac{3\times 6}{3\times x}=\frac{2\times x}{3\times x}$ {to achieve a common denominator}

$\therefore 18=2x$ {equating numerators}

$\therefore x=9$ {dividing both sides by 2}



Example 17 Self Tutor

Solve for x: $\frac{5}{x+2}=\frac{2}{x-1}$

$\frac{5}{x+2}=\frac{2}{x-1}$ {LCD $=(x+2)(x-1)$}

$\therefore \frac{5\times(x-1)}{(x+2)\times(x-1)}=\frac{2\times(x+2)}{(x-1)\times(x+2)}$ {to achieve a common denominator}

$\therefore 5(x-1)=2(x+2)$ {equating numerators}

$\therefore 5x-5=2x+4$ {expanding brackets}

$\therefore 3x-5=4$ {subtracting $2x$ from both sides}

$\therefore 3x=9$ {adding 5 to both sides}

$\therefore x=3$ {dividing both sides by 3}

EXERCISE 8E

1 Solve for x:

a $\frac{3}{x} = \frac{1}{5}$ **b** $\frac{3}{x} = \frac{2}{3}$ **c** $\frac{2}{7} = \frac{5}{x}$

d $\frac{4}{9} = \frac{1}{x}$ **e** $\frac{1}{2x} = \frac{4}{3}$ **f** $\frac{7}{3x} = -\frac{4}{5}$

g $-5 = \frac{2}{3x}$ **h** $\frac{x-3}{x} = \frac{3}{5}$ **i** $\frac{x+5}{2x} = -\frac{8}{5}$

2 Solve for x:

a $\frac{2}{x} = \frac{8}{x+6}$ **b** $\frac{1}{x+1} = \frac{7}{x-1}$ **c** $\frac{8}{2x+3} = \frac{9}{x-1}$

REVIEW SET 8A

1 If $p = 5$, $q = -3$, and $r = 6$, evaluate:

a $\frac{r}{q}$ **b** $\frac{p-q}{p+q}$ **c** $\frac{\sqrt{p^2-16}}{r-q}$ **d** $\frac{p+2q-2r}{r^2-p^2}$

2 Simplify:

a $\frac{(2t)^2}{6t}$ **b** $\frac{x(x-4)}{3(x-4)}$ **c** $\frac{16a+8b}{6a+3b}$ **d** $\frac{8}{4x+8}$

3 Simplify:

a $\frac{3(x+2)^2}{x(x+2)}$ **b** $\frac{3x(x-2)}{(3x+1)(x-2)}$ **c** $\frac{2x+6}{x^2-9}$

4 Simplify:

a $\frac{2a-2b}{b-a}$ **b** $\frac{5x-15}{3x-x^2}$ **c** $\frac{4x-x^2}{2x-8}$

5 Simplify:

a $\frac{a}{b} \times \frac{b}{3}$ **b** $\frac{a}{b} \div \frac{b}{3}$ **c** $\frac{a}{b} + \frac{b}{3}$ **d** $\frac{a}{b} - \frac{b}{3}$

6 Simplify:

a $\frac{7x-14}{x} \times \frac{3}{x-2}$ **b** $\frac{t^2-3t}{6t+6} \times \frac{t+1}{4t-12}$

7 Simplify:

a $\frac{9}{n} \div 6$ **b** $\frac{7}{3x-6} \div \frac{x+5}{x^2-2x}$

8 Write as a single fraction:

a $\frac{2x}{3} + \frac{x}{4}$ **b** $2 + \frac{x}{7}$ **c** $\frac{x}{4} - 1$

9 Simplify:

a $\frac{x}{3} + \frac{x-1}{4}$ **b** $\frac{x+2}{3} - \frac{2-x}{6}$ **c** $\frac{2x+1}{5} - \frac{x-1}{10}$

10 Simplify:

a $\dfrac{1}{x+1}+\dfrac{2}{x-2}$ **b** $\dfrac{5}{x-1}-\dfrac{4}{x+1}$ **c** $\dfrac{1}{x^2}+\dfrac{1}{x+1}$

11 Solve for x:

a $\dfrac{2}{x}=-\dfrac{5}{2}$ **b** $\dfrac{6}{x}=\dfrac{5}{11-x}$ **c** $\dfrac{10}{x-3}=\dfrac{7}{x+6}$

REVIEW SET 8B

1 If $m=-4$, $n=3$, and $p=6$, evaluate:

a $\dfrac{p}{m+n}$ **b** $\dfrac{p-2n}{m+n}$ **c** $\dfrac{p-m}{\sqrt{m^2+n^2}}$

2 Simplify:

a $\dfrac{(3x)^2}{6x^3}$ **b** $\dfrac{3a+6b}{3}$ **c** $\dfrac{(x+2)^2}{x^2+2x}$

3 Simplify:

a $\dfrac{a+b}{3b+3a}$ **b** $\dfrac{2x^2-4x}{x+2}$ **c** $\dfrac{(x-3)^2}{4(3-x)}$

4 Simplify:

a $\dfrac{m}{n}\times\dfrac{2}{n}$ **b** $\dfrac{m}{n}\div\dfrac{2}{n}$ **c** $m^2\div\dfrac{n}{m}$

5 Simplify:

a $\dfrac{3}{x}+\dfrac{5}{2x}$ **b** $\dfrac{6}{y}-\dfrac{a}{b}$ **c** $\dfrac{8}{3x}+\dfrac{1}{4x}$

6 Simplify:

a $\dfrac{3x}{7}-\dfrac{x}{14}$ **b** $\dfrac{4}{3x}+\dfrac{3}{x^2}$

7 Write as a single fraction:

a $5+\dfrac{x}{2}$ **b** $3-\dfrac{y}{x}$ **c** $2a+\dfrac{1}{a}$

8 Simplify:

a $\dfrac{y^2-5y}{y+2}\times\dfrac{3}{2y-10}$ **b** $\dfrac{9-3x}{4x+16}\div\dfrac{x^2-3x}{x+4}$

9 Simplify:

a $\dfrac{x}{4}-\dfrac{2-x}{8}$ **b** $\dfrac{x+5}{2}+\dfrac{2x+1}{5}$ **c** $\dfrac{3-x}{6}-\dfrac{2x}{9}$

10 Simplify:

a $\dfrac{2}{x-1}-\dfrac{3}{x+2}$ **b** $\dfrac{1}{x-1}-\dfrac{2}{x^2}$ **c** $\dfrac{x}{x+2}-\dfrac{2}{x}$

11 Solve for x:

a $\dfrac{5}{2x}=\dfrac{3}{4}$ **b** $\dfrac{3}{x}=\dfrac{5}{x-6}$ **c** $\dfrac{5}{3x-2}=\dfrac{4}{x-3}$

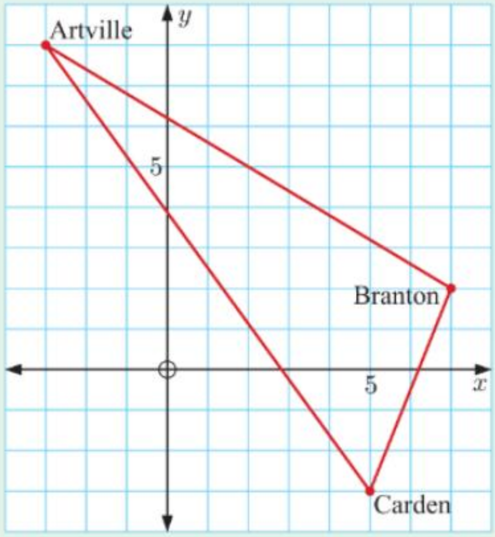
Chapter 9

Coordinate geometry

Contents:

OPENING PROBLEM

The towns Artville, Branton, and Carden are joined by straight roads. On a road map Artville is at $(-3, 8)$, Branton is at $(7, 2)$, and Carden is at $(5, -3)$. The grid units are kilometres.



Things to think about:

- **a** How far is it from Artville to Branton?
- **b** What point is halfway between Branton and Carden?
- **c** Are any of the roads perpendicular to each other?
- **d** **i** Can you find the *equation* of the road connecting Artville and Carden?
  ii Does the point $(2, 1)$ lie on this road?

HISTORICAL NOTE

History shows that the two Frenchmen **René Descartes** and **Pierre de Fermat** arrived at the idea of **analytical geometry** at about the same time. Descartes' work *La Geometrie* was published first, in 1637, while Fermat's *Introduction to Loci* was not published until after his death.

René Descartes

Pierre de Fermat

Today, they are considered the co-founders of this important branch of mathematics which links algebra and geometry.

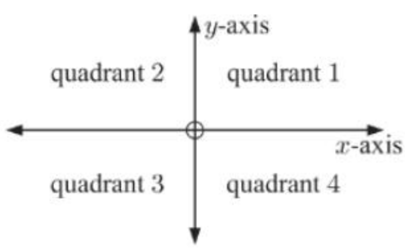
The initial approaches used by these mathematicians were quite opposite. Descartes began with a line or curve and then found the equation which described it. Fermat, to a large extent, started with an equation and investigated the shape of the curve it described.

Analytical geometry and its use of coordinates enabled **Isaac Newton** to later develop another important branch of mathematics called **calculus**. Newton humbly stated: *"If I have seen further than Descartes, it is because I have stood on the shoulders of giants."*

The **number plane** consists of two perpendicular axes which intersect at the **origin**, O.

The **x-axis** is horizontal and the **y-axis** is vertical.

The axes divide the number plane into four **quadrants**.



The number plane is also known as either the **2-dimensional plane**, or the **Cartesian plane** after **René Descartes**.

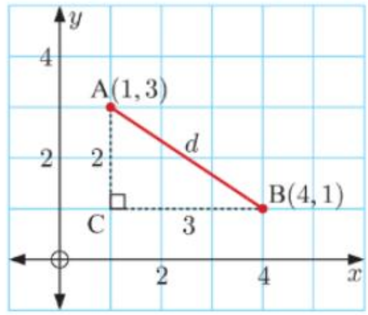
The position of any point in the number plane can be specified in terms of an **ordered pair** of numbers (x, y), where:

- x is the **horizontal step** from O, and is the x-coordinate of the point
- y is the **vertical step** from O, and is the y-coordinate of the point.

DEMO

A THE DISTANCE BETWEEN TWO POINTS

Suppose we want to find the distance d between the points A(1, 3) and B(4, 1).



By drawing line segments [AC] and [BC] along the grid lines, we form a right angled triangle with hypotenuse [AB].

$\therefore\ d^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore\ d^2 = 13$

$\therefore\ d = \sqrt{13}$ {as $d > 0$}

So, the distance between A and B is $\sqrt{13}$ units.

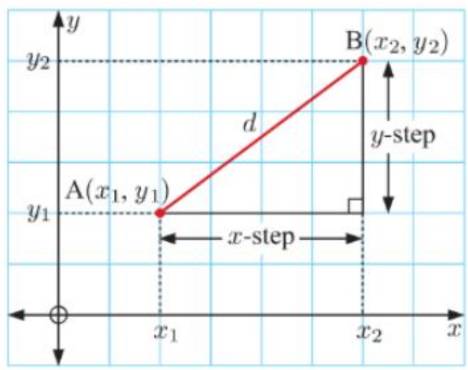
While this approach is effective, it is time-consuming because a diagram is needed.

To make the process quicker, we can develop a formula.

To go from $A(x_1, y_1)$ to $B(x_2, y_2)$, we find the

x-step $= x_2 - x_1$

and y-step $= y_2 - y_1$.



Using Pythagoras' theorem,

$(AB)^2 = (x\text{-step})^2 + (y\text{-step})^2$

$\therefore\ AB = \sqrt{(x\text{-step})^2 + (y\text{-step})^2}$

$\therefore\ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The distance d between two points (x_1, y_1) and (x_2, y_2) is given by

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

Example 1 — Self Tutor

Find the distance between A(−2, 1) and B(3, 4).

A(−2, 1) B(3, 4)

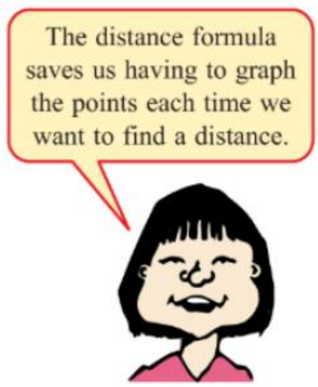
x_1 y_1 x_2 y_2

$AB = \sqrt{(3 - -2)^2 + (4 - 1)^2}$

$= \sqrt{5^2 + 3^2}$

$= \sqrt{25 + 9}$

$= \sqrt{34}$ units



EXERCISE 9A

1 Find the distance between:

- **a** A(3, 1) and B(5, 3)
- **b** C(−1, 2) and D(6, 2)
- **c** O(0, 0) and P(−2, 4)
- **d** E(8, 0) and F(2, −3)
- **e** G(0, −2) and H(0, 5)
- **f** I(2, 0) and J(0, −1)
- **g** R(1, 2) and S(−2, 3)
- **h** W(1, −1) and Z($\frac{1}{2}$, −2).

2 In the map below, the grid lines are 10 km apart.

© OpenStreetMap contributors

Find the direct distance between:

- **a** Dalgety Bay and Edinburgh
- **b** Coatbridge and Dalgety Bay
- **c** Coatbridge and Edinburgh.

Example 2 — Self Tutor

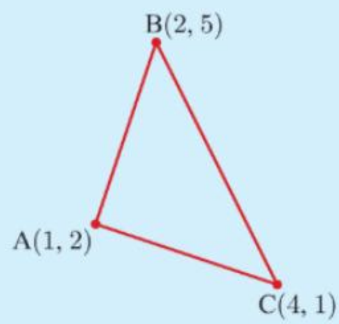
Consider the triangle formed by the points A(1, 2), B(2, 5), and C(4, 1).

- **a** Use the distance formula to classify the triangle as equilateral, isosceles, or scalene.
- **b** Determine whether the triangle is right angled.

a $AB = \sqrt{(2-1)^2 + (5-2)^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ units

$AC = \sqrt{(4-1)^2 + (1-2)^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$BC = \sqrt{(4-2)^2 + (1-5)^2} = \sqrt{2^2 + (-4)^2} = \sqrt{20}$ units



Since AB = AC, the triangle is isosceles.

b The shortest sides are [AB] and [AC].

Now $AB^2 + AC^2 = 10 + 10 = 20 = BC^2$

Using the converse of Pythagoras' theorem, the triangle is right angled. The right angle is at A, opposite the longest side.

3 Use the distance formula to classify triangle ABC as either equilateral, isosceles, or scalene:

- **a** $A(3, -1)$, $B(1, 8)$, $C(-6, 1)$
- **b** $A(1, 0)$, $B(3, 1)$, $C(4, 5)$
- **c** $A(-1, 0)$, $B(2, -2)$, $C(4, 1)$
- **d** $A(\sqrt{2}, 0)$, $B(-\sqrt{2}, 0)$, $C(0, -\sqrt{5})$
- **e** $A(\sqrt{3}, 1)$, $B(-\sqrt{3}, 1)$, $C(0, -2)$
- **f** $A(a, b)$, $B(-a, b)$, $C(0, 2)$

4 Determine whether the following triangles are right angled. If there is a right angle, state the vertex where it occurs.

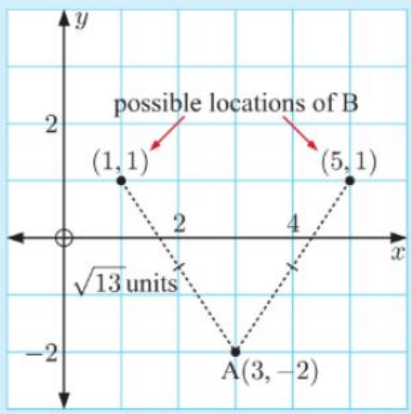
- **a** $A(-2, -1)$, $B(3, -1)$, $C(3, 3)$
- **b** $A(-1, 2)$, $B(4, 1)$, $C(4, -5)$
- **c** $A(1, -2)$, $B(3, 0)$, $C(-3, 2)$
- **d** $A(3, -4)$, $B(-2, -5)$, $C(-1, 1)$

Example 3 **Self Tutor**

Find b given that $A(3, -2)$ and $B(b, 1)$ are $\sqrt{13}$ units apart.
Explain your result using a diagram.

From A to B, x-step $= b - 3$
y-step $= 1 - -2 = 3$

$$\therefore \sqrt{(b-3)^2 + 3^2} = \sqrt{13}$$
$$\therefore (b-3)^2 + 9 = 13 \quad \{\text{squaring both sides}\}$$
$$\therefore (b-3)^2 = 4$$
$$\therefore b - 3 = \pm 2$$
$$\therefore b = 3 \pm 2$$
$$\therefore b = 5 \text{ or } 1$$



Point B could be at two possible locations:
$(5, 1)$ or $(1, 1)$.

5 For each of the cases below, find a and explain the result using a diagram:

- **a** $P(2, 3)$ and $Q(a, -1)$ are 4 units apart
- **b** $P(-1, 1)$ and $Q(a, -2)$ are 5 units apart
- **c** $X(a, a)$ is $\sqrt{8}$ units from the origin
- **d** $A(0, a)$ is equidistant from $P(3, -3)$ and $Q(-2, 2)$.

6 Find the relationship between x and y if the point $P(x, y)$ is always:

- **a** 3 units from $O(0, 0)$
- **b** 2 units from $A(1, 3)$.

B MIDPOINTS

The **midpoint** of line segment [AB] is the point which lies midway between points A and B.

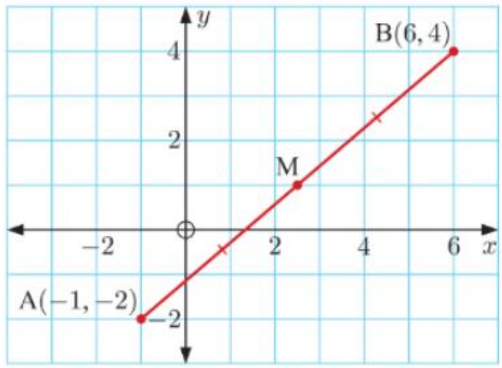
Consider the points A$(-1, -2)$ and B$(6, 4)$. From the diagram we see that the midpoint of [AB] is M$(2\frac{1}{2}, 1)$.

The x-coordinate of M is the *average* of the x-coordinates of A and B.

$\therefore$ the x-coordinate of M $= \dfrac{-1+6}{2} = \frac{5}{2} = 2\frac{1}{2}$

The y-coordinate of M is the *average* of the y-coordinates of A and B.

$\therefore$ the y-coordinate of M $= \dfrac{-2+4}{2} = 1$



If A(x_1, y_1) and B(x_2, y_2) are two points, then the **midpoint** of [AB] has coordinates $\left(\boldsymbol{\dfrac{x_1+x_2}{2}}, \boldsymbol{\dfrac{y_1+y_2}{2}}\right)$.

DEMO

Example 4 **Self Tutor**

Find the midpoint of [AB] given A$(-1, 3)$ and B$(4, 7)$.

$$\begin{array}{cc} \text{A}(-1, & 3) \\ \uparrow & \uparrow \\ x_1 & y_1 \end{array} \qquad \begin{array}{cc} \text{B}(4, & 7) \\ \uparrow & \uparrow \\ x_2 & y_2 \end{array}$$

The x-coordinate of the midpoint $= \dfrac{x_1+x_2}{2} = \dfrac{-1+4}{2} = \frac{3}{2} = 1\frac{1}{2}$

The y-coordinate of the midpoint $= \dfrac{y_1+y_2}{2} = \dfrac{3+7}{2} = 5$

So, the midpoint is $(1\frac{1}{2}, 5)$.

Example 5 **Self Tutor**

M is the midpoint of [AB]. A is $(1, 3)$ and M is $(4, -2)$. Find the coordinates of B.

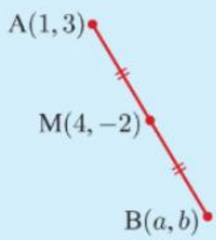
Suppose B has coordinates (a, b).

$\therefore \dfrac{a+1}{2} = 4$ and $\dfrac{b+3}{2} = -2$

$\therefore a+1 = 8$ and $b+3 = -4$

$\therefore a = 7$ and $b = -7$

$\therefore$ B is $(7, -7)$.



EXERCISE 9B

1 Find the coordinates of the midpoint of the line segment joining:

a $(8, 1)$ and $(2, 5)$ **b** $(2, -3)$ and $(0, 1)$ **c** $(3, 0)$ and $(0, 6)$

d $(-1, 4)$ and $(1, 4)$ **e** $(5, -3)$ and $(-1, 0)$ **f** $(5, 9)$ and $(-3, -4)$.

2 M is the midpoint of [AB]. Find the coordinates of B for:

a A$(6, 4)$ and M$(3, -1)$ **b** A$(-5, 0)$ and M$(0, -1)$

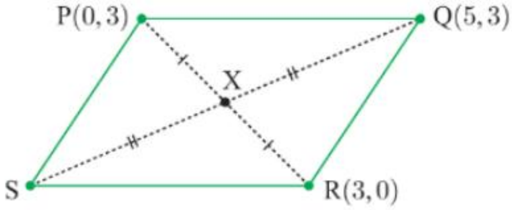
c A$(3, -2)$ and M$(1\frac{1}{2}, 2)$ **d** A$(-1, -2)$ and M$(-\frac{1}{2}, 2\frac{1}{2})$

e A$(7, -3)$ and M$(0, 0)$ **f** A$(3, -1)$ and M$(0, -\frac{1}{2})$.

3 [AB] is a diameter of a circle with centre C. If A is $(3, -2)$ and B is $(-1, -4)$, find the coordinates of C.

4 [PQ] is a diameter of a circle with centre $(3, -\frac{1}{2})$. If Q is $(-1, 2)$, find the coordinates of P.

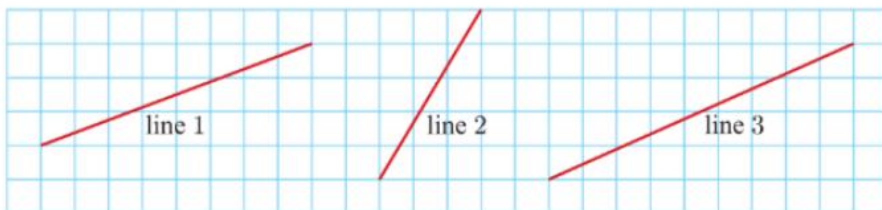
5 The diagonals of parallelogram PQRS bisect each other at X. Find the coordinates of S.



6 Triangle ABC has vertices $A(-1, 3)$, $B(1, -1)$, and $C(5, 2)$. Find the length of the line segment from A to the midpoint of [BC].

C GRADIENT

Consider the lines shown:



We can see that line 2 rises much faster than the other two lines, so line 2 is steepest.

However, most people would find it hard to tell which of lines 1 and 3 is steeper just by looking at them. We therefore need a more precise way to measure the steepness of a line.

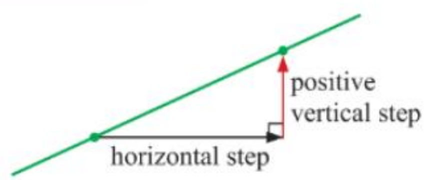
The **gradient** of a line is a measure of its steepness.

To calculate the gradient of a line, we first choose any two distinct points on the line. We can move from one point to the other by making a positive **horizontal step** followed by a **vertical step**.

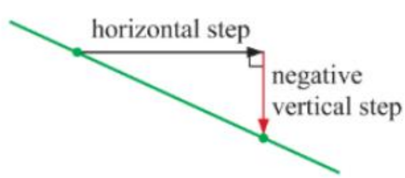
The gradient is calculated by dividing the vertical step by the horizontal step.

The **gradient** of a line $= \dfrac{\text{vertical step}}{\text{horizontal step}}$ or $\dfrac{y\text{-step}}{x\text{-step}}$.

If the line is sloping upwards, then both steps are positive, so the line has a **positive gradient**.

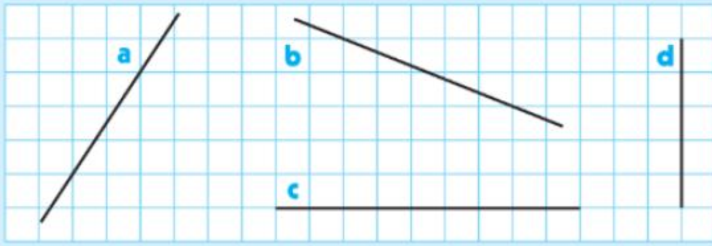


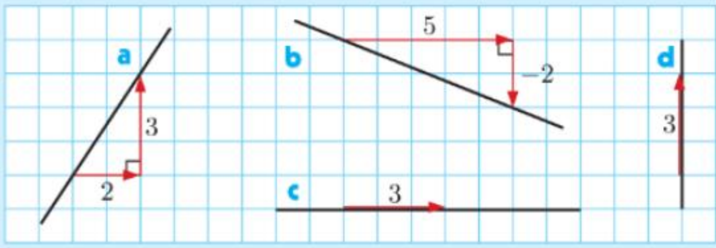
If the line is sloping downwards, the horizontal step is positive and the vertical step is negative, so the line has a **negative gradient**.



Example 6 **Self Tutor**

Find the gradient of each line segment:





a gradient $= \frac{3}{2}$

b gradient $= \frac{-2}{5} = -\frac{2}{5}$

c gradient $= \frac{0}{3} = 0$

d gradient $= \frac{3}{0}$ which is undefined

From the previous **Example**, we can see that:

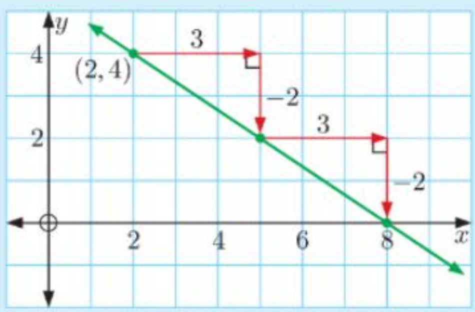
- The gradient of all **horizontal** lines is **0**, since the vertical step is 0.
- The gradient of all **vertical** lines is **undefined**, since the horizontal step is 0.

Example 7 **Self Tutor**

Draw a line with gradient $-\frac{2}{3}$, through the point $(2, 4)$.

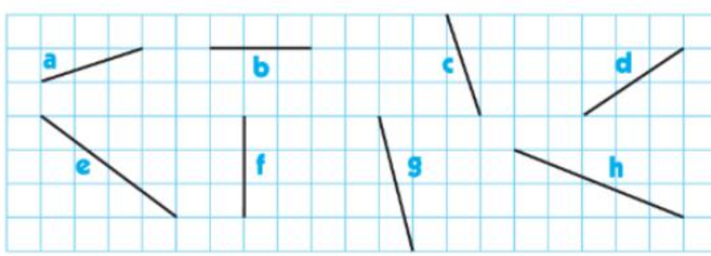
Plot the point $(2, 4)$.

gradient $= -\frac{2}{3} = \frac{-2}{3}$ ← y-step, ← x-step



EXERCISE 9C.1

1 Find the gradient of each line segment:



2 On grid paper, draw a line segment with gradient:

a $\frac{3}{4}$ **b** $-\frac{1}{2}$ **c** 2 **d** -3 **e** 0 **f** $-\frac{2}{5}$

3 Draw a line with gradient $\frac{1}{2}$, through the point $(3, -1)$.

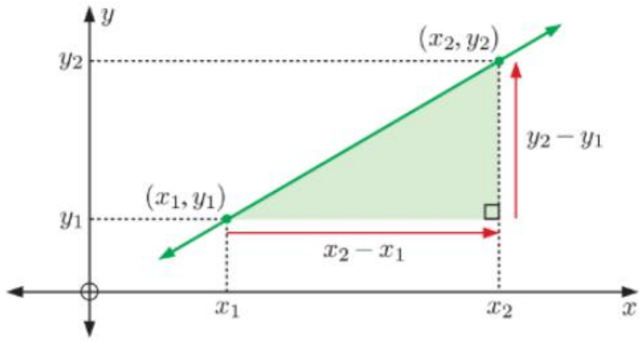
4 Draw a line with gradient $-\frac{3}{4}$, through the point $(-1, 3)$.

5 On the same set of axes, draw lines through $(2, 3)$ with gradients $\frac{1}{3}$, $\frac{3}{4}$, 2, and 4.

6 On the same set of axes, draw lines through $(-1, 2)$ with gradients 0, $-\frac{2}{5}$, -2, and -5.

THE GRADIENT FORMULA

The **gradient** of the line through (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$.



Example 8 **Self Tutor**

Find the gradient of the line through $(3, -2)$ and $(6, 4)$.

$(3, -2)$: $x_1 = 3$, $y_1 = -2$ $(6, 4)$: $x_2 = 6$, $y_2 = 4$

$$\begin{aligned}\text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - -2}{6 - 3} \\ &= \frac{6}{3} \\ &= 2\end{aligned}$$

Example 9 **Self Tutor**

Find t given that the line segment joining $(5, -2)$ and $(9, t)$ has gradient $\frac{2}{3}$.

The line segment joining $(5, -2)$ and $(9, t)$ has gradient $= \dfrac{t - -2}{9 - 5} = \dfrac{t+2}{4}$.

$$\begin{aligned}\therefore\ \frac{t+2}{4} &= \frac{2}{3} \\ \therefore\ 3(t+2) &= 8 \\ \therefore\ 3t + 6 &= 8 \\ \therefore\ 3t &= 2 \\ \therefore\ t &= \tfrac{2}{3}\end{aligned}$$

EXERCISE 9C.2

1 Use the gradient formula to find the gradient of the line through $A(-2, -3)$ and $B(5, 1)$. Plot the line segment [AB] on a set of axes to illustrate your answer.

2 Find the gradient of the line segment joining:

a $(2, 3)$ and $(7, 4)$ **b** $(5, 7)$ and $(1, 6)$ **c** $(1, -2)$ and $(3, 6)$

d $(5, 5)$ and $(-1, 5)$ **e** $(3, -1)$ and $(3, -4)$ **f** $(5, -1)$ and $(-2, -3)$

g $(-5, 2)$ and $(2, 0)$ **h** $(0, -1)$ and $(-2, -3)$ **i** $(-1, 7)$ and $(11, -9)$.

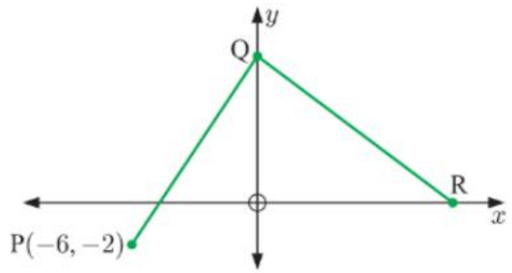
3 Find t given that the line segment joining:

a $(-3, 5)$ and $(4, t)$ has gradient 2 **b** $(5, t)$ and $(10, 12)$ has gradient $-\frac{1}{2}$

c $(3, -6)$ and $(t, -2)$ has gradient 3 **d** $(t, 9)$ and $(4, 7)$ has gradient $-\frac{3}{5}$

e $(2, 5)$ and (t, t) has gradient $\frac{4}{7}$ **f** $(t, 2t)$ and $(-3, 12)$ has gradient $-\frac{1}{4}$.

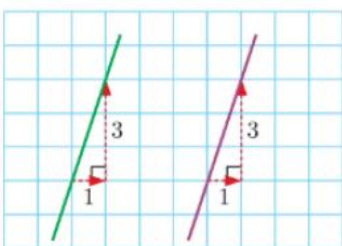
4



The gradient of [PQ] is $\frac{3}{2}$, and the gradient of [QR] is $-\frac{3}{4}$. Find the coordinates of R.

D PARALLEL AND PERPENDICULAR LINES

PARALLEL LINES



The given lines are parallel, and both of them have a gradient of 3.

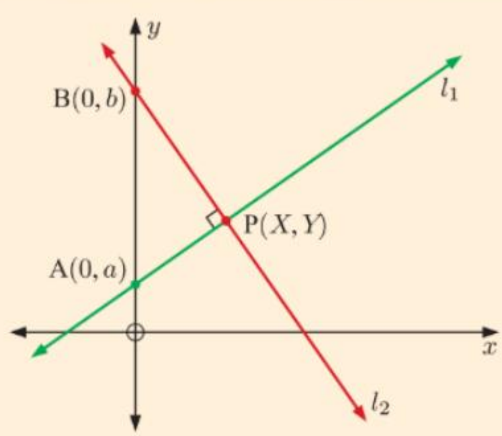
- If two lines are **parallel**, then they have **equal gradient**.
- If two lines have **equal gradient**, then they are **parallel**.

PERPENDICULAR LINES

INVESTIGATION PERPENDICULAR LINES

Consider two lines l_1 and l_2 which intersect at right angles at point $P(X, Y)$.

If l_1 and l_2 are not horizontal or vertical, then both lines will cut the y-axis. We suppose line l_1 cuts the y-axis at $A(0, a)$, and line l_2 cuts the y-axis at $B(0, b)$.

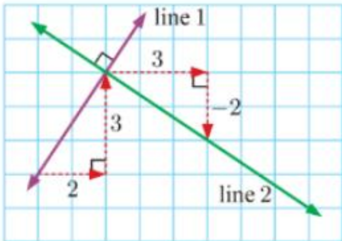


What to do:

1. Explain why $(AP)^2 + (BP)^2 = (AB)^2$.
2. Hence show that $X^2 + (Y - a)^2 + X^2 + (Y - b)^2 = (b - a)^2$.
3. By expanding the brackets and simplifying, show that $Y^2 - (a + b)Y + ab = -X^2$.

4 Hence show that $\dfrac{Y-a}{X} \times \dfrac{Y-b}{X} = -1$.

5 Explain the significance of the result in **4**.



Line 1 and line 2 are perpendicular.

Line 1 has gradient $\frac{3}{2}$.

Line 2 has gradient $\frac{-2}{3} = -\frac{2}{3}$.

We see that the gradients are *negative reciprocals* of each other, and their product is $\frac{3}{2} \times -\frac{2}{3} = -1$.

For lines which are not horizontal or vertical:

- if the lines are **perpendicular**, then their gradients are **negative reciprocals**
- if the gradients are **negative reciprocals**, then the lines are **perpendicular**.

DEMO

Example 10 **Self Tutor**

Find the gradient of all lines perpendicular to a line with a gradient of:

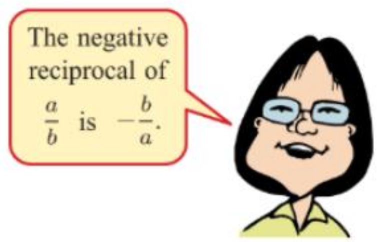
a $\frac{2}{7}$ **b** -5

a The negative reciprocal of $\frac{2}{7}$ is $-\frac{7}{2}$.

$\therefore$ the gradient of any perpendicular line is $-\frac{7}{2}$.

b The negative reciprocal of $-5 = \frac{-5}{1}$ is $\frac{1}{5}$.

$\therefore$ the gradient of any perpendicular line is $\frac{1}{5}$.



EXERCISE 9D.1

1 Find the gradient of all lines perpendicular to a line with a gradient of:

a $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** 3 **d** 7

e $-\frac{2}{5}$ **f** $-\frac{7}{2}$ **g** $-1\frac{1}{3}$ **h** -1

2 The gradients of two lines are listed below. Which of the line pairs are perpendicular?

a $\frac{1}{3}, 3$ **b** $5, -5$ **c** $\frac{3}{7}, -2\frac{1}{3}$ **d** $4, -\frac{1}{4}$

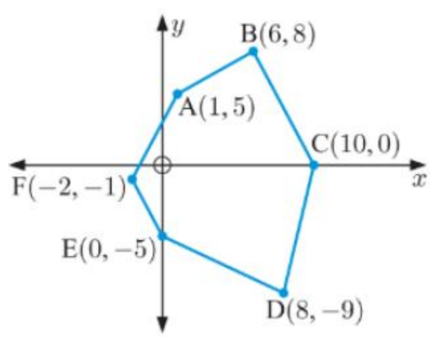
e $6, -\frac{5}{6}$ **f** $\frac{2}{3}, -\frac{3}{2}$ **g** $\frac{p}{q}, \frac{q}{p}$ **h** $\frac{a}{b}, -\frac{b}{a}$

3 Consider the hexagon alongside.

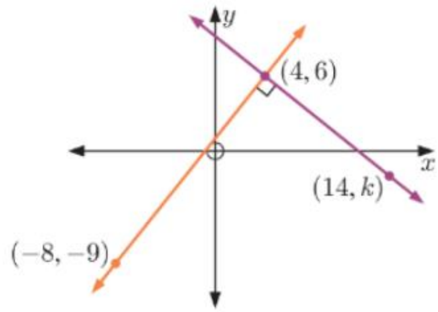
a Calculate the gradient of each side of the hexagon.

b Which sides are:

i parallel **ii** perpendicular?



4 Find the value of k:



5 Consider the points $A(1, 4)$, $B(-1, 0)$, $C(6, 3)$, and $D(t, -1)$. Find t if:

a [AB] is parallel to [CD]

b [AC] is parallel to [DB]

c [AB] is perpendicular to [CD]

d [AD] is perpendicular to [BC].

6 Consider the points $P(1, 5)$, $Q(5, 7)$, and $R(3, 1)$.

a Show that triangle PQR is isosceles.

b Find the midpoint M of [QR].

c Use gradients to verify that [PM] is perpendicular to [QR].

d Draw a sketch to illustrate what you have found.

7 For the points $A(-1, 1)$, $B(1, 5)$, and $C(5, 1)$, M is the midpoint of [AB], and N is the midpoint of [BC]. Show that [MN] is:

a parallel to [AC]

b half the length of [AC].

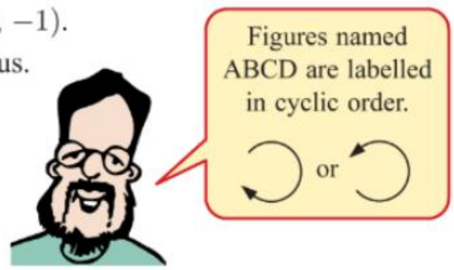
8 Consider the points $A(1, 3)$, $B(6, 3)$, $C(3, -1)$, and $D(-2, -1)$.

a Use the distance formula to show that ABCD is a rhombus.

b Find the midpoints of [AC] and [BD].

c Show that [AC] and [BD] are perpendicular.

d Sketch your findings.



9 The sketch of quadrilateral ABCD is not drawn to scale. P, Q, R, and S are the midpoints of [AB], [BC], [CD], and [DA] respectively.

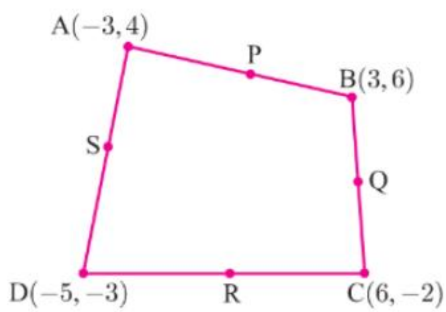
a Find the coordinates of:

i P **ii** Q **iii** R **iv** S.

b Find the gradient of:

i [PQ] **ii** [QR] **iii** [RS] **iv** [SP].

c What can be deduced about quadrilateral PQRS?



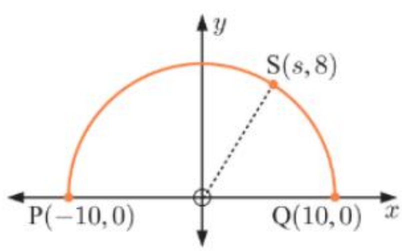
10 $S(s, 8)$ lies on a semi-circle as shown.

a Find s.

b Find the gradient of:

i [PS] **ii** [SQ].

c Hence show that angle PSQ is a right angle.

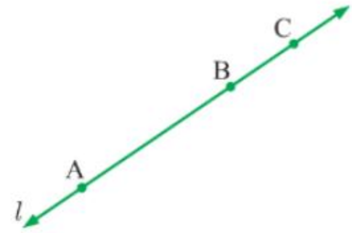


COLLINEAR POINTS

Three or more points are **collinear** if they lie on the same straight line.

Consider the three collinear points A, B, and C, which all lie on the line l.

gradient of [AB] = gradient of [BC] = gradient of l



Three points A, B, and C are **collinear** if gradient of [AB] = gradient of [BC].

Example 11 Self Tutor

Show that the points A(1, −1), B(6, 9), and C(3, 3) are collinear.

Gradient of [AB] $= \dfrac{9 - -1}{6-1} = \frac{10}{5} = 2.$ Gradient of [BC] $= \dfrac{3-9}{3-6} = \frac{-6}{-3} = 2.$

$\therefore$ [AB] is parallel to [BC], and point B is common to both line segments.

$\therefore$ A, B, and C are collinear.

EXERCISE 9D.2

1 Determine whether the following sets of points are collinear:

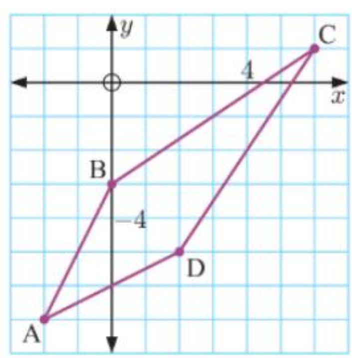
- **a** A(1, 2), B(4, 6), and C(−4, −4)
- **b** P(−6, −6), Q(−1, 0), and R(4, 6)
- **c** R(5, 2), S(−6, 5), and T(0, −4)
- **d** A(0, −2), B(−1, −5), and C(3, 7).

2 Find c given that these three points are collinear:

- **a** A(−4, −2), B(0, 2), and C(c, 5)
- **b** P(3, −2), Q(4, c), and R(−1, 10).

3 The points A(−2, −7), B(0, −3), C(6, 1), and D(2, −5) form a kite.

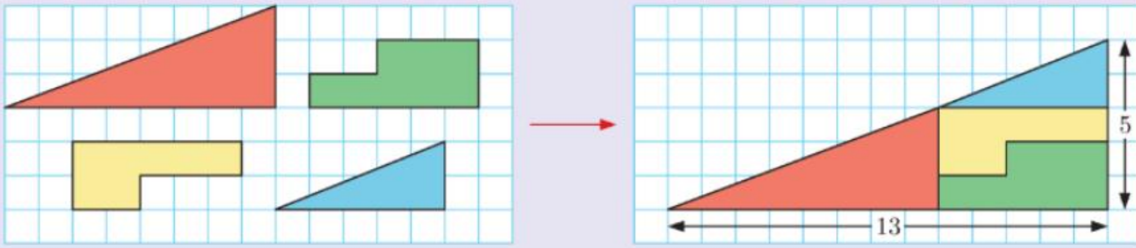
- **a** Find the midpoint M of [BD].
- **b** Show that A, M, and C are collinear.
- **c** Show that [AC] is perpendicular to [BD].



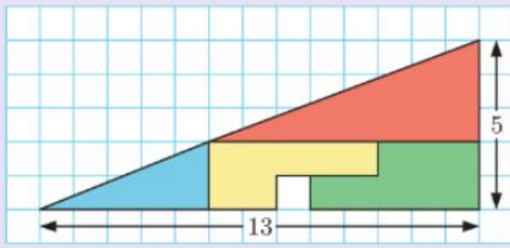
PUZZLE — THE MISSING SQUARE

Stephanie presents the following puzzle to her friend Courtney:

"I can arrange these four shapes to form a right angled triangle which is 13 units long and 5 units high."



"I can then rearrange the shapes to form a triangle of the same size. However, there is now a missing square."



Can you explain why there is a missing square?

E THE EQUATION OF A LINE

The **equation of a line** is a rule which connects the x and y-coordinates of **all** points on the line.

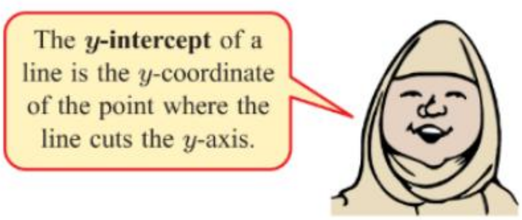
The equation of a line is commonly written in either **gradient-intercept form** or in **general form**.

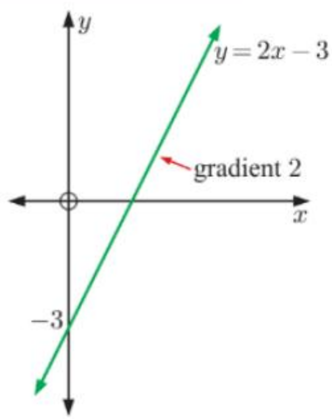
GRADIENT-INTERCEPT FORM

$y = mx + c$ is called the **gradient-intercept form** of an equation of a line.

The line with equation $y = mx + c$ has gradient m and y-intercept c.

For example, the line with equation $y = 2x - 3$ has gradient 2 and y-intercept -3.





GENERAL FORM

$Ax + By = C$ is called the **general form** of the equation of a line.

For example, the equations $2x + 3y = 5$ and $x - 6y = -7$ are in general form.

Equations in general form are usually written with a positive coefficient of x.

FINDING THE EQUATION OF A LINE

If we are given enough information about a line, we can determine its equation.

To determine the equation of a line, we need to know either:

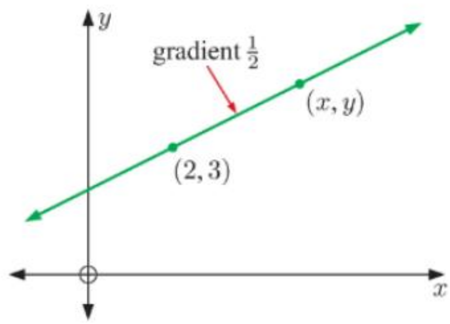
- its gradient and at least one point which lies on the line, or
- two points which lie on the line.

Suppose that a line has gradient $\frac{1}{2}$, and passes through the point $(2, 3)$.

For any point (x, y) which lies on the line, the gradient between $(2, 3)$ and (x, y) is $\frac{y-3}{x-2}$.

$\therefore$ the line has equation $\frac{y-3}{x-2} = \frac{1}{2}$

which can be written as $y - 3 = \frac{1}{2}(x - 2)$.



We can rearrange this to find the equation of the line in either gradient-intercept form or general form:

Gradient-intercept form

$$y - 3 = \tfrac{1}{2}(x - 2)$$
$$\therefore\ y - 3 = \tfrac{1}{2}x - 1$$
$$\therefore\ y = \tfrac{1}{2}x + 2$$

General form

$$y - 3 = \tfrac{1}{2}(x - 2)$$
$$\therefore\ 2(y - 3) = 1(x - 2)$$
$$\therefore\ 2y - 6 = x - 2$$
$$\therefore\ x - 2y = -4$$

If a straight line has gradient m and passes through (a, b), then it has equation

$$\boldsymbol{\frac{y-b}{x-a} = m} \quad \text{or} \quad \boldsymbol{y - b = m(x - a)}.$$

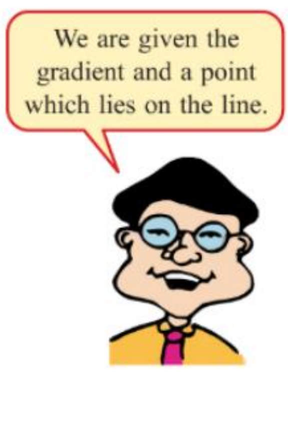
We can rearrange the equation into either **gradient-intercept form** or **general form**.

Example 12 **Self Tutor**

Find, in *gradient-intercept form*, the equation of the line with gradient 5 that passes through $(-1, 3)$.

The equation of the line is $y - 3 = 5(x - -1)$

$$\therefore\ y - 3 = 5(x + 1)$$
$$\therefore\ y - 3 = 5x + 5$$
$$\therefore\ y = 5x + 8$$



EXERCISE 9E.1

1 Find, in *gradient-intercept form*, the equation of the line with:

a gradient 2, passing through $(1, 3)$

b gradient -1, passing through $(-1, 2)$

c gradient $\frac{2}{3}$, passing through $(-3, 1)$

d gradient $-\frac{4}{5}$, passing through $(4, -2)$

e gradient $-\frac{3}{4}$, passing through $(6, -5)$.

Example 13 **Self Tutor**

Find, in *general form*, the equation of the line with gradient $\frac{3}{4}$ that passes through $(5, -2)$.

The equation of the line is $y - -2 = \frac{3}{4}(x - 5)$

$$\therefore\ 4(y + 2) = 3(x - 5)$$
$$\therefore\ 4y + 8 = 3x - 15$$
$$\therefore\ 3x - 4y = 23$$

2 Find, in *general form*, the equation of the line with:

a gradient 4, passing through $(3, 5)$

b gradient $-\frac{3}{5}$, passing through $(-2, 1)$

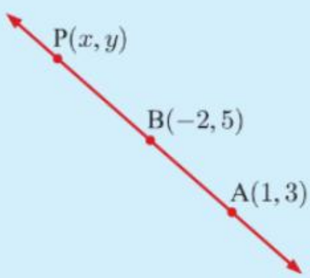
c gradient $\frac{1}{3}$, passing through $(1, 4)$

d gradient $-\frac{3}{4}$, passing through $(0, 6)$

e gradient $\frac{2}{7}$, passing through $(-5, -5)$.

Example 14 **Self Tutor**

Find, in *gradient-intercept form*, the equation of the line which passes through $A(1, 3)$ and $B(-2, 5)$.

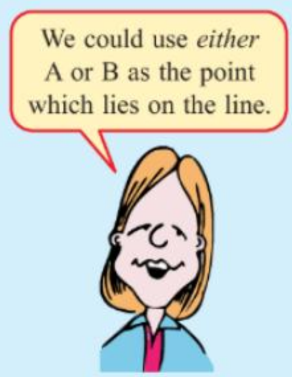


The line has gradient $= \dfrac{5-3}{-2-1} = \dfrac{2}{-3} = -\dfrac{2}{3}$,

and passes through the point $A(1, 3)$.

$\therefore$ the equation of the line is

$$y - 3 = -\tfrac{2}{3}(x - 1)$$
$$\therefore\ y - 3 = -\tfrac{2}{3}x + \tfrac{2}{3}$$
$$\therefore\ y = -\tfrac{2}{3}x + \tfrac{11}{3}$$



3 Find, in *gradient-intercept form*, the equation of the line which passes through:

a $A(8, 4)$ and $B(5, 1)$

b $A(5, -1)$ and $B(4, 0)$

c $A(-2, 4)$ and $B(-3, -2)$

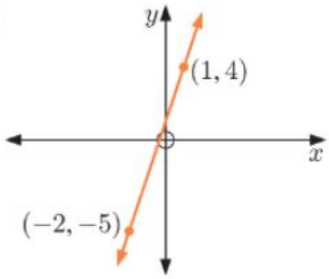
d $P(-4, 6)$ and $Q(2, 9)$

e $M(-1, -2)$ and $N(5, -4)$

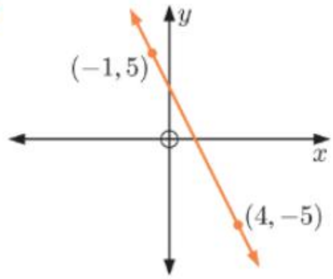
f $R(2, -4)$ and $S(7, -7)$.

4 Find, in *general form*, the equation of each of the following lines:

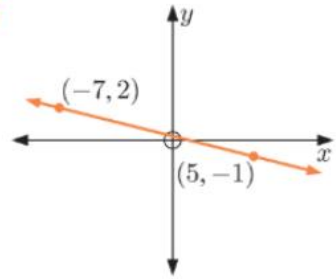
a



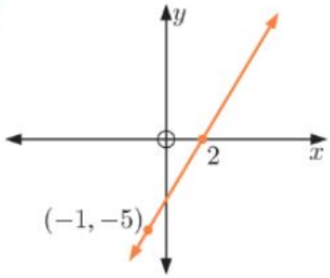
b



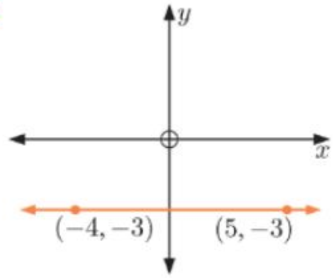
c



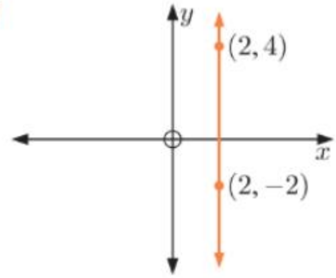
d



e



f



5 **a** Find, in general form, the equation of the line through $A(-3, 5)$ and $B(2, 1)$.

b Show that the point $C(12, -7)$ also lies on this line.

6 Find the equation of the line which:

a cuts the x-axis at 5 and the y-axis at -2

b cuts the x axis at -1, and passes through $(-3, 4)$

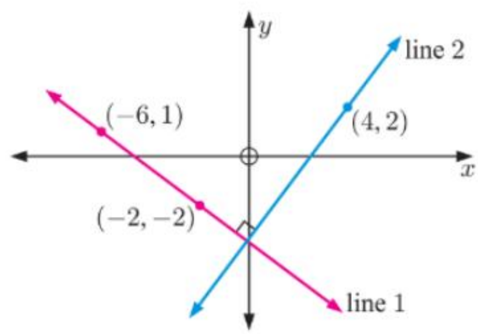
c is parallel to a line with gradient 2, and passes through the point $(-1, 4)$

d is perpendicular to a line with gradient $\frac{3}{4}$, and cuts the x-axis at 5

e is perpendicular to a line with gradient -2, and passes through $(-2, 3)$.

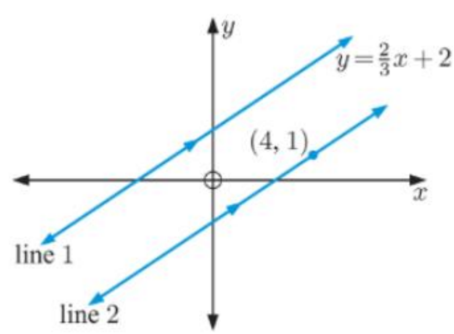
7 **a** Find the gradient of line 1.

b Hence, find the equation of line 2.



8 **a** Find, in gradient-intercept form, the equation of line 2.

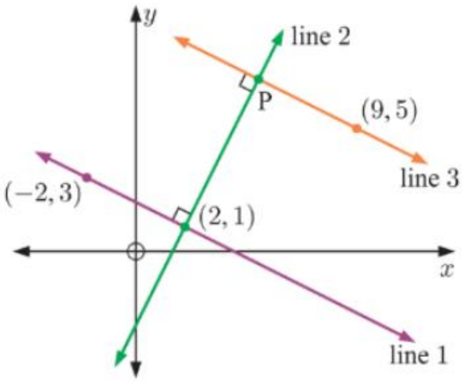
b Hence, find the y-intercept of line 2.



9 Find the equation of the line through $(-1, 7)$, which is parallel to the line through $(-3, -4)$ and $(2, 3)$.

10 Find the equation of the line through $(2, 0)$, which is perpendicular to the line through $(-5, 3)$ and $(4, -3)$.

11



a Find, in gradient-intercept form, the equation of:
   i line 1   **ii** line 2   **iii** line 3.

b Show that the coordinates of P are $(5, 7)$.

FINDING THE GENERAL FORM OF A LINE QUICKLY

If a line has gradient $\frac{3}{4}$, its equation has the form $y = \frac{3}{4}x + c$

$$\therefore\ 4y = 3x + 4c$$

$$\therefore\ 3x - 4y = C \quad \text{for some constant } C.$$

Similarly, if a line has gradient $-\frac{3}{4}$, its equation has the form $3x + 4y = C$.

- The equation of a line with gradient $\dfrac{A}{B}$ has the general form $Ax - By = C$.
- The equation of a line with gradient $-\dfrac{A}{B}$ has the general form $Ax + By = C$.

The constant term C is obtained by substituting the coordinates of any point which lies on the line.

Example 15 — Self Tutor

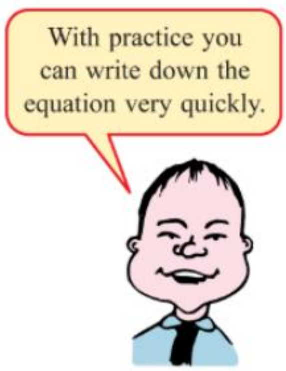
Find the equation of the line:

a with gradient $\frac{3}{4}$, which passes through $(5, -2)$

b with gradient $-\frac{3}{4}$, which passes through $(1, 7)$.

a The equation is $3x - 4y = 3(5) - 4(-2)$
$\therefore\ 3x - 4y = 23$

b The equation is $3x + 4y = 3(1) + 4(7)$
$\therefore\ 3x + 4y = 31$



EXERCISE 9E.2

1 Find the equation of the line:

a through $(4, 1)$ with gradient $\frac{1}{2}$

b through $(-2, 5)$ with gradient $\frac{2}{3}$

c through $(5, 0)$ with gradient $\frac{3}{4}$

d through $(3, -2)$ with gradient 3

e through $(1, 4)$ with gradient $-\frac{1}{3}$

f through $(2, -3)$ with gradient $-\frac{3}{4}$

g through $(3, -2)$ with gradient -2

h through $(0, 4)$ with gradient -3.

2 Find the gradient of the line with equation:

- **a** $2x + 3y = 8$
- **b** $3x - 7y = 11$
- **c** $6x - 11y = 4$
- **d** $5x + 6y = -1$
- **e** $3x + 6y = -1$
- **f** $15x - 5y = 17$

3 Explain why:

- **a** any line parallel to $3x + 5y = 2$ has the form $3x + 5y = C$
- **b** any line perpendicular to $3x + 5y = 2$ has the form $5x - 3y = C$.

4 Find the equation of the line which is:

- **a** parallel to the line $3x + 4y = 6$ and which passes through $(2, 1)$
- **b** perpendicular to the line $5x + 2y = 10$ and which passes through $(-1, -1)$
- **c** perpendicular to the line $x - 3y + 6 = 0$ and which passes through $(-4, 0)$
- **d** parallel to the line $x - 3y = 11$ and which passes through $(0, 0)$.

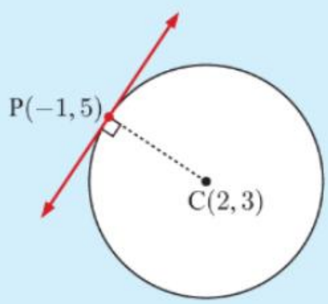
5 $2x - 3y = 6$ and $6x + ky = 4$ are two straight lines.

- **a** Write down the gradient of each line.
- **b** Find k such that the lines are parallel.
- **c** Find k such that the lines are perpendicular.

6 Answer the **Opening Problem** on page **158**.

Example 16 **Self Tutor**

A circle has centre $(2, 3)$. Find the equation of the tangent to the circle with point of contact $(-1, 5)$.



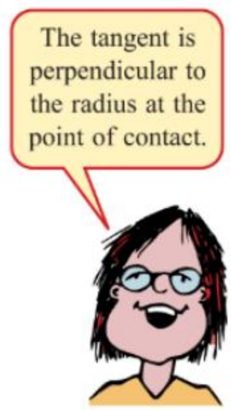
The gradient of [CP] is $\dfrac{5-3}{(-1)-2} = \dfrac{2}{-3}$
$= -\frac{2}{3}$

$\therefore$ the gradient of the tangent at P is $\frac{3}{2}$

$\therefore$ the equation of the tangent is

$3x - 2y = 3(-1) - 2(5)$

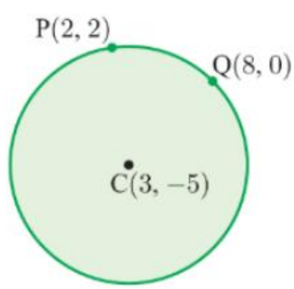
which is $3x - 2y = -13$.



7 Find the equation of the tangent to the circle:

- **a** with centre $(0, 2)$ if the point of contact is $(-1, 5)$
- **b** with centre $(0, 0)$ if the point of contact is $(3, -2)$
- **c** with centre $(3, -1)$ if the point of contact is $(-1, 1)$
- **d** with centre $(2, -2)$ if the point of contact is $(5, -2)$.

8



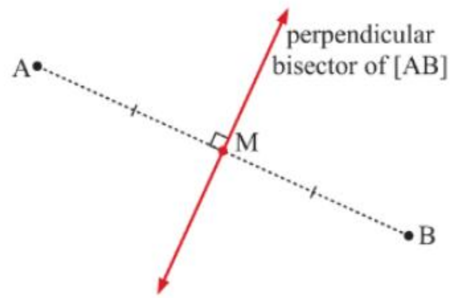
- **a** Find the equation of the tangent to the circle at:
  - **i** P
  - **ii** Q.
- **b** Show that the point $R(\frac{11}{2}, \frac{5}{2})$ lies on both tangents.
- **c** Show that $PR = QR$.

F PERPENDICULAR BISECTORS

If A and B are two points, the **perpendicular bisector** of [AB] is the line perpendicular to [AB], passing through the midpoint of [AB].

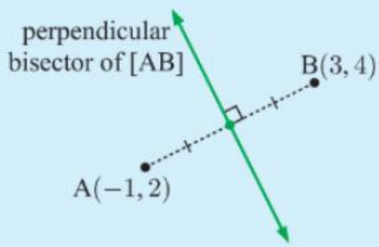
The perpendicular bisector of [AB] divides the number plane into two regions. On one side of the line are points that are closer to A than to B, and on the other side are points that are closer to B than to A.

Points on the perpendicular bisector of [AB] are **equidistant** from A and B.



Example 17 Self Tutor

Given $A(-1,\ 2)$ and $B(3,\ 4)$, find the equation of the perpendicular bisector of [AB].



M is $\left(\frac{-1+3}{2}, \frac{2+4}{2}\right)$ or $(1,\ 3)$.

The gradient of [AB] is $\frac{4-2}{3--1} = \frac{2}{4} = \frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{2}{1}$

$\therefore$ the equation of the perpendicular bisector is $2x + y = 2(1) + (3)$
which is $2x + y = 5$.

EXERCISE 9F

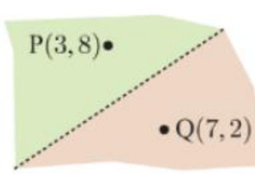
1 Find the equation of the perpendicular bisector of [AB] for:

- **a** $A(3,\ -3)$ and $B(1,\ -1)$
- **b** $A(1,\ 3)$ and $B(-3,\ 5)$
- **c** $A(3,\ 1)$ and $B(-3,\ 6)$
- **d** $A(4,\ -2)$ and $B(4,\ 4)$.

2 Suppose A is $(-1,\ -4)$ and B is $(3,\ 2)$.

- **a** Find the equation of the perpendicular bisector of [AB].
- **b** Show that $C(-5,\ 3)$ lies on the perpendicular bisector.
- **c** Show that C is equidistant from A and B.

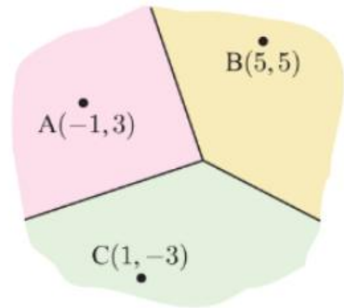
3 Two Post Offices are located at $P(3,\ 8)$ and $Q(7,\ 2)$ on a Council map. Find the equation of the line which should form the boundary between the two regions serviced by the Post Offices.



4 The **Voronoi** diagram alongside shows the location of three Post Offices and the corresponding regions of closest proximity. The Voronoi edges are the perpendicular bisectors of [AB], [BC], and [CA] respectively. Find:

a the equations of the Voronoi edges

b the coordinates of the point where the Voronoi edges meet.



REVIEW SET 9A

1 Find the midpoint of the line segment joining A(-2, 3) to B(-4, 3).

2 Find the distance from C(-3, -2) to D(0, 5).

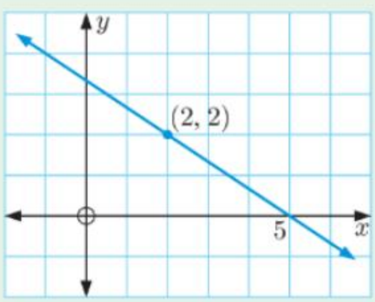
3 Find the gradient of all lines perpendicular to a line with gradient $\frac{2}{3}$.

4 K(-3, 2) and L(3, m) are $\sqrt{52}$ units apart. Find m.

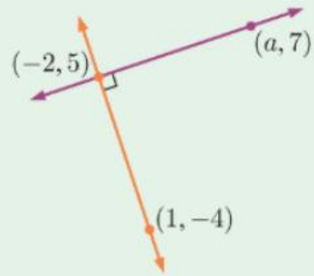
5 Find t given that the line joining (-1, t) and (5, -3) has gradient $\frac{4}{3}$.

6 Show that A(1, -2), B(4, 4), and C(5, 6) are collinear.

7 Find the equation of the line:



8 Find the value of a:



9 Consider the points A(-3, 1), B(1, 4), and C(4, 0).

a Show that triangle ABC is right angled and isosceles.

b Find the midpoint X of [AC].

c Use gradients to verify that [BX] is perpendicular to [AC].

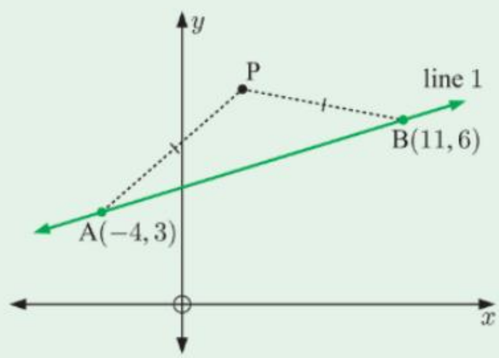
10 a Find, in general form, the equation of line 1.

b Point P has x-coordinate 3, and is equidistant from A and B. Find the coordinates of P.

c Find the equation of line 2, which is perpendicular to line 1, and passes through P.

d i Find the midpoint M of [AB].

ii Show that M lies on line 2.



11 Find the equation of the:

a tangent to the circle with centre $(-1, 2)$ at the point $(3, 1)$

b perpendicular bisector of [AB] for $A(2, 6)$ and $B(5, -2)$.

REVIEW SET 9B

1 Consider the points $S(7, -2)$ and $T(-1, 1)$.

a Find the distance ST.

b Determine the midpoint of [ST].

2 Find, in general form, the equation of the line passing through $P(-3, 2)$ and $Q(3, -1)$.

3 **a** Find the gradient of all lines perpendicular to a line with gradient $-\frac{1}{2}$.

b Determine whether the line $2x + y = 3$ is perpendicular to a line with gradient $-\frac{1}{2}$.

4 $X(-2, 3)$ and $Y(a, -1)$ are $\sqrt{17}$ units apart. Find the value of a.

5 Find b given that $A(-6, 2)$, $B(b, 0)$, and $C(3, -4)$ are collinear.

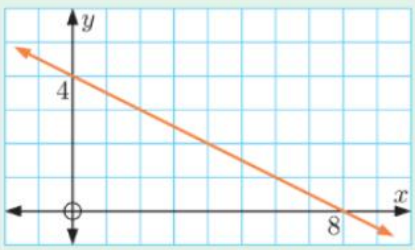
6 Find the equation of the line:

a with gradient -2 and y-intercept 7

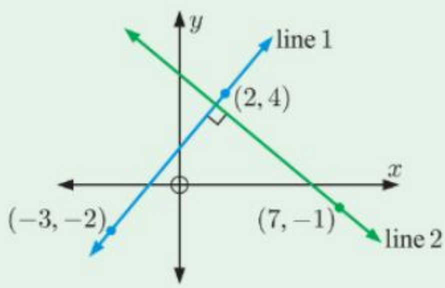
b passing through $(-1, 3)$ and $(2, 1)$

c parallel to a line with gradient $\frac{3}{2}$, and passing through $(5, 0)$.

7 Determine the equation of the line:



8 Find the equation of line 2.



9 Find, in gradient-intercept form, the equation of the line passing through $(1, -2)$ and $(3, 4)$.

10 $A(-3, 2)$, $B(2, 3)$, $C(4, -1)$, and $D(-1, -2)$ are the vertices of quadrilateral ABCD.

a **i** Find the gradient of each side of the quadrilateral.

ii What can you deduce about quadrilateral ABCD?

b **i** Find the midpoints of the diagonals [AC] and [BD].

ii What property of parallelograms does this check?

c **i** Find the gradients of the diagonals [AC] and [BD].

ii What does the product of these gradients tell us about quadrilateral ABCD?

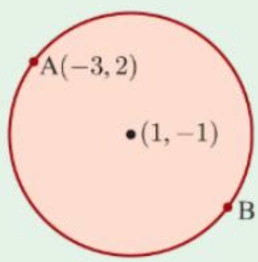
11 [AB] is a diameter of a circle with centre $(1, -1)$. A has coordinates $(-3, 2)$.

a Find the radius of the circle.

b Find the equation of the tangent at A.

c Find the coordinates of B.

d Find the equation of the tangent at B.



Chapter 10

Congruence and similarity

Contents:

Two figures are **congruent** if they are identical in every respect except for position.

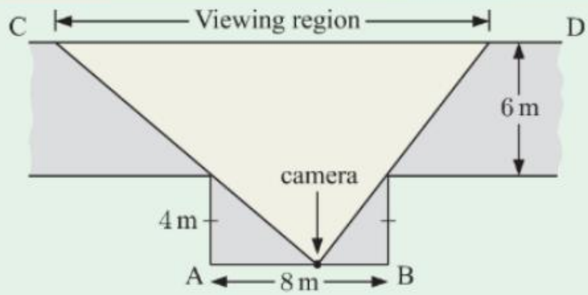
Two figures are **similar** if one figure is an enlargement of the other.

OPENING PROBLEM

In an art gallery, a security camera is being installed along the wall [AB], to view the opposite wall [CD].

Peter is wondering which location for the camera along [AB] maximises the viewing region on the opposite wall.

"It doesn't matter," says Linda, "no matter where the camera is placed, the size of the viewing region will be the same."



Things to think about:

Can you use similar triangles to determine whether Linda is correct?

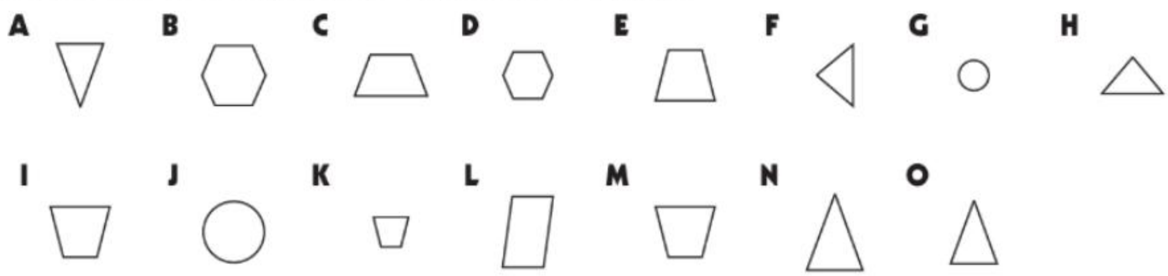
A CONGRUENCE

Two objects are **congruent** if they are identical in every respect except for position. The objects have the same size *and* the same shape.

EXERCISE 10A

1 Which of the following figures are congruent?

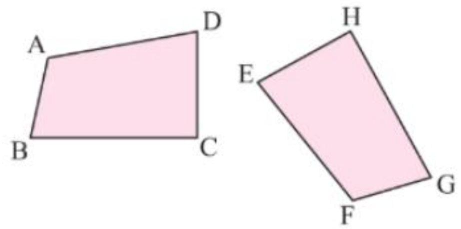
2 Which of the following geometric figures are congruent?



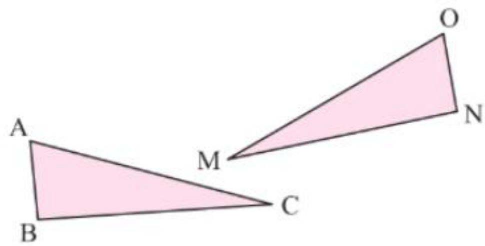
3 For each pair of congruent figures:

- **i** Identify the side in the second figure corresponding to side [AB] in the first figure.
- **ii** Identify the angle in the second figure corresponding to $A\widehat{B}C$ in the first figure.

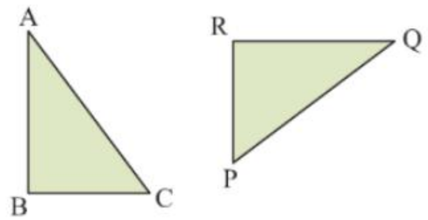
a



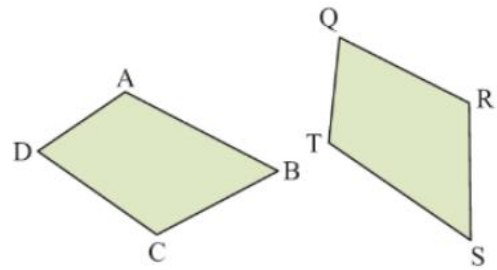
b



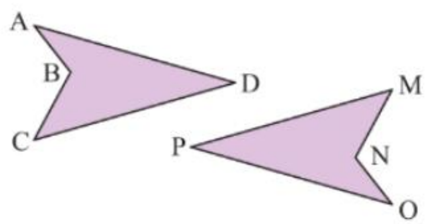
c



d



e



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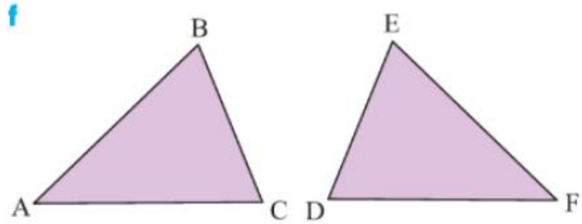
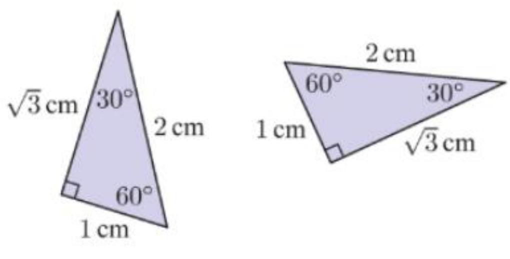


B CONGRUENT TRIANGLES

Two triangles are **congruent** if they are identical in every respect except for position.



The triangles alongside are congruent.

They have identical side lengths and angles.

If we are given sufficient information about a triangle, there will be only one way in which it can be drawn. Any two triangles which have this information in common must be **congruent**.

TESTS FOR TRIANGLE CONGRUENCE

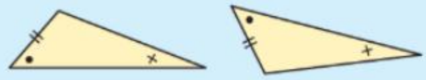
There are four acceptable tests for the congruence of two triangles.

Two triangles are **congruent** if one of the following is true:

- All corresponding sides are equal in length. **(SSS)**

- Two sides and the **included** angle are equal. **(SAS)**

- Two angles and a pair of **corresponding sides** are equal. **(AAcorS)**



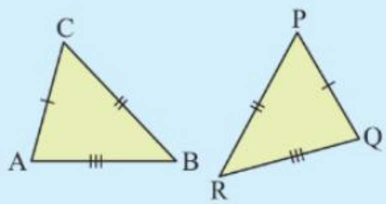
- For right angled triangles, the hypotenuses and one other pair of sides are equal. **(RHS)**

The information we are given will help us decide which congruence test to use. The diagrams in the following Exercise are sketches only and are *not* drawn to scale. However, the information marked on them is correct.

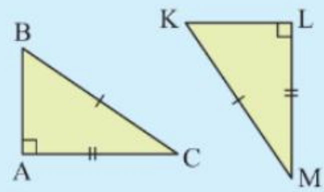
Example 1 Self Tutor

Are the following pairs of triangles congruent? If so, state the congruence relationship and give a brief reason.

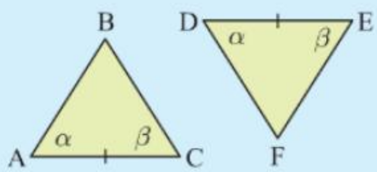
a



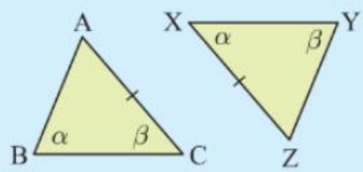
b



c



d

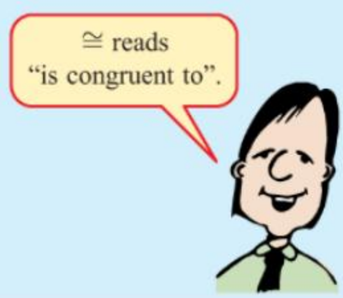


a $\triangle ABC \cong \triangle QRP$ {SSS}

b $\triangle ABC \cong \triangle LKM$ {RHS}

c $\triangle ABC \cong \triangle DFE$ {AAcorS}

d The two angles α and β are common, but although AC equals XZ, these sides are *not* corresponding. [AC] is opposite α whereas [XZ] is opposite β.
So, the triangles are not congruent.

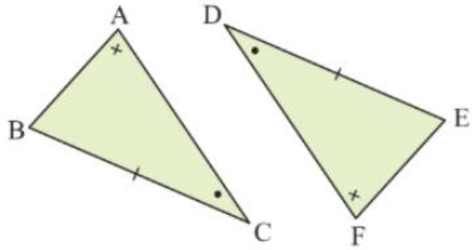


When we describe congruent triangles, we label the vertices in corresponding positions in the same order. In **Example 1** part **a** above, A and Q are opposite two tick marks, B and R are opposite one tick mark, and C and P are opposite three tick marks. So we write $\triangle ABC \cong \triangle QRP$, not $\triangle ABC \cong \triangle PQR$.

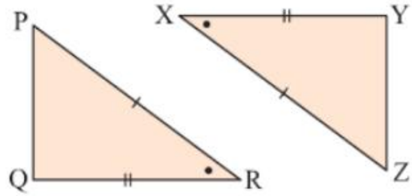
EXERCISE 10B

1 Are the following pairs of triangles congruent? If so, state the congruence relationship and give a brief reason.

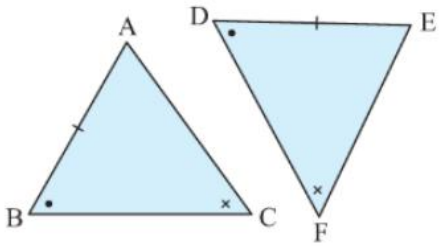
a



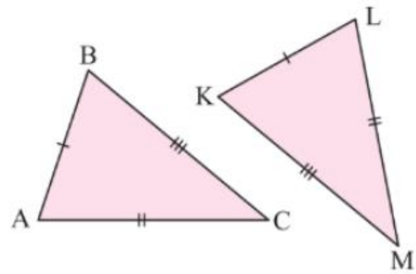
b



c



d



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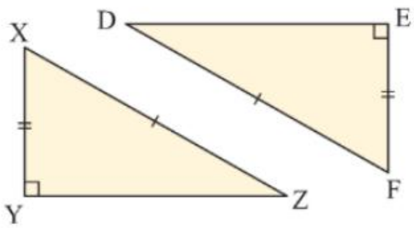

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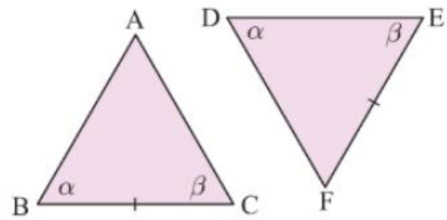
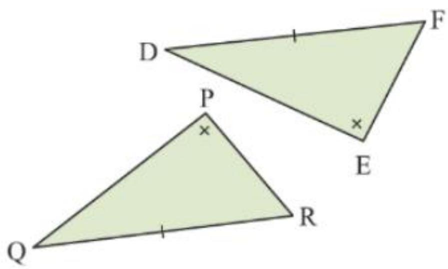
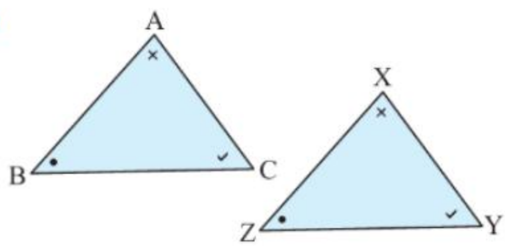


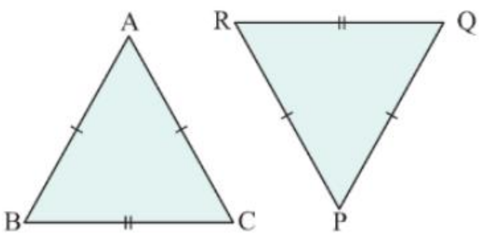
g



h



i



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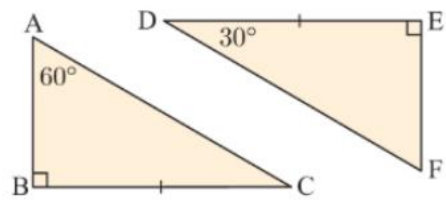
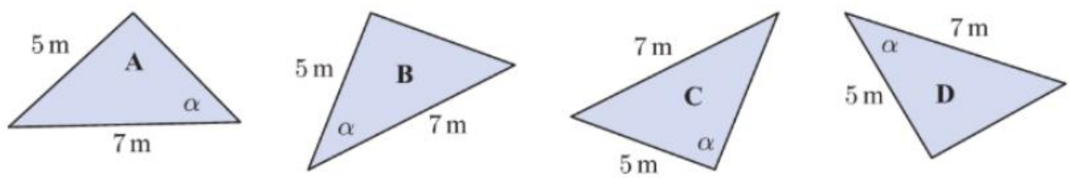
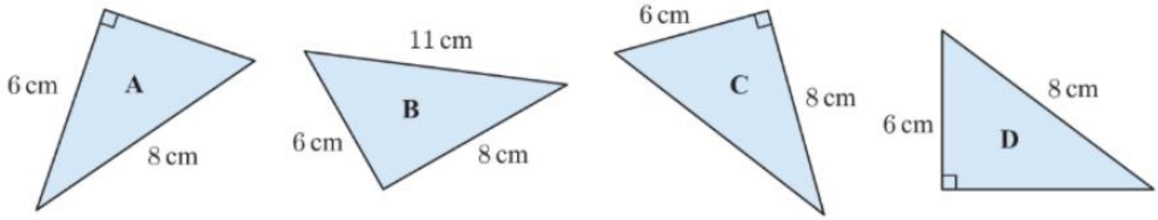


2 For the following groups of triangles, determine which two triangles are congruent. Give reasons for your answers. The triangles are not drawn to scale, but contain correct information.

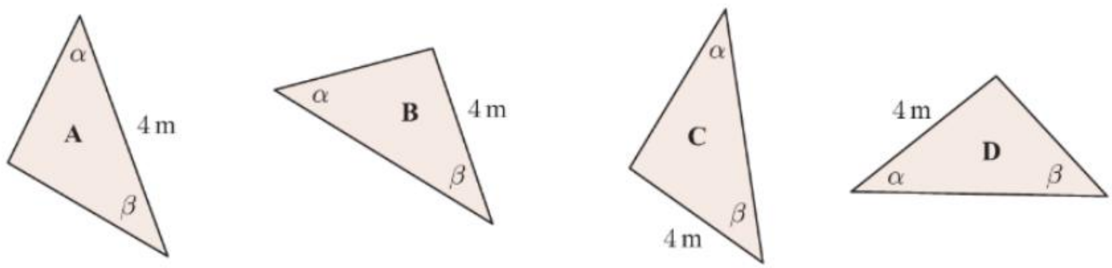
a



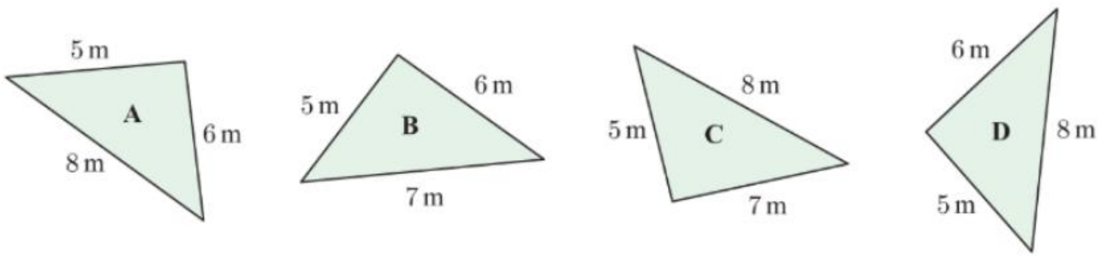
b



c



d

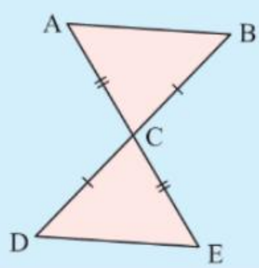


C PROOF USING CONGRUENCE

Once we have proven that two triangles are congruent, we can deduce that the remaining corresponding sides and angles of the triangles are equal. We can therefore use congruence to prove facts about geometric figures.

Example 2 **Self Tutor**

In the figure alongside, show that [AB] is parallel to [DE].

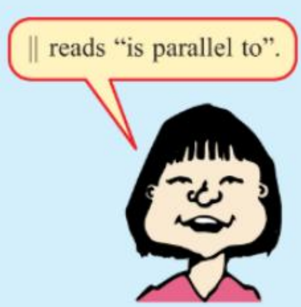


In triangles ABC and EDC:

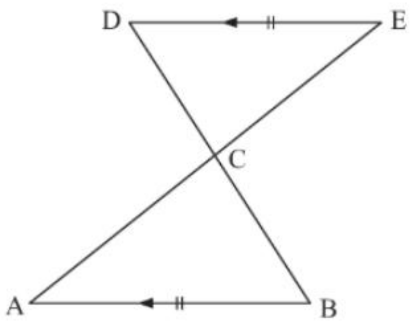
- BC = DC {given}
- AC = EC {given}
- $\widehat{ACB} = \widehat{ECD}$ {vertically opposite angles}

So, $\triangle ABC \cong \triangle EDC$ {SAS}

Therefore, $\widehat{ABC} = \widehat{EDC}$ {equating angles}
and so [AB] ∥ [DE] {equal alternate angles}



EXERCISE 10C

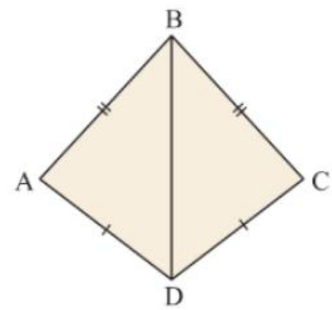
1 

In the given figure, [DE] is parallel to [AB] and DE = AB.
Show that the triangles are congruent.

2 **a** Show that triangles ABD and CBD are congruent.

b Given that $A\widehat{B}D = 47°$ and $B\widehat{A}D = 82°$, find the size of:

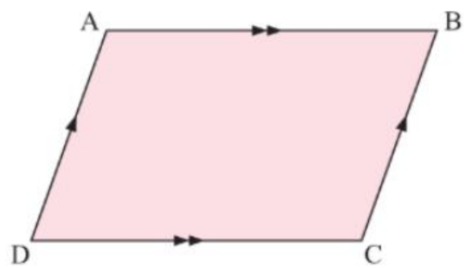
i $C\widehat{B}D$ **ii** $C\widehat{D}B$.

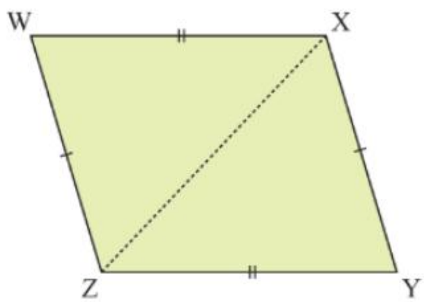


3 Consider the quadrilateral ABCD alongside. [AB] is parallel to [DC], and [AD] is parallel to [BC].

a Use congruence to show that the opposite sides are equal in length.

b Hence, show that the diagonals of a parallelogram bisect each other.



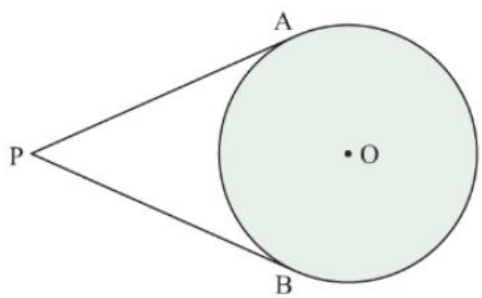
4 

WXYZ is a quadrilateral with opposite sides equal. [XZ] is added to the figure.

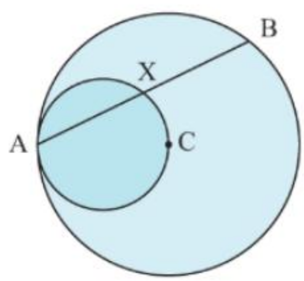
a Show that the two triangles created are congruent.

b Hence deduce that WXYZ is a parallelogram.

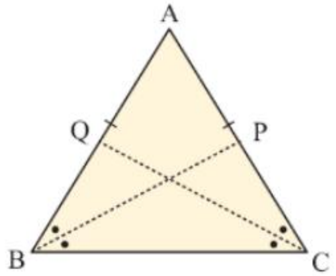
5 The tangents to a circle at A and B intersect at P.
Show that AP = BP.



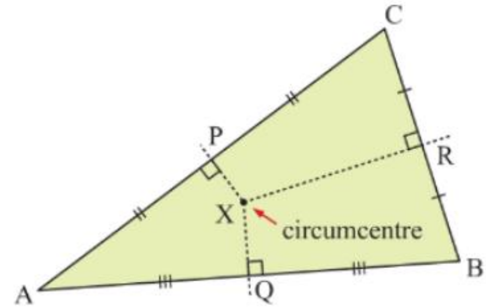
6 [AC] is a radius of the large circle, and a diameter of the small circle. A line through A cuts the small circle at X and the large circle at B.
Show that X is the midpoint of [AB].



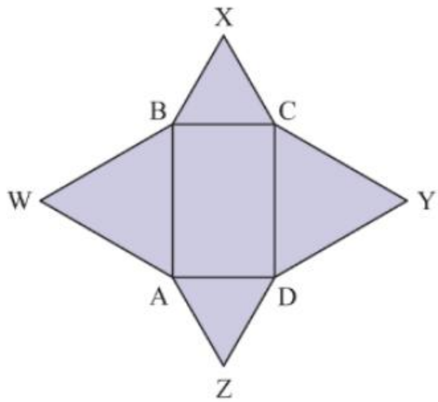
7 Triangle ABC is isosceles, with AB = AC. The angle bisectors of B and C are drawn, meeting the triangle at P and Q respectively. Show that AP = AQ.



8 The perpendicular bisectors of a triangle's edges meet at a point called the **circumcentre** of the triangle.
Prove that the circumcentre is equidistant from each vertex of the triangle.



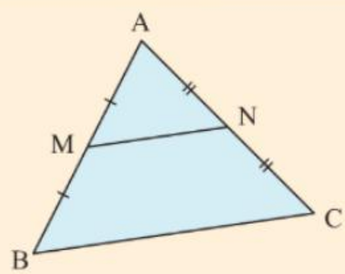
9 ABCD is a rectangle. Equilateral triangles are drawn from each side of the rectangle, with apexes W, X, Y, and Z.
Show that WXYZ is a rhombus.



INVESTIGATION — THE MIDPOINT THEOREM

In triangle ABC, M is the midpoint of [AB], and N is the midpoint of [AC].

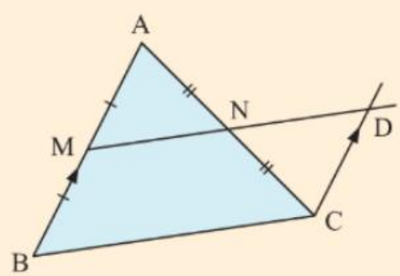
The **midpoint theorem** states that the line [MN] is parallel to [BC], and half its length.



Proving the midpoint theorem

What to do:

Suppose we extend [MN], and draw a line through C parallel to [AB]. We let these lines meet at D.



1 Show that triangles AMN and CDN are congruent.

2 Hence show that:

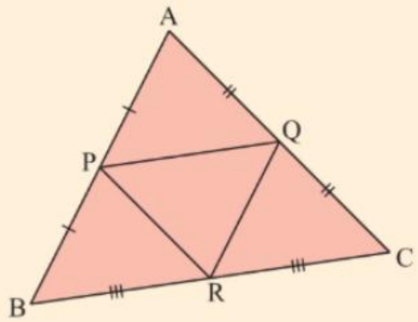
a MN = DN **b** BM = CD.

3 Show that BCDM is a parallelogram.

4 Hence, show that: **a** [MN] is parallel to [BC] **b** $MN = \frac{1}{2}BC$.

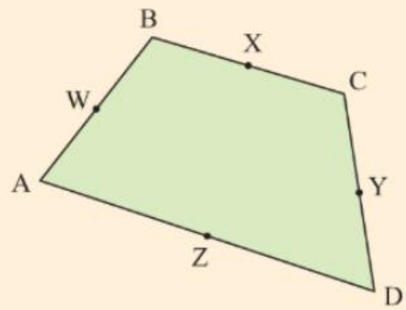
Using the midpoint theorem

What to do:

1 

In the diagram alongside, P, Q, and R are the midpoints of [AB], [AC], and [BC] respectively. Use the midpoint theorem to show that the four small triangles are all congruent.

2 For any quadrilateral ABCD, let W, X, Y, and Z be the midpoints of [AB], [BC], [CD], and [DA] respectively. Use the midpoint theorem to show that WXYZ is a parallelogram.

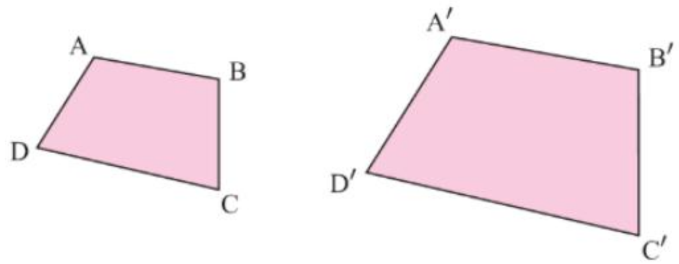


D SIMILARITY

Two figures are **similar** if one is an enlargement of the other, regardless of orientation.

If two figures are similar then their corresponding sides are **in proportion**. The lengths of their sides will be increased (or decreased) by the **same ratio** from one figure to the next. This ratio is called the **enlargement factor**.

Consider the enlargement below for which the enlargement factor k is 1.5.



Since $k = 1.5$, $\frac{\text{A}'\text{B}'}{\text{AB}} = \frac{\text{B}'\text{C}'}{\text{BC}} = \frac{\text{C}'\text{D}'}{\text{CD}} = \frac{\text{D}'\text{A}'}{\text{DA}} = \frac{\text{B}'\text{D}'}{\text{BD}} = ... = 1.5$.

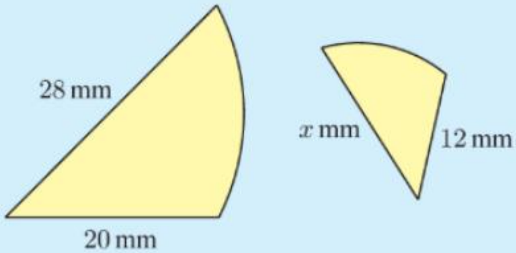
When a figure is enlarged or reduced, the size of its angles does not change. The figures are therefore **equiangular**.

Two figures are **similar** if:

- the figures are **equiangular** *and*
- the corresponding sides are in the **same ratio**.

Example 3 Self Tutor

These figures are similar. Find the value of x.



Since the figures are similar, their corresponding sides are in the same ratio.

$$\therefore \quad \frac{x}{28} = \frac{12}{20}$$

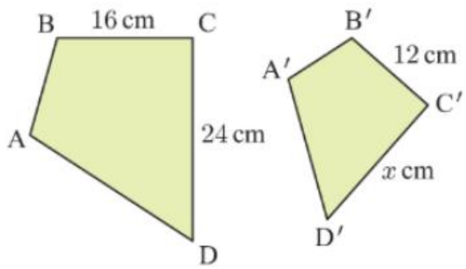
$$\therefore \quad x = \frac{\cancel{12}^{\,3}}{\cancel{20}_{\,5}} \times 28$$

$$\therefore \quad x = \frac{84}{5} = 16.8$$

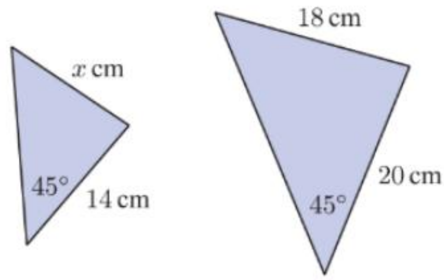
EXERCISE 10D

1 For each pair of similar figures, find the value of x:

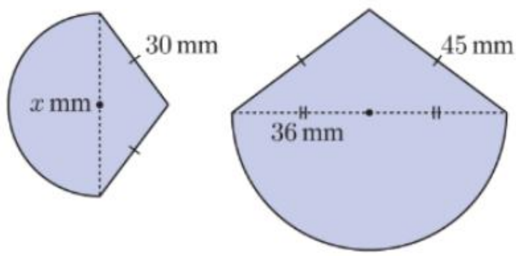
a



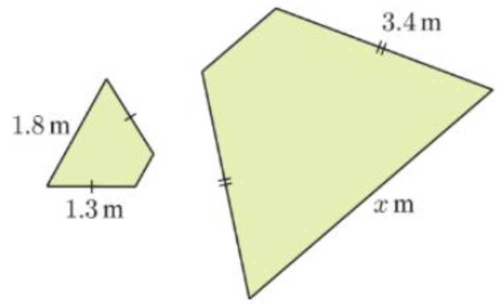
b



c



d



2 Comment on the truth of the following statements. For any statement which is false, justify your answer with an illustration.

a All equilateral triangles are similar.

b All isosceles triangles are similar.

c All rhombuses are similar.

d All circles are similar.

e All regular pentagons are similar.

f All ellipses are similar.

g All equiangular triangles are similar.

h All equiangular quadrilaterals are similar.

E SIMILAR TRIANGLES

When we are dealing with triangles, if *either* the triangles are equiangular *or* their corresponding sides are in the same ratio, then both these conditions must be true. Therefore, when testing for similar triangles, we only need to check that *one* of the conditions is true.

Two triangles are similar if *either*:

- they are equiangular *or*
- their side lengths are in the same ratio.

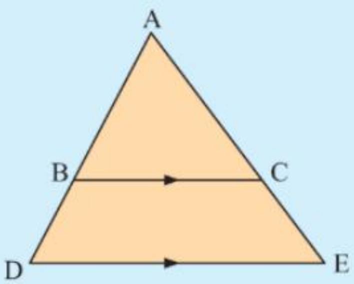
If we can show that two of the angles in one triangle are equal in size to two of the angles in another triangle, then the remaining angles must also be equal, since the angles in each triangle sum to $180°$.

Once we have established that two triangles are similar, we can use the fact that corresponding sides are in the same ratio to find unknown lengths.

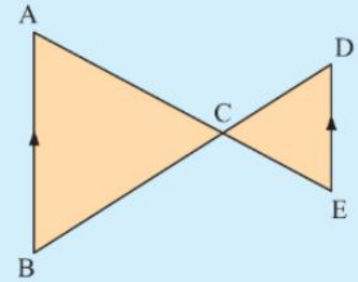
Example 4 **Self Tutor**

Show that the following figures possess similar triangles:

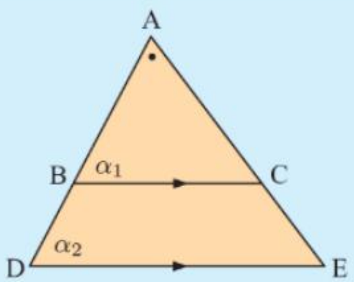
a



b



a

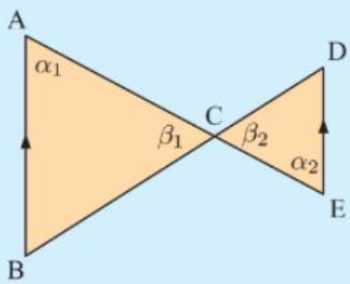


$\triangle$s ABC and ADE are equiangular as:

- $\alpha_1 = \alpha_2$ {equal corresponding angles}
- $\widehat{A}$ is common to both triangles

$\therefore$ the triangles are similar.

b



$\triangle$s ABC and EDC are equiangular as:

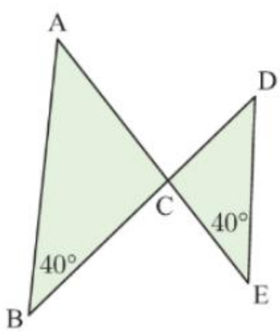
- $\alpha_1 = \alpha_2$ {equal alternate angles}
- $\beta_1 = \beta_2$ {vertically opposite angles}

$\therefore$ the triangles are similar.

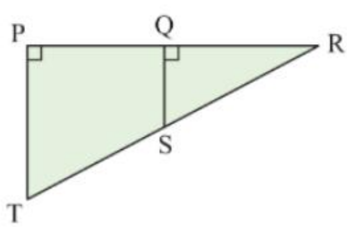
EXERCISE 10E

1 Show that the following figures possess similar triangles:

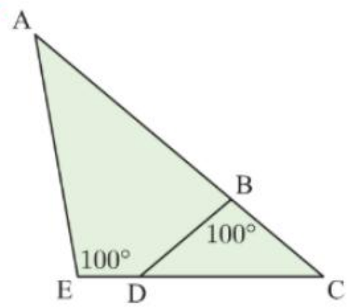
a



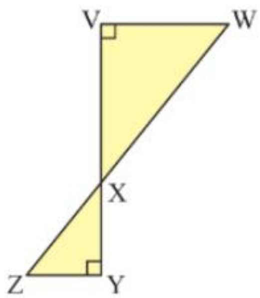
b



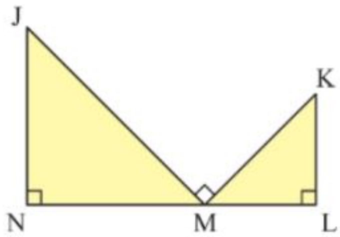
c



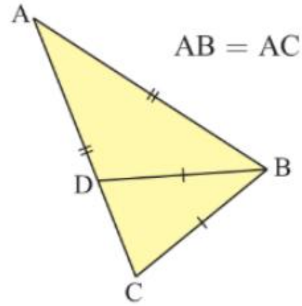
d



e

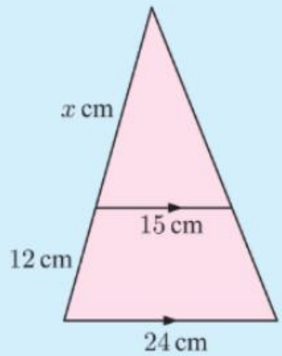


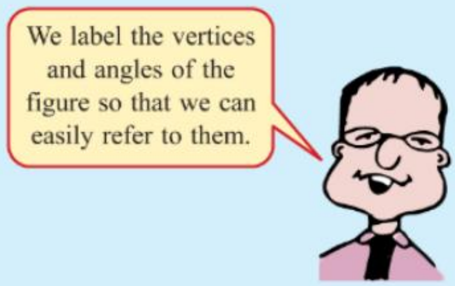
f

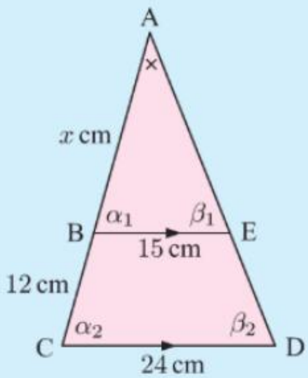


Example 5 **Self Tutor**

Establish that a pair of triangles is similar, and hence find x:







$\triangle$s ABE and ACD are equiangular since

$\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {equal corresponding angles}

$\therefore$ $\triangle$s ABE and ACD are similar.

Corresponding sides must be in the same ratio.

$$\therefore \frac{\text{AC}}{\text{AB}} = \frac{\text{CD}}{\text{BE}}$$

$$\therefore \frac{x+12}{x} = \frac{24}{15}$$

$$\therefore 1 + \frac{12}{x} = \frac{8}{5}$$

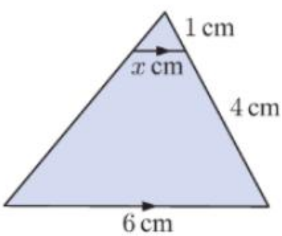
$$\therefore \frac{12}{x} = \frac{3}{5}$$

$$\therefore \frac{x}{12} = \frac{5}{3}$$

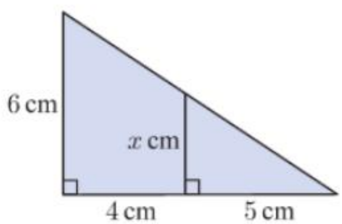
$$\therefore x = 20$$

2 For the following figures, establish that a pair of triangles is similar, and hence find x:

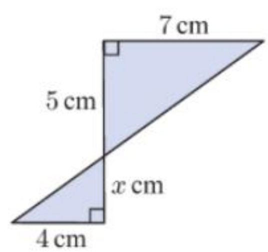
a



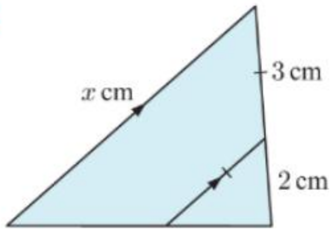
b



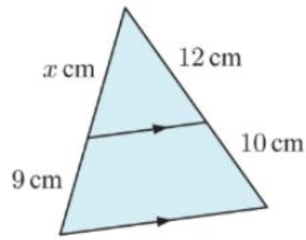
c



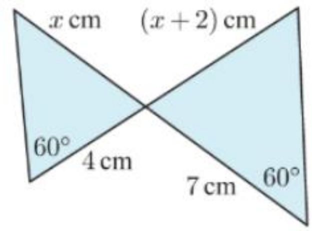
d



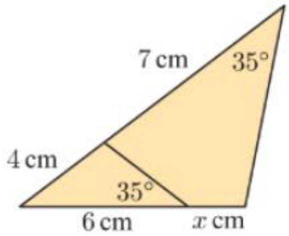
e



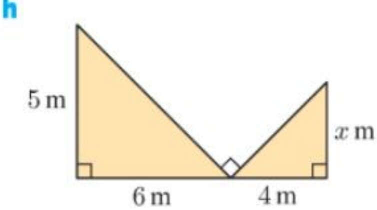
f



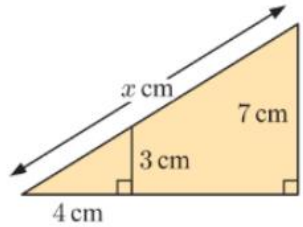
g



h



i

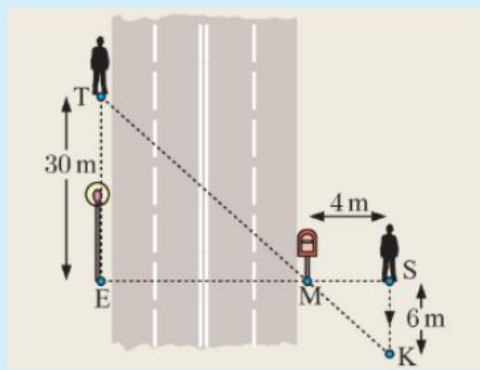


Example 6 **Self Tutor**

An electric light post E is directly opposite a mail box M on the other side of a straight road. Taj walks 30 metres along the road away from E to point T.

Kanvar is 4 metres away from M at point S, so that E, M, and S are in a straight line. Kanvar walks 6 metres parallel to the road in the opposite direction to Taj, to K. Now T, M, and K are in a straight line.

Find the width of the road.



Let the width of the road be x m.

$\triangle$s TEM and KSM are equiangular as:

- $T\widehat{E}M = K\widehat{S}M = 90°$
- $E\widehat{M}T = S\widehat{M}K$ {vertically opposite angles}

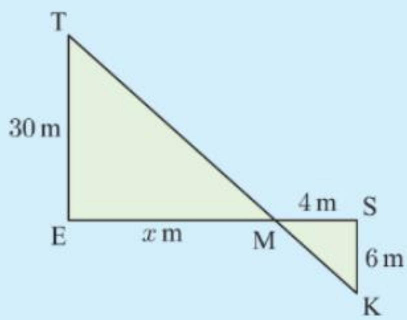
$\therefore$ $\triangle$s TEM and KSM are similar.

Corresponding sides must be in the same ratio.

$$\therefore \frac{EM}{SM} = \frac{TE}{KS}$$

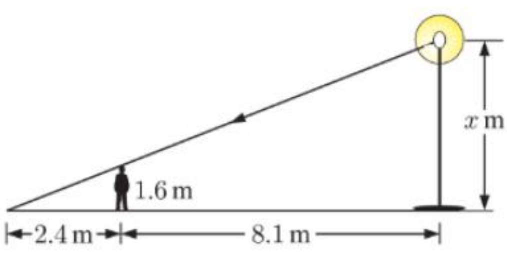
$$\therefore \frac{x}{4} = \frac{30}{6}$$

$$\therefore x = 5 \times 4 = 20$$



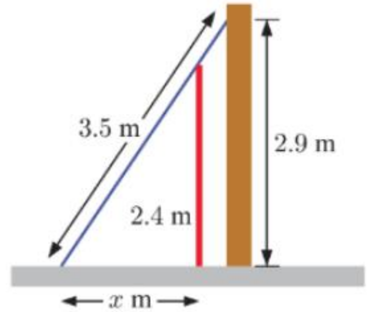
So, the road is 20 metres wide.

3

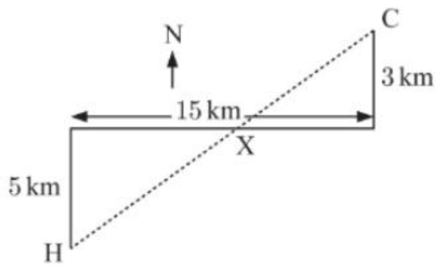


A boy who is 1.6 m tall stands 8.1 m from the base of an electric light pole. He casts a shadow 2.4 m long. How high above the ground is the light globe?

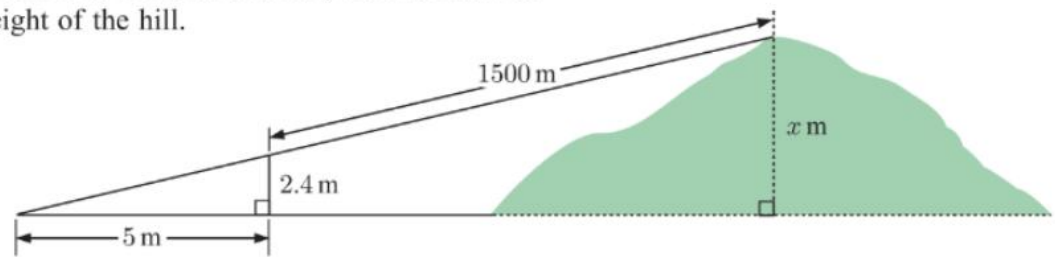
4 A 3.5 m ladder leans on a 2.4 m high fence. One end is on the ground and the other end touches a vertical wall 2.9 m from the ground.
How far is the bottom of the ladder from the fence?



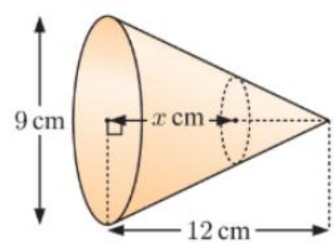
5 A hospital H receives a report about a serious road accident at C. An ambulance reaches the scene by travelling 5 km north, 15 km east, then 3 km north. A helicopter travels directly from H to C. Their paths intersect at X.
Find the distance from the hospital to X.



6 Two surveyors estimate the height of a nearby hill. One stands 5 m away from the other on horizontal ground holding a measuring stick vertically. The other surveyor finds a "line of sight" to the top of the hill, and observes that this line passes the vertical stick at a height of 2.4 m. They measure the distance from the stick to the top of the hill to be 1500 m using laser equipment.
Find, correct to the nearest metre, their estimate for the height of the hill.

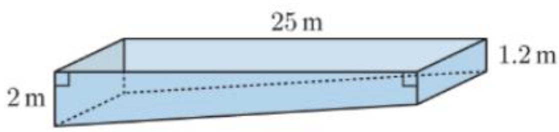


7 Mitchell pushes a coin of diameter 3 cm into a cone with diameter 9 cm and height 12 cm. How far into the cone can Mitchell push the coin before it gets stuck?

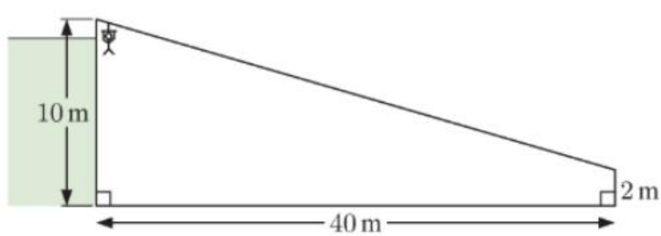


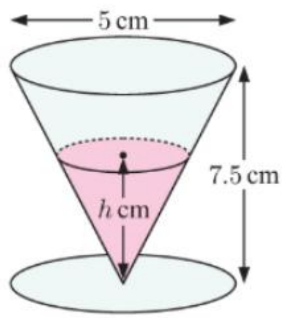
8 Answer the **Opening Problem** on page **180**.

9 A swimming pool is 1.2 m deep at one end, and 2 m deep at the other end. The pool is 25 m long. Isaac jumps into the pool 10 m from the shallow end. How deep is the pool at this point?



10 It is safe to let go of the flying fox shown alongside when you are 3 m above the ground. How far can you travel along the flying fox before letting go?

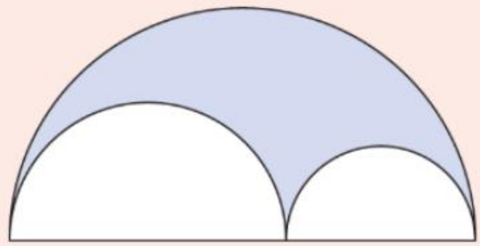


11 

The conical medicine glass alongside is filled with 20 mL of medicine.
To what height does the medicine level rise?

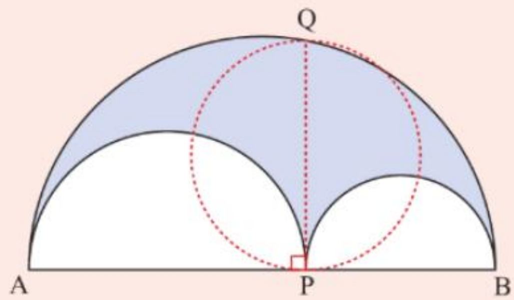
ACTIVITY — THE SHOEMAKER'S KNIFE

The shaded area alongside is called an **arbelos**, a Greek word meaning "shoemaker's knife". It is formed by drawing two smaller semi-circles inside a large semi-circle.



Archimedes showed that if a line is drawn from P perpendicular to [AB], meeting the arbelos again at Q, then the area of the arbelos is equal to the area of the circle with diameter [PQ].

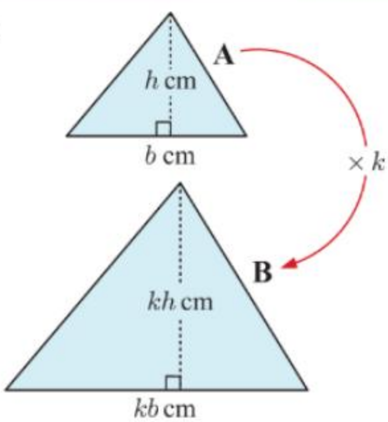
Can you use similarity to prove this fact?

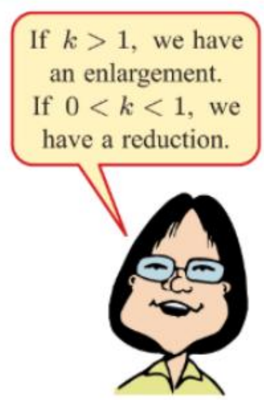


F AREAS AND VOLUMES

Triangle **A** has base b cm and height h cm.

Suppose it is enlarged with scale factor k to produce a *similar* triangle **B**.





Area of triangle **B**
$= \frac{1}{2}(kb)(kh)$
$= k^2(\frac{1}{2}bh)$
$= k^2 \times$ area of triangle **A**.

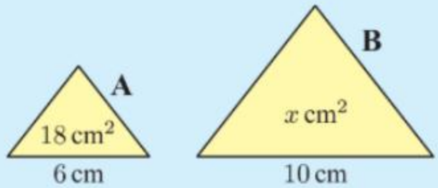
This suggests that:

If a figure is enlarged with scale factor k to produce a similar figure, then the new area $= k^2 \times$ the old area.

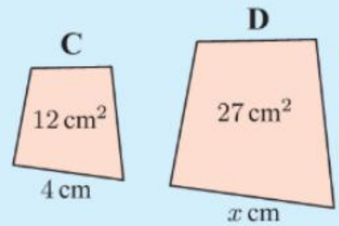
Example 7 **Self Tutor**

For each pair of similar figures, find x:

a



b



a **A** is enlarged with scale factor k to give **B**.

$\therefore \quad k = \frac{10}{6} = \frac{5}{3}$

Area of **B** $= k^2 \times$ area of **A**

$\therefore \quad x = (\frac{5}{3})^2 \times 18$

$\therefore \quad x = 50$

b **C** is enlarged with scale factor k to give **D**.

Area of **D** $= k^2 \times$ area of **C**

$\therefore \quad 27 = k^2 \times 12$

$\therefore \quad \frac{9}{4} = k^2$

$\therefore \quad k = \frac{3}{2} \quad \{\text{as } k > 0\}$

Since the sides are in the same ratio,

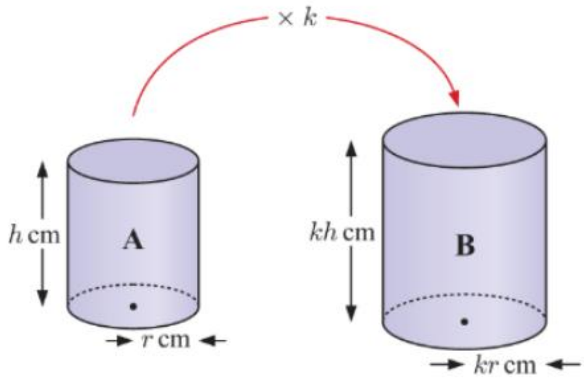
$x = \frac{3}{2} \times 4$

$\therefore \quad x = 6$

VOLUME

The cylinder **A** has radius r cm and height h cm. Suppose it is enlarged with scale factor k to produce a *similar* cylinder **B**.

The radius of cylinder **B** will be kr, and its height will be kh.



Volume of cylinder **B**

$= \pi(kr)^2(kh)$

$= \pi(k^2r^2)(kh)$

$= k^3(\pi r^2 h)$

$= k^3 \times$ volume of cylinder **A**

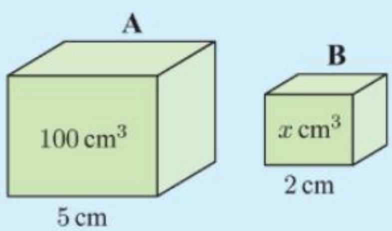
This suggests that:

If a 3-dimensional figure is enlarged with scale factor k to produce a similar figure, then the new volume $= k^3 \times$ the old volume.

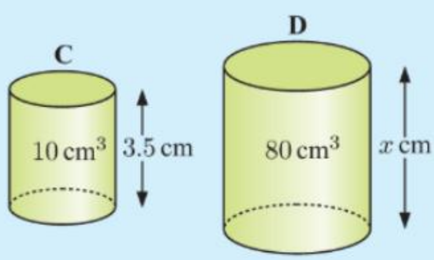
Example 8 **Self Tutor**

For each pair of similar figures, find x:

a



b



a **A** is reduced with scale factor k to give **B**.

$\therefore \quad k = \frac{2}{5}$

Volume of $\mathbf{B} = k^3 \times$ volume of **A**

$\therefore \quad x = (\frac{2}{5})^3 \times 100$

$\therefore \quad x = 6.4$

b **C** is enlarged with scale factor k to give **D**.

Volume of $\mathbf{D} = k^3 \times$ volume of **C**

$\therefore \quad 80 = k^3 \times 10$

$\therefore \quad 8 = k^3$

$\therefore \quad k = 2$

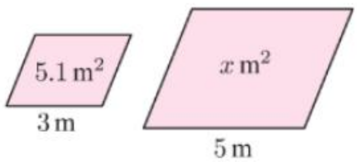
So, $\quad x = 2 \times 3.5$

$\therefore \quad x = 7$

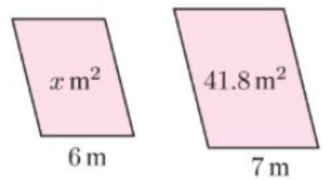
EXERCISE 10F

1 For each pair of similar figures, find x:

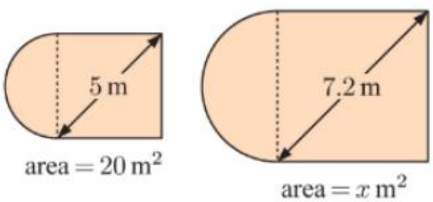
a



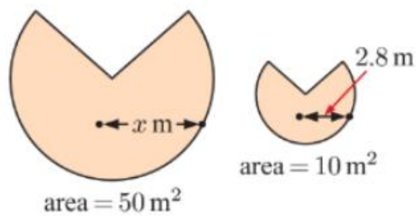
b



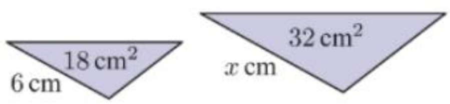
c



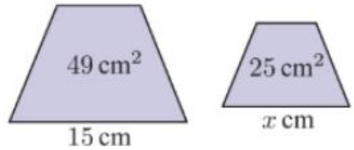
d



e

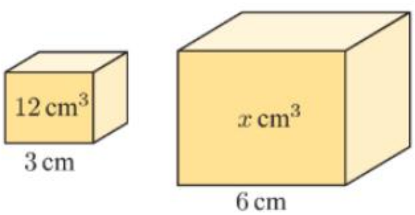


f

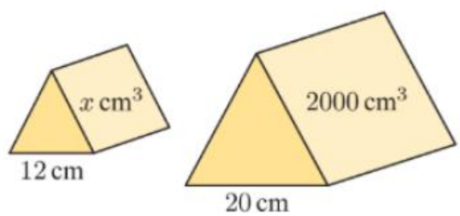


2 For each pair of similar figures, find x:

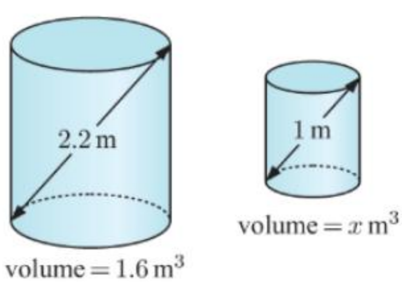
a



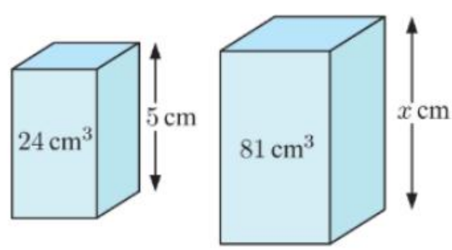
b



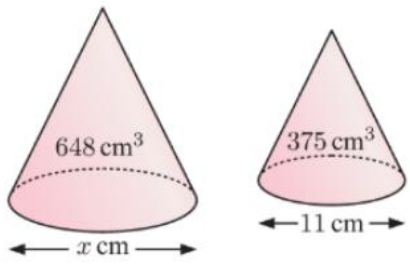
c



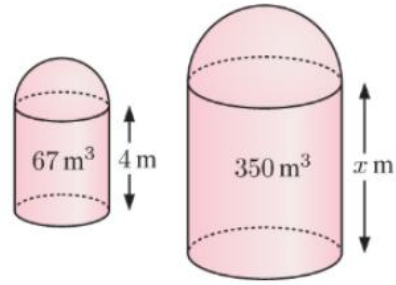
d



e

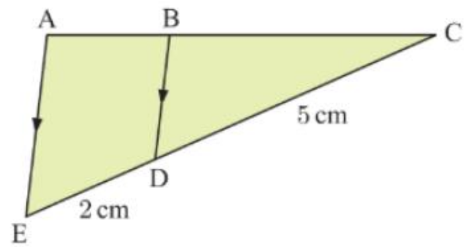


f



3 The area of $\triangle$BCD is 6.4 cm^2.
Find the area of:

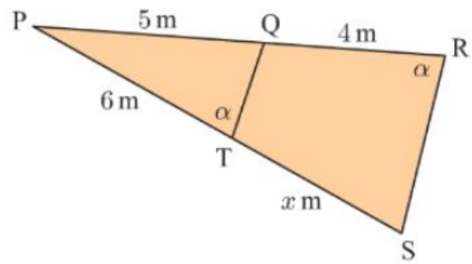
a $\triangle$ACE **b** quadrilateral ABDE.



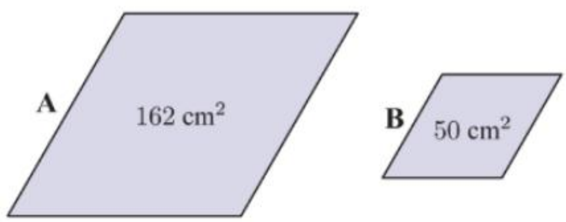
4 Quadrilateral QRST has area 22 m^2.

a Find the value of x.

b Find the area of $\triangle$PQT.



5 Rhombuses **A** and **B** are similar.
Given that each side of **A** is 13.5 cm long, find the perimeter of **B**.



6 The *density* of an object is its ratio of mass to volume. Objects made of the same material have the same density, so their mass is in proportion to their volume.

a What will happen to the mass of a sphere if its radius is:

i doubled **ii** increased by 20%?

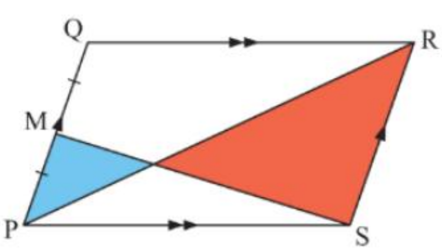
b What will happen to the mass of a cylinder if its radius and height are both:

i halved **ii** increased by 50%?

c Two similar cones made from the same material have surface areas 192 cm^2 and 75 cm^2. The volume of the larger cone is 200 cm^3. The mass of the smaller cone is 320 g.

i Find the volume of the smaller cone. **ii** Find the mass of the larger cone.

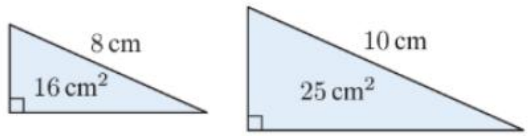
7



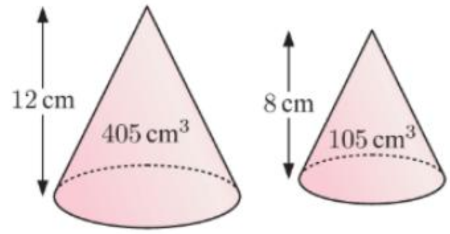
In parallelogram PQRS, M is the midpoint of [PQ].
Show that the red area is 4 times larger than the blue area.

8 Determine whether each of the pairs of figures below are similar.

a



b



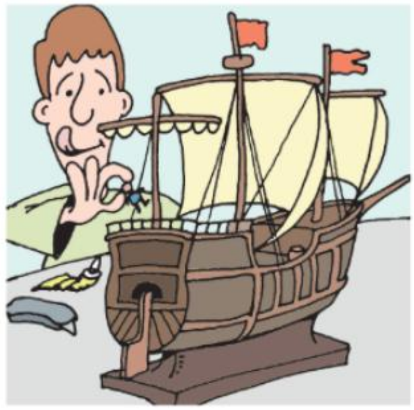
9 A scale model is made of a 300 year old sailing ship. The model is a 1 : 200 reduction of the original. Find:

a the height of the mast in the model if the original mast was 20 m high

b the area of a sail in the model if the original sail was 120 m^2

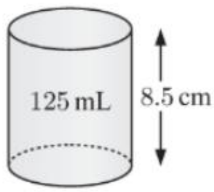
c the height and radius of a keg in the model if the original was 1.2 m high and 0.9 m in diameter

d the capacity of the water tank in the model if the capacity of the original was 10 000 litres.

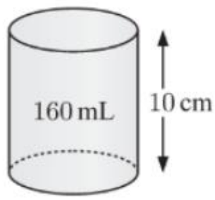


10 A glassware company manufactures cylindrical drinking glasses in six different sizes. Their heights and capacities are given below. Which two of the glasses are similar?

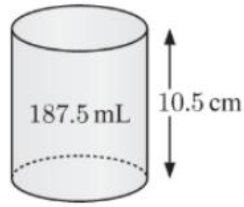
A



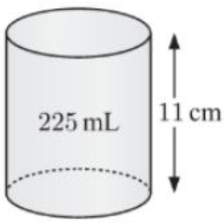
B



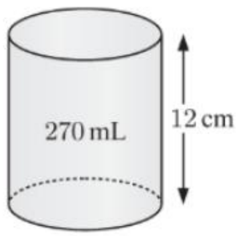
C



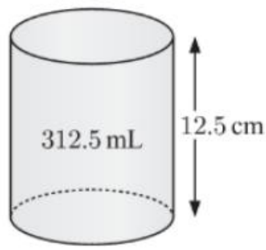
D



E



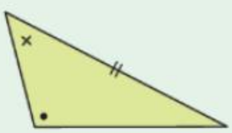
F



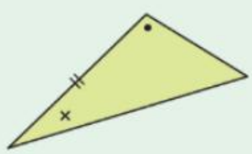
REVIEW SET 10A

1 In each set of three triangles, two are congruent. State which pair is congruent, giving a reason for your answer. The triangles are not drawn to scale, but contain correct information.

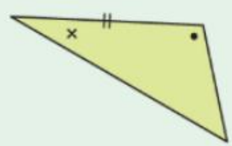
a **A**



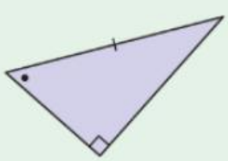
B



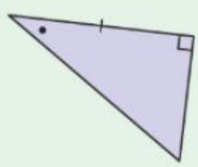
C



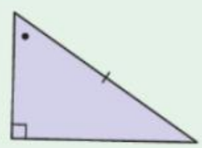
b **A**



B

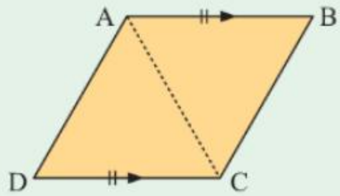


C



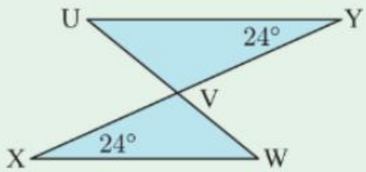
2 Consider the quadrilateral ABCD.

- **a** Show that triangles ABC and CDA are congruent.
- **b** Hence deduce that ABCD is a parallelogram.

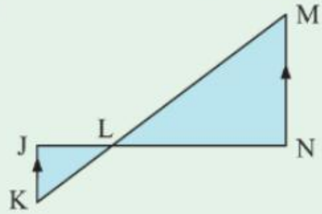


3 Show that the following figures possess similar triangles.

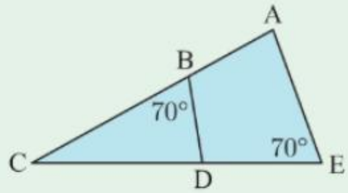
a



b

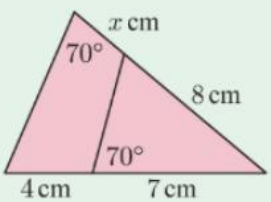


c

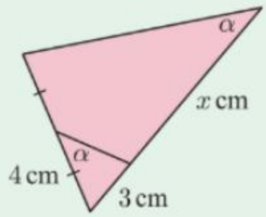


4 In each of the following figures, establish that a pair of triangles is similar, and hence find x:

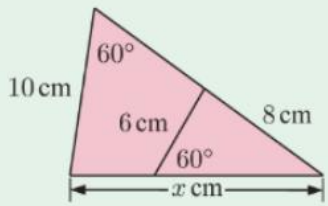
a



b

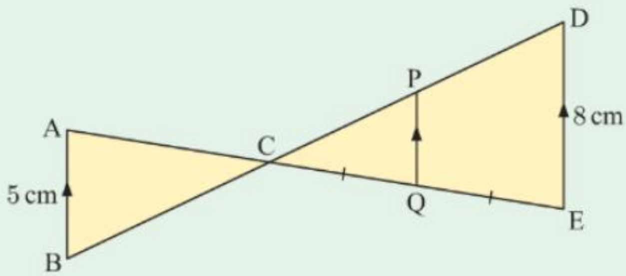


c

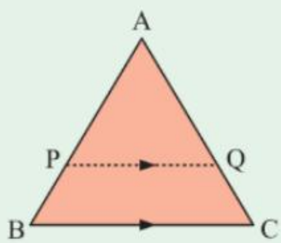


5 $\triangle$ABC has an area of 15 cm^2.

- **a** Find the area of $\triangle$CDE.
- **b** Find the area of PQED.

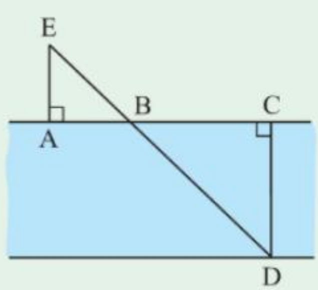


6

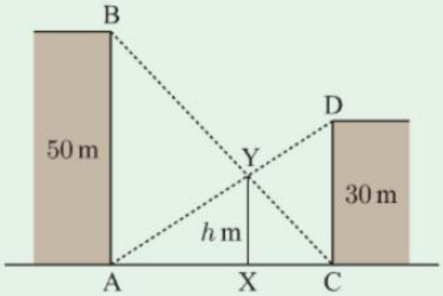


Triangle ABC is isosceles, with AB = AC.
[PQ] is parallel to [BC].
Show that CP = BQ.

7 A, B, and C are pegs on the bank of a canal which has parallel straight sides. C and D are directly opposite each other. AB = 30 m and BC = 140 m.
When I walk from A directly away from the bank, I reach a point E, 25 m from A, such that E, B, and D line up. How wide is the canal?



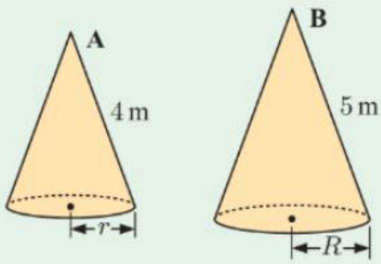
8 A sphere of lead with radius 10 cm is melted into 125 identical smaller spheres. Find the radius of each new sphere.

9 

The vertical walls of two buildings are 50 m and 30 m tall. A vertical flagpole [XY] stands between the buildings such that B, Y, and C are collinear, and A, Y, and D are collinear.

a Show that $\frac{h}{50} + \frac{h}{30} = 1$.

b Hence, find the height of the flagpole.

10 

The slant heights of two similar cones are 4 m and 5 m respectively.

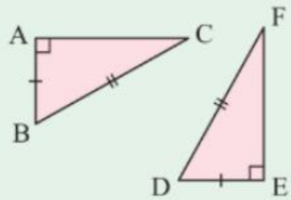
a Find the ratio $R : r$.

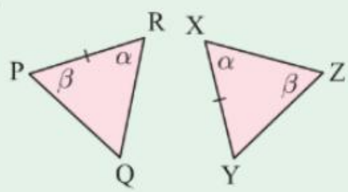
b Find the ratio of the surface areas for the curved part of each figure.

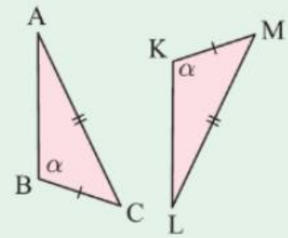
c Find the ratio of the volumes of the cones.

REVIEW SET 10B

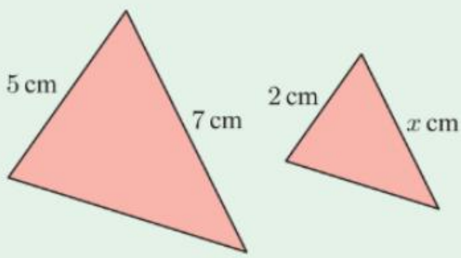
1 Are these triangles congruent? If so, state the congruence relationship and give a brief reason.

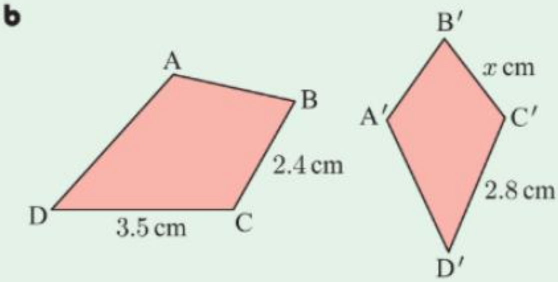
a 

b 

c 

2 For each of the following pairs of similar figures, find x:

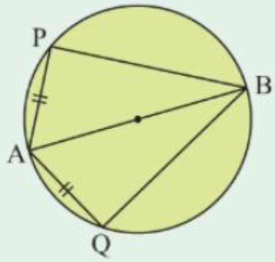
a 

b 

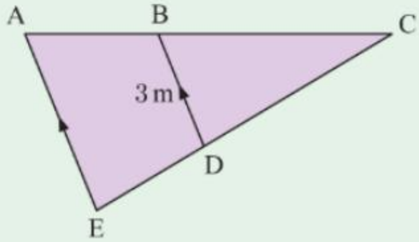
3 [AB] is a diameter of the circle.

a Show that the figure contains congruent triangles.

b What other facts can then be deduced about the figure?

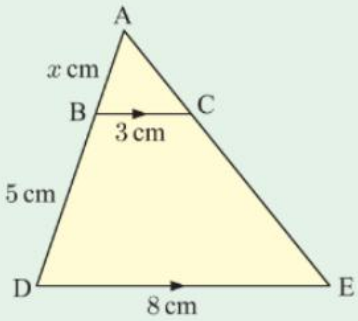


4 Triangle BCD has area 8 m^2, and quadrilateral ABDE has area 12 m^2. Find the length of [AE].

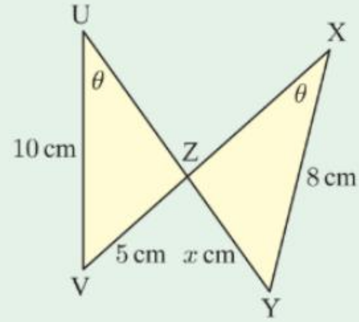


5 In the following figures, establish that a pair of triangles is similar, then find x:

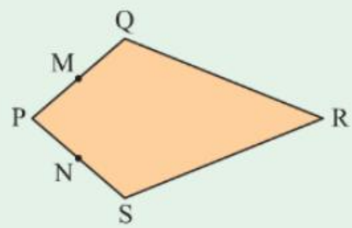
a



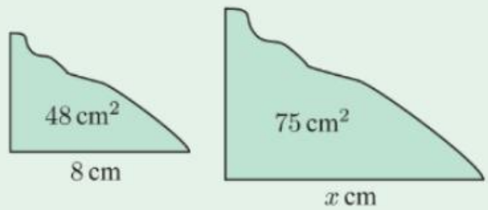
b



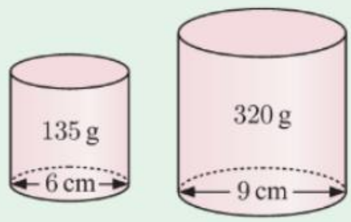
6 PQRS is a kite, with PQ = PS and QR = SR. M and N are the midpoints of [PQ] and [PS] respectively. Prove that triangle MNR is isosceles.



7 For the following similar figures, find x.



8 The cylinders below are made from the same material, so their densities are the same. Determine whether the cylinders are similar.

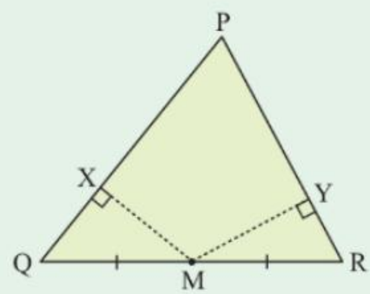


9 In $\triangle$PQR, M is the midpoint of [QR]. [MX] is drawn perpendicular to [PQ], and [MY] is drawn perpendicular to [PR].

Suppose these perpendiculars are equal in length.

a Prove that $\triangle$MQX is congruent to $\triangle$MRY.

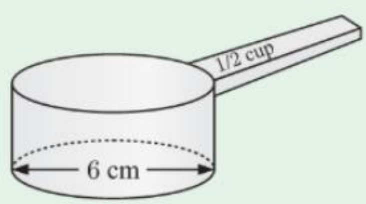
b Hence, prove that $\triangle$PQR is isosceles.



10 In a measuring cup set, the $\frac{1}{2}$ cup measure is 6 cm wide. The set also contains a 1 cup measure and a $\frac{1}{3}$ cup measure, both of which are similar in shape to the $\frac{1}{2}$ cup measure. Find the width of:

a the 1 cup measure

b the $\frac{1}{3}$ cup measure.



Chapter 11

Statistics

Contents:

OPENING PROBLEM

Roland owns two hotels, one in New York and one in Miami. He wants to find out whether there is a difference in the number of nights guests stay at the hotels.

He therefore inspects the last 40 reservations placed for each hotel, and records the number of nights the guests stayed.

New York

2	3	1	2	4	2	6	3	4	5
8	3	1	3	4	2	1	2	4	5
3	6	2	3	2	1	3	6	2	4
8	1	5	7	2	1	8	5	3	2

Miami

2	4	4	5	3	6	2	3	1	7
2	3	4	3	5	6	5	2	4	7
3	2	8	1	7	3	1	2	5	6
4	5	6	4	5	4	8	1	3	7

Things to think about:

a What is the best way to organise this data?

b How can the data be displayed?

c What is the most common length of stay at each hotel?

d How can Roland best measure:

  i the average length of stay for each hotel **ii** the spread of each data set?

e Can a reliable conclusion be drawn from the data? What factors could affect the reliability of the conclusion?

f How could Roland improve the accuracy of his investigation?

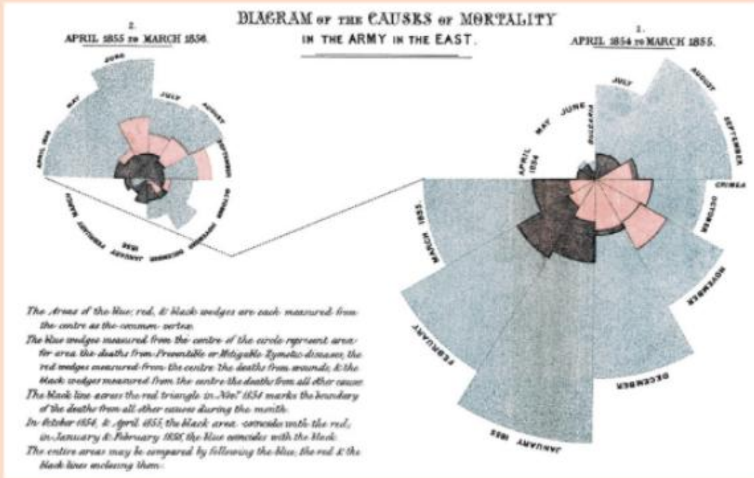
HISTORICAL NOTE

Florence Nightingale (1820 - 1910) was a British nurse in Turkey during the Crimean War. She worked in very difficult conditions, with overcrowding, poor sanitation, little food, and few basic supplies. Nightingale provided a statistical argument for the British government to provide improved facilities. By the time the war ended in 1856, the hospitals were well-run and efficient, with mortality rates no greater than civilian hospitals in England. Nightingale had earned an extraordinary reputation, along with the label "the lady with the lamp".

Florence Nightingale

After returning from the war, Nightingale compiled vast tables of statistics about how many soldiers died, where and why. Many of her findings shocked her. She discovered that in peacetime, soldiers in England died at twice the rate of civilians, even though they were strong young men. She recognised that the problems with the military health service extended far beyond the hospitals during war-time. The statistics also made Nightingale realise that poor sanitation had been the principal cause of most of the deaths in Turkey. Work conducted in March 1855 by the Turkish Sanitary Commission led to a dramatic decrease in deaths due to disease. However, Nightingale worried that Queen Victoria would not properly consider the data presented in the tables, so she found ways to present the data in charts, to persuade the Queen of the need for action.

Nightingale's best-known chart was a variation of a pie graph called the **polar area diagram**. It showed the number of deaths each month and their causes. Each month is represented as a twelfth of a circle. Months with more deaths were shown with longer wedges, and the area of each wedge represented the number of deaths in that month from wounds, disease, or other causes. Nightingale used blue wedges to represent disease, red wedges for wounds, and black wedges for other causes. Using this diagram, Nightingale illustrated the dramatic effect of the Sanitary Commission's work in 1855, as the wedges were far smaller in the following months.



Nightingale's work had a lasting effect. By the end of the century, Army mortality was lower than civilian mortality. She wrote, "To understand God's thoughts we must study statistics, for these are the measure of his purpose."

In **statistics** we collect and analyse data to give us an understanding of the world around us.

Most nations conduct a census at regular intervals to gain information about their populations. The United Nations gives assistance to developing countries to help them with census procedures, so that accurate and comparable worldwide statistics can be collected.

A DISCRETE DATA

A **discrete variable** takes exact number values, and is often a result of **counting**.

For example, in the **Opening Problem**, the *number of nights stayed* is a discrete variable. It can only take an exact value such as 1, 2, 3, 4, 5, ...

ORGANISATION AND DISPLAY OF DISCRETE DATA

A **tally and frequency table** can be used to organise numerical data.

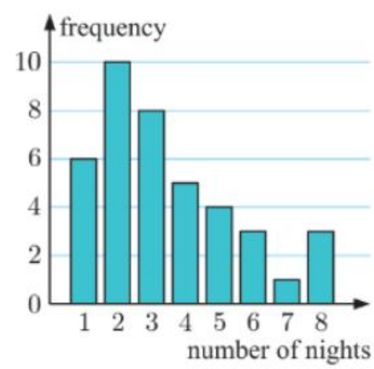
The data can then be displayed using a **column graph** or **dot plot**.

For the New York hotel data, we have:

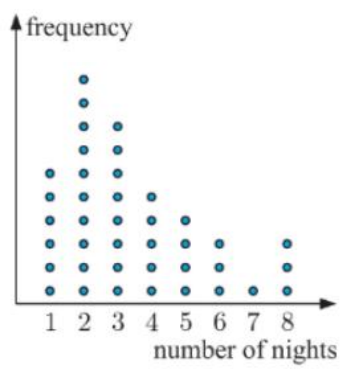
Tally and frequency table

Nights	*Tally*	*Frequency*
1	卌 |	6
2	卌 卌	10
3	卌 |||	8
4	卌	5
5	||||	4
6	|||	3
7	|	1
8	|||	3

Column graph



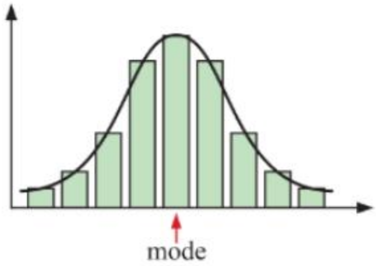
Dot plot



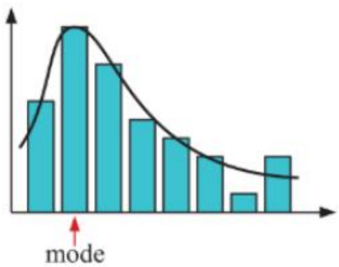
DESCRIBING THE DISTRIBUTION OF THE DATA SET

Many data sets show **symmetry** or **partial symmetry** about the **mode**, which is the most frequently occurring value.

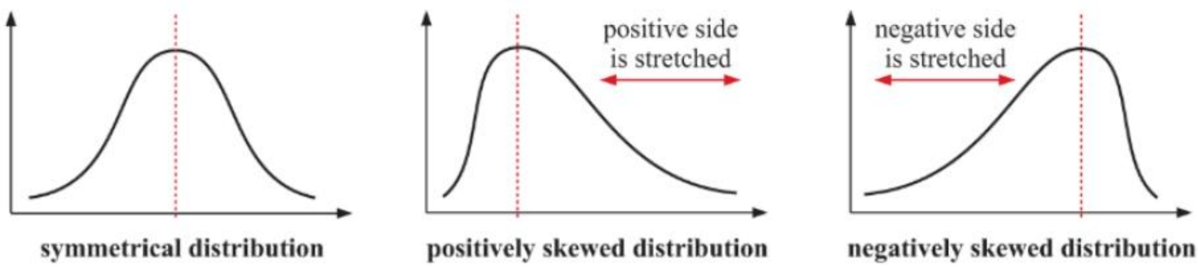
If we place a curve over the column graph alongside, we see that this curve shows symmetry. We say that we have a **symmetrical distribution**.



The distribution for the New York hotel data is shown alongside. It is said to be **positively skewed** because, by comparison with the symmetrical distribution, it has been 'stretched' on the right or positive side of the mode.



So, we have:



OUTLIERS

Outliers are data values that are either much larger or much smaller than the general body of data. Outliers appear separated from the body of data on a frequency graph.

For example, in the data set 3, 1, 7, 6, 8, 18, 2, 6, 7, 7, the data value 18 is an outlier. If outliers are genuine pieces of data, then they should be included in an analysis of the whole data set. However, if outliers occur due to human recording error, they should not be included when the data is analysed.

GROUPED DISCRETE DATA

In situations where there are lots of different numerical values recorded, it may not be practical to use an ordinary tally and frequency table, or to display the data using a dot plot or column graph. Instead, we group the data into **class intervals**.

For example, a local hardware store is studying the number of people visiting the store at lunch time. Over 30 consecutive weekdays they recorded the data:

37, 30, 17, 13, 46, 23, 40, 28, 38, 24, 23, 22, 18, 29, 16,
35, 24, 18, 24, 44, 32, 54, 31, 39, 32, 38, 41, 38, 24, 32.

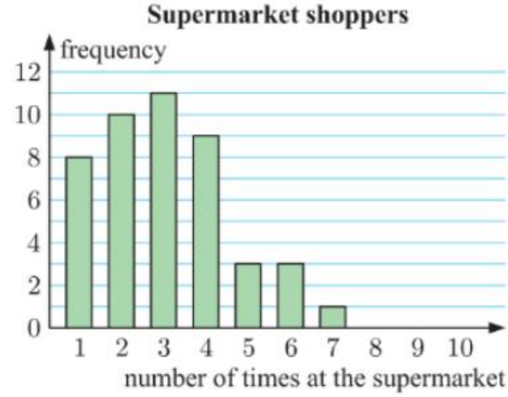
In this case, we group the data into class intervals of length 10. The tally and frequency table is shown alongside.

We can now use this table to draw a column graph for the data. However, we must remember that the individual data values are no longer seen.

Number of people	*Tally*	*Frequency*
10 to 19	𝍸	5
20 to 29	𝍸 ||||	9
30 to 39	𝍸 𝍸 |	11
40 to 49	||||	4
50 to 59	|	1
	Total	30

EXERCISE 11A

1 A randomly selected sample of shoppers was asked, "How many times did you shop at a supermarket in the past week?" A column graph was constructed for the results.



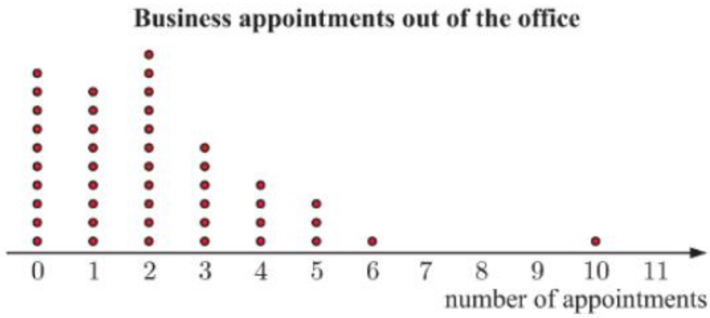
a How many shoppers gave data in the survey?

b How many of the shoppers shopped once or twice?

c What percentage of the shoppers shopped more than four times?

d Describe the distribution of the data.

2 Employees of a company were asked how many times they left the office on business appointments during one week. The following dot plot was constructed from the data:



a How many employees did not leave the office?

b What percentage of the employees left the office more than 5 times?

c Describe the distribution of the data.

d How would you describe the data value '10'?

3 20 students were asked "How many TV sets do you have in your household?" The following data was collected:

2 1 0 3 1 2 1 3 4 0 0 2 2 0 1 1 0 1 0 1

a Construct a dot plot to display the data.

b How would you describe the distribution of the data? Are there any outliers?

c How many households had no TV sets?

d What percentage of the households had three or more TV sets?

4 The number of toothpicks in a box is stated as 50, but the actual number of toothpicks has been found to vary. To investigate this, the number of toothpicks in a box was counted for a sample of 60 boxes. The results were:

50 52 51 50 50 51 52 49 50 48 51 50 47 50 52 48 50 49 51 50
49 50 52 51 50 50 52 50 53 48 50 51 50 50 49 48 51 49 52 50
49 49 50 52 50 51 49 52 52 50 49 50 49 51 50 50 51 50 53 48

a Use a tally and frequency table to organise this data.

b Display the data using a column graph.

c Describe the distribution of the data.

d What percentage of the boxes contained exactly 50 toothpicks?

5 Consider the data for the Miami hotel in the **Opening Problem** on page **202**.

a Organise the data in a tally and frequency table.

b Draw a column graph of the data.

c Are there any outliers?

d Describe the distribution of the data.

e Compare your column graph with that for the New York hotel on page **203**. In which hotel do guests generally stay longer?

6 The data below are the test scores (out of 100) for a Science test for 50 students.

92	29	78	67	68	58	80	89	92
69	66	56	88	81	70	73	63	55
67	64	62	74	56	75	90	56	47
59	64	89	39	51	87	89	76	59
72	80	95	68	80	64	53	43	61
71	38	44	88	62				

a Construct a tally and frequency table for this data using class intervals 20 - 29, 30 - 39, ..., 90 - 100.

b What percentage of the students scored 80 or more for the test?

c What percentage of students scored less than 50 for the test?

d Copy and complete the following:
More students had a test score in the interval ...... than in any other interval.

e Describe the distribution of the data.

7 A test score out of 60 marks is recorded for a group of 45 students:

34	37	44	51	53	39	33	58	40	42	43	43	47	37	35
41	43	48	50	55	44	44	52	54	59	39	31	29	44	57
45	34	29	27	18	49	41	42	37	42	43	43	45	34	51

a Organise the data in a tally and frequency table, using the test score ranges 15 - 19, 20 - 24, and so on.

b Draw a column graph for the data.

c Describe the distribution of the data.

d An A is awarded to students who scored 50 or more in the test. What percentage of students scored an A?

B CONTINUOUS DATA

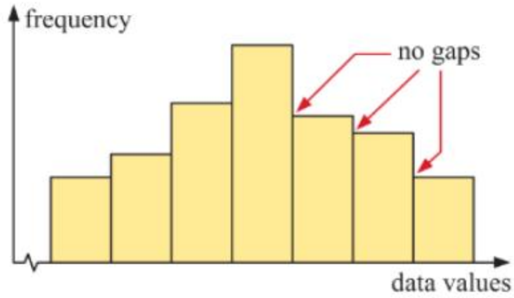
A **continuous variable** takes values within a certain continuous range, and is usually a result of **measuring**.

When data is recorded for a continuous variable, there will be many different values. The data is therefore organised using **class intervals**. A **frequency histogram** is used to display the data.

A histogram is similar to a column graph, but because the data is continuous, the columns are joined together.

An example is given alongside.

The **modal class** is the class of values that appears most often. On a histogram, the modal class has the highest column.



Example 1 **Self Tutor**

The weights of parcels sent on a given day from a post office were, in kilograms:

4.07	1.63	3.52	2.91	3.24	3.47	5.29	4.63
3.11	2.85	3.76	4.92	3.44	1.39	2.58	2.22

a Organise the data into class intervals.

b Draw a frequency histogram to display the data.

c Find the modal class.

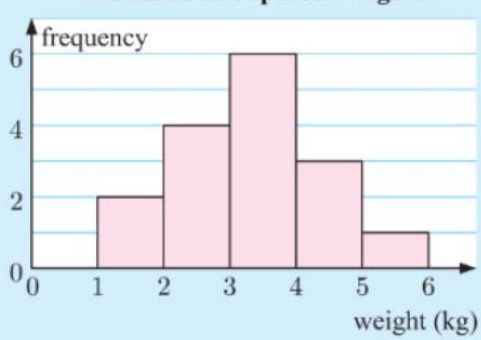
d Describe the distribution of the data.

e Over the next month, 564 parcels are sent from the post office. Estimate the number which weigh more than 4 kg.

a The lowest weight recorded was 1.39 kg and the highest was 5.29 kg, so we will use class intervals of 1 kg. We suppose w is the weight of a parcel.

Weight w (kg)	*Frequency*
$1 \leqslant w < 2$	2
$2 \leqslant w < 3$	4
$3 \leqslant w < 4$	6
$4 \leqslant w < 5$	3
$5 \leqslant w < 6$	1

b **Distribution of parcel weights**



c The modal class is $3 \leqslant w < 4$ kg.

d The distribution is approximately symmetrical.

e Of the 16 parcels sent on the one day, $\frac{4}{16} = \frac{1}{4}$ of them weighed more than 4 kg.

$\therefore$ for the next month we expect $\frac{1}{4} \times 564 = 141$ parcels to weigh more than 4 kg.

EXERCISE 11B

1 A frequency table for the weights w of players in a volleyball squad is given alongside.

Weight w (kg)	*Frequency*
$75 \leqslant w < 80$	2
$80 \leqslant w < 85$	5
$85 \leqslant w < 90$	8
$90 \leqslant w < 95$	7
$95 \leqslant w < 100$	5
$100 \leqslant w < 105$	1

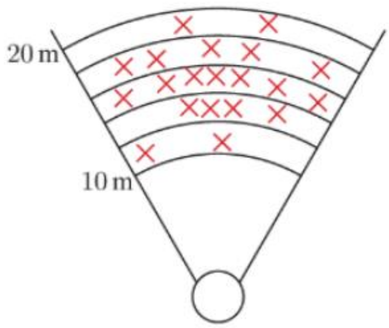
- **a** Explain why *weight* is a continuous variable.
- **b** Construct a frequency histogram to display the data.
- **c** Find and interpret the modal class.
- **d** Describe the distribution of the data.

2 A plant inspector takes a random sample of seedlings from a nursery, and measures their height h in millimetres. The results are shown in the table alongside.

Height h (mm)	*Frequency*
$20 \leqslant h < 40$	4
$40 \leqslant h < 60$	17
$60 \leqslant h < 80$	15
$80 \leqslant h < 100$	8
$100 \leqslant h < 120$	2
$120 \leqslant h < 140$	4

- **a** How many of the seedlings are 100 mm or more?
- **b** What percentage of the seedlings are between 60 and 80 mm?
- **c** Represent the data on a frequency histogram.
- **d** Find the modal class.
- **e** Describe the distribution of the data.
- **f** There are 857 seedlings in the nursery. Estimate the number of seedlings which measure:
  - **i** less than 100 mm
  - **ii** between 40 and 100 mm.

3



During a training session, Daniel performed 20 throws of the shot put. The results of the throws are shown alongside.

- **a** Organise the data into class intervals.
- **b** Draw a histogram to display the data.
- **c** Find the modal class.
- **d** Describe the distribution of the data.

4 A group of athletics students obtained the following times, in seconds, for running 200 metres:

26.57	25.22	27.09	26.44	24.13	27.83	25.72	26.40
23.12	27.44	24.76	25.09	28.70	26.13	23.94	27.23
26.35	28.91	26.30	27.02	24.19	25.27	27.45	26.45
27.40	27.22	25.88	23.50	26.49	27.19	28.37	25.17
28.08	26.80	28.14	26.82	27.66	25.41	24.89	27.92

- **a** Organise the data into class intervals.
- **b** What percentage of the students obtained a time faster than 25 seconds?
- **c** Draw a histogram to display the data.
- **d** Find the modal class.
- **e** Describe the distribution of the data.

C MEASURING THE CENTRE

We can get a better understanding of a data set if we can locate the **middle** or **centre** of the data, and get an indication of its **spread**. Knowing one of these without the other is often of little use.

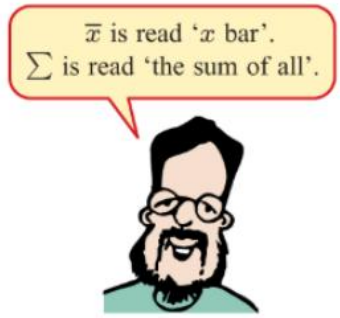
There are three statistics that are used to measure the **centre** of a data set. These are:
the **mean**, the **median**, and the **mode**.

THE MEAN

The **mean** $\overline{x}$ of a data set is the statistical name for its arithmetic average.

$$\textbf{mean} = \frac{\textbf{the sum of the data values}}{\textbf{the number of data values}}$$

or $\overline{x} = \dfrac{\sum x}{n}$ where $\sum x$ is the sum of the data.

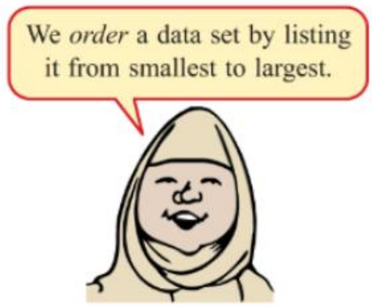


The mean is not necessarily a member of the data set.

THE MEDIAN

The **median** is the *middle value* of an ordered data set.

The median splits an ordered data set in halves. Half of the data are less than or equal to the median, and half are greater than or equal to it.



If there are n data, the median is the $\left(\dfrac{n+1}{2}\right)$th data value.

For example:

If $n = 13$, $\dfrac{n+1}{2} = 7$ so the median is the 7th ordered data value.

If $n = 14$, $\dfrac{n+1}{2} = 7.5$ so the median is the average of 7th and 8th ordered data values.

If there is an even number of data values, the median is not necessarily a member of the data set.

THE MODE

The **mode** is the most frequently occurring value in the data set.

If there are two values which occur most frequently, we say the data set is **bimodal**.

If a data set has more than two modes, we do not use the mode as a measure of the centre of the data set.

Example 2 **Self Tutor**

The number of small aeroplanes flying into a remote airstrip over a 15-day period is given below:

$$5 \quad 7 \quad 0 \quad 3 \quad 4 \quad 6 \quad 4 \quad 0 \quad 5 \quad 3 \quad 6 \quad 9 \quad 4 \quad 2 \quad 8$$

For this data set, find the: **a** mean **b** median **c** mode.

a $\text{mean} = \dfrac{\overbrace{5+7+0+3+4+6+4+0+5+3+6+9+4+2+8}^{\sum x}}{\underbrace{15}_{n}}$

$= \dfrac{66}{15}$

$= 4.4$ aeroplanes

b The ordered data set is: $\cancel{0 \;\; 0 \;\; 2 \;\; 3 \;\; 3 \;\; 4 \;\; 4} \;\; \boxed{4} \;\; \cancel{5 \;\; 5 \;\; 6 \;\; 6 \;\; 7 \;\; 8 \;\; 9}$

Since $n = 15$, $\dfrac{n+1}{2} = 8$

$\therefore$ the median is the 8th data value

$\therefore$ the median $= 4$ aeroplanes.

c 4 is the score which occurs the most often

$\therefore$ the mode $= 4$ aeroplanes.

Equal or approximately equal values of the mean, mode, and median *may* indicate a *symmetrical distribution* of data. However, we should always check using a graph before calling a data set symmetric.

EXERCISE 11C.1

1 For each of the following data sets, find the:

i mean **ii** median **iii** mode.

GRAPHICS CALCULATOR INSTRUCTIONS

a 12, 17, 20, 24, 30, 30, 42

b 8, 8, 8, 10, 11, 11, 12, 12, 16, 20, 20, 24

c 7.9, 8.5, 9.1, 9.2, 9.9, 10.0, 11.1, 11.2, 11.2, 12.6, 12.9

d 427, 423, 415, 405, 445, 433, 442, 415, 435, 448, 429, 427, 403, 430, 446, 440, 425, 424, 419, 428, 441

2 Consider the following two data sets:

Data set A: 5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 12

Data set B: 5, 6, 6, 7, 7, 7, 8, 8, 9, 10, 20

a Find the mean for both *Data set A* and *Data set B*.

b Find the median for both *Data set A* and *Data set B*.

c Explain why the mean of *Data set A* is less than the mean of *Data set B*.

d Explain why the median of *Data set A* is the same as the median of *Data set B*.

3 The selling price of nine houses are:

\$158 000, \$290 000, \$290 000, \$1.1 million, \$900 000,
\$395 000, \$925 000, \$420 000, \$760 000

a Find the mean, median, and modal selling prices.

b Explain why the mode is an unsatisfactory measure of the middle in this case.

c Is the median a satisfactory measure of the middle of this data set?

4 The following raw data is the daily rainfall (to the nearest millimetre) for the month of February 2014 in a city in China:

0, 4, 1, 0, 0, 0, 2, 9, 3, 0, 0, 0, 8, 27, 5, 0, 0, 0, 0, 8, 1, 3, 0, 0, 15, 1, 0, 0

a Find the mean, median, and mode for the data.

b Explain why the **i** median **ii** mode is not a suitable measure of centre for this data.

c Identify any outliers in the data set. Do you think the outliers are errors or genuine data? Should they be removed before finding the measures of centre? Explain your answer.

5 A basketball team scored 38, 52, 43, 54, 41, and 36 points in their first six matches.

a Find the mean number of points scored for the first six matches.

b What score does the team need to shoot in their next match to maintain the same mean score?

c The team scores only 20 points in their seventh match. Find the mean number of points scored for the seven matches.

d The team scores 42 points in their eighth and final match.

i Will their previous mean score increase or decrease?

ii Find the mean score for all eight matches.

Example 3 Self Tutor

Each student in a class of 20 is assigned a number between 1 and 10 to indicate his or her fitness. The results are: 7, 9, 8, 8, 10, 9, 8, 7, 8, 6, 9, 5, 6, 8, 9, 7, 7, 8, 10, 8

Group the data in a table, and hence calculate the:

a mean **b** median **c** mode.

a

Score	*Tally*	*Number of students*	*Product*
5	|	1	5
6	||	2	12
7	||||	4	28
8	~~||||~~ ||	7	56
9	||||	4	36
10	||	2	20
Total		20	157

The mean score

$= \dfrac{\text{total of scores}}{\text{number of scores}}$

$= \dfrac{157}{20}$

$= 7.85$

b There are 20 scores, so the median is the average of the 10th and 11th ordered scores.

Score	*Number of students*	
5	1	1st student
6	2	2nd and 3rd student
7	4	4th, 5th, 6th, and 7th student
8	7	8th, 9th, **10th**, **11th**, 12th, 13th, 14th student
9	4	
10	2	

GRAPHICS CALCULATOR INSTRUCTIONS

STATISTICS PACKAGE

The 10th and 11th students both scored 8, so the median $= 8$.

c Looking down the 'number of students' column, the highest frequency is 7. This corresponds to a score of 8, so the mode $= 8$.

6 3 coins were tossed simultaneously 40 times, and the number of heads for each toss was recorded.

Calculate the:

a mode **b** median

c mean.

Number of heads	*Frequency*
0	6
1	16
2	14
3	4
Total	40

7 The frequency column graph gives the value of donations for an overseas aid organisation, collected in a particular street.

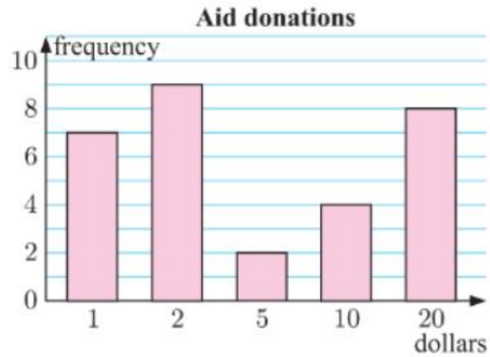
a Construct a frequency table from the graph.

b Determine the total number of donations.

c Find the:

i mean **ii** median **iii** mode.

d Which of the measures of centre can be found easily from the graph only?



8 Hui breeds ducks. The number of ducklings surviving for each pair after one month is recorded in the table.

a Calculate the:

i mean **ii** mode **iii** median.

b Is the data skewed?

c How does the skewness of the data affect the measures of the middle of the distribution?

Number of survivors	*Frequency*
0	1
1	2
2	5
3	9
4	20
5	30
6	9
Total	76

9 Consider the **Opening Problem** on page **202**.

a For each data set, find the: **i** mean **ii** mode **iii** median.

b At which hotel do guests generally stay longer?

Example 4 **Self Tutor**

Linda has taken four Mathematics tests so far this year. Each test has been out of 20 marks, and her average mark has been 15.

What mark does Linda need in the 5th test to raise her average to 16?

Average mark $= \dfrac{\text{sum of marks}}{4} = 15$

$\therefore$ sum of marks $= 15 \times 4 = 60$

Let Linda's mark for the 5th test be x.

$\therefore$ we require $\dfrac{60+x}{5} = 16$

$\therefore \ 60 + x = 80$

$\therefore \ x = 20$

So, Linda needs a mark of 20 in the 5th test.

10 Jackie has played 8 games of netball this season, scoring an average of 17 goals per game. How many goals does she need to score in the next game to increase her average to 18?

11 A sample of 12 measurements has a mean of 8.5, and a sample of 20 measurements has a mean of 7.5. Find the mean of all 32 measurements.

12 

On Saturday, Derek picked pears from 32 trees. He picked an average of 17 pears per tree. On Sunday he picked some more pears, averaging 12 pears per tree. Over the whole weekend he picked an average of 14 pears per tree. How many trees did Derek pick from on Sunday?

13 Find x if:

a 7, 15, 6, 10, 4, and x have a mean of 9

b 10, x, 15, 20, x, x, 17, 7, and 15 have a mean of 12.

14 The mean, median, and mode of seven numbers are 8, 7, and 6 respectively. Two of the numbers are 8 and 10. The smallest of the numbers is 4. Find the largest of the numbers.

15 Consider four numbers a, b, c, and d. The mean of a and b is 7, the mean of b and c is 10, and the mean of c and d is 9. Find the mean of a and d.

DISCUSSION

Develop at least two examples to show how the measures of centre are affected by outliers.

Which of the measures of centre is most affected by the presence of an outlier?

Which of the measures of centre are unaffected by the presence of an outlier?

ESTIMATING THE MEAN OF GROUPED DATA

When data is presented in **class intervals**, the actual data values are not known. This makes it impossible to calculate the exact mean of the data set.

To *estimate* the mean of the data, we use the **midpoint** of an interval to represent all of the scores within the interval.

For example, if we have the interval $50 \text{ km} \leqslant d < 100 \text{ km}$, we *estimate* that all of the data in that interval corresponds to the distance 75 km.

Example 5 Self Tutor

The transport department collected data regarding the distance each of its trams travelled in one day. This is shown in the table.

Estimate the mean distance travelled by the trams.

Distance d (km)	*Frequency*
$50 \leqslant d < 100$	10
$100 \leqslant d < 150$	15
$150 \leqslant d < 200$	16
$200 \leqslant d < 250$	9

Distance d (km)	*Frequency*	*Interval midpoint*	*Product*
$50 \leqslant d < 100$	10	75	750
$100 \leqslant d < 150$	15	125	1875
$150 \leqslant d < 200$	16	175	2800
$200 \leqslant d < 250$	9	225	2025
Total	50		7450

$\therefore$ mean

$= \dfrac{\text{sum of data values}}{\text{the number of data values}}$

$\approx \dfrac{7450}{50}$

≈ 149 km

EXERCISE 11C.2

1 The daily maximum temperatures for Manila over a one year period are given below.

Maximum temperature t (°C)	*Frequency*
$24 \leqslant t < 26$	1
$26 \leqslant t < 28$	8
$28 \leqslant t < 30$	32
$30 \leqslant t < 32$	107
$32 \leqslant t < 34$	174
$34 \leqslant t < 36$	43

Estimate the mean maximum temperature.

2 Nick served a tennis ball 200 times. The speeds of the serves are summarised in the table alongside.

Speed s (km h^{-1})	*Frequency*
$150 \leqslant s < 155$	18
$155 \leqslant s < 160$	28
$160 \leqslant s < 165$	35
$165 \leqslant s < 170$	43
$170 \leqslant s < 175$	41
$175 \leqslant s < 180$	35

a Find the modal class of the data.

b If possible, find the:

i number of serves faster than 170 km h^{-1}

ii number of serves slower than 162 km h^{-1}

iii percentage of serves between 155 km h^{-1} and 175 km h^{-1}.

c Estimate the mean speed of the serves.

3 The table alongside shows the number of runs scored by Clive during his team's cricket season.

Number of runs	*Frequency*
0 - 9	3
10 - 19	4
20 - 29	9
30 - 39	5
40 - 49	2

a How many times did Clive bat?

b How many times did Clive score at least 20 runs?

c Estimate the mean number of runs scored by Clive.

4 The male and female Year 10 students of a school were asked how long they slept for last night. The responses are shown below.

Male

Time slept, t (hours)	*Frequency*
$4 \leqslant t < 5$	6
$5 \leqslant t < 6$	10
$6 \leqslant t < 7$	13
$7 \leqslant t < 8$	9
$8 \leqslant t < 9$	7

Female

Time slept, t (hours)	*Frequency*
$4 \leqslant t < 5$	4
$5 \leqslant t < 6$	8
$6 \leqslant t < 7$	10
$7 \leqslant t < 8$	11
$8 \leqslant t < 9$	8

a Draw a histogram to display each data set.

b Describe the distribution of each data set.

c Estimate the mean of each data set.

d Which group do you suspect slept on average for longer? Comment on the reliability of your answer.

5 The amounts of petrol bought in one hour by customers at a service station are shown below, in litres:

41.59	33.09	37.21	58.85	47.20	26.01	31.12	41.11	56.21	43.59
31.77	44.56	23.15	46.67	44.43	58.55	40.09	37.51	43.72	27.56
28.90	36.82	47.19	59.23	39.08	47.81	29.95	55.91	34.11	44.75
46.12	27.09	57.85	33.13	51.05	34.80	56.14	47.33	51.91	57.76
37.10	52.39	48.52	41.08	22.09	49.91	38.10	58.77	25.87	39.21

Suppose the amount of petrol is p litres.

a Find the mean $\overline{p}$ of the data.

b **i** Organise the data into the class intervals $20 \leqslant p < 30$, $30 \leqslant p < 40$, $40 \leqslant p < 50$, and $50 \leqslant p < 60$.

ii Use these intervals to estimate the mean of the data.

c **i** Organise the data into the class intervals $20 \leqslant p < 25$, $25 \leqslant p < 30$, $30 \leqslant p < 35$, ..., $55 \leqslant p < 60$.

ii Use these intervals to estimate the mean of the data.

d Compare the estimates of the mean in **b** and **c** with the actual mean found in **a**.

e In general, would you expect the use of larger or smaller intervals to produce a more accurate estimate of the mean? Explain your answer.

D CUMULATIVE DATA

It is sometimes useful to know the number or proportion of scores that lie above or below a particular value. To determine this we construct a **cumulative frequency table**, and draw a **cumulative frequency graph**.

The **cumulative frequency** gives a *running total* of the number of data less than a particular value.

Example 6 **Self Tutor**

The table summarises the weights of 60 male rugby players.

a Construct a cumulative frequency table for the data.

b Represent the data on a cumulative frequency graph.

c Use your graph to estimate the:

i median weight

ii number of men weighing less than 83 kg

iii number of men weighing more than 102 kg.

Weight w (kg)	*Frequency*
$75 \leqslant w < 80$	3
$80 \leqslant w < 85$	11
$85 \leqslant w < 90$	12
$90 \leqslant w < 95$	14
$95 \leqslant w < 100$	13
$100 \leqslant w < 105$	4
$105 \leqslant w < 110$	2
$110 \leqslant w < 115$	1

a

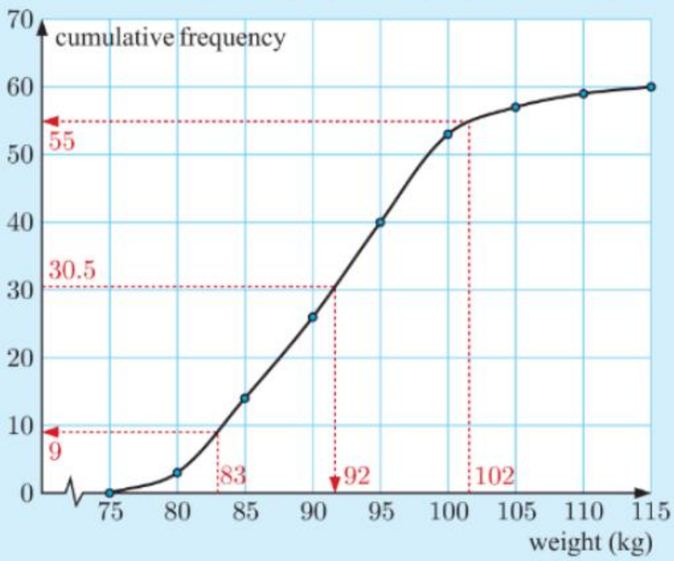
Weight w (kg)	*Frequency*	*Cumulative frequency*
$75 \leqslant w < 80$	3	3
$80 \leqslant w < 85$	11	14
$85 \leqslant w < 90$	12	26
$90 \leqslant w < 95$	14	40
$95 \leqslant w < 100$	13	53
$100 \leqslant w < 105$	4	57
$105 \leqslant w < 110$	2	59
$110 \leqslant w < 115$	1	60

this is $3 + 11$

there are $3 + 11 + 12 = 26$ players who weigh less than 90 kg

b

Cumulative frequency graph of rugby player's weights



c **i** The median is the average of the 30th and 31st weights. Reading from the graph, the median $\approx$ 92 kg.

ii There are 9 men who weigh less than 83 kg.

iii There are $60 - 55 = 5$ men who weigh more than 102 kg.

STATISTICS PACKAGE

EXERCISE 11D

1 The cumulative frequency graph alongside shows the lengths of trout caught during a fishing competition.

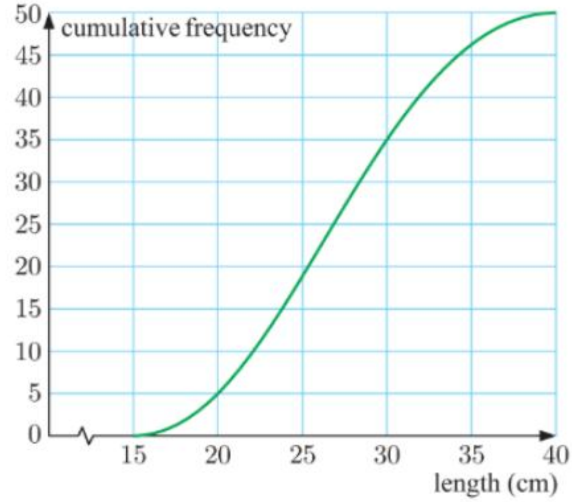
a How many trout were caught during the competition?

b How many of the trout were:

i shorter than 20 cm

ii longer than 30 cm?

c Estimate the median length of trout caught.



2 In an examination, a group of students achieved the percentages shown in the table.

a Construct a cumulative frequency table for the data.

b Draw a cumulative frequency graph of the data.

c Use your graph to estimate:

i the median examination mark

ii the number of students who scored less than 65%

iii the mark required to be awarded a credit, given that this is awarded to the top 25% of students.

Score x (%)	*Frequency*
$10 \leqslant x < 20$	1
$20 \leqslant x < 30$	3
$30 \leqslant x < 40$	6
$40 \leqslant x < 50$	15
$50 \leqslant x < 60$	14
$60 \leqslant x < 70$	28
$70 \leqslant x < 80$	18
$80 \leqslant x < 90$	11
$90 \leqslant x < 100$	4

3

Time t (min)	*Frequency*
$30 \leqslant t < 35$	7
$35 \leqslant t < 40$	13
$40 \leqslant t < 45$	18
$45 \leqslant t < 50$	25
$50 \leqslant t < 55$	12
$55 \leqslant t < 60$	5

In a running race, the times of 80 competitors were recorded. They are summarised in the table shown.

a Construct a cumulative frequency table for the data.

b Draw a cumulative frequency graph of the data.

c Use your graph to estimate:

i the median time

ii the number of runners whose time was less than 38 minutes

iii the time in which the fastest 30 runners completed the course.

4 The table alongside is a summary of the distance a baseball was thrown by a number of students.

a Estimate the mean distance the ball was thrown.

b Draw a cumulative frequency graph of the data.

c Use your graph to estimate:

i the median distance thrown by the students

ii the number of students who threw the ball less than 35 m

iii the distance required to be in the top 20% of students.

Distance d (m)	*Frequency*
$20 \leqslant d < 30$	2
$30 \leqslant d < 40$	6
$40 \leqslant d < 50$	26
$50 \leqslant d < 60$	12
$60 \leqslant d < 70$	3
$70 \leqslant d < 80$	1

E MEASURING THE SPREAD

Knowing the middle of a data set can be quite useful, but for a more complete picture of the data set we also need to know its **spread** or **variation**.

Three commonly used statistics that indicate the spread of a set of data are:

- the **range**
- the **interquartile range**
- the **standard deviation**.

THE RANGE

The **range** is the difference between the **maximum** data value and the **minimum** data value.

range = maximum data value − minimum data value

Example 7 Self Tutor

Find the range of the data set: 5, 3, 8, 4, 9, 7, 5, 6, 2, 3, 6, 8, 4.

range = maximum data value − minimum data value = $9 - 2 = 7$

THE INTERQUARTILE RANGE

The median divides an ordered data set in halves. By finding the median of each half, we divide the original data into **quartiles**.

The **lower quartile** is the median of the lower half, and is 25% of the way through the ordered data set.

The **upper quartile** is the median of the upper half, and is 75% of the way through the ordered data set.

The data set is therefore divided into quarters by the lower quartile $\mathbf{Q_1}$, the median $\mathbf{Q_2}$, and the upper quartile $\mathbf{Q_3}$.

The **interquartile range** is the range of the middle half (50%) of the data.

interquartile range = upper quartile − lower quartile

or $\mathbf{IQR = Q_3 - Q_1}$

Example 8 Self Tutor

For the data set 6, 7, 3, 7, 9, 8, 5, 5, 4, 6, 6, 8, 7, 6, 6, 5, 4, 5, 6, find the:

a median **b** lower and upper quartiles **c** interquartile range.

The ordered data set is: $\cancel{3\ 4\ 4\ 5\ 5\ 5\ 5\ 6\ 6}\ \boxed{6}\ \cancel{6\ 6\ 6\ 7\ 7\ 7\ 8\ 8\ 9}$ {19 data values}

a Since $n = 19$, $\dfrac{n+1}{2} = \dfrac{19+1}{2} = 10$

$\therefore$ the median is the 10th data value, which is 6.

b As there are an odd number of data values, we ignore the middle value and split the remaining data into two groups.

$\cancel{3\ 4\ 4\ 5}\ \boxed{5}\ \cancel{5\ 5\ 6\ 6}$ $\qquad$ $\cancel{6\ 6\ 6\ 7}\ \boxed{7}\ \cancel{7\ 8\ 8\ 9}$

Q_1 = median of lower half = 5 $\qquad$ Q_3 = median of upper half = 7

c $IQR = Q_3 - Q_1 = 2$

Example 9 — **Self Tutor**

For the data set 9, 8, 2, 3, 7, 6, 5, 4, 5, 4, 6, 8, 9, 5, 5, 5, 4, 6, 6, 8, find the:

a median **b** lower and upper quartiles **c** interquartile range.

The ordered data set is:

~~2 3 4 4 4 5 5 5 5~~ 5 6 ~~6 6 6 7 8 8 8 9 9~~ {20 data values}

a Since $n = 20$, $\frac{n+1}{2} = \frac{21}{2} = 10.5$

$\therefore$ the median $= \frac{\text{10th value} + \text{11th value}}{2} = \frac{5+6}{2} = 5.5$

b As there are an even number of data values, we split the original data into two equal groups of 10.

~~2 3 4 4~~ 4 5 ~~5 5 5 5~~ ~~6 6 6 6~~ 7 8 ~~8 8 9 9~~

$\therefore$ $Q_1 = 4.5$ $\therefore$ $Q_3 = 7.5$

c $\text{IQR} = Q_3 - Q_1 = 3$

You can use the **statistics package** or your **calculator** to find the measures of spread of a data set.

STATISTICS PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 11E

1 For each of the following data sets, make sure the data is ordered and then find:

i the range **ii** the median

iii the lower and upper quartiles **iv** the interquartile range.

a 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 10, 10, 11, 11, 11, 12, 12

b 11, 13, 16, 13, 25, 19, 20, 19, 19, 16, 17, 21, 22, 18, 19, 17, 23, 15

c 23.8, 24.4, 25.5, 25.5, 26.6, 26.9, 27, 27.3, 28.1, 28.4, 31.5

2 While Jarrod was eating a bag of mandarins, he counted the number of seeds in each. The numbers he counted were:

3 4 7 11 2 6 3 14 10 6 9

Calculate the range and interquartile range of the data.

3 The table alongside shows the number of tows performed by a tow truck driver each day over a 45 day period.

Find the:

a mean **b** median

c range **d** interquartile range.

Number of tows	*Frequency*
3	2
4	5
5	6
6	9
7	12
8	8
9	3

4 Kylie and Chris were asked to listen to 20 songs, and give each a rating out of 20. The results are shown below.

Kylie

14	11	16	8	10
7	10	5	20	13
12	3	19	6	11
4	15	10	19	16

Chris

15	11	9	12	16
14	10	14	9	17
18	13	12	12	16
11	12	18	14	10

a Find the:

i median of each data set

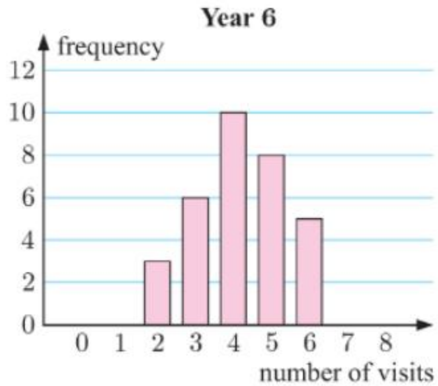
ii range of each data set

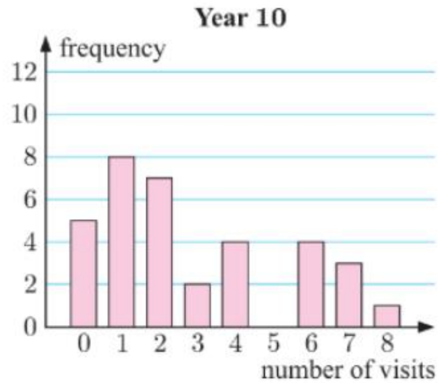
iii interquartile range of each data set.

b In general, who gave the higher ratings?

c Who had the greater variation in their ratings?

5 The Year 6 and Year 10 students at a school were asked how many times they visited their grandparents in the last month. The results are shown in the graphs below.





a For each data set, find the:

i median **ii** range **iii** interquartile range.

b Which class:

i generally visited their grandparents more often

ii had greater variation in their number of visits?

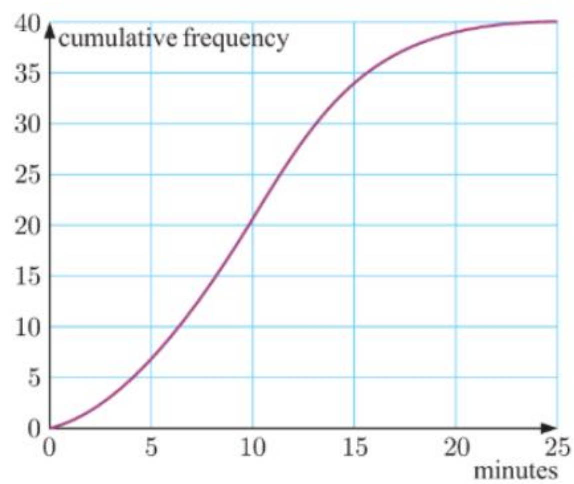
6 In one month, Milton made 40 phone calls. The cumulative frequency graph alongside shows the lengths of the calls.
Using the graph, estimate the:

a median of the data

b lower quartile of the data

c upper quartile of the data

d interquartile range of the data.



F BOX-AND-WHISKER PLOTS

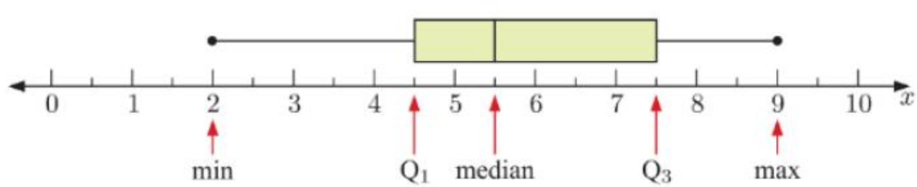
A **box-and-whisker plot** is a visual display of some of the descriptive statistics of a data set. It shows:

- the minimum value (min)
- the lower quartile (Q_1)
- the median (Q_2)
- the upper quartile (Q_3)
- the maximum value (max)

These five numbers form the **five-number summary** of a data set.

For **Example 9**, the five-number summary and corresponding box-and-whisker plot are:

minimum $= 2$
$Q_1 = 4.5$
median $= 5.5$
$Q_3 = 7.5$
maximum $= 9$



Notice that:
- the rectangular box represents the 'middle' half of the data set
- the lower whisker represents the 25% of the data with the lowest values
- the upper whisker represents the 25% of the data with the highest values.

Example 10 Self Tutor

Consider the data set: 5 1 6 8 1 7 4 5 6 11 3 4 4 2 5 5

a Construct the five-number summary for the data.

b Draw a box-and-whisker plot for the data.

c Find the **i** range **ii** interquartile range.

d Find the percentage of data values less than 4.

---

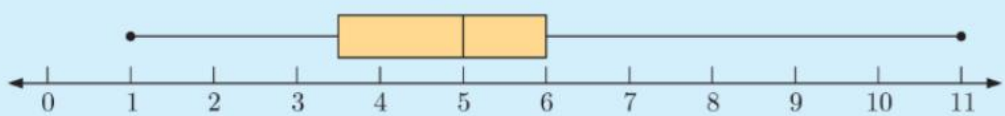
a The ordered data set is:

~~1 1 2~~ 3 4 ~~4 4~~ 5 5 ~~5 5~~ 6 6 ~~7 8 11~~ {16 data values}

$Q_1 = 3.5$ median $= 5$ $Q_3 = 6$

The five-number summary is: $\begin{cases} \text{min} = 1 & Q_1 = 3.5 \\ \text{median} = 5 & Q_3 = 6 \\ \text{max} = 11 & \end{cases}$

b



c **i** range $= \text{max} - \text{min}$
$= 11 - 1$
$= 10$

ii IQR $= Q_3 - Q_1$
$= 6 - 3.5$
$= 2.5$

STATISTICS PACKAGE

d 25% of the data values are less than 4.

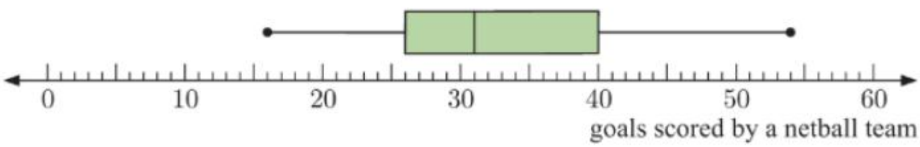
You can use your **calculator** or the **statistics package** to draw box-and-whisker plots.

GRAPHICS CALCULATOR INSTRUCTIONS

STATISTICS PACKAGE

EXERCISE 11F.1

1



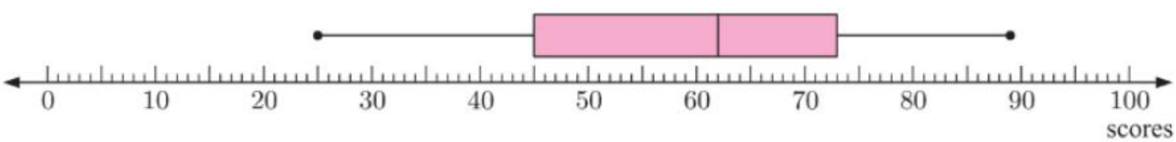
a This box-and-whisker plot summarises the goals scored by a netball team. Find the:

- **i** median
- **ii** maximum value
- **iii** minimum value
- **iv** upper quartile
- **v** lower quartile.

b Calculate:

- **i** the range
- **ii** the interquartile range.

2



The box-and-whisker plot shown summarises the points scored by a basketball team in their matches during a season.

a Copy and complete the following statements about their results:

- **i** The highest score was ...... points.
- **ii** The lowest score was ...... points.
- **iii** Half of the scores were greater than or equal to ...... points.
- **iv** The top 25% of the scores were at least ...... points.
- **v** The middle half of the scores were between ...... and ...... points.

b Find the range of the data set.

c Find the interquartile range of the data set.

3 For each of the following data sets:

- **i** Construct a five-number summary for the data.
- **ii** Draw a box-and-whisker plot for the data.
- **iii** Find the range.
- **iv** Find the interquartile range.

a 5, 5, 10, 9, 4, 2, 8, 6, 5, 8, 6, 7, 9, 6, 10, 3, 11

b 7, 0, 4, 6, 8, 8, 9, 5, 6, 8, 8, 8, 9, 8, 1, 8, 3, 7, 2, 7, 4, 5, 9, 4

4 The weight, in kilograms, of a particular brand of bags of firewood is stated to be 20 kg. However, some bags weigh more than this and some weigh less. A sample of bags is carefully weighed, and the measurements are given in the ordered stem-and-leaf plot shown.

Stem	*Leaf*
18	8
19	5 7 7 8 8 9
20	1 1 1 2 2 5 6 8
21	0 1 1 2 4 6
22	3

20 | 5 represents 20.5 kg

a Locate the median, upper and lower quartiles, and maximum and minimum weights for the sample.

b Draw a box-and-whisker plot for the data.

c Find: **i** the interquartile range **ii** the range.

d Copy and complete the following statements about the distribution of weights for the bags of firewood in this sample:

i Half of the bags of firewood weighed at least ...... kg.

ii ......% of the bags weighed less than 20 kg.

iii The weights of the middle 50% of the bags were spread over ...... kg.

iv The lightest 25% of the bags weighed ...... kg or less.

e Is the distribution of weights in this sample symmetrical, or positively or negatively skewed?

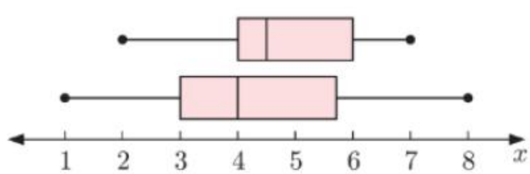
ACTIVITY

Click on the icon to practice matching box-and-whisker plots with their graphs.

GAME

PARALLEL BOX-AND-WHISKER PLOTS

If we have two data sets that we want to compare, we can draw a box-and-whisker plot for each data set on the same scale. This is known as a **parallel box-and-whisker plot**. It can be drawn horizontally or vertically.



Parallel box-and-whisker plots enable us to make a *visual comparison* of the statistics and distributions of two data sets.

Example 11 Self Tutor

An office worker has the choice of travelling to work by car or bus. He has collected data giving the travel times from recent journeys using both of these methods. He is interested to know which type of transport is the quickest to get him to work, and which is the most reliable.

Car (minutes):	13	14	18	18	19	21	22	22	24	25	27	28	30	33	43
Bus (minutes):	16	16	16	17	17	18	18	18	20	20	21	21	23	28	30

a Draw a parallel box-and-whisker plot for the data sets.

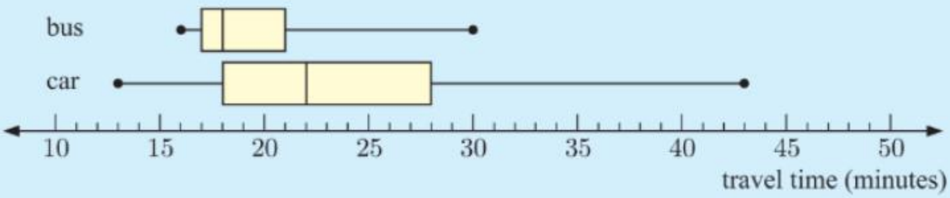
b Hence, determine which method of transport is:

i quicker **ii** more reliable.

a We first construct the five-number summary for each data set.

Car: min $= 13$ $\quad Q_1 = 18$ $\quad$ median $= 22$ $\quad Q_3 = 28$ $\quad$ max $= 43$

Bus: min $= 16$ $\quad Q_1 = 17$ $\quad$ median $= 18$ $\quad Q_3 = 21$ $\quad$ max $= 30$

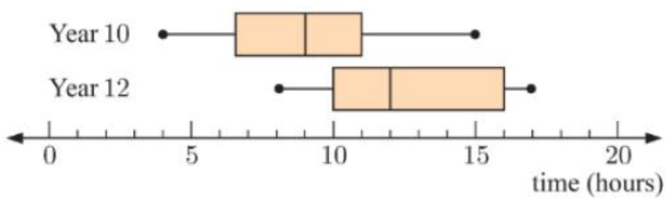


b **i** Looking at the parallel box-and-whisker plot, the bus travel times are generally lower than the car travel times. So, the bus is generally quicker.

ii The car travel times have greater spread than the bus travel times. So, the bus is also more reliable.

EXERCISE 11F.2

1 This parallel box-and-whisker plot compares the times students in Year 10 and Year 12 spend on homework over a one week period.

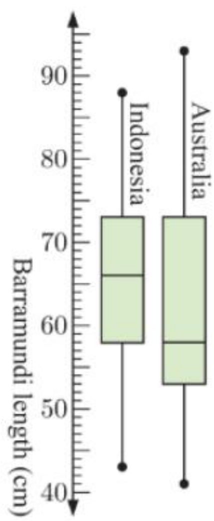


a Find the five-number summaries for each year group.

b For each group, determine the:

i range $\qquad$ **ii** interquartile range.

2 Barramundi is a species of fish caught in the tropical waters of northern Australia and Indonesia. Alongside is a parallel box-and-whisker plot of the lengths of barramundi caught in Indonesian and Australian waters.



a For each region, find the:

i greatest length $\qquad$ **ii** shortest length

iii range $\qquad$ **iv** interquartile range.

b The legal length for a barramundi to be kept is 58 cm. What percentage of fish caught were of legal length in:

i Indonesia $\qquad$ **ii** Australia?

c Copy and complete:

i The fish caught in ...... generally have greater length.

ii The fish caught in ...... have greater length variability.

3 Consider the **Opening Problem** on page **202**.

a Construct the five-number summary for each data set.

b Draw a parallel box-and-whisker plot to display the data sets.

c Describe the distribution of each data set.

d Determine the hotel in which the guests generally stay longer.

4 The heights, in centimetres, of boys and girls in a Year 10 class are:

Boys	165	171	169	169	172	171	171	180	168	168	166	168	170
	165	171	173	187	181	175	174	165	167	163	160	169	167
	172	174	177	188	177	185	167	160					
Girls	162	171	156	166	168	163	170	171	177	169	168	165	156
	159	165	164	154	171	172	166	152	169	170	163	162	165
	163	168	155	175	176	170	166						

a Find the five-number summary for each of the data sets.

b Draw a parallel box-and-whisker plot to display the data sets.

c Compare the centre and spread of the data sets.

5 Samples of lobster were caught in two adjacent bays on the coast of California. The following data shows the average weights in pounds for the two bays over a 20 day catching period.

Bay 1:	2.6	2.5	2.7	2.4	2.9	2.7	2.6	2.7	2.8	2.5
	2.7	2.6	2.8	2.6	2.5	2.8	2.5	2.4	2.7	2.3
Bay 2:	2.7	3.0	2.6	2.9	2.7	2.8	2.9	2.6	2.7	2.7
	2.9	3.1	2.6	2.7	2.7	2.8	3.2	2.7	2.8	2.8

a Find the five-number summary for each of the data sets.

b Construct a parallel box-and-whisker plot to display the data sets.

c Compare the centre and spread of the data sets.

Global context

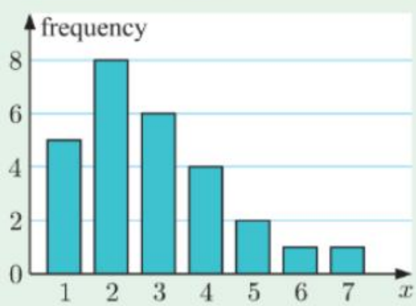
click here

Air passenger numbers

Statement of inquiry:	Analysing data can help us to identify changes over time.
Global context:	Orientation in space and time
Key concept:	Logic
Related concepts:	Quantity, Change
Objectives:	Communicating, Applying mathematics in real-life contexts
Approaches to learning:	Communication, Self-management

REVIEW SET 11A

1 Describe the data distribution shown:



2 A class of 20 students was asked "How many bedrooms are there in your house?"
The following data was collected:

3 2 3 2 2 4 3 4 2 3 2 1 2 2 3 2 4 2 3 2

a Is the data discrete or continuous? **b** Are there any outliers in the data?

c Construct a dot plot to display the data.

3 Consider the set of data: 17 14 9 12 23 14 12 18 9 15 6 14 21 13 10

a Find the:

i mode **ii** mean **iii** median **iv** range

v upper and lower quartiles **vi** interquartile range.

b Draw a box-and-whisker plot to display the data.

4 The masses of eggs in a carton marked '50 g eggs' are recorded alongside.

Mass m (g)	*Frequency*
$48 \leqslant m < 49$	1
$49 \leqslant m < 50$	1
$50 \leqslant m < 51$	16
$51 \leqslant m < 52$	4
$52 \leqslant m < 53$	3

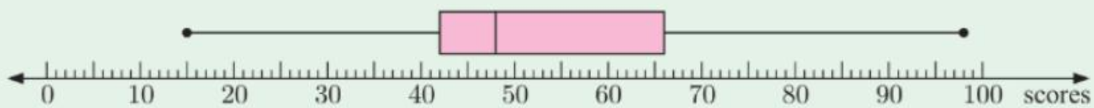
a Construct a frequency histogram for the data.

b What is the modal class? Explain what this means.

c Describe the distribution of the data.

d Estimate the mean mass of an egg in the carton.

5 The scores out of 100 for an exam are displayed in the box-and-whisker plot below.



a State the:

i median score **ii** maximum score **iii** minimum score

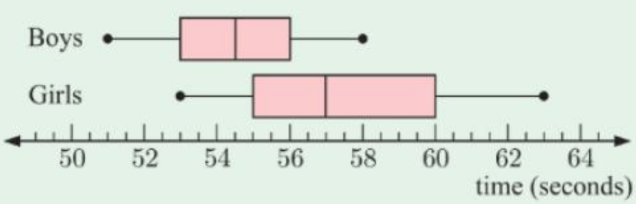
iv upper quartile **v** lower quartile.

b Calculate the:

i range **ii** interquartile range of scores.

6 Nine scores have an average of 8. Scores of x and $x + 1$ are added, and these increase the average to 9. Find x.

7 The given parallel box-and-whisker plot represents the 100-metre swim times for the members of a swimming squad.



Copy and complete the following:

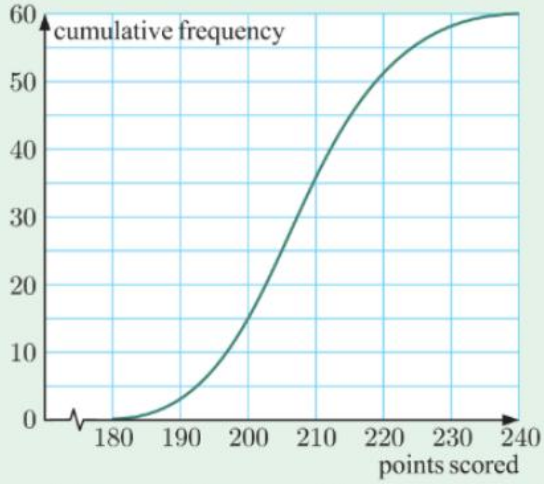
a Comparing the median swim times for girls and boys shows that, in general, the ...... swim ...... seconds faster than the ......

b The range of the girls' swim times is ...... seconds compared to the range of ...... seconds for the boys.

c The fastest 25% of the boys swim as fast, or faster than ......% of the girls.

d ......% of the boys swim faster than 60 seconds whereas ......% of the girls swim faster than 60 seconds.

8



The cumulative frequency graph illustrates the points scored by competitors in a ski aerials competition.

a How many competitors took part in the competition?

b What percentage of competitors scored less than 200 points?

c Estimate the median score.

9 The Davis and Douglas families kept a record of the amount they spent at the supermarket each week for 10 weeks:

Davis:	\$102.50	\$115.95	\$107.60	\$122.15	\$131.05
	\$111.15	\$120.50	\$127.55	\$100.95	\$113.40
Douglas:	\$109.80	\$86.75	\$94.50	\$129.75	\$72.05
	\$133.05	\$121.05	\$97.60	\$73.80	\$105.35

a Find the mean and interquartile range of each data set.

b Which family generally spent more at the supermarket each week?

c Which family had the greater variation in the amount they spent each week?

10 The lengths of newborn babies at a hospital were recorded over a one month period. The results are shown in the table.

Length l (cm)	*Frequency*
$48 \leqslant l < 49$	1
$49 \leqslant l < 50$	3
$50 \leqslant l < 51$	9
$51 \leqslant l < 52$	10
$52 \leqslant l < 53$	16
$53 \leqslant l < 54$	4
$54 \leqslant l < 55$	5
$55 \leqslant l < 56$	2

a Display the data using a frequency histogram.

b How many babies were 52 cm or more?

c What percentage of babies had lengths in the interval $50 \text{ cm} \leqslant l < 53$ cm?

d Draw a cumulative frequency graph for the data.

e Use your graph to estimate the:

i median length

ii number of babies with length less than 51.5 cm.

REVIEW SET 11B

1 For the data set alongside, find the:

a mean **b** median **c** mode.

13	16	15	17	14	13	13	15	16	14
16	14	15	15	15	13	17	14	12	14

2 A sample of 15 measurements has a mean of 14.2, and a sample of 10 measurements has a mean of 12.6. Find the mean of the combined sample of 25 measurements.

3 As punishment for misbehaving, a class of students had to pick up litter during recess. The table shows the number of litter pieces picked up by the students.

Pieces of litter	*Frequency*
6	1
7	3
8	2
9	6
10	7
11	8
12	5

a How many students were in the class?

b For this data, find the:

i mean **ii** mode **iii** median.

c Draw a dot plot to display the data.

d Describe the distribution of the data.

4 The numbers of people at a judo class each week were:

10 8 10 9 7 11 9 11 10
10 9 8 9 9 11 8 10 9
10 11 10 7 9 11 10 8

a Draw a column graph to display the data.

b Describe the distribution of the data.

5 The data below show how many people used the swimming pool at a gym each day over 40 days:

22 37 54 31 43 47 65 60 51 42 34 30 26 38
45 55 49 32 49 53 68 47 32 45 58 37 42 53
41 59 64 51 37 23 25 41 30 28 34 53

a Organise the data into a tally and frequency table, grouping the data appropriately.

b Draw a column graph for the data.

c On what percentage of the days did at least 50 people use the pool?

6 Draw a box-and-whisker plot to display the data set:

5 7 8 9 12 15 15 17 19 20 22

7 A class consists of 30 students. The mean height of students in the class is 172 cm. The mean height of the males is 176 cm, and the mean height of the females is 166 cm. How many males are in the class?

8 Find the interquartile range of the data set: 21 39 27 43 51 37 18 31 44

9 The table alongside shows the prices of houses listed for sale in a particular suburb.

Price p ($\$ \times 1000$)	*Frequency*
$250 \leqslant p < 300$	3
$300 \leqslant p < 350$	6
$350 \leqslant p < 400$	11
$400 \leqslant p < 450$	7
$450 \leqslant p < 500$	5

a How many houses are listed for sale in the suburb?

b Estimate the mean house price.

10 The students at a touch typing course were tested to see how many words they could correctly type in one minute. The test was given both before and after the practice modules in the course.

Before:	47	52	32	41	37	39	42	27	36	41	33	29	36	40	25
After:	59	67	42	49	52	60	59	48	53	62	55	44	50	48	47

a Construct a five-number summary for each data set.

b Draw a parallel box-and-whisker plot to display the data.

c Did the course improve the typing speeds of the students? Explain your answer.

Chapter 12

Quadratic equations

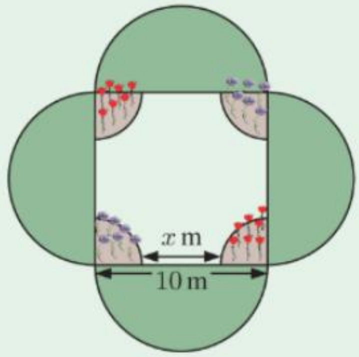
Contents:

OPENING PROBLEM

A square garden gazebo is surrounded by 4 semi-circles of lawn as shown.

For a wedding, quarter circles of flowers are planted in each corner of the gazebo. The lawn covers 4 times as much area as the flower beds.

Suppose the flower beds are x metres apart as shown.



Things to think about:

- **a** Explain why the radius of each flower bed is $\left(5 - \frac{x}{2}\right)$ metres.
- **b** Hence, show that $x^2 - 20x + 50 = 0$.
- **c** By solving this equation, can you find the distance between the flower beds?

In the past we have solved many equations of the form $ax + b = 0$, $a \neq 0$. These are called **linear equations**, and have *only one* solution.

For example, $3x - 2 = 0$ is a linear equation which has the solution $x = \frac{2}{3}$.

A **quadratic equation** is an equation which can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are constants, $a \neq 0$.

A quadratic equation may have *two*, *one*, or *no* real solutions.

For example: $x^2 + 3x - 10 = 0$ is a quadratic equation.

If $x = 2$,
$$\begin{aligned} &x^2 + 3x - 10 \\ &= 2^2 + 3 \times 2 - 10 \\ &= 4 + 6 - 10 \\ &= 0 \end{aligned}$$

If $x = -5$,
$$\begin{aligned} &x^2 + 3x - 10 \\ &= (-5)^2 + 3 \times (-5) - 10 \\ &= 25 - 15 - 10 \\ &= 0 \end{aligned}$$

$x = 2$ and $x = -5$ both satisfy the equation $x^2 + 3x - 10 = 0$, so we say that they are both **solutions** or **roots** of the equation.

In contrast, the quadratic equation $x^2 + 2x + 1 = 0$ has only the one solution $x = -1$,
and the quadratic equation $x^2 + 1 = 0$ has no real solutions.

A SOLUTION BY FACTORISATION

One method for solving quadratic equations is to **factorise** the quadratic and then apply the **Null Factor law**.

The **Null Factor law** states that:

When the product of two or more numbers is zero, then *at least one* of them must be zero.

So, if $\boldsymbol{ab = 0}$ then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$.

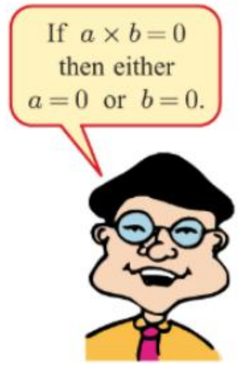
To solve quadratic equations by factorisation, we follow these steps:

Step 1: If necessary, rearrange the equation so one side is **zero**.
Step 2: **Fully factorise** the other side (usually the LHS).
Step 3: Apply the **Null Factor law**.
Step 4: **Solve** the resulting linear equations.

Example 1 Self Tutor

Solve for x: $x^2 = 3x$

$$\begin{aligned} x^2 &= 3x \\ \therefore\ x^2 - 3x &= 0 && \{\text{making the RHS} = 0\} \\ \therefore\ x(x-3) &= 0 && \{\text{factorising the LHS}\} \\ \therefore\ x = 0 \ \text{ or } \ x - 3 &= 0 && \{\text{Null Factor law}\} \\ \therefore\ x &= 0 \ \text{ or } \ 3 \end{aligned}$$



WARNING ON INCORRECT CANCELLING

Let us reconsider the equation $x^2 = 3x$ from **Example 1**.

We notice that there is a common factor of x on both sides.

If we cancel x from both sides, we will have $\frac{x^2}{x} = \frac{3x}{x}$ and thus $x = 3$.

Consequently, we will 'lose' the solution $x = 0$.

From this example we conclude that:

We must never cancel a variable that is a common factor from both sides of an equation unless we know that the factor cannot be zero.

Example 2 Self Tutor

Solve for x: $x^2 + 3x = 28$

$$\begin{aligned} x^2 + 3x &= 28 \\ \therefore\ x^2 + 3x - 28 &= 0 && \{\text{making the RHS} = 0\} \\ \therefore\ (x+7)(x-4) &= 0 && \{\text{the numbers } +7 \text{ and } -4 \text{ have sum 3 and product } -28\} \\ \therefore\ x + 7 = 0 \ \text{ or } \ x - 4 &= 0 && \{\text{Null Factor law}\} \\ \therefore\ x &= -7 \ \text{ or } \ 4 \end{aligned}$$

Check: If $x = -7$, then $x^2 + 3x = (-7)^2 + 3(-7) = 49 - 21 = 28$ ✓
If $x = 4$, then $x^2 + 3x = 4^2 + 3(4) = 16 + 12 = 28$ ✓

EXERCISE 12A

1 Solve for x:

a $x^2 - 7x = 0$ **b** $x^2 - 5x = 0$ **c** $x^2 = 8x$
d $x^2 = 4x$ **e** $3x^2 + 6x = 0$ **f** $2x^2 + 5x = 0$
g $4x^2 - 3x = 0$ **h** $4x^2 = 5x$ **i** $3x^2 = 9x$

2 Solve for x:

a $x^2 + 3x + 2 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $x^2 - 10x + 25 = 0$
d $x^2 + 5x + 6 = 0$ **e** $x^2 + 6 = 5x$ **f** $x^2 + 7x = -6$
g $x^2 + 14 = -9x$ **h** $x^2 + 11x = -30$ **i** $x^2 = 15 - 2x$
j $x^2 + 4x = 12$ **k** $x^2 = 11x - 24$ **l** $x^2 = 14x - 49$

Example 3 **Self Tutor**

Solve for x: $5x^2 = 3x + 2$

$$5x^2 = 3x + 2$$
$$\therefore\ 5x^2 - 3x - 2 = 0 \qquad \{\text{making the RHS} = 0\}$$

We need two numbers with sum -3 and product -10. These are -5 and $+2$.

$$\therefore\ 5x^2 - 5x + 2x - 2 = 0 \qquad \{\text{‘splitting’ the middle term}\}$$
$$\therefore\ 5x(x-1) + 2(x-1) = 0$$
$$\therefore\ (x-1)(5x+2) = 0$$
$$\therefore\ x - 1 = 0 \text{ or } 5x + 2 = 0 \qquad \{\text{Null Factor law}\}$$
$$\therefore\ x = 1 \text{ or } -\tfrac{2}{5}$$

3 Solve for x:

a $2x^2 + 18x + 36 = 0$ **b** $-x^2 - 11x - 28 = 0$ **c** $3x^2 + 6x = 24$
d $2x^2 + 2x = 24$ **e** $4x^2 + 24 = 20x$ **f** $5x^2 + 20 = 20x$
g $3x^2 = 3x + 18$ **h** $-x^2 = 7x - 60$ **i** $140 + 6x = 2x^2$
j $2x^2 - 5x + 2 = 0$ **k** $3x^2 + 8x - 3 = 0$ **l** $3x^2 + 17x + 20 = 0$
m $2x^2 + 5x = 3$ **n** $2x^2 + 5 = 11x$ **o** $2x^2 + 7x + 5 = 0$

4 Solve for x:

a $3x^2 + 13x + 4 = 0$ **b** $5x^2 - 6 = 13x$ **c** $2x^2 + 17x = 9$
d $2x^2 = 3x + 5$ **e** $3x^2 = 8 - 2x$ **f** $2x^2 = 18 - 9x$
g $-6x^2 + 17x + 3 = 0$ **h** $-2x^2 - 5x + 12 = 0$ **i** $12x^2 - 22x + 6 = 0$
j $9x^2 + 6x - 48 = 0$ **k** $28x^2 - 8 = 2x$ **l** $36x^2 + 39x = 12$

5 Solve for x:

a $x(x+5) + 2(x+6) = 0$ **b** $x(1+x) + x = 3$ **c** $(x-1)(x+9) = 8x$
d $3x(x+2) - 5(x-3) = 17$ **e** $4x(x+1) = -1$ **f** $2x(x-6) = x - 20$
g $x(8-x) + 20 = -13$ **h** $(8x+1)(x-2) = 13 - x$

Example 4 Self Tutor

Solve for x: $\dfrac{x-2}{x} = \dfrac{6+x}{2}$

$$\frac{x-2}{x} = \frac{6+x}{2} \qquad \{\text{LCD} = 2x\}$$

$$\therefore \frac{2\times(x-2)}{2\times x} = \frac{x\times(6+x)}{x\times 2} \qquad \{\text{to achieve a common denominator}\}$$

$$\therefore 2(x-2) = x(6+x) \qquad \{\text{equating numerators}\}$$

$$\therefore 2x-4 = 6x+x^2$$

$$\therefore x^2+4x+4=0$$

$$\therefore (x+2)^2=0$$

$$\therefore x+2=0$$

$$\therefore x=-2$$

Check: If $x=-2$ then

$$\text{LHS} = \frac{(-2)-2}{(-2)} = \tfrac{-4}{-2} = 2$$

and $$\text{RHS} = \frac{6+(-2)}{2} = \tfrac{4}{2} = 2 \quad \checkmark$$

6 Solve for x:

a $\dfrac{x}{4} = \dfrac{1}{x}$ **b** $\dfrac{5}{x} = \dfrac{x}{2}$ **c** $\dfrac{x}{8} = \dfrac{2}{x}$

d $\dfrac{x+1}{4} = \dfrac{1}{2x}$ **e** $\dfrac{x+4}{2} = \dfrac{6}{x}$ **f** $\dfrac{x+2}{x} = x$

g $\dfrac{x-1}{x+2} = \dfrac{2}{x}$ **h** $\dfrac{x}{1+2x} = \dfrac{1}{3x}$ **i** $\dfrac{3x+1}{2x} = x+2$

Example 5 Self Tutor

Solve for x: $\dfrac{1}{x} + \dfrac{4}{x+6} = 1$

$$\frac{1}{x} + \frac{4}{x+6} = \frac{1}{1} \qquad \{\text{LCD} = x(x+6)\}$$

$$\therefore \frac{1\times(x+6)}{x\times(x+6)} + \frac{4\times x}{(x+6)\times x} = \frac{1\times x(x+6)}{1\times x(x+6)} \qquad \{\text{to achieve a common denominator}\}$$

$$\therefore x+6+4x = x(x+6) \qquad \{\text{equating numerators}\}$$

$$\therefore 5x+6 = x^2+6x$$

$$\therefore 0 = x^2+x-6$$

$$\therefore (x+3)(x-2)=0$$

$$\therefore x=-3 \text{ or } 2$$

7 Solve for x:

a $\dfrac{2}{x} + \dfrac{3}{x+2} = 1$ **b** $\dfrac{5}{x} + \dfrac{x}{x-6} = 2$ **c** $\dfrac{4}{x+1} + \dfrac{3}{x+3} = -1$

d $\dfrac{5}{x+2} - \dfrac{6}{x-1} = -2$ **e** $\dfrac{4}{x-2} + \dfrac{3}{x-1} = 3$ **f** $\dfrac{x}{x-2} - \dfrac{x+3}{x} = -5$

g $\dfrac{1}{x+3} + \dfrac{1}{x-3} = x$ **h** $\dfrac{4}{x+2} - \dfrac{2}{x+1} = x$

B COMPLETING THE SQUARE

Some quadratic equations, such as $x^2+4x-7=0$, cannot be solved using the factorisation methods already practised. This is because the solutions are irrational.

Instead, we use a method called **completing the square**. This method relies on rearranging the equation so there is a perfect square on the left hand side. We can then apply the rule:

$$\text{If} \quad \boldsymbol{x^2 = k} \quad \text{then} \quad \begin{cases} \boldsymbol{x = \pm\sqrt{k}} & \text{if } k > 0 \\ \boldsymbol{x = 0} & \text{if } k = 0 \\ \text{there are } \textbf{no real solutions} & \text{if } k < 0. \end{cases}$$

Example 6 — Self Tutor

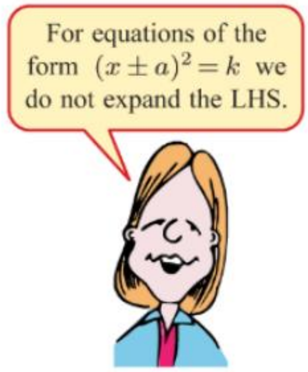
Solve for x:

a $(x-3)^2 = 16$

b $(x+2)^2 = 11$

a
$$\begin{aligned} (x-3)^2 &= 16 \\ \therefore\ x-3 &= \pm\sqrt{16} \\ \therefore\ x-3 &= \pm 4 \\ \therefore\ x &= 3 \pm 4 \\ \therefore\ x &= 7 \text{ or } -1 \end{aligned}$$

b
$$\begin{aligned} (x+2)^2 &= 11 \\ \therefore\ x+2 &= \pm\sqrt{11} \\ \therefore\ x &= -2 \pm \sqrt{11} \end{aligned}$$



EXERCISE 12B.1

1 Solve for x:

a $x^2 = 64$ b $x^2 = -25$ c $7x^2 = 0$

2 Solve for x:

a $(x-1)^2 = 9$ b $(x+4)^2 = 16$ c $(x+2)^2 = -1$

d $(x-4)^2 = 5$ e $(x-6)^2 = -4$ f $(x+2)^2 = 0$

g $(2x-5)^2 = 0$ h $(3x+2)^2 = 4$ i $(3x+1)^2 = 81$

j $(2x+1)^2 = 48$ k $(3-2x)^2 = 7$ l $\frac{1}{3}(2x+3)^2 = 2$

FORMING A PERFECT SQUARE

Most quadratic equations are not presented to us with a perfect square already present. Instead we need to form it for ourselves.

The process of creating a **perfect square** on the left hand side is called **completing the square**.

Consider the following solution to the equation $x^2+4x-7=0$.

$$\begin{aligned} x^2+4x-7 &= 0 \\ \therefore\ x^2+4x &= 7 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2+4x+4 &= 7+4 && \{\text{completing the square}\} \\ \therefore\ (x+2)^2 &= 11 \\ \therefore\ x+2 &= \pm\sqrt{11} \\ \therefore\ x &= -2 \pm \sqrt{11} \end{aligned}$$

Notice that we began by moving the constant term to the RHS. We then added the same constant to both sides of the equation in order to complete the square.

DISCUSSION

- Why is the constant added to *both* sides of the equation?
- How do we know what constant to add?

From our previous study of perfect squares, we observe that:

$$(x+3)^2 = x^2 + 2\times 3\times x + 3^2$$
$$(x-5)^2 = x^2 + 2\times(-5)\times x + (-5)^2$$
and in general that $$(x+a)^2 = x^2 + 2ax + a^2.$$

To complete a perfect square, the number we must add to both sides is found by **halving the coefficient of x, then squaring this value**.

Example 7 Self Tutor

For each of the following equations:

i Find what must be added to both sides of the equation to create a perfect square on the LHS.

ii Write the equation in the form $(x+p)^2 = k$.

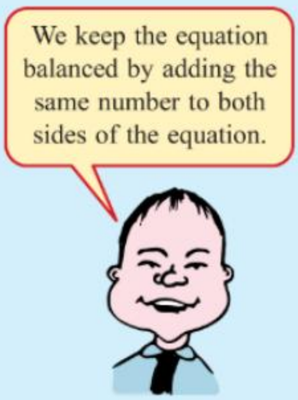
a $x^2 + 8x = -5$ b $x^2 - 6x = 13$

a i In $x^2+8x=-5$, half the coefficient of x is $\frac{8}{2}=4$.
So, we add 4^2 to both sides.

ii The equation becomes $x^2 + 8x + 4^2 = -5 + 4^2$
$\therefore (x+4)^2 = -5 + 16$
$\therefore (x+4)^2 = 11$

b i In $x^2-6x=13$, half the coefficient of x is $\frac{-6}{2}=-3$.
So, we add $(-3)^2 = 3^2$ to both sides.

ii The equation becomes $x^2 - 6x + 3^2 = 13 + 3^2$
$\therefore (x-3)^2 = 13 + 9$
$\therefore (x-3)^2 = 22$



EXERCISE 12B.2

1 For each of the following equations:

i Find what must be added to both sides of the equation to create a perfect square on the LHS.

ii Write each equation in the form $(x+p)^2 = k$.

a $x^2 + 2x = 5$ b $x^2 - 2x = -7$ c $x^2 + 6x = 2$

d $x^2 - 6x = -3$ e $x^2 + 10x = 1$ f $x^2 - 8x = 5$

g $x^2 + 12x = 13$ h $x^2 + 5x = -2$ i $x^2 - 7x = 4$

Example 8 **Self Tutor**

Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2+2x-2=0$ **b** $x^2-5x+3=0$

a
$$\begin{aligned} x^2+2x-2&=0 \\ \therefore\ x^2+2x\quad &=2 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2+2x+1^2&=2+1^2 && \{\text{adding } (\tfrac{2}{2})^2=1^2 \text{ to both sides}\} \\ \therefore\ (x+1)^2&=3 \\ \therefore\ x+1&=\pm\sqrt{3} \\ \therefore\ x&=-1\pm\sqrt{3} \end{aligned}$$

b
$$\begin{aligned} x^2-5x+3&=0 \\ \therefore\ x^2-5x\quad &=-3 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2-5x+(\tfrac{5}{2})^2&=-3+(\tfrac{5}{2})^2 && \{\text{adding } (\tfrac{-5}{2})^2=(\tfrac{5}{2})^2 \text{ to both sides}\} \\ \therefore\ (x-\tfrac{5}{2})^2&=-3+\tfrac{25}{4} \\ \therefore\ (x-\tfrac{5}{2})^2&=\tfrac{13}{4} \\ \therefore\ x-\tfrac{5}{2}&=\pm\sqrt{\tfrac{13}{4}} \\ \therefore\ x&=\tfrac{5}{2}\pm\tfrac{\sqrt{13}}{2} \\ \therefore\ x&=\frac{5\pm\sqrt{13}}{2} \end{aligned}$$

If $(x-a)^2=k$ where $k>0$, then $x=a\pm\sqrt{k}$.

2 Solve for x by completing the square, leaving answers in simplest radical form:

a $x^2-4x+1=0$ **b** $x^2-2x-2=0$ **c** $x^2-4x-3=0$

d $x^2+2x-1=0$ **e** $x^2+4x+1=0$ **f** $x^2+6x+3=0$

g $x^2+3x+2=0$ **h** $x^2+8x+14=0$ **i** $x^2-3x-1=0$

Example 9 **Self Tutor**

Solve for x by completing the square: $x^2-4x+6=0$

$$\begin{aligned} x^2-4x+6&=0 \\ \therefore\ x^2-4x\quad &=-6 && \{\text{moving the constant term to the RHS}\} \\ \therefore\ x^2-4x+2^2&=-6+2^2 && \{\text{adding } (-\tfrac{4}{2})^2=2^2 \text{ to both sides}\} \\ \therefore\ (x-2)^2&=-2 \end{aligned}$$

which is impossible as $(x-2)^2$ cannot be <0.

$\therefore$ no real solutions exist.

If $(x-a)^2=k$ where $k<0$, then there are no real solutions.

3 If possible, solve for x by completing the square:

a $x^2+2x+4=0$ **b** $x^2-5x+6=0$ **c** $x^2-6x+11=0$

d $x^2+x-1=0$ **e** $x^2+5x-2=0$ **f** $x^2-7x+13=0$

Example 10 **Self Tutor**

Solve the equation $3x^2+6x-2=0$ by completing the square.

$$\begin{aligned} 3x^2+6x-2&=0 \\ \therefore\ x^2+2x-\tfrac{2}{3}&=0 && \{\text{dividing both sides by } 3\} \\ \therefore\ x^2+2x&=\tfrac{2}{3} \\ \therefore\ x^2+2x+1^2&=\tfrac{2}{3}+1^2 && \{\text{adding } (\tfrac{2}{2})^2=1^2 \text{ to both sides}\} \\ \therefore\ (x+1)^2&=\tfrac{5}{3} \\ \therefore\ x+1&=\pm\sqrt{\tfrac{5}{3}} \\ \therefore\ x&=-1\pm\frac{\sqrt{5}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} && \{\text{writing the RHS with integer denominator}\} \\ \therefore\ x&=\frac{-3\pm\sqrt{15}}{3} \end{aligned}$$

4 Solve by completing the square:

a $2x^2+4x-1=0$ **b** $3x^2-12x+7=0$ **c** $5x^2-10x+3=0$

HISTORICAL NOTE BABYLONIAN ALGEBRA

The mathematics used by the **Babylonians** was recorded on clay tablets in cuneiform. One such tablet which has been preserved is called *Plimpton 322*, written around 1600 BC.

Plimpton 322

The Ancient Babylonians were able to solve difficult equations using the rules we use today, such as transposing terms, multiplying both sides by like quantities to remove fractions, and factorisation.

They could, for example, add $4xy$ to $(x-y)^2$ to obtain $(x+y)^2$.

This was all achieved without the use of letters for unknown quantities. Instead, they often used words for the unknown.

Consider the following example from about 4000 years ago:

Problem: *"I have subtracted the side of my square from the area and the result is 870. What is the side of the square?"*

Solution: Take half of 1, which is $\frac{1}{2}$, and multiply $\frac{1}{2}$ by $\frac{1}{2}$ which is $\frac{1}{4}$.
Add this to 870 to get $870\frac{1}{4}$. This is the square of $29\frac{1}{2}$.
Now add $\frac{1}{2}$ to $29\frac{1}{2}$ and the result is 30, the side of the square.

Using our modern symbols, the equation is $x^2-x=870$, and one of the solutions is $x=\sqrt{(\frac{1}{2})^2+870}+\frac{1}{2}=30$.

C THE QUADRATIC FORMULA

Many quadratic equations cannot be solved easily by factorisation, and completing the square is rather tedious. Consequently, the **quadratic formula** has been developed.

If $\boldsymbol{ax^2+bx+c=0}$ where $\boldsymbol{a\neq 0}$, then $\boldsymbol{x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}}$.

Proof:

$$\begin{aligned}
&\text{If} & ax^2+bx+c&=0 \\
&\text{then} & x^2+\frac{b}{a}x+\frac{c}{a}&=0 && \{\text{dividing each term by } a, \text{ as } a\neq 0\} \\
&\therefore & x^2+\frac{b}{a}x&=-\frac{c}{a} \\
&\therefore & x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2&=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2 && \{\text{completing the square on the LHS}\} \\
&\therefore & \left(x+\frac{b}{2a}\right)^2&=-\frac{c}{a}\left(\frac{4a}{4a}\right)+\frac{b^2}{4a^2} \\
&\therefore & \left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2} \\
&\therefore & x+\frac{b}{2a}&=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\
&\therefore & x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \\
&\therefore & x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Example 11 **Self Tutor**

Solve for x:

a $x^2-2x-2=0$

b $2x^2+3x-4=0$

a $x^2-2x-2=0$ has $a=1,\ b=-2,\ c=-2$

$$\therefore\ x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}$$

$$\therefore\ x=\frac{2\pm\sqrt{4+8}}{2}$$

$$\therefore\ x=\frac{2\pm\sqrt{12}}{2}$$

$$\therefore\ x=\frac{2\pm 2\sqrt{3}}{2}$$

$$\therefore\ x=1\pm\sqrt{3}$$

b $2x^2+3x-4=0$ has $a=2,\ b=3,\ c=-4$

$$\therefore\ x=\frac{-3\pm\sqrt{3^2-4(2)(-4)}}{2(2)}$$

$$\therefore\ x=\frac{-3\pm\sqrt{9+32}}{4}$$

$$\therefore\ x=\frac{-3\pm\sqrt{41}}{4}$$

EXERCISE 12C

1 Solve the following equations using:

i factorisation **ii** the quadratic formula.

a $x^2 + 6x + 8 = 0$ **b** $x^2 - 10x + 25 = 0$ **c** $3x^2 - 7x - 6 = 0$

2 Use the quadratic formula to solve for x:

a $x^2 + x - 5 = 0$ **b** $x^2 - 5x + 5 = 0$ **c** $x^2 - 4x - 1 = 0$

d $3x^2 + 5x - 1 = 0$ **e** $-2x^2 + x + 7 = 0$ **f** $5x^2 - 8x + 1 = 0$

g $x^2 + 1 = 3x$ **h** $2x^2 = 2x + 3$ **i** $9x^2 = 6x + 1$

j $7x^2 = 5x + 1$ **k** $3x^2 + 2x = 2$ **l** $25x^2 + 1 = 20x$

3 Use the quadratic formula to solve for x:

GRAPHICS CALCULATOR INSTRUCTIONS

a $(x + 2)(x - 1) = 5$ **b** $(x + 1)^2 = 3 - x^2$ **c** $\dfrac{x - 1}{x} = \dfrac{x}{3}$

d $x + \dfrac{1}{x + 2} = 4$ **e** $3x - \dfrac{4}{x + 1} = 10$ **f** $\dfrac{x + 2}{x - 1} = \dfrac{3x}{x + 1}$

D PROBLEM SOLVING

When practical problems are converted into algebraic form, a quadratic equation may result. To solve these problems, follow these steps:

Step 1: Carefully **read the question** until you understand the problem. A **sketch** may be useful.

Step 2: Decide on the **unknown** quantity and label it x, say.

Step 3: Use the information given to write an **equation**.

Step 4: **Solve** the equation.

Step 5: **Check** that any solutions satisfy the equation and are reasonable.

Step 6: Write your answer to the question in **sentence form**.

Example 12 Self Tutor

The sum of a number and its square is 42. Find the number.

Let the number be x. Therefore its square is x^2.

$$x + x^2 = 42$$

$$\therefore\ x^2 + x - 42 = 0 \quad \{\text{rearranging}\}$$

$$\therefore\ (x + 7)(x - 6) = 0 \quad \{\text{factorising}\}$$

$$\therefore\ x = -7 \text{ or } x = 6$$

So, the number is -7 or 6.

Check: The sum of -7 and its square is $-7 + (-7)^2 = -7 + 49 = 42$ ✓

The sum of 6 and its square is $6 + 6^2 = 6 + 36 = 42$ ✓

EXERCISE 12D

1 The sum of a number and its square is 110. Find the number.

2 When 24 is subtracted from the square of a number, the result is five times the original number. Find the number.

3 The sum of two numbers is 6, and the sum of their squares is 28. Find these numbers exactly.

4 Two numbers differ by 7, and the sum of their squares is 29. Find the numbers.

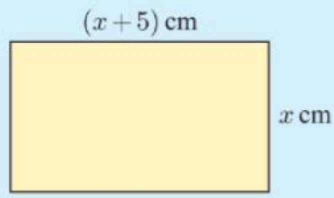
Example 13 **Self Tutor**

A rectangle has length 5 cm greater than its width, and its area is 84 cm^2. Find the dimensions of the rectangle.

Let the width of the rectangle be x cm.

$\therefore$ the length of the rectangle is $(x+5)$ cm.

$$\begin{aligned} \text{Now}\quad \text{area} &= 84\ \text{cm}^2 \\ \therefore\ x(x+5) &= 84 \\ \therefore\ x^2+5x &= 84 \\ \therefore\ x^2+5x-84 &= 0 \\ \therefore\ (x+12)(x-7) &= 0 \\ \therefore\ x &= -12 \text{ or } 7 \end{aligned}$$



But $x > 0$ as lengths must be positive, so $x = 7$.

$\therefore$ the rectangle is 7 cm by 12 cm.

5 A rectangle has length 4 cm greater than its width. The area of the rectangle is 96 cm^2. Find its width.

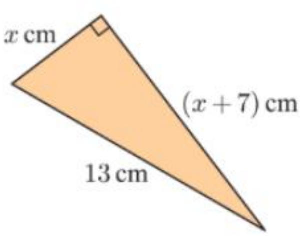
6 The base of a triangle is 4 m longer than its altitude. The area of the triangle is 70 m^2. Find the triangle's altitude.

7 A rectangular enclosure is made from 60 m of fencing. The area enclosed is 216 m^2. Find the dimensions of the enclosure.

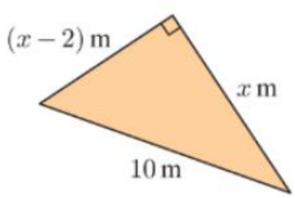
8 A rectangular garden bed was built against an existing brick wall. 24 m of edging was used to enclose 60 m^2. Find the dimensions of the garden bed to the nearest centimetre.

9 Find the value of x:

a



b



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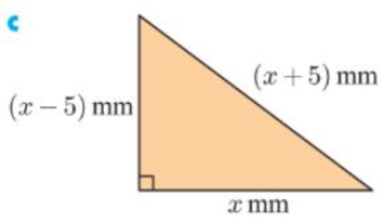
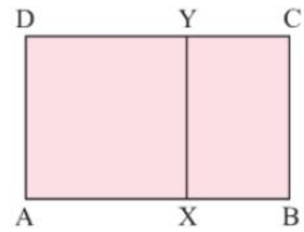
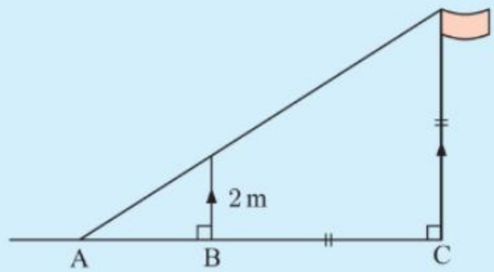


10 A right angled triangle has sides 3 cm and 8 cm respectively less than its hypotenuse. Find the length of the hypotenuse to the nearest millimetre.

11 ABCD is a rectangle in which $AB = 21$ cm.
The square AXYD is removed and the remaining rectangle has area 80 cm^2.
Find the length of [BC].



Example 14 **Self Tutor**



Given that [AB] is 3 m shorter than [BC], find the height of the flagpole.

Let the height of the flagpole be x m.

$\therefore$ $BC = x$ m and $AB = (x-3)$ m

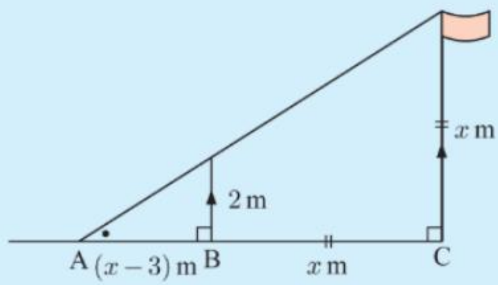
The triangles are equiangular, so they are similar.

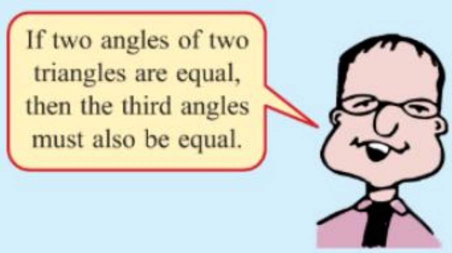
$$\therefore \quad \frac{x}{2} = \frac{(x-3)+x}{x-3}$$
$$\therefore \quad x(x-3) = 2(2x-3)$$
$$\therefore \quad x^2 - 3x = 4x - 6$$
$$\therefore \quad x^2 - 7x + 6 = 0$$
$$\therefore \quad (x-1)(x-6) = 0$$
$$\therefore \quad x = 1 \text{ or } 6$$

But $x - 3 > 0$ as lengths must be positive

$\therefore$ $x = 6$

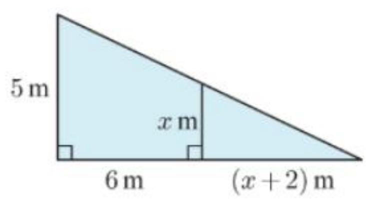
So, the flagpole is 6 m high.



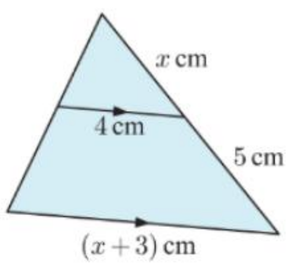


12 Find x in:

a



b



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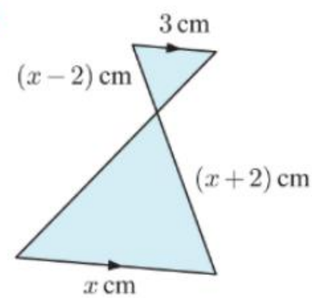
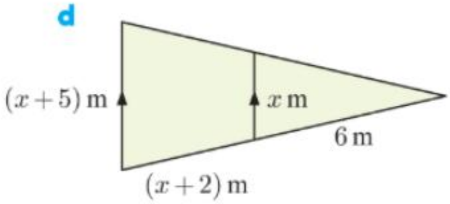
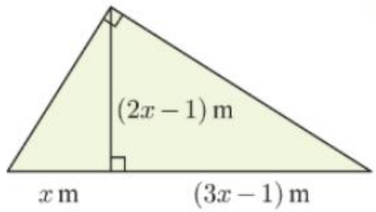


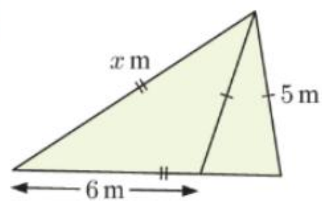
d



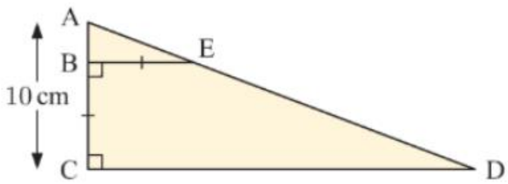
e



f

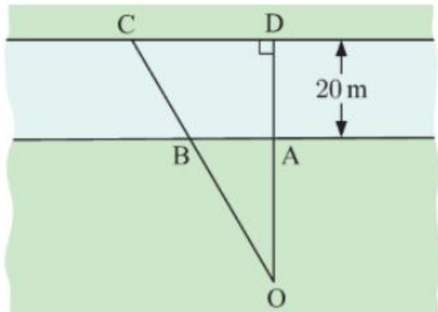


13

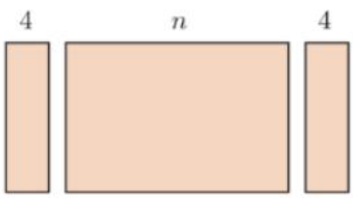


In the figure alongside, [BC] is the same length as [BE], and [CD] is 3 cm more than twice the length of [BE].
Find the length of [BE].

14 A, B, C, and D are posts on the banks of a 20 m wide canal. A and B are 2 m apart, and OA = CD.
Find the exact distance between C and D.



15



A theatre contains a central block of seats with n seats per row. Blocks on either side contain 4 seats per row. The number of rows is 5 less than the total number of seats per row. In total there are 126 seats in the theatre. Find the value of n.

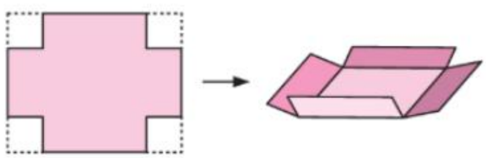
16 The numerator of a fraction is 3 less than the denominator. If the numerator is increased by 6 and the denominator is increased by 5, the fraction is doubled in value. Find the original fraction.

17 At a fruit market, John bought some oranges for a total of \$20. When Jenny visited a different stall, she bought 10 more oranges than John for the same total amount. Given that the difference in price per orange was 10 cents, how many oranges did John purchase?

18 The sum of a number and its reciprocal is $2\frac{1}{12}$. Find the number.

19 Two numbers have a sum of 4, and the sum of their reciprocals is 8. Find the numbers.

20 A sheet of cardboard is 15 cm long and 10 cm wide. It is to be made into an open box with base area 66 cm^2, by cutting out equal squares from the four corners and then bending the edges upwards.
Find the size of the squares to be cut out.



21 Answer the **Opening Problem** on page **230**.

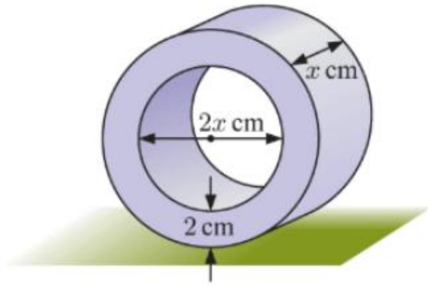
22 A rectangular swimming pool is 12 m long by 6 m wide. It is surrounded by a pavement of uniform width. The area of the pavement is $\frac{7}{8}$ of the area of the pool.

a If the pavement is x m wide, show that the area of the pavement is $(4x^2+36x)$ m^2.

b Hence, show that $4x^2+36x-63=0$.

c How wide is the pavement?

23 A circular magnet has an inner radius of x cm, an outer radius 2 cm larger, and its depth is the same as the inner radius. The total volume of the magnet is 120π cm^3. Find x.

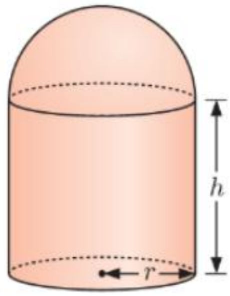


24 A takeaway milkshake container is cylindrical, with a hemispherical lid on top.

The height h of the container is 7 cm greater than its base radius r.

The surface area of the container and lid is 96π cm^2.

Find the base radius of the container.



REVIEW SET 12A

1 Solve for x: **a** $-x^2+11=0$ **b** $7-2x^2=-25$

2 Solve for x: **a** $(x-4)^2=25$ **b** $(x+1)^2-1=0$

3 Solve for x:

a $x^2-4x-21=0$ **b** $4x^2-25=0$ **c** $6x^2-x-2=0$

4 Solve for x:

a $3x^2-6x-72=0$ **b** $5x^2+30x+45=0$ **c** $4x^2-18x+8=0$

5 Solve by completing the square: $x^2+6x+4=0$

6 Solve for x:

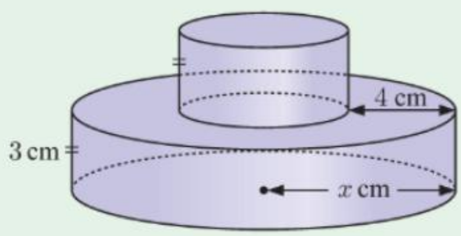
a $x^2+24x=11$ **b** $(x+6)(x-3)=10x$ **c** $10x^2-6=11x$

7 The sum of a number and its reciprocal is $2\frac{1}{6}$. Find the number.

8 Use the quadratic formula to solve for x:

a $2x^2-3x-2=0$ **b** $3x^2+4x-5=0$ **c** $\dfrac{x+4}{x-2}=\dfrac{5x}{x-1}$

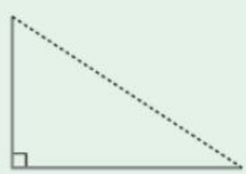
9 The volume of the solid alongside is 174π cm^3. Find x.



10 A straight length of wire is 20 cm long. It is bent at right angles to form the two shorter sides of a right angled triangle.

If the triangle's area is 30 cm^2, find:

a the length of each side **b** the triangle's perimeter.



REVIEW SET 12B

1 Solve for x: **a** $8-3x^2=-10$ **b** $2x^2-5=-3$

2 Solve for x: **a** $(x+3)^2-19=0$ **b** $(3x-1)^2=17$

3 Solve for x: **a** $x^2-8x-33=0$ **b** $8x^2+2x-3=0$

4 Solve by completing the square: $x^2-2x=100$

5 Solve for x: **a** $x^2-45=4x$ **b** $6x^2+26x=20$

6 Solve for x: **a** $\frac{x}{5}=\frac{7}{x}$ **b** $\frac{x}{3-2x}=\frac{x+1}{3-x}$

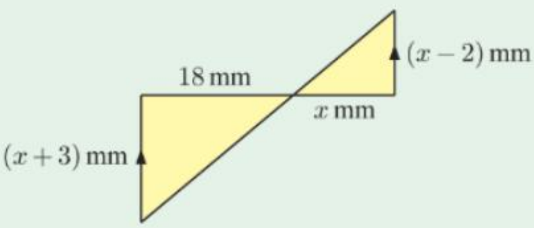
7 The hypotenuse of a right angled triangle is one centimetre more than twice the length of the shortest side. The other side is 7 cm longer than the shortest side. Find the length of each side of the triangle.

8 Use the quadratic formula to solve:

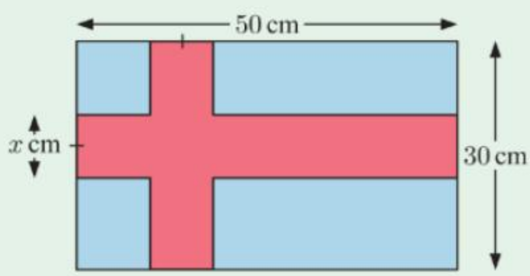
a $2x^2+2x-1=0$ **b** $\frac{1}{x}-\frac{1}{1-x}=2$

9 A group of friends hires a bus for a day for \$480, agreeing to share the cost equally. At the last minute, two more people decide to go on the trip, and as a result each person pays \$8 less.
How many people go on the trip and how much does each person pay?

10 Find the value of x:



11 In the flag below, the area of the red stripes is 700 cm^2. Find the width of the red stripes.



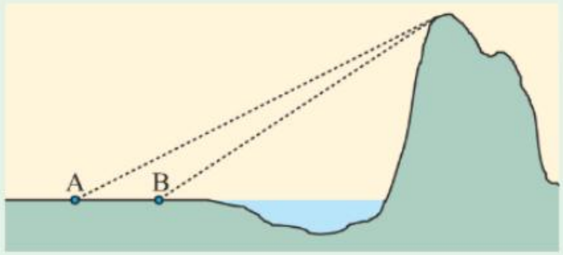
Chapter 13

Trigonometry

Contents:

OPENING PROBLEM

A surveyor is standing on horizontal ground, and wishes to find the height of a mountain on the other side of a lake. He uses a theodolite to accurately measure:



- the angle of elevation from the horizontal ground at A up to the top of the mountain, to be 33.7°
- the angle of elevation from the horizontal ground at B up to the top of the mountain, to be 41.6°
- the distance from A to B to be 400 m.

Things to think about:

a Can you draw a labelled diagram of the situation showing all information given?

b Can you find the height of the mountain?

Trigonometry is a branch of mathematics that deals with the relationship between the side lengths and angles of triangles.

We can apply trigonometry in engineering, astronomy, architecture, navigation, surveying, the building industry, and in many other branches of applied science.

HISTORICAL NOTE — ASTRONOMY AND TRIGONOMETRY

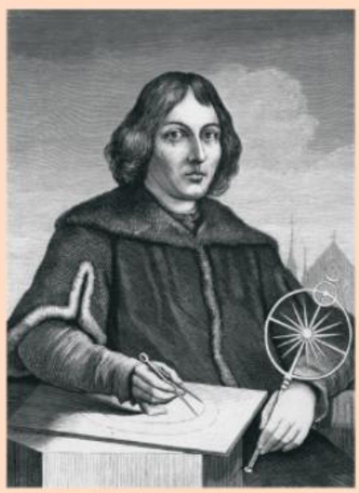
The Greek astronomer **Hipparchus** (140 BC) is credited with being the founder of trigonometry. To aid his astronomical calculations, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest*, which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 5th and 6th Century AD, constructed a table of the lengths of half-chords of a circle with radius one unit. This was the first table of **sine** values.

In the late 16th century, **Georg Joachim de Porris**, also known as **Rheticus**, produced comprehensive and remarkably accurate tables of all six trigonometric ratios, three of which you will learn about in this chapter. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

Rheticus was the only student of **Nicolaus Copernicus**, and helped his tutor publish his work *De revolutionibus orbium coelestium* (On the Revolutions of the Heavenly Spheres).

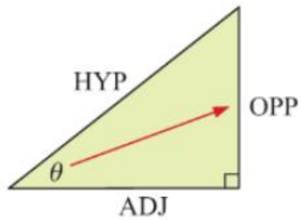


Nicolaus Copernicus

A TRIGONOMETRIC RATIOS

For the right angled triangle with angle θ:

- the **hypotenuse (HYP)** is the longest side
- the **opposite (OPP)** side is opposite θ
- the **adjacent (ADJ)** side is adjacent to θ.



In previous years we have defined the following basic trigonometric ratios for right-angled triangles:

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \quad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \quad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

We can use these ratios to find unknown side lengths and angles in right angled triangles.

INVESTIGATION 1 COMPLEMENTARY ANGLES

Two angles are **complementary** if their sum is 90°. We say that θ and $(90^\circ - \theta)$ are **complements** of each other.

PRINTABLE WORKSHEET

Your task is to determine whether a relationship exists between the sines and cosines of an angle and its complement.

What to do:

1. Use your calculator to complete a table like the one shown. Include some angles of your choice.

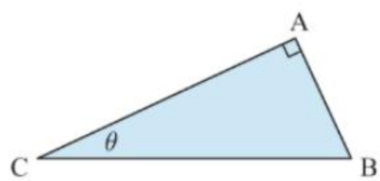
GRAPHICS CALCULATOR INSTRUCTIONS

θ	$\sin\theta$	$\cos\theta$	$90^\circ - \theta$	$\sin(90^\circ - \theta)$	$\cos(90^\circ - \theta)$
17°			73°		
38°					
59°					

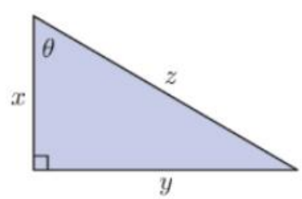
2. Write down your observations from the tabled values.

EXERCISE 13A.1

1. For the triangle given, name the:
   a. hypotenuse
   b. side opposite θ
   c. side adjacent to θ.



2. a. Name:
   i. the hypotenuse
   ii. the side opposite θ
   iii. the side adjacent to θ.



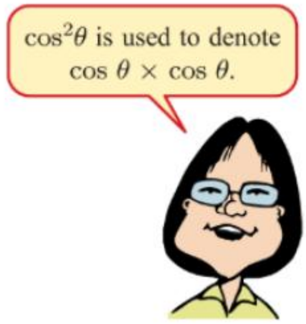
b Hence write an algebraic fraction for:

i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$

c Hence show that:

i $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

ii $\cos^2\theta + \sin^2\theta = 1$. **Hint:** Use Pythagoras.

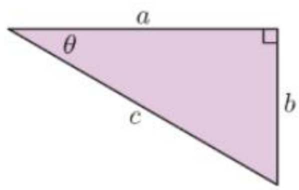


3 **a** Name:

i the hypotenuse

ii the side opposite θ

iii the side adjacent to θ.



b Hence write an algebraic fraction for:

i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$

c Explain why the third angle of the triangle measures $(90^\circ - \theta)$.

d Hence write an algebraic fraction for:

i $\sin(90^\circ - \theta)$ **ii** $\cos(90^\circ - \theta)$ **iii** $\tan(90^\circ - \theta)$

e Show that:

i $\sin(90^\circ - \theta) = \cos\theta$ **ii** $\cos(90^\circ - \theta) = \sin\theta$ **iii** $\tan(90^\circ - \theta) = \dfrac{1}{\tan\theta}$

FINDING SIDE LENGTHS

Suppose we are given the angles of a right angled triangle, and the length of a side. We can use the trigonometric ratios to find the other side lengths.

Step 1: Redraw the figure and mark on it HYP, OPP, and ADJ relative to a given angle.

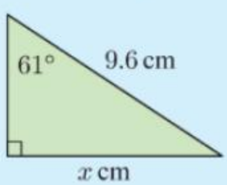
Step 2: Choose an appropriate trigonometric ratio, and construct an equation.

Step 3: Solve the equation to find the unknown side length.

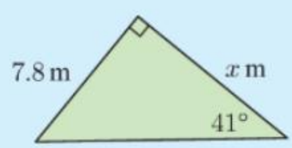
Example 1 **Self Tutor**

Find x, rounded to 3 significant figures:

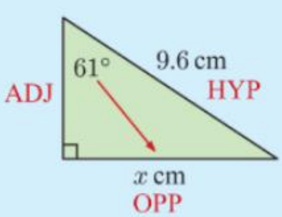
a



b



a The relevant sides are OPP and HYP, so we use the *sine* ratio.

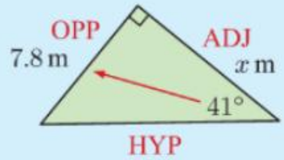


$$\sin 61^\circ = \frac{x}{9.6} \quad \left\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\right\}$$

$$\therefore\ \sin 61^\circ \times 9.6 = x \quad \{\text{multiplying both sides by } 9.6\}$$

$$\therefore\ x \approx 8.40 \quad \{\text{calculator}\}$$

b The relevant sides are OPP and ADJ, so we use the *tangent* ratio.



$$\tan 41° = \frac{7.8}{x} \qquad \left\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$$

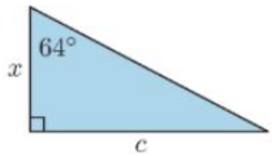
$$\therefore\ x \times \tan 41° = 7.8 \qquad \{\text{multiplying both sides by } x\}$$

$$\therefore\ x = \frac{7.8}{\tan 41°} \qquad \{\text{dividing both sides by } \tan 41°\}$$

$$\therefore\ x \approx 8.97 \qquad \{\text{calculator}\}$$

EXERCISE 13A.2

1 Construct a trigonometric equation connecting the angle with the sides given:

a


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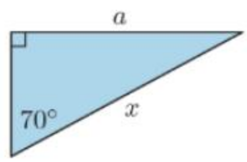
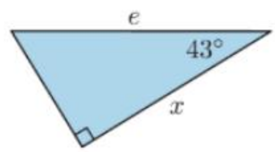


c


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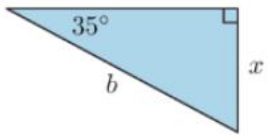

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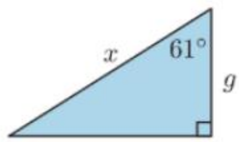

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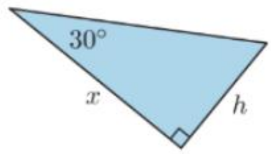
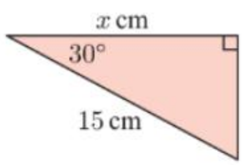


2 Find x, rounded to 2 decimal places:

a


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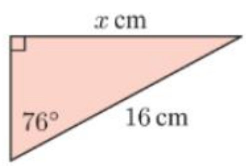

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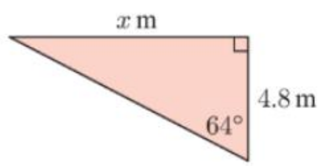

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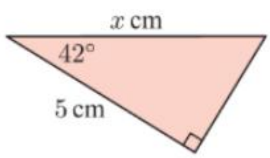

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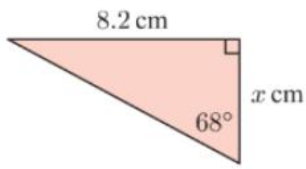

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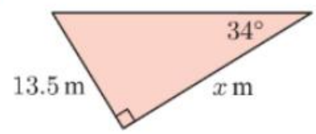

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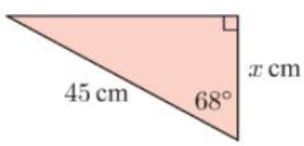

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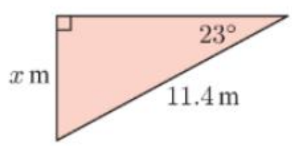
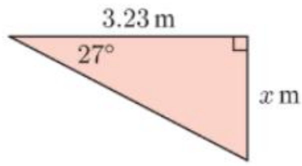


i


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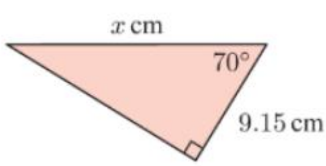

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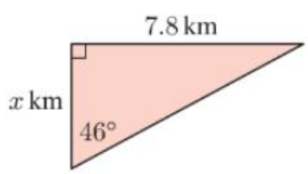
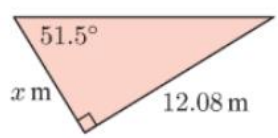
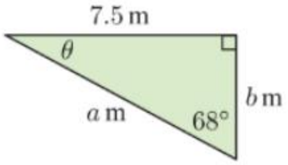


l


3 Find *all* the unknown angles and sides of:

a



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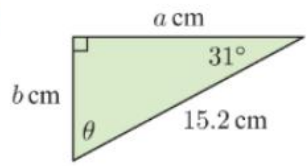

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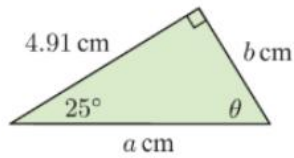
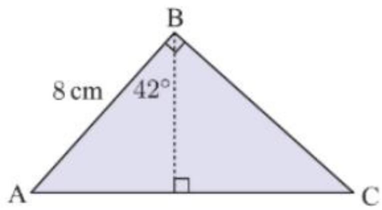


4 Find the perimeter of triangle ABC.

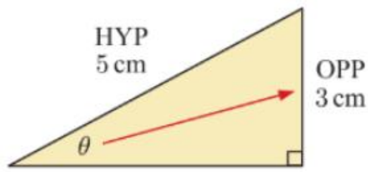


FINDING ANGLES

If we know two side lengths of a right angled triangle, we can use trigonometry to find the angles.

In the right angled triangle shown, $\sin\theta = \frac{3}{5}$.

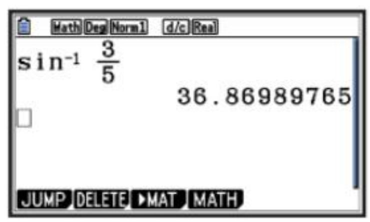
We say that θ is the **inverse sine** of $\frac{3}{5}$, and write $\theta = \sin^{-1}(\frac{3}{5})$.



We can use a calculator to evaluate inverse sines. Click on the icon for instructions.

For the right angled triangle above, we find $\theta \approx 36.9°$.

GRAPHICS CALCULATOR INSTRUCTIONS

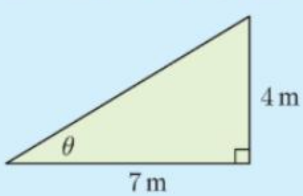


We define **inverse cosine** and **inverse tangent** in a similar way.

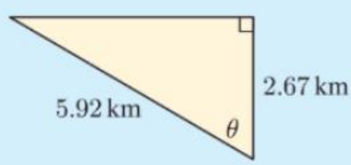
Example 2 — Self Tutor

Find the measure of the angle marked θ:

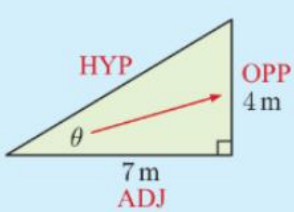
a



b



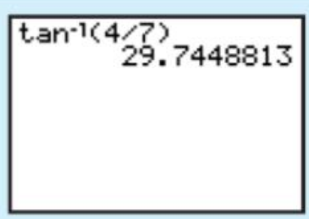
a



$\tan\theta = \frac{4}{7}$ $\quad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \theta = \tan^{-1}(\frac{4}{7})$

$\therefore\ \theta \approx 29.7°$ $\quad$ {calculator}



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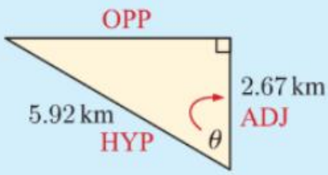
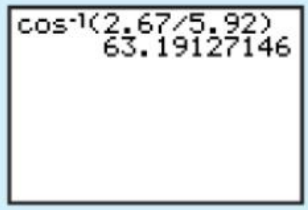


$\cos\theta = \dfrac{2.67}{5.92}$ $\quad \left\{\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}}\right\}$

$\therefore \ \theta = \cos^{-1}\left(\dfrac{2.67}{5.92}\right)$

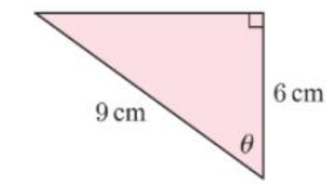
$\therefore \ \theta \approx 63.2°$ $\quad$ {calculator}



EXERCISE 13A.3

1 Find the measure of the angle marked θ:

a



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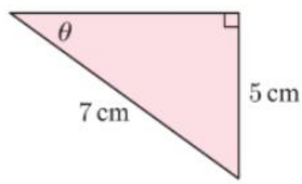

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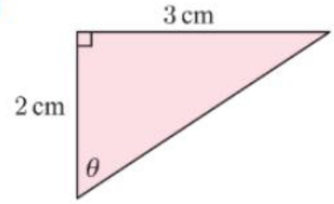

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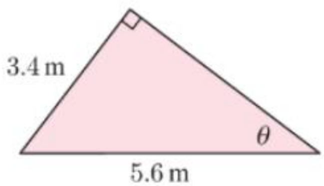

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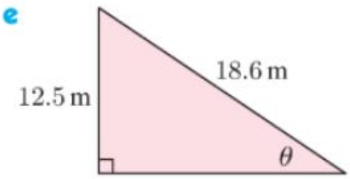

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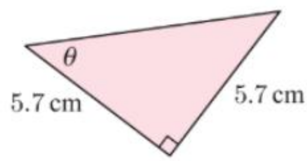

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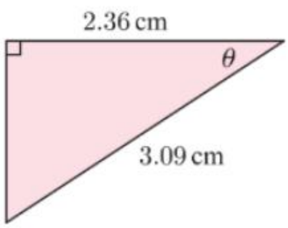

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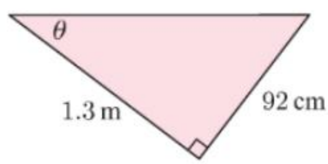
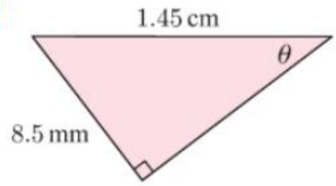
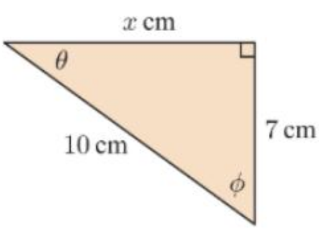


i



2 Find, using trigonometry, all the unknown angles and sides in the following triangles. Check your answers for x using Pythagoras' theorem.

a



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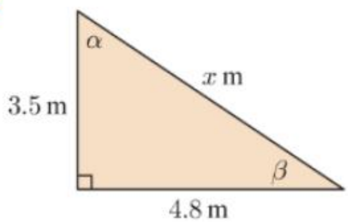

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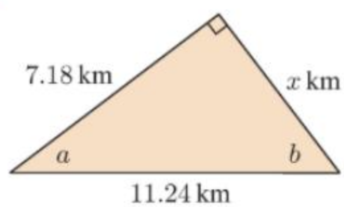
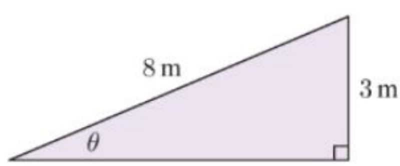


3 Consider the triangle alongside.



a Copy and complete, stating exact values:

i $\sin\theta = ...$ **ii** $\cos\theta = ...$

iii $\tan\theta = ...$

b Use each of these equations to find θ. Check that you get the same answer each time.

B PROBLEM SOLVING USING TRIGONOMETRY

The trigonometric ratios can be used to solve a wide variety of problems involving right angled triangles. When solving these problems it is important to follow the steps below:

Step 1: Draw a **diagram** to illustrate the situation.

Step 2: Mark on the diagram the **unknown** angle or side that needs to be calculated. We often use x for a length and θ for an angle.

Step 3: Locate a **right angled triangle** in your diagram.

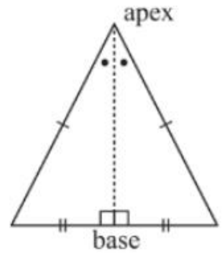
Step 4: Write an **equation** connecting an angle and two sides of the triangle using an appropriate trigonometric ratio.

Step 5: **Solve** the equation to find the unknown.

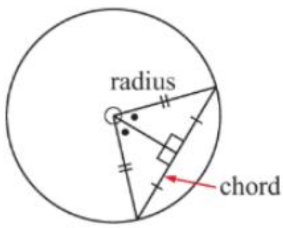
Step 6: **Write** your answer in sentence form.

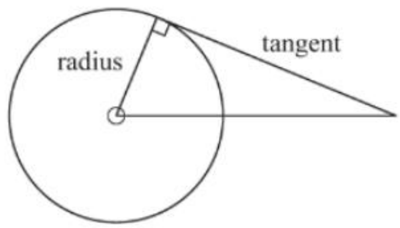
When solving problems using trigonometry, we often use:

- the properties of isosceles triangles



- the properties of circles and tangents



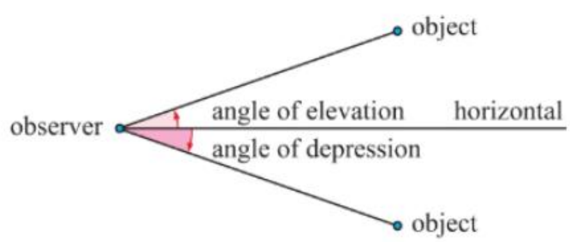


- angles of elevation and depression.

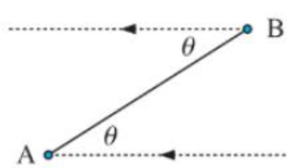
ANGLES OF ELEVATION AND DEPRESSION

When an object is **higher** than an observer, the **angle of elevation** is the angle from the horizontal **up** to the object.

When an object is **lower** than an observer, the **angle of depression** is the angle from the horizontal **down** to the object.

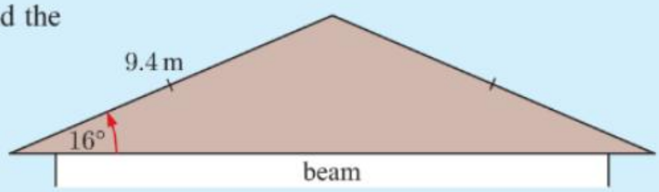


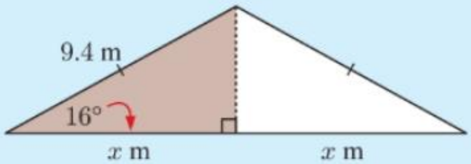
If the angle of elevation from A to B is θ, then the angle of depression from B to A is also θ.



Example 3 **Self Tutor**

The roof alongside has a pitch of 16°. Find the length of the horizontal beam.





$\cos 16° = \dfrac{x}{9.4}$ $\quad\left\{\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}}\right\}$

$\therefore\ x = 9.4 \times \cos 16°$

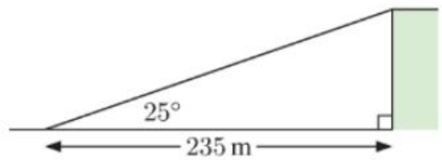
$\therefore\ x \approx 9.036$ $\quad$ {calculator}

$\therefore$ the length of the beam $\approx 2 \times 9.036$ m

≈ 18.1 m

EXERCISE 13B

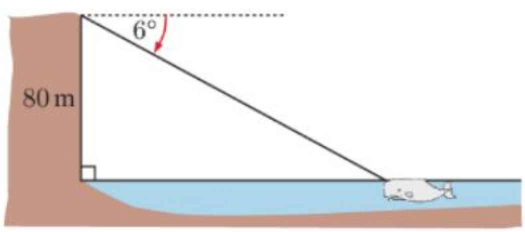
1 From a point 235 m from the base of a cliff, the angle of elevation to the cliff top is 25°. Find the height of the cliff.



2 A 5 m ladder reaches 4.2 m up a wall. What angle does the ladder make with the wall?

3 The angle of elevation from a row boat to the top of a lighthouse 25 m above sea-level is 6°. Calculate the horizontal distance from the boat to the lighthouse.

4

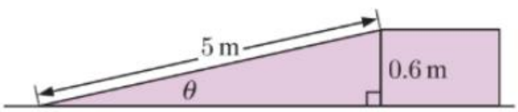


From a vertical cliff 80 m above sea level, a whale is observed at an angle of depression of 6°.

Find the distance between the observer at the top of the cliff, and the whale.

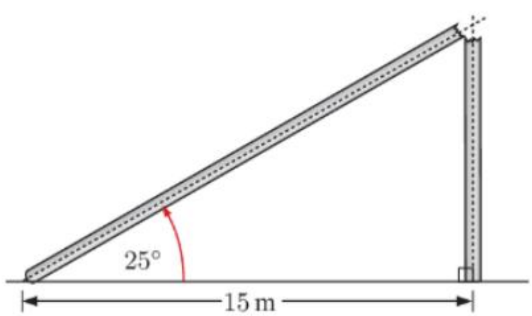
5 A train travelling up an incline of 4° travels a horizontal distance of 4 km. How much altitude has the train gained?

6



At the entrance to a building there is a ramp for wheelchair access. The length of the ramp is 5 metres, and it rises to a height of 0.6 metres. Find the angle θ that the ramp makes with the ground.

7 A goal post was hit by lightning and snapped in two. The top of the post is now resting 15 m from its base, at an angle of 25°. Find the height of the goal post before it snapped.



Example 4 **Self Tutor**

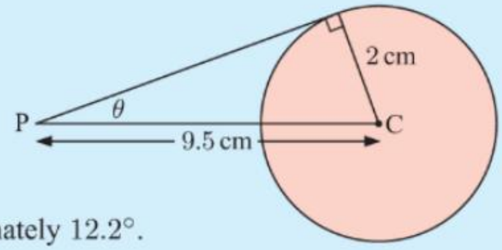
A point P is 9.5 cm from the centre C of a circle with radius 2 cm.

Find the angle between [PC] and the tangent to the circle from P.

$\sin\theta = \frac{2}{9.5}$ $\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

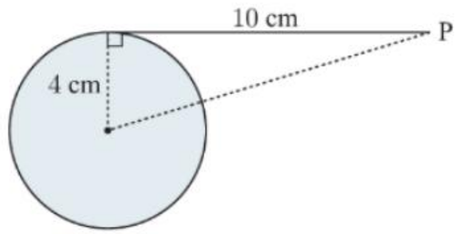
$\therefore \theta = \sin^{-1}\left(\frac{2}{9.5}\right)$

$\therefore \theta \approx 12.2°$ {calculator}



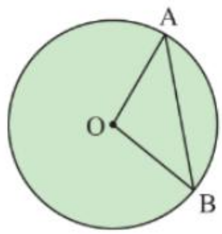
The angle between [PC] and the tangent is approximately 12.2°.

8 A tangent from point P to a circle of radius 4 cm is 10 cm long. Find the angle between the tangent and the line joining P to the centre of the circle.

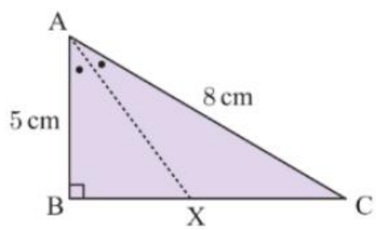


9 A rhombus has sides of length 10 cm, and the angle between two adjacent sides is 76°. Find the length of the longer diagonal of the rhombus.

10 [AB] is a chord of a circle with centre O and radius 5 cm. [AB] has length 8 cm. Find the size of $A\widehat{O}B$.



11



In triangle ABC, the angle bisector at A meets [BC] at X. Is X the midpoint of [BC]? If not, what is the distance between X and the midpoint?

12 In an isosceles triangle, the equal sides are $\frac{2}{3}$ of the length of the base. Determine the measure of the base angles.

13 An isosceles triangle is drawn with base angles 24° and base 28 cm. Find the base angles of the isosceles triangle with the same base length but with treble the area.

14 The angle of elevation from a marker on level ground to the top of a building 100 m high is 22°. Find the distance:

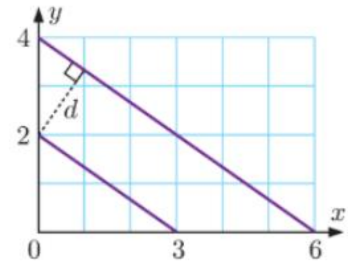
- **a** from the marker to the base of the building
- **b** the marker must be moved towards the building so that the angle of elevation becomes 40°.

15 An observer notices an aeroplane flying directly overhead. Two minutes later the aeroplane is at an angle of elevation of 27°. Assuming the aeroplane is travelling with constant speed and altitude, what will be its angle of elevation after another two minutes?

16 A surveyor standing on a horizontal plain can see a volcano in the distance. The angle of elevation to the top of the volcano is $23°$. If the surveyor moves 750 m closer, the angle of elevation is now $37°$. Determine the height of the volcano above the plain.

17 Find the shortest distance d between the two parallel lines using:

a trigonometry **b** areas.

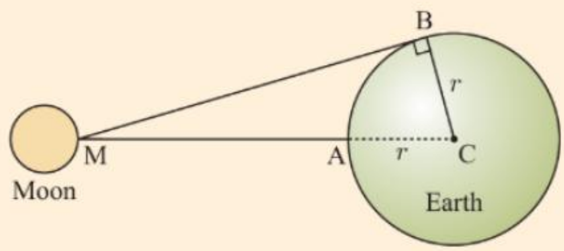


INVESTIGATION 2 HIPPARCHUS AND THE UNIVERSE

Hipparchus was a Greek astronomer and mathematician born in Nicaea in the 2nd century BC. He is considered among the greatest astronomers of all time.

Part 1: How Hipparchus measured the distance to the moon

Consider two points A and B on the Earth's equator. The moon is directly overhead A. From B the moon is just visible, since [MB] is a tangent to the Earth and is therefore perpendicular to [BC]. Angle BCM is the difference in longitude between A and B, which Hipparchus calculated to be approximately $89°$.



What to do:

1 Given $r = 6378$ km and $B\widehat{C}M = 89°$, estimate the distance from the centre of the Earth C to the moon.

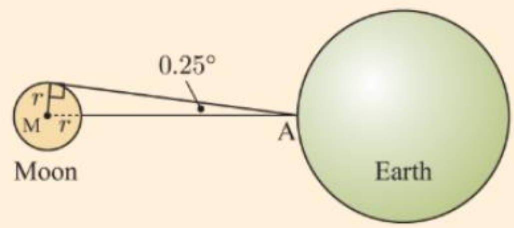
2 Now calculate the distance AM *between* the Earth and the moon.

3 In calculating just one distance between the Earth and the moon, Hipparchus was assuming that the orbit of the moon was circular. In fact it is not. Research the shortest and longest distances to the moon. How were these distances determined? How do they compare with the distance obtained using Hipparchus' method?

Part 2: How Hipparchus measured the radius of the moon

From point A on the Earth's surface, the angle between an imaginary line to the centre of the moon and a tangent to the moon is about $0.25°$.

The average distance from the Earth to the moon is about 384 403 km.



What to do:

1 Confirm from the diagram that $\sin 0.25° = \dfrac{r}{r + 384\,403}$.

2 Solve this equation to find r, the radius of the moon.

3 Research the actual radius of the moon, and if possible find out how it was calculated. Compare your answer in **2** to the actual radius.

C TRUE BEARINGS

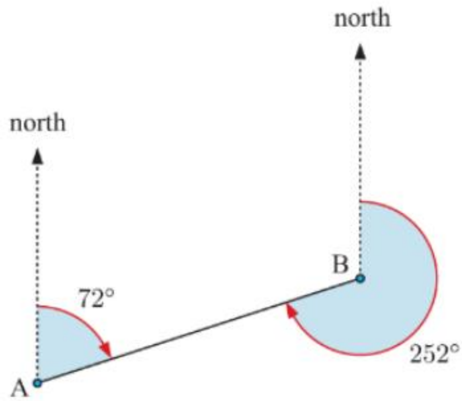
We can describe a direction by comparing it with the **true north direction**. We call this a **true bearing**.

Imagine you are standing at point A, facing north. You turn **clockwise** through an angle until you face B. The **bearing of B from A** is the angle through which you have turned.

So, the bearing of B from A is the clockwise measure of the angle between the 'north' line through A, and [AB].

In the diagram alongside, the bearing of B from A is 72° from true north. We write this as 72°T or 072°.

To find the **bearing of A from B**, we place ourselves at point B, face north, then turn clockwise until we face A. The true bearing of A from B is 252°.



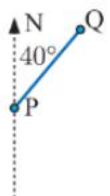
Note:
- A true bearing is always written using three digits. For example, we write 072° rather than 72°.
- The bearing of A from B, and the bearing of B from A, always differ by 180°. You should be able to explain this using angle pair properties for parallel lines.

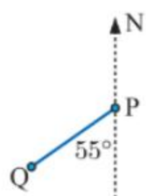
EXERCISE 13C

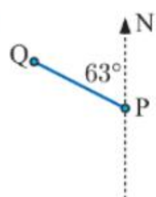
1 Draw diagrams to represent bearings from O of:

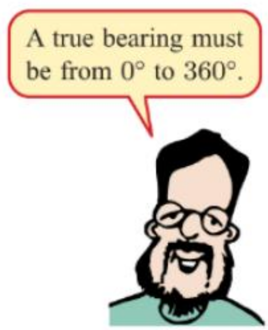
a 136° **b** 240° **c** 051° **d** 327°

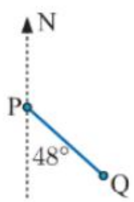
2 Write the bearing of Q from P in each diagram:

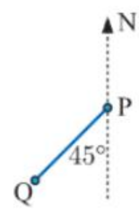
a


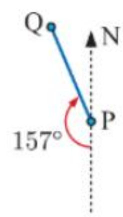
b


c




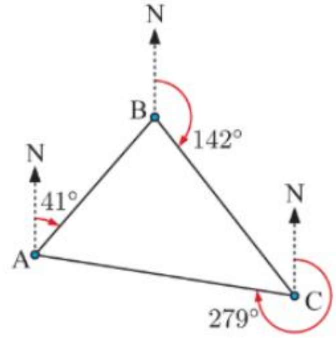
d


e


f


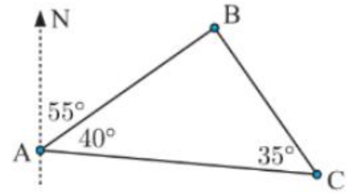
3 A, B, and C are the checkpoints of a triangular orienteering course. Find the bearing of:

a B from A **b** C from B **c** A from C
d B from C **e** A from B **f** C from A.



4 Find the bearing of:

- **a** B from A
- **b** A from B
- **c** C from A
- **d** A from C
- **e** C from B
- **f** B from C.



Example 5 — Self Tutor

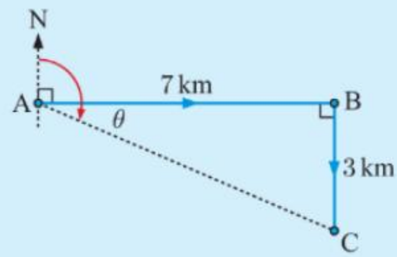
Starting from A, a motorbike travels 7 km east to B, then 3 km south to C.
Find the bearing of C from A.

$\tan\theta = \frac{3}{7}$ $\qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \theta = \tan^{-1}(\frac{3}{7})$

$\therefore\ \theta \approx 23.2°$

So, the bearing of C from A $\approx 90° + 23.2°$
$\approx 113°$



5 A bush-walker walks 14 km east and then 9 km north. Find the bearing of his finishing point from his starting point.

6 Starting from A, a truck travels 10 km north to B, then 13 km west to C. Find the bearing of C from A.

7 A kayaker paddles due west for 1.5 km. He then turns due south and paddles a further 800 m.

- **a** Draw a diagram of the situation.
- **b** How far is the kayaker from his starting point?
- **c** In what direction must he travel to return to his starting point?

8 Runners A and B leave the same point at the same time. Runner A runs at 10 km h^{-1} due north. Runner B runs at 12 km h^{-1} due east. Find the distance and bearing of runner B from runner A after 30 minutes.

Example 6 — Self Tutor

A rally driver travels on a bearing of 145° for 28.5 km.
How far east of the starting position is the rally driver?

$\cos 55° = \frac{x}{28.5}$ $\qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ x = 28.5 \times \cos 55°$

$\therefore\ x \approx 16.3$

The driver is about 16.3 km east of his starting position.

9 A ship sails for 60 km on a bearing 040°. How far north of its starting point is the ship?

10 An athlete ran for $2\frac{1}{2}$ hours in the direction 164° at a speed of 14 km/h.

a Draw a fully labelled diagram of the situation.

b Find the distance travelled by the athlete.

c How far **i** east **ii** south of the starting point is the athlete?

11 A hiker walks in the direction 215° and stops when she is 2 km south of her starting point. How far did she walk?

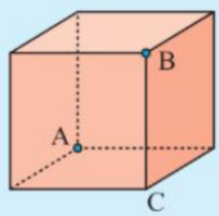
12 An aeroplane travels on a bearing of 295° until it is 200 km west of its starting point. How far has it travelled on this bearing?

D 3-DIMENSIONAL PROBLEM SOLVING

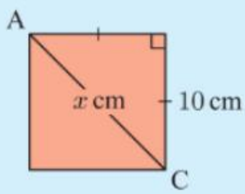
Right angled triangles occur frequently in 3-dimensional figures. We can use Pythagoras' theorem and trigonometry to find unknown angles and lengths.

Example 7 **Self Tutor**

A cube has sides of length 10 cm. Find the angle between the diagonal [AB] and the edge [BC].



Let AC $= x$ cm.

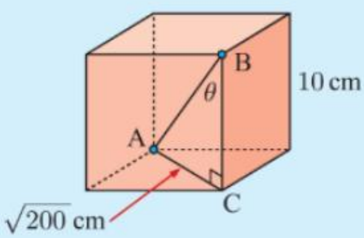


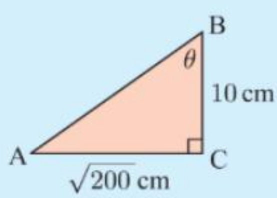
Using Pythagoras, $x^2 = 10^2 + 10^2$

$\therefore\ x^2 = 200$

$\therefore\ x = \sqrt{200}$

The required angle is A$\widehat{\text{B}}$C. We let this angle be θ.





$\tan\theta = \frac{\sqrt{200}}{10}$ $\quad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \theta = \tan^{-1}\left(\frac{\sqrt{200}}{10}\right)$

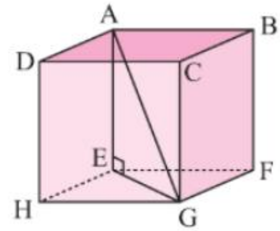
$\therefore\ \theta \approx 54.7°$

The angle between the diagonal [AB] and the edge [BC] is about 54.7°.

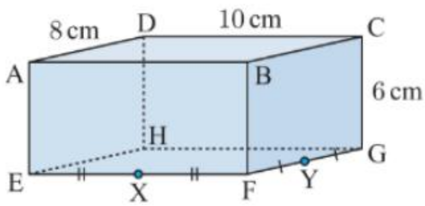
EXERCISE 13D

1 The figure alongside is a cube with sides of length 15 cm. Find:

a EG **b** $A\widehat{G}E$.



2 The figure alongside is a rectangular prism. X and Y are the midpoints of [EF] and [FG] respectively. Find:

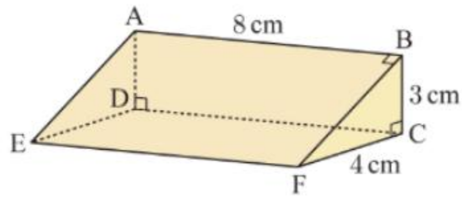


a HX **b** $D\widehat{X}H$

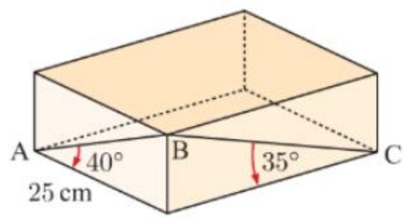
c HY **d** $D\widehat{Y}H$.

3 In this triangular prism, find:

a DF **b** $A\widehat{F}D$.



4 [AB] and [BC] are wooden support struts on a crate. Find the total length of wood required to make the two struts.

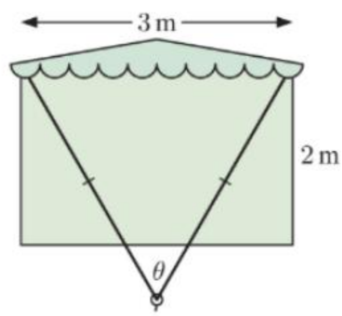


5 All edges of a square-based pyramid are 12 m in length.

a Find the angle between a slant edge and a base diagonal.

b Show that this angle is the same for any square-based pyramid with all edge lengths equal.

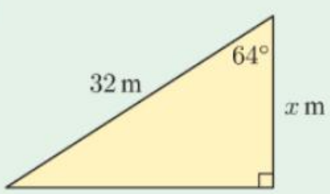
6 Each side of a tent is fixed to the ground by ropes as shown. The peg is 1.5 m from the side of the tent. Find the angle θ between the ropes.



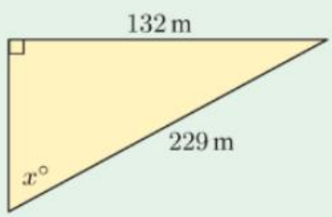
REVIEW SET 13A

1 Find the value of x:

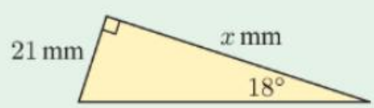
a



b

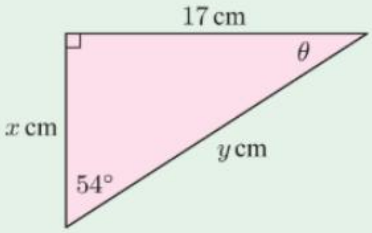


c

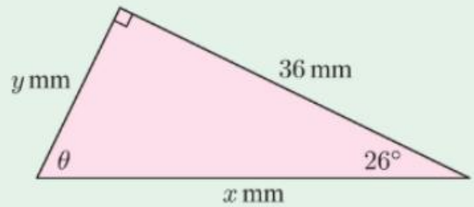


2 Find the measure of all unknown sides and angles in:

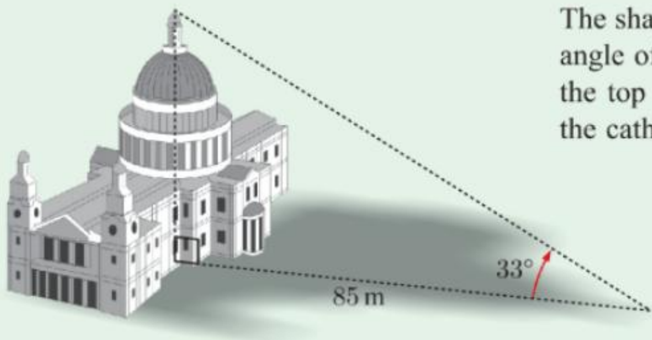
a



b

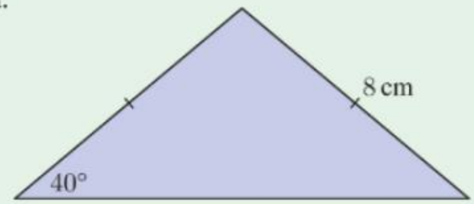


3



The shadow of a cathedral is 85 m in length. The angle of elevation from the end of the shadow to the top of the steeple is 33°. Find the height of the cathedral.

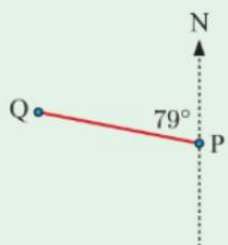
4 An isosceles triangle has the measurements shown. Find the area of the triangle.



5 Find the bearing of Q from P in the following diagrams:

a

b

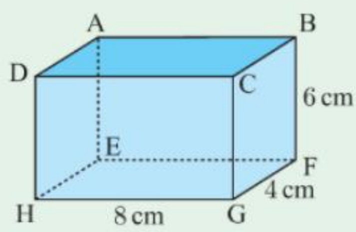


6 A taxi travels 8 km south, then 3 km west. Find the bearing of the taxi's finishing point from its starting point.

7 A 2.3 m long ladder is resting against a wall. It makes an angle of 82° with the ground. How high up the wall does the ladder reach?

8 Find the measure of:

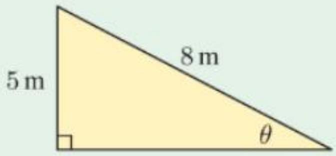
a $B\widehat{G}F$ **b** $A\widehat{G}E$.



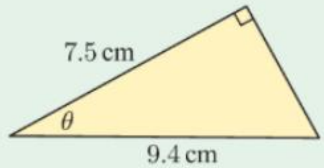
REVIEW SET 13B

1 Find the measure of the angle marked θ:

a

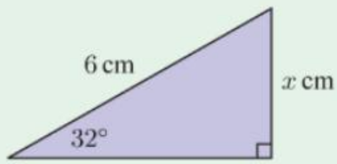


b

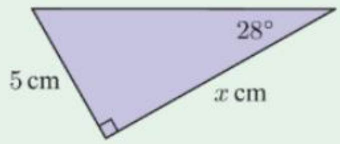


2 Find the value of x, rounding your answer to 2 decimal places:

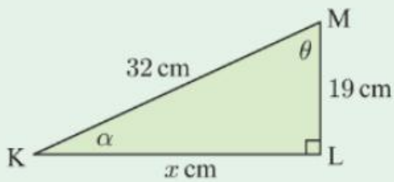
a



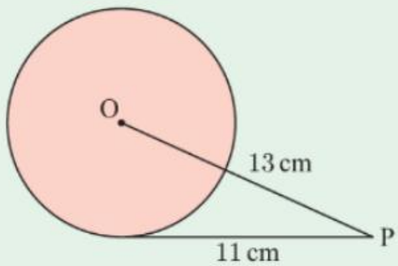
b



3 Find the measure of all unknown sides and angles in triangle KLM.



4 Point P is 13 cm from the centre of a circle. The tangent from P to the circle is 11 cm long. Find the angle between the tangent and the line from P to the centre of the circle.



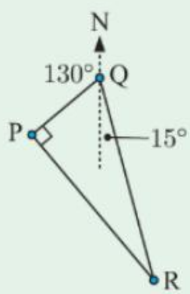
5 An aeroplane takes off at an angle of $22°$ to the horizontal runway. At the time when it has flown 500 m, what is the altitude of the plane? Give your answer correct to the nearest metre.

6 Find the bearing of:

a P from Q

b R from Q

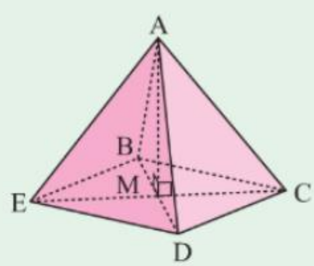
c R from P.



7 The edges of a square-based pyramid are all 20 cm long. Find:

a $\widehat{ADM}$

b $\widehat{ACD}$.



8 A cruiseboat leaves port at 10 am. It sails at 42 km/h on the bearing $152°$.

  a At noon, how far east of the port is the boat?

  b The boat's destination is 105 km south of the port, on the bearing $152°$. At what time will the boat reach its destination?

Chapter 14

Probability

Contents:

OPENING PROBLEM

When Karla dropped some metal nuts she noticed that they finished either on their ends or on their sides. She was interested to know how likely it was that a nut would finish on its end. So, she tossed a nut 200 times, and found that it finished on its end 137 times.

Her friend Sam repeated the experiment, and the nut finished on its end 145 times.

Things to think about:

a What would Karla's best estimate be of the chance or probability that the nut will finish on its end?

b What would Sam's estimate be?

c How can we obtain a more accurate estimate of the chance of the nut finishing on its end?

d Hilda said that the best estimate would be obtained when the nut is tossed thousands of times. Do you agree with Hilda?

Consider these statements:

"The Wildcats will probably beat the Tigers on Saturday."
"It is unlikely that it will rain today."
"I will probably make the team."

Each of these statements indicates a **likelihood** or **chance** of a particular event happening.

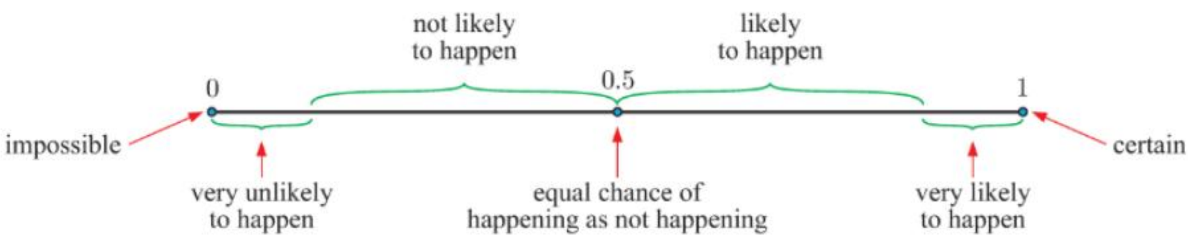
For every event we can assign a number to indicate the chance that event will happen. We call this a **probability**.

An **impossible** event has 0% chance of happening, and is assigned the probability 0.

A **certain** event has 100% chance of happening, and is assigned the probability 1.

All other events can be assigned a probability between 0 and 1.

A number close to 0% indicates the event is **very unlikely** to occur, whereas a number close to 100% means that it is **very likely** to occur.



We can determine probabilities by either:

- observing the results of an experiment (experimental probability), or
- using arguments of symmetry (theoretical probability).

HISTORICAL NOTE

- **Girolamo Cardano** (1501 to 1576) admitted in his autobiography that he gambled "not only every year, but .... every day, and with the loss at once of thought, of substance, and of time". He wrote a handbook on gambling, with tips on cheating and how to detect it. His book included discussions on equally likely events, frequency tables for dice probabilities, and expectations.
- **Pierre-Simon Laplace** (1749 - 1827) once described the theory of probability as "nothing but common sense reduced to calculation".
- **Blaise Pascal** (1623 - 1662) invented the first mechanical digital calculator. Pascal and his friend **Pierre de Fermat** (1601 - 1665) were the first to develop probability theory as we know it today. Pascal also developed the syringe and the hydraulic press, and he wrote a large number of articles on Christian beliefs and ethics.

Girolamo Cardano

A EXPERIMENTAL PROBABILITY

In experiments involving chance, we use the following terms to describe what we are doing and the results we are obtaining:

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome divided by the total number of trials. It is expressed as either a fraction, a decimal, or a percentage.

For example, suppose a tin can is tossed in the air 250 times, and it comes to rest on an end 29 times. We say:

- the number of trials is 250
- the outcomes are *ends* and *sides*
- the frequency of *ends* is 29, and of *sides* is 221
- the relative frequency of *ends* $= \frac{29}{250} = 0.116$
- the relative frequency of *sides* $= \frac{221}{250} = 0.884$.

In the absence of any further data, the relative frequency of each outcome is our best estimate of its probability.

The **experimental probability** is the **relative frequency** of the outcome.

The larger the number of trials, the more accurate the estimate will be.

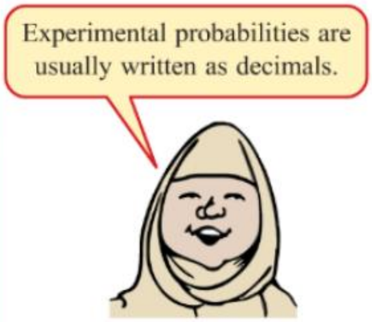
Example 1 **Self Tutor**

Larissa took 43 shots at a netball goal, and scored 29 times. Estimate the probability that Larissa will miss with her next shot.

From 43 shots, Larissa scored 29 times and missed 14 times.

$\therefore$ P(misses next shot) $\approx \frac{14}{43}$

≈ 0.326



EXERCISE 14A

1. Clem fired 200 arrows at a target and hit the target 168 times. Estimate the probability that Clem will hit the target with his next shot.
2. Ivy has free-range hens. Out of the first 123 eggs that they laid, she found that 11 had double-yolks. Estimate the probability that the next egg laid will have a double-yolk.
3. Jackson leaves for work at the same time each day. Over a period of 227 working days, he had to wait for a train at the railway crossing on 58 days. Estimate the probability that Jackson will not have to wait for a train tomorrow.
4. Ravi has a circular spinner marked P, Q, and R. When the spinner was twirled 417 times, it finished on P 138 times, and on Q 107 times. Estimate the probability that the next spin will finish on R.
5. Answer the **Opening Problem** on page **264**.

B PROBABILITIES FROM TABLED DATA

If we are given a table of frequencies then we can use **relative frequencies** to estimate the probability of each event.

$$\textbf{relative frequency} = \frac{\textbf{frequency}}{\textbf{number of trials}}$$

Example 2 **Self Tutor**

A marketing company surveys 80 people to discover what brand of shoe cleaner they use. The results are shown in the table alongside.

Estimate the probability that a randomly selected community member uses:

a Brite **b** Cleano or No scuff.

Brand	*Frequency*
Shine	27
Brite	22
Cleano	20
No scuff	11

a P(Brite) $\approx \frac{22}{80} \approx 0.275$

b P(Cleano or No scuff) $\approx \dfrac{20+11}{80} \approx 0.388$

PROBABILITIES FROM TWO-WAY TABLES

Two-way tables are tables comparing two variables. They usually result from a survey.

For example, the Year 10 students in a small school were asked whether they were good at mathematics. The results are summarised in the following two-way table:

	Boy	Girl
Good at mathematics	17	19
Not good at mathematics	8	9

← 9 girls said they were not good at mathematics.

In this case the variables are *ability in mathematics* and *gender*.

We can use two-way tables to estimate probabilities.

Example 3 — Self Tutor

To investigate the breakfast habits of teenagers, a survey was conducted amongst some students from a high school. The results are shown alongside.

	Male	Female
Regularly eats breakfast	87	53
Does not regularly eat breakfast	68	92

Use this table to estimate the probability that a randomly selected student from the school:

a is male **b** is male *and* regularly eats breakfast

c is female *or* regularly eats breakfast.

We extend the table to include totals:

	Male	Female	*Total*
Regularly eats breakfast	87	53	140
Does not regularly eat breakfast	68	92	160
Total	155	145	300

a There are 155 males amongst the 300 students surveyed.

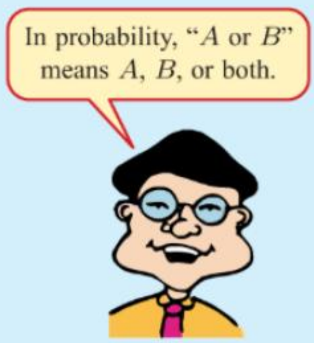
$\therefore$ P(male) $\approx \frac{155}{300} \approx 0.517$

b 87 of the 300 students are male and regularly eat breakfast.

$\therefore$ P(male *and* regularly eats breakfast) $\approx \frac{87}{300} \approx 0.29$

c $53 + 92 + 87 = 232$ out of the 300 are female or regularly eat breakfast.

$\therefore$ P(female *or* regularly eats breakfast) $\approx \frac{232}{300} \approx 0.773$



EXERCISE 14B

1 A marketing company surveyed people to discover which brand of soap they use. The results of the survey are given alongside.

a How many people were surveyed?

b Estimate the probability that a randomly selected person uses:

i Just Soap **ii** Indulgence or Silktouch.

Brand	*Frequency*
Silktouch	125
Super	107
Just Soap	93
Indulgence	82

2 This table shows the flavour of ice creams sold in a café one afternoon.

Flavour	*Frequency*
Chocolate	21
Strawberry	17
Lemon	4
Vanilla	15

a How many ice creams were sold?

b Estimate the probability that the next ice cream sold will be:

i strawberry **ii** chocolate or vanilla.

3

Councillor	*Frequency*
Mr Tony Trimboli	216
Mrs Andrea Sims	72
Mrs Sara Chong	238
Mr John Henry	
Total	

Results from a poll for a local Council election are shown in the table. It is known that 600 people were surveyed in the poll.

a Copy and complete the table.

b Estimate the probability that a randomly selected person from this electorate will vote for:

i John Henry **ii** a female councillor.

4 310 students at a high school in South Africa were surveyed on the question "Do you like watching rugby on TV?". The results are shown in the two-way table.

	Like	Dislike
Junior students	87	38
Senior students	129	56

a Copy and complete the table to include 'totals'.

b Estimate the probability that a randomly selected student:

i likes watching rugby on TV and is a junior student

ii likes watching rugby on TV and is a senior student

iii dislikes watching rugby on TV.

c Find the total of the probabilities found in **b**. Explain your answer.

5 A random selection of students in a youth club were asked whether they wore glasses, contact lenses, or neither. The results were further categorised by gender.

	Glasses	Contact lenses	Neither
Male	15	6	26
Female	14	8	31

a How many students were surveyed?

b Estimate the probability that a randomly chosen student in the club:

i wears glasses **ii** is female and wears contact lenses

iii is male and wears neither **iv** is female or wears glasses.

6 The table alongside describes the types of cars advertised for sale in a newspaper.

Estimate the probability that a randomly selected car for sale:

	Manual	Automatic
Hatchback	26	27
Sedan	30	39
4WD	9	16

a is a sedan **b** is a manual hatchback

c is automatic, but not a sedan.

7 A random selection of hotels in Paris are given a gold star rating for quality, and a green star rating for environmental friendliness.
Estimate the probability that a randomly selected Paris hotel is given:

a 2 gold stars and 4 green stars
b 3 gold stars or higher
c the same number of gold stars as green stars
d more green stars than gold stars.

		Green star			
		★	★★	★★★	★★★★
Gold star	☆	5	2	1	1
	☆☆	4	10	4	3
	☆☆☆	2	8	13	8
	☆☆☆☆	1	7	9	4

8 The table alongside gives the age distribution of inmates in a prison on December 31, 2011.
A new prisoner entered the prison on January 1, 2012.
Estimate the probability that:

a the prisoner was male
b the prisoner was aged from 17 to 19
c the prisoner was 19 or under and was female
d the prisoner was aged from 30 to 49 and was male.

Age distribution of prison inmates			
Age	*Female*	*Male*	*Total*
15	0	6	6
16	5	23	28
17 - 19	26	422	448
20 - 24	41	1124	1165
25 - 29	36	1001	1037
30 - 34	32	751	783
35 - 39	31	520	551
40 - 49	24	593	617
50 - 59	16	234	250
60+	5	148	153
Total	216	4822	5038

Global context

click here

How much time do we have?

Statement of inquiry: Collecting and interpreting data can help us to understand our place in the world.

Global context: Identities and relationships

Key concept: Relationships

Related concepts: Change, Representation

Objectives: Knowing and understanding, Applying mathematics in real-life contexts

Approaches to learning: Communication, Self-management

C SAMPLE SPACE

A **sample space** is the set of all possible outcomes of an experiment.

Possible ways of representing sample spaces are:

- listing them
- using a 2-dimensional grid
- using a tree diagram
- using a Venn diagram.

Example 4 — Self Tutor

When two coins are tossed, the possible outcomes are:

two heads | head and tail | tail and head | two tails

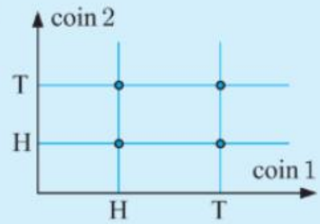
Represent the sample space for tossing two coins using:

a a list **b** a 2-D grid **c** a tree diagram.

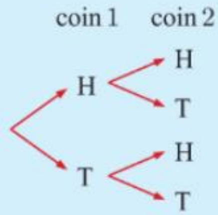
We let H represent a 'head' and T represent a 'tail'.

a {HH, HT, TH, TT}

b



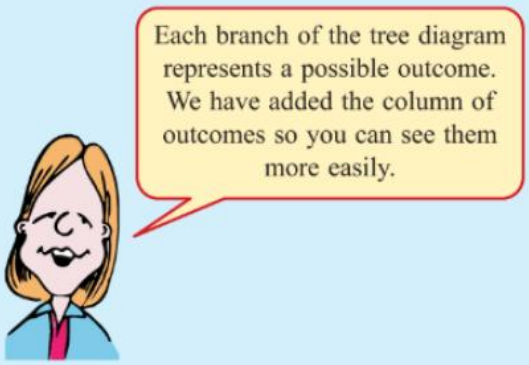
c

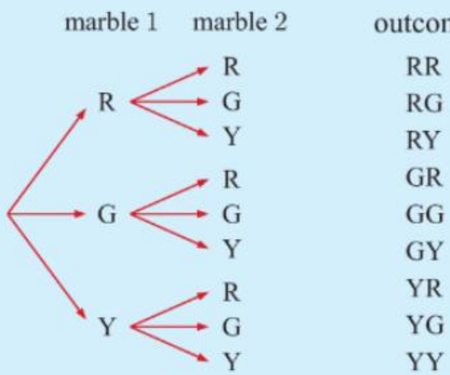


Example 5 — Self Tutor

Illustrate, using a tree diagram, the possible outcomes when drawing two marbles from a bag containing red, green, and yellow marbles.

Let R be the event of getting a red,
G be the event of getting a green, and
Y be the event of getting a yellow.





EXERCISE 14C

1 List the sample space for the following:

a twirling a square spinner labelled A, B, C, D

b the genders of a 2-child family

c the order in which 4 blocks A, B, C, and D can be lined up

d the 8 different 3-child families

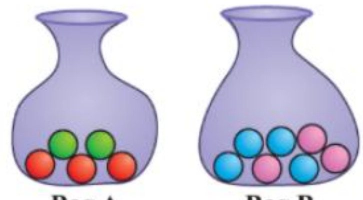
e tossing a coin **i** twice **ii** three times **iii** four times.

2 Use a 2-dimensional grid to illustrate the sample space for:

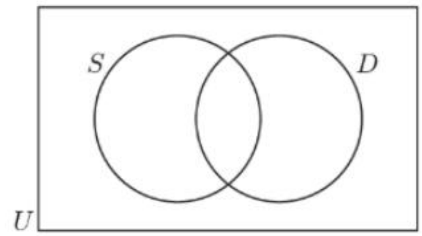
- **a** rolling a die and tossing a coin simultaneously
- **b** rolling two dice
- **c** rolling a die and spinning a spinner with sides A, B, C, D
- **d** twirling two square spinners, one labelled A, B, C, D, and the other 1, 2, 3, 4.

3 Illustrate on a tree diagram the sample space for:

- **a** tossing a 5-cent and 10-cent coin simultaneously
- **b** tossing a coin and twirling an equilateral triangular spinner labelled A, B, and C
- **c** twirling two equilateral triangular spinners labelled 1, 2, 3, and X, Y, Z respectively
- **d** drawing two tickets from a hat containing pink, blue, and white tickets
- **e** selecting bag A or bag B, then drawing a ball from that bag.



4 There are 48 students in the school production. 26 have singing parts, 23 have dance parts, and 10 have neither. Illustrate this information on a Venn diagram like the one given.



5 Draw a Venn diagram to show a class of 20 students in which 7 study History and Geography, 10 study History, and 15 study Geography.

D THEORETICAL PROBABILITY

Once we have represented the sample space of an experiment, we can use it to calculate probabilities.

If a sample space has n outcomes which are **equally likely** to occur when the experiment is performed once, then each outcome has probability $\frac{1}{n}$ of occurring.

EVENTS

An **event** occurs when we obtain an outcome with a particular property or feature.

For example, suppose two coins are tossed simultaneously. The possible outcomes are {HH, HT, TH, TT}.

The event *exactly one head* is satisfied by the two outcomes HT and TH.

So, the probability of the event is $\frac{2}{4} = \frac{1}{2}$.

When the outcomes of an experiment are equally likely, the probability of an event E occurring is given by:

$$\mathbf{P}(E) = \frac{\textbf{number of outcomes corresponding to } E}{\textbf{number of outcomes in the sample space}}$$

Example 6 Self Tutor

Suppose three coins are tossed simultaneously. Find the probability of getting:

a three heads **b** at least one head.

The possible outcomes are:

{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}.

There are 8 possible outcomes.

a $\text{P(three heads)} = \frac{1}{8}$ ← three heads only occurs in the outcome HHH; ← 8 possible outcomes

b $\text{P(at least one head)} = \frac{7}{8}$ ← all outcomes except TTT contain at least one head; ← 8 possible outcomes

The **complement** of event E is the event that E does *not* occur. We denote the complement E'.

For any event E, $\text{P}(E') = 1 - \text{P}(E)$.

EXERCISE 14D

1 The three letters O, D, and G are placed at random in a row. Find the probability of:

a spelling DOG **b** O appearing first

c O not appearing first **d** spelling DOG or GOD.

2 Determine the probability that a randomly selected 3-child family consists of:

a all boys **b** all girls **c** boy, then girl, then girl

d two girls and a boy **e** a girl for the eldest **f** at least one boy.

3 A bag contains 8 tokens of the same shape. 5 tokens are green, 2 are red, and 1 is purple. One token is randomly selected from the bag. Let E be the event that the token is red.

a Explain the meaning of the event E'.

b Find $\text{P}(E)$.

c Find $\text{P}(E')$.

4 Four friends Alex, Bodi, Claire, and Daniel sit randomly in a row. Determine the probability that:

a Alex is on one of the ends

b Claire and Daniel are on the ends

c Bodi is on an end, and Claire is seated next to him

d Alex and Claire sit next to each other.

Example 7 — Self Tutor

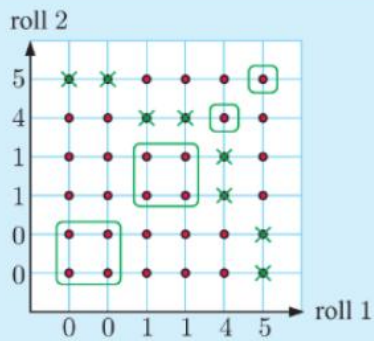
A die has the numbers 0, 0, 1, 1, 4, and 5. It is rolled twice.

a Illustrate the possible outcomes using a 2-dimensional grid.

b Hence find the probability of getting:

i a total of 5 **ii** two numbers which are the same.

a



b There are $6 \times 6 = 36$ possible outcomes.

i P(total of 5)

$= \frac{8}{36}$ {those with a $\times$ }

$= \frac{2}{9}$

ii P(same numbers)

$= \frac{10}{36}$ {those circled}

$= \frac{5}{18}$

5 **a** Draw a grid to illustrate the sample space when a 10-cent and a 50-cent coin are tossed simultaneously.

b Hence, determine the probability of getting:

i two heads **ii** two tails

iii exactly one head **iv** at least one head.

6 A coin and a pentagonal spinner with sectors 1, 2, 3, 4, and 5 are tossed and spun respectively.

a Draw a grid to illustrate the sample space of possible outcomes.

b Use your grid to determine the chance of getting:

i a head and a 4 **ii** a tail and an odd number

iii an even number **iv** a tail or a 3.

7 **a** Use a grid to display the possible outcomes when a pair of dice is rolled.

b Hence, determine the probability of:

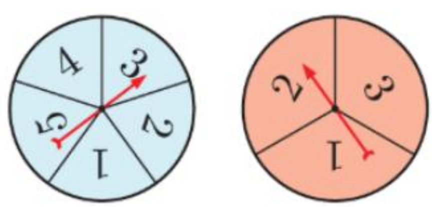
i one die showing a 4 and the other a 5

ii both dice showing the same number

iii at least one of the dice showing a 3 **iv** either a 4 or 6 being displayed

v both dice showing even numbers **vi** the sum of the numbers being 7.

8 The spinners shown are spun once, and the numbers spun are multiplied together.



a Find the probability that the result is:

i 9 **ii** 6

iii greater than 5 **iv** prime.

b Is the result more likely to be even or odd?

Example 8 **Self Tutor**

In a library group of 50 readers, 31 like science fiction, 20 like detective stories, and 12 dislike both.

a Draw a Venn diagram to represent this information.

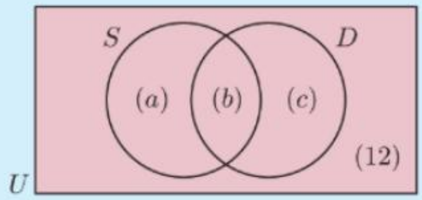
b If a reader is randomly selected, find the probability that he or she:

i likes science fiction and detective stories

ii likes exactly one of science fiction and detective stories.

a Let S represent readers who like science fiction, and D represent readers who like detective stories.

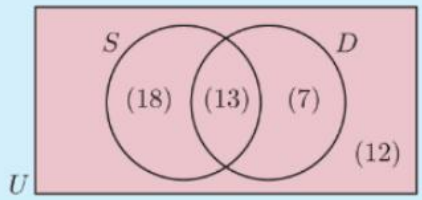
We are given that
$$a+b=31$$
$$b+c=20$$
$$a+b+c=50-12=38$$
$$\therefore\ c=38-31=7$$
$$\therefore\ b=13 \text{ and } a=18$$



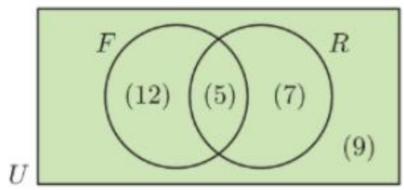
b **i** P(likes both) $= \frac{13}{50}$

ii P(likes exactly one) $= \dfrac{18+7}{50}$

$= \frac{1}{2}$



9 The Venn diagram shows the sports played by the students in a Year 10 class.



$F \equiv$ football
$R \equiv$ rugby

a How many students are in the class?

b A student is chosen at random. Find the probability that the student:

i plays football

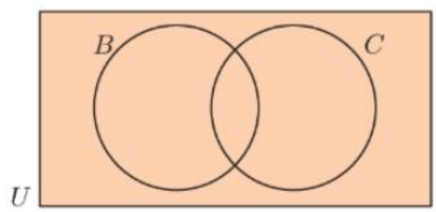
ii plays both codes

iii plays football or rugby

iv plays exactly one of these sports.

10 In a class of 24 students, 10 study Biology, 12 study Chemistry, and 5 study neither Biology nor Chemistry.

a Copy and complete the Venn diagram.



b Find the probability that a student picked at random from the class studies:

i Chemistry, but not Biology

ii both Chemistry and Biology.

11 50 tourists went on a 'thrill seekers' holiday. 40 went white-water rafting, 21 went paragliding, and each tourist did at least one of these activities.

Find the probability that a randomly selected tourist:

a participated in both activities

b went white-water rafting but not paragliding.

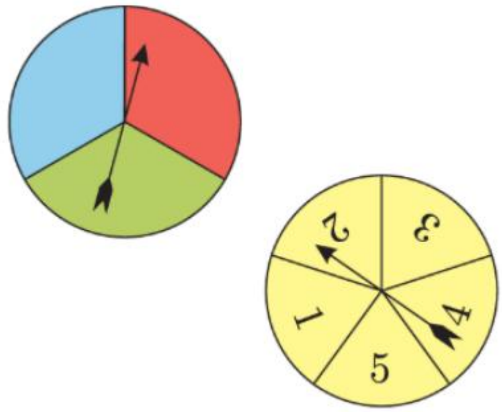
E COMPOUND EVENTS

We will now look at calculating probabilities for **combined events**, which are also called **compound events**. We will consider both **independent events** and **dependent events**.

INDEPENDENT EVENTS

Two events are **independent** if the occurrence of each event does not affect the occurrence of the other.

For example, suppose the spinners alongside are spun simultaneously.



Let A be the event *the first spinner lands on red*, and B be the event *the second spinner lands on an even number*.

Whether A occurs or not does not affect the probability of B occurring. Likewise, whether B occurs or not does not affect the probability of A occurring. Therefore, A and B are independent events.

Now, we know that $\mathrm{P}(A) = \frac{1}{3}$, and $\mathrm{P}(B) = \frac{2}{5}$.

By listing each of the possible outcomes, we can see that $\mathrm{P}(A \text{ and } B) = \frac{2}{15}$.

{R1, **R2**, R3, **R4**, R5, G1, G2, G3, G4, G5, B1, B2, B3, B4, B5}

We notice that $\frac{2}{15} = \frac{1}{3} \times \frac{2}{5}$, so $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

If two events A and B are **independent**, then $\mathrm{P}(A \textbf{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

Example 9 **Self Tutor**

A coin is tossed and a die rolled simultaneously. Find the probability that a tail and a '2' result.

'Getting a tail' and 'rolling a 2' are independent events.

$\therefore$ P(a tail **and** a '2') = P(a tail) × P(a '2')

$$= \frac{1}{2} \times \frac{1}{6}$$

$$= \frac{1}{12}$$

This rule can be extended to any number of independent events.

For example: If A, B, and C are all independent events, then

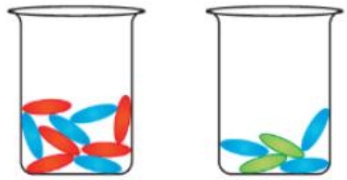
$$\mathrm{P}(A \textbf{ and } B \textbf{ and } C) = \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C).$$

EXERCISE 14E.1

1 A die is rolled, and the spinner alongside is spun. Find the probability of obtaining:

- **a** a '5' and a green
- **b** an even number and a non-red.

2 A disc is selected from each of the containers alongside. Find the probability of selecting:



A B

- **a** a red disc from **A** and a green disc from **B**
- **b** a blue disc from both containers.

3 A school has two photocopiers. On any one day, machine A has an 8% chance of having a paper jam, and machine B has a 12% chance of having a paper jam.
Determine the probability that, on any one day, both machines will:

- **a** have paper jams
- **b** work uninterrupted.

4 A boy and a girl were each asked what day of the week they were born. Find the probability that:

- **a** the boy was born on a Monday and the girl was born on a Wednesday
- **b** the boy was born on a weekend, but the girl was born on a weekday
- **c** both children were born on a weekday.

5 Each day, Steve attempts the easy, medium, and hard crosswords in the newspaper. He has probability 0.84 of completing the easy crossword, probability 0.59 of completing the medium crossword, and probability 0.11 of completing the hard crossword.
Find the probability that, on any given day, Steve will:

- **a** complete all 3 crosswords
- **b** leave all 3 crosswords incomplete
- **c** complete the easy and medium crosswords, but not the hard crossword
- **d** complete the medium crossword, but not the other two crosswords.

6 A drawing pin was tossed into the air 600 times. It landed on its back  243 times and on its side  for the remainder.

- **a** Estimate the probability that, when tossed once, this drawing pin will land on its:
  - **i** back
  - **ii** side.
- **b** Suppose the drawing pin is tossed twice. Estimate the probability that:
  - **i** it lands on its back both times
  - **ii** it lands on its side both times.

7 A biased coin is flipped 200 times. It lands on heads 143 times, and on tails for the remainder. If the coin is flipped 3 times, estimate the probability of getting:

- **a** all heads
- **b** all tails.

Example 10 — Self Tutor

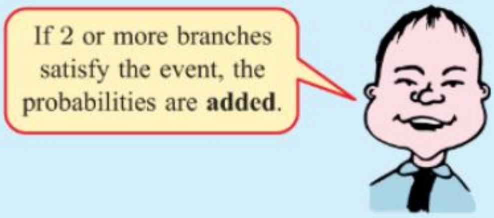
A marble is selected at random from each of the bags alongside.

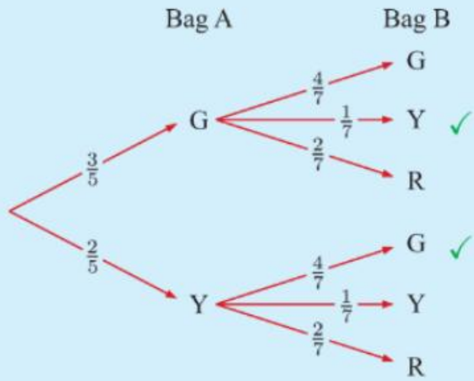
a Draw a tree diagram to display the possible outcomes.

b Find the probability of obtaining a green marble and a yellow marble.

Bag A **Bag B**

a Let G represent a green marble,
Y represent a yellow marble, and
R represent a red marble.

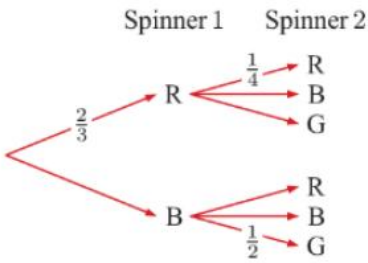


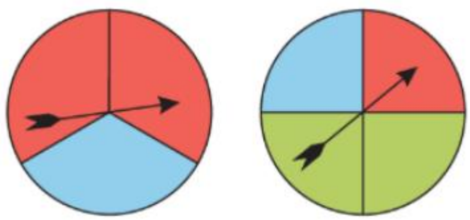


b P(a green and a yellow)
$= P(GY \text{ or } YG)$ {branches marked ✓}
$= \frac{3}{5} \times \frac{1}{7} + \frac{2}{5} \times \frac{4}{7}$
$= \frac{11}{35}$

8 Each of the spinners alongside is spun once.

a Copy and complete this tree diagram for the possible outcomes:





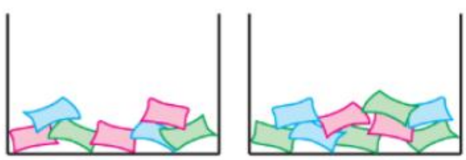
Spinner 1 **Spinner 2**

b Find the probability that:

i the spinners land on the same colour **ii** the spinners land on different colours

iii exactly one of the spinners lands on blue.

9 A ticket is selected at random from each of the boxes alongside.



Box A **Box B**

a Draw a tree diagram to display the possible outcomes.

b Find the probability of obtaining:

i a blue ticket and a green ticket **ii** two tickets of the same colour

iii exactly one pink ticket.

c Find the sum of the probabilities in **b**. Explain your result.

10 Sharon, Christine, and Keith have agreed to meet at a restaurant for lunch. Sharon has probability 0.8 of being on time, Christine has probability 0.7 of being on time, and Keith has probability 0.4 of being on time.

a Draw a tree diagram to display the possible outcomes.

b Determine the probability that at least 2 people will arrive on time.

DEPENDENT EVENTS

Dependent events are events for which the occurrence of one event *does affect* the occurrence of the other event.

Suppose a bag contains 5 blue discs and 3 yellow discs. One disc is selected at random, and put to one side. A second disc is then selected from the bag.

Let A be the event that the *first* disc is blue, and B be the event that the *second* disc is blue.

Now, if A occurs, then $\text{P}(B) = \frac{4}{7}$ ← 4 blue discs remaining / ← 7 discs left to choose from

However, if A does not occur, then $\text{P}(B) = \frac{5}{7}$ ← 5 blue discs remaining / ← 7 discs left to choose from

The occurrence of A *does* affect the probability of B occurring. Therefore A and B are **dependent events**.

The rule for finding compound event probabilities for dependent events is different from the rule for independent events.

If A and B are dependent events, then $\text{P}(A \text{ and } B) = \text{P}(A) \times \text{P}(B \text{ given that } A \text{ has occurred})$.

Example 11 **Self Tutor**

A fruit bowl contains 3 apples and 5 oranges. Callan selects a piece of fruit at random, and eats it. His sister Michelle then selects a piece of fruit for herself.

Find the probability that:

a both pieces of fruit selected are apples

b Callan selects an orange and Michelle selects an apple.

a P(both are apples)

= P(Callan selects an apple *and* Michelle selects an apple)

= P(Callan selects an apple)

× P(Michelle selects an apple *given* that Callan selects an apple)

$= \frac{3}{8} \times \frac{2}{7}$ ← 2 apples remaining / ← 7 pieces of fruit to choose from

$= \frac{6}{56}$

$= \frac{3}{28}$

b P(Callan selects an orange *and* Michelle selects an apple)

$= $ P(Callan selects an orange)

$\times$ P(Michelle selects an apple *given* that Callan selects an orange)

$= \frac{5}{8} \times \frac{3}{7}$ ← 3 apples remaining ← 7 pieces of fruit to choose from

$= \frac{15}{56}$

EXERCISE 14E.2

1 A bucket contains 2 orange, 5 white, and 3 yellow ping pong balls. Two balls are selected at random from the bucket, the second being selected without replacing the first. Find the probability that:

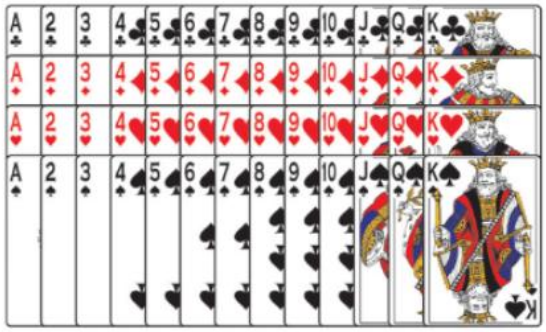
a both balls are orange **b** the first ball is yellow, and the second ball is white.

2 Two cards are selected, without replacement, from a standard deck of 52 cards. Find the probability that:

a both of the cards are red

b the first card is a club, and the second card is a diamond

c both of the cards are aces.



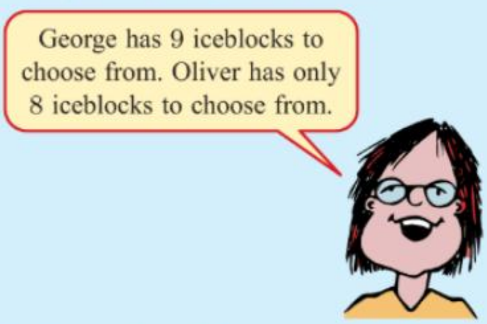
Example 12 **Self Tutor**

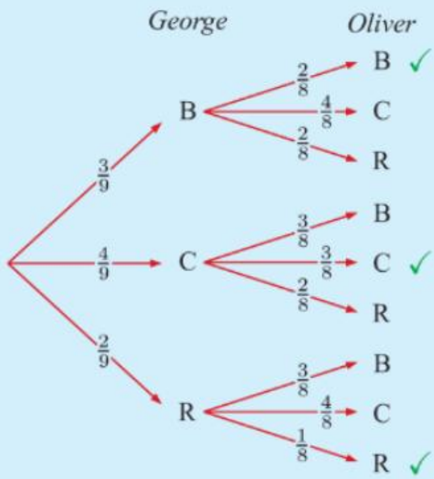
A container holds 3 banana iceblocks, 4 chocolate iceblocks, and 2 raspberry iceblocks. George randomly selects an iceblock from the container and eats it. His brother Oliver then randomly selects an iceblock from the container.

a Draw a tree diagram to display the possible outcomes.

b Find the probability that the brothers selected the same type of iceblock.

a Let B represent a banana iceblock,
C represent a chocolate iceblock, and
R represent a raspberry iceblock.





b P(brothers selected the same type of iceblock)

$= $ P(BB or CC or RR) {those marked ✓}

$= \frac{3}{9} \times \frac{2}{8} + \frac{4}{9} \times \frac{3}{8} + \frac{2}{9} \times \frac{1}{8}$

$= \frac{20}{72}$

$= \frac{5}{18}$

3 A drawer contains 5 blue pens, 3 red pens, and 2 green pens. A pen is selected at random from the drawer and put to one side. A second pen is then selected at random from the drawer.

- **a** Draw a tree diagram to display the possible outcomes.
- **b** Find the probability that:
  - **i** the selected pens are the same colour
  - **ii** exactly one of the pens is green.

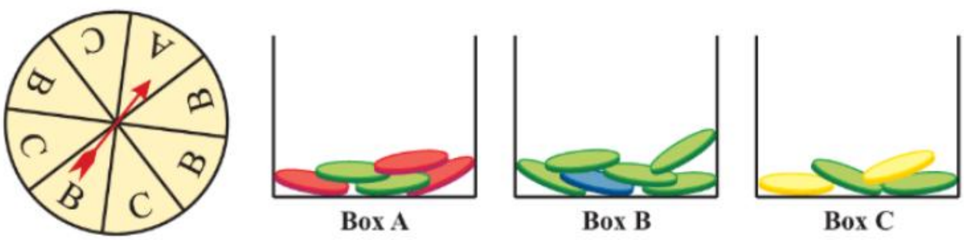
4 Matt is the best player in his lacrosse team. He has an injured knee, and has only a 60% chance of playing the next game. The team has a 70% chance of winning the next game if Matt plays, but only a 45% chance of winning if he does not play.

- **a** Represent this information on a tree diagram.
- **b** Find the probability that the team will win the next game.

5 In a netball team, the Goal Shooter takes 65% of the team's shots, and scores 70% of the time. The Goal Attack takes the remainder of the shots, and scores 60% of the time.
Find the probability that the team will score with their next shot.

6 A fair coin is tossed. If the result is heads, one marble is selected from the bag alongside. If the result is tails, two marbles are selected, without replacement. Find the probability that at least one red marble will be selected.

7 The spinner below is used to select box A, B, or C. Two discs are then randomly selected without replacement from that box.



Find the probability that exactly one green disc is selected.

F MUTUALLY EXCLUSIVE AND INDEPENDENT EVENTS

MUTUALLY EXCLUSIVE EVENTS

> Two events are **mutually exclusive** or **disjoint** if they have no common outcomes.
>
> If A and B are mutually exclusive events then $P(A \text{ and } B) = 0$.

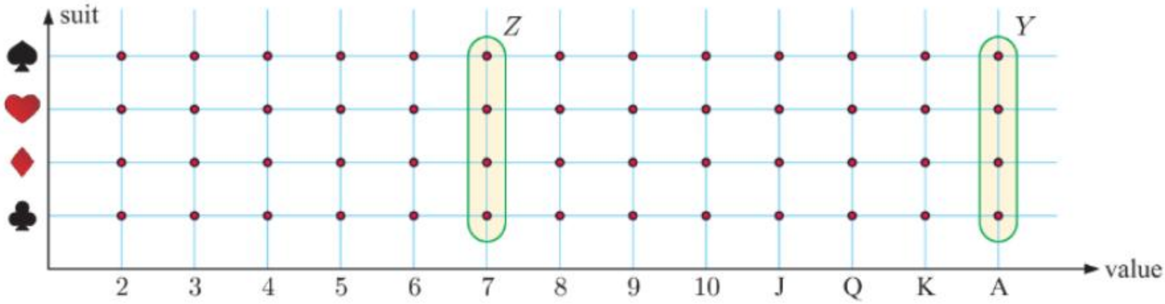
For example, suppose we select a card at random from a normal pack of 52 playing cards. Consider these events:

Event X: the card is a heart *Event Y*: the card is an ace *Event Z*: the card is a 7

Notice that:

- X and Y have a common outcome, the Ace of hearts
- X and Z have a common outcome, the 7 of hearts
- Y and Z do not have a common outcome, so they are mutually exclusive.

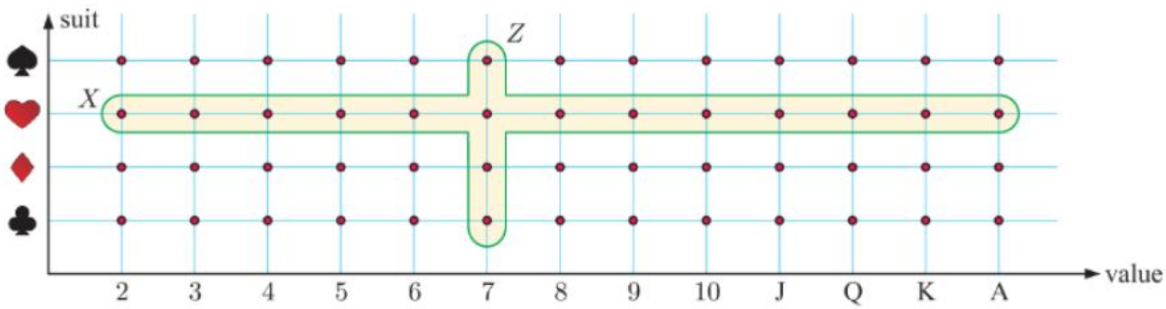
The events Y and Z are shown on the 2-dimensional grid below:



Notice that $\text{P}(Y \textbf{ or } Z) = \frac{8}{52}$ and $\text{P}(Y) + \text{P}(Z) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$.

> If two events A and B are **mutually exclusive**, then
> $$\text{P}(A \textbf{ or } B) = \text{P}(A) + \text{P}(B)$$

Now consider the events X and Z, which are *not* mutually exclusive since they have a common outcome: the 7 of hearts.

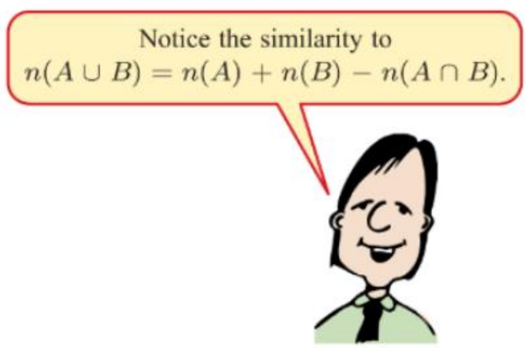


Notice that $\text{P}(X \textbf{ or } Z) = \frac{16}{52}$ and $\text{P}(X) + \text{P}(Z) = \frac{13}{52} + \frac{4}{52} = \frac{17}{52}$.

These values are not the same, since when we find $\text{P}(X) + \text{P}(Z)$ the common outcome is counted *twice*.

Actually, $\text{P}(X \textbf{ or } Z) = \text{P}(X) + \text{P}(Z) - \text{P}(X \textbf{ and } Z)$.

> If two events A and B are **not mutually exclusive** then
> $$\text{P}(A \textbf{ or } B) = \text{P}(A) + \text{P}(B) - \text{P}(A \textbf{ and } B).$$



Example 13 Self Tutor

Suppose $\mathrm{P}(A) = 0.2$, $\mathrm{P}(B) = 0.6$, and $\mathrm{P}(A \text{ and } B) = 0.1$.

a Are A and B mutually exclusive events? Explain your answer.

b Find $\mathrm{P}(A \text{ or } B)$.

a A and B are *not* mutually exclusive, since $\mathrm{P}(A \text{ and } B) \neq 0$.

b $$\begin{aligned}\mathrm{P}(A \text{ or } B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \text{ and } B)\\ &= 0.2 + 0.6 - 0.1\\ &= 0.7\end{aligned}$$

INDEPENDENT EVENTS

In **Section E**, we saw that two events are **independent** if the occurrence of each event does not affect the occurrence of the other.

If A and B are independent events, then $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

Example 14 Self Tutor

Suppose $\mathrm{P}(X) = 0.6$ and $\mathrm{P}(Y) = 0.2$. Find $\mathrm{P}(X \text{ and } Y)$ given that X and Y are:

a mutually exclusive **b** independent.

a If X and Y are mutually exclusive, then $\mathrm{P}(X \text{ and } Y) = 0$.

b If X and Y are independent, then $$\begin{aligned}\mathrm{P}(X \text{ and } Y) &= \mathrm{P}(X) \times \mathrm{P}(Y)\\ &= 0.6 \times 0.2\\ &= 0.12\end{aligned}$$

EXERCISE 14F

1 An ordinary die with faces 1, 2, 3, 4, 5, and 6 is rolled once. Consider these events:

A: rolling a 1
B: rolling a 3
C: rolling an odd number
D: rolling an even number
E: rolling a prime number
F: rolling a result greater than 3.

List the pairs of events which are mutually exclusive.

2 A coin and an ordinary die are tossed and rolled simultaneously.

a Draw a grid showing the 12 possible outcomes.

b Let A be the event 'tossing a head' and B be the event 'rolling a 5'.

i Are A and B mutually exclusive? Explain your answer.

ii Find $\mathrm{P}(A \text{ or } B)$ and $\mathrm{P}(A \text{ and } B)$.

iii Check that $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \text{ and } B)$.

3 Suppose $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.2$, and $\mathrm{P}(A \text{ and } B) = 0.15$.

a Are A and B mutually exclusive events? Explain your answer.

b Find $\mathrm{P}(A \text{ or } B)$.

4 Suppose $P(X) = 0.3$ and $P(X \text{ or } Y) = 0.7$. Given that X and Y are mutually exclusive, find $P(Y)$.

5 Suppose $P(C) = 0.6$ and $P(D) = 0.7$. Explain why C and D are not mutually exclusive.

6 Let $P(A) = 0.4$ and $P(B) = 0.25$. Find $P(A \text{ and } B)$ given that A and B are:

a mutually exclusive

b independent.

7 X and Y are independent events such that $P(X) = 0.5$ and $P(Y) = 0.3$. Find $P(X \text{ or } Y)$.

8 A and B are independent events such that $P(A \text{ or } B) = 0.9$ and $P(A \text{ and } B) = 0.4$. Find $P(A)$ and $P(B)$ given that $P(A) > P(B)$.

REVIEW SET 14A

1 Donna kept records of the number of clients she interviewed over consecutive days.

a For how many days did Donna keep records?

b Estimate the probability that tomorrow Donna will interview:

i no clients

ii four or more clients

iii less than three clients.

Number of clients	*Frequency*
0	1
1	6
2	12
3	8
4	6
5	3
6	0
7	2

2 A coin and a pentagonal spinner with sides labelled A, B, C, D, and E are tossed and spun simultaneously. Illustrate the possible outcomes using a 2-dimensional grid.

3 When a box of drawing pins was dropped onto the floor, 49 pins landed on their backs, and 32 landed on their sides. Estimate, to 2 decimal places, the probability of a drawing pin landing:

a on its back

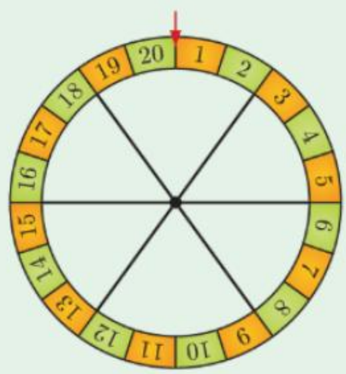
b on its side.

4 A wheel numbered 1 to 20 is spun. Find the probability that the result is:

a 13

b a multiple of 3

c greater than 11.



5 On a particular day, 500 people visited a carnival. 300 people rode the Ferris wheel, and 350 people rode the roller coaster. Each person rode at least one of these attractions.

a Display this information on a Venn diagram.

b Find the probability that a randomly chosen person rode:

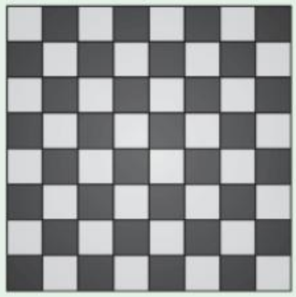
i the roller coaster, but not the Ferris wheel

ii the Ferris wheel and the roller coaster.

6 A bag contains 4 green and 3 red marbles. Two marbles are randomly selected from the bag, the first being put to one side before the second is drawn. Find the probability that:

a both are green **b** they are different in colour.

7 A chess piece is placed on a random square of an 8×8 chess board. A second piece is then placed at random on one of the unoccupied squares.



Find the probability that the two pieces lie:

a on the same row

b on the same row or column.

8 A married couple own a large car and a small car. Glen uses the small car 30% of the time. When he goes to the shops, the probability that he can park in the car park is 80% if he has the small car, and 60% if he has the large car.

Find the probability that, on a given day, Glen is able to park in the shop's car park.

9 The two-way table alongside describes the books on Elizabeth's bookshelf.

	Biography	Novel	Reference
Softcover	3	22	6
Hardcover	5	7	4

a How many books are on Elizabeth's bookshelf?

b Find the probability that a randomly selected book is:

i a softcover book **ii** a hardcover reference book.

10 Tyson has one of each type of Australian coin in his pocket. He randomly selects two coins from his pocket, replacing the first coin before selecting the second.

a Draw a 2-dimensional grid to display the possible outcomes.

b Find the probability that:

i both coins are silver

ii at least one of the coins is the 50 cent coin

iii the value of the coins is at least 65 cents.

11 Brothers Paul, Cameron, and Bruce play in the same rugby team. Their probabilities of getting injured during the season are 0.4, 0.3, and 0.2 respectively. What is the most likely number of injured brothers during the season?

12 Let $P(A) = 0.2$ and $P(B) = 0.7$. Find $P(A \text{ and } B)$ given that A and B are:

a mutually exclusive **b** independent.

REVIEW SET 14B

1 Pierre conducted a survey to determine the ages of people walking through a shopping mall. The results are shown in the table alongside. Estimate the probability that the next person Pierre meets in the shopping mall will be:

Age	*Frequency*
0 - 19	22
20 - 39	43
40 - 59	39
60+	14

a between 20 and 39 years of age

b less than 40 years of age **c** at least 20 years of age.

2 Use a tree diagram to illustrate the sample spaces for the following:

a Bags A, B, and C contain green and yellow tickets. A bag is selected and then a ticket taken from it.

b Martin and Justin play tennis. The first to win three sets wins the match.

3 The two-way table alongside shows the results from asking the question "Do you like the school uniform?".
If a student is randomly selected from these year groups, estimate the probability that the student:

	Likes	Dislikes
Year 8	129	21
Year 9	108	42
Year 10	81	69

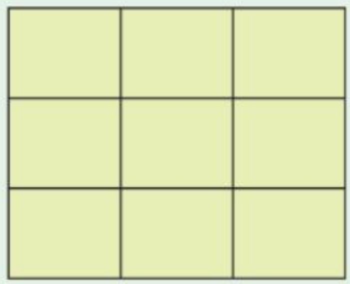
a likes the school uniform

b dislikes the school uniform

c is in Year 8 and dislikes the uniform.

4 The digits 1, 6, and 9 are placed in random order to create a 3 digit number. Find the probability that this number will be a perfect square.

5 A farmer fences his rectangular property into 9 rectangular paddocks as shown.
A paddock is selected at random. Find the probability that it has:



a no fences on the boundary of the property

b one fence on the boundary of the property

c two fences on the boundary of the property.

6 A paper plate is tossed in the air 50 times. It lands face up 37 times, and lands face down 13 times.

a Estimate the probability that the paper plate will land face up next time it is thrown.

b If the plate is tossed in the air three times, estimate the probability that the plate will land face down on all three occasions.

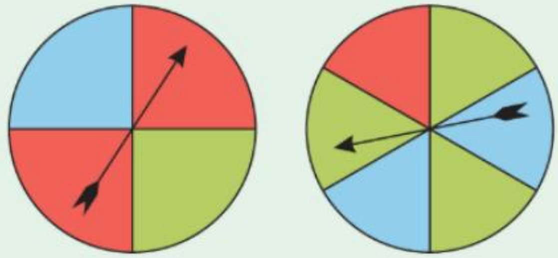
7 Bag X contains three white and two red marbles. Bag Y contains one white and three red marbles. A bag is randomly chosen, and two marbles are drawn from it.

a Illustrate the given information on a tree diagram.

b Determine the probability of drawing two marbles of the same colour.

8 Matthew is taking a mathematics test. There is a 2% chance that his calculator will not work. If his calculator works, Matthew has a 70% probability of passing the test. If his calculator does not work, Matthew has a 55% probability of passing the test.
Find the probability that Matthew passes the test.

9 The spinners alongside are each spun once. Find the probability that the spins are the same colour.



10 X and Y are independent events such that $\mathrm{P}(X) = 0.6$ and $\mathrm{P}(X \text{ and } Y) = 0.3$. Find $\mathrm{P}(X \text{ or } Y)$.

11 Shelley draws 3 cards without replacement from the container alongside. She will win a prize if all 3 cards are the same colour.

a Find the probability that Shelley will win a prize.

b Suppose the rules change so that Shelley now draws her cards *with* replacement.

i Do you think this increases or decreases her probability of winning? Explain your answer.

ii Find the probability of Shelley winning.

12 In a weightlifting competition, 3 competitors are given one chance to lift a weight of their choosing. The table alongside shows the weights chosen by each competitor, and their probability of lifting it.

The winner is the competitor who successfully lifts the greatest weight. If nobody lifts their weight, the competition is a tie.

Competitor	*Weight*	*Probability*
Ihor	210 kg	0.5
Ruslan	225 kg	0.4
Behdad	240 kg	0.3

a Find the probability that:

i Ihor **ii** Ruslan **iii** Behdad

will win the competition.

b Ihor is considering increasing his weight to 230 kg, so he is lifting more than Ruslan.

However, the probability that he can lift this weight is only 0.34.

Would this strategy increase or decrease Ihor's probability of winning the competition?

Chapter 15

Formulae

Contents:

OPENING PROBLEM

In Gaelic football games, teams can score goals worth 3 points each, as well as individual points.

While watching a football game, Josh noticed something unusual about Mayo's score.

Mayo had scored 2 goals and 6 points, which is a total of 12 points, but he also recognised that $2 \times 6 = 12$.

Josh wondered whether there were other football scores with this property.

Things to think about:

- **a** For a score of g goals and p points to have this property, can you explain why $3g + p = gp$?
- **b** Can you *rearrange* this formula to make p the subject?
- **c** By *substituting* different values for g, can you find other scores which have this property?

A **formula** is an equation which connects two or more variables.

For example, the formula $s = \dfrac{d}{t}$ relates the three variables *speed* (s), *distance travelled* (d), and *time taken* (t).

We usually write a formula with one variable on its own on the left hand side. The other variable(s) and constants are written on the right hand side.

The variable on its own is called the **subject** of the formula. We say this variable is written *in terms of* the other variables.

A FORMULA CONSTRUCTION

When we try to construct a formula to connect related variables, we often start with numerical examples. They are useful to help us understand the situation before we generalise the result.

Example 1 Self Tutor

Write a formula for the amount $\$A$ in a person's bank account if initially the balance was:

- **a** \$5000, and \$200 was withdrawn each week for 10 weeks
- **b** \$5000, and \$200 was withdrawn each week for w weeks
- **c** \$5000, and $\$x$ was withdrawn each week for w weeks
- **d** $\$B$, and $\$x$ was withdrawn each week for w weeks.

a $A = 5000 - 200 \times 10$

b $A = 5000 - 200 \times w$
$\therefore\ A = 5000 - 200w$

c $A = 5000 - x \times w$
$\therefore\ A = 5000 - xw$

d $A = B - x \times w$
$\therefore\ A = B - xw$

We do not simplify the amount in **a** because we want to see how the formula is put together.

EXERCISE 15A

1 Write a formula for the amount €A earnt for working:

- **a** 5 hours at €15 per hour
- **b** 5 hours at €p per hour
- **c** t hours at €p per hour.

2 Write a formula for the amount $\$A$ in a bank account if the initial balance was:

- **a** \$2000, and then \$150 was deposited each week for 8 weeks
- **b** \$2000, and then \$150 was deposited each week for w weeks
- **c** \$2000, and then $\$d$ was deposited each week for w weeks
- **d** $\$P$, and then $\$d$ was deposited each week for w weeks.

3 Write a formula for the total cost £C of hiring a plumber given a fixed call-out fee of:

- **a** £40, plus £60 per hour for 5 hours of work
- **b** £40, plus £60 per hour for t hours of work
- **c** £40, plus £x per hour for t hours of work
- **d** £F, plus £x per hour for t hours of work.

4 In a multiple choice mathematics competition, students are awarded 3 points for each question answered correctly, and penalised 1 point for each question answered incorrectly. Write a formula for the number of points P scored by a student who:

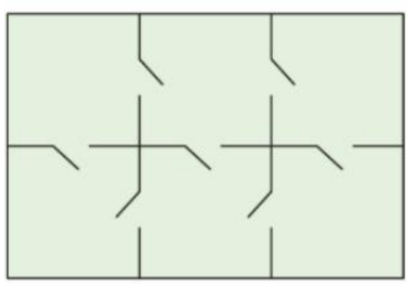
- **a** answers 15 questions and gets 10 of them correct
- **b** answers 20 questions and gets c of them correct
- **c** answers a questions and gets c of them correct.

5 A musical recital consists of performances by a number of musicians, with a short break between each performance. Write a formula for the duration D minutes of a recital consisting of:

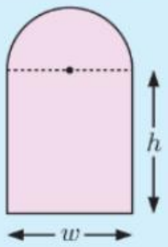
- **a** 4 performances of 6 minutes each, with a 2 minute break between performances
- **b** 5 performances of m minutes each, with a 3 minute break between performances
- **c** 8 performances of m minutes each, with a b minute break between performances
- **d** p performances of m minutes each, with a b minute break between performances.

6 A rectangular paddock is fenced into a rectangular array of yards so that each yard is connected by a gate to each adjacent yard. A 2×3 arrangement of yards is shown alongside.
Write a formula for the number of gates G for:

- **a** a 2×3 arrangement
- **b** a 3×5 arrangement
- **c** a 4×4 arrangement
- **d** an $m \times n$ arrangement.



Example 2 **Self Tutor**



The illustrated door consists of a semi-circle and a rectangle. Find a formula for the area of the door in terms of the width w and height h of the rectangular part.

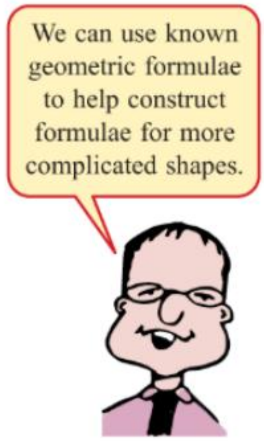
The area of a rectangle = height × width
$= hw$

The radius of the semi-circle is $\frac{w}{2}$.

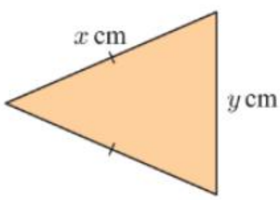
$\therefore$ the area of the semi-circle $= \frac{1}{2} \times$ (area of full circle)

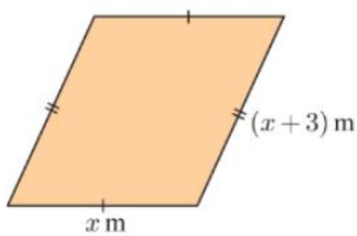
$$= \frac{1}{2} \times \pi r^2$$
$$= \frac{1}{2} \times \pi \times \left(\frac{w}{2}\right)^2$$
$$= \frac{1}{2} \times \pi \times \frac{w^2}{4}$$
$$= \frac{1}{8}\pi w^2$$

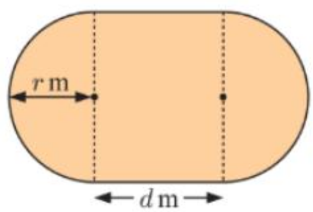
$\therefore$ the total area is $A = hw + \frac{1}{8}\pi w^2$



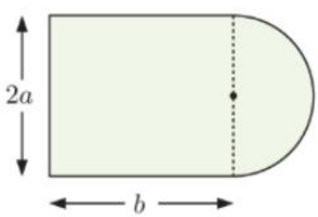
7 Write a formula for the perimeter P of the following figures:

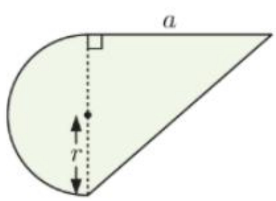
a


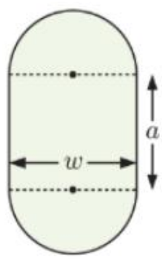
b


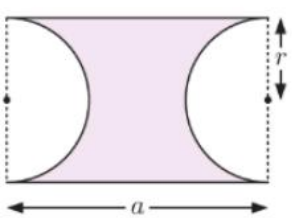
c


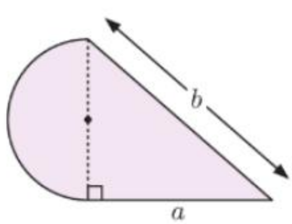
8 Find a formula for the area A of each of the shaded regions:

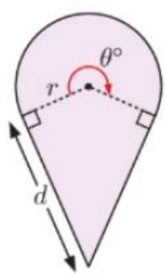
a


b


c


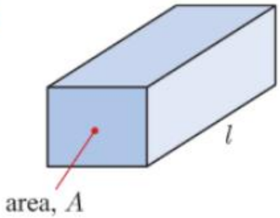
d


e


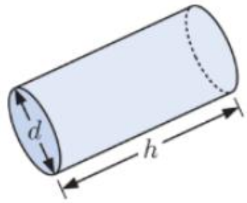
f


9 Find a formula for the volume V of each of the following objects:

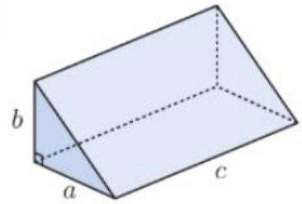
a



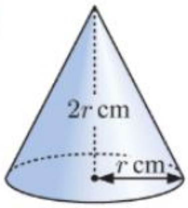
b



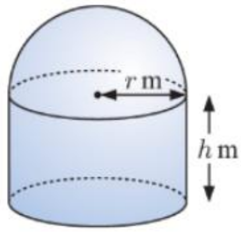
c



d

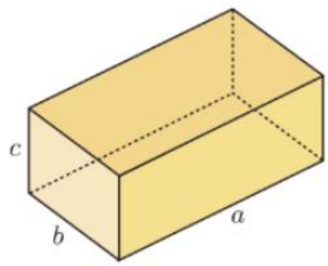


e

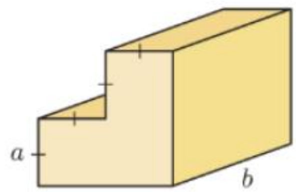


10 Find a formula for the surface area A of each of the following:

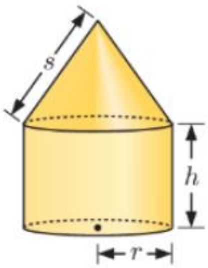
a



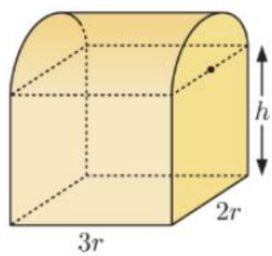
b



c

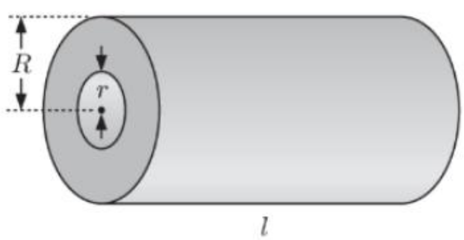


d

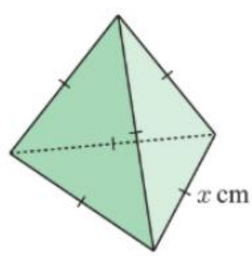


11 A cylindrical pipe has outside radius R, inside radius r, and length l. Show that the volume of concrete used to make the pipe is given by

$$V = \pi l(R+r)(R-r).$$



12 Write a formula for the volume of this tetrahedron.



B SUBSTITUTING INTO FORMULAE

Suppose a formula contains two or more variables, and we know the value of all but one of them. We can **substitute** the known values into the formula to find the corresponding value of the unknown variable.

Step 1: Write down the formula.
Step 2: State the values of the known variables.
Step 3: Substitute the known values into the formula to form a one variable equation.
Step 4: Solve the equation for the unknown variable.

Example 3 — Self Tutor

When a stone is dropped from a cliff, the total distance fallen after t seconds is given by the formula $D = \frac{1}{2}gt^2$ metres, where $g = 9.8$ m/s^2. Find:

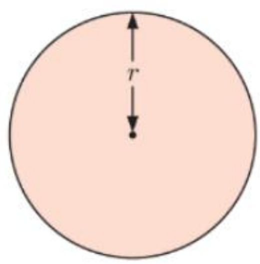
a the distance fallen after 4 seconds
b the time, to the nearest $\frac{1}{100}$th second, taken for the stone to fall 200 metres.

a $D = \frac{1}{2}gt^2$ where $g = 9.8$ and $t = 4$
$\therefore$ $D = \frac{1}{2} \times 9.8 \times 4^2 = 78.4$
$\therefore$ the stone has fallen 78.4 metres.

b $D = \frac{1}{2}gt^2$ where $D = 200$ and $g = 9.8$
$\therefore$ $\frac{1}{2} \times 9.8 \times t^2 = 200$
$\therefore$ $4.9t^2 = 200$
$\therefore$ $t^2 = \frac{200}{4.9}$
$\therefore$ $t = \sqrt{\frac{200}{4.9}}$ {t must be positive}
$\therefore$ $t \approx 6.39$
$\therefore$ the time taken is about 6.39 seconds.

EXERCISE 15B

1 The formula for finding the circumference C of a circle with radius r is $C = 2\pi r$. Find:



a the circumference of a circle of radius 4.2 cm
b the radius of a circle with circumference 112 cm
c the diameter of a circle with circumference 400 metres.

2 When a stone is dropped from the top of a cliff, the distance fallen after t seconds is given by the formula $D = \frac{1}{2}gt^2$ metres, where $g = 9.8$ m/s^2. Find:

a the distance fallen in the first 2 seconds
b the time taken for the stone to fall 100 metres.

3 The area A of a circle with radius r is $A = \pi r^2$. Find:

a the area of a circle with radius 6.4 cm

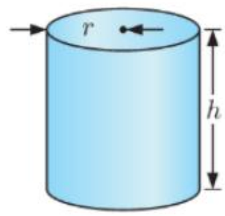
b the radius of a circular swimming pool which has an area of 160 m^2.

4 The volume of a cylinder with radius r and height h is given by $V = \pi r^2 h$. Find:

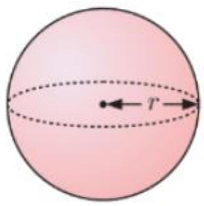
a the volume of a cylindrical tin can with radius 8 cm and height 21.2 cm

b the height of a cylinder with radius 6 cm and volume 120 cm^3

c the radius, in mm, of a copper pipe with volume 470 cm^3 and length 6 m.



5



The formula for the surface area A of a sphere with radius r is $A = 4\pi r^2$. Find:

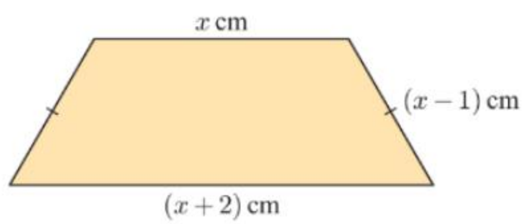
a the surface area of a sphere with radius 7.5 cm

b the radius, in cm, of a spherical balloon which has a surface area of 2 m^2.

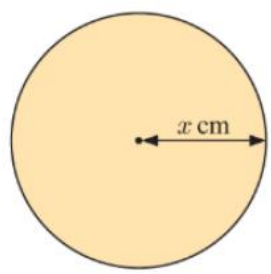
6 The perimeter of each of the following figures is 12 cm.
For each figure:

i Write a formula for the perimeter P. **ii** Find the value of x.

a



b



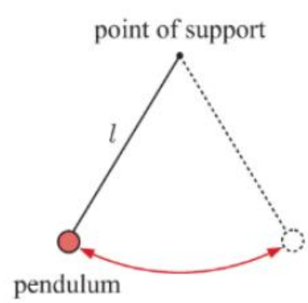
7 **a** Write a formula for the area of an equilateral triangle with sides of length x m.

b An equilateral triangle has area $16\sqrt{3}$ cm^2. Find the length of its sides.

8 The *period* or time taken for one complete swing of a simple pendulum is given approximately by $T = \frac{1}{5}\sqrt{l}$ seconds, where l is the length of the pendulum in centimetres. Find:

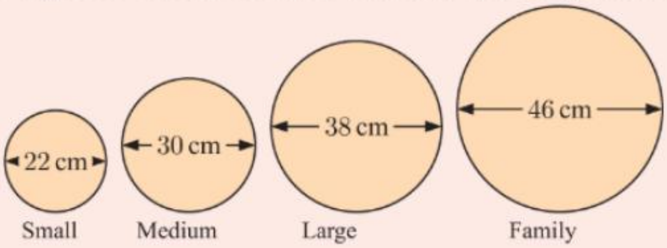
a the time for one complete swing of a pendulum with length 45 cm

b the length of a pendulum which has a period of 1.8 seconds.



ACTIVITY — PIZZA PRICING

Luigi's Pizza Parlour has a 'Seafood Special' pizza advertised this week.



LUIGI'S PIZZAS

Free Delivery!

"Seafood Special"	
Small	€8.00
Medium	€10.60
Large	€14.00
Family	€18.20

Sasha, Enrico, and Bianca each attempted to find Luigi's formula for the price €P of each pizza size. The formulae they worked out for a pizza of radius r cm were:

Sasha: $P = \dfrac{17r - 27}{20}$ Enrico: $P = \sqrt{\dfrac{33r - 235}{2}}$ Bianca: $P = 5 + \dfrac{r^2}{40}$.

What to do:

1. Investigate the suitability of each formula.
2. Luigi is introducing a Party size pizza of diameter 54 cm. What do you think his price will be?

C REARRANGING FORMULAE

In the formula $D = xt + p$, D is expressed in terms of the other variables, x, t, and p. We say that D is the **subject** of the formula.

We can rearrange a formula to make one of the other variables the subject. However, we must do this carefully to ensure that the formula is still true.

We **rearrange** formulae using the same processes which we use to solve equations. Anything we do to one side we must also do to the other.

Example 4 — Self Tutor

Make y the subject of:

a $2x + 3y = 12$ **b** $x = 5 - cy$

a $2x + 3y = 12$

$\therefore\ 3y = 12 - 2x$ {subtracting $2x$ from both sides}

$\therefore\ y = \dfrac{12 - 2x}{3}$ {dividing both sides by 3}

$\therefore\ y = 4 - \frac{2}{3}x$

b $x = 5 - cy$

$\therefore\ x + cy = 5$ {adding cy to both sides}

$\therefore\ cy = 5 - x$ {subtracting x from both sides}

$\therefore\ y = \dfrac{5 - x}{c}$ {dividing both sides by c}

EXERCISE 15C

1 Make y the subject of:

a $2x + 5y = 10$ **b** $3x + 4y = 20$ **c** $2x - y = 8$
d $2x + 7y = 14$ **e** $5x + 2y = 20$ **f** $2x - 3y = -12$

2 Make x the subject of:

a $p + x = r$ **b** $xy = z$ **c** $3x + a = d$
d $5x + 2y = d$ **e** $ax + by = p$ **f** $y = mx + c$
g $2 + tx = s$ **h** $p + qx = m$ **i** $6 = a + bx$

3 Make y the subject of:

a $z = t - 5y$ **b** $c - 2y = p$ **c** $a - 3y = t$
d $n - ky = 5$ **e** $a - by = n$ **f** $p = a - ny$

Example 5 **Self Tutor**

Make z the subject of $c = \dfrac{m}{z}$.

$$c = \frac{m}{z}$$

$\therefore\ cz = m$ {multiplying both sides by z}

$\therefore\ z = \dfrac{m}{c}$ {dividing both sides by c}

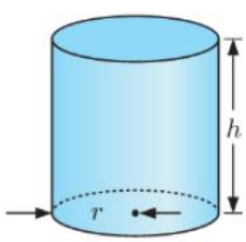
4 Make z the subject of:

a $az = \dfrac{b}{c}$ **b** $\dfrac{a}{z} = d$ **c** $\dfrac{3}{d} = \dfrac{2}{z}$
d $\dfrac{z}{2} = \dfrac{a}{z}$ **e** $\dfrac{b}{z} = \dfrac{z}{n}$ **f** $\dfrac{m}{z} = \dfrac{z}{a-b}$

5 Make:

a a the subject of $F = ma$
b r the subject of $C = 2\pi r$
c d the subject of $V = ldh$
d K the subject of $A = \dfrac{b}{K}$
e h the subject of $A = \dfrac{bh}{2}$
f T the subject of $I = \dfrac{PRT}{100}$

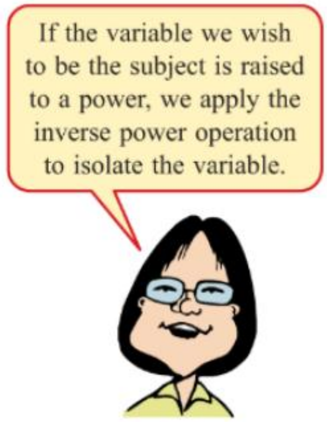
6 The surface area of a cylinder with radius r and height h is given by $A = 2\pi r^2 + 2\pi rh$.
Rearrange this formula to make h the subject.



Example 6 — Self Tutor

Make t the subject of $s = \frac{1}{2}gt^2$, given that $t > 0$.

$$\begin{aligned} \tfrac{1}{2}gt^2 &= s && \{\text{rewriting with } t^2 \text{ on the LHS}\} \\ \therefore\ gt^2 &= 2s && \{\text{multiplying both sides by } 2\} \\ \therefore\ t^2 &= \frac{2s}{g} && \{\text{dividing both sides by } g\} \\ \therefore\ t &= \sqrt{\frac{2s}{g}} && \{\text{as } t > 0\} \end{aligned}$$



7 Make:

- **a** r the subject of $A = \pi r^2$, $r > 0$
- **b** x the subject of $N = \dfrac{x^2}{a}$
- **c** k the subject of $M = 5k^2$
- **d** x the subject of $D = \dfrac{n}{x^3}$
- **e** x the subject of $y = 4x^2 - 7$, $x < 0$
- **f** Q the subject of $P^2 = Q^2 + R^2$

Example 7 — Self Tutor

Make x the subject of $T = \dfrac{a}{\sqrt{x}}$.

$$\begin{aligned} T &= \frac{a}{\sqrt{x}} \\ \therefore\ T^2 &= \left(\frac{a}{\sqrt{x}}\right)^2 && \{\text{squaring both sides}\} \\ \therefore\ T^2 &= \frac{a^2}{x} \\ \therefore\ T^2x &= a^2 && \{\text{multiplying both sides by } x\} \\ \therefore\ x &= \frac{a^2}{T^2} && \{\text{dividing both sides by } T^2\} \end{aligned}$$

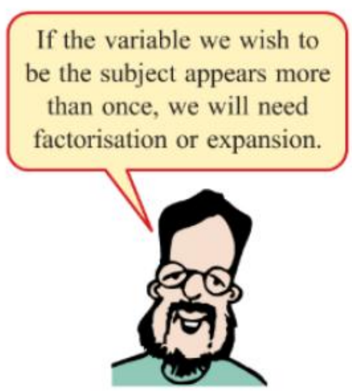
8 Make:

- **a** a the subject of $d = \dfrac{\sqrt{a}}{n}$
- **b** l the subject of $T = \frac{1}{5}\sqrt{l}$
- **c** a the subject of $c = \sqrt{a^2 - b^2}$
- **d** d the subject of $\dfrac{k}{a} = \dfrac{5}{\sqrt{d}}$
- **e** l the subject of $T = 2\pi\sqrt{\dfrac{l}{g}}$
- **f** b the subject of $A = 4\sqrt{\dfrac{a}{b}}$

Example 8 — Self Tutor

Make x the subject of $ax + 3 = bx + d$.

$$\begin{aligned} ax + 3 &= bx + d \\ \therefore\ ax - bx &= d - 3 && \{\text{writing terms containing } x \text{ on the LHS}\} \\ \therefore\ x(a - b) &= d - 3 && \{x \text{ is a common factor on the LHS}\} \\ \therefore\ x &= \frac{d-3}{a-b} && \{\text{dividing both sides by } (a-b)\} \end{aligned}$$



9 Make x the subject of:

a $3x + a = bx + c$ **b** $ax = c - bx$ **c** $mx + a = nx - 2$

d $8x + a = -bx$ **e** $a - x = b - cx$ **f** $rx + d = e - sx$

Example 9 Self Tutor

Make x the subject of $T = \dfrac{a}{x-b}$.

$$T = \frac{a}{x-b}$$

$\therefore\ T(x-b) = a$ {multiplying both sides by $(x-b)$}

$\therefore\ Tx - Tb = a$

$\therefore\ Tx = a + Tb$ {adding Tb to both sides}

$\therefore\ x = \dfrac{a+Tb}{T}$ {dividing both sides by T}

10 Make:

a a the subject of $P = \dfrac{2}{a+b}$ **b** r the subject of $T = \dfrac{8}{q+r}$

c q the subject of $A = \dfrac{B}{p-q}$ **d** x the subject of $A = \dfrac{3}{2x+y}$

Example 10 Self Tutor

Make x the subject of $y = \dfrac{3x+2}{x-1}$.

$$y = \frac{3x+2}{x-1}$$

$\therefore\ y(x-1) = 3x+2$ {multiplying both sides by $(x-1)$}

$\therefore\ xy - y = 3x + 2$ {expanding the brackets}

$\therefore\ xy - 3x = y + 2$ {writing the terms containing x on the LHS}

$\therefore\ x(y-3) = y+2$ {x is a common factor}

$\therefore\ x = \dfrac{y+2}{y-3}$

11 Make x the subject of:

a $y = \dfrac{x}{x+1}$ **b** $y = \dfrac{x-3}{x+2}$ **c** $y = \dfrac{3x-1}{x+3}$

d $y = \dfrac{4x-1}{2-x}$ **e** $y = 1 + \dfrac{2}{x-3}$ **f** $y = -2 + \dfrac{5}{x+4}$

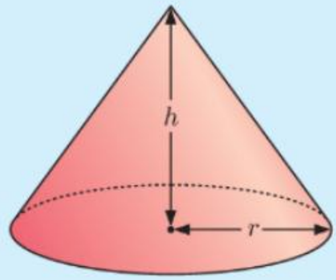
D REARRANGEMENT AND SUBSTITUTION

In the section on formula substitution, the known variables were replaced by numbers, and we then solved an equation to find the unknown.

In situations when we need to perform this process several times, it is quicker to **rearrange** the formula first, and then **substitute**.

Example 11 Self Tutor

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$, where r is the base radius and h is the height.



a Rearrange this formula to make r the subject.

b Hence, find the base radius of a cone with:

  i height 6 cm and volume 100 cm^3
  ii height 10 cm and volume 200 cm^3
  iii height 15 cm and volume 150 cm^3.

a $V = \frac{1}{3}\pi r^2 h$

$\therefore\ 3V = \pi r^2 h$ {multiplying both sides by 3}

$\therefore\ \frac{3V}{\pi h} = r^2$ {dividing both sides by πh}

$\therefore\ r = \sqrt{\frac{3V}{\pi h}}$ {as r must be positive}

b **i** When $h = 6$ and $V = 100$, $r = \sqrt{\frac{3 \times 100}{\pi \times 6}} = \sqrt{\frac{50}{\pi}} \approx 3.99$

So, the base radius is approximately 3.99 cm.

ii When $h = 10$ and $V = 200$, $r = \sqrt{\frac{3 \times 200}{\pi \times 10}} = \sqrt{\frac{60}{\pi}} \approx 4.37$

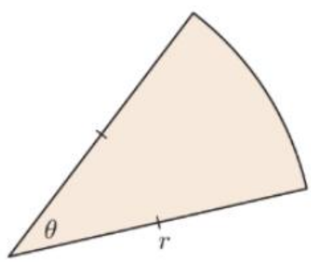
So, the base radius is approximately 4.37 cm.

iii When $h = 15$ and $V = 150$, $r = \sqrt{\frac{3 \times 150}{\pi \times 15}} = \sqrt{\frac{30}{\pi}} \approx 3.09$

So, the base radius is approximately 3.09 cm.

EXERCISE 15D

1 The area of a sector with radius r and angle θ is given by the formula $A = \frac{\theta}{360} \times \pi r^2$.



a Rearrange this formula to make θ the subject.

b Hence, find the angle of a sector with:

  i radius 3 cm and area 5 cm^2
  ii radius 7 cm and area 45 cm^2
  iii radius 8.5 cm and area 135 cm^2.

2 **a** Make a the subject of the formula $K = \dfrac{d^2}{2ab}$.

b Find the value of a when:

i $K = 112, \quad d = 24, \quad b = 2$ **ii** $K = 400, \quad d = 72, \quad b = 0.4$

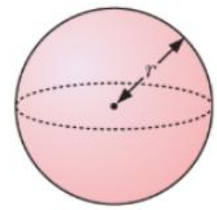
3 The height of a bush after t years is given by the formula $H = 1 + \sqrt{t}$ metres.

a Rearrange this formula to make t the subject.

b How long will it take for the bush to reach a height of:

i 2 m **ii** 3 m **iii** 3.5 m?

4 The formula for the volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.



a Make r the subject of the formula.

b Find the radius of a sphere which has volume:

i 40 cm^3 **ii** 800 cm^3 **iii** 1 000 000 cm^3.

5 An object with constant acceleration a m/s^2 travels s m. Its initial speed is u m/s and its final speed is v m/s. The variables are connected by the formula $v^2 - u^2 = 2as$.

a Rearrange the formula to make v the subject, where $v \geqslant 0$.

b Find the final speed of an object which travels:

i 100 m with initial speed 5 m/s and constant acceleration 2 m/s^2

ii 1.5 km with initial speed 10 m/s and constant acceleration 0.9 m/s^2.

6 The *winning percentage* of a tennis player who has won w matches and lost l matches is given by the formula $P = \dfrac{w}{w+l} \times 100\%$.

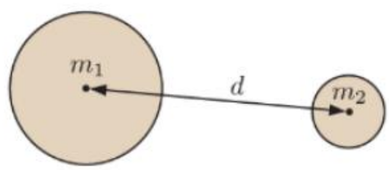
a Find the winning percentage of a player who has won 10 matches and lost 7 matches.

b Rearrange the formula to make w the subject.

c This year Mary has lost 15 matches, with a winning percentage of 37.5%. How many matches has she won?

d Over his career, Claude has won 84 matches and lost 49 matches. His aim is to increase his winning percentage to 65%. How many consecutive matches must he win to reach his target?

7 Consider two objects with masses m_1 kg and m_2 kg, which are d m apart. The gravitational force between the objects is given by the formula



$$F = G\frac{m_1 m_2}{d^2} \text{ Newtons}$$

where $G \approx 6.67 \times 10^{-11}$ is the universal gravitational constant.

a The Earth has mass 5.97×10^{24} kg, and the Moon has mass 7.35×10^{22} kg. Given that the Earth and the Moon are approximately 3.82×10^8 m apart, find the gravitational force between them. Give your answer in scientific notation.

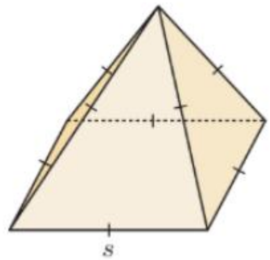
b Rearrange the formula so that d is the subject.

c **i** The Sun has mass 1.99×10^{30} kg. Given that the gravitational force between the Sun and the Earth is 3.54×10^{22} N, find the distance between the Sun and the Earth.

ii Two planets each have mass 2.32×10^{26} kg, and the gravitational force between them is 1.76×10^{14} N. Find the distance between the planets.

8 Answer the **Opening Problem** on page **288**.

9 Consider a square-based pyramid whose edges are all s units long.



a Show that the surface area of the pyramid is given by $A = s^2\left(1+\sqrt{3}\right)$.

b Rearrange this formula to show that $s = \sqrt{\frac{A}{2}\left(\sqrt{3}-1\right)}$.

c Find the side length if the pyramid has surface area:

i 50 cm^2 **ii** 150 cm^2 **iii** 600 cm^2.

Global context

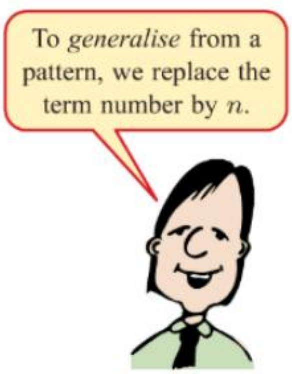
Newtonian mechanics

Statement of inquiry:	Formulating mathematical laws can help us to understand the world.
Global context:	Scientific and technical innovation
Key concept:	Relationships
Related concepts:	Equivalence, System
Objectives:	Knowing and understanding
Approaches to learning:	Thinking, Research

E PREDICTING FORMULAE

We can often predict a formula for a general situation by examining simple cases and looking for a pattern.

For example, the set of even numbers is $\{2, 4, 6, 8, 10, 12, ...\}$.



We observe that: the 1st term is 2×1
the 2nd term is 2×2
the 3rd term is 2×3, and so on.

We see from the pattern that the 13th term will be 2×13.

So, we generalise by saying that "the nth even number is $2n$".

The coefficient 2 in $2n$ indicates that the terms increase by 2 each time n is increased by 1.

Example 12 **Self Tutor**

Examine the matchstick pattern:  , ...

How many matches are needed to make:

a the first diagram **b** the second diagram **c** the third diagram
d the 4th diagram **e** the nth diagram?

a 3 matches **b** 5 matches **c** 7 matches

d The fourth diagram is , which contains 9 matches.

e So far, the set of numbers is $\{3, 5, 7, 9, ...\}$. We are adding 2 matches each time, so the formula must involve $2n$.
The expression $2n$ would generate the set $\{2, 4, 6, 8, ...\}$, whereas our set of numbers is always 1 more than these values.
$\therefore$ there are $2n + 1$ matches in the nth diagram.
Check: If $n = 1$, $2(1) + 1 = 3$ ✓
If $n = 2$, $2(2) + 1 = 5$ ✓
If $n = 3$, $2(3) + 1 = 7$ ✓

EXERCISE 15E

1 Examine the matchstick pattern: 

How many matchsticks make up the:

a 1st, 2nd, 3rd, 4th, and 5th diagrams **b** 10th diagram **c** nth diagram?

2 Examine the matchstick pattern: 

How many matchsticks make up the:

a 1st, 2nd, 3rd, 4th, and 5th diagrams **b** 10th diagram **c** nth diagram?

3 **a** Find:
i $1+3$ **ii** $1+3+5$ **iii** $1+3+5+7$ **iv** $1+3+5+7+9$

b If S_n is the sum of the first n positive odd numbers, then $S_1 = 1$, $S_2 = 4$, and $S_3 = 9$. Predict a formula for S_n.

4 **a** Find:
i $1+2$ **ii** $1+2+4$ **iii** $1+2+4+8$ **iv** $1+2+4+8+16$

b Predict a formula for the sum of the first n terms of the number set $\{1, 2, 4, 8, 16, ...\}$.

5 Consider the pattern:

$$S_1 = \frac{1}{1\times 2}, \quad S_2 = \frac{1}{1\times 2} + \frac{1}{2\times 3}, \quad S_3 = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4}, \quad ....$$

a Find the values of S_1, S_2, S_3, and S_4.

b Predict the value of:
i S_{10} **ii** S_n.

6 Consider the pattern: $S_1 = 1^2$, $S_2 = 1^2 + 2^2$, $S_3 = 1^2 + 2^2 + 3^2$, ...

a Check that the formula $S_n = \dfrac{n(n+1)(2n+1)}{6}$ is correct for $n = 1$, 2, 3, and 4.

b Assuming the formula in **a** is always true, find the sum of $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + ... + 100^2$, which is the sum of the squares of the first 100 integers.

REVIEW SET 15A

1 **a** A trough is initially empty. Write a formula for the volume of water V in the trough if:

i six 8-litre buckets of water are poured into it

ii n 8-litre buckets of water are poured into it

iii n l-litre buckets of water are poured into it.

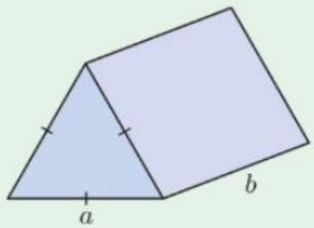
b A trough initially contains 25 L of water. Write a formula for the volume of water V in the trough if n buckets of water, each containing l litres, are poured into it.

2 The average speed of an object which travels d km in t hours is given by the formula $s = \dfrac{d}{t}$ km/h.

a Find the average speed of a truck which travels 540 km in 6 hours.

b Find the distance travelled by an aeroplane which flies for $6\frac{1}{2}$ hours at an average speed of 600 km/h.

3 Find a formula for the surface area A of the solid alongside.



4 Make x the subject of:

a $mx + n = 3p$ **b** $\dfrac{7}{y} = \dfrac{5}{x}$

5 Make k the subject of:

a $T = \sqrt{k - l^2}$ **b** $P = 2k^2 - r, \quad k < 0$

6 1.5 m^3 of garden soil is dumped on a flat surface. It forms a cone whose diameter equals its height.

a Suppose the diameter is x m. Write a formula for the volume of the heap in terms of x.

b Find the diameter of the heap of soil.

7 The electric current in a circuit with voltage E volts, resistance r ohms, and load resistance R ohms, is given by the formula $I = \dfrac{E}{r + R}$ amperes.

a Find the current in a circuit with voltage 24 V, resistance 0.5 ohms, and load resistance 2.5 ohms.

b Rearrange the formula to make r the subject.

c Find the resistance of a circuit with current 1.5 amperes, voltage 7.725 V, and load resistance 5 ohms.

8 For the following matchstick pattern, find the number of matches M in the:

a 1st, 2nd, and 3rd diagram **b** nth diagram.

9 To convert temperatures from degrees Fahrenheit (°F) to Kelvin (K), we use the formula $K = \frac{5}{9}(F - 32) + 273.15$.

- **a** Convert the following temperatures to Kelvin, correct to 1 decimal place:
  - **i** 50°F
  - **ii** -130°F
  - **iii** 150°F
- **b** Rearrange the formula to make F the subject.
- **c** Convert the following temperatures to degrees Fahrenheit:
  - **i** 313.15 K
  - **ii** 0 K
  - **iii** 200 K

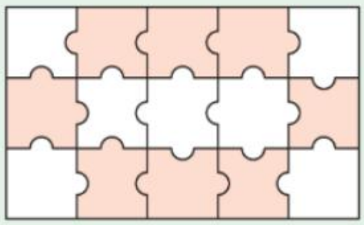
REVIEW SET 15B

1 Write a formula for the bill \$$B$ at a restaurant if there is a charge of:

- **a** \$15 for corkage, plus \$25 per person for 5 people
- **b** \$$c$ for corkage, plus \$25 per person for p people
- **c** \$$c$ for corkage, plus \$$m$ per person for p people.

2 Write a formula for the number of edge pieces E (excluding corner pieces) in a:

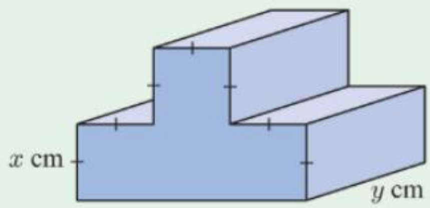
- **a** 3×5 jigsaw puzzle
- **b** 4×8 jigsaw puzzle
- **c** $m \times n$ jigsaw puzzle.



3 Consider the formula $M = p - qr$. Find:

- **a** M when $p = 19$, $q = -3$, and $r = 6$
- **b** r when $M = -2$, $p = 14$, and $q = 2$.

4 Find a formula for the volume V of the solid of uniform cross-section shown.



5 Make a the subject of:

- **a** $B = ad - f$
- **b** $\dfrac{Q}{\sqrt{a}} = \dfrac{t}{3}$
- **c** $G = \sqrt{\dfrac{5}{a+1}}$

6 Amy is trying to find pairs of numbers which have the same sum and product. In other words, she is looking for number pairs a and b such that $ab = a + b$.

- **a** Rearrange this formula to make b the subject.
- **b** Find b given that $a = 3$. Check your answer by finding the sum and product of the numbers.

7 Make x the subject of $y = \dfrac{4x - 3}{3x + 2}$.

8 Examine the matchstick pattern:

How many matchsticks make up the nth diagram?

9 **a** Find:

i $2+4$ **ii** $2+4+6$ **iii** $2+4+6+8$ **iv** $2+4+6+8+10$

b Hence write a formula for the sum of the first n positive even numbers.

10 The kinetic energy of an object with mass m kg which is moving with speed v m/s, is given by the formula $E = \frac{1}{2}mv^2$ joules, $v \geqslant 0$.

a Find the kinetic energy of a person with mass 80 kg moving at 5 m/s.

b Rearrange the formula to make v the subject.

c A running wombat with mass 25 kg has 800 joules of kinetic energy. Find the speed of the wombat.

Chapter 16

Simultaneous equations

Contents:

OPENING PROBLEM

A farmer has only hens and goats in an enclosure. He said to his daughter Susan, "You know we have only hens and goats. I counted 48 heads altogether and 122 legs. Can you tell me how many of each animal we have?"

It was too dark for Susan to go out and count them, so she decided to let the number of hens be x and the number of goats be y.

Things to think about:

a Can you explain why:
  i $x + y = 48$ **ii** $2x + 4y = 122$?

b How many solutions are there to the equation:
  i $x + y = 48$ **ii** $2x + 4y = 122$?

c Can you find values for x and y which make *both* equations true?

To answer the **Opening Problem** we need values for x and y which satisfy both $x + y = 48$ and $2x + 4y = 122$ *at the same time*.

We say that $\begin{cases} x + y = 48 \\ 2x + 4y = 122 \end{cases}$ is a set of **simultaneous equations**.

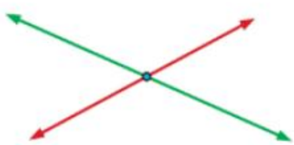
A **solution** to a set of simultaneous equations needs to satisfy *both* equations at the same time.

A GRAPHICAL SOLUTION

One way to solve simultaneous equations is to draw the graph of both equations on the same set of axes. Any **point of intersection** corresponds to a solution to the simultaneous equations.

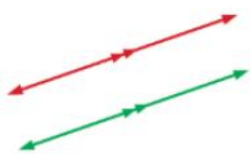
A set of two linear simultaneous equations can be represented by two straight lines. The simultaneous equations may have either one, zero, or infinitely many solutions.

intersecting lines



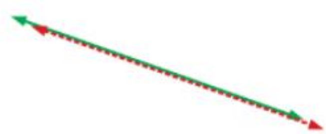
one solution

parallel lines



no solutions

coincident lines



infinitely many solutions

Example 1 **Self Tutor**

Solve the simultaneous equations graphically: $\begin{cases} y = 2x + 1 \\ y = -3x + 11 \end{cases}$

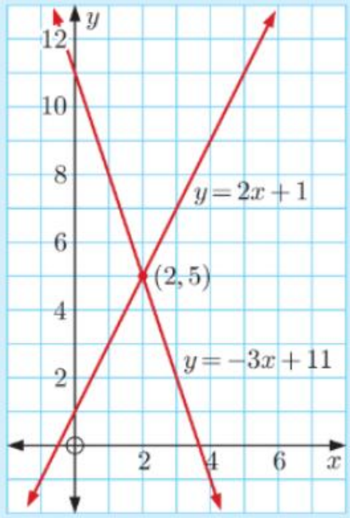
For $y = 2x + 1$:
when $x = 0$, $y = 1$ and when $y = 0$, $x = -\frac{1}{2}$.

For $y = -3x + 11$:
when $x = 0$, $y = 11$ and when $y = 0$, $x = \frac{11}{3}$.

We draw the graphs of $y = 2x + 1$ and $y = -3x + 11$ on the same set of axes.

The graphs meet at the point $(2, 5)$.

$\therefore$ the solution is $x = 2$, $y = 5$.

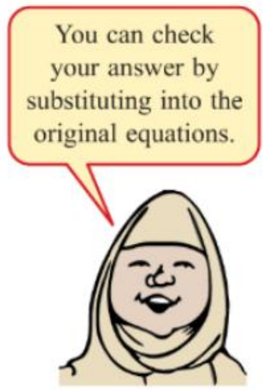


EXERCISE 16A.1

1 Solve the simultaneous equations graphically:

a $\begin{cases} y = x \\ y = 4 - x \end{cases}$ **b** $\begin{cases} y = x - 4 \\ y = -3x \end{cases}$ **c** $\begin{cases} y = 2x + 3 \\ y = x + 3 \end{cases}$

d $\begin{cases} y = -3x + 1 \\ y = -x - 1 \end{cases}$ **e** $\begin{cases} y = 5 - 2x \\ y = x - 1 \end{cases}$ **f** $\begin{cases} y = 3x + 4 \\ y = 2x + 1 \end{cases}$



2 Solve the simultaneous equations graphically:

a $\begin{cases} x - y = 2 \\ 3x - y = -6 \end{cases}$ **b** $\begin{cases} x - y = -1 \\ x + y = 7 \end{cases}$ **c** $\begin{cases} x + y = 5 \\ x - y = -4 \end{cases}$

d $\begin{cases} x + y = -2 \\ 6x - y = 9 \end{cases}$ **e** $\begin{cases} 3x - y = -2 \\ 2x - y = -3 \end{cases}$ **f** $\begin{cases} 5x - y = -1 \\ 2x - y = 5 \end{cases}$

3 Try to solve the following simultaneous equations graphically. State the number of solutions in each case.

a $\begin{cases} y = 3x + 2 \\ y = 3x - 1 \end{cases}$ **b** $\begin{cases} x + 2y = 4 \\ y = -\frac{1}{2}x + 2 \end{cases}$

USING TECHNOLOGY

You can also use your **graphics calculator** or the **graphing package** to graph the equations and find the intersection of the graphs. This is particularly useful if the solutions to the set of simultaneous equations are not integers.

GRAPHICS CALCULATOR INSTRUCTIONS

GRAPHING PACKAGE

When using your graphics calculator, any equations in the general form $Ax + By = C$ will need to be rearranged to make y the subject.

Example 2 **Self Tutor**

Use technology to solve the following simultaneous equations: $\begin{cases} y = 2x + 7 \\ x + 2y = 5 \end{cases}$

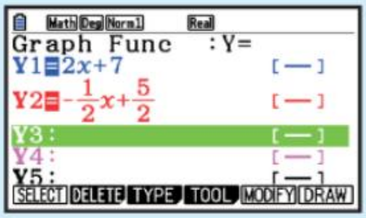
We need to rearrange the second equation to make y the subject.

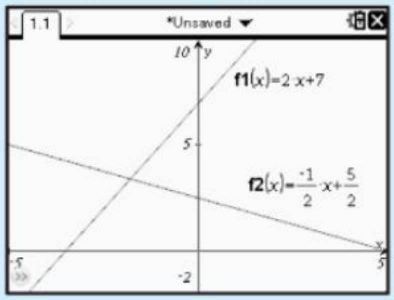
$$\begin{aligned} x + 2y &= 5 \\ \therefore\ x + 2y - x &= 5 - x \qquad \{\text{subtracting } x \text{ from both sides}\} \\ \therefore\ 2y &= 5 - x \\ \therefore\ y &= \frac{5}{2} - \frac{x}{2} \qquad \{\text{dividing both sides by } 2\} \end{aligned}$$

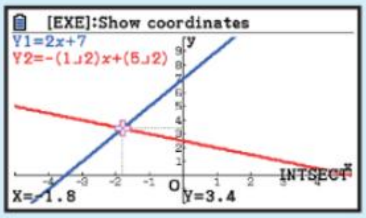
The system is now: $\begin{cases} y = 2x + 7 \\ y = -\frac{1}{2}x + \frac{5}{2} \end{cases}$

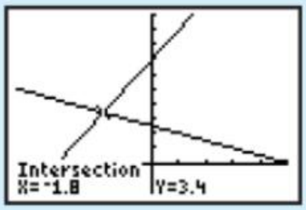
Rearranging an equation does not change its solutions.

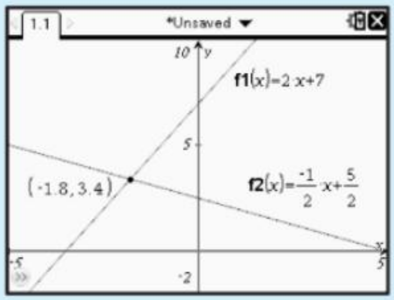
Casio fx-CG20 **TI-84 Plus** **TI-$n$spire**











The solution is $x = -1.8,\ y = 3.4$.

EXERCISE 16A.2

1 Use technology to solve the following simultaneous equations:

a $\begin{cases} y = 2x - 3 \\ y = x + 1 \end{cases}$ **b** $\begin{cases} y = 5x + 1 \\ y = 4 - x \end{cases}$ **c** $\begin{cases} y = x + 4 \\ y = 3x + 2 \end{cases}$

2 Use technology to solve the following simultaneous equations:

a $\begin{cases} 2x + y = 30 \\ x - 3y = 22 \end{cases}$ **b** $\begin{cases} x - y = -19 \\ 2x + 3y = -13 \end{cases}$ **c** $\begin{cases} x + 2y = 39 \\ 3x - 2y = 45 \end{cases}$

d $\begin{cases} 2x + 3y = 35 \\ 3x - y = -30 \end{cases}$ **e** $\begin{cases} 5x + 7y = 20 \\ 3x + 5y = 13 \end{cases}$ **f** $\begin{cases} 6x - y = 78 \\ 7x - 2y = 80 \end{cases}$

B SOLUTION BY EQUATING VALUES OF y

We will now consider some **algebraic** methods for solving linear simultaneous equations. Algebraic methods are often quicker than graphing, and more accurate if the solutions are not integers.

If y is the subject of *both* equations, we equate the y values and solve for x.

Example 3 Self Tutor

Solve the simultaneous equations: $\begin{cases} y = x + 1 \\ y = 2x - 3 \end{cases}$

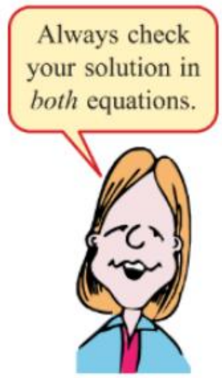
If $y = x + 1$ and $y = 2x - 3$, then

$$\begin{aligned} x + 1 &= 2x - 3 && \{\text{equating values of } y\} \\ \therefore\ x + 1 - x &= 2x - 3 - x && \{\text{subtracting } x \text{ from both sides}\} \\ \therefore\ 1 &= x - 3 \\ \therefore\ 1 + 3 &= x - 3 + 3 && \{\text{adding 3 to both sides}\} \\ \therefore\ 4 &= x \end{aligned}$$

Now $y = x + 1$
$\therefore\ y = 4 + 1$
$\therefore\ y = 5$

The solution is $x = 4$, $y = 5$.

Check: In $y = 2x - 3$, $y = 2 \times 4 - 3 = 8 - 3 = 5$ ✓



Example 4 Self Tutor

Find the point of intersection of the two lines $y = 2x + 5$ and $y = -x - 1$.

The lines meet when

$$\begin{aligned} 2x + 5 &= -x - 1 && \{\text{equating values of } y\} \\ \therefore\ 2x + 5 + x &= -x - 1 + x && \{\text{adding } x \text{ to both sides}\} \\ \therefore\ 3x + 5 &= -1 && \{\text{collecting like terms}\} \\ \therefore\ 3x + 5 - 5 &= -1 - 5 && \{\text{subtracting 5 from both sides}\} \\ \therefore\ 3x &= -6 && \{\text{collecting like terms}\} \\ \therefore\ x &= -2 && \{\text{dividing both sides by 3}\} \end{aligned}$$

When $x = -2$, $y = 2 \times (-2) + 5$ $\{\text{using } y = 2x + 5\}$
$\therefore\ y = -4 + 5$
$\therefore\ y = 1$

So, the lines meet at $(-2, 1)$.

EXERCISE 16B

1 Solve simultaneously by equating values of y:

a $\begin{cases} y = 3x + 2 \\ y = 2x + 3 \end{cases}$ **b** $\begin{cases} y = x + 2 \\ y = 2x - 3 \end{cases}$ **c** $\begin{cases} y = 6x - 6 \\ y = x + 4 \end{cases}$

d $\begin{cases} y = 2x + 1 \\ y = x - 3 \end{cases}$ **e** $\begin{cases} y = 5x + 2 \\ y = 3x - 2 \end{cases}$ **f** $\begin{cases} y = x - 2 \\ y = 3x + 6 \end{cases}$

2 Find the point of intersection of the two lines:

a $y = x + 1$ and $y = 7 - x$ **b** $y = x + 4$ and $y = 5 - x$

c $y = 2x - 5$ and $y = 3 - 2x$ **d** $y = x - 4$ and $y = -2x - 4$

e $y = 3x + 2$ and $y = -2x - 3$ **f** $y = 4x + 6$ and $y = 6 - 2x$

C SOLUTION BY SUBSTITUTION

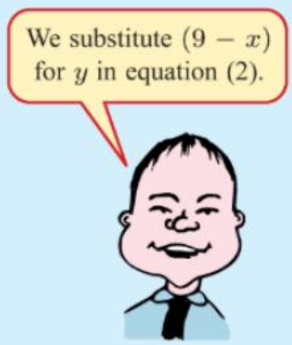
The method of **solution by substitution** is used when at least one equation is given with either x or y as the **subject** of the formula. We substitute an expression for this variable into the other equation.

Example 5 **Self Tutor**

Solve simultaneously by substitution: $\begin{cases} y = 9 - x \\ 2x + 3y = 21 \end{cases}$

$y = 9 - x$ .... (1)
$2x + 3y = 21$ .... (2)

Substituting (1) into (2) gives $2x + 3(9 - x) = 21$
$\therefore\ 2x + 27 - 3x = 21$
$\therefore\ 27 - x = 21$
$\therefore\ x = 6$



Substituting $x = 6$ into (1) gives $y = 9 - 6$
$\therefore\ y = 3$

The solution is $x = 6$, $y = 3$.

Check: In (1): $3 = 9 - 6$ ✓
In (2): $2(6) + 3(3) = 12 + 9 = 21$ ✓

EXERCISE 16C

1 Solve simultaneously by substitution:

a $\begin{cases} y = 3 + x \\ 5x - 2y = 0 \end{cases}$ **b** $\begin{cases} y = x - 2 \\ x + 3y = 6 \end{cases}$ **c** $\begin{cases} y = 5 - x \\ 4x + y = 5 \end{cases}$

d $\begin{cases} y = 2x - 1 \\ 3x - y = 6 \end{cases}$ **e** $\begin{cases} y = 3x + 4 \\ 2x + 3y = 12 \end{cases}$ **f** $\begin{cases} y = 5 - 2x \\ 5x - 2y = 8 \end{cases}$

Example 6 **Self Tutor**

Solve simultaneously by substitution: $\begin{cases} 2y - x = 2 \\ x = 1 + 8y \end{cases}$

$2y - x = 2$ .... (1)
$x = 1 + 8y$ .... (2)

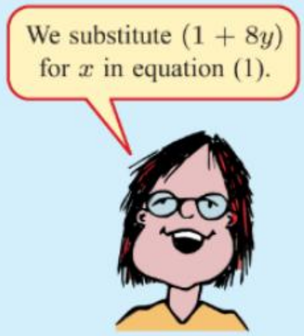
Substituting (2) into (1) gives

$$\begin{aligned} 2y - (1 + 8y) &= 2 \\ \therefore\ 2y - 1 - 8y &= 2 \\ \therefore\ -6y &= 3 \\ \therefore\ y &= -\tfrac{1}{2} \end{aligned}$$

Substituting $y = -\frac{1}{2}$ into (2) gives
$x = 1 + 8 \times -\frac{1}{2} = -3$

The solution is $x = -3$, $y = -\frac{1}{2}$.

Check: (1) $2(-\frac{1}{2}) - (-3) = -1 + 3 = 2$ ✓
(2) $1 + 8(-\frac{1}{2}) = 1 - 4 = -3$ ✓



2 Solve simultaneously by substitution:

a $\begin{cases} x = y + 2 \\ 3x - 2y = 9 \end{cases}$ **b** $\begin{cases} x = -1 + 5y \\ x = 3 - 5y \end{cases}$ **c** $\begin{cases} x = 6 - 3y \\ 3x - 3y = 2 \end{cases}$

d $\begin{cases} x = 1 - 2y \\ 2x + 3y = 4 \end{cases}$ **e** $\begin{cases} x = -4 - 2y \\ 2y - 3x = 8 \end{cases}$ **f** $\begin{cases} x = -y - 8 \\ 2x - 4y = 5 \end{cases}$

3 Try to solve by substitution:

a $\begin{cases} y = 2x + 5 \\ 4x - 2y = -14 \end{cases}$ **b** $\begin{cases} y = 4x + 3 \\ 8x - 2y = -6 \end{cases}$

Comment on your results.

D SOLUTION BY ELIMINATION

Consider the simultaneous equations $\begin{cases} 3x + 4y = 10 \\ 5x - 4y = 6 \end{cases}$.

Neither equation has x or y given as the subject, so solution by substitution is tedious. We instead use the method of **elimination**.

In this method, we make the coefficients of x (or y) the **same size** but **opposite in sign**. We then **add** the equations, which has the effect of **eliminating** one of the variables.

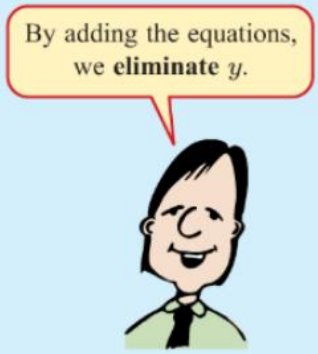
Example 7 **Self Tutor**

Solve simultaneously by elimination: $\begin{cases} 3x + 4y = 10 \\ 5x - 4y = 6 \end{cases}$

The coefficients of y are the same size but opposite in sign.

We **add** the LHSs and the RHSs to get an equation which contains x only.

$$\begin{array}{rll} & 3x + 4y = 10 & .... \ (1) \\ & 5x - 4y = 6 & .... \ (2) \\ \hline \text{Adding,} & 8x \qquad = 16 & \\ & \therefore \ x = 2 & \end{array}$$



Substituting $x = 2$ into (1) gives
$$\begin{aligned} 3(2) + 4y &= 10 \\ \therefore \ 6 + 4y &= 10 \\ \therefore \ 4y &= 4 \\ \therefore \ y &= 1 \end{aligned}$$

The solution is $x = 2, \ \ y = 1$.

Check: In (2): $5(2) - 4(1) = 10 - 4 = 6$ ✓

In problems where the coefficients of x (or y) are **not** the **same size** or **opposite in sign**, we must first **multiply** one or both equations by a constant.

Example 8 **Self Tutor**

Solve by elimination: $\begin{cases} 3x + 4y = 14 \\ 4x + 5y = 17 \end{cases}$

$3x + 4y = 14 \quad .... \ (1)$
$4x + 5y = 17 \quad .... \ (2)$

To make the coefficients of x the same size but opposite in sign, we multiply (1) by 4 and (2) by -3.

$$\begin{array}{rll} & 12x + 16y = 56 & .... \ \{(1) \times 4\} \\ & -12x - 15y = -51 & .... \ \{(2) \times -3\} \\ \hline \text{Adding,} & y = 5 & \end{array}$$

Substituting $y = 5$ into (2) gives
$$\begin{aligned} 4x + 5(5) &= 17 \\ \therefore \ 4x + 25 &= 17 \\ \therefore \ 4x &= -8 \\ \therefore \ x &= -2 \end{aligned}$$

The solution is $x = -2, \ \ y = 5$.

Check: In (1): $3(-2) + 4(5) = (-6) + 20 = 14$ ✓

EXERCISE 16D

1 What equation results when the following are added vertically?

a $3x + y = 12$
$x - y = 8$

b $x - 2y = -3$
$2x + 2y = 6$

c $3x + 2y = 4$
$-3x - y = 1$

2 Solve simultaneously by elimination:

a $\begin{cases} 2x + y = 6 \\ x - y = 6 \end{cases}$ **b** $\begin{cases} -2x + y = 18 \\ 2x - 3y = 2 \end{cases}$ **c** $\begin{cases} 3x + 2y = 0 \\ -3x + y = 9 \end{cases}$

d $\begin{cases} 5x + y = -1 \\ 3x - y = -7 \end{cases}$ **e** $\begin{cases} -4x - 3y = 1 \\ 4x + y = -3 \end{cases}$ **f** $\begin{cases} 4x - 3y = 8 \\ -x + 3y = 1 \end{cases}$

3 Give the equation that results when both sides of the equation:

a $2x + 3y = 1$ are multiplied by 2
b $-x + 2y = 7$ are multiplied by -2
c $4x - y = -2$ are multiplied by 3
d $3x - 4y = -3$ are multiplied by -5.

4 Solve simultaneously by elimination:

a $\begin{cases} 3x - 9y = -3 \\ -x + 4y = 2 \end{cases}$ **b** $\begin{cases} x - 2y = -5 \\ 2x + y = -5 \end{cases}$ **c** $\begin{cases} 3x + 2y = 1 \\ 5x - 4y = -13 \end{cases}$

d $\begin{cases} 3x + 2y = 11 \\ 9x - 5y = 22 \end{cases}$ **e** $\begin{cases} 5x - 4y = -12 \\ x + 3y = -10 \end{cases}$ **f** $\begin{cases} 4x - 3y = 6 \\ 7x + 12y = 45 \end{cases}$

5 Solve simultaneously by elimination:

a $\begin{cases} 3x + 4y = 8 \\ -2x + 3y = -11 \end{cases}$ **b** $\begin{cases} 5x - 2y = 15 \\ 4x + 3y = 12 \end{cases}$ **c** $\begin{cases} 7x + 2y = -10 \\ 3x - 5y = -16 \end{cases}$

d $\begin{cases} -4x + 3y = -5 \\ 3x + 2y = -9 \end{cases}$ **e** $\begin{cases} -5x + 4y = 3 \\ 3x + 5y = 50 \end{cases}$ **f** $\begin{cases} 2x - 7y = -44 \\ -5x + 3y = 23 \end{cases}$

E PROBLEM SOLVING

Many problems can be described mathematically by a **pair of linear equations**. The **Opening Problem** is one example.

Once the equations are formed, they can be solved simultaneously. We can then answer the original problem.

Step 1: Decide on two unknowns such as x and y. Do not forget the units.
Step 2: Write down **two** equations connecting x and y.
Step 3: Solve the equations simultaneously.
Step 4: Check your solutions with the original data given.
Step 5: Give your answer in sentence form.

The form of the original equations will help you decide whether to use substitution, elimination, or technology to solve them simultaneously.

Example 9 — Self Tutor

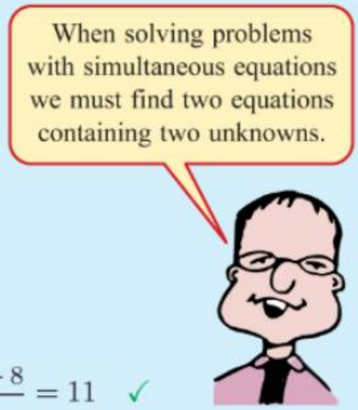
Two numbers have a difference of 6 and a mean of 11. Find the numbers.

Let x and y be the unknown numbers, where $x > y$.

The difference between x and y is $x - y = 6$ .... (1)

The mean of x and y is $\dfrac{x+y}{2} = 11$ .... (2)

$$\begin{array}{rll} & x - y = 6 & \{(1)\} \\ & x + y = 22 & \{(2) \times 2\} \\ \hline \text{Adding,} & 2x = 28 & \\ & \therefore\ x = 14 & \end{array}$$



Substituting $x = 14$ into (1) gives $14 - y = 6$

$\therefore\ y = 14 - 6$

$\therefore\ y = 8$

The numbers are 14 and 8.

Check: In (1): $14 - 8 = 6$ ✓ In (2): $\dfrac{14+8}{2} = 11$ ✓

EXERCISE 16E

1 The sum of two numbers is 33, and their difference is 15. Find the numbers.

2 The difference between two numbers is 11. The larger number is 1 more than twice the smaller number. Find the two numbers.

3 Two numbers have a difference of 17 and a mean of 32. Find the numbers.

Example 10 — Self Tutor

I have 8 coins in my pocket. They are \$1 and \$2 coins, and their total value is \$11. How many of each type of coin do I have?

Suppose I have x \$1 coins and y \$2 coins.

The number of coins is $x + y = 8$ .... (1)

The value of the coins is $x + 2y = 11$ .... (2)

$$\begin{array}{rll} & x + 2y = 11 & \{(2)\} \\ & -x - y = -8 & \{-(1)\} \\ \hline \text{Adding,} & y = 3 & \end{array}$$

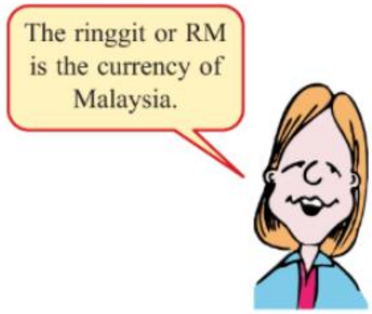
Substituting $y = 3$ into (1) gives $x + 3 = 8$

$\therefore\ x = 5$

$\therefore$ I have 5 \$1 coins and 3 \$2 coins in my pocket.

Check: In (2), $x + 2y = 5 + 2(3) = 11$ ✓

4 My friend has 12 coins in her pocket. They are all either £1 or £2 coins, and the total value of her coins is £17. How many of each type does she have?



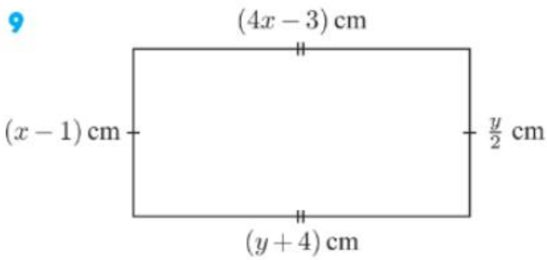
5 Answer the **Opening Problem** on page **306**.

6 At a music store, all CDs have one price and all DVDs have another price. Kim bought 4 CDs and 3 DVDs for a total of RM99. Li bought 3 CDs and 2 DVDs for a total of RM70. Find the cost of each of these items.

7 A carpenter is making chairs with 4 legs, and stools with 3 legs. He has 23 seats and 86 legs which can be used for stools and chairs. If he uses all of the legs and seats, how many of each item does he make?

8 17 small bags of potatoes and 13 large bags of potatoes weigh a total of 99 kg. 15 small bags and 21 large bags weigh a total of 135 kg. Find the weight of each size of bag.

9



a Find x and y.

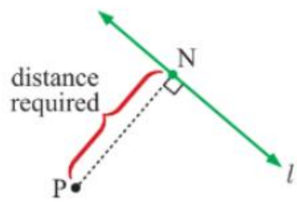
b Hence find the perimeter of the rectangle.

10 Five years ago, a woman was four times as old as her daughter. The woman was 25 when she gave birth. How old is the daughter now?

INVESTIGATION DISTANCE FROM A POINT TO A LINE

When we talk about the distance from a point to a line, we actually mean the *shortest* distance from the point to the line.

The distance from a point P to a line l is the distance from P to N, where N is the point on l such that [NP] is perpendicular to l.



To find the shortest distance from a point P to a line l we follow these steps:

Step 1: Find the gradient of the line l, and hence the gradient of [NP].

Step 2: Find the equation of the line segment [NP].

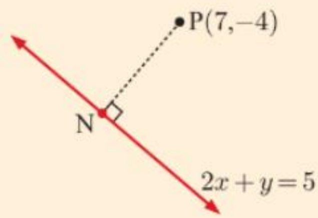
Step 3: Find the coordinates of N by solving simultaneously the equations of line l and line segment [NP].

Step 4: Find the distance NP using the distance formula.

For example, suppose we want to find the distance from $P(7, -4)$ to the line with equation $2x + y = 5$.

Step 1: The gradient of $2x + y = 5$ is $-\frac{2}{1}$

$\therefore$ the gradient of [NP] is $\frac{1}{2}$



Step 2: The equation of [NP] is

$x - 2y = (7) - 2(-4)$

which is $x - 2y = 15$

Step 3: We now solve simultaneously: $\begin{cases} 2x + y = 5 & \text{.... (1)} \\ x - 2y = 15 & \text{.... (2)} \end{cases}$

$$\begin{aligned} 4x + 2y &= 10 \quad \{(1) \times 2\} \\ x - 2y &= 15 \quad \{(2)\} \\ \text{Adding,}\quad 5x \quad &= 25 \\ \therefore\ x \quad &= 5 \end{aligned}$$

When $x = 5$, $2(5) + y = 5$

$\therefore\ 10 + y = 5$

$\therefore\ y = -5$

$\therefore$ N is $(5, -5)$.

Step 4: $\text{NP} = \sqrt{(7-5)^2 + (-4 - -5)^2}$

$= \sqrt{2^2 + 1^2}$

$= \sqrt{5}$ units

What to do:

1 Find the distance from:

a $(7, -4)$ to $y = 3x - 5$

b $(-6, 0)$ to $y = 3 - 2x$

c $(8, -5)$ to $y = -2x - 4$

d $(-10, 9)$ to $y = -4x + 3$

e $(-2, 8)$ to $3x - y = 6$

f $(1, 7)$ to $4x - 3y = 8$.

2 Find the distance between the following pairs of parallel lines:

a $y = 3x + 2$ and $y = 3x - 8$

b $3x + 4y = 4$ and $3x + 4y = -16$

Hint: Find any point on one of the lines, then find the distance from this point to the other line.

3 A straight water pipeline passes through two points with map references $(3, 2)$ and $(7, -1)$. The shortest spur pipe from the pipeline to the farm at $P(9, 7)$ is [NP].

a Find the coordinates of N.

b Find the length of the pipeline [NP] given that the grid reference scale is $1 \text{ unit} \equiv 0.5$ km.

DISCUSSION

- If you are presented with two linear equations to be solved simultaneously, what features suggest the method that should be used?
- Which methods could be used to solve simultanteous equations which are not linear?

- Here is a method for solving simultaneously the linear equation $2x - y = -4$ and the quadratic equation $y = x^2 - 3x - 2$:

$$\begin{aligned}
&\text{Using } 2x - y = -4,\\
&\quad \therefore\ 2x = -4 + y && \{\text{adding } y \text{ to both sides}\}\\
&\quad \therefore\ 2x + 4 = y && \{\text{adding } 4 \text{ to both sides}\}\\
&\text{Equating values for } y, \text{ we find that}\\
&\quad x^2 - 3x - 2 = 2x + 4\\
&\quad \therefore\ x^2 - 5x - 2 = 4 && \{\text{subtracting } 2x \text{ from both sides}\}\\
&\quad \therefore\ x^2 - 5x - 6 = 0 && \{\text{subtracting } 4 \text{ from both sides}\}\\
&\therefore\ (x-6)(x+1) = 0\\
&\quad \therefore\ x = -1 \text{ or } 6\\
&\text{Using } y = 2x + 4:\\
&\quad \text{If } x = -1 \text{ then } y = 2(-1) + 4 = 2 && \text{If } x = 6 \text{ then } y = 2(6) + 4 = 16\\
&\text{The simultaneous solutions are}\\
&\quad x = -1,\ y = 2 \quad \text{and} \quad x = 6,\ y = 16
\end{aligned}$$

How many simultaneous solutions could there be for:
- a linear equation and a quadratic equation
- two quadratic equations?

REVIEW SET 16A

1 Solve these simultaneous equations graphically:

a $\begin{cases} y = 2x \\ y = x - 2 \end{cases}$ **b** $\begin{cases} y = 2x - 4 \\ y = 1 - 3x \end{cases}$ **c** $\begin{cases} 2x + y = 7 \\ -x + 3y = 7 \end{cases}$

2 Use technology to solve:

a $\begin{cases} 7x - 5y = 92 \\ -3x + 18y = -87 \end{cases}$ **b** $\begin{cases} 5x - 7y = 13 \\ 2x + 5y = -4 \end{cases}$

3 Solve by equating values of y:

a $\begin{cases} y = 6x - 5 \\ y = 2x + 3 \end{cases}$ **b** $\begin{cases} y = 3x + 4 \\ y = -2x - 6 \end{cases}$

4 Solve by substitution:

a $\begin{cases} y = 3x - 4 \\ 2x - y = 8 \end{cases}$ **b** $\begin{cases} y - 5x = 8 \\ y = 3x + 6 \end{cases}$

5 Try to solve $\begin{cases} x - 2y = 5 \\ x - 2y = 7 \end{cases}$ simultaneously. Interpret your result.

6 Solve by elimination:

a $\begin{cases} 3x + 2y = 4 \\ 2x - y = 5 \end{cases}$ **b** $\begin{cases} 3x - 2y = 3 \\ 4x + 3y = 4 \end{cases}$

7 The larger of two numbers is 2 more than three times the smaller number. The difference between the numbers is 12. Find the numbers.

8 At the local grocer, 3 apples and 5 oranges cost a total of $6.20. 7 apples and 4 oranges cost a total of $7.95. What is the cost of one apple and one orange?

REVIEW SET 16B

1 Solve these simultaneous equations graphically:

a $\begin{cases} y = -x + 5 \\ y = 3x + 1 \end{cases}$ **b** $\begin{cases} y = -2x + 1 \\ y = 2x - 5 \end{cases}$ **c** $\begin{cases} 3x - y = 16 \\ x + 4y = 1 \end{cases}$

2 Use technology to solve:

a $\begin{cases} -2x + 3y = 64 \\ 3x + y = 3 \end{cases}$ **b** $\begin{cases} 2x + 3y = 2.7 \\ -x + 7y = 5.2 \end{cases}$

3 Find the point of intersection of:

a $y = 5x - 2$ and $y = 2x + 10$ **b** $y = -3x + 2$ and $y = 2x + 7$

4 Solve by substitution:

a $\begin{cases} -2x + y = 9 \\ x = 3y - 7 \end{cases}$ **b** $\begin{cases} y = 4x - 11 \\ -2x + 9y = 3 \end{cases}$

5 Try to solve $\begin{cases} 2x + 4y = 2 \\ x = 1 - 2y \end{cases}$ simultaneously. Interpret your result.

6 Solve by elimination:

a $\begin{cases} 3x + y = 16 \\ -2x - y = -7 \end{cases}$ **b** $\begin{cases} 5x - 2y = 9 \\ -2x + 3y = -10 \end{cases}$

7 While Morgan was on safari, he saw giraffes and ostriches. He counted 39 heads and 114 legs. How many giraffes and how many ostriches did he see?

8 The drinks van at a carnival sells small and large bottles of soda. Susan bought 3 small bottles and 2 large bottles for a total of €11.55. Thao bought 2 small bottles and 3 large bottles for a total of €12.95. Find the price of each size bottle.

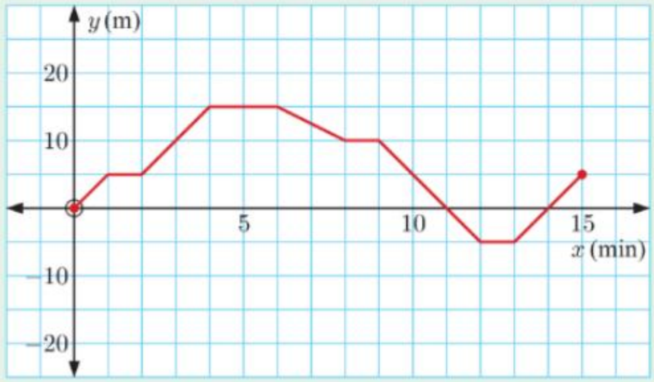
Chapter 17

Relations and functions

Contents:

OPENING PROBLEM

Fernando and Gerard are competing in a cycling race. The graph alongside shows how far Fernando is ahead of Gerard after x minutes.



Things to think about:

a What variables are connected by this graph?

b For what values of x are there corresponding values of y? What does this tell us about the race?

c Who won the race? How do you know?

d During the race, which cyclist had the bigger lead? How far ahead was he?

A RELATIONS

Sue-Ellen wants to send a parcel to a friend. The cost of posting the parcel a fixed distance is determined by the weight of the parcel, as shown in the table.

Weight w (kg)	*Cost* \$$C$
$1 \leqslant w < 2$	\$5.00
$2 \leqslant w < 5$	\$8.00
$5 \leqslant w < 10$	\$12.00
$10 \leqslant w < 15$	\$16.00
$15 \leqslant w < 20$	\$20.00

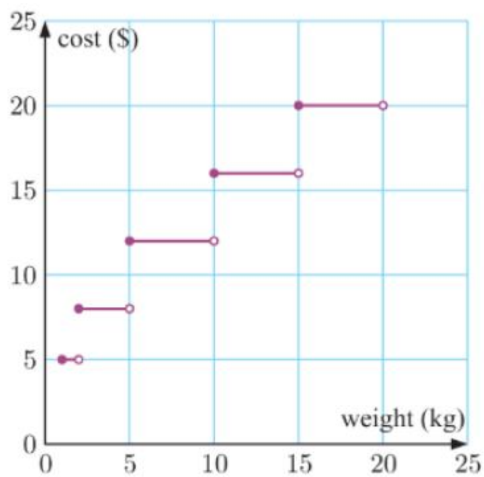
For example, it will cost \$8.00 to post a parcel weighing at least 2 kg but less than 5 kg. It will therefore cost \$8.00 to post a parcel weighing 2 kg or 3.6 kg or 4.955 kg.

We can illustrate the postal charges on a graph.

An end point that is included has a filled in circle.

An end point that is not included has an open circle.

There is a *relationship* between the variables *weight* and *cost*, so the table of costs is an example of a **relation**.



A relation may be a finite number of ordered pairs, for example $\{(2, 8), (3, 8), (4, 8), (5, 12)\}$, or an infinite number of ordered pairs, such as the relation between the variables *weight* and *cost* in the postal charges above.

A **relation** is any set of points which connects two variables.

We often use the graph of a relation to help us understand the connection between the variables.

Example 1 — **Self Tutor**

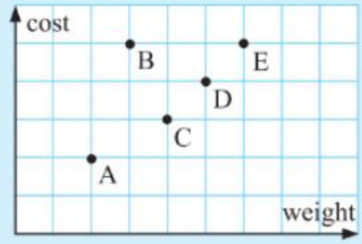
The given graph shows the weight and cost of 5 statues in a garden shop.

a What are the variables?

b Which statue is the heaviest?

c Which statues are the most expensive?

d Which statue is the least expensive?



a The variables are the *weight* of a statue and the *cost* of a statue.

b E has the largest value on the weight axis. $\therefore$ E is the heaviest.

c B and E are the highest points on the cost axis.
$\therefore$ B and E are the most expensive.

d A is the lowest point on the cost axis. $\therefore$ A is the least expensive.

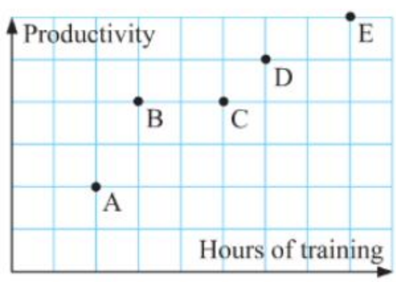
EXERCISE 17A

1 The graph shows productivity for various hours of training for five employees.

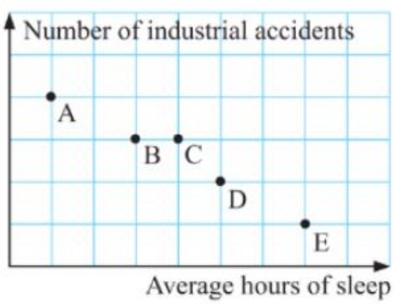
a What are the variables?

b Who received the least hours of training?

c Who is the most productive?



2 The workers of five factories were surveyed on their average hours of sleep. The results were graphed against the number of industrial accidents in the factories.



a Which factory had the most accidents?

b The workers of which factory averaged the least number of hours of sleep?

c Did any factories have the same number of accidents?

d Comment on the relationship between the average hours of sleep of the workers, and the number of industrial accidents.

3 Managers of a retail store conduct a customer count to help them decide how to roster sales staff. The results are shown in the line graph.

a What are the variables?

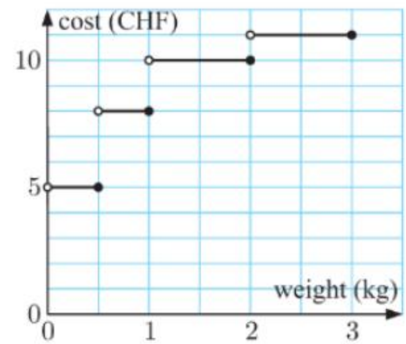
b At what time was the number of people in the store:

i greatest **ii** least?

c Describe what happened in the store between 3 pm and 4 pm.

d Use the graph to estimate the number of people in the store at 9:30 am.

4 A business in Switzerland frequently sends small parcels to the UK. The rates are given by the step graph alongside.



a Find the cost of sending a parcel weighing:

i 325 grams **ii** 1.2 kg.

b What is the heaviest parcel that could be sent for CHF10?

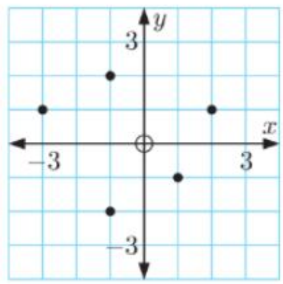
c Write out the information given in the graph in table form.

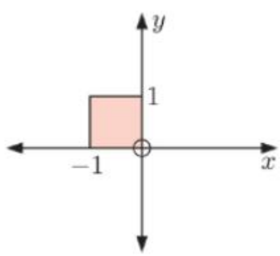
B DOMAIN AND RANGE

The relation between two variables x and y can be represented on a graph by a set of points with coordinates (x, y).

For example:

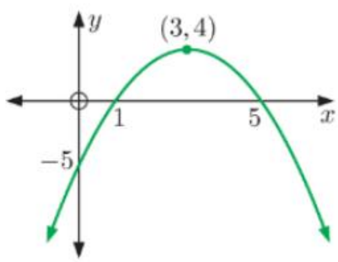
- This set of 5 points is a relation.
  There is no equation connecting the variables x and y in this case, but we can list the set of points:
  $\{(-3, 1), (-1, 2), (2, 1), (1, -1), (-1, -2)\}$.



- 

  The set of all points on and within the illustrated square is a relation. It is the set of all points (x, y) such that $-1 \leqslant x \leqslant 0$ and $0 \leqslant y \leqslant 1$.

- The set of all points on this curve is a relation.
  It is the set of all points (x, y) such that
  $y = -x^2 + 6x - 5$.

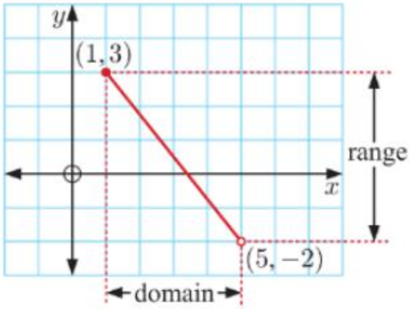


DOMAIN AND RANGE

> The **domain** of a relation is the set of possible values that x may have.
> The **range** of a relation is the set of possible values that y may have.

The domain and range of a relation are often described using **interval notation**.

Consider the following examples:

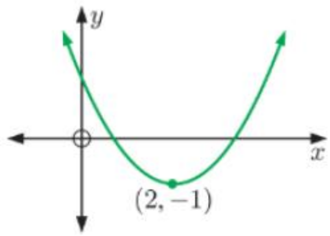
- 

The domain consists of all real x such that $1 \leqslant x < 5$. We write this as:

$$\{x \mid 1 \leqslant x < 5\}.$$

the set of all   such that

The range is $\{y \mid -2 < y \leqslant 3\}$.

- 

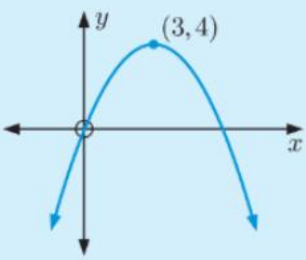
The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \geqslant -1\}$.

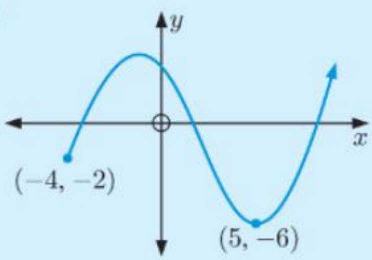
• indicates the point is included.
∘ indicates the point is excluded.

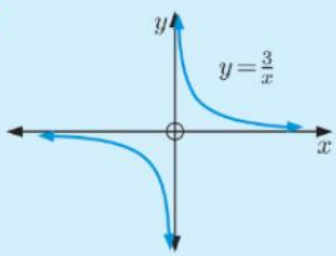
$\mathbb{R}$ is the set of all real numbers. It includes all numbers on the number line.

Example 2 **Self Tutor**

For each of the following graphs, state the domain and range:

a 

b 

c 

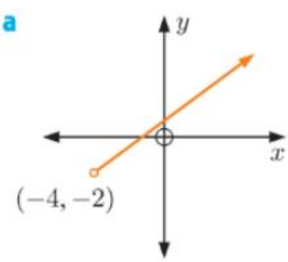
a Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \leqslant 4\}$.

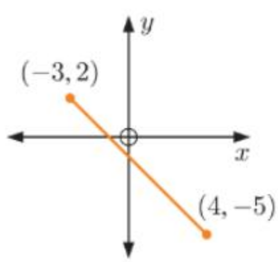
b Domain is $\{x \mid x \geqslant -4\}$.
Range is $\{y \mid y \geqslant -6\}$.

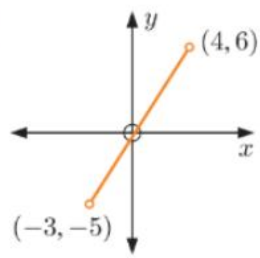
c Domain is $\{x \mid x \neq 0\}$.
Range is $\{y \mid y \neq 0\}$.

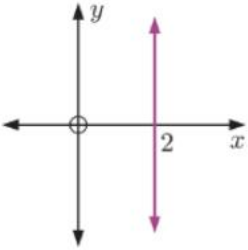
EXERCISE 17B

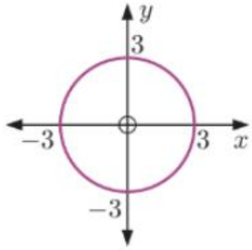
1 For each of the following graphs, state the domain and range:

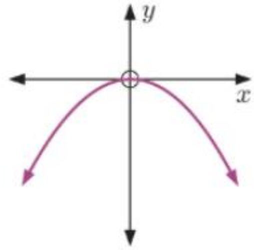
a 

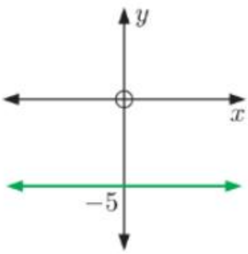
b 

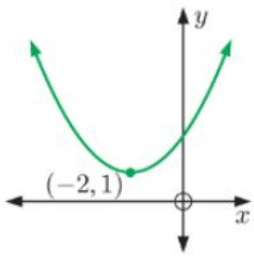
c 

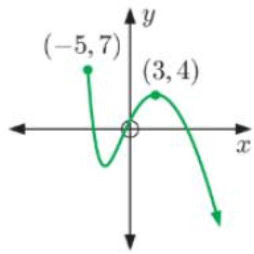
d


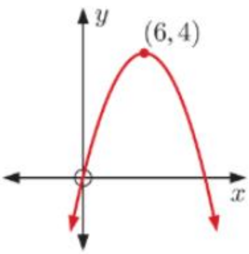
e


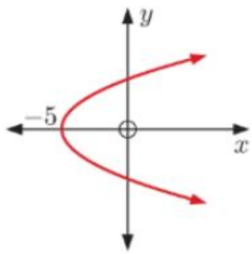
f


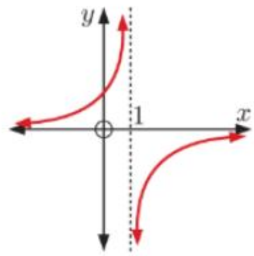
g


h


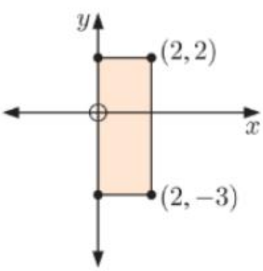
i


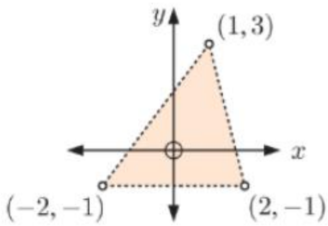
j


k


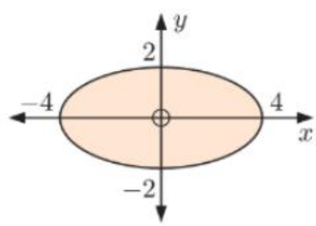
l


2 State the domain and range of:

a


b


(boundaries not included)

c


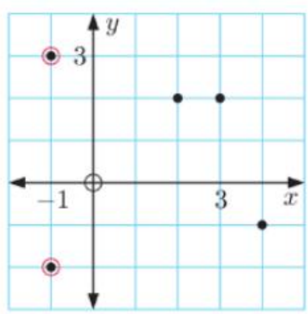
C FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same first member.

The graph alongside shows the set of points:

$\{(-1, 3), (2, 2), (-1, -2), (3, 2), (4, -1)\}$.

The two circled points $(-1, -2)$ and $(-1, 3)$ have the same first member, so the set of points is a relation but *not* a function.



GEOMETRIC TEST FOR FUNCTIONS: "VERTICAL LINE TEST"

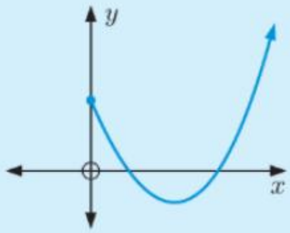
Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If *any* line cuts the graph more than once, then the relation is *not* a function.

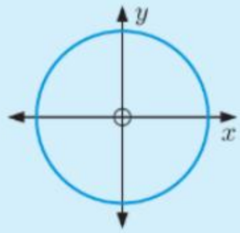
Example 3 **Self Tutor**

Which of these relations are functions?

a

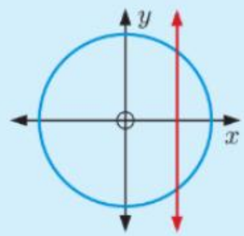


b



a Every vertical line we could draw cuts the graph at most once.
$\therefore$ the relation is a function.

b This vertical line cuts the graph twice.
$\therefore$ the relation is not a function.



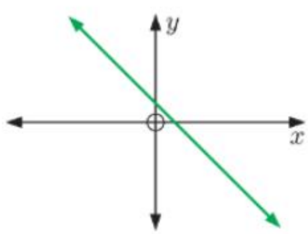
EXERCISE 17C

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

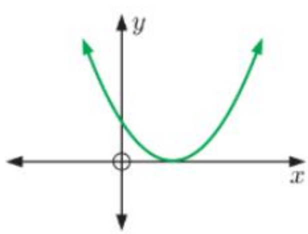
a $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
b $\{(-1, 2), (-3, 2), (3, 2), (1, 2)\}$
c $\{(2, 5), (-1, 4), (-3, 7), (2, -3)\}$
d $\{(3, -2), (3, 0), (3, 2), (3, 4)\}$
e $\{(-7, 0), (-5, 0), (-3, 0), (-1, 0)\}$
f $\{(0, 5), (0, 1), (2, 1), (2, -5)\}$

2 Use the vertical line test to determine which of the following relations are functions:

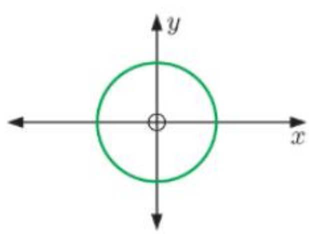
a



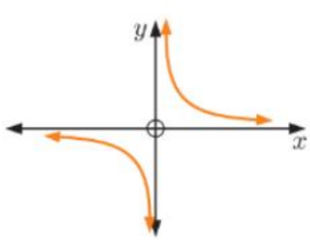
b



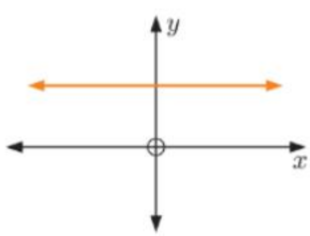
c



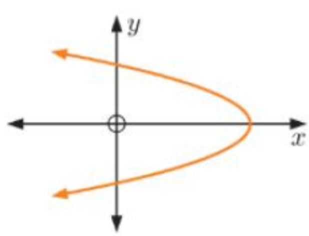
d



e



f



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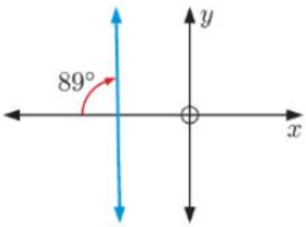
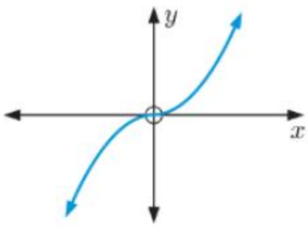
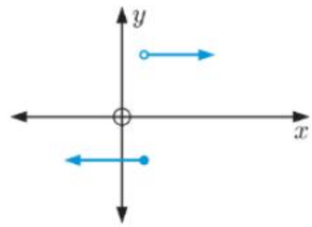


h



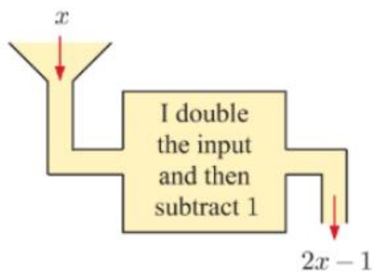
i



3 Will the graph of a straight line always be a function? Explain your answer.

D FUNCTION NOTATION

The 'function machine' alongside has been programmed to perform a particular function. Whatever number is fed into the machine, the machine will double the number and then subtract 1.



For example, if 3 is fed into the machine, then $2(3) - 1 = 5$ comes out.

If f is used to represent this particular function, we can write:

f is the function that will convert x into $2x - 1$.

This function can be written as: $\boldsymbol{f : x \mapsto 2x - 1}$

function f such that x is converted into $2x - 1$

or more simply as $f(x) = 2x - 1$.

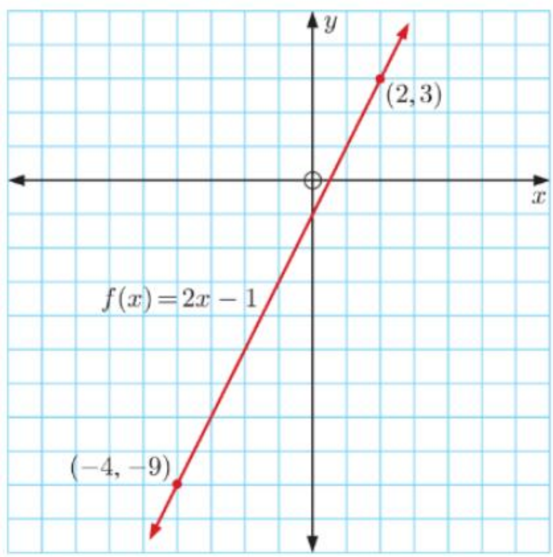
For any function f, the value of the function when $x = a$ is given by $f(a)$.

For $f(x) = 2x - 1$, $f(2) = 2(2) - 1 = 3$.

This indicates that the point (2, 3) lies on the graph of the function.

Likewise, $f(-4) = 2(-4) - 1 = -9$.

So, the point $(-4, -9)$ also lies on the graph.



Note that:

- $f(x)$ is read as "f of x", and is the value of the function at any value of x.
- f is the function which converts x into $f(x)$, so $f : x \mapsto f(x)$.
- $f(x)$ is sometimes called the **image** of x.

Example 4 — Self Tutor

For $f : x \mapsto 3x^2 - 4x$, find the value of:

a $f(2)$ **b** $f(-5)$

$f(x) = 3x^2 - 4x$

a $f(2)$
$= 3(2)^2 - 4(2)$ {replacing x by (2)}
$= 3 \times 4 - 8$
$= 4$

b $f(-5)$
$= 3(-5)^2 - 4(-5)$ {replacing x by (-5)}
$= 3(25) + 20$
$= 95$

Example 5 — Self Tutor

For $f(x) = 4 - 3x - x^2$, find in simplest form:

a $f(-x)$ **b** $f(x+2)$

a $f(-x) = 4 - 3(-x) - (-x)^2$ {replacing x by $(-x)$}
$= 4 + 3x - x^2$

b $f(x+2) = 4 - 3(x+2) - (x+2)^2$ {replacing x by $(x+2)$}
$= 4 - 3x - 6 - [x^2 + 4x + 4]$
$= -x^2 - 7x - 6$

EXERCISE 17D.1

1 For $f : x \mapsto 2x + 3$, find:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{2})$

2 For $g(x) = -5x + 3$, find:

a $g(1)$ **b** $g(4)$ **c** $g(-2)$ **d** $g(-x)$ **e** $g(x+4)$

3 For $f(x) = 2x^2 - 3x + 2$, find:

a $f(0)$ **b** $f(3)$ **c** $f(-4)$ **d** $f(-x)$ **e** $f(x+1)$

4 For $P : x \mapsto 4x^2 + 4x - 3$, find:

a $P(3)$ **b** $P(-1)$ **c** $P(\frac{1}{2})$ **d** $P(x-3)$ **e** $P(2x)$

5 Consider the function $R(x) = \dfrac{2x-3}{x+2}$.

a Evaluate:

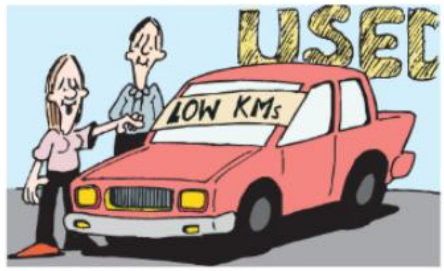
i $R(0)$ **ii** $R(1)$ **iii** $R(-\frac{1}{2})$

b Find a value of x such that $R(x)$ does not exist.

c Find $R(x-2)$ in simplest form.

d Find x such that $R(x) = -5$.

6 The value of a car t years after purchase is given by $V(t) = 28\,000 - 4000t$ dollars.

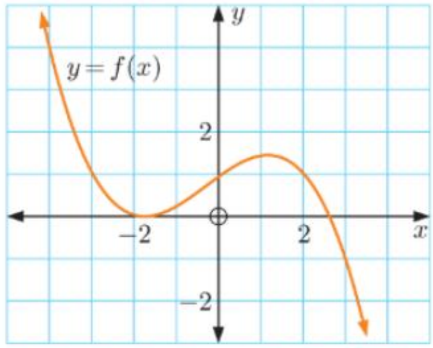


a Find $V(4)$, and state what this value means.

b Find t when $V(t) = 8000$, and explain what this value means.

c Find the original purchase price of the car.

7 The graph of $y = f(x)$ is shown alongside.

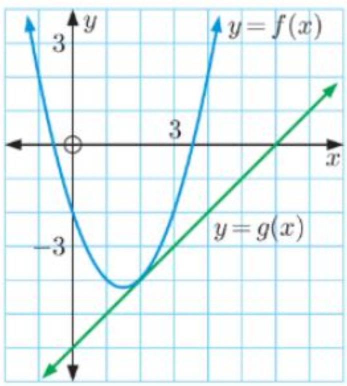


a Find:

i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.

8



Consider the graphs of $y = f(x)$ and $y = g(x)$ shown.

a Find:

i $f(4)$ **ii** $g(0)$ **iii** $g(5)$

b Find the *two* values of x such that $f(x) = -2$.

c Find the value of x such that $f(x) = g(x)$.

d Show that $g(x) = x - 6$.

9 Draw a graph of $y = f(x)$ such that $f(-2) = 5$, $f(1) = 0$, and $f(4) = 3$.

10 The graph of $y = f(x)$ is a straight line passing through $(-3, -5)$ and $(1, 7)$.

a Draw the graph of $y = f(x)$. **b** Find $f(-3)$ and $f(1)$.

c Find $f(x)$.

THE DOMAIN OF A FUNCTION

To find the domain of a function, we need to consider what values of the variable make the function undefined.

For example:

- the domain of $f(x) = \sqrt{x}$ is $\{x \mid x \geqslant 0,\ x \in \mathbb{R}\}$, since $\sqrt{x}$ has meaning only when $x \geqslant 0$
- the domain of $f(x) = \frac{1}{x}$ is $\{x \mid x \neq 0,\ x \in \mathbb{R}\}$, since $\frac{1}{0}$ is undefined
- the domain of $f(x) = \frac{1}{\sqrt{x-1}}$ is $\{x \mid x > 1,\ x \in \mathbb{R}\}$ since, when $x - 1 = 0$ we are 'dividing by zero', and when $x - 1 < 0$, $\sqrt{x-1}$ is undefined.

Example 6 **Self Tutor**

Find the domain of:

a $f(x) = \dfrac{1}{2-x}$ **b** $f(x) = \dfrac{2}{\sqrt{x+3}}$

a $f(x) = \dfrac{1}{2-x}$ is defined when $2 - x \neq 0$

$\therefore\ x \neq 2$

So, the domain of $f(x)$ is $\{x \mid x \neq 2,\ x \in \mathbb{R}\}$.

b $f(x) = \dfrac{2}{\sqrt{x+3}}$ is defined when $x + 3 > 0$

$\therefore\ x > -3$

So, the domain of $f(x)$ is $\{x \mid x > -3,\ x \in \mathbb{R}\}$.

EXERCISE 17D.2

1 Find the domain of:

a $f(x) = 2x$ **b** $f(x) = \dfrac{1}{3x}$ **c** $f(x) = \dfrac{1}{x-3}$

d $f(x) = \dfrac{1}{(x-1)(x+2)}$ **e** $f(x) = \dfrac{x}{x^2-9}$ **f** $f(x) = \dfrac{3}{x^2-5x+4}$

2 Find the domain of:

a $f(x) = \sqrt{x-2}$ **b** $f(x) = \sqrt{3-x}$

c $f(x) = \dfrac{1}{\sqrt{x}}$ **d** $f(x) = \dfrac{1}{\sqrt{4-x}}$

Check your answers using the graphing package.

GRAPHING PACKAGE

E THE MODULUS FUNCTION

MODULUS

The **modulus** or **absolute value** of a real number is its size, ignoring its sign. We denote the modulus of x by $|x|$.

For example, the modulus of 7 is 7, and the modulus of -7 is also 7, so we write $|7| = 7$ and $|-7| = 7$.

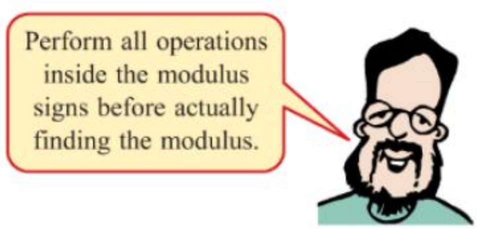
Example 7 **Self Tutor**

If $a = -7$ and $b = 3$, find:

a $|a+b|$ **b** $|ab|$

a
$$\begin{aligned} &|a+b| \\ &= |-7+3| \\ &= |-4| \\ &= 4 \end{aligned}$$

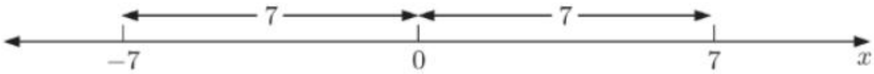
b
$$\begin{aligned} &|ab| \\ &= |-7 \times 3| \\ &= |-21| \\ &= 21 \end{aligned}$$



GEOMETRIC DEFINITION OF MODULUS

> $|x|$ is the distance of x from 0 on the number line.
> Because the modulus is a distance, it cannot be negative.

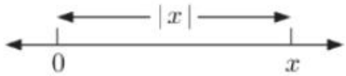
For example:



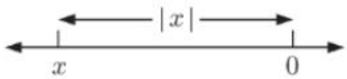
$|-7| = |7| = 7$

ALGEBRAIC DEFINITION OF MODULUS

If $x \geqslant 0$, the distance from 0 to x on the number line is simply x.
So, if $x \geqslant 0$, then $|x| = x$.



If $x < 0$, the distance from 0 to x on the number line is $-x$.
So, if $x < 0$, then $|x| = -x$.



INVESTIGATION ALGEBRAIC DEFINITION OF MODULUS

What to do:

1 Suppose $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$

a Copy and complete:

x	-3	-2	-1	0	1	2	3	4
$\lvert x \rvert$								

b Plot the points from the table and hence graph $y = |x|$.

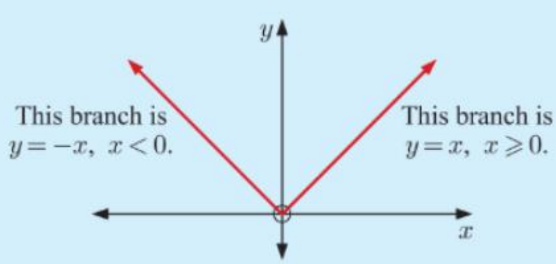
2 Copy and complete:

x	-3	-2	-1	0	1	2	3	4
$\sqrt{x^2}$								

3 What can you conclude from **1** and **2**?

From the **Investigation**, you should have found that:

- $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$ or $|x| = \sqrt{x^2}$
- $y = |x|$ has graph:

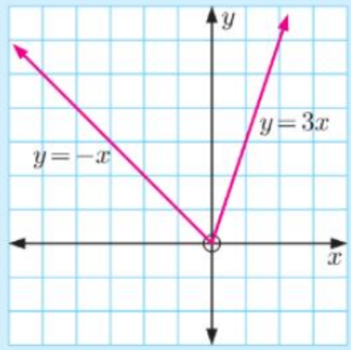


To draw graphs involving $|x|$, we must consider the cases $x \geqslant 0$ and $x < 0$ separately. This allows us to write the function without the modulus sign.

Example 8 **Self Tutor**

Draw the graph of $f(x) = x + 2|x|$.

$$f(x) = x + 2|x|$$
$$= \begin{cases} x + 2(x) & \text{if } x \geqslant 0 \\ x + 2(-x) & \text{if } x < 0 \end{cases}$$
$$= \begin{cases} 3x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$



EXERCISE 17E

1 If $x = -4$, find the value of:

a $|x+6|$ **b** $|x-6|$ **c** $|2x+3|$ **d** $|7-x|$

e $|x-7|$ **f** $|x^2-6x|$ **g** $|6x-x^2|$ **h** $\dfrac{|x|}{x+2}$

2 If $a = 5$ and $b = -2$, find the value of:

a $|a+b|$ **b** $|ab|$ **c** $|b-a|$ **d** $|a|+b$

e $|3a+b|$ **f** $\dfrac{|b-8|}{a}$ **g** $\left|\dfrac{a}{b}\right|$ **h** $\dfrac{|b^2|}{|a|}$

3 **a** Copy and complete:

x	-3	-2	-1	0	1	2	3
x^2							
$\|x\|^2$							

b What can you conclude from **a**?

4 By replacing $|x|$ with x for $x \geqslant 0$ and $(-x)$ for $x < 0$, write the following functions without the modulus sign. Hence, graph each function.

a $f(x) = -|x|$ **b** $f(x) = |x| + x$

c $f(x) = |x| + 2$ **d** $f(x) = x - 2|x|$

e $f(x) = 3|x| + 1$ **f** $f(x) = 5 - |x|$

g $f(x) = |x|^2 - 4$ **h** $f(x) = \dfrac{|x|}{x}$

i $f(x) = \sqrt{|x|}$

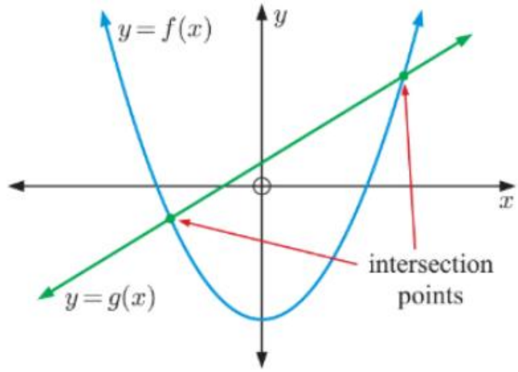
GRAPHING PACKAGE

F WHERE FUNCTIONS MEET

To find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$, we solve the equation $f(x) = g(x)$.

The solutions of this equation give us the x-coordinates of the intersection points.

The y-coordinates are then found by substituting the x-coordinates into one of the functions.



Example 9 Self Tutor

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x + 3$ and $y = 2x + 7$.

The graphs meet when $x^2 - x + 3 = 2x + 7$

$$\therefore\ x^2 - 3x - 4 = 0$$
$$\therefore\ (x+1)(x-4) = 0$$
$$\therefore\ x = -1 \text{ or } 4$$

Substituting into $y = 2x + 7$: when $x = -1$, $y = 2(-1) + 7 = 5$
when $x = 4$, $y = 2(4) + 7 = 15$

$\therefore$ the graphs meet at $(-1, 5)$ and $(4, 15)$.

EXERCISE 17F

1 Find the coordinates of the point of intersection of the graphs with equations:

a $y = 5x - 1$ and $y = 2x + 5$
b $y = 12 - x$ and $y = 3x + 7$
c $y = \dfrac{1}{x}$ and $y = \dfrac{2}{x+5}$
d $y = x^2 - x + 3$ and $y = x^2 + 5x - 3$.

2 Find the coordinates of the point(s) of intersection of the graphs with equations:

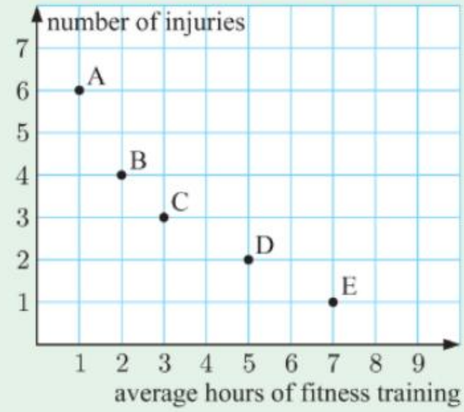
a $y = x^2 + 2x - 1$ and $y = x + 5$
b $y = \dfrac{2}{x}$ and $y = x - 1$
c $y = 3x^2 + 4x - 1$ and $y = x^2 - 3x - 4$
d $y = \dfrac{1}{x}$ and $y = 5x - 4$.

3 Use a **graphing package** or a **graphics calculator** to find the coordinates, correct to 2 decimal places, of the points of intersection of the graphs with equations:

a $y = x^2 + 3x + 1$ and $y = 2x + 2$
b $y = x^2 - 5x + 2$ and $y = \dfrac{3}{x}$
c $y = -x^2 - 2x + 5$ and $y = x^2 + 7$
d $y = x^2 - 1$ and $y = x^3$.

REVIEW SET 17A

1 Five hockey clubs were surveyed to find the average number of hours of general fitness training their players did, and the number of injuries to their players during a season.



a What are the variables?

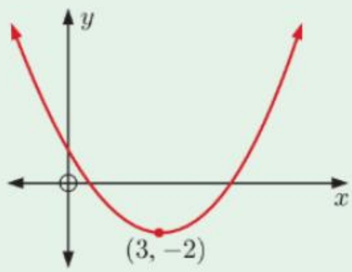
b Which club had the lowest number of injuries?

c Which club's players did the lowest number of hours of fitness training?

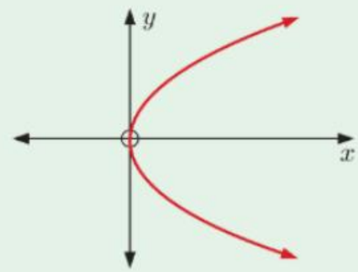
d Write a sentence describing the general trend of the graph.

2 Find the domain and range of the following relations:

a



b



3 For $f(x) = 3x - x^2$, find:

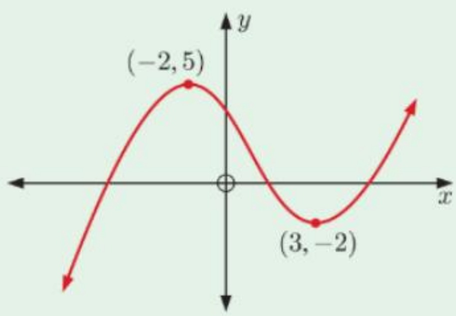
a $f(2)$ **b** $f(-1)$ **c** $f(x-3)$

4 Consider the function $g(x) = x^2 + 2x$. Find:

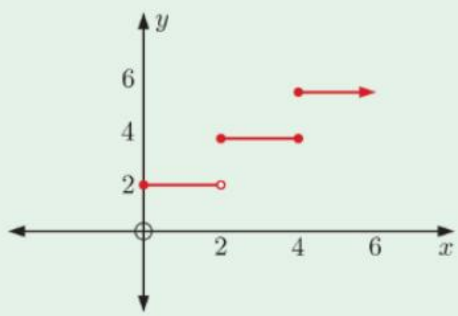
a $g(2)$ **b** $g(3x)$ **c** x such that $g(x) = 15$.

5 Determine whether the following relations are functions:

a



b



6 Draw a graph of $y = f(x)$ such that $f(-3) = 2$, $f(1) = -4$, and $f(4) = 5$.

7 Find the domain of:

a $f(x) = \dfrac{2}{x+1}$ **b** $f(x) = \dfrac{1}{x^2-4}$ **c** $f(x) = \dfrac{1}{\sqrt{x-1}}$

8 If $x = -3$, find the value of:

a $|x-4|$ **b** $|x| - 4$ **c** $\left|x^2 + 3x\right|$

9 Draw the graph of $y = f(x)$ for:

a $f(x) = |x| + 3x$ **b** $f(x) = 2|x| - 4$

10 Find the coordinates of the points of intersection of the graphs with equations:

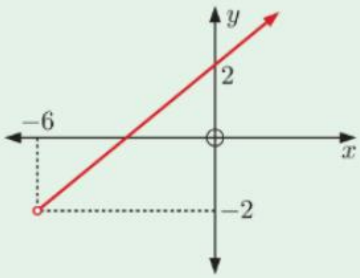
a $y = 3x - 2$ and $y = -x + 6$ **b** $y = x^2 + 4x - 2$ and $y = 2x + 1$

REVIEW SET 17B

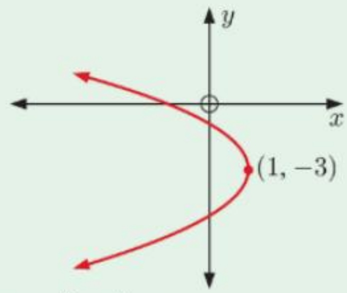
1 Answer the **Opening Problem** on page **320**.

2 Find the domain and range of the following relations:

a



b



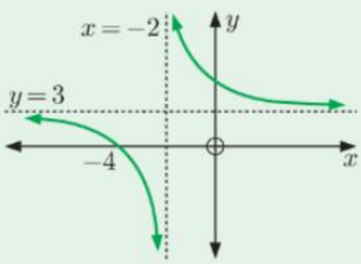
3 Determine whether the following sets of ordered pairs are functions:

a $\{(-3, 5), (1, 7), (-1, 7), (2, 5)\}$

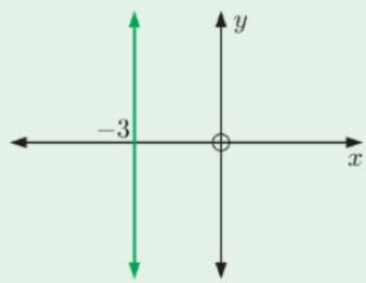
b $\{(-4, -5), (-1, 3), (5, 4), (0, 3), (-1, 2)\}$

4 Determine whether the following relations are functions:

a



b



5 For $f(x) = 5x - x^2$, find:

a $f(-3)$ **b** $f(-x)$ **c** $f(x+1)$

6 Find the domain of:

a $f(x) = \dfrac{1}{x+4}$ **b** $f(x) = \dfrac{x}{x^2 + 4x - 5}$ **c** $f(x) = \sqrt{6 - x}$

7 The graph of $y = f(x)$ is a straight line passing through $(-1, 5)$ and $(3, -3)$.

a Draw the graph of $y = f(x)$. **b** Find $f(-1)$ and $f(3)$.

c Find $f(x)$.

8 If $a = -4$ and $b = 9$, find the value of:

a $|ab|$ **b** $|2a - b| + a$ **c** $\dfrac{|a^2 - b|}{|a|}$

9 Draw the graph of:

a $f(x) = 7 - |x|$ **b** $f(x) = \dfrac{x}{|x|} + 3$

10 Find the coordinates of the points of intersection of the graphs with equations:

a $y = 2x + 1$ and $y = 3x + 2$ **b** $y = \dfrac{3}{x}$ and $y = 2x - 1$

Chapter 18

Exponential functions

Contents:

OPENING PROBLEM

A lotus plant initially covers an area of 40 cm^2. The area it covers increases by 20% each week.

Things to think about:

- **a** Does the area covered by the plant increase by a constant amount each week?
- **b** Can you explain why the area covered by the lotus plant after n weeks is given by the function $A(n) = 40 \times 1.2^n \text{ cm}^2$?
- **c** What does the graph of $A(n)$ look like?
- **d** What area is covered by the lotus plant after 3 weeks?

When a quantity increases or decreases by a **fixed percentage** each time period, the quantity can be described using an **exponential function**.

In this chapter, we will study exponential functions and their graphs, and solve **exponential equations** using algebraic methods and technology.

A EXPONENTIAL FUNCTIONS

An **exponential function** is a function in which the variable occurs as part of the index or exponent.

Examples of exponential functions are $f(x) = 3^x$, $g(x) = 2^{x-4}$, and $h(x) = 6 + 5^{-x}$.

Example 1 Self Tutor

For the function $f(x) = 3 - 2^{-x}$, find:

a $f(0)$ **b** $f(3)$ **c** $f(-x)$

a $f(0) = 3 - 2^0 = 3 - 1 = 2$

b $f(3) = 3 - 2^{-3} = 3 - \frac{1}{8} = 2\frac{7}{8}$

c $f(-x) = 3 - 2^{-(-x)} = 3 - 2^x$

EXERCISE 18A

1 Determine whether the following are exponential functions:

a $f(x) = 7^x$ **b** $f(x) = x^4$ **c** $f(x) = 5 - 3^{x-2}$

d $f(x) = 10 \times 2^{\frac{x}{2}}$ **e** $f(x) = 9x - x^6$ **f** $f(x) = -2 - 5^{3x}$

2 For the function $f(x) = 2^x$, find:

a $f(-3)$ **b** $f(-2)$ **c** $f(-1)$ **d** $f(0)$

e $f(1)$ **f** $f(2)$ **g** $f(3)$ **h** $f(2x)$

3 For the function $f(x) = 3^x + 2$, find:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(2x)$

4 For the function $f(x) = 5^{-x} - 3$, find:

a $f(0)$ **b** $f(1)$ **c** $f(-2)$ **d** $f(-x)$

5 For the function $g(x) = 3^{x-2}$, find:

a $g(0)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(x+5)$

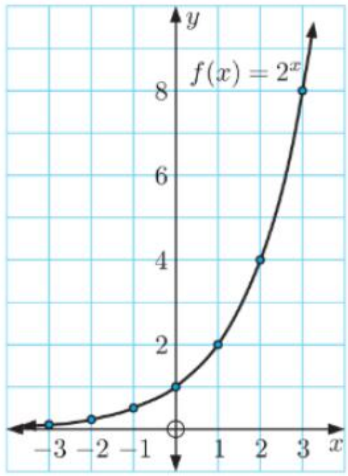
B GRAPHS OF EXPONENTIAL FUNCTIONS

GRAPHS OF THE FORM $f(x) = a^x$, $a > 0$, $a \neq 1$

We can draw graphs of exponential functions using a table of values.

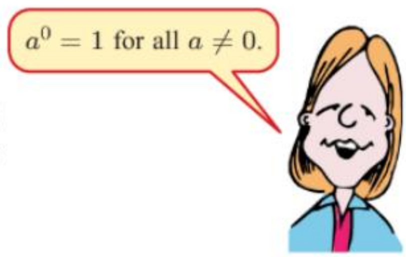
For example, the graph of $f(x) = 2^x$ is shown alongside.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Notice that:

- the y-intercept of the function is 1
- the graph lies entirely above the y-axis
- as the values of x get smaller, the values of y get closer and closer to zero, but never actually reach zero. We say that the line $y = 0$ is a **horizontal asymptote** of the function.



INVESTIGATION 1 GRAPHS OF THE FORM $f(x) = a^x$, $a > 0$, $a \neq 1$

What to do:

1 Use the **graphing package** or your **graphics calculator** to draw the graph of:

a $f(x) = 1.3^x$ **b** $f(x) = 2^x$

c $f(x) = 3^x$ **d** $f(x) = 5^x$

e $f(x) = 0.8^x$ **f** $f(x) = \left(\frac{1}{2}\right)^x$

g $f(x) = 0.3^x$ **h** $f(x) = 0.1^x$

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

2 What do you notice about:

a the y-intercept of each graph **b** the horizontal asymptote of each graph?

3 For the graph of $f(x) = a^x$, what effect does a have on:

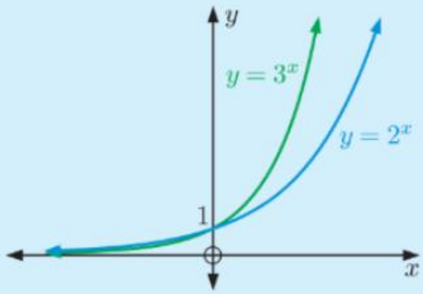
a whether the graph is increasing or decreasing

b the steepness of the graph?

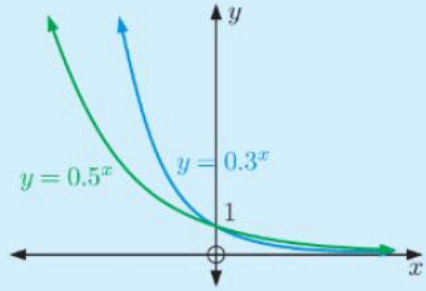
In the previous **Investigation** you should have made the following discoveries:

For all exponential functions of the form $f(x) = a^x$, $a > 0$, $a \neq 1$:

- The y-intercept is 1, since $f(0) = a^0 = 1$.
- The graph has the horizontal asymptote $y = 0$.
- If $a > 1$, the graph is **increasing**.



- If $0 < a < 1$, the graph is **decreasing**.



- The graph of $f(x) = a^x$ becomes steeper as a moves further away from 1.

EXERCISE 18B.1

1 Use a table of values from $x = -3$ to $x = 3$ to help sketch each of the following exponential functions:

a $f(x) = 3^x$ **b** $f(x) = 4^x$ **c** $f(x) = \left(\frac{1}{2}\right)^x$ **d** $f(x) = \left(\frac{1}{3}\right)^x$

2 Use technology to graph the following functions on the same set of axes:

a $y = 5^x$ **b** $y = 1.8^x$ **c** $y = 0.7^x$ **d** $y = \left(\frac{2}{5}\right)^x$

3 Match each function with the correct graph:

a $y = 3.6^x$

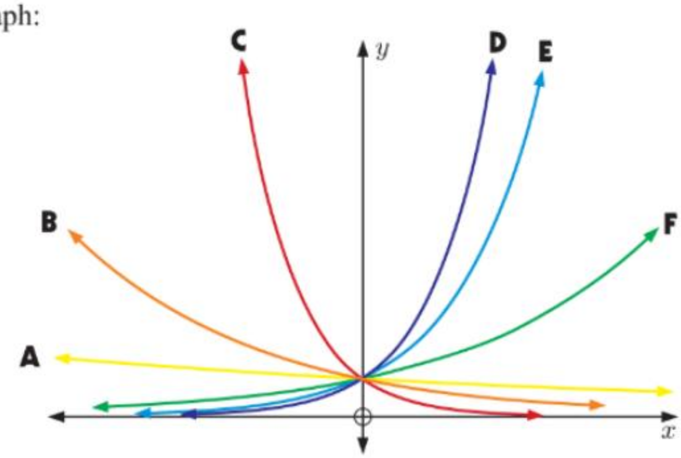
b $y = 0.9^x$

c $y = 1.5^x$

d $y = \left(\frac{1}{4}\right)^x$

e $y = \left(\frac{2}{3}\right)^x$

f $y = \left(\frac{5}{2}\right)^x$



USING TRANSFORMATIONS TO GRAPH EXPONENTIAL FUNCTIONS

In previous years, we have studied different transformations of figures including translations, reflections, rotations, and enlargements. We can use this knowledge to apply transformations to functions.

INVESTIGATION 2 TRANSFORMATIONS OF $y = a^x$

In this Investigation we will learn how to graph more complicated exponential functions by applying **transformations** to graphs of the form $y = a^x$.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

What to do:

1 **a** Use the **graphing package** or your **graphics calculator** to draw the graph of:

i $y = 3^x$ **ii** $y = 3^x + 2$ **iii** $y = 3^x - 4$

b For each graph in **a**, find the y-intercept and the equation of the horizontal asymptote.

c Copy and complete:

The graph of $y = 3^x + k$ is obtained from $y = 3^x$ by a .... with translation vector $\begin{pmatrix} \ldots\ldots \\ \ldots\ldots \end{pmatrix}$.

2 **a** Draw the graph of:

i $y = 2^x$ **ii** $y = 2^{x-1}$ **iii** $y = 2^{x+4}$

b Copy and complete:

The graph of $y = 2^{x-k}$ is obtained from $y = 2^x$ by a .... with translation vector $\begin{pmatrix} \ldots\ldots \\ \ldots\ldots \end{pmatrix}$.

3 **a** Draw the graph of:

i $y = 5^x$ **ii** $y = -5^x$ **iii** $y = 5^{-x}$

iv $y = \left(\frac{1}{2}\right)^x$ **v** $y = -\left(\frac{1}{2}\right)^x$ **vi** $y = \left(\frac{1}{2}\right)^{-x}$

b Copy and complete:

i The graph of $y = -a^x$ is obtained from $y = a^x$ by a .... in the ....

ii The graph of $y = a^{-x}$ is obtained from $y = a^x$ by a .... in the ....

4 In previous years we have seen enlargements and reductions. These are also known as **dilations**. When we deal with functions, it is useful to consider dilating or stretching the graph in *one direction only*, either vertical or horizontal. We call these vertical and horizontal dilations.

a Draw the graph of:

i $y = 4^x$ **ii** $y = 2 \times 4^x$ **iii** $y = \frac{1}{3} \times 4^x$

b Find the y-intercept of each graph in **a**.

c Copy and complete:

The graph of $y = k \times 4^x$ is obtained from $y = 4^x$ by a .... dilation with scale factor ....

d Draw the graph of:

i $y = 3^x$ **ii** $y = 3^{\frac{x}{2}}$ **iii** $y = 3^{2x}$

e Copy and complete:

The graph of $y = 3^{\frac{x}{k}}$ is obtained from $y = 3^x$ by a .... dilation with scale factor ....

From the **Investigation** you should have found that:

- Graphs of the form $y = a^x + k$ are obtained by a translation of $y = a^x$ with translation vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$.
  - If $k > 0$, the graph moves upwards.
  - If $k < 0$, the graph moves downwards.
- Graphs of the form $y = a^{x-k}$ are obtained by a translation of $y = a^x$ with translation vector $\begin{pmatrix} k \\ 0 \end{pmatrix}$.
  - If $k > 0$, the graph moves right.
  - If $k < 0$, the graph moves left.
- Graphs of the form $y = -a^x$ are obtained by a reflection of $y = a^x$ in the x-axis.
- Graphs of the form $y = a^{-x}$ are obtained by a reflection of $y = a^x$ in the y-axis.
- Graphs of the form $y = k \times a^x$ are obtained by a vertical dilation of $y = a^x$ with scale factor k.
  - If $k > 1$, the graph moves away from the x-axis.
  - If $0 < k < 1$, the graph moves towards the x-axis.
- Graphs of the form $y = a^{\frac{x}{k}}$ are obtained by a horizontal dilation of $y = a^x$ with scale factor k.
  - If $k > 1$, the graph moves away from the y-axis.
  - If $0 < k < 1$, the graph moves towards the y-axis.

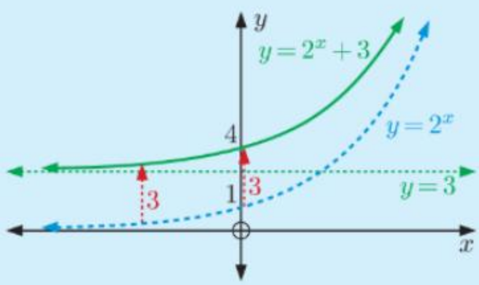
Example 2 **Self Tutor**

Sketch the graph of:

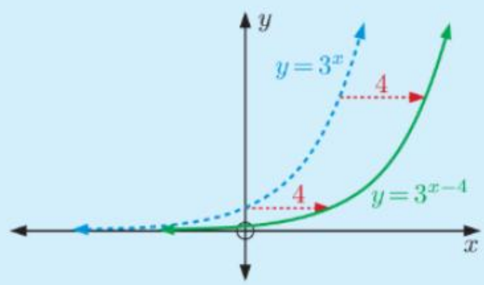
a $y = 2^x + 3$

b $y = 3^{x-4}$

a The graph of $y = 2^x + 3$ is found by translating $y = 2^x$ upwards by 3 units.



b The graph of $y = 3^{x-4}$ is found by translating $y = 3^x$ to the right by 4 units.



EXERCISE 18B.2

1 For each of the following functions:

i sketch the graph
ii state the equation of the horizontal asymptote
iii find the range
iv find the y-intercept.

a $y = 2^x - 1$
b $y = 3^x + 2$
c $y = \left(\frac{1}{2}\right)^x - 3$

2 Sketch the graph of:

a $y = 2^{x-3}$ **b** $y = 5^{x+2}$ **c** $y = \left(\frac{1}{3}\right)^{x-1}$

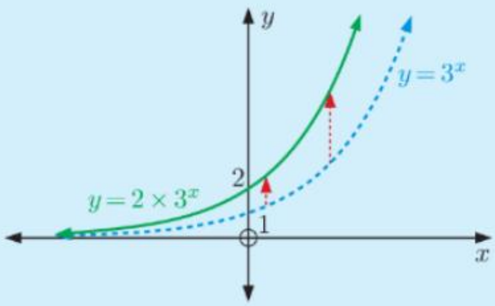
Example 3

Self Tutor

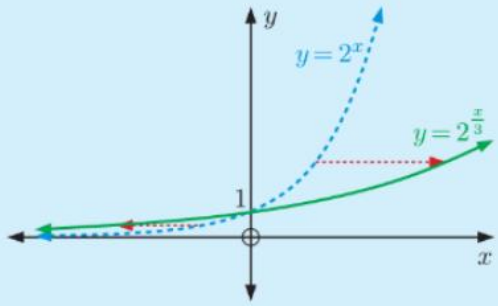
Sketch the graph of:

a $y = 2 \times 3^x$ **b** $y = 2^{\frac{x}{3}}$

a The graph of $y = 2 \times 3^x$ is found by vertically dilating $y = 3^x$ with scale factor 2.



b The graph of $y = 2^{\frac{x}{3}}$ is found by horizontally dilating $y = 2^x$ with scale factor 3.



3 Sketch the graph of:

a $y = 3 \times 2^x$ **b** $y = \frac{1}{2} \times 5^x$ **c** $y = 4 \times \left(\frac{1}{2}\right)^x$

4 Sketch the graph of:

a $y = 5^{\frac{x}{2}}$ **b** $y = 2^{3x}$ **c** $y = \left(\frac{2}{3}\right)^{2x}$

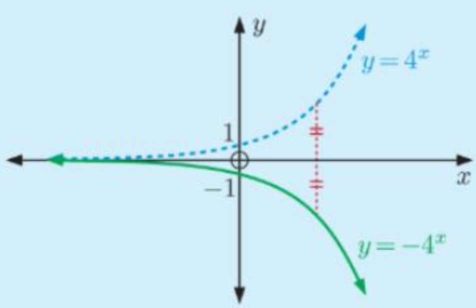
Example 4

Self Tutor

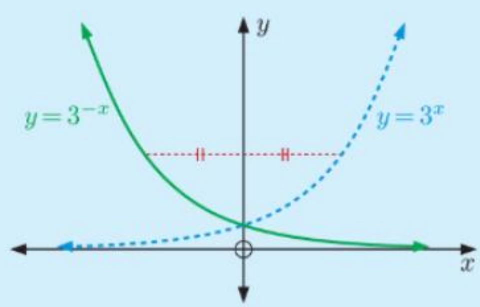
Sketch the graph of:

a $y = -4^x$ **b** $y = 3^{-x}$

a The graph of $y = -4^x$ is found by reflecting $y = 4^x$ in the x-axis.



b The graph of $y = 3^{-x}$ is found by reflecting $y = 3^x$ in the y-axis.



5 Sketch the graph of:

a $y = -2^x$ **b** $y = -5^x$ **c** $y = -\left(\frac{1}{3}\right)^x$

6 Sketch the graph of:

a $y = 4^{-x}$ **b** $y = 2^{-x}$ **c** $y = \left(\frac{1}{3}\right)^{-x}$

C GROWTH AND DECAY

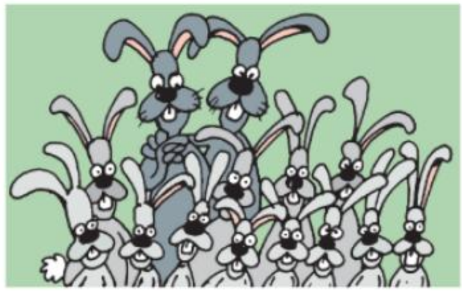
In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay**, and occur frequently in the world around us.

An exponential function of the form $f(x) = k \times a^x$ exhibits:

- **growth** if $a > 1$
- **decay** if $0 < a < 1$.

GROWTH

Under favourable conditions, a population of rabbits will grow exponentially.



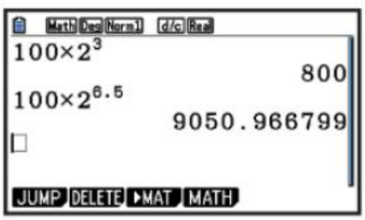
Suppose the population is initially 100 rabbits, and the population doubles every month. The population after t months can be described by the exponential function $P = 100 \times 2^t$.

We can use this relationship to answer questions about the rabbit population.

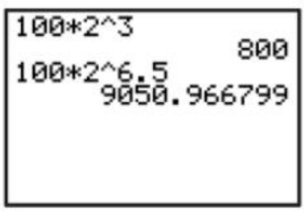
For example:

- To find the population after 3 months, we substitute $t = 3$ and find $P = 100 \times 2^3$ $= 800$ rabbits.
- To find the population after $6\frac{1}{2}$ months, we substitute $t = 6.5$ and find $P = 100 \times 2^{6.5}$ ≈ 9051 rabbits.

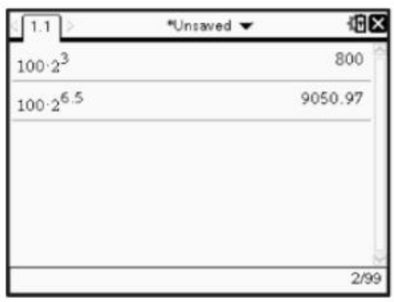
Casio fx-CG20



TI-84 Plus



TI-$n$spire



Clearly, the population cannot continue to grow exponentially in the long term because eventually the rabbits will run out of food. Nevertheless, an exponential function can be valuable for modelling the population in the short term.

Example 5 **Self Tutor**

During a locust plague, the area of land eaten n weeks after the initial observation, is given by $A = 8000 \times 1.4^n$ hectares.

a Find the size of the area eaten:

i initially **ii** after 4 weeks **iii** after 7 weeks.

b Hence sketch the graph of A against n.

a **i** Initially, $n = 0$

$\therefore \quad A = 8000 \times 1.4^0$

$= 8000$ hectares

ii When $n = 4$,

$A = 8000 \times 1.4^4$

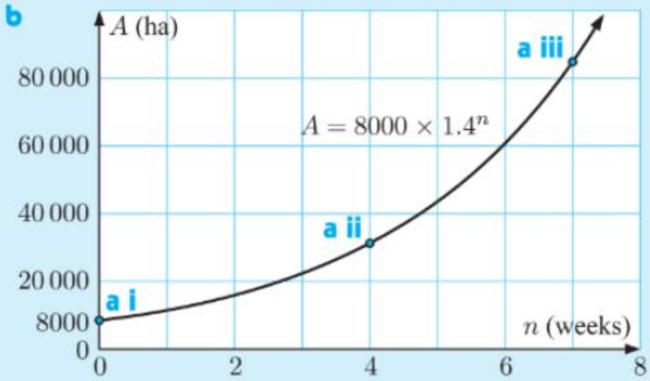
$\approx 30\,700$ ha

iii When $n = 7$,

$A = 8000 \times 1.4^7$

$\approx 84\,300$ ha

b



EXERCISE 18C.1

1 A local zoo starts a breeding program to ensure the survival of a species of mongoose. Using results from a previous program, the expected population in n years' time is given by $P = 40 \times 1.15^n$.

a Find the initial population purchased by the zoo.

b Find the expected population after:

i 3 years **ii** 10 years **iii** 30 years.

c Sketch the graph of P against n using **a** and **b**.

2 In Tasmania, a reserve is set aside for the breeding of echidnas. The expected population after t years is given by $P = 50 \times 1.26^t$.

a Find the initial breeding colony size.

b Find the expected colony size after:

i 3 years **ii** 9 years **iii** 20 years.

c Sketch the graph of P against t using **a** and **b**.

3 In Uganda, the number of breeding females in an endangered population of gorillas is G_0. Biologists predict that the number of breeding females G in n years' time will grow according to $G = G_0 \times 1.01^n$.

a There are currently 28 breeding females in the colony. Find G_0.

b Predict the number of breeding females after:

i 5 years **ii** 10 years **iii** 20 years.

c Sketch the graph of G against n using **a** and **b**.

4 A tip contains 3000 tonnes of rubbish. The amount of rubbish in the tip increases by 5% each year.

a Explain why the amount of rubbish in the tip after n years is given by $A(n) = 3000 \times 1.05^n$ tonnes.

b Find the amount of rubbish in the tip after:

i 5 years **ii** 10 years.

c Sketch the graph of A against n.

5 Answer the **Opening Problem** on page **336**.

DECAY

When the value of a variable decreases exponentially over time, we call it **decay**.

Examples of decay include:

- the cooling of a cup of tea or coffee
- radioactive decay
- the drop in current when an electrical appliance is turned off.

Example 6 Self Tutor

The current I flowing through the electric circuit in a fan, t milliseconds after it is switched off, is given by $I = 320 \times 0.7^t$ milliamps.

a Find the initial current in the circuit.

b Find the current after: **i** 4 milliseconds **ii** 10 milliseconds.

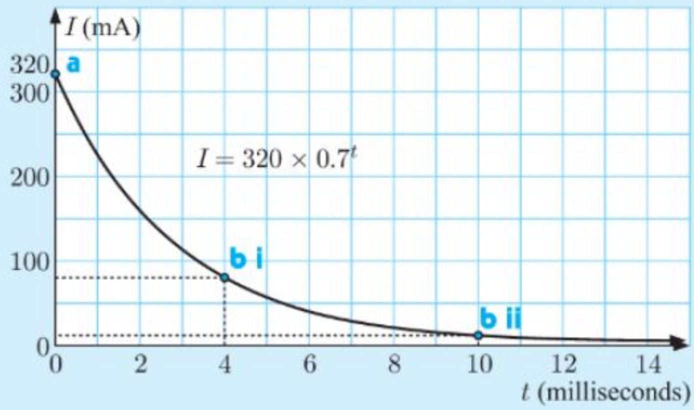
c Sketch the graph of I against t using **a** and **b**.

a When $t = 0$,
$I = 320 \times 0.7^0$
$= 320$ milliamps

b **i** When $t = 4$,
$I = 320 \times 0.7^4$
≈ 76.8 milliamps

ii When $t = 10$,
$I = 320 \times 0.7^{10}$
≈ 9.04 milliamps

c



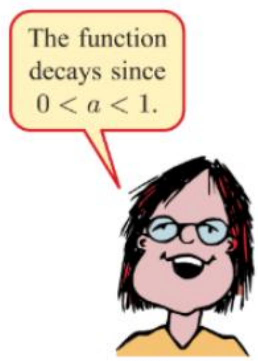
EXERCISE 18C.2

1 Boiling water is left in a pot to cool. After t minutes, its temperature is given by $T = 100 \times 0.84^t$ °C.

a Find the initial temperature of the water.

b Find the water temperature after:
i 2 minutes **ii** 10 minutes **iii** 20 minutes.

c Sketch the graph of T against t using **a** and **b**.



2 The weight of radioactive material left in an ore sample after t years is given by $W = 2.3 \times 0.96^t$ grams.

a Find the initial weight.

b Find the weight after: **i** 20 years **ii** 40 years **iii** 60 years.

c Sketch the graph of W against t using **a** and **b**.

d Find the percentage loss in weight from $t = 0$ to $t = 20$ years.

3 There are currently 500 possums in a section of bushland. The population is expected to fall by 8% each year.

a Write an exponential function for the expected possum population P after t years.

b Find the expected possum population after:

i 1 year **ii** 5 years **iii** 10 years.

c Sketch the graph of P against t.

D EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

To solve exponential equations, we try to write both sides of the equation with the **same base**. We can then **equate indices**.

If $\boldsymbol{a^x = a^k}$, then $\boldsymbol{x = k}$.

Example 7 **Self Tutor**

Solve for x:

a $2^x = 32$ **b** $3^{x-2} = \frac{1}{9}$

a $2^x = 32$

$\therefore\ 2^x = 2^5$

$\therefore\ x = 5$ {equating indices}

b $3^{x-2} = \frac{1}{9}$

$\therefore\ 3^{x-2} = 3^{-2}$

$\therefore\ x - 2 = -2$ {equating indices}

$\therefore\ x = 0$

To solve exponential equations, we need to apply the **index laws**.

EXERCISE 18D

1 Solve for x:

a $3^x = 3$ **b** $3^x = 9$ **c** $2^x = 8$ **d** $5^x = 1$

e $3^x = \frac{1}{3}$ **f** $5^x = \frac{1}{5}$ **g** $2^x = \frac{1}{16}$ **h** $5^{x+2} = 25$

2 Solve for x:

a $2^{x+2} = \frac{1}{4}$ **b** $3^{x-1} = \frac{1}{27}$ **c** $2^{x-1} = 32$ **d** $3^{1-2x} = \frac{1}{27}$

e $4^{2x+1} = \frac{1}{2}$ **f** $9^{x-3} = 3$ **g** $\left(\frac{1}{2}\right)^{x-1} = 2$ **h** $\left(\frac{1}{3}\right)^{2-x} = 9$

Example 8 **Self Tutor**

Solve for x:

a $6 \times 3^x = 54$

b $4^{x-1} = \left(\frac{1}{2}\right)^{1-3x}$

a $6 \times 3^x = 54$
$\therefore\ 3^x = 9$
$\therefore\ 3^x = 3^2$
$\therefore\ x = 2$ {equating indices}

b $4^{x-1} = \left(\frac{1}{2}\right)^{1-3x}$
$\therefore\ (2^2)^{x-1} = (2^{-1})^{1-3x}$
$\therefore\ 2^{2x-2} = 2^{3x-1}$
$\therefore\ 2x - 2 = 3x - 1$ {equating indices}
$\therefore\ -2 + 1 = 3x - 2x$
$\therefore\ x = -1$

3 Solve for x:

a $5 \times 2^x = 40$ **b** $6 \times 2^{x+2} = 24$ **c** $3 \times \left(\frac{1}{2}\right)^x = 12$

d $4 \times 5^x = 500$ **e** $54 \times 3^{x+2} = 2$ **f** $7 \times \left(\frac{1}{3}\right)^x = 63$

g $2^{2-5x} = 4^x$ **h** $5^{x-1} = \left(\frac{1}{25}\right)^x$ **i** $9^{x-2} = \left(\frac{1}{3}\right)^{3x-1}$

4 Consider the rabbit population described on page **342**, which is given by the function $P = 100 \times 2^t$ after t months. How long will it take for the rabbit population to reach 3200?

5 Solve for x:

a $\dfrac{3^{2x+1}}{3^x} = 9^x$ **b** $\dfrac{25^x}{5^{x+4}} = 25^{1-x}$ **c** $\dfrac{4^x}{2^{x+2}} = \dfrac{2^{x+1}}{8^x}$

INVESTIGATION 3 SOLVING EXPONENTIAL EQUATIONS GRAPHICALLY

Consider the exponential equation $3^x = 6$. We cannot easily write 6 as a power of 3, so we cannot solve this equation by equating indices. However, since $3^1 = 3$ and $3^2 = 9$, the solution for x must lie between 1 and 2.

We can solve this equation graphically using either a **graphics calculator** or the **graphing package**.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

What to do:

1 Draw the graphs of $y = 3^x$ and $y = 6$ on the same set of axes.

2 Find the coordinates of the point of intersection of the graphs.

3 Solve for x, rounded to 3 decimal places:

a $3^x = 10$ **b** $3^x = 30$

c $3^x = 100$ **d** $2^x = 12$

e $5^x = 40$ **f** $7^x = 42$

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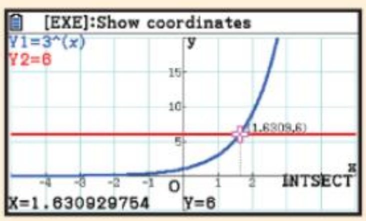


REVIEW SET 18A

1 Consider the function $f(x) = 3^x - 1$. Find:

a $f(0)$ **b** $f(3)$ **c** $f(-1)$ **d** $f(2x)$

2 Use technology to help sketch the following on the same set of axes:

a $y = 2.7^x$ **b** $y = 1.6^x$ **c** $y = 0.8^x$

3 Use transformations to sketch the graph of:

a $y = -3^x$ **b** $y = 2^x + 5$

4 Find the equation of the horizontal asymptote of $y = 2^{-x} - \frac{3}{2}$.

5 The weight of a radioactive substance remaining after t years is given by $W = 1000 \times (0.98)^t$ grams.

a Find the initial weight present.

b Find the weight after:

i 10 years **ii** 50 years **iii** 100 years.

c Sketch the graph of W against t using **a** and **b**.

6 Solve for x without using a calculator:

a $27^x = 3$ **b** $8^{x+1} = 16^{4-x}$ **c** $4^{x+1} = \left(\frac{1}{8}\right)^x$

d $8 \times 2^{x+1} = 4$ **e** $3 \times 25^{3-x} = 15$ **f** $\dfrac{3^{x+2}}{9^{3-x}} = \dfrac{27^{1-2x}}{3^{2x}}$

7 The population of a colony of seals after t years is given by $P = 50 \times 3^t$.

a Find the population after 2 years.

b How long will it take for the population to reach 4050?

REVIEW SET 18B

1 **a** Explain why $f(x) = 5^{x-2}$ is an exponential function.

b What transformation can be applied to $y = 5^x$ to obtain $y = f(x)$?

c Sketch the graphs of $y = 5^x$ and $y = f(x)$ on the same set of axes.

2 Suppose $P(x) = 2 \times 3^{-x}$. Find:

a $P(0)$ **b** $P(2)$ **c** $P(x+4)$

3 Use transformations to sketch the graph of:

a $y = 2^{x+2}$ **b** $y = 5^{-x}$

4 Find the range of the function:

a $y = -2^x$ **b** $y = 4^{-x}$ **c** $y = 3^x - 2$ **d** $y = 5^{x-2} + 3$

5 Solve for x without using your calculator:

a $5^{1-x} = 125$ **b** $9^x = 27^{2-2x}$ **c** $16^{x+1} = 32^{2-x}$

6 Kelly has started taking Spanish lessons. The number of Spanish words she knows after n weeks is given by $W(n) = 2 \times 1.9^n$.

a How many Spanish words does Kelly know:

i before she starts the lessons
ii after 3 weeks
iii after 5 weeks?

b Sketch the graph of W against n.

7 Danielle writes an online blog. The number of people following her blog after t weeks is given by $N(t) = 10 \times 1.3^t$.

a How many people are following Danielle's blog:

i initially **ii** after 4 weeks
iii after 8 weeks?

b Sketch the graph of N against t.

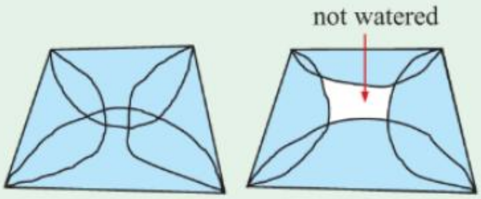
Chapter 19

Deductive geometry

Contents:

OPENING PROBLEM

Market gardener Joe has four long straight pipes of different lengths. He places the pipes on the ground, and joins them with rubber hose to form a garden bed in the shape of a quadrilateral. A sprinkler which casts water in semi-circles of diameter equal to the length of a pipe is placed at the midpoint of each pipe.



Joe's sketch Clinton's sketch

Joe draws a rough sketch of the watering system, and decides that his sprinklers will water the whole of the garden. His son Clinton is sceptical of his father's idea, and draws his own sketch which suggests that there will be an unwatered patch in the centre.

Things to think about:

a By drawing an accurate diagram of this situation, can you determine whether Joe or Clinton is correct?

b Can you prove that your answer is true using geometric theorems?

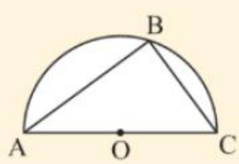
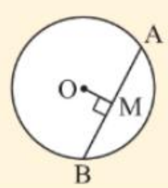
The geometry of triangles, quadrilaterals, and circles has been used for at least 3000 years in art, design, and architecture. Many amazing discoveries have been made by mathematicians and non-mathematicians who were simply drawing figures with rulers and compasses.

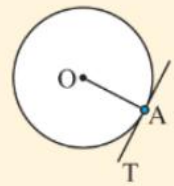
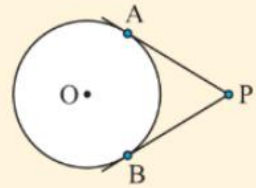
A CIRCLE THEOREMS

In geometry, we use logical reasoning to prove that certain observations about geometrical figures are true. We do this using special results called **theorems**.

CIRCLE THEOREMS

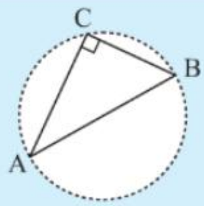
You should be familiar with the following **circle theorems**:

Name of theorem	*Statement*	*Diagram*
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$\widehat{ABC} = 90^\circ$ 
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	$AM = BM$ 

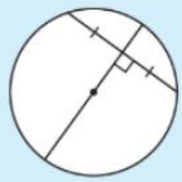
Name of theorem	*Statement*	*Diagram*
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	$O\widehat{A}T = 90°$ 
Tangents from an external point	Tangents from an external point are equal in length.	$AP = BP$ 

Two useful **converses** are:

- If line segment [AB] subtends a right angle at C, then the circle through A, B, and C has diameter [AB].

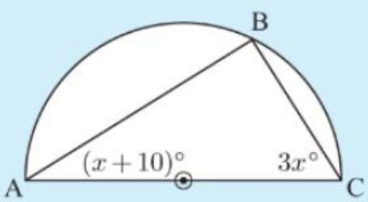


- The perpendicular bisector of a chord of a circle passes through its centre.



Example 1 **Self Tutor**

Find x, giving brief reasons for your answer.



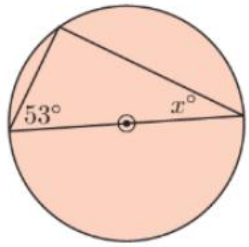
$A\widehat{B}C$ measures 90° {angle in a semi-circle}

$$\begin{aligned} \therefore\ (x+10)+3x+90 &= 180 \quad \text{\{angles in a triangle\}} \\ \therefore\ 4x+100 &= 180 \\ \therefore\ 4x &= 80 \\ \therefore\ x &= 20 \end{aligned}$$

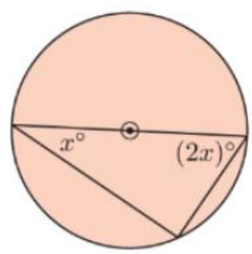
EXERCISE 19A

1 Find x, giving brief reasons for your answers:

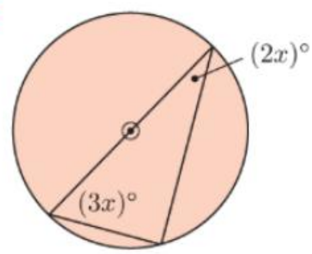
a



b

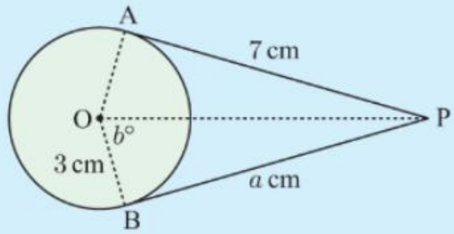


c



Example 2 **Self Tutor**

In the figure alongside, find a and b.



$\text{AP} = \text{BP}$ {tangents from an external point}

$\therefore\ a = 7$

Now, $\text{O}\widehat{\text{B}}\text{P} = 90°$ {radius-tangent theorem}

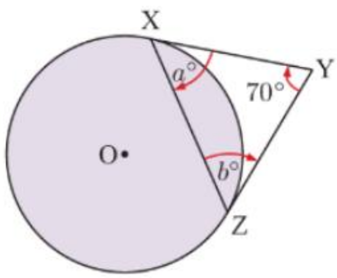
$\therefore$ in $\triangle\text{OBP}$, $\tan b° = \frac{7}{3}$

$\therefore\ b = \tan^{-1}(\frac{7}{3})$

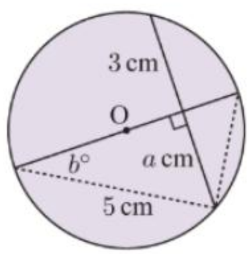
$\therefore\ b \approx 66.8$

2 Find a and b in the following figures:

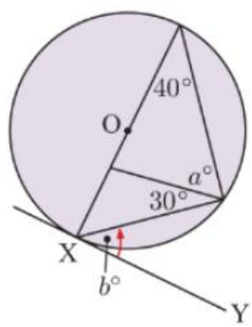
a



b



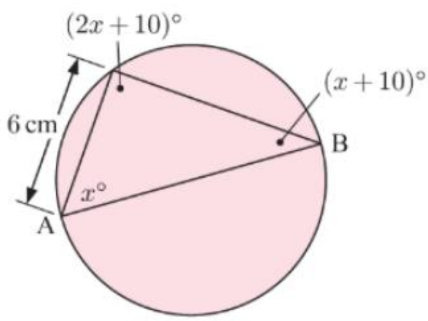
c



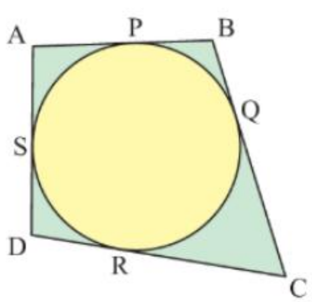
3 **a** Find x.

b Hence show that [AB] is a diameter of the circle.

c Find the length of the diameter.



4

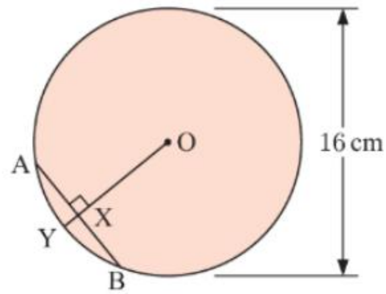


A circle is drawn, and four tangents to it are constructed as shown.

Deduce that $\text{AB} + \text{CD} = \text{BC} + \text{AD}$.

5 The chord [AB] is equal in length to the radius of the circle. Find the length of:

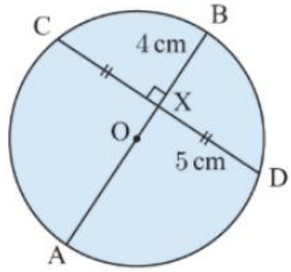
a [AX] **b** [OX] **c** [XY]



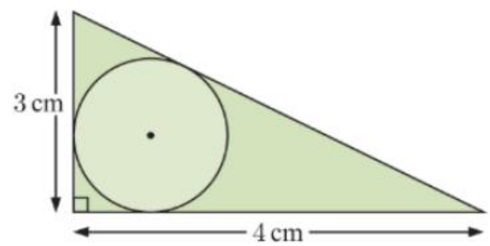
6 [AB] is the perpendicular bisector of the chord [CD]. Suppose $\text{OX} = x$ cm.

a Explain why $(x+4)^2 = x^2 + 5^2$.

b Find the diameter of the circle.



7 A circle touches the three sides of the triangle as shown. Find the radius of the circle.



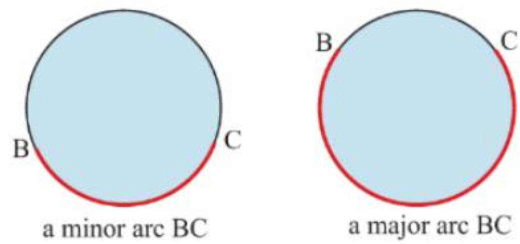
B FURTHER CIRCLE THEOREMS

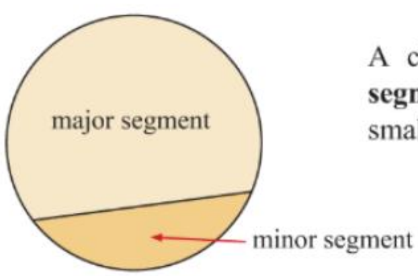
Before we can explore additional circle theorems, we need some more terminology for describing the parts of a circle.

Any continuous part of a circle is called an **arc**.

If the arc is less than half the circle, it is called a **minor arc**.

If the arc is greater than half the circle, it is called a **major arc**.

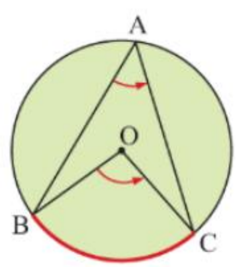




A chord divides the interior of a circle into two regions called **segments**. The larger region is called a **major segment**, and the smaller region is called a **minor segment**.

In the diagram opposite:

- the minor arc BC **subtends** the angle BAC, where A is on the circle
- the minor arc BC also subtends angle BOC at the centre of the circle.

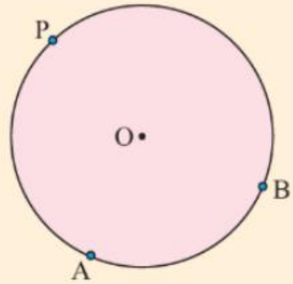


INVESTIGATION 1 FURTHER CIRCLE THEOREMS

The use of the **geometry package** is recommended, but this Investigation can also be done using a ruler, compass, and protractor.

Part 1: Angle at the centre theorem

GEOMETRY PACKAGE



1 Draw a large circle with centre O. Mark on it points A, B, and P.

2 Join [AO], [BO], [AP], and [BP].

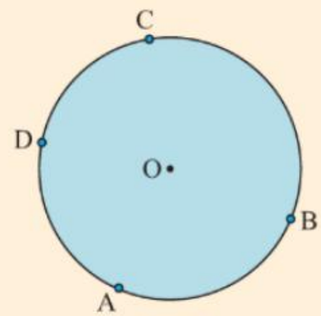
3 Measure angles AOB and APB. What do you notice?

4 Repeat the above steps with another circle.

5 Copy and complete:
"The angle at the centre of a circle is .... the angle on the circle subtended by the same arc."

Part 2: Angles subtended by the same arc theorem

GEOMETRY PACKAGE



1 Draw a circle with centre O. Mark on it points A, B, C, and D.

2 Join [AC], [BC], [AD], and [BD].

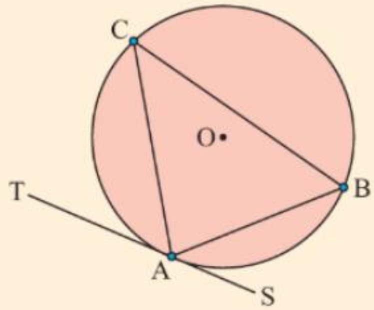
3 Measure angles ACB and ADB. What do you notice?

4 Repeat the above steps with another circle.

5 Copy and complete:
"Angles subtended by an arc on the circle are .... in size."

Part 3: Angle between a tangent and a chord theorem

GEOMETRY PACKAGE



1 Draw a circle and mark on it points A, B, and C.

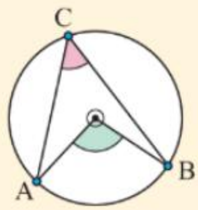
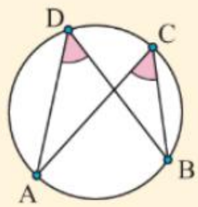
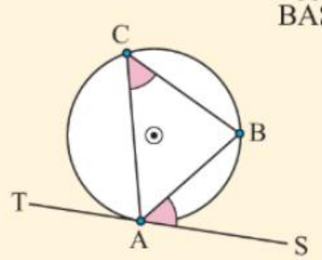
2 Draw tangent TAS at A, and join [AB], [BC], and [CA].

3 Measure angles BAS and ACB. What do you notice?

4 Repeat the above steps with another circle.

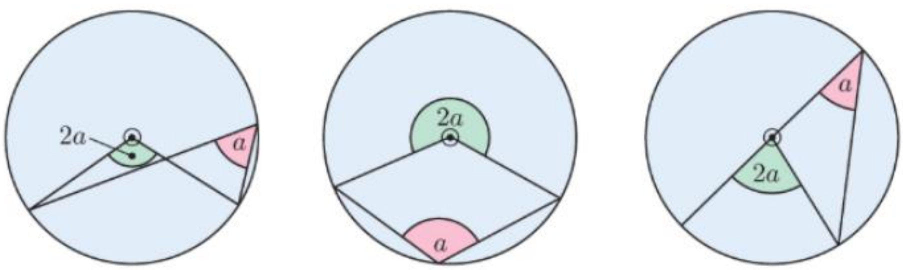
5 Copy and complete:
"The angle between a tangent and a chord at the point of contact is .... to the angle subtended by the chord in the alternate segment."

From the **Investigation** you should have discovered the following theorems:

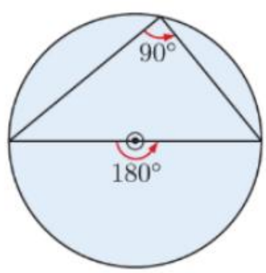
Name of theorem	*Statement*	*Diagram*
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	$A\widehat{O}B = 2 \times A\widehat{C}B$ 
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	$A\widehat{D}B = A\widehat{C}B$ 
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	$B\widehat{A}S = A\widehat{C}B$ 

Note:
- The following diagrams show other cases of **the angle at the centre theorem**. These cases can be shown using the **geometry package**.

GEOMETRY PACKAGE



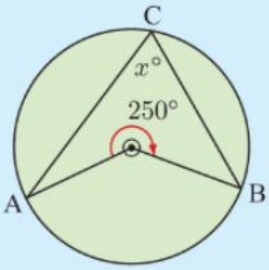
- The **angle in a semi-circle theorem** is a special case of the angle at the centre theorem.



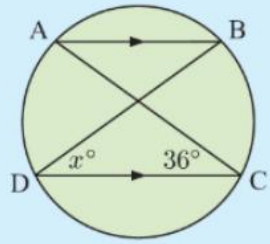
Example 3 **Self Tutor**

Find x:

a



b



a Obtuse $\widehat{AOB} = 360° - 250°$ {angles at a point}

$\therefore\ \widehat{AOB} = 110°$

$\therefore\ 2x = 110$ {angle at the centre}

$\therefore\ x = 55$

b $\widehat{ABD} = 36°$ {angles on the same arc}

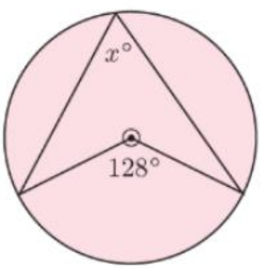
and $\widehat{BDC} = \widehat{ABD}$ {equal alternate angles}

$\therefore\ x = 36$

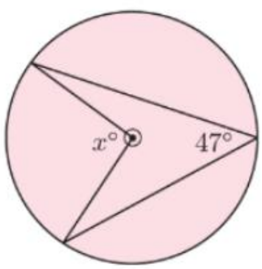
EXERCISE 19B

1 Find, giving reasons, the value of x:

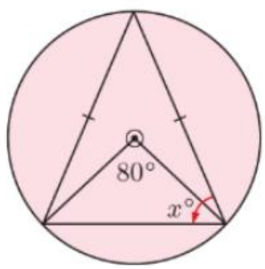
a



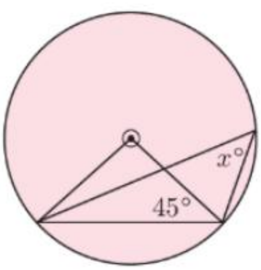
b



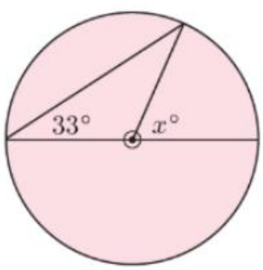
c



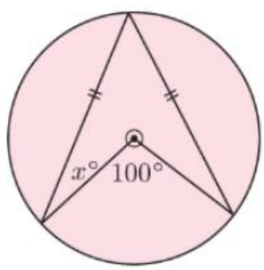
d



e

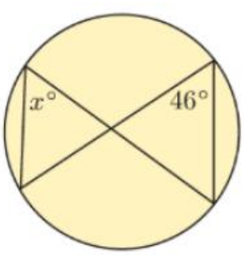


f

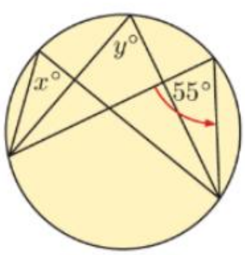


2 Find, giving reasons, the value of each pronumeral:

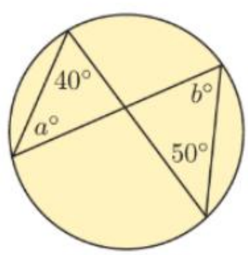
a



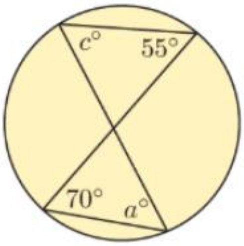
b



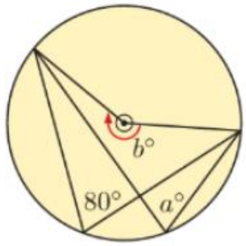
c



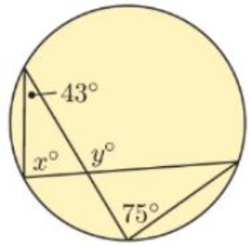
d



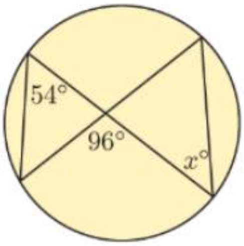
e



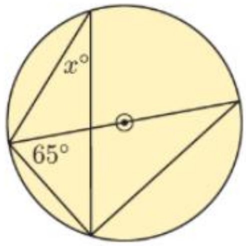
f



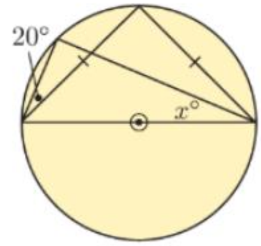
g



h

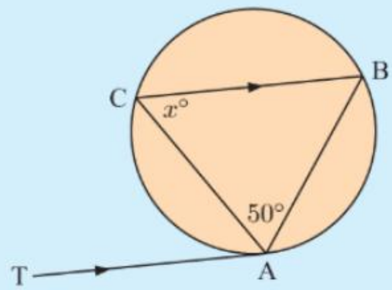


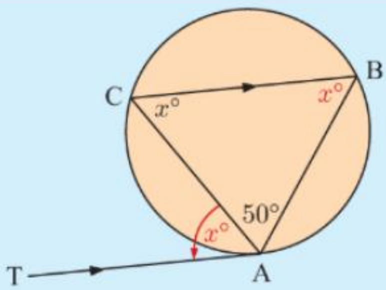
i



Example 4 **Self Tutor**

[AT] is a tangent to the circle. Find x.

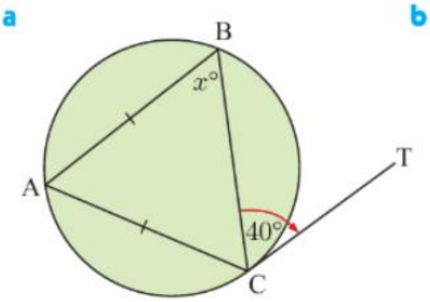




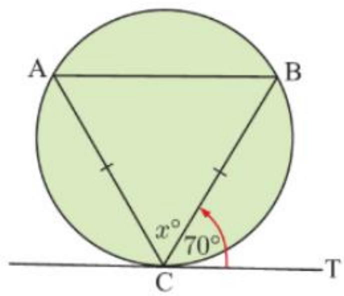
$$\widehat{CAT} = x° \quad \{\text{equal alternate angles}\}$$
$$\therefore\ \widehat{ABC} = \widehat{CAT} = x° \quad \{\text{angle between a tangent and a chord}\}$$
$$\therefore\ x + x + 50 = 180 \quad \{\text{angles in a triangle}\}$$
$$\therefore\ 2x = 130$$
$$\therefore\ x = 65$$

3 In each diagram, C is the point of contact of tangent [CT]. Find x, giving reasons for your answer:

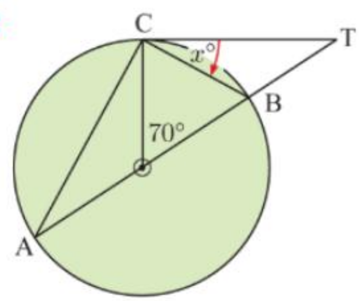
a



b



c

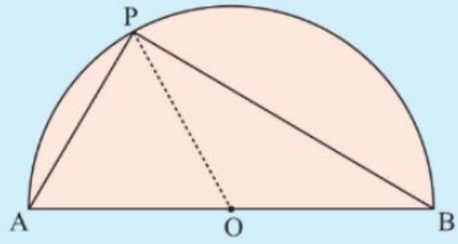


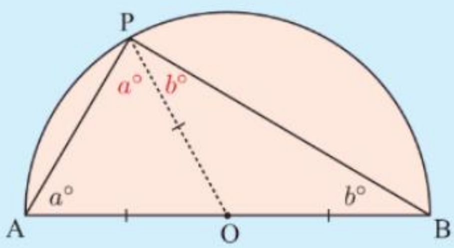
C GEOMETRIC PROOF

The circle theorems and other geometric facts can be formally **proven** using mathematical tools we already possess, such as the isosceles triangle theorem and congruence.

Example 5 Self Tutor

Use the given figure to prove the *angle in a semi-circle theorem*.





Let $P\widehat{A}O = a°$ and $P\widehat{B}O = b°$.

Now $OA = OP = OB$ {equal radii}

$\therefore$ $\triangle$s OAP and OBP are isosceles.

$\therefore$ $O\widehat{P}A = a°$ and $B\widehat{P}O = b°$ {isosceles $\triangle$s}

Now $a + (a + b) + b = 180$ {angles of $\triangle$APB}

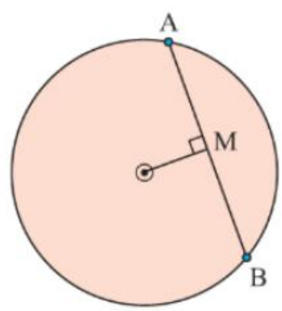
$\therefore$ $2a + 2b = 180$

$\therefore$ $a + b = 90$

So, $A\widehat{P}B$ is a right angle.

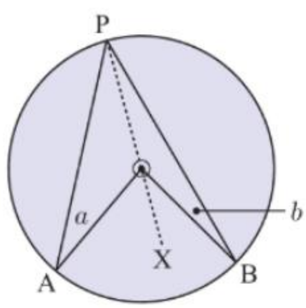
EXERCISE 19C

1 In this question we prove the *chords of a circle* theorem.



- **a** For the given figure, join [OA] and [OB], and show that triangles OAM and OBM are congruent.
- **b** Hence show that $AM = BM$.

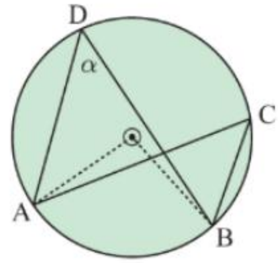
2



In this question we prove the *angle at the centre* theorem.

- **a** Explain why $\triangle$s OAP and OBP are isosceles.
- **b** Find the measure of the following angles in terms of a and b:
  - **i** $A\widehat{P}O$
  - **ii** $B\widehat{P}O$
  - **iii** $A\widehat{O}X$
  - **iv** $B\widehat{O}X$
  - **v** $A\widehat{P}B$
  - **vi** $A\widehat{O}B$
- **c** What can be deduced from **b v** and **b vi**?

3 In this question we prove the *angles subtended by the same arc* theorem.

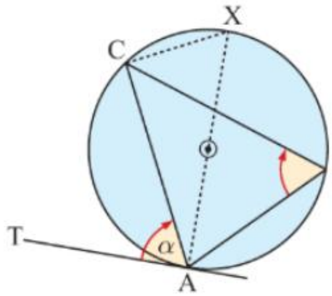


a Using the results of **2**, find the size of $A\widehat{O}B$ in terms of α.

b Hence find the size of $A\widehat{C}B$ in terms of α.

c State the relationship between $A\widehat{D}B$ and $A\widehat{C}B$.

4



In this question we prove the *angle between a tangent and a chord* theorem.

a We draw diameter [AX] and join [CX].

Find the size of: **i** $T\widehat{A}X$ **ii** $A\widehat{C}X$

b If $T\widehat{A}C = \alpha$, find in terms of α:

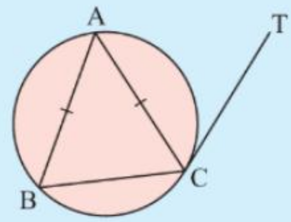
i $C\widehat{A}X$ **ii** $C\widehat{X}A$ **iii** $C\widehat{B}A$

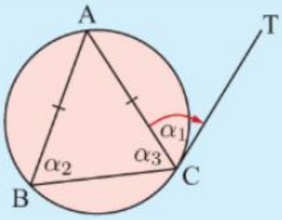
c State the relationship between $T\widehat{A}C$ and $C\widehat{B}A$.

Example 6 Self Tutor

Isosceles $\triangle$ABC is inscribed in a circle.
[TC] is a tangent to the circle.

Prove that [AC] bisects $B\widehat{C}T$.





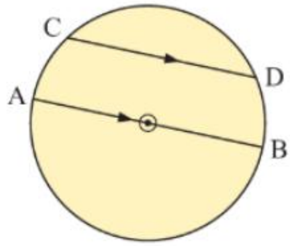
$\alpha_1 = \alpha_2$ {tangent and chord theorem}

and $\alpha_2 = \alpha_3$ {isosceles $\triangle$ theorem}

$\therefore \alpha_1 = \alpha_3$

So, [AC] bisects $B\widehat{C}T$.

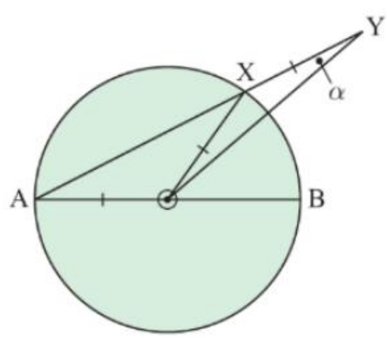
5



[AB] is a diameter of a circle with centre O.
[CD] is a chord parallel to [AB].

Prove that [BC] bisects $D\widehat{C}O$.

6 [AB] is the diameter of a circle with centre O. X is a point on the circle, and [AX] is produced to Y such that OX = XY.



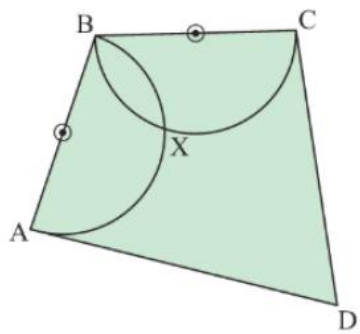
a If $X\widehat{Y}O = \alpha$, find in terms of α:

i $Y\widehat{O}X$ **ii** $A\widehat{X}O$ **iii** $X\widehat{A}O$

iv $X\widehat{O}B$ **v** $B\widehat{O}Y$

b What is the relationship between $B\widehat{O}Y$ and $Y\widehat{O}X$?

7 Revisit the **Opening Problem** on page **350**. Consider the two semi-circles in the figure alongside.



a Find the measure of $B\widehat{X}A$ and $B\widehat{X}C$.

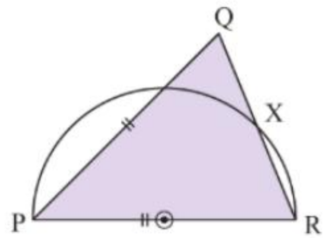
b What does **a** tell us about the points A, X, and C?

c Do the two illustrated sprinklers water all of the area on one side of the diagonal [AC]?

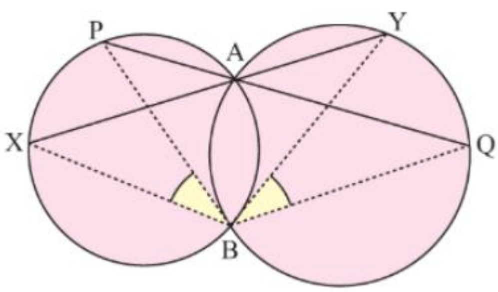
d Will Joe's four sprinklers water the whole garden? Explain your answer.

8 P is any point on a circle. [QR] is a chord of the circle parallel to the tangent at P. Prove that triangle PQR is isosceles.

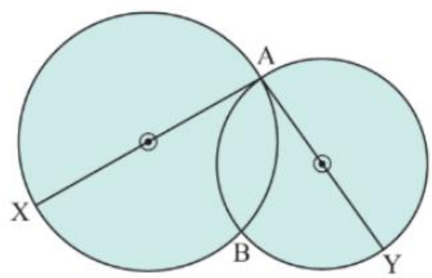
9 Triangle PQR is isosceles with $PQ = PR$.
A semi-circle with diameter [PR] is drawn to cut [QR] at X.
Prove that X is the midpoint of [QR].



10 Two circles intersect at A and B. Straight lines [PQ] and [XY] are drawn through A to meet the circles as shown.
Show that $X\widehat{B}P = Y\widehat{B}Q$.

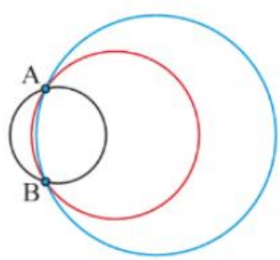


11 Two circles intersect at A and B. [AX] and [AY] are diameters as shown. Prove that X, B, and Y are collinear.

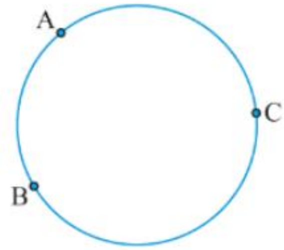


D CYCLIC QUADRILATERALS

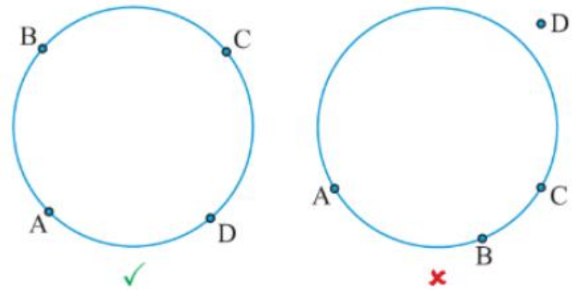
Suppose we are given two points A and B, and we must draw a circle passing through these points. There are infinitely many circles that we can draw.



If we are given three points A, B, and C which are not collinear, there is a unique circle which passes through the points.

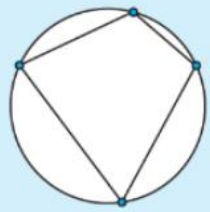


If we are given four points A, B, C, and D, no three of which are collinear, there may or may not be a circle which passes through the points. To see this, we draw the unique circle which passes through A, B, and C. The fourth point D may or may not lie on this circle.



Four points are said to be **concyclic** if a circle can be drawn through them.

If four concyclic points are joined to form a convex quadrilateral, then the quadrilateral is called a **cyclic quadrilateral**.



GEOMETRY PACKAGE

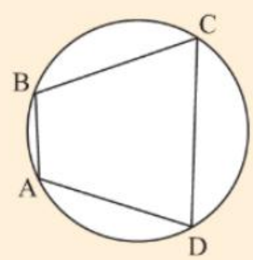
INVESTIGATION 2 CYCLIC QUADRILATERALS

This Investigation can be done using a compass, ruler, and protractor, or you can use the **geometry package** by clicking on the icon.

GEOMETRY PACKAGE

What to do:

1 Draw several circles, and on each circle draw a different cyclic quadrilateral with vertices A, B, C, and D. Make sure the quadrilaterals are large enough for you to measure the angles with a protractor.



2 Measure all angles to the nearest degree, and record your results in a table like this:

Figure	$\widehat{A}$	$\widehat{B}$	$\widehat{C}$	$\widehat{D}$	$\widehat{A}+\widehat{C}$	$\widehat{B}+\widehat{D}$
1						
2						
⋮						

PRINTABLE QUADRILATERALS

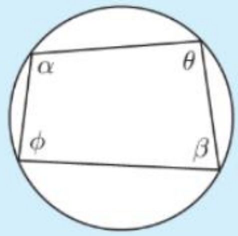
3 Write a sentence to summarise your results.

From the **Investigation** you should have discovered the following theorem:

OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL THEOREM

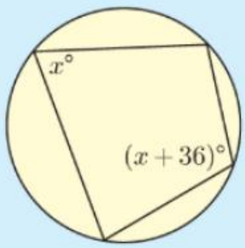
The opposite angles of a cyclic quadrilateral are **supplementary**.

$\alpha + \beta = 180°$ and $\theta + \phi = 180°$



Example 7 Self Tutor

Find x:

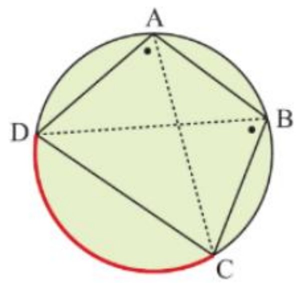


The angles given are opposite angles of a cyclic quadrilateral.

$$\therefore \quad x + (x+36) = 180$$
$$\therefore \quad 2x + 36 = 180$$
$$\therefore \quad 2x = 144$$
$$\therefore \quad x = 72$$

We can use the *angles subtended by the same arc* theorem to discover another property of cyclic quadrilaterals.

For the cyclic quadrilateral ABCD, notice that $D\widehat{A}C = D\widehat{B}C$.



The converses of these two properties give us two useful tests for determining whether a quadrilateral is cyclic.

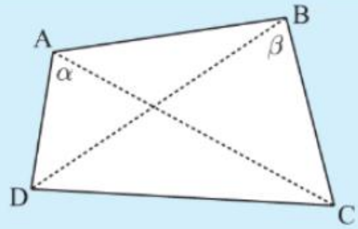
TESTS FOR CYCLIC QUADRILATERALS

A quadrilateral is a **cyclic quadrilateral** if one of the following is true:

- one pair of opposite angles is supplementary

If $\alpha + \beta = 180°$ then ABCD is a cyclic quadrilateral.

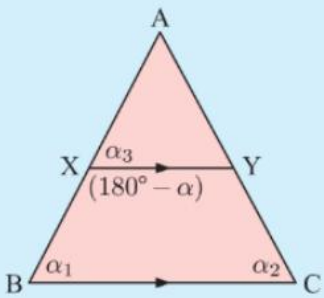
- one side subtends equal angles at the other two vertices



If $\alpha = \beta$ then ABCD is a cyclic quadrilateral.

Example 8 **Self Tutor**

Triangle ABC is isosceles with $AB = AC$. X and Y lie on [AB] and [AC] respectively such that [XY] is parallel to [BC]. Prove that XYCB is a cyclic quadrilateral.



$\triangle$ABC is isosceles with $AB = AC$.

$\therefore \alpha_1 = \alpha_2$ {equal base angles}

Since $[XY] \parallel [BC]$, $\alpha_1 = \alpha_3$ {equal corresponding angles}

$\therefore Y\widehat{X}B = 180° - \alpha$

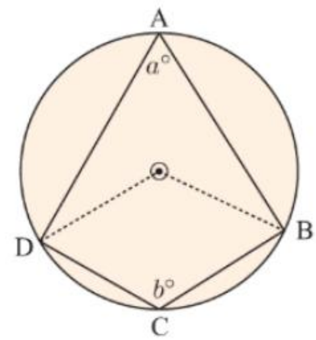
$\therefore Y\widehat{X}B + Y\widehat{C}B = 180° - \alpha + \alpha$

$= 180°$

$\therefore$ XYCB is a cyclic quadrilateral {opposite angles are supplementary}

EXERCISE 19D

1 In this question we prove the *opposite angles of a cyclic quadrilateral* theorem.



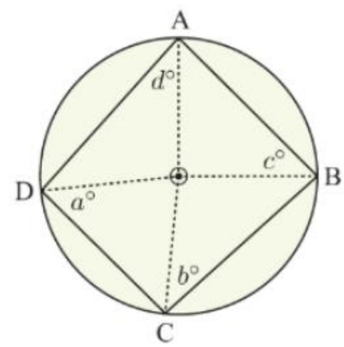
a For the given figure, find:

i $D\widehat{O}B$ in terms of a

ii reflex $D\widehat{O}B$ in terms of b.

b Hence, show that $a + b = 180$.

2 An alternative method for establishing the *opposite angles of a cyclic quadrilateral* theorem is to use the figure alongside.

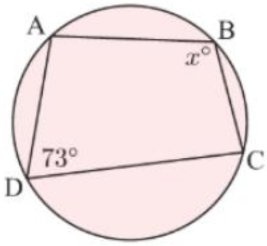


a Show that $2a + 2b + 2c + 2d = 360$.

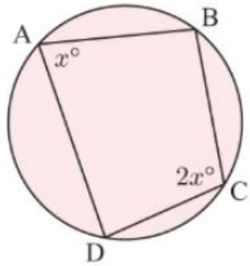
b Hence prove the opposite angles of a cyclic quadrilateral theorem.

3 Find x, giving reasons:

a



b



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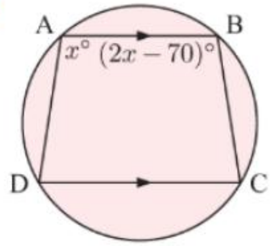
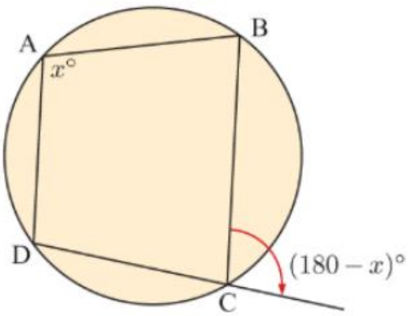
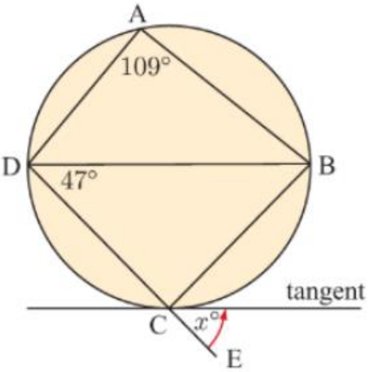


d

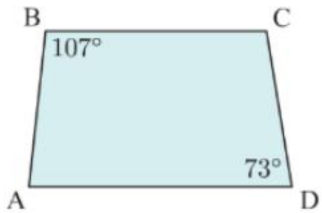


e



4 Is ABCD a cyclic quadrilateral? Explain your answer.

a



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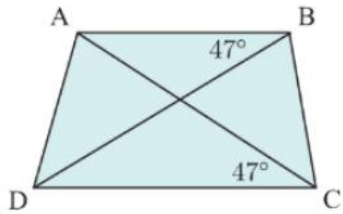

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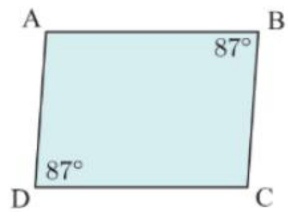
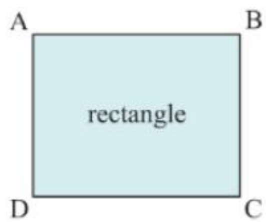


d



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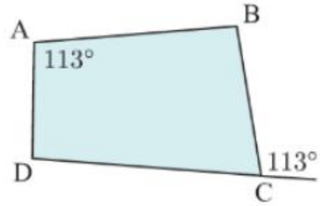

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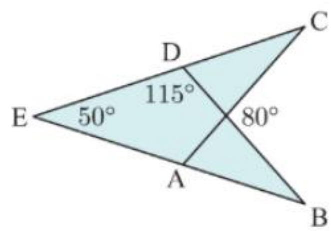
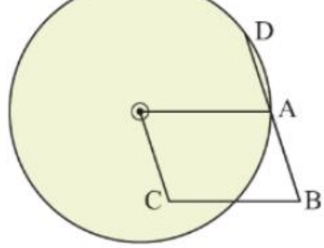
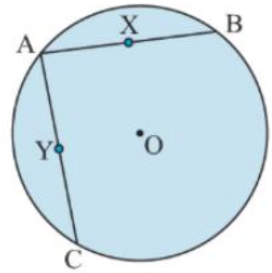


5 A parallelogram is inscribed in a circle. Prove that the parallelogram must be a rectangle.

6 OABC is a parallelogram.
A circle with centre O and radius [OA] is drawn.
[BA] produced meets the circle at D.
Prove that DOCB is a cyclic quadrilateral.



7 [AB] and [AC] are chords of a circle with centre O. X and Y are the midpoints of [AB] and [AC] respectively. Prove that OXAY is a cyclic quadrilateral.

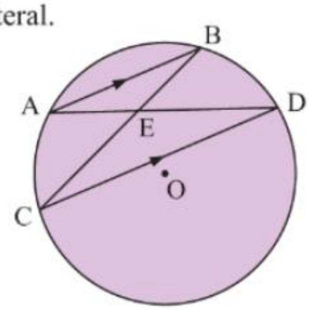


8 ABCD is a cyclic quadrilateral, and X is any point on diagonal [CA]. [XY] is drawn parallel to [CB] to meet [AB] at Y. [XZ] is drawn parallel to [CD] to meet [AD] at Z. Prove that XYAZ is a cyclic quadrilateral.

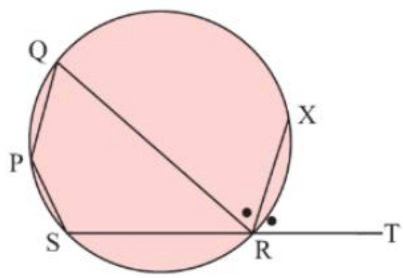
9 ABC is an isosceles triangle with $AB = AC$. The angle bisectors at B and C meet the sides [AC] and [AB] at X and Y respectively. Show that BCXY is a cyclic quadrilateral.

10 The non-parallel sides of a trapezium have equal length. Prove that the trapezium is a cyclic quadrilateral.

11 [AB] and [CD] are two parallel chords of a circle with centre O. [AD] and [BC] meet at E. Prove that AEOC is a cyclic quadrilateral.



12 [RX] is the bisector of angle QRT. Prove that [PX] bisects angle QPS.

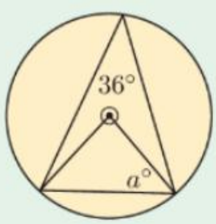


13 Two circles meet at points X and Y. Line segment [AXB] meets one circle at A and the other at B. Line segment [CYD] meets one circle at C and the other at D. Prove that [AC] is parallel to [BD].

REVIEW SET 19A

1 Find the value of a:

a



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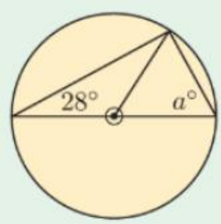

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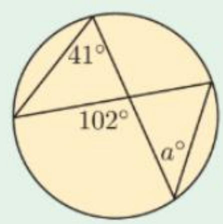

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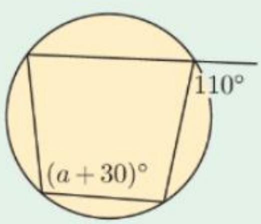

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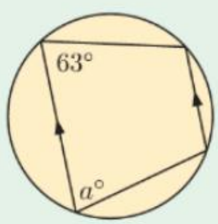

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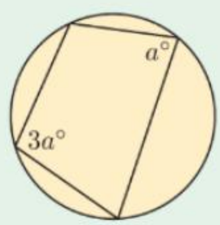
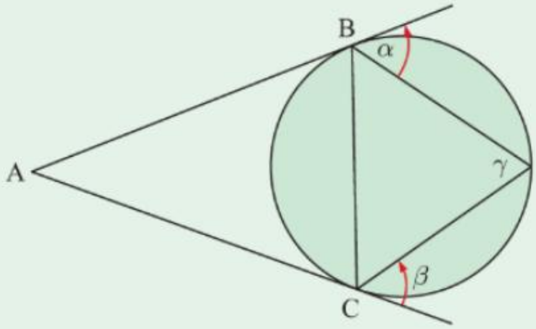
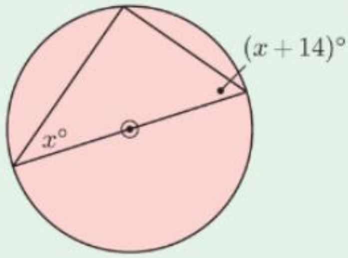


2 [AB] and [AC] are tangents to the circle.
Find an equation connecting α, β, and γ.



3 Find the value of x:

a



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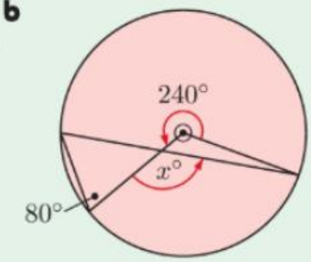

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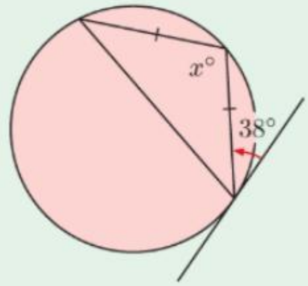
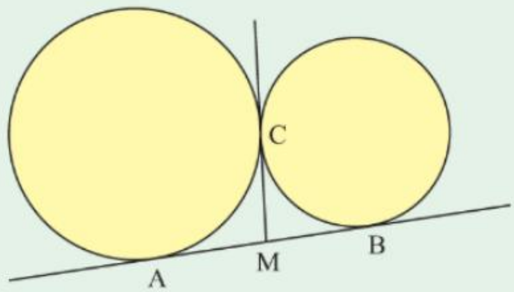


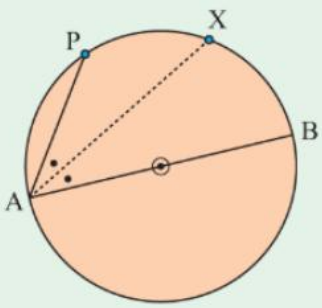
4 [AB] and [CM] are common tangents to two touching circles. Show that:

a M is the midpoint of [AB]

b $\widehat{ACB}$ is a right angle.



5



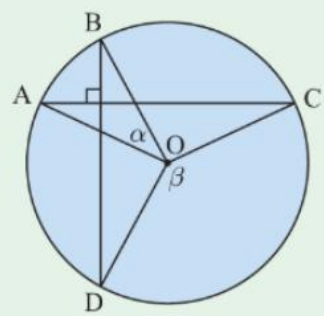
The circle alongside has diameter [AB], and P is another point on the circle. The angle bisector of $\widehat{PAB}$ meets the circle at X. Show that the tangent at X is parallel to [PB].

6 O is the centre of the circle alongside. The chords [AC] and [BD] are perpendicular.

a Join [BC]. Use the angle at the centre theorem to find:

i $\widehat{DBC}$ **ii** $\widehat{ACB}$

b Hence show that $\alpha + \beta = 180°$.

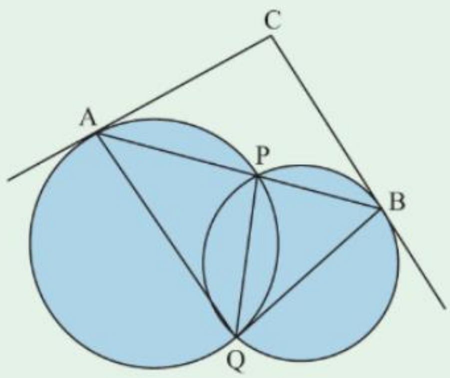


7 **a** Copy and complete: "*The angle between a tangent and a chord through the point of contact is equal to ......*"

b Two circles intersect at points P and Q. A line segment [APB] is drawn through P, and the tangents at A and B meet at C.

i Let $\widehat{ABC} = \alpha$ and $\widehat{BAC} = \beta$. Write expressions for $\widehat{PQB}$, $\widehat{PQA}$, and $\widehat{AQB}$ in terms of α and β.

ii Hence, show that ACBQ is a cyclic quadrilateral.



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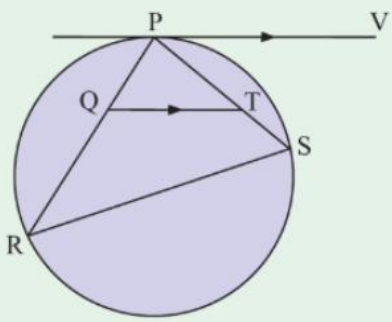
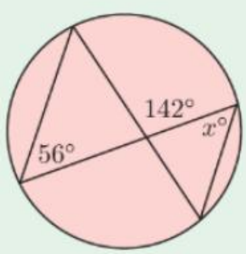


[PV] is a tangent to the circle, and [QT] is parallel to [PV].
Prove that QRST is a cyclic quadrilateral.

REVIEW SET 19B

1 Find the value of x:

a



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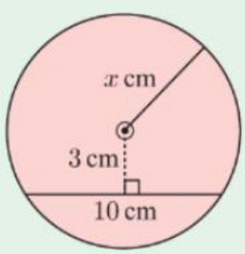

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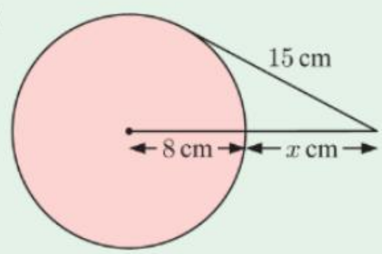
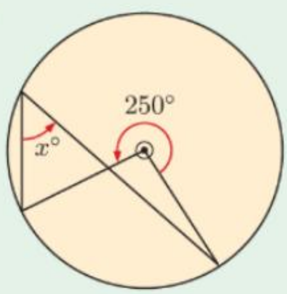


2 Find the value of x:

a



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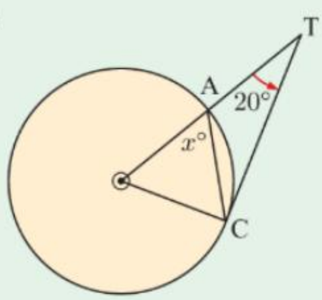

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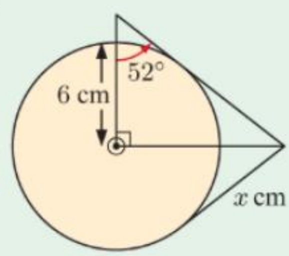
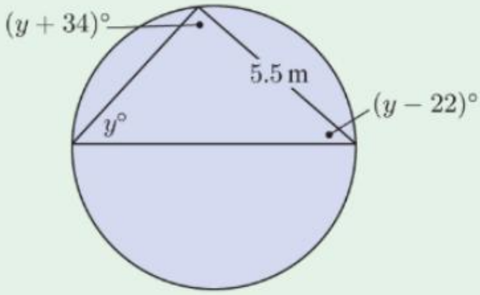
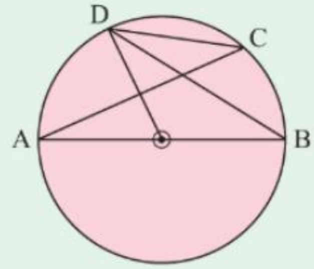


3 Find the diameter of the circle:



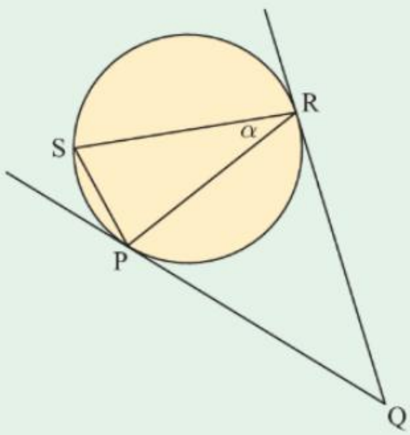
4 [AB] is the diameter of a circle with centre O. [AC] and [BD] are any two chords.

Show that $B\widehat{D}O = A\widehat{C}D$.



5 A, B, and C are three points on a circle. The bisector of angle BAC cuts [BC] at P, and the circle at Q. Prove that $A\widehat{P}C = A\widehat{B}Q$.

6



[QP] and [QR] are tangents to a circle. S is a point on the circle such that $P\widehat{S}R$ and $P\widehat{Q}R$ are equal, and both are double $P\widehat{R}S$.

Let $P\widehat{R}S$ be α.

a Find, in terms of α:

i $P\widehat{S}R$ **ii** $P\widehat{Q}R$

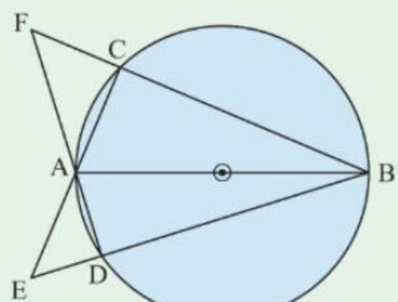
iii $P\widehat{R}Q$ **iv** $Q\widehat{P}R$

b Use triangle PQR to show that $\alpha = 30°$.

c Find the measure of $Q\widehat{R}S$.

d What can you conclude about [RS]?

7 [AB] is the diameter of a circle, and C and D are two other points on the circle. [AC] and [BD] meet at E, and [AD] and [BC] meet at F. Show that CDEF is a cyclic quadrilateral.

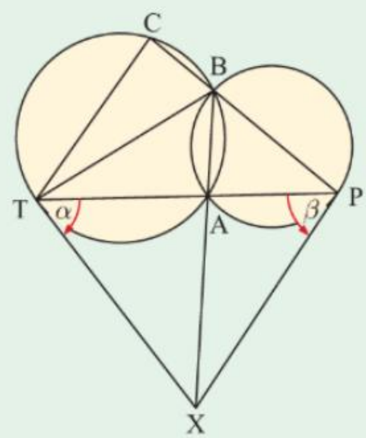


8 In the figure alongside, [XT] and [XP] are tangents, $A\widehat{T}X = \alpha$, and $A\widehat{P}X = \beta$.

a Prove that BTXP is a cyclic quadrilateral.

b Show that $\alpha = \beta$.

c Show that [PT] bisects $C\widehat{T}X$.



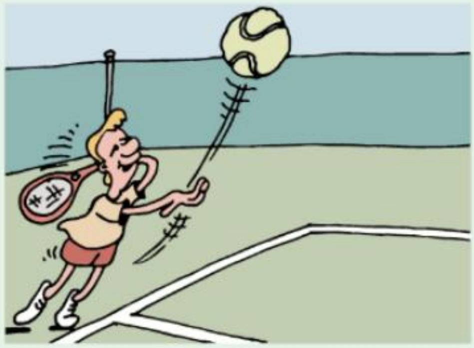
Chapter 20

Quadratic functions

Contents:

OPENING PROBLEM

Tennis player Bradley tosses the ball in the air before he serves it. The ball's height above the ground t seconds after it is tossed is given by the function $H(t) = -5t^2 + 6t + 2$ metres.



Things to think about:

a How high is the ball when it is released?

b What is the maximum height reached by the tennis ball?

c Bradley hits the ball when it is 3 metres above the ground, on its way down. How long after Bradley releases the ball does he hit it?

In this chapter we consider relationships between variables which are **quadratic** in nature. These relationships are described algebraically using **quadratic functions**.

A QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables which can be written in the form $y = ax^2 + bx + c$ where x and y are the variables, and a, b, and c are constants, $a \neq 0$.

Using function notation, $y = ax^2 + bx + c$ can be written as $f(x) = ax^2 + bx + c$.

FINDING y GIVEN x

For any value of x, the corresponding value of y can be found by substitution into the function equation.

Example 1 Self Tutor

Suppose $y = 2x^2 + 4x - 5$. Find the value of y when:

a $x = 0$ **b** $x = 3$

a When $x = 0$,
$$\begin{aligned} y &= 2(0)^2 + 4(0) - 5 \\ &= 0 + 0 - 5 \\ &= -5 \end{aligned}$$

b When $x = 3$,
$$\begin{aligned} y &= 2(3)^2 + 4(3) - 5 \\ &= 18 + 12 - 5 \\ &= 25 \end{aligned}$$

FINDING x GIVEN y

When we substitute a value for y, we are left with a quadratic equation which we need to solve for x. Since the equation is quadratic, there may be 0, 1, or 2 possible values for x.

Example 2 — Self Tutor

Suppose $y = x^2 - 6x + 8$. Find the value(s) of x for which:

a $y = 15$ **b** $y = -1$

a When $y = 15$,
$$x^2 - 6x + 8 = 15$$
$$\therefore\ x^2 - 6x - 7 = 0$$
$$\therefore\ (x+1)(x-7) = 0$$
$$\therefore\ x = -1 \quad \text{or} \quad x = 7$$

b When $y = -1$,
$$x^2 - 6x + 8 = -1$$
$$\therefore\ x^2 - 6x + 9 = 0$$
$$\therefore\ (x-3)^2 = 0$$
$$\therefore\ x = 3$$

EXERCISE 20A

1 Which of the following are quadratic functions?

a $y = 15x - 8$ **b** $y = \frac{1}{3}x^2 + 6$ **c** $3y + 2x^2 - 7 = 0$ **d** $y = 15x^3 + 2x - 16$

2 Suppose $y = 3x^2 + 7$. Find the value of y when:

a $x = 2$ **b** $x = 5$ **c** $x = -3$

3 For each of the following functions, find the value of y for the given value of x:

a $y = x^2 + 5x - 14$ when $x = 2$ **b** $y = 2x^2 + 9x$ when $x = -5$

c $y = -2x^2 + 3x - 6$ when $x = 3$ **d** $y = 4x^2 + 7x + 10$ when $x = -2$

4 State whether the following quadratic functions are satisfied by the given ordered pairs:

a $y = 6x^2 - 10$ $(0, 4)$ **b** $y = 2x^2 - 5x - 3$ $(4, 9)$

c $y = -4x^2 + 6x$ $(-\frac{1}{2}, -4)$ **d** $y = -7x^2 + 9x + 11$ $(-1, -6)$

e $y = 3x^2 - 11x + 20$ $(2, -10)$ **f** $y = -3x^2 + x + 6$ $(\frac{1}{3}, 4)$

5 Suppose $y = x^2 - 2x - 3$. Find the value(s) of x for which:

a $y = 0$ **b** $y = -4$ **c** $y = 5$

6 For each of the following quadratic functions, find the value(s) of x for the given value of y:

a $y = x^2 + 6x + 10$ when $y = 1$ **b** $y = x^2 + 5x + 8$ when $y = 2$

c $y = x^2 - 5x + 1$ when $y = -3$ **d** $y = 3x^2$ when $y = -3$

7 Find the value(s) of x for which:

a $f(x) = 3x^2 - 3x + 6$ takes the value 6

b $f(x) = x^2 - 2x - 7$ takes the value -4

c $f(x) = -2x^2 - 13x + 3$ takes the value -4

d $f(x) = 2x^2 - 10x + 1$ takes the value -11

Example 3 **Self Tutor**

A stone is thrown into the air. Its height above the ground is given by the function $h(t) = -5t^2 + 30t + 2$ metres, where t is the time in seconds from when the stone is thrown.

a How high is the stone above the ground after 3 seconds?

b From what height above the ground was the stone released?

c At what times is the stone 27 m above the ground?

a $h(3) = -5(3)^2 + 30(3) + 2$
$= -45 + 90 + 2$
$= 47$
$\therefore$ the stone is 47 m above the ground.

b The stone was released when $t = 0$ s.
Now $h(0) = -5(0)^2 + 30(0) + 2 = 2$
$\therefore$ the stone was released from 2 m above ground level.

c When $h(t) = 27$,
$$-5t^2 + 30t + 2 = 27$$
$$\therefore -5t^2 + 30t - 25 = 0$$
$$\therefore t^2 - 6t + 5 = 0$$
$$\therefore (t-1)(t-5) = 0$$
$$\therefore t = 1 \text{ or } 5$$
$\therefore$ the stone is 27 m above the ground after 1 second and after 5 seconds.

8 An object is projected into the air with a velocity of 80 m s^{-1}. Its height after t seconds is given by the function $h(t) = 80t - 5t^2$ metres.

a Calculate the height of the object after:
i 1 second **ii** 3 seconds **iii** 5 seconds.

b Calculate the time(s) at which the height of the object is:
i 140 m **ii** 0 m.

c Explain your answers in part **b**.

9 A cake manufacturer finds that the profit from making x cakes per day is given by the function $P(x) = -\frac{1}{2}x^2 + 36x - 40$ dollars.

a Calculate the profit if **i** 0 cakes **ii** 20 cakes are made per day.

b How many cakes must be made per day to achieve a profit of $270?

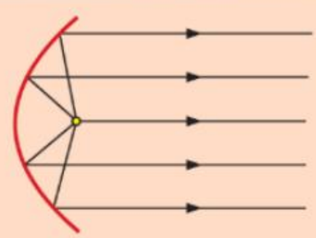
B GRAPHS OF QUADRATIC FUNCTIONS

The graph of a quadratic function is called a **parabola**. The parabola is one of the **conic sections**.

HISTORICAL NOTE **CONIC SECTIONS**

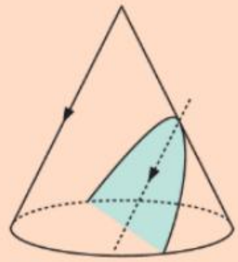
Conic sections are curves which can be obtained by cutting a cone with a plane. The Ancient Greek mathematicians were fascinated by conic sections.

The name parabola comes from the Greek word for **thrown** because when an object is thrown, its path makes a parabolic arc.



There are many other examples of parabolas in everyday life. For example, parabolic mirrors are used in car headlights, heaters, satellite dishes, and radio telescopes, because of their special geometric properties.

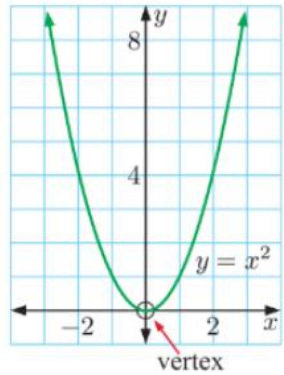
You may like to explore the conic sections for yourself by cutting an icecream cone. Cutting parallel to the side produces a parabola, as shown in the diagram.



The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

We can see that the graph has a minimum turning point at $(0, 0)$. We call this the **vertex** of the parabola.



Example 4 **Self Tutor**

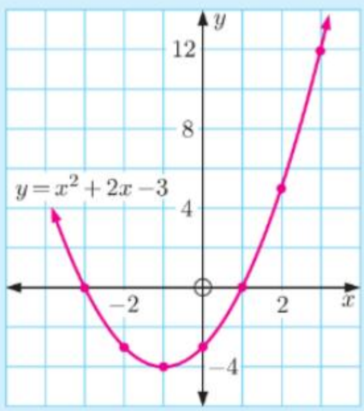
Draw the graph of $y = x^2 + 2x - 3$ using a table of values from $x = -3$ to $x = 3$.

Consider $f(x) = x^2 + 2x - 3$

Now, $f(-3) = (-3)^2 + 2(-3) - 3$
$= 9 - 6 - 3$
$= 0$

We can do the same for the other values of x:

x	-3	-2	-1	0	1	2	3
y	0	-3	-4	-3	0	5	12



EXERCISE 20B.1

1 Using a table of values from $x = -3$ to $x = 3$, draw the graph of:

a $y = x^2 - 2x + 8$
b $f(x) = -x^2 + 2x + 1$
c $y = 2x^2 + 3x$
d $y = -2x^2 + 4$
e $y = x^2 + x + 4$
f $f(x) = -x^2 + 4x - 9$

Use the **graphing package** or a **graphics calculator** to check your graphs.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

2 **a** Use tables of values to graph $y = 2x^2 - x - 3$ and $y = -x^2 + 2x + 3$ on the same set of axes. *Hence* find the values of x for which $2x^2 - x - 3 = -x^2 + 2x + 3$.

b Solve algebraically: $2x^2 - x - 3 = -x^2 + 2x + 3$.

USING TRANSFORMATIONS TO GRAPH QUADRATIC FUNCTIONS

By observing how a quadratic function is related to $f(x) = x^2$, we can transform the graph of $y = x^2$ to produce the graph of the function.

INVESTIGATION 1 GRAPHS OF QUADRATIC FUNCTIONS

In this Investigation we consider different forms of quadratic functions, and how the form of the quadratic affects its graph. You can use either the graphing package or your graphics calculator to draw the graphs.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

Part 1: Graphs of the form $y = x^2 + k$

What to do:

1 Graph each pair of functions on the same set of axes, and observe the coordinates of the vertex of each function.

a $y = x^2$ and $y = x^2 + 2$ **b** $y = x^2$ and $y = x^2 - 2$

c $y = x^2$ and $y = x^2 + 4$ **d** $y = x^2$ and $y = x^2 - 4$

2 What effect does the value of k have on:

a the position of the graph **b** the shape of the graph?

3 Copy and complete:

"*The graph of* $y = x^2 + k$ *is obtained from* $y = x^2$ *by a* .... *with translation vector* $\begin{pmatrix} \cdots\cdots \\ \cdots\cdots \end{pmatrix}$."

Part 2: Graphs of the form $y = (x - h)^2$

What to do:

1 Graph each pair of functions on the same set of axes, and observe the coordinates of the vertex of each function.

a $y = x^2$ and $y = (x - 2)^2$ **b** $y = x^2$ and $y = (x + 2)^2$

c $y = x^2$ and $y = (x - 4)^2$ **d** $y = x^2$ and $y = (x + 4)^2$

2 What effect does the value of h have on:

a the position of the graph **b** the shape of the graph?

3 Copy and complete:

"*The graph of* $y = (x - h)^2$ *is obtained from* $y = x^2$ *by a* .... *with translation vector* $\begin{pmatrix} \cdots\cdots \\ \cdots\cdots \end{pmatrix}$."

Part 3: Graphs of the form $y = (x-h)^2 + k$

What to do:

1 Graph each pair of functions on the same set of axes, and observe the coordinates of the vertex of each function.

a $y = x^2$ and $y = (x-2)^2 + 3$
b $y = x^2$ and $y = (x+4)^2 - 1$
c $y = x^2$ and $y = (x-5)^2 - 2$
d $y = x^2$ and $y = (x+1)^2 + 5$

2 Copy and complete:

- "*The graph of* $y = (x-h)^2 + k$ *is the same shape as the graph of* ...."
- "*The graph of* $y = (x-h)^2 + k$ *is obtained from* $y = x^2$ *by a* .... *with translation vector* $\begin{pmatrix} \ldots\ldots \\ \ldots\ldots \end{pmatrix}$."
- "*The vertex of the graph of* $y = (x-h)^2 + k$ *is at* (......, ...)."

Part 4: Graphs of the form $y = ax^2, \quad a \neq 0$

What to do:

1 Graph each pair of functions on the same set of axes, and observe the coordinates of the vertex of each function.

a $y = x^2$ and $y = 2x^2$
b $y = x^2$ and $y = 4x^2$
c $y = x^2$ and $y = \frac{1}{2}x^2$
d $y = x^2$ and $y = -x^2$
e $y = x^2$ and $y = -2x^2$
f $y = x^2$ and $y = -\frac{1}{2}x^2$

2 These functions are all members of the family $y = ax^2$. What effect does a have on:

a the position of the graph
b the shape of the graph
c the direction in which the graph opens?

Part 5: Graphs of the form $y = a(x-h)^2 + k, \quad a \neq 0$

What to do:

1 Graph each pair of functions on the same set of axes, and observe the coordinates of the vertex of each function.

a $y = 2x^2$ and $y = 2(x-1)^2 + 3$
b $y = -x^2$ and $y = -(x+2)^2 - 1$
c $y = \frac{1}{2}x^2$ and $y = \frac{1}{2}(x-3)^2 - 2$
d $y = -3x^2$ and $y = -3(x+1)^2 + 4$

2 Copy and complete:

- "*The graph of* $y = a(x-h)^2 + k$ *has the same shape and opens in the same direction as the graph of* ...."
- "*The graph of* $y = a(x-h)^2 + k$ *is obtained from* $y = ax^2$ *by a* .... *with translation vector* $\begin{pmatrix} \ldots\ldots \\ \ldots\ldots \end{pmatrix}$."

From the **Investigation** you should have discovered the following important facts:

- Graphs of the form $y = x^2 + k$ have the same shape as the graph of $y = x^2$.
  The graph of $y = x^2$ is translated through $\begin{pmatrix} 0 \\ k \end{pmatrix}$ to give the graph of $y = x^2 + k$.
- Graphs of the form $y = (x-h)^2$ have the same shape as the graph of $y = x^2$.
  The graph of $y = x^2$ is translated through $\begin{pmatrix} h \\ 0 \end{pmatrix}$ to give the graph of $y = (x-h)^2$.
- Graphs of the form $y = (x-h)^2 + k$ have the same shape as the graph of $y = x^2$.
  The graph of $y = x^2$ is translated through $\begin{pmatrix} h \\ k \end{pmatrix}$ to give the graph of $y = (x-h)^2 + k$.
  The vertex is shifted to (h, k).
- If $a > 0$, $y = ax^2$ opens upwards.

  If $a < 0$, $y = ax^2$ opens downwards.

  If $a < -1$ or $a > 1$, then $y = ax^2$ is 'thinner' than $y = x^2$.
  If $-1 < a < 1$, $a \neq 0$, then $y = ax^2$ is 'wider' than $y = x^2$.
- $a > 0$
  $a < 0$

  vertical translation of k units:
  if $k > 0$ it goes up
  if $k < 0$ it goes down

  $$y = a(x-h)^2 + k$$

  $a < -1$ or $a > 1$, thinner than $y = x^2$
  $-1 < a < 1$, $a \neq 0$, wider than $y = x^2$

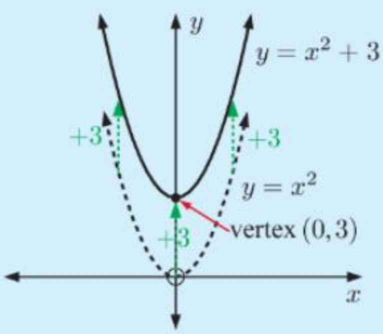
  horizontal translation of h units:
  if $h > 0$ it goes right
  if $h < 0$ it goes left

Example 5 **Self Tutor**

Sketch each of the following functions on the same set of axes as $y = x^2$. In each case state the coordinates of the vertex.

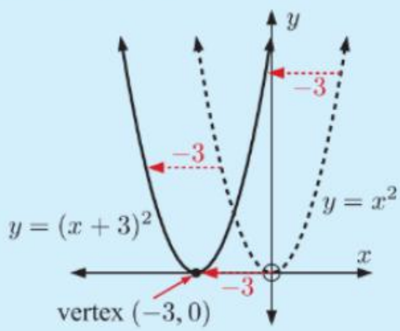
a $y = x^2 + 3$ **b** $y = (x+3)^2$

a We draw $y = x^2$, then translate it 3 units upwards.



The vertex is at $(0, 3)$.

b We draw $y = x^2$, then translate it 3 units to the left.



The vertex is at $(-3, 0)$.

EXERCISE 20B.2

1 Sketch each of the following functions on the same set of axes as $y = x^2$. Use a separate set of axes for each part, and in each case state the coordinates of the vertex.

GRAPHING PACKAGE

a $y = x^2 - 3$ **b** $y = x^2 - 1$ **c** $y = x^2 + 2$

d $y = x^2 - 5$ **e** $y = x^2 + 5$ **f** $y = x^2 - \frac{1}{2}$

Use a **graphics calculator** or **graphing package** to check your answers.

2 Sketch each of the following functions on the same set of axes as $y = x^2$. Use a separate set of axes for each part, and in each case state the coordinates of the vertex.

a $y = (x - 3)^2$ **b** $y = (x + 1)^2$ **c** $y = (x - 2)^2$

d $y = (x - 5)^2$ **e** $y = (x + 5)^2$ **f** $y = (x - \frac{3}{2})^2$

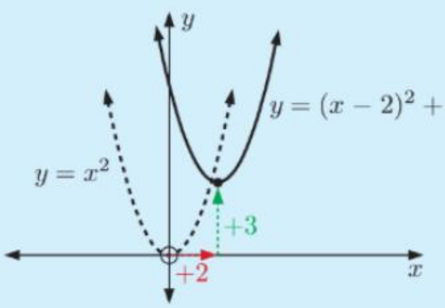
Use a **graphics calculator** or **graphing package** to check your answers.

Example 6 Self Tutor

Sketch each of the following functions on the same set of axes as $y = x^2$. In each case state the coordinates of the vertex.

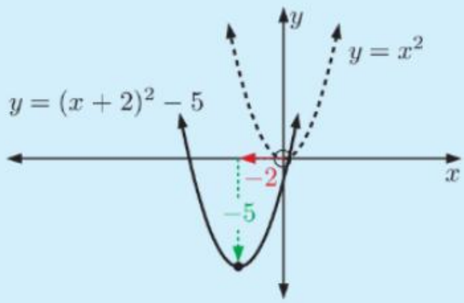
a $y = (x - 2)^2 + 3$ **b** $y = (x + 2)^2 - 5$

a We draw $y = x^2$, then translate it 2 units to the right and 3 units upwards.



The vertex is at (2, 3).

b We draw $y = x^2$, then translate it 2 units to the left and 5 units downwards.



The vertex is at $(-2, -5)$.

3 Sketch each of the following functions on the same set of axes as $y = x^2$. Use a separate set of axes for each part, and in each case state the coordinates of the vertex.

a $y = (x - 1)^2 + 3$ **b** $y = (x - 2)^2 - 1$ **c** $y = (x + 1)^2 + 4$

d $y = (x + 2)^2 - 3$ **e** $y = (x + 3)^2 - 2$ **f** $y = (x - 3)^2 + 3$

Use a **graphics calculator** or **graphing package** to check your answers.

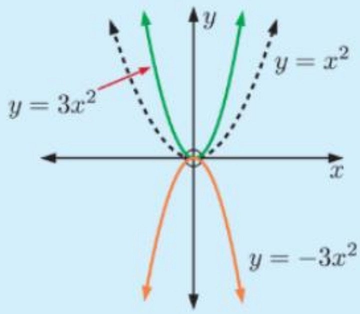
Example 7 Self Tutor

Sketch $y = x^2$ on a set of axes and hence sketch:

a $y = 3x^2$ **b** $y = -3x^2$

a $y = 3x^2$ is 'thinner' than $y = x^2$.

b $y = -3x^2$ has the same shape as $y = 3x^2$, but opens downwards.



4 Sketch each of the following functions on the same set of axes as $y = x^2$. Comment on the shape of the graph, and the direction in which the graph opens.

GRAPHING PACKAGE

a $y = 5x^2$ **b** $y = -5x^2$ **c** $y = \frac{1}{3}x^2$

d $y = -\frac{1}{3}x^2$ **e** $y = -4x^2$ **f** $y = \frac{1}{4}x^2$

Use a **graphics calculator** or **graphing package** to check your answers.

Example 8 Self Tutor

Sketch the graph of $y = -(x-2)^2 - 3$ from the graph of $y = x^2$, and hence state the coordinates of its vertex.

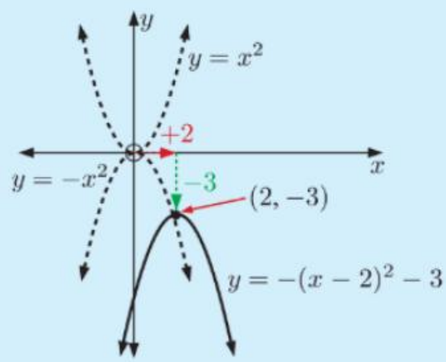
$y = -(x-2)^2 - 3$

reflect in x-axis; horizontal translation 2 units right; vertical translation 3 units down

We start with $y = x^2$, then reflect it in the x-axis to give $y = -x^2$.

We then translate $y = -x^2$ 2 units to the right and 3 units down.

The vertex of $y = -(x-2)^2 - 3$ is $(2, -3)$.



5 Sketch each of the following functions on the same set of axes as $y = x^2$. In each case, state the coordinates of the vertex.

a $y = -(x-1)^2 + 3$ **b** $y = 2x^2 + 4$ **c** $y = -(x-2)^2 + 4$

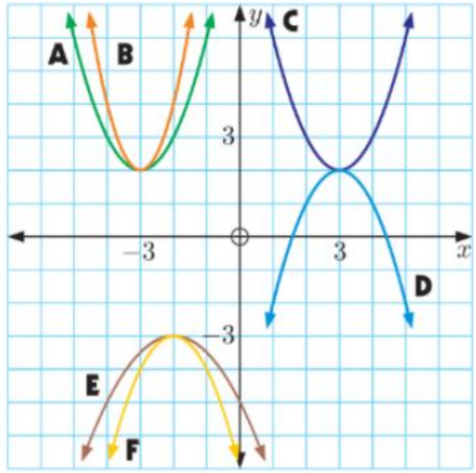
d $y = 3(x+1)^2 - 4$ **e** $y = \frac{1}{2}(x+3)^2$ **f** $y = -\frac{1}{2}(x+3)^2 + 1$

g $y = -2(x+4)^2 + 3$ **h** $y = 2(x-3)^2 + 5$ **i** $y = \frac{1}{2}(x-2)^2 - 1$

Use a **graphics calculator** or **graphing package** to check your answers.

6 Match each quadratic function with its graph:

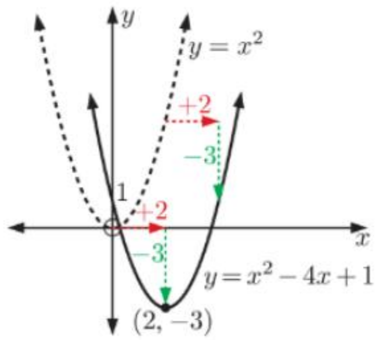
  a $y = -(x+2)^2 - 3$
  b $y = (x-3)^2 + 2$
  c $y = 2(x+3)^2 + 2$
  d $y = -(x-3)^2 + 2$
  e $y = -\frac{1}{2}(x+2)^2 - 3$
  f $y = (x+3)^2 + 2$



COMPLETING THE SQUARE

Suppose we want to graph the quadratic function $y = x^2 - 4x + 1$. This function is not written in the form $y = (x-h)^2 + k$, but we can convert it into this form by **completing the square**.

Consider
$$y = x^2 - 4x + 1$$
$$\therefore\ y = \underbrace{x^2 - 4x + 2^2} + 1 - 2^2$$
$$\therefore\ y = (x-2)^2 - 3$$



So, the graph of $y = x^2 - 4x + 1$ can be found by translating $y = x^2$ 2 units to the right and 3 units down.

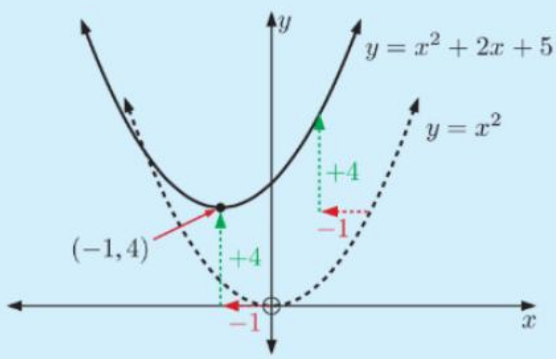
Example 9 **Self Tutor**

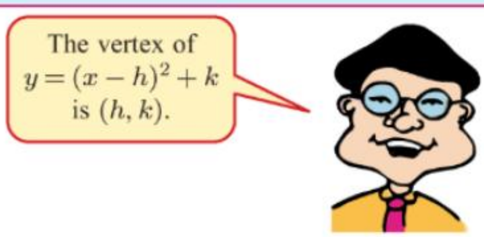
Write $y = x^2 + 2x + 5$ in the form $y = (x-h)^2 + k$.
Hence sketch $y = x^2 + 2x + 5$, stating the coordinates of the vertex.

$$y = x^2 + 2x + 5$$
$$\therefore\ y = x^2 + 2x + 1^2 + 5 - 1^2$$
$$\therefore\ y = (x+1)^2 + 4$$

translate 1 unit left   translate 4 units up

The vertex is at $(-1, 4)$.





EXERCISE 20B.3

GRAPHING PACKAGE

1 Write the following quadratics in the form $y = (x-h)^2 + k$.
Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 + 2x + 4$ **b** $y = x^2 - 6x + 3$
c $y = x^2 + 4x - 1$ **d** $y = x^2 - 2x + 5$
e $y = x^2 - 2x$ **f** $y = x^2 + 5x$
g $y = x^2 + 5x - 3$ **h** $y = x^2 - 3x + 3$ **i** $y = x^2 - 5x + 2$

Use a **graphics calculator** or **graphing package** to check your answers.

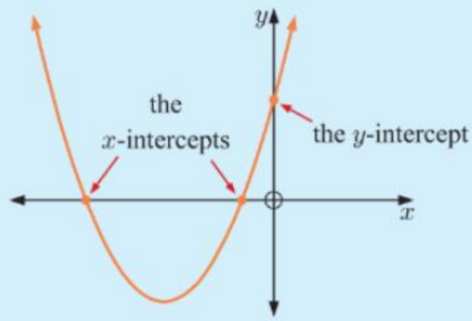
2 Write the following quadratics in the form $y = a(x-h)^2 + k$.
Hence sketch each function, stating the coordinates of the vertex.

a $y = 2x^2 + 10x + 8$ **b** $y = -x^2 + x + 6$ **c** $y = 3x^2 - 6x - 24$
d $y = -2x^2 + 6x + 8$ **e** $y = 2x^2 - 8x - 3$ **f** $y = -3x^2 - 6x + 2$

Use a **graphics calculator** or **graphing package** to check your answers.

C AXES INTERCEPTS

- An **x-intercept** of a function is a value of x where its graph meets the x-axis.
  x-intercepts are found by letting y be 0 in the equation of the function.
- A **y-intercept** of a function is a value of y where its graph meets the y-axis.
  y-intercepts are found by letting x be 0 in the equation of the function.



INVESTIGATION 2 AXES INTERCEPTS

What to do:

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

1 For each of the following quadratic functions, use a **graphing package** or **graphics calculator** to:

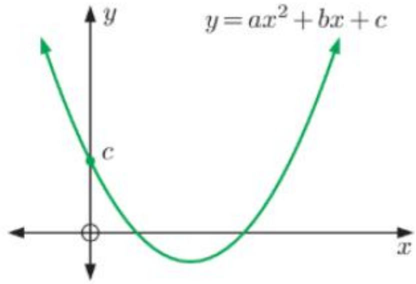
i draw the graph **ii** find the y-intercept
iii find any x-intercepts.

a $y = x^2 - 3x - 4$ **b** $y = -x^2 + 2x + 8$ **c** $y = 2x^2 - 3x$
d $y = -2x^2 + 2x - 3$ **e** $y = (x-1)(x-3)$ **f** $y = -(x+2)(x-3)$
g $y = 3(x+1)(x+4)$ **h** $y = 2(x-2)^2$ **i** $y = -3(x+1)^2$

2 **a** State the y-intercept of a quadratic function in the form $y = ax^2 + bx + c$.
b State the x-intercepts of a quadratic function in the form:
i $y = a(x-\alpha)(x-\beta)$ **ii** $y = a(x-\alpha)^2$

THE y-INTERCEPT

For a quadratic function in the form $y = ax^2 + bx + c$, the y-intercept is the constant term c.

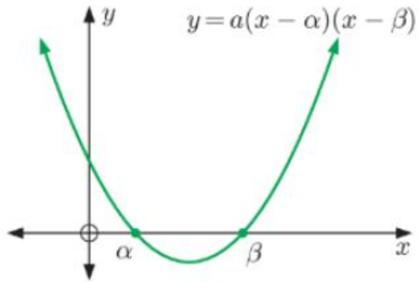


Proof:

If $x = 0$ then $y = a(0)^2 + b(0) + c$
$\therefore \; y = c$

THE x-INTERCEPTS

For a quadratic function in the form $y = a(x-\alpha)(x-\beta)$, the x-intercepts are α and β.



Proof:

If $y = 0$ then $a(x-\alpha)(x-\beta) = 0$
$\therefore \; x = \alpha$ or β {since $a \neq 0$}

x-intercepts are therefore easy to find when the quadratic is in **factorised** form.

Example 10 — Self Tutor

Find the x-intercepts of:

a $y = 2(x-3)(x+2)$

b $y = -(x-4)^2$

a When $y = 0$,
$2(x-3)(x+2) = 0$
$\therefore \; x = 3$ or $x = -2$
$\therefore$ the x-intercepts are 3 and -2.

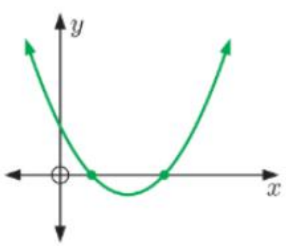
b When $y = 0$,
$-(x-4)^2 = 0$
$\therefore \; x = 4$
$\therefore$ the x-intercept is 4.

If a quadratic function has only one x-intercept then its graph must **touch** the x-axis.

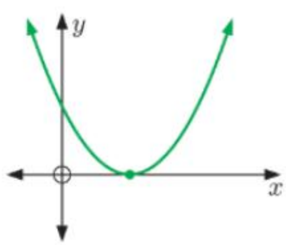
FACTORISING TO FIND x-INTERCEPTS

For any quadratic function of the form $y = ax^2 + bx + c$, the x-intercepts can be found by solving the equation $ax^2 + bx + c = 0$.

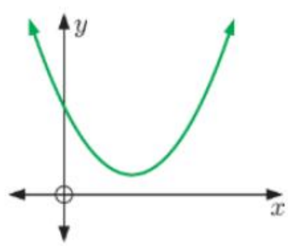
You will recall from **Chapter 12** that quadratic equations may have *two solutions*, *one solution*, or *no solutions*. These solutions correspond to the *two x-intercepts*, *one x-intercept*, or *no x-intercepts* found when the graph of the corresponding quadratic function is drawn.



two x-intercepts



one x-intercept



no x-intercepts

Example 11 — Self Tutor

Find the x-intercept(s) of the quadratic function:

a $y = x^2 - 6x + 9$ **b** $y = -x^2 - x + 6$

a When $y = 0$,
$x^2 - 6x + 9 = 0$
$\therefore\ (x-3)^2 = 0$
$\therefore\ x = 3$
$\therefore$ the x-intercept is 3.

b When $y = 0$,
$-x^2 - x + 6 = 0$
$\therefore\ x^2 + x - 6 = 0$
$\therefore\ (x+3)(x-2) = 0$
$\therefore\ x = -3$ or 2
$\therefore$ the x-intercepts are -3 and 2.

EXERCISE 20C

1 State the y-intercept of:

a $y = x^2 + 3x + 3$ **b** $y = x^2 - 5x + 2$ **c** $f(x) = 2x^2 + 7x - 8$
d $y = 3x^2 - x + 1$ **e** $f(x) = -x^2 + 3x + 6$ **f** $y = -2x^2 + 5 - x$
g $y = 6 - x - x^2$ **h** $f(x) = 8 + 2x - 3x^2$ **i** $y = 5x - x^2 - 2$

2 Find the x-intercepts of:

a $y = (x-3)(x+1)$ **b** $f(x) = -(x-2)(x-4)$ **c** $y = 2(x+3)(x+2)$
d $y = -3(x-4)(x-5)$ **e** $y = 2(x+3)^2$ **f** $f(x) = -5(x-1)^2$

3 Find the x-intercepts of:

a $y = x^2 - 9$ **b** $y = 25 - x^2$ **c** $y = x^2 - 6x$
d $f(x) = x^2 + 7x + 10$ **e** $y = x^2 + x - 12$ **f** $y = 4x - x^2$
g $y = -x^2 - 6x - 8$ **h** $f(x) = -2x^2 - 4x - 2$ **i** $y = 4x^2 - 24x + 36$

Example 12 — Self Tutor

Use the quadratic formula to find the x-intercepts of the quadratic function $y = x^2 - 2x - 5$.

When $y = 0$,

$x^2 - 2x - 5 = 0$

$\therefore\ x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$

$\therefore\ x = \dfrac{2 \pm \sqrt{4+20}}{2}$

$\therefore\ x = \dfrac{2 \pm \sqrt{24}}{2}$

$\therefore\ x = \dfrac{2 \pm 2\sqrt{6}}{2}$

$\therefore\ x = 1 \pm \sqrt{6}$

$\therefore$ the x-intercepts are $1 + \sqrt{6}$ and $1 - \sqrt{6}$.

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

4 Use the quadratic formula to find the x-intercepts of:

a $y = x^2 - 4x + 1$ **b** $y = x^2 + 4x - 3$ **c** $y = -x^2 + 6x - 4$

d $f(x) = 3x^2 - 7x - 2$ **e** $f(x) = 2x^2 - x - 5$ **f** $f(x) = -4x^2 + 9x - 3$

Example 13 **Self Tutor**

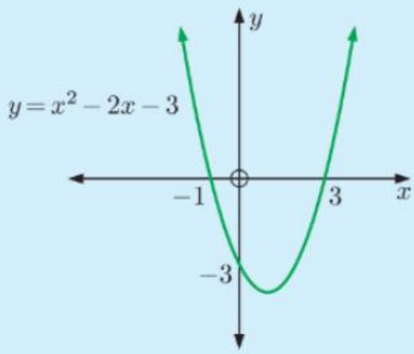
Sketch the graph of each quadratic function by considering:

i the value of a **ii** the y-intercept **iii** the x-intercepts.

a $y = x^2 - 2x - 3$ **b** $y = -2(x+1)(x-2)$ **c** $y = 2(x-3)^2$

a $y = x^2 - 2x - 3$

i $a = 1$ which is > 0, so the parabola opens upwards.



ii When $x = 0$, $y = -3$

$\therefore$ the y-intercept is -3.

iii When $y = 0$,

$x^2 - 2x - 3 = 0$

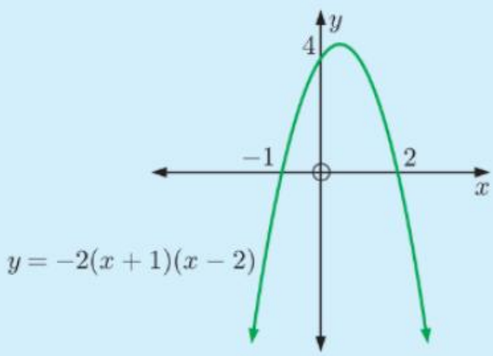
$\therefore (x-3)(x+1) = 0$

$\therefore x = 3$ or $x = -1$

$\therefore$ the x-intercepts are 3 and -1.

b $y = -2(x+1)(x-2)$

i $a = -2$ which is < 0, so the parabola opens downwards.



ii When $x = 0$,

$y = -2(0+1)(0-2)$

$= -2 \times 1 \times -2$

$= 4$

$\therefore$ the y-intercept is 4.

iii When $y = 0$,

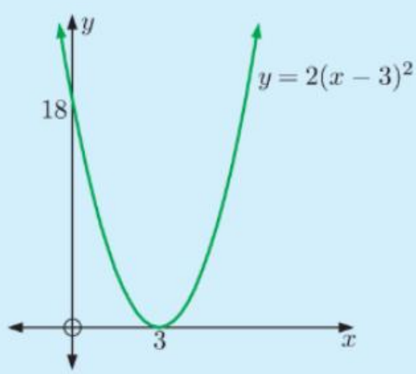
$-2(x+1)(x-2) = 0$

$\therefore x = -1$ or $x = 2$

$\therefore$ the x-intercepts are -1 and 2.

c $y = 2(x-3)^2$

i $a = 2$ which is > 0, so the parabola opens upwards.



ii When $x = 0$, $y = 2(0-3)^2 = 18$

$\therefore$ the y-intercept is 18.

iii When $y = 0$, $2(x-3)^2 = 0$

$\therefore x = 3$

$\therefore$ the x-intercept is 3.

There is only one x-intercept, which means the graph *touches* the x-axis.

5 Sketch the graph of the quadratic function which has:

a x-intercepts -1 and 1, and y-intercept -1

b x-intercepts -3 and 1, and y-intercept 2

c x-intercepts 2 and 5, and y-intercept -4

d x-intercept 2 and y-intercept 4.

6 Sketch the graph of each quadratic function by considering:

i the value of a **ii** the y-intercept **iii** the x-intercepts.

a $y = x^2 - 4x + 4$ **b** $f(x) = (x-1)(x+3)$ **c** $y = 2(x+2)^2$

d $f(x) = -(x-2)(x+1)$ **e** $y = -3(x+1)^2$ **f** $y = -3(x-4)(x-1)$

g $y = 2(x+3)(x+1)$ **h** $y = -2x^2 - 3x + 5$ **i** $f(x) = -x^2 + 8x - 10$

ACTIVITY

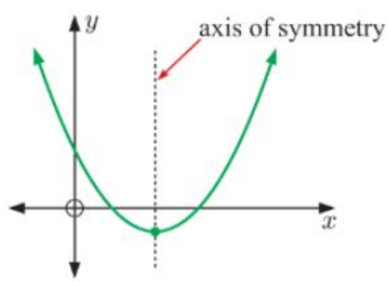
Click on the icon to practise matching a quadratic function with its graph.

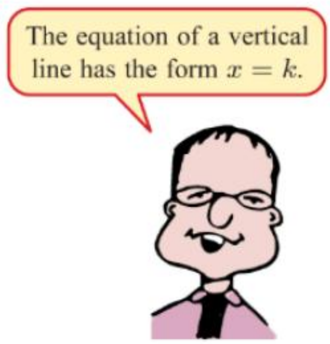
QUADRATIC FUNCTIONS

D AXIS OF SYMMETRY

The graphs of all quadratic functions are symmetrical about a vertical line passing through the vertex. This line is called the **axis of symmetry**.

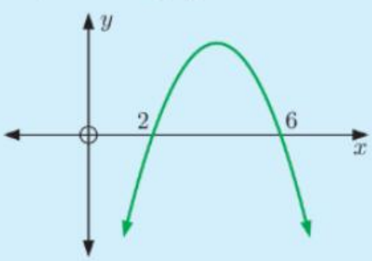
If the graph has two x-intercepts, then the axis of symmetry is midway between them.



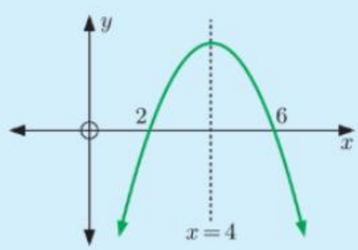


Example 14 **Self Tutor**

Find the equation of the axis of symmetry for the quadratic graph below.



The x-intercepts are 2 and 6, and 4 is midway between 2 and 6.



$\therefore$ the axis of symmetry is $x = 4$.

If the quadratic does not have any x-intercepts, or if we do not know the x-intercepts, we can use the following rule to find the axis of symmetry:

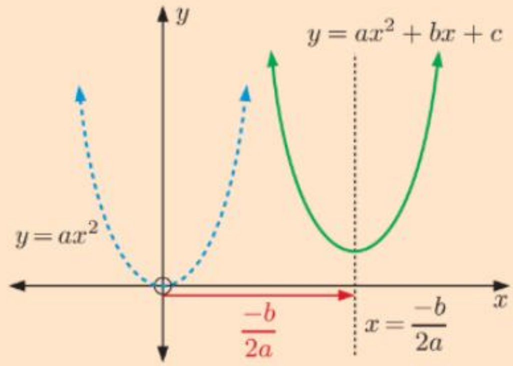
The equation of the axis of symmetry of $y = ax^2 + bx + c$ is $x = \dfrac{-b}{2a}$.

Proof:

$$y = ax^2 + bx + c$$
$$\therefore\ y = a\left(x^2 + \frac{b}{a}x\right) + c$$
$$\therefore\ y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$
$$\therefore\ y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

The graph of $y = ax^2 + bx + c$ is a translation of $y = ax^2$ with translation vector $\begin{pmatrix} -\frac{b}{2a} \\ c - \frac{b^2}{4a} \end{pmatrix}$.

$\therefore$ the axis of symmetry has equation $x = \dfrac{-b}{2a}$.



Example 15 Self Tutor

Find the equation of the axis of symmetry of $y = 2x^2 + 3x + 1$.

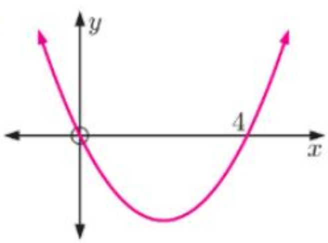
$y = 2x^2 + 3x + 1$ has $a = 2$, $b = 3$, $c = 1$.

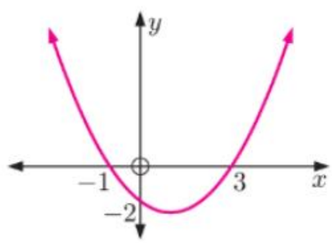
Now $\dfrac{-b}{2a} = \dfrac{-3}{2 \times 2} = -\dfrac{3}{4}$

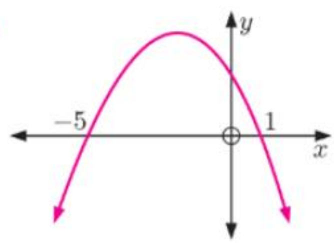
$\therefore$ the axis of symmetry has equation $x = -\frac{3}{4}$.

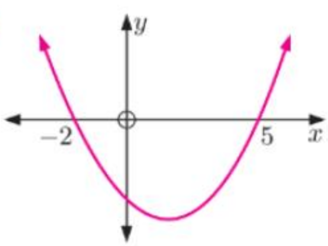
EXERCISE 20D

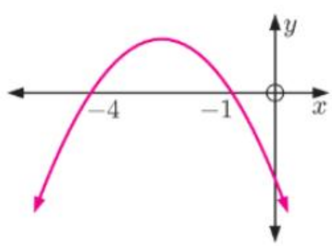
1 For each of the following graphs, find the equation of the axis of symmetry:

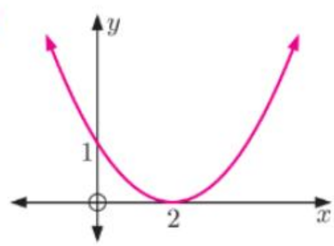
a


b


c


d


e


f


2 For each of the following quadratic functions, find the equation of the axis of symmetry:

a $y=(x-2)(x-4)$ **b** $y=-(x+1)(x-5)$ **c** $y=2(x+3)(x-3)$

d $y=x(x+5)$ **e** $y=-3(x+4)^2$ **f** $y=4(x+6)(x-9)$

3 For each of the following quadratic functions, find the equation of the axis of symmetry:

a $y=x^2+4x+1$ **b** $y=2x^2-6x+3$ **c** $f(x)=3x^2+4x-1$

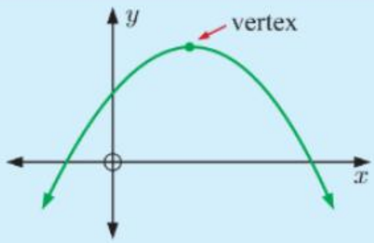
d $y=-x^2-4x+5$ **e** $y=-2x^2+5x+1$ **f** $f(x)=\frac{1}{2}x^2-10x+2$

g $y=\frac{1}{3}x^2+4x$ **h** $f(x)=100x-4x^2$ **i** $y=-\frac{1}{10}x^2+30x$

E VERTEX

The **vertex** or **turning point** of the quadratic function $y=ax^2+bx+c$ is the point at which the function has:

- a **maximum value** for $a<0$, *or*
- a **minimum value** for $a>0$.



Since the vertex lies on the axis of symmetry, its x-coordinate will be $\dfrac{-b}{2a}$.

The y-coordinate can be found by substituting this value for x into the function.

Example 16 **Self Tutor**

Consider the quadratic function $y=-x^2+2x+3$.

a Find the axes intercepts. **b** Find the equation of the axis of symmetry.

c Find the coordinates of the vertex.

d Sketch the function, showing all important features.

a When $x=0$, $y=3$

$\therefore$ the y-intercept is 3.

When $y=0$, $-x^2+2x+3=0$

$\therefore\ x^2-2x-3=0$

$\therefore\ (x-3)(x+1)=0$

$\therefore\ x=3$ or -1

$\therefore$ the x-intercepts are 3 and -1.

c When $x=1$,

$y=-(1)^2+2(1)+3$

$=-1+2+3$

$=4$

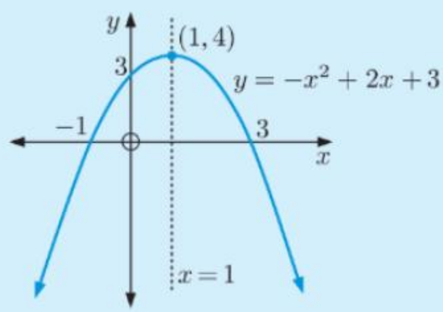
$\therefore$ the vertex is $(1, 4)$.

b $a=-1$, $b=2$, $c=3$

$\therefore\ \dfrac{-b}{2a}=\dfrac{-2}{-2}=1$

$\therefore$ the axis of symmetry is $x=1$.

d



EXERCISE 20E

1 For each of the following quadratic functions:

- **i** Find the coordinates of the vertex.
- **ii** Determine whether the vertex is a maximum or minimum turning point.
- **iii** Find the range of the function.

a $y = x^2 - 4x + 2$ **b** $y = x^2 + 2x - 3$
c $f(x) = 2x^2 + 4$ **d** $y = -3x^2 + 1$
e $y = -x^2 - 4x - 4$ **f** $y = 2x^2 - 10x + 3$

The vertex is called the **maximum turning point** or the **minimum turning point**, depending on whether the graph opens downwards or upwards.

2 For each of the following quadratic functions:

- **i** Find the axes intercepts.
- **ii** Find the equation of the axis of symmetry.
- **iii** Find the coordinates of the vertex.
- **iv** Hence, sketch the graph of the function.

a $y = x^2 - 2x - 8$ **b** $y = 4x - x^2$ **c** $y = x^2 + 3x$
d $f(x) = x^2 + 4x + 4$ **e** $y = x^2 + 3x - 4$ **f** $y = -x^2 + 2x - 1$
g $y = 2x^2 + 5x - 3$ **h** $f(x) = -3x^2 - 4x + 4$ **i** $y = x^2 - 6x + 3$

Example 17

Self Tutor

Consider the quadratic function $y = 2(x-2)(x+4)$.

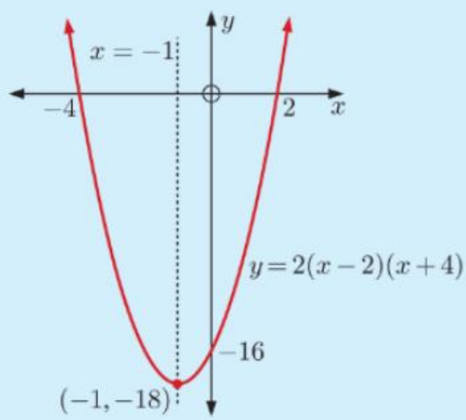
a Find the axes intercepts.
b Find the equation of the axis of symmetry.
c Find the coordinates of the vertex.
d Sketch the function, showing all important features.

a When $x = 0$, $y = 2 \times -2 \times 4 = -16$
$\therefore$ the y-intercept is -16.
When $y = 0$, $2(x-2)(x+4) = 0$
$\therefore$ $x = 2$ or $x = -4$
$\therefore$ the x-intercepts are 2 and -4.

b The axis of symmetry is halfway between the x-intercepts, and -1 is halfway between 2 and -4.
$\therefore$ the axis of symmetry is $x = -1$.

c When $x = -1$,
$$\begin{aligned} y &= 2(-1-2)(-1+4) \\ &= 2 \times -3 \times 3 \\ &= -18 \end{aligned}$$
$\therefore$ the vertex is $(-1, -18)$.

d



3 For each of the following quadratic functions:

- **i** Find the axes intercepts.
- **ii** Find the equation of the axis of symmetry.
- **iii** Find the coordinates of the vertex.
- **iv** Hence, sketch the graph of the function.

a $f(x) = x(x-2)$ **b** $y = 2(x-3)^2$ **c** $y = -(x-1)(x+3)$

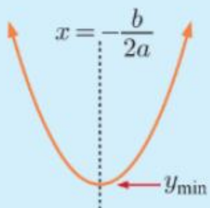
d $y = -2(x-1)^2$ **e** $f(x) = -5(x+2)(x-2)$ **f** $y = 2(x+1)(x+4)$

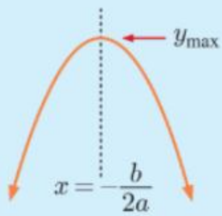
F QUADRATIC OPTIMISATION

The process of finding the **maximum** or **minimum** value of a function is called **optimisation**.

For the quadratic function $y = ax^2 + bx + c$:

- If $a > 0$, the **minimum** value of y occurs when $x = -\frac{b}{2a}$.
- If $a < 0$, the **maximum** value of y occurs when $x = -\frac{b}{2a}$.





Optimisation is a very useful tool when looking at such issues as:

- maximising profits
- minimising costs
- maximising heights reached.

Example 18 **Self Tutor**

The height of a rocket t seconds after it is fired upwards is given by $H(t) = 100t - 5t^2$ metres, $t \geqslant 0$.

a How long does the rocket take to reach its maximum height?

b Find the maximum height reached by the rocket.

c How long does it take for the rocket to fall back to earth?

a $H(t) = 100t - 5t^2$

$\therefore\ H(t) = -5t^2 + 100t$

Now $a = -5$ which is < 0, so the shape of the graph is  .

The maximum height is reached when $t = \frac{-b}{2a} = \frac{-100}{2(-5)} = 10$

$\therefore$ the maximum height is reached after 10 seconds.

b $H(10) = 100(10) - 5(10)^2$

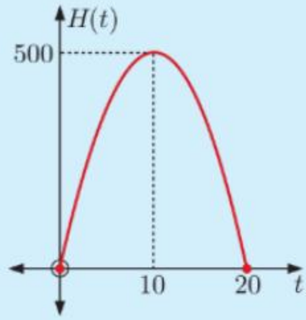
$= 500$

$\therefore$ the maximum height reached is 500 m.

c The rocket falls back to earth when $H(t) = 0$

$$\therefore \quad -5t^2 + 100t = 0$$
$$\therefore \quad -5t(t-20) = 0$$
$$\therefore \quad t = 0 \text{ or } 20$$

$\therefore$ the rocket falls back to earth after 20 seconds.



EXERCISE 20F

1 The height of a ball t seconds after it is kicked upwards is given by $H(t) = 20t - 5t^2$ metres.

a How long does the ball take to reach its maximum height?

b Find the maximum height reached by the ball.

c How long does it take for the ball to hit the ground?

2 A manufacturer finds that the profit €P from assembling x bicycles per day is given by $P(x) = -x^2 + 50x - 200$.

a How many bicycles should be assembled per day to maximise the profit?

b Find the maximum profit.

c What is the loss made if no bicycles are assembled in a day? Suggest why this loss would be made.

3 The driver of a car travelling downhill applied the brakes. The speed of the car, t seconds after the brakes were applied, is given by $s(t) = -6t^2 + 12t + 60 \text{ km h}^{-1}$.

a How fast was the car travelling when the driver applied the brakes?

b After how many seconds did the car reach its maximum speed?

c Find the maximum speed reached.

4 The hourly profit obtained from operating a fleet of n taxis is given by $P(n) = 120n - 200 - 2n^2$ dollars.

a What number of taxis gives the maximum hourly profit?

b Find the maximum hourly profit.

c How much money is lost per hour if no taxis are on the road?

5 The temperature T°C in a greenhouse t hours after 7:00 pm is given by $T(t) = \frac{1}{4}t^2 - 6t + 25$ for $t \leqslant 20$.

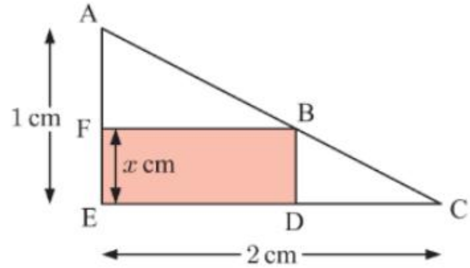
a Find the temperature in the greenhouse at 7:00 pm.

b At what time is the temperature at a minimum?

c Find the minimum temperature in the greenhouse for $0 \leqslant t \leqslant 20$.

6 Answer the questions posed in the **Opening Problem** on page **370**.

7 Infinitely many rectangles may be inscribed within the triangle ACE shown. One of them is illustrated. Suppose EF $= x$ cm.



a Show that triangles ABF and ACE are similar.

b Show that BF $= 2(1-x)$ cm.

c Show that the area of rectangle BDEF is given by $A = -2x^2 + 2x$ cm^2.

d **i** Find x such that the area of the rectangle is maximised.

ii What is the maximum area?

Global context

click here

Arches

Statement of inquiry:	Mathematical models can describe the characteristics of the architecture we see around us.
Global context:	Personal and cultural expression
Key concept:	Form
Related concepts:	Model, Space
Objectives:	Knowing and understanding
Approaches to learning:	Thinking, Self-management

REVIEW SET 20A

1 For the quadratic function $y = x^2 - 3x - 15$, find:

a the value of y when $x = 4$ **b** the values of x when $y = 3$.

2 Sketch each of the following functions on the same set of axes as $y = x^2$:

a $y = 3x^2$ **b** $y = (x-2)^2 + 1$ **c** $y = -(x+3)^2 - 2$

3 **a** Write the quadratic $y = x^2 - 4x + 10$ in the form $y = (x-h)^2 + k$.

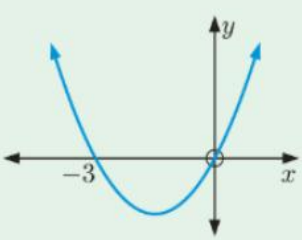
b Hence sketch the function, stating the coordinates of the vertex.

4 Find the x-intercepts of:

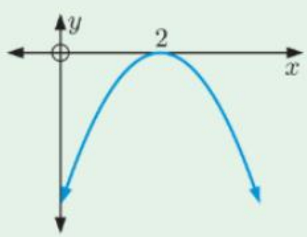
a $y = 5x(x+4)$ **b** $y = 2x^2 + 6x - 56$

5 Find the equation of the axis of symmetry for:

a



b



6 Find the vertex of each of the following quadratic functions:

a $y = x^2 - 8x - 3$ **b** $y = -4x^2 + 4x - 3$

7 Consider the quadratic function $y = -2(x-1)(x+3)$.

a Find the:

i direction the parabola opens
ii y-intercept
iii x-intercepts
iv equation of the axis of symmetry.

b Sketch the function, showing all of the above features.

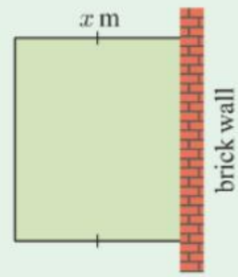
8 Consider the function $y = x^2 - 2x - 15$.

a Find the:

i y-intercept
ii x-intercepts
iii equation of the axis of symmetry
iv coordinates of the vertex.

b Sketch the function, showing all of the above features.

9 A vegetable gardener has 40 m of fencing to enclose a rectangular garden plot where one side is an existing brick wall. Suppose the plot is x m wide as shown.



a Show that the area enclosed is given by $A = -2x^2 + 40x$ m^2.

b Find x such that the vegetable garden has the maximum possible area.

c Find the maximum possible area.

REVIEW SET 20B

1 Find the values of x for which $f(x) = x^2 + x - 12$ takes the value 30.

2 Determine whether the ordered pair (2, 5) satisfies the quadratic function $f(x) = x^2 - 3x + 8$.

3 Draw the graphs of $y = x^2$ and $y = (x+2)^2 + 5$ on the same set of axes.

4 Use the quadratic formula to find the x-intercepts of:

a $y = 3x^2 - x - 5$
b $y = -x^2 + 2x + 6$

5 Draw the graph of $y = 3(x-2)^2$, showing the axes intercepts and the coordinates of the vertex.

6 Find the equation of the axis of symmetry for the quadratic function:

a $f(x) = (x+3)(x-5)$
b $f(x) = 3x^2 - 5x + 2$

7 **a** Find the vertex of the quadratic function $f(x) = -x^2 + 4x - 7$.

b Hence, find the range of the function.

8 Consider the quadratic function $f(x) = (x-1)(x-4)$.

a Find $f(-1)$.

b Find the axes intercepts.

c Find the equation of the axis of symmetry.

d Find the coordinates of the vertex.

e Sketch the graph of $y = f(x)$, showing all of the above features.

9 Suppose $f(x) = (x+2)(x+6)$ and $g(x) = -x^2 - 8x - 20$.

- **a** Find the axes intercepts of each function.
- **b** Show that the two functions have the same vertex.
- **c** State the range of each function.
- **d** Sketch the functions on the same set of axes.

10 The height of a javelin t seconds after it is thrown, is given by $H(t) = 1.6 + 20.3t - 4.9t^2$ metres.

- **a** From what height is the javelin released?
- **b** Find the maximum height reached by the javelin.
- **c** For how long will the javelin be in the air before it hits the ground?

Chapter 21

Advanced trigonometry

Contents:

OPENING PROBLEM

A steamroller has a spot of paint on its roller. As the steamroller moves, the spot rotates around the axle.

Things to think about:

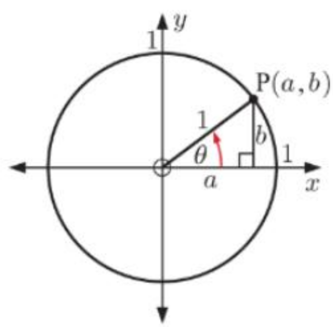
- **a** How does the *height* of the spot above ground level change over time?
  What would the graph of the spot's height over time look like?
- **b** Suppose the roller has a radius of 1 metre.
  How would this be shown in the graph?
- **c** Suppose the roller completes one full revolution every 10 seconds.
  How would this be shown in the graph?
- **d** Can we use a function involving a trigonometric ratio to determine the height of the spot over time?

DEMO

In this chapter we extend our knowledge of trigonometry. We will consider the trigonometric ratios in more detail, including the relationship between sine and cosine, and the ratios for some special angles. We will also see how the trigonometric ratios are used in **trigonometric functions** which model real world situations.

A THE UNIT CIRCLE

Consider a circle of radius 1 unit with its centre at the origin O. This circle is called the **unit circle**. It has equation $x^2 + y^2 = 1$.



Suppose P lies on the circle so that [OP] makes angle θ with the positive x-axis. θ is always measured in the anticlockwise direction.

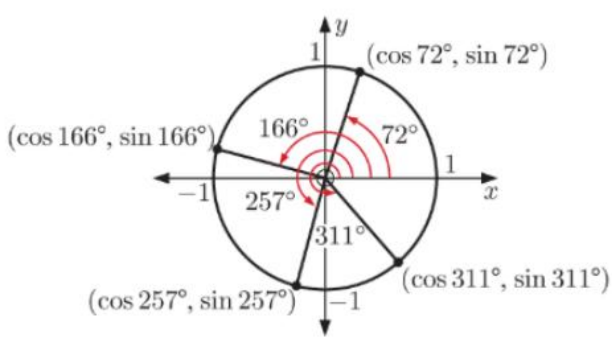
For any acute angle θ, notice that $\cos\theta = \frac{a}{1} = a$ and $\sin\theta = \frac{b}{1} = b$.

In fact, for any angle θ:

> The x-coordinate of P is called the **cosine** of angle θ, written $\cos\theta$.
>
> The y-coordinate of P is called the **sine** of angle θ, written $\sin\theta$.

Notice that:

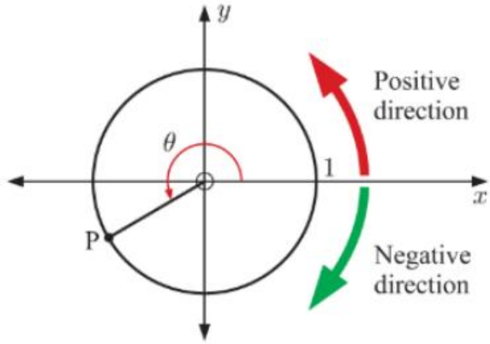
- The coordinates of the point on the unit circle at angle θ to the positive x-axis are $(\cos\theta, \sin\theta)$.



- $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$ for all points on the unit circle, so

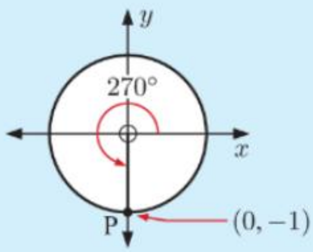
$$\mathbf{-1 \leqslant \cos\theta \leqslant 1} \quad \text{and} \quad \mathbf{-1 \leqslant \sin\theta \leqslant 1} \quad \text{for all } \theta.$$

- θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



Example 1 — Self Tutor

From a unit circle diagram, find $\cos 270°$ and $\sin 270°$.

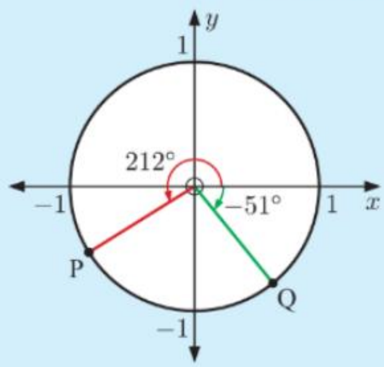


For an angle of $270°$, P is $(0, -1)$.

$\therefore\ \cos 270° = 0$ {the x-coordinate of P}
$\quad \sin 270° = -1$ {the y-coordinate of P}

Example 2 — Self Tutor

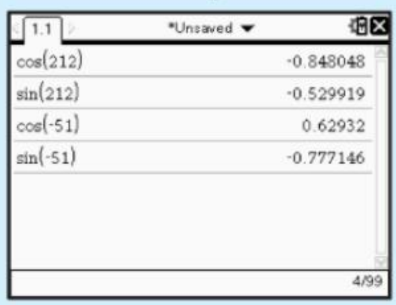
Find the coordinates of P and Q, rounded to 3 decimal places.



P is $(\cos 212°, \sin 212°)$, which is approximately $(-0.848, -0.530)$.

Q is $(\cos(-51°), \sin(-51°))$, which is approximately $(0.629, -0.777)$.

TI-*n*spire

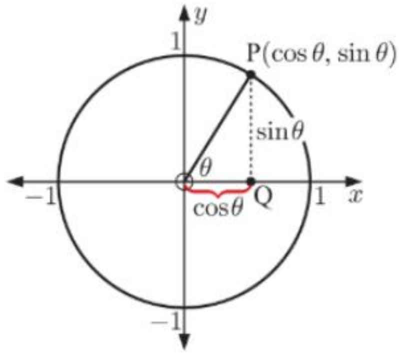


TAN θ

Consider the point $P(\cos\theta, \sin\theta)$ on the unit circle, where θ is acute.

Using $\tan\theta = \frac{\text{OPP}}{\text{ADJ}}$ in $\triangle$OPQ, we observe that:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\tan\theta$ is the **gradient** of [OP].



These facts are true for all angles θ.

By identifying the point P on the unit circle corresponding to angle θ, we can determine whether $\cos\theta$, $\sin\theta$, and $\tan\theta$ are positive or negative.

Example 3 **Self Tutor**

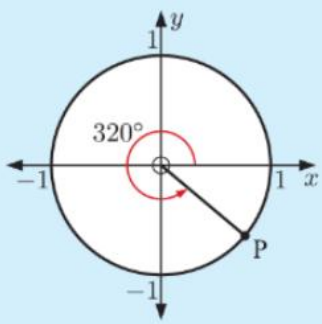
For $\theta = 320°$, determine whether $\cos\theta$, $\sin\theta$, and $\tan\theta$ are positive or negative.

The point P on the unit circle corresponding to $\theta = 320°$ is in quadrant 4.

The x-coordinate of P is positive, so $\cos\theta$ is positive.

The y-coordinate of P is negative, so $\sin\theta$ is negative.

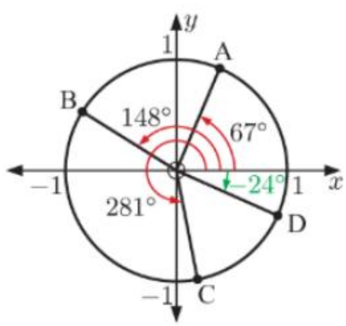
The gradient of [OP] is negative, so $\tan\theta$ is negative.

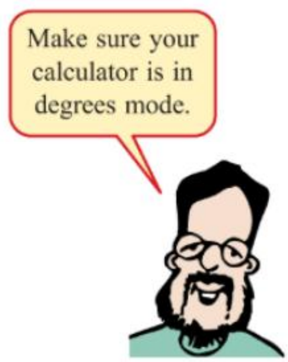


EXERCISE 21A

1 **a** Write down the exact coordinates of points A, B, C, and D.

b Use your calculator to state the coordinates of A, B, C, and D, rounded to 3 significant figures.





2 Use a unit circle to explain why:

a $\cos 380° = \cos 20°$

b $\sin 413° = \sin 53°$

c $\cos(-52°) = \cos 308°$

d $\tan 25° = \tan 205°$

3 Use a unit circle diagram to find the values of:

a $\cos 0°$ and $\sin 0°$

b $\cos 90°$ and $\sin 90°$

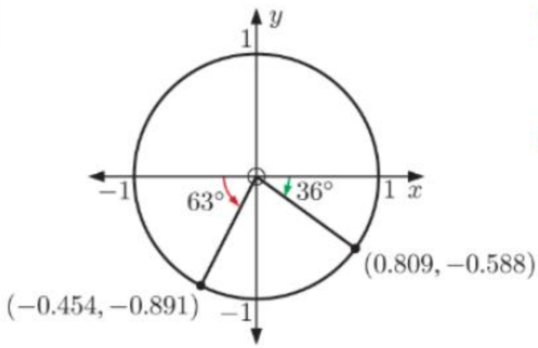
c $\cos 360°$ and $\sin 360°$

d $\cos 450°$ and $\sin 450°$

e $\cos(-90°)$ and $\sin(-90°)$

f $\cos(-180°)$ and $\sin(-180°)$

4



a Use the unit circle alongside to find the value of:

i $\cos 243°$ **ii** $\sin 243°$

iii $\cos 324°$ **iv** $\sin 324°$

b Hence, find the value of:

i $\tan 243°$ **ii** $\tan 324°$

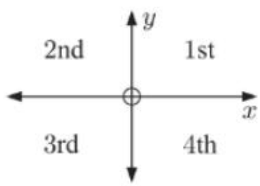
5 Determine whether $\cos\theta$, $\sin\theta$, and $\tan\theta$ are positive or negative for:

a $\theta = 135°$ **b** $\theta = 30°$ **c** $\theta = 300°$ **d** $\theta = 210°$

e $\theta = 49°$ **f** $\theta = 158°$ **g** $\theta = 207°$ **h** $\theta = 296°$

Use your calculator to check your answers.

6 The diagram alongside shows the 4 quadrants. They are numbered anticlockwise.



a Copy and complete:

Quadrant	*Degree measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	positive	positive	positive
2				
3				
4				

b Determine the quadrants in which:

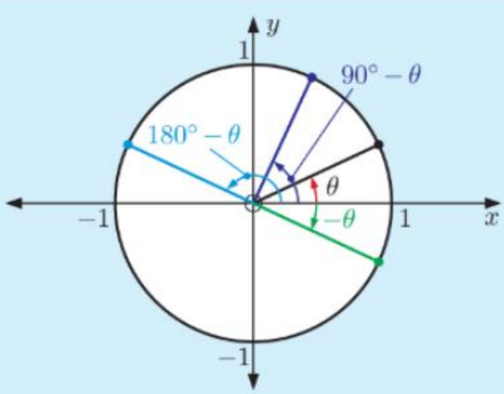
i $\sin\theta$ is negative **ii** $\cos\theta$ is positive

iii $\cos\theta$ and $\sin\theta$ are both negative **iv** $\cos\theta$ is positive and $\tan\theta$ is negative.

B NEGATIVE, COMPLEMENTARY, AND SUPPLEMENTARY ANGLES

For any given angle θ:

- the **negative** of θ is $-\theta$
- the **complement** of θ is $(90° - \theta)$
- the **supplement** of θ is $(180° - \theta)$.



INVESTIGATION 1 TRIGONOMETRIC FORMULAE

In this Investigation we consider formulae for the trigonometric ratios of the negative, complement, and supplement of a given angle.

What to do:

1 Copy and complete the following table, giving answers correct to four decimal places:

θ	$\cos\theta$	$\sin\theta$	$\cos(-\theta)$	$\sin(-\theta)$	$\cos(90° - \theta)$	$\sin(90° - \theta)$	$\cos(180° - \theta)$	$\sin(180° - \theta)$
$18°$								
$27°$								
$53°$								
$70°$								
$125°$								

2 Use your table to predict a relationship between:

a $\cos(-\theta)$ and $\cos\theta$
b $\sin(-\theta)$ and $\sin\theta$
c $\cos(90° - \theta)$ and $\sin\theta$
d $\sin(90° - \theta)$ and $\cos\theta$
e $\cos(180° - \theta)$ and $\cos\theta$
f $\sin(180° - \theta)$ and $\sin\theta$.

PRINTABLE TABLE

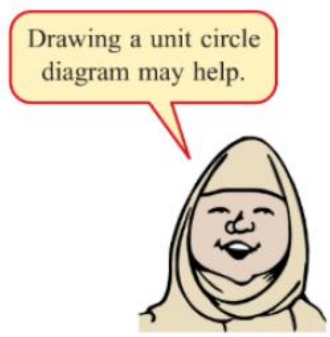
From the **Investigation**, you should have discovered the following relationships:

For any angle θ:

$$\cos(-\theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta \qquad \cos(180° - \theta) = -\cos\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \sin(90° - \theta) = \cos\theta \qquad \sin(180° - \theta) = \sin\theta$$

EXERCISE 21B



1 Find another angle which has the same cosine as:

a $42°$ **b** $61°$ **c** $-18°$ **d** $117°$

2 Find an angle whose sine is the negative of:

a $\sin 24°$ **b** $\sin 70°$ **c** $\sin(-5°)$ **d** $\sin 157°$

3 Find an acute angle whose sine has the same value as:

a $\cos 40°$ **b** $\cos 58°$ **c** $\cos 16°$ **d** $\cos 85°$

4 Find an acute angle whose cosine has the same value as:

a $\sin 50°$ **b** $\sin 35°$ **c** $\sin 79°$ **d** $\sin 21°$

5 Find the obtuse angle which has the same sine as:

a $26°$ **b** $45°$ **c** $69°$ **d** $86°$

6 Find the acute angle which has the same sine as:

a $98°$ **b** $127°$ **c** $156°$ **d** $168°$

7 Find the obtuse angle whose cosine is the negative of:

a $\cos 26°$ **b** $\cos 45°$ **c** $\cos 69°$ **d** $\cos 86°$

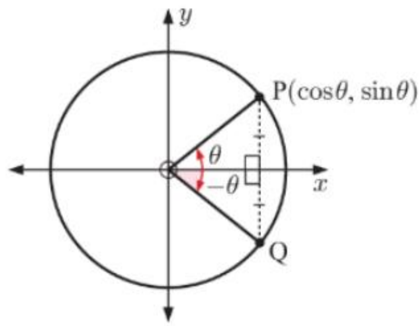
8 Find the acute angle whose cosine is the negative of:

a $\cos 98°$ **b** $\cos 127°$ **c** $\cos 156°$ **d** $\cos 168°$

9 Use a unit circle to explain why:

a $\sin 160° = \sin 20°$ **b** $\cos 160° = -\cos 20°$ **c** $\cos 310° = \cos 50°$

10 In the unit circle alongside, Q corresponds to the angle $-\theta$.



a Write down the coordinates of Q in terms of $\cos(-\theta)$ and $\sin(-\theta)$.

b Q is the reflection of P in the x-axis. Use this fact to write down:

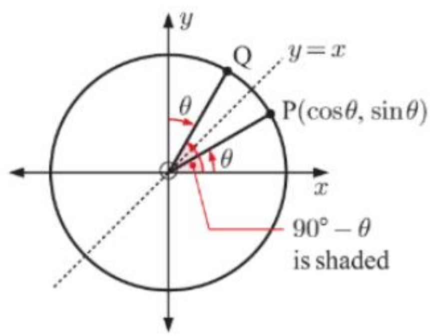
i the x-coordinate of Q in terms of $\cos\theta$

ii the y-coordinate of Q in terms of $\sin\theta$.

c Hence establish the negative angle formulae.

d Write $\tan(-\theta)$ in terms of $\tan\theta$.

11 In the unit circle alongside, Q corresponds to the angle $(90° - \theta)$.

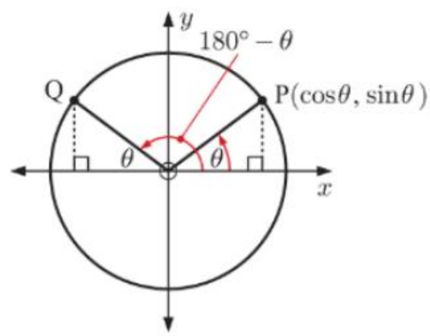


a Write down the coordinates of Q in terms of $\cos(90° - \theta)$ and $\sin(90° - \theta)$.

b Q is the reflection of P in the line $y = x$, so Q has coordinates $(\sin\theta, \cos\theta)$. Hence establish the complementary angle formulae.

c Write $\tan(90° - \theta)$ in terms of $\tan\theta$.

12 In the unit circle alongside, Q corresponds to the angle $(180° - \theta)$.



a Write down the coordinates of Q in terms of $\cos(180° - \theta)$ and $\sin(180° - \theta)$.

b Q is the reflection of P in the y-axis. Use this fact to write down:

i the x-coordinate of Q in terms of $\cos\theta$

ii the y-coordinate of Q in terms of $\sin\theta$.

c Hence establish the supplementary angle formulae.

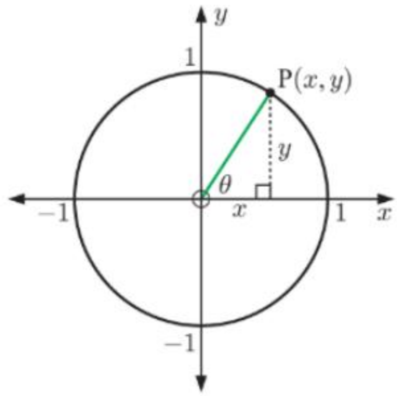
d Write $\tan(180° - \theta)$ in terms of $\tan\theta$.

C THE RELATIONSHIP BETWEEN $\sin\theta$ AND $\cos\theta$

For any point $P(x, y)$ on the unit circle, $x^2 + y^2 = 1$ {Pythagoras}.

Since $x = \cos\theta$ and $y = \sin\theta$, we find that for any angle θ, $(\cos\theta)^2 + (\sin\theta)^2 = 1$. We commonly write this as:

$$\cos^2\theta + \sin^2\theta = 1$$

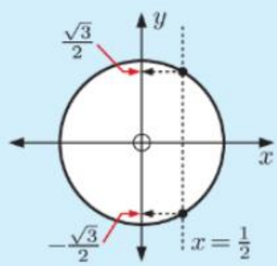


If we are given one of the trigonometric ratios $\cos\theta$ or $\sin\theta$, we can use this relationship to find the possible values of the other ratio.

Example 4 | Self Tutor

Find the possible values of $\sin\theta$ when $\cos\theta = \frac{1}{2}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \left(\tfrac{1}{2}\right)^2 + \sin^2\theta = 1$$
$$\therefore \tfrac{1}{4} + \sin^2\theta = 1$$
$$\therefore \sin^2\theta = \tfrac{3}{4}$$
$$\therefore \sin\theta = \pm\tfrac{\sqrt{3}}{2}$$

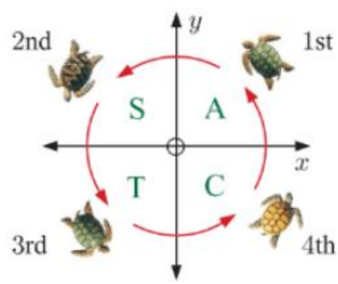


If we know the quadrant in which θ lies, we can determine the sign of the unknown ratio.

In the diagram alongside, the letters show which trigonometric ratios are positive in each quadrant. The 'A' stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.



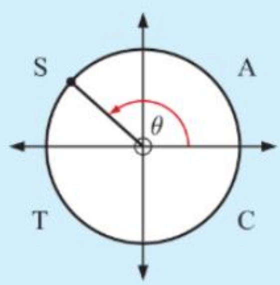
Example 5 | Self Tutor

Find the exact value of $\cos\theta$ if $\sin\theta = \frac{3}{4}$ and $90° < \theta < 180°$.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos^2\theta + \left(\tfrac{3}{4}\right)^2 = 1$$
$$\therefore \cos^2\theta + \tfrac{9}{16} = 1$$
$$\therefore \cos^2\theta = \tfrac{7}{16}$$
$$\therefore \cos\theta = \pm\tfrac{\sqrt{7}}{4}$$

But θ lies in quadrant 2 where $\cos\theta$ is *negative*.

$$\therefore \cos\theta = -\tfrac{\sqrt{7}}{4}$$



EXERCISE 21C

1 Using a unit circle to illustrate your answers, find the possible values of $\sin\theta$ when:

a $\cos\theta = \frac{3}{5}$ **b** $\cos\theta = -\frac{1}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

2 Using a unit circle to illustrate your answers, find the possible values of $\cos\theta$ when:

a $\sin\theta = \frac{12}{13}$ **b** $\sin\theta = -1$ **c** $\sin\theta = 0$ **d** $\sin\theta = -\frac{3}{5}$

3 Find the exact value of $\cos\theta$ if:

a $\sin\theta = \frac{2}{3}$ and $0 < \theta < 90°$ **b** $\sin\theta = \frac{4}{5}$ and $90° < \theta < 180°$

c $\sin\theta = -\frac{1}{3}$ and $180° < \theta < 270°$ **d** $\sin\theta = -\frac{5}{13}$ and $270° < \theta < 360°$

4 Find the exact value of $\sin\theta$ if:

a $\cos\theta = \frac{3}{5}$ and $0° < \theta < 90°$ **b** $\cos\theta = \frac{1}{4}$ and $270° < \theta < 360°$

c $\cos\theta = -\frac{3}{4}$ and $90° < \theta < 180°$ **d** $\cos\theta = -\frac{5}{13}$ and $180° < \theta < 270°$

5 Suppose $180° < \theta < 270°$ and $\sin\theta = -\frac{3}{8}$.

a Find the exact value of $\cos\theta$. **b** Hence, find the exact value of $\tan\theta$.

D THE MULTIPLES OF $30°$ AND $45°$

MULTIPLES OF $45°$

Consider $\theta = 45°$.

Angle OPB also measures $45°$, so triangle OBP is isosceles.

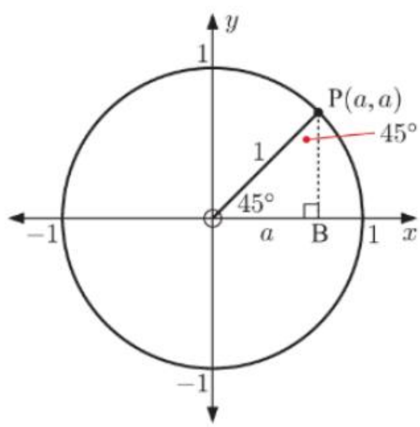
$\therefore$ we let OB = BP = a

Now $a^2 + a^2 = 1^2$ {Pythagoras}

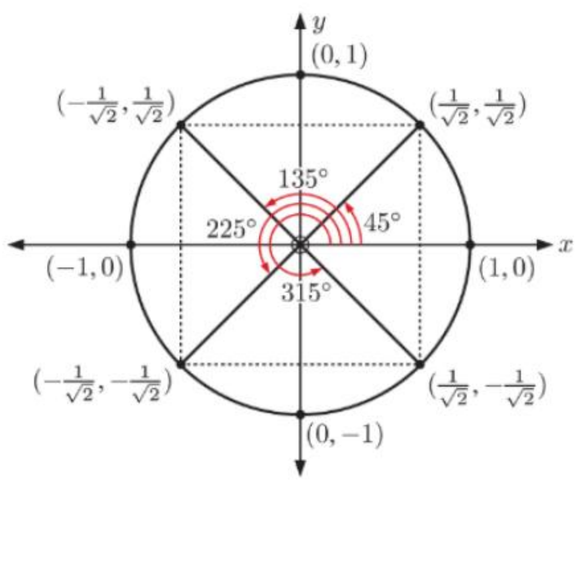
$\therefore a^2 = \frac{1}{2}$

$\therefore a = \frac{1}{\sqrt{2}}$ {since $a > 0$}

$\therefore$ P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.7$.

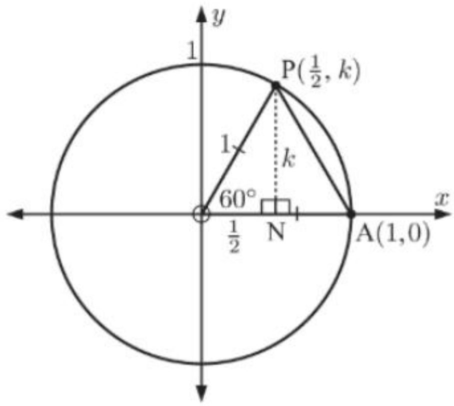


We can now find the coordinates of all points on the unit circle corresponding to multiples of $45°$ by using rotations and reflections.



MULTIPLES OF 30°

Consider $\theta = 60^\circ$.



Since OA = OP, triangle OAP is isosceles.

Now $A\widehat{O}P = 60^\circ$, so the remaining angles are therefore also 60°. Triangle AOP is therefore equilateral.

The altitude [PN] bisects base [OA] and apex $O\widehat{P}A$.

So, $ON = \frac{1}{2}$ and $O\widehat{P}N = 30^\circ$.

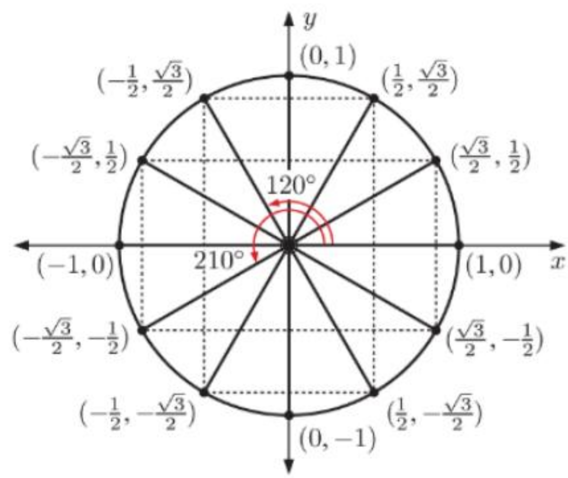
If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

$\therefore\ k^2 = \frac{3}{4}$

$\therefore\ k = \frac{\sqrt{3}}{2}$ {since $k > 0$}

$\therefore$ $\boxed{\text{P is } (\frac{1}{2}, \frac{\sqrt{3}}{2}) \text{ where } \frac{\sqrt{3}}{2} \approx 0.9.}$

We can now find the coordinates of all points on the unit circle corresponding to multiples of 30° by using rotations and reflections.



Summary:

- If θ is a **multiple of 90°**, the coordinates of the points on the unit circle involve 0 and ± 1.
- If θ is a **multiple of 45°**, but not a multiple of 90°, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- If θ is a **multiple of 30°**, but not a multiple of 90°, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

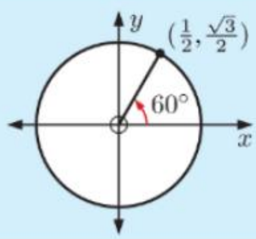
We can calculate the **tangent** of any angle θ using $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$.

Example 6 **Self Tutor**

Use a unit circle diagram to find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for:

a $\theta = 60^\circ$ **b** $\theta = 150^\circ$ **c** $\theta = 225^\circ$

a

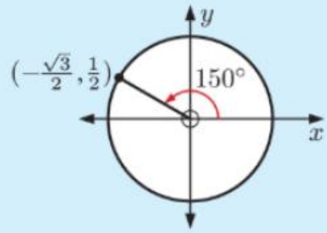


$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$

$\tan 60^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

b

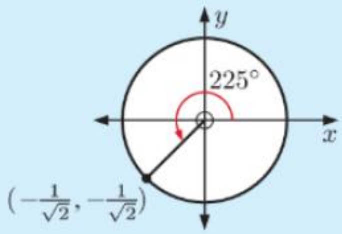


$\sin 150° = \frac{1}{2}$

$\cos 150° = -\frac{\sqrt{3}}{2}$

$\tan 150° = \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

c



$\sin 225° = -\frac{1}{\sqrt{2}}$

$\cos 225° = -\frac{1}{\sqrt{2}}$

$\tan 225° = 1$

EXERCISE 21D

1 Use a unit circle to find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for:

a $\theta = 30°$ **b** $\theta = 45°$ **c** $\theta = 0°$ **d** $\theta = 135°$
e $\theta = 90°$ **f** $\theta = 120°$ **g** $\theta = 270°$ **h** $\theta = 180°$
i $\theta = 210°$ **j** $\theta = 240°$ **k** $\theta = 330°$ **l** $\theta = 360°$
m $\theta = 300°$ **n** $\theta = 315°$ **o** $\theta = 450°$ **p** $\theta = 540°$

2 Without using a calculator, find the exact value of:

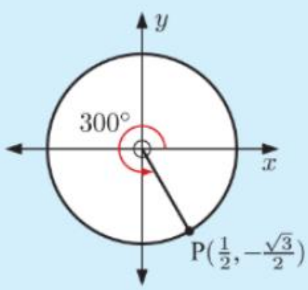
a $\sin^2 45°$ **b** $\cos^2 60°$ **c** $\tan^2 30°$
d $\cos^3(-30°)$ **e** $\sin^2 150°$ **f** $\tan^2 300°$

Check your answers using a calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 7 **Self Tutor**

Use a unit circle diagram to find the angle between $0°$ and $360°$ which has a cosine of $\frac{1}{2}$ and a sine of $-\frac{\sqrt{3}}{2}$.



P is at $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Since $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$ are involved, the angle is a multiple of $30°$.

The angle is $300°$.

3 Use a unit circle diagram to find the angle between $0°$ and $360°$ which has:

a a cosine of $-\frac{\sqrt{3}}{2}$ and a sine of $\frac{1}{2}$
b a cosine of $-\frac{1}{\sqrt{2}}$ and a sine of $-\frac{1}{\sqrt{2}}$
c a cosine of $\frac{\sqrt{3}}{2}$ and a tangent of $\frac{1}{\sqrt{3}}$
d a sine of $-\frac{1}{\sqrt{2}}$ and a tangent of -1.

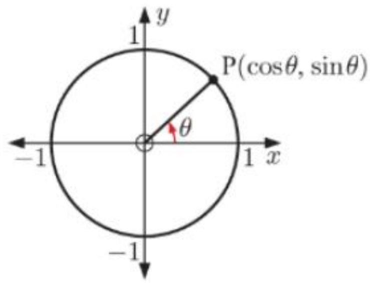
E TRIGONOMETRIC FUNCTIONS

A **trigonometric function** is a function which involves one of the trigonometric ratios.

Consider the point $P(\cos\theta, \sin\theta)$ on the unit circle.

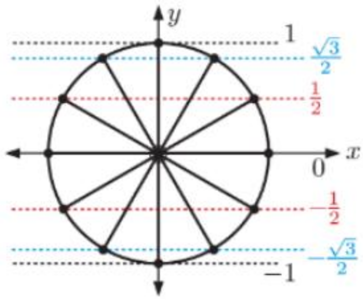
As θ increases, the point P moves anticlockwise around the unit circle, and the values of $\cos\theta$ and $\sin\theta$ change.

We can draw the graphs of $y = \sin\theta$ and $y = \cos\theta$ by plotting the values of $\sin\theta$ and $\cos\theta$ against the angle θ.



THE GRAPH OF $y = \sin\theta$

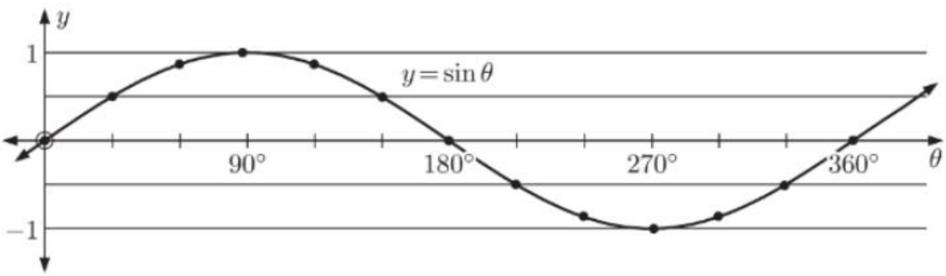
The diagram alongside gives the y-coordinates for all points on the unit circle at intervals of $30°$.

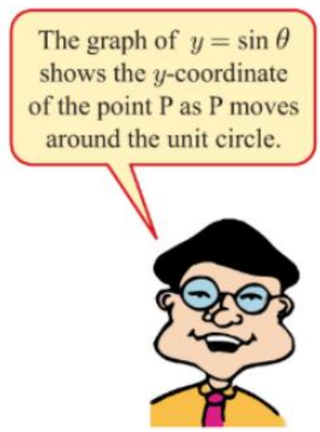


A table for $\sin\theta$ can be constructed from these values:

θ	0	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Plotting $\sin\theta$ against θ gives:





Once we reach $360°$, P has completed a full revolution of the unit circle, and so this pattern repeats itself.

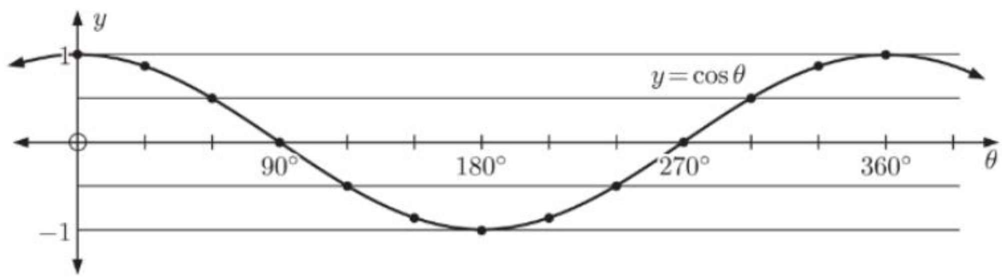
DEMO

THE GRAPH OF $y = \cos\theta$

By considering the x-coordinates of the points on the unit circle at intervals of $30°$, we can create a table of values for $\cos\theta$:

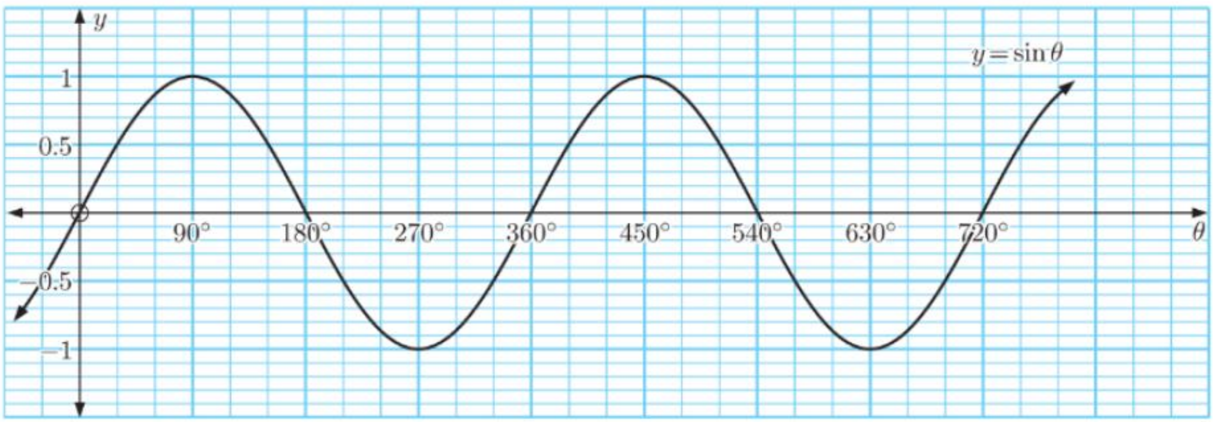
θ	0	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

Plotting $\cos\theta$ against θ gives:



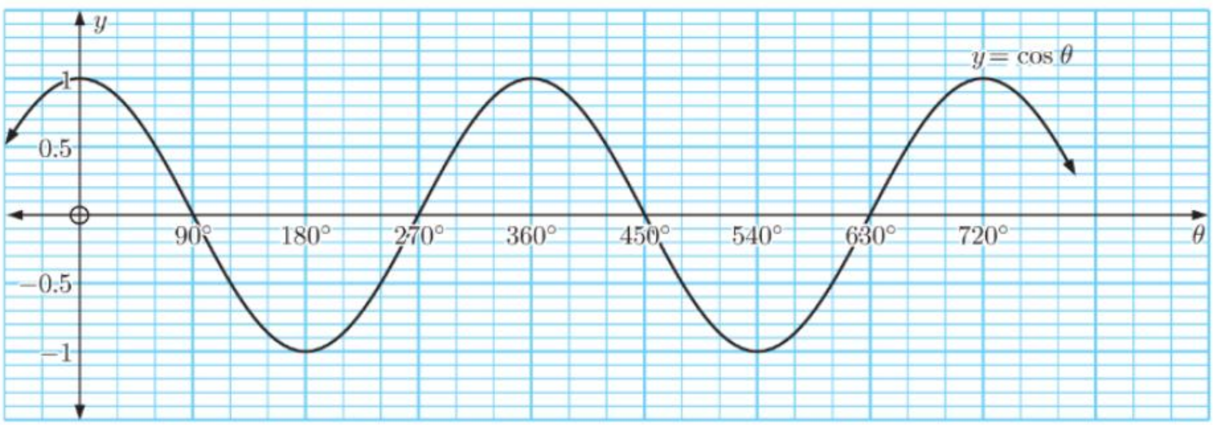
EXERCISE 21E.1

1 Below is an accurate graph of $y = \sin\theta$.



- **a** Find the y-intercept of the graph.
- **b** Find the values of θ on $0° \leqslant \theta \leqslant 720°$ for which:
  - **i** $\sin\theta = 0$ **ii** $\sin\theta = -1$ **iii** $\sin\theta = \frac{1}{2}$ **iv** $\sin\theta = \frac{\sqrt{3}}{2}$
- **c** Use the graph to estimate the values of θ on $0° \leqslant \theta \leqslant 720°$ for which $\sin\theta = 0.3$.
- **d** Find the intervals on $0° \leqslant \theta \leqslant 720°$ where $\sin\theta$ is:
  - **i** positive **ii** negative.
- **e** Find the range of the function.

2 Below is an accurate graph of $y = \cos\theta$.



- **a** Find the y-intercept of the graph.

b Find the values of θ on $0° \leqslant \theta \leqslant 720°$ for which:

i $\cos\theta = 0$ **ii** $\cos\theta = 1$ **iii** $\cos\theta = -\frac{1}{2}$ **iv** $\cos\theta = -\frac{1}{\sqrt{2}}$

c Use the graph to estimate the values of θ on $0° \leqslant \theta \leqslant 720°$ for which $\cos\theta = 0.3$.

d Find the intervals on $0° \leqslant \theta \leqslant 720°$ where $\cos\theta$ is:

i positive **ii** negative.

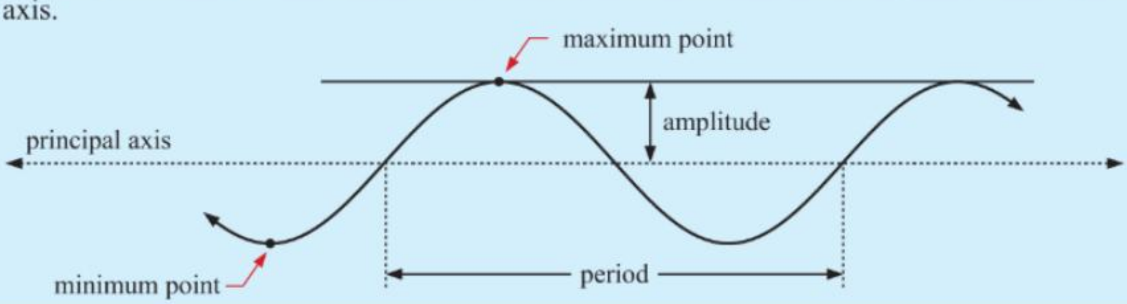
e Find the range of the function.

USING TRANSFORMATIONS TO GRAPH TRIGONOMETRIC FUNCTIONS

Once we are familiar with the graphs of $y = \sin\theta$ and $y = \cos\theta$, we can use transformations to graph more complicated trigonometric functions.

Before we consider the graphs of these functions in detail, we need to learn appropriate language for describing them:

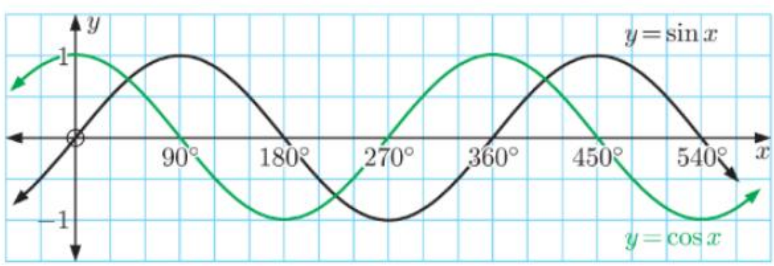
- A **periodic function** is one which repeats itself over and over in a horizontal direction.
- The **period** of a periodic function is the length of one repetition or cycle.
- The graph oscillates about a horizontal line called the **principal axis** or **mean line**.
- A **maximum point** occurs at the top of a crest.
- A **minimum point** occurs at the bottom of a trough.
- The **amplitude** is the vertical distance between a maximum or minimum point and the principal axis.



Instead of using θ, we will now use x to represent the angle variable. This is just for convenience, so we are dealing with the familiar function form $y = f(x)$.

For the graphs of $y = \sin x$ and $y = \cos x$:

- The **period** is $360°$.
- The **amplitude** is 1.
- The **principal axis** is the line $y = 0$.



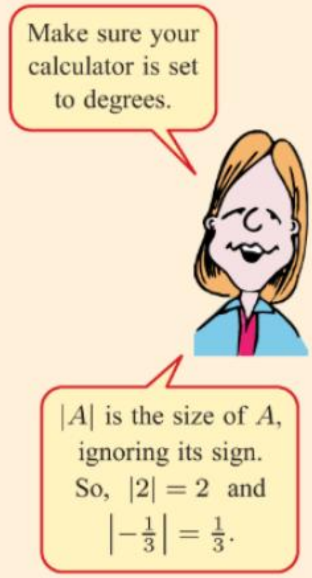
INVESTIGATION 2 FAMILIES OF TRIGONOMETRIC FUNCTIONS

In this Investigation, we will use technology to draw graphs related to $y = \sin x$.

What to do:

1. **a** Use the graphing package or your calculator to graph, on the same set of axes:

   i $y = \sin x$ **ii** $y = 2\sin x$
   iii $y = \frac{1}{2}\sin x$ **iv** $y = -\sin x$
   v $y = -\frac{1}{3}\sin x$ **vi** $y = -\frac{3}{2}\sin x$



   b For graphs of the form $y = A\sin x$, comment on the significance of:

   i the sign of A **ii** the size of A, or $|A|$.

2. **a** Graph, on the same set of axes:

   i $y = \sin x$ **ii** $y = \sin 2x$
   iii $y = \sin\left(\frac{1}{2}x\right)$ **iv** $y = \sin 3x$

   b For graphs of the form $y = \sin Bx$, $B > 0$, what is the *period* of the graph?

3. **a** Graph on the same set of axes:

   i $y = \sin x$ **ii** $y = \sin x + 2$ **iii** $y = \sin x - 2$

   b Copy and complete:
   "*The graph of* $y = \sin x + C$ *is a* .... *of* $y = \sin x$ *with translation vector* $\begin{pmatrix} \cdots\cdots \\ \cdots\cdots \end{pmatrix}$."

   c What is the *principal axis* of $y = \sin x + C$?

From the **Investigation** you should have observed the following properties about the **sine function** $y = A\sin Bx + C$:

$$\boldsymbol{y = A\sin Bx + C}$$

affects **amplitude** affects **period** affects **vertical translation**

- The amplitude is $|A|$.
- The period is $\dfrac{360°}{B}$ for $B > 0$.
- The principal axis is $y = C$.

Click on the icon to obtain a demonstration for the general sine function.

DEMO

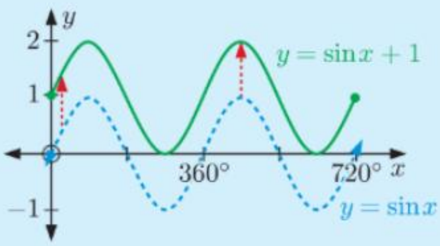
The properties of the **cosine function** $y = A\cos Bx + C$ are the same as those of the sine function.

Example 8 **Self Tutor**

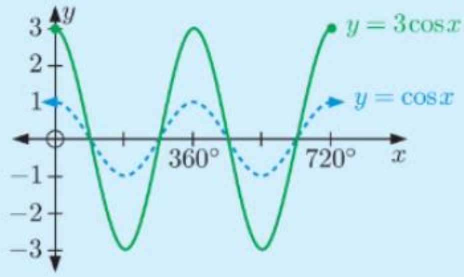
Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = \sin x + 1$ **b** $y = 3\cos x$ **c** $y = \sin 2x$

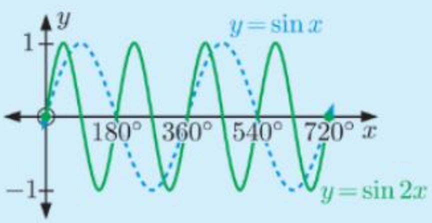
a We translate $y = \sin x$ 1 unit upwards.

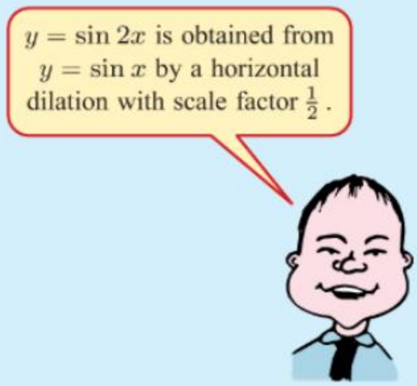


b $y = 3\cos x$ is obtained from $y = \cos x$ by increasing the amplitude to 3.



c $y = \sin 2x$ has amplitude 1 and period $\dfrac{360°}{2} = 180°$.





EXERCISE 21E.2

1 Find the amplitude of:

a $y = 4\sin x$ **b** $y = -2\cos x + 1$ **c** $y = 3\sin 2x$

2 Find the period of:

a $y = \cos 3x$ **b** $y = 5\sin 4x + 2$ **c** $y = -\cos(\frac{x}{2})$

3 Find the principal axis of:

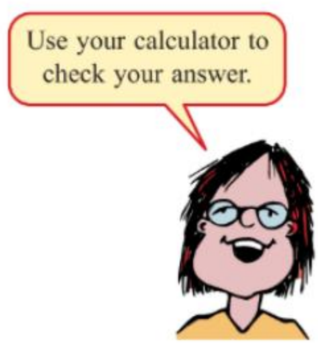
a $y = \sin x - 3$ **b** $y = -2\cos x + 5$ **c** $y = \frac{1}{2}\sin 4x$

4 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = 3\sin x$ **b** $y = \sin x - 2$

c $y = -2\sin x$ **d** $y = \sin 3x$

e $y = \sin(\frac{x}{2})$



5 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = 2\cos x$ **b** $y = \cos x + 3$

c $y = -\frac{1}{3}\cos x$ **d** $y = \cos 2x$

e $y = \cos(\frac{3x}{2})$

6 Sketch the graph of the following for $0° \leqslant x \leqslant 360°$:

a $y = 2\cos x + 1$ **b** $y = \sin 2x + 3$ **c** $y = \frac{1}{2}\cos 3x$ **d** $y = 3\sin 4x + 7$

7 **a** Sketch the graph of $y = 6\sin x + 10$ for $0° \leqslant x \leqslant 720°$.

b Find the value of y when $x = 30°$.

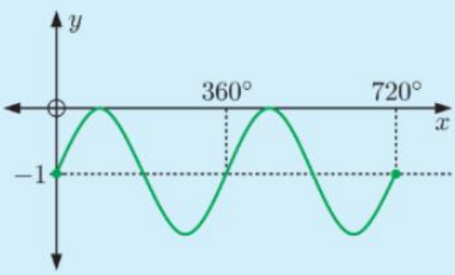
c Find the maximum value of y, and the values of x at which the maximum occurs.

d Find the minimum value of y, and the values of x at which the minimum occurs.

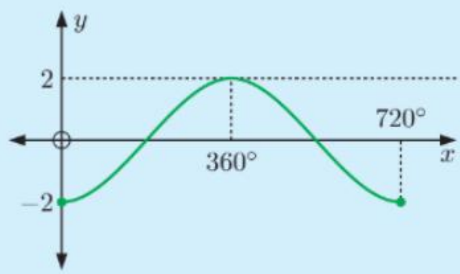
Example 9 **Self Tutor**

Identify the function in the graph:

a



b



a When $x = 0°$, the function is at its principal axis.

$\therefore$ it is a sine function.

The principal axis is $y = -1$, so $C = -1$.

The amplitude is 1, and A is positive, so $A = 1$.

The period is $360°$, so $B = 1$.

$\therefore$ the function is $y = \sin x - 1$ with domain $0° \leqslant x \leqslant 720°$.

b When $x = 0°$, the function is at a minimum.

$\therefore$ it is a cosine function which has been reflected in the x-axis, and A is negative.

The principal axis is $y = 0$, so $C = 0$.

The amplitude is 2, so $A = -2$.

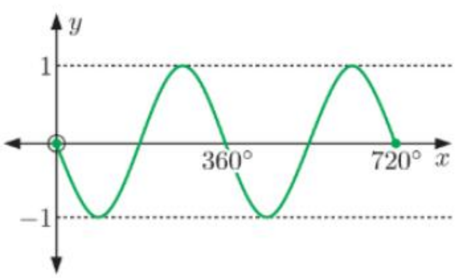
The period is $720°$, so $\frac{360}{B} = 720$

$\therefore B = \frac{1}{2}$

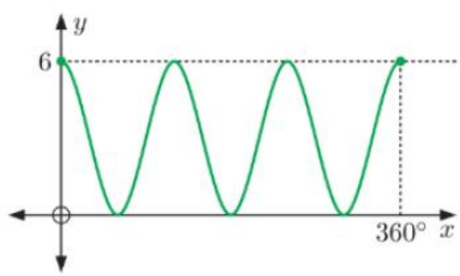
$\therefore$ the function is $y = -2\cos(\frac{1}{2}x)$ with domain $0° \leqslant x \leqslant 720°$.

8 Identify the function in the graph:

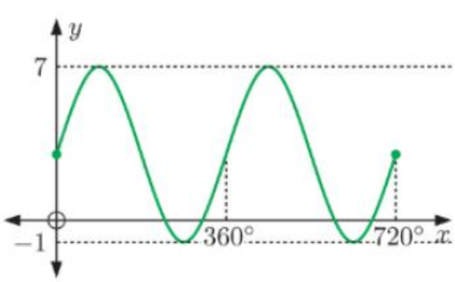
a



b



c



d

Example 10 **Self Tutor**

The average daytime temperature for a city is given by the function $D(t) = 5\cos(30t)^\circ + 20$ °C, where t is the time in months after January.

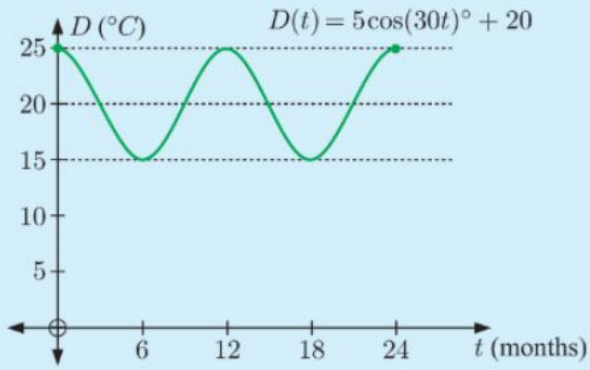
a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the average daytime temperature during May.

c Find the minimum average daytime temperature, and the month in which it occurs.

a For $D(t) = 5\cos(30t)^\circ + 20$:

- the amplitude is 5
- the period is $\frac{360}{30} = 12$ months
- the principal axis is $D = 20$.



b May is 4 months after January.

When $t = 4$, $D = 5 \times \cos(120^\circ) + 20$
$= 5 \times (-\frac{1}{2}) + 20$
$= 17.5$

So, the average daytime temperature during May is 17.5°C.

c The minimum average daytime temperature is $20 - 5 = 15$°C, which occurs when $t = 6$ or 18.

So, the minimum average daytime temperature occurs during July.

9 The temperature inside Vanessa's house t hours after midday is given by the function $T(t) = 6\sin(15t)^\circ + 26$ °C.

a Find the:

i amplitude **ii** principal axis **iii** period of the function.

b Sketch the graph of T against t for $0 \leqslant t \leqslant 24$.

c Find the temperature inside Vanessa's house at:

i midnight **ii** 2 pm.

d Find the maximum temperature inside Vanessa's house, and the time at which it occurs.

10 The depth of water in a harbour t hours after midnight is $D(t) = 4\cos(30t)^\circ + 6$ metres.

a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the highest and lowest depths of the water, and the times at which they occur.

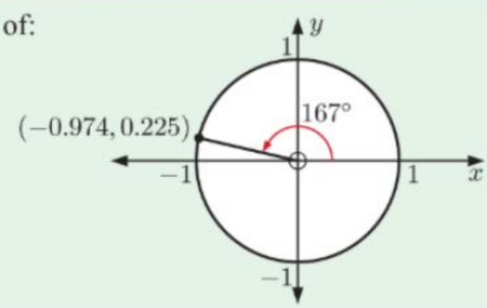
c A boat requires a water depth of 5 metres to sail in. Will the boat be able to enter the harbour at 8 pm?

REVIEW SET 21A

1 **a** Use the unit circle alongside to find the value of:

i $\cos 167^\circ$ **ii** $\sin 167^\circ$

b Hence find the value of $\tan 167^\circ$.



2 **a** Find another angle which has the same cosine as $23°$.

b Find an angle whose sine is the negative of $\sin 62°$.

3 Find an acute angle whose:

a sine has the same value as $\cos 74°$ **b** cosine has the same value as $\sin 7°$.

4 Using a unit circle to illustrate your answers, find the possible values of $\sin\theta$ when:

a $\cos\theta = \frac{2}{3}$ **b** $\cos\theta = -\frac{3}{\sqrt{13}}$

5 For $\theta = 315°$, determine whether $\cos\theta$, $\sin\theta$, and $\tan\theta$ are positive or negative.

6 Find the exact value of:

a $\sin\theta$ if $\cos\theta = \frac{1}{\sqrt{5}}$ and $0° \leqslant \theta \leqslant 90°$

b $\cos\theta$ if $\sin\theta = -\frac{3}{7}$ and $270° \leqslant \theta \leqslant 360°$

7 Use a unit circle diagram to find the angle between $0°$ and $360°$ which has a sine of $\frac{1}{2}$ and a tangent of $-\frac{1}{\sqrt{3}}$.

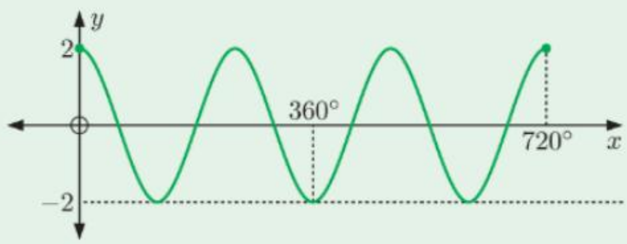
8 Find the exact value of:

a $\sin^2 120°$ **b** $\tan^2 225°$

9 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = 4\sin x$ **b** $y = 2\cos x - 3$ **c** $y = \sin 3x$

10 Identify the function in the graph:



11 As the tip of a windmill's blade rotates, its height above ground is given by $H(t) = 10\cos(30t)° + 20$ metres, where t is the time in seconds.

a Sketch the graph of H against t for $0 \leqslant t \leqslant 36$.

b Find the height of the blade's tip after 9 seconds.

c Find the minimum height of the blade's tip.

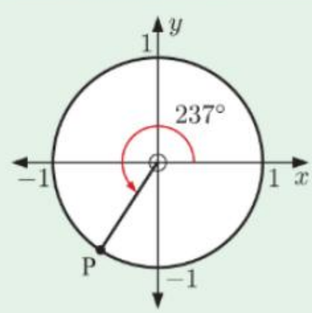
d How long does the blade take to complete a full revolution?

REVIEW SET 21B

1 Find:

a the exact coordinates of P

b the approximate coordinates of P, rounded to 3 significant figures.



2 Find the:

a acute angle which has the same sine as $115°$

b obtuse angle which has the same sine as $44°$

c obtuse angle whose cosine is the negative of $\cos 13°$

d acute angle whose cosine is the negative of $\cos 130°$.

3 Use a unit circle to explain why:

a $\sin(-153°) = \sin 207°$ **b** $\tan(-24°) = \tan 156°$

c $\cos 184° = \cos 176°$

4 Using a unit circle to illustrate your answers, find the possible values of $\cos\theta$ when:

a $\sin\theta = \frac{1}{4}$ **b** $\sin\theta = -\frac{2}{\sqrt{5}}$

5 Suppose $90° < \theta < 180°$ and $\cos\theta = -\frac{3}{7}$. Find the exact value of:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(180° - \theta)$

6 Without using a calculator, find the exact value of:

a $\cos^2 135°$ **b** $\tan^2 120°$

7 Find the amplitude and period of:

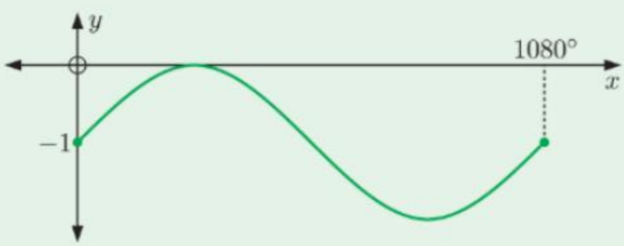
a $y = 5\cos 2x + 3$ **b** $y = -4\cos(\frac{x}{2}) - 1$

8 Find the exact value of $\cos\theta$ if $\sin\theta = -\frac{\sqrt{5}}{4}$ and $\tan\theta$ is negative.

9 Sketch the graph of the following for $0° \leqslant x \leqslant 720°$:

a $y = 3\cos x$ **b** $y = -\sin 2x$ **c** $y = \frac{3}{2}\cos x + \frac{5}{2}$

10 Identify the function in the graph:



11 The fraction of the Moon which is illuminated each night is given by the function $M(t) = \frac{1}{2}\cos(12t)° + \frac{1}{2}$, where t is the time in days after January 1st.

a Sketch the graph of M against t for $0 \leqslant t \leqslant 60$.

b Find the fraction of the Moon which is illuminated on the night of:

i January 6th **ii** January 21st **iii** January 27th **iv** February 19th.

c How often does a full moon occur?

d On what dates during January and February is the Moon not illuminated at all?

Chapter 22

Number sequences

Contents:

OPENING PROBLEM

Jasmine is attempting to cycle 2000 km in 30 days to raise money for charity. She cycled 40 km on the first day, and each day she will cycle 2 km further than the day before.

Things to think about:

a How far will Jasmine cycle on the:
  i 2nd day **ii** 10th day **iii** 20th day?

b How far will Jasmine have cycled in total after:
  i 2 days **ii** 10 days **iii** 20 days?

c Can we find the answers to **a** and **b** without having to calculate the distance cycled on each of the days?

d Will Jasmine achieve her goal of cycling 2000 km in 30 days?

In this chapter, we will study **number sequences**. We will see that, for certain types of sequences, there are rules which allow us to find any member of the sequence. We can also calculate the sum of the members of the sequence without having to add all the members individually.

A NUMBER SEQUENCES

Consider the illustrated pattern of balls:

The first layer has just one blue ball.
The second layer has three pink balls.
The third layer has five black balls.
The fourth layer has seven green balls.

If we let u_n represent the number of balls in the nth layer, then $u_1 = 1$, $u_2 = 3$, $u_3 = 5$, and $u_4 = 7$.

The pattern could be continued forever, generating the **sequence** of numbers:
1, 3, 5, 7, 9, 11, 13, 15, 17, ...

The string of dots indicates that the pattern continues forever.

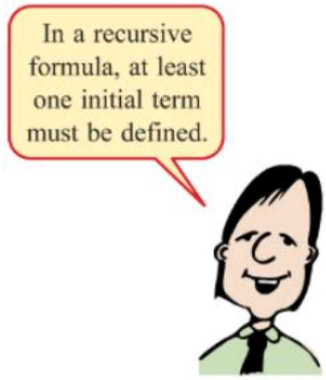
> A **number sequence** is an ordered list of numbers defined by a rule.
>
> The numbers in a sequence are called the **terms** of the sequence.

We will look at three ways of describing a number sequence.

(1) Using **words**.
The sequence for the pattern of balls can be described as "starting at 1, and increasing by 2 each time".

(2) Using an **explicit formula**, which gives the nth term of the sequence u_n in terms of n.
The explicit formula for the pattern of balls is $u_n = 2n - 1$. u_n is called the ***n*th term** or the **general term**.
We can use this formula to find, for example, the 10th term of the sequence, which is $u_{10} = 2(10) - 1 = 19$.

(3) Using a **recursive formula**, which gives the nth term of the sequence in terms of one or more of the preceding terms.
The recursive formula for the pattern of balls is $u_1 = 1$, $u_n = u_{n-1} + 2$ for $n \geqslant 2$.
So, we have $u_1 = 1$
$u_2 = u_1 + 2 = 1 + 2 = 3$
$u_3 = u_2 + 2 = 3 + 2 = 5$, and so on.



EXERCISE 22A

1 Consider the number sequence 2, 4, 7, 11, 16, 22, ... Find:
a u_2 **b** u_5 **c** $u_1 + u_2 + u_3$.

2 Write down the first five terms of the sequence:
a starting at 3, and increasing by 4 each time
b starting at 40, and decreasing by 5 each time
c whose nth term is the nth prime number.

3 Describe the sequence 5, 8, 11, 14, 17, 20, ... using:
a words **b** a recursive formula.

4 Describe the sequence 1, 4, 9, 16, 25, 36, ... using:
a words **b** an explicit formula.

5 Find the 5th term of the sequence with explicit formula:
a $u_n = 3n + 4$ **b** $u_n = 54 - 7n$ **c** $u_n = 3 \times 2^n$ **d** $u_n = n^2 + 2$

6 Consider the sequence with explicit formula $u_n = 50 - n^2$. Find:
a the 6th term **b** the sum of the first 3 terms
c the first term of the sequence which is negative.

7 Find the first four terms of the sequence with recursive formula:
a $u_1 = 7$, $u_n = u_{n-1} + 3$, $n \geqslant 2$ **b** $u_1 = 25$, $u_n = u_{n-1} - 4$, $n \geqslant 2$
c $u_1 = 5$, $u_n = 3 \times u_{n-1}$, $n \geqslant 2$ **d** $u_1 = 100$, $u_n = \frac{u_{n-1}}{2}$, $n \geqslant 2$
e $u_1 = 3$, $u_n = 2 \times u_{n-1} - 1$, $n \geqslant 2$ **f** $u_1 = 4$, $u_n = 10 - u_{n-1}$, $n \geqslant 2$
g $u_1 = 3$, $u_2 = 4$, $u_n = u_{n-1} \times u_{n-2}$, $n \geqslant 3$

INVESTIGATION 1 THE FIBONACCI SEQUENCE

Perhaps the most famous example of a recursive relationship is that which generates the **Fibonacci sequence**:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ ....$$

The sequence starts with two 1s, and after that each term is obtained by adding the two terms preceding it:

$$\begin{aligned}1+1&=2\\1+2&=3\\2+3&=5\\3+5&=8\\&\vdots\end{aligned}$$

Leonardo Fibonacci

We can hence write down a **recursive formula** for the sequence:
$u_1 = u_2 = 1, \quad u_n = u_{n-1} + u_{n-2}$ for $n \geqslant 3$.

Check: If $n = 3$, $u_3 = u_2 + u_1 = 1 + 1 = 2$ ✓
If $n = 4$, $u_4 = u_3 + u_2 = 2 + 1 = 3$ ✓
If $n = 5$, $u_5 = u_4 + u_3 = 3 + 2 = 5$ ✓

Leonardo Fibonacci (1170 - 1250) noticed that the sequence 1, 1, 2, 3, 5, 8, ... occurred frequently in the natural world.

For example, he noticed that flowers of particular species have the same number of petals, and that these numbers are often members of the Fibonacci sequence.

3 petals	lily, iris
5 petals	buttercup
8 petals	delphinium
13 petals	cineraria
21 petals	aster
34 petals	pyrethrum

Fibonacci also observed the number sequence in:

- the number of leaves arranged about the stem in plants
- the seed patterns of a sunflower
- the seed pattern on a pine cone.

The recursive formula is very useful for dealing with the Fibonacci sequence because its explicit formula is very complicated. In fact, it is not obvious from the explicit formula

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n = 1, 2, 3, 4, 5, ...$$

that the sequence generated will be integers.

What to do:

1 Show that the explicit formula is correct for $n = 1$ and $n = 2$.

2 Use the recursive formula to find the next five terms of the Fibonacci sequence after 55.

3 Can you see a pattern of even and odd numbers in the Fibonacci sequence? Do you think the 100th term of the sequence will be even or odd?

4 Research other occurrences of the Fibonacci sequence in nature.

B ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number. We call this number the **common difference**.

For example:

- the sequence 1, 5, 9, 13, 17, ... is arithmetic with common difference 4, since
  $5-1=4$
  $9-5=4$
  $13-9=4$, and so on
- the sequence 42, 37, 32, 27, ... is arithmetic with common difference -5, since
  $37-42=-5$
  $32-37=-5$
  $27-32=-5$, and so on.

THE GENERAL TERM OF AN ARITHMETIC SEQUENCE

Notice in the sequence 1, 5, 9, 13, 17, ... that

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1+4 &&= 1+1(4) \\ u_3 &= 1+4+4 &&= 1+2(4) \\ u_4 &= 1+4+4+4 &&= 1+3(4), \quad \text{and so on.} \end{aligned}$$

For an **arithmetic sequence** with **first term u_1** and **common difference d**, the **general term** or **nth term** is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 1 **Self Tutor**

Consider the sequence 3, 9, 15, 21, 27, ...

a Show that the sequence is arithmetic.
b Find a formula for the general term u_n.
c Find the 100th term of the sequence.
d Is i 489 ii 1592 a member of the sequence?

a $9-3=6$, $15-9=6$, $21-15=6$, $27-21=6$
Assuming that the pattern continues, consecutive terms differ by 6.
$\therefore$ the sequence is arithmetic with $u_1=3$ and common difference $d=6$.

b $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 3+6(n-1)$
$\therefore\ u_n = 6n-3$

c $u_{100} = 6(100)-3$
$= 597$

d **i** Let $u_n = 489$
$\therefore\ 6n - 3 = 489$
$\therefore\ 6n = 492$
$\therefore\ n = 82$
$\therefore$ 489 is the 82nd term of the sequence.

ii Let $u_n = 1592$
$\therefore\ 6n - 3 = 1592$
$\therefore\ 6n = 1595$
$\therefore\ n = 265\frac{5}{6}$
But n must be an integer, so 1592 is not a term of the sequence.

EXERCISE 22B

1 Determine whether the following sequences are arithmetic:

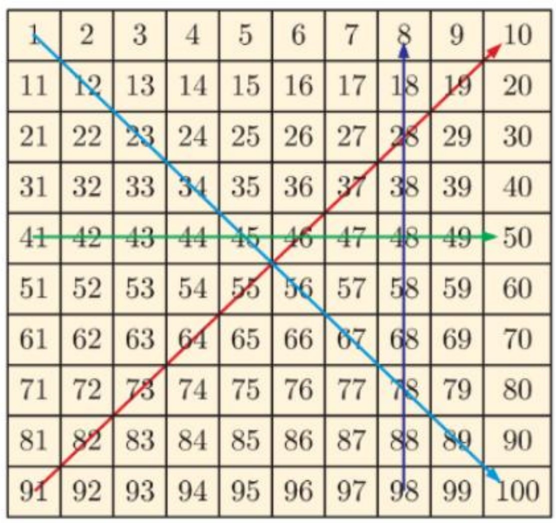
a 2, 5, 8, 11, 14, ...
b 5, 9, 13, 18, 22, ...
c 29, 23, 16, 10, 4, ...
d 11, 4, −3, −10, −17, ...

2 Find the unknowns given that the following sequences are arithmetic:

a 4, 10, □, 22, 28, ...
b 13, 20, 27, □, 41, ...
c 19, □, 11, 7, △, ...
d 22, □, 4, △, −14, ...

3 Write down the first term u_1 and common difference d for the arithmetic sequence of numbers given by:

a the green arrow
b the blue arrow
c the purple arrow
d the red arrow.



4 Consider the sequence 4, 11, 18, 25, 32, ...

a Show that the sequence is arithmetic.
b Find a formula for the general term u_n.
c Find the 30th term of the sequence.
d Is 340 a member of the sequence?
e Is 738 a member of the sequence?

5 Consider the sequence 67, 63, 59, 55, ...

a Show that the sequence is arithmetic.
b Find a formula for the general term u_n.
c Find the 60th term of the sequence.
d Is −143 a member of the sequence?
e Is 85 a member of the sequence?

6 An arithmetic sequence is defined by $u_n = 11n - 7$.

a Find u_1 and d.
b Find the 37th term.
c What is the least term of the sequence which is greater than 250?

Example 2 **Self Tutor**

Find k given that $k+5$, -1, and $2k-1$ are consecutive terms of an arithmetic sequence.

Since the terms are consecutive,

$$\begin{aligned} -1-(k+5) &= (2k-1)-(-1) \quad \{\text{equating common differences}\} \\ \therefore\ -1-k-5 &= 2k-1+1 \\ \therefore\ -k-6 &= 2k \\ \therefore\ -6 &= 3k \\ \therefore\ k &= -2 \end{aligned}$$

7 Find k given the consecutive terms of an arithmetic sequence:

a 31, k, 13 **b** k, 8, $k+11$ **c** $k+2$, $2k+3$, 17

8 **a** Find k given that $k-1$, 13, and $3k+3$ are consecutive terms of an arithmetic sequence.

b Write down the next two terms of the sequence.

Example 3 **Self Tutor**

Find the general term u_n for an arithmetic sequence given that $u_3=4$ and $u_7=-24$.

$$\begin{aligned} & u_3 = u_1+2d = 4 \quad \ldots\ (1) \\ \text{and}\ & u_7 = u_1+6d = -24 \quad \ldots\ (2) \\ & \therefore\ -4d = 28 \quad \{(1)-(2)\} \\ & \therefore\ d = -7 \end{aligned}$$

Substituting into (1), $u_1+2(-7)=4$

$\therefore\ u_1-14=4$

$\therefore\ u_1=18$

$$\begin{aligned} u_n &= u_1+(n-1)d \\ \therefore\ u_n &= 18+(n-1)(-7) \\ \therefore\ u_n &= 18-7n+7 \\ \therefore\ u_n &= 25-7n \end{aligned}$$

Check:

$u_3 = 25-7(3) = 4$ ✓

$u_7 = 25-7(7) = -24$ ✓

9 Find the general term u_n for an arithmetic sequence given that:

a $u_4=37$ and $u_{10}=67$ **b** $u_5=-10$ and $u_{12}=-38$

c $u_4=-4$ and $u_{15}=29$ **d** $u_{10}=-16$ and $u_6=-13$.

10 An arithmetic sequence has $u_3=10$ and $u_{11}=58$.

a Find the general term for the sequence. **b** Find the 30th term of the sequence.

C GEOMETRIC SEQUENCES

A sequence is **geometric** if each term can be obtained from the previous one by multiplying by the same non-zero constant. This constant is called the **common ratio**.

For example: 2, 6, 18, 54, ... is a geometric sequence with common ratio 3, since

$2 \times 3 = 6$ and $6 \times 3 = 18$ and $18 \times 3 = 54$.

THE GENERAL TERM OF A GEOMETRIC SEQUENCE

Notice in the sequence 2, 6, 18, 54, ... that

$$\begin{aligned} u_1 &= 2 \\ u_2 &= 2 \times 3 \\ u_3 &= 2 \times 3 \times 3 = 2 \times 3^2 \\ u_4 &= 2 \times 3 \times 3 \times 3 = 2 \times 3^3, \quad \text{and so on.} \end{aligned}$$

For a **geometric sequence** with **first term** $\boldsymbol{u_1}$ and **common ratio** $\boldsymbol{r}$, the **general term** or $\boldsymbol{n}$**th term** is $\boldsymbol{u_n = u_1 r^{n-1}}$.

r is called the common ratio because $\dfrac{u_{n+1}}{u_n} = r$ for all n.

Example 4 Self Tutor

Consider the sequence 16, 8, 4, 2, 1, ...

a Show that the sequence is geometric.

b Find the general term u_n.

c Find the 10th term as a fraction.

a $\frac{8}{16} = \frac{1}{2}$, $\frac{4}{8} = \frac{1}{2}$, $\frac{2}{4} = \frac{1}{2}$, $\frac{1}{2} = \frac{1}{2}$

Assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 16$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$ $\therefore$ $u_n = 16 \times \left(\frac{1}{2}\right)^{n-1}$

c $u_{10} = 16 \times (\frac{1}{2})^9 = \dfrac{2^4}{2^9} = \dfrac{1}{2^5} = \frac{1}{32}$

EXERCISE 22C

1 Determine whether the following sequences are geometric:

a 5, 10, 20, 40, ...

b 4, 12, 36, 72, ...

c 45, 15, 5, $\frac{5}{3}$, ...

d 1, -4, 16, 64, ...

2 Write down the common ratio for these geometric sequences:

a 2, 10, 50, 250, ...

b 60, 30, 15, 7.5, ...

c 3, -6, 12, -24, ...

d 4000, -400, 40, -4, ...

3 For the geometric sequence with first two terms given, find b and c:

a 3, 6, b, c, ...

b 8, 2, b, c, ...

c 15, -5, b, c, ...

4 **a** Show that the sequence 1, 3, 9, 27, ... is geometric.
b Find the general term u_n.
c Find the 10th term of the sequence.

5 **a** Show that the sequence 40, -20, 10, -5, ... is geometric.
b Find the general term u_n.
c Find the 12th term as a fraction.

6 **a** Show that the sequence 16, -4, 1, -0.25, ... is geometric.
b Find the 8th term as a decimal.

7 Find the general term of the geometric sequence: $3, 3\sqrt{2}, 6, 6\sqrt{2}, ...$

Example 5 **Self Tutor**

$k-1$, $k+2$, and $3k$ are consecutive terms of a geometric sequence.
Find the possible values of k.

Equating common ratios gives $\dfrac{3k}{k+2} = \dfrac{k+2}{k-1}$

$$\therefore\ 3k(k-1) = (k+2)^2$$
$$\therefore\ 3k^2 - 3k = k^2 + 4k + 4$$
$$\therefore\ 2k^2 - 7k - 4 = 0$$
$$\therefore\ (k-4)(2k+1) = 0$$
$$\therefore\ k = 4 \text{ or } -\tfrac{1}{2}$$

Check: If $k = 4$, the terms are: 3, 6, 12. ✓ $\{r = 2\}$
If $k = -\frac{1}{2}$, the terms are: $-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}$. ✓ $\{r = -1\}$

8 Find k given the consecutive terms of a geometric sequence:

a $k, 2, 6$
b $4, 6, k$
c $k,\ 2\sqrt{2},\ k^2$
d $3, k, 27$
e $k,\ 3k,\ 10k+7$
f $k,\ k+4,\ 8k+2$

Example 6 **Self Tutor**

A geometric sequence has $u_2 = -5$ and $u_5 = 40$. Find its general term.

For a geometric sequence, $u_n = u_1 r^{n-1}$

$\therefore\ u_2 = u_1 r = -5$ .... (1)
and $u_5 = u_1 r^4 = 40$ .... (2)

$$\therefore\ \frac{u_1 r^4}{u_1 r} = \frac{40}{-5} \qquad \{(2) \div (1)\}$$
$$\therefore\ r^3 = -8$$
$$\therefore\ r = \sqrt[3]{-8}$$
$$\therefore\ r = -2$$

Substituting into (1), $u_1(-2) = -5$
$$\therefore\ u_1 = \tfrac{5}{2}$$
Thus $u_n = \frac{5}{2} \times (-2)^{n-1}$

9 Find the general term u_n of the geometric sequence which has:

a $u_3 = 16$ and $u_4 = 48$

b $u_3 = 32$ and $u_6 = -4$

c $u_2 = 10$ and $u_4 = 250$

d $u_3 = 3$ and $u_7 = \frac{3}{4}$.

10 A geometric sequence has general term u_n with $u_3 = 48$ and $u_7 = 3$. Find u_{12}.

INVESTIGATION 2 SEQUENCES IN FINANCE

Arithmetic and geometric sequences are observed in financial calculations such as simple interest, compound interest, and depreciation.

What to do:

1 \$1000 is invested at a **simple interest** rate of 7% per year with the interest paid at the end of each year. For the case of simple interest, interest is only paid on the amount originally invested.

After 1 year, the value is \$1000 × 1.07 {to increase by 7% we multiply by 107%}

After 2 years, the value is \$1000 × 1.14 {an increase of $2 \times 7\% = 14\%$ on the original}

a Find the value of the investment at the end of:

i 3 years **ii** 4 years **iii** 5 years.

b Do the amounts form an arithmetic sequence, geometric sequence, or neither? Explain your answer.

c Write an explicit formula for the sequence.

d Write a recursive formula for the sequence.

2 \$6000 is invested at a fixed rate of 7% p.a. **compound interest** over a lengthy period. Interest is paid at the end of each year. For the case of compound interest, interest is paid on the current value of the investment.

After 1 year, the value is \$6000 × 1.07

After 2 years, the value is $(\$6000 \times 1.07) \times 1.07 = \$6000 \times (1.07)^2$

a Explain why the amount after 3 years is given by $\$6000 \times (1.07)^3$.

b Write down, in the same form, the amount after:

i 4 years **ii** 5 years **iii** 6 years.

c Do the amounts at the end of each year form an arithmetic sequence, geometric sequence, or neither? Explain your answer.

d Write an explicit formula for the sequence.

e Write a recursive formula for the sequence.

3 A photocopier originally cost \$12 000 and it depreciates in value by 20% each year.

After one year, its value is $\$12\,000 \times 0.8$.

a Find its value at the end of:

i two years **ii** three years.

b Do the resulting annual values form an arithmetic sequence, or a geometric sequence?

c Give an explicit formula for the value at the end of the nth year.

d Give a recursive formula for the value.

D SERIES

A **series** is the sum of the terms of a sequence.

For a finite sequence $u_1, u_2, u_3, ..., u_n$ with n terms, the corresponding series is $u_1 + u_2 + u_3 + ... + u_n$.

The sum of this series is $S_n = u_1 + u_2 + u_3 + ... + u_n$.

For example, consider the sequence 1, 3, 6, 10, 15, which has 5 terms. The corresponding series is $1 + 3 + 6 + 10 + 15$, and the sum of the series is $S_5 = 35$.

EXERCISE 22D

1 Consider the sequence 3, 5, 8, 11, 19. Find:

a u_3 **b** S_3 **c** u_5 **d** S_5.

2 For each of the following sequences, find:

i the first 4 terms of the sequence **ii** S_4.

a $u_n = 2n + 3$ **b** $u_n = n^2 - 6$

c $u_n = 7 \times 2^{n-1}$ **d** $u_1 = 3,\ u_n = 3 \times u_{n-1} - 4,\ n \geqslant 2$

3 A sequence has $S_5 = 21$ and $S_6 = 33$. Find the value of u_6.

E ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

Consider the arithmetic sequence 5, 10, 15, ..., 90, 95, 100. The first term is $u_1 = 5$ and the last term is $u_{20} = 100$.

The sum of the terms of this sequence is $S_{20} = 5 + 10 + 15 + ... + 90 + 95 + 100$

However, we can also write $S_{20} = 100 + 95 + 90 + ... + 15 + 10 + 5$ {reversing the terms}

Adding these equations gives $2 \times S_{20} = \underbrace{105 + 105 + 105 + ... + 105 + 105 + 105}_{\text{20 of these}}$

$$\therefore\ 2 \times S_{20} = 20 \times 105$$
$$\therefore\ S_{20} = \tfrac{20}{2} \times 105 = 1050$$

The sum of a finite arithmetic series with first term u_1, common difference d, and last term u_n, is

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$\textit{or}\quad S_n = \frac{n}{2}(2u_1 + (n-1)d) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$$

Example 7 **Self Tutor**

An arithmetic series has 8 terms. The first term is 3 and the last term is 17. Find the sum of the series.

The series is arithmetic with $u_1 = 3$ and $u_8 = 17$.

Now $S_n = \frac{n}{2}(u_1 + u_n)$

$\therefore\ S_8 = \frac{8}{2}(3 + 17)$

$= 80$

EXERCISE 22E

1 Find the value of $5 + 8 + 11 + 14 + 17 + 20$:

a by adding the terms directly

b using $S_n = \frac{n}{2}(u_1 + u_n)$

c using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

2 An arithmetic series has 12 terms. The first term is 10 and the last term is 65. Find the sum of the series.

3 An arithmetic series has 20 terms. The first term is 30 and the last term is -8. Find the sum of the series.

Example 8 **Self Tutor**

Find the sum of $1 + 5 + 9 + 13 + ...$ to 30 terms.

The series is arithmetic with $u_1 = 1$, $d = 4$, and $n = 30$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore\ S_{30} = \frac{30}{2}(2 \times 1 + 29 \times 4)$

$= 1770$

4 Find the sum of:

a $3 + 7 + 11 + 15 + ...$ to 10 terms

b $40 + 35 + 30 + 25 + ...$ to 15 terms

c $8 + 11 + 14 + 17 + ...$ to 50 terms

d $-6 + 3 + 12 + 21 + ...$ to 40 terms

e $21 + 19 + 17 + 15 + ...$ to 60 terms

f $7 + 1 + (-5) + (-11) + ...$ to 30 terms

g $5 + 5\frac{1}{2} + 6 + 6\frac{1}{2} + ...$ to 25 terms

h $20 + 19\frac{1}{2} + 19 + 18\frac{1}{2} + ...$ to 50 terms.

5 An arithmetic sequence has $S_1 = 4$ and $S_2 = 11$. Find S_{40}.

6 An arithmetic sequence has $S_1 = -7$ and $S_3 = 0$. Find S_{30}.

Example 9 Self Tutor

Find the sum of $6+10+14+18+....+102$.

The series is arithmetic with $u_1=6$, $d=4$, and $u_n=102$.

First we need to find n.

$$u_n = u_1+(n-1)d$$
$$\therefore\ 102 = 6+(n-1)(4)$$
$$\therefore\ 6+4(n-1) = 102$$
$$\therefore\ 4n = 100$$
$$\therefore\ n = 25$$

Now $S_n = \frac{n}{2}(u_1+u_n)$

$$\therefore\ S_{25} = \tfrac{25}{2}(6+102) = 1350$$

7 Consider the series $4+9+14+19+....+119$.

a How many terms are in the series?

b Find the sum of the series.

8 Find the sum of:

a $7+9+11+13+....+55$

b $10+13+16+19+....+100$

c $87+83+79+75+....+15$

d $-5+1+7+13+....+109$

e $12+7+2+(-3)+....+(-58)$

f $6+7\frac{1}{2}+9+10\frac{1}{2}+....+30$

9 **a** Show that the sum of the first n multiples of 4 is $2n(n+1)$.

b Hence, find $4+8+12+16+....+80$.

10 Jim is saving money to buy a car. He puts €20 in the bank in the first week, then €25 in the second week, then €30 in the third week, and so on.

a How much will Jim put into the bank in the 10th week?

b How much money will Jim have saved in total after 20 weeks?

11 Consider the sequence $3, 5, 7, 9, ...$

The sum of the first n terms of the sequence is 120. Find the value of n.

DISCUSSION

Is it possible to find the sum of an infinite arithmetic series?

F GEOMETRIC SERIES

A **geometric series** is the sum of the terms of a geometric sequence.

For example: $1, 2, 4, 8, 16, ..., 1024$ is a geometric sequence.

$1+2+4+8+16+....+1024$ is a geometric series.

If we are adding the first n terms of a geometric sequence, we say we have a **finite geometric series**.

If we are adding all of the terms of a geometric sequence which goes on and on forever, we say we have an **infinite geometric series**.

SUM OF A FINITE GEOMETRIC SERIES

The sum of the first n terms of a geometric series is $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-1}$

For a finite geometric series with $\boldsymbol{r \neq 1}$,

$$S_n = \frac{u_1(r^n - 1)}{r-1} \quad or \quad S_n = \frac{u_1(1-r^n)}{1-r}.$$

It is easiest to use the first formula for $r > 1$ and the second formula for $r < 1$.

Proof:

$$\text{If } S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1} \quad (*)$$

$$\text{then } rS_n = (u_1r + u_1r^2 + u_1r^3 + u_1r^4 + ... + u_1r^{n-1}) + u_1r^n$$

$$\therefore rS_n = (S_n - u_1) + u_1r^n \quad \{\text{from } (*)\}$$

$$\therefore rS_n - S_n = u_1r^n - u_1$$

$$\therefore S_n(r-1) = u_1(r^n - 1)$$

$$\therefore S_n = \frac{u_1(r^n-1)}{r-1} \quad \text{or} \quad \frac{u_1(1-r^n)}{1-r} \quad \text{provided } r \neq 1.$$

Example 10 | **Self Tutor**

Find the sum of:

a $3+6+12+24+....$ to 10 terms

b $8-4+2-1+....$ to 7 terms.

a The series is geometric with $u_1 = 3$ and $r = 2$.

$$S_n = \frac{u_1(r^n-1)}{r-1}$$

$$\therefore S_{10} = \frac{3(2^{10}-1)}{2-1} = 3069$$

b The series is geometric with $u_1 = 8$ and $r = -\frac{1}{2}$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore S_7 = \frac{8(1-(-\frac{1}{2})^7)}{1-(-\frac{1}{2})} = \frac{8(\frac{129}{128})}{\frac{3}{2}} = \frac{43}{8}$$

EXERCISE 22F.1

1 Find $5+10+20+40+80+160$:

a by adding the terms directly

b using $S_n = \dfrac{u_1(r^n-1)}{r-1}$.

2 Find the sum of:

a $1+3+9+27+ ....$ to 8 terms

b $2+10+50+250+ ....$ to 10 terms

c $24+12+6+3+ ....$ to 9 terms

d $1+\sqrt{2}+2+2\sqrt{2}+ ....$ to 12 terms.

3 Find the sum of:

a $4-8+16-32+ ....$ to 10 terms

b $81-27+9-3+ ....$ to 8 terms

c $11\sqrt{11}+11+\sqrt{11}+1+ ....$ to 9 terms

d $16-4+1-\frac{1}{4}+ ....$ to 10 terms.

4 A geometric series has $S_1 = 2$ and $S_2 = 8$.

a Find the first term u_1 and the common ratio r.

b Find the sum of the first 10 terms of the series.

5 A geometric series has first term u_1 and common ratio $r=-1$.

Show that $S_n = \begin{cases} u_1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.

6 Doug is marooned on a desert island with only 2000 mL of fresh water.

He drinks 500 mL on the first day, but realises he will soon run out of water if he drinks that much each day. He decides that each day he will drink $\frac{3}{4}$ of the amount he drank the previous day.

a How much water will Doug drink on the:

i 5th day **ii** 10th day **iii** 15th day?

b Describe what happens to the amount of water Doug drinks each day.

c How much water will Doug have drunk in total after:

i 10 days **ii** 20 days **iii** 30 days?

SUM OF AN INFINITE GEOMETRIC SERIES

An **infinite geometric series** is the sum of the terms of a geometric sequence which continues indefinitely.

Examples of infinite geometric series include $2+8+32+128+ ....$ and $10+5+2\frac{1}{2}+1\frac{1}{4}+ .....$

If $\boldsymbol{r>1}$ **or** $\boldsymbol{r<-1}$, the terms in the series get larger and larger. The sum of the series becomes infinitely large, and cannot be found. We say that the series **diverges**.

If $\boldsymbol{-1<r<1}$, the terms in the series get smaller and smaller, and the sum of the series **converges** to a finite value.

Consider $S_n = \dfrac{u_1(1-r^n)}{1-r}$ as n gets very large.

For $-1<r<1$, r^n approaches zero, so S_n approaches the value $\dfrac{u_1}{1-r}$.

If $-1 < r < 1$, the sum of an infinite geometric series with first term u_1 and common ratio r is $S = \dfrac{u_1}{1-r}$.

Example 11 Self Tutor

Find the sum of $2 + 1 + \frac{1}{2} + \frac{1}{4} + ...$

This is an infinite geometric series with $u_1 = 2$ and $r = \frac{1}{2}$.

$-1 < r < 1$, so the series converges, and the sum is
$$\begin{aligned} S &= \frac{u_1}{1-r} \\ &= \frac{2}{1-\frac{1}{2}} \\ &= 4 \end{aligned}$$

EXERCISE 22F.2

1 Decide whether the following infinite geometric series will converge or diverge:

a $7 + 14 + 28 + 56 + ...$
b $6 + 3 + 1\frac{1}{2} + \frac{3}{4} + ...$
c $1 - \sqrt{3} + 3 - 3\sqrt{3} + ...$
d $80 - 8 + 0.8 - 0.08 + ...$

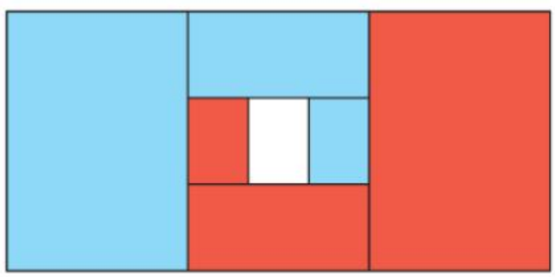
2 Consider the infinite geometric series $9 + 6 + 4 + \frac{8}{3} + ...$

a Find: **i** S_5 **ii** S_{10} **iii** S_{20}.
b Predict the sum of the infinite geometric series.
c Check your answer to **b** by finding the sum $S = \dfrac{u_1}{1-r}$.

3 Find the sum of:

a $16 + 8 + 4 + 2 + ...$
b $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + ...$
c $36 - 12 + 4 - \frac{4}{3} + ...$
d $32 + 24 + 18 + \frac{27}{2} + ...$
e $72 - 12 + 2 - \frac{1}{3} + ...$
f $0.6 + 0.06 + 0.006 + 0.0006 + ...$

4 Consider a rectangle with area 1 unit^2.
The rectangle is divided into thirds, with one third coloured blue, and another third coloured red. The rectangle is then rotated $90°$, and the process is repeated on the remaining unshaded third.
Suppose this process continues indefinitely.



a Show that the total blue shaded area $= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$
b Explain why the blue shaded area = red shaded area.
c Hence, explain why $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... = \frac{1}{2}$.
d Check this fact using $S = \dfrac{u_1}{1-r}$.

DEMO

Global context

click here

Sustainable farming

Statement of inquiry:	It is important that humans farm in a responsible manner to minimise the impact on the environment.
Global context:	Globalisation and sustainability
Key concept:	Relationships
Related concepts:	Model, Pattern
Objectives:	Investigating patterns, Applying mathematics in real-life contexts
Approaches to learning:	Thinking, Communication

REVIEW SET 22A

1 Write down the first five terms of the sequence:

a starting at 8, and increasing by 5 each time

b starting at 19, and decreasing by 7 each time

c whose nth term is the nth cubic number.

2 Find the first four terms of the sequence with recursive formula $u_1 = 2, \; u_n = 2u_{n-1} + 4, \; n \geqslant 2$.

3 Find the unknowns given that the following sequences are arithmetic:

a 9, 17, $\square$, 33, 41, ...

b 27, $\square$, 15, $\triangle$, 3, ...

4 Find k given that $k-2$, $2k-1$, and $4k-6$ are consecutive terms of an arithmetic sequence.

5 Write down the common ratio for these geometric sequences:

a 250, 200, 160, 128, ...

b 4, -12, 36, -108, ...

6 Find the general term u_n of the geometric sequence with $u_4 = 24$ and $u_7 = -192$.

7 Consider the sequence defined by $u_n = n^2 + 2n$. Find:

a the first 4 terms of the sequence

b S_4.

8 Find the sum of:

a $21 + 25 + 29 + 33 + ...$ to 20 terms

b $40 + 34 + 28 + 22 + ...$ to 30 terms.

9 Find the sum of:

a $5 + 20 + 80 + 320 + ...$ to 10 terms

b $18 - 12 + 8 - \frac{16}{3} + ...$ to 9 terms.

10 Find the sum of:

a $25 + 5 + 1 + \frac{1}{5} + ...$

b $24 - 12 + 6 - 3 + ...$

REVIEW SET 22B

1 Find the 8th term of the sequence with explicit formula:

a $u_n = 61 - 4n$

b $u_n = n^2 - 10$

2 Consider the sequence $-6, -2, 2, 6, ...$

a Show that the sequence is arithmetic.

b Find a formula for the general term u_n.

c Find the 100th term of the sequence.

d Find the largest term that is less than 500.

3 **a** Show that the sequence $64, -32, 16, -8, 4, ...$ is geometric.

b Find the 16th term of the sequence as a fraction.

4 **a** Find the first 5 terms of the sequence defined by the recursive formula $u_1 = 5, \; u_n = 12 - u_{n-1}, \; n \geqslant 2$.

b Hence, state the value of u_{100}.

5 An arithmetic series has 15 terms. The first term is -10, and the last term is 32. Find the sum of the series.

6 Find the general term u_n for the arithmetic sequence with $u_3 = 24$ and $u_{11} = -36$.

7 Find the sum of:

a $6 + 11 + 16 + 21 + ... + 101$

b $80 + 73 + 66 + 59 + ...$ to 20 terms

c $17 + 14 + 11 + 8 + ... + (-31)$

d $16 + 24 + 36 + 54 + ...$ to 10 terms.

8 $2k + 7, \; 1 - k,$ and $k - 7$ are consecutive terms of a geometric sequence.

a Find the possible values of k.

b For each possible value of k, find the corresponding common ratio.

9 A geometric series has $S_1 = 6$ and $S_2 = 3$.

a Find the first term u_1 and the common ratio r.

b Find the sum of the first 10 terms.

10 Answer the **Opening Problem** on page **414**.

Chapter 23

Inequalities

Contents:

In this course so far, we have mostly dealt with **equations** in which two expressions are separated by the equality sign $=$.

In this chapter we consider **inequalities** in which two expressions are separated by one of the four inequality signs $<$, $\leqslant$, $>$, or $\geqslant$.

We will learn how to **solve** linear inequalities using algebra, and how to **apply** linear inequalities to real world problems.

This leads us to **linear programming**, which is a method of solving problems in which at least two people or products compete for limited resources. From the **constraints** imposed by the limited resources, we can determine a set of **feasible solutions** to the problem. We then use an algorithm to identify the **optimal solution** according to a particular objective, such as maximising profit or minimising cost.

OPENING PROBLEM

To solve the inequality $8x - 4 < -2(x + 2)$, Trent performed these steps:

$$\begin{aligned} 8x - 4 &< -2(x+2) \\ \therefore\ 8x - 4 &< -2x - 4 \quad \{\text{expanding the brackets}\} \\ \therefore\ 8x &< -2x \quad \{\text{simplifying}\} \\ \therefore\ 8 &< -2 \quad \{\text{dividing both sides by } x\} \end{aligned}$$

which is clearly false, so there are no solutions.

His friend Donna pointed out that $x = -1$ is a solution to the inequality, so Trent must have done something wrong.

Things to think about:

a At what step was Trent's method wrong?

b What is the correct solution to this inequality?

A INTERVAL NOTATION

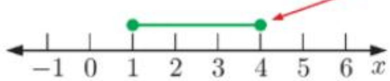
In **Chapter 2**, we used **interval notation** to describe a set of numbers.

For example, the set of real numbers from 1 to 4 inclusive can be represented by

$\{x \mid 1 \leqslant x \leqslant 4, \quad x \in \mathbb{R}\}$

the set of all such that x is real.

The filled circle shows that 4 is included.



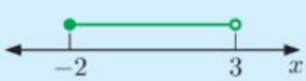
We normally assume we are dealing with real x, so the set can be represented simply as $\{x \mid 1 \leqslant x \leqslant 4\}$.

Example 1 Self Tutor

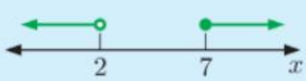
Draw a number line graph to display:

a $\{x \mid -2 \leqslant x < 3\}$ **b** $\{x \mid x < 2 \text{ or } x \geqslant 7\}$ **c** $\{x \mid x < 0 \text{ or } 1 \leqslant x < 4\}$

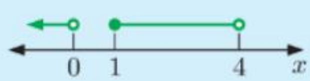
a



b



c



SQUARE BRACKET NOTATION

An alternative to using inequality signs is to use **square bracket notation**.

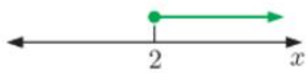
The endpoints of the interval are written within square brackets. The bracket is reversed if the endpoint is not included.

- $[a, b]$ represents the interval $\{x \mid a \leqslant x \leqslant b\}$

- $[a, b[$ represents the interval $\{x \mid a \leqslant x < b\}$

- $]a, b]$ represents the interval $\{x \mid a < x \leqslant b\}$

- $]a, b[$ represents the interval $\{x \mid a < x < b\}$



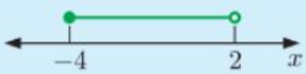
For intervals which extend to infinity, we use the symbol ∞. We always use an 'outwards' bracket for infinity. So, $[2, \infty[$ represents the interval $\{x \mid x \geqslant 2\}$.

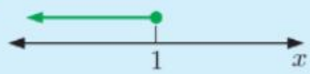
In square bracket notation, we use the union symbol $\cup$ to replace 'or'.

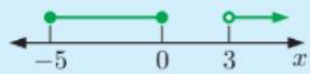
So, for $\{x \mid 1 \leqslant x < 3$ or $x \geqslant 5\}$ we would write $[1, 3[\cup [5, \infty[$.

Example 2 — Self Tutor

Use square bracket notation to describe:

a


b


c


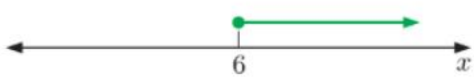
a $[-4, 2[$ **b** $]-\infty, 1]$ **c** $[-5, 0] \cup]3, \infty[$

EXERCISE 23A

1 Draw a number line graph to display:

a $\{x \mid x > 4\}$ **b** $\{x \mid x \leqslant -5\}$ **c** $\{x \mid -2 \leqslant x \leqslant 3\}$
d $\{x \mid 0 < x \leqslant 7\}$ **e** $\{x \mid x < 1$ or $x > 3\}$ **f** $\{x \mid x \leqslant 2$ or $x > 6\}$
g $[-1, 6]$ **h** $]4, 9]$ **i** $[-5, 0[$
j $]2, 8[$ **k** $[5, \infty[$ **l** $]-\infty, 6]$
m $]-3, \infty[$ **n** $[0, 4] \cup [7, \infty[$ **o** $]-\infty, 2[\cup [5, 11[$

2 Use interval notation to describe:

a


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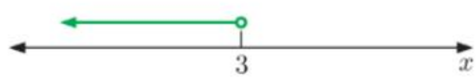
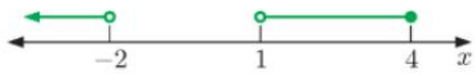


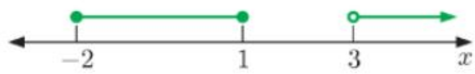
c


d


e 

f 

g 

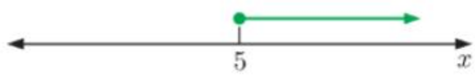
h 

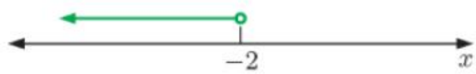
3 Write these number sets using square bracket notation:

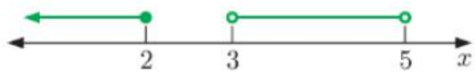
a $\{x \mid -1 \leqslant x \leqslant 6\}$
b $\{x \mid 0 < x < 5\}$
c $\{x \mid -4 < x \leqslant 7\}$
d $\{x \mid 4 \leqslant x < 8\}$
e $\{x \mid x > -7\}$
f $\{x \mid x \leqslant 0\}$
g $\{x \mid x \leqslant 2 \text{ or } x \geqslant 5\}$
h $\{x \mid x < -3 \text{ or } x > 4\}$
i $\{x \mid -1 < x \leqslant 1 \text{ or } x \geqslant 2\}$
j $\{x \mid x < -4 \text{ or } 2 \leqslant x < 7\}$

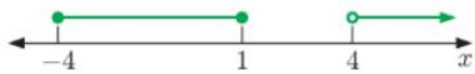
k 

l 

m 

n 

o 

p 

q 

r 

B LINEAR INEQUALITIES

Linear inequalities take the same form as linear equations, except they contain an inequality sign instead of an 'equals' sign.

$2x < 7$ and $3x + 5 \geqslant -10$ are examples of linear inequalities.

RULES FOR SOLVING LINEAR INEQUALITIES

Notice that $5 > 3$ and $3 < 5$,

and that $-3 < 2$ and $2 > -3$.

This suggests that if we **interchange** the LHS and RHS of an inequality, then we must **reverse** the inequality sign. $>$ is the reverse of $<$, and $\geqslant$ is the reverse of $\leqslant$.

You may also remember from previous years that:

- If we **add** or **subtract** the same number to both sides, the inequality sign is *maintained.*
  For example, if $5 > 3$, then $5 + 2 > 3 + 2$.
- If we **multiply** or **divide** both sides by a **positive** number, the inequality sign is *maintained.*
  For example, if $5 > 3$, then $5 \times 2 > 3 \times 2$.
- If we **multiply** or **divide** both sides by a **negative** number, the inequality sign is *reversed.*
  For example, if $5 > 3$, then $5 \times -1 < 3 \times -1$.

So, the method of solving linear inequalities is the same as that for linear equations, except that:

- **interchanging** the sides **reverses** the inequality sign
- **multiplying** or **dividing** both sides by a **negative** number **reverses** the inequality sign.

We should not multiply or divide both sides of an inequality by an expression involving the variable, unless we are certain that the expression is always positive or always negative. This is the mistake which Trent made in the **Opening Problem**.

Example 3 **Self Tutor**

Solve for x and graph the solution:

a $3x - 4 \leqslant 2$ **b** $3 - 2x < 7$

a $3x - 4 \leqslant 2$

$\therefore\ 3x \leqslant 6$ {adding 4 to both sides}

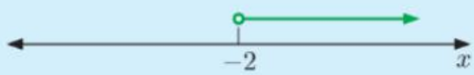
$\therefore\ x \leqslant 2$ {dividing both sides by 3}

Check: If $x = 1$ then $3x - 4 = 3 \times 1 - 4 = -1$ and $-1 \leqslant 2$ ✓

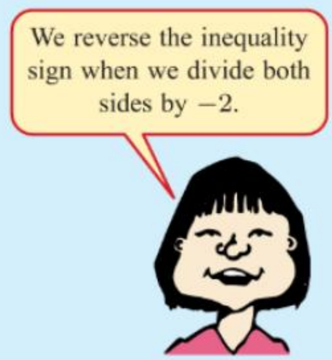
b $3 - 2x < 7$

$\therefore\ -2x < 4$ {subtracting 3 from both sides}

$\therefore\ x > -2$ {dividing both sides by -2, so reverse the sign}



Check: If $x = 3$ then $3 - 2x = 3 - 2 \times 3 = -3$ and $-3 < 7$ ✓



Example 4 **Self Tutor**

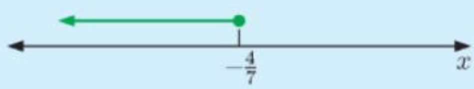
Solve for x and graph the solution: $3 - 5x \geqslant 2x + 7$

$3 - 5x \geqslant 2x + 7$

$\therefore\ 3 - 7x \geqslant 7$ {subtracting $2x$ from both sides}

$\therefore\ -7x \geqslant 4$ {subtracting 3 from both sides}

$\therefore\ x \leqslant -\frac{4}{7}$ {dividing both sides by -7, so reverse the sign}



EXERCISE 23B

1 Solve for x and graph the solution:

a $3x + 2 < 0$ **b** $5x - 7 > 2$ **c** $2 - 3x \geqslant 1$

d $5 - 2x \leqslant 11$ **e** $2(3x - 1) < 4$ **f** $5(1 - 3x) \geqslant 8$

2 Solve for x and graph the solution:

a $7 \geqslant 2x - 1$ **b** $-13 < 3x + 2$ **c** $20 > -5x$

d $-3 \geqslant 4 - 3x$ **e** $3 < 5 - 2x$ **f** $2 \leqslant 5(1 - x)$

3 Solve for x and graph the solution:

a $3x + 2 > x - 5$ **b** $2x - 3 < 5x - 7$ **c** $5 - 2x \geqslant x + 4$

d $7 - 3x \leqslant 5 - x$ **e** $3x - 2 > 2(x - 1) + 5x$ **f** $1 - (x - 3) \geqslant 2(x + 5) - 1$

C FEASIBLE REGIONS

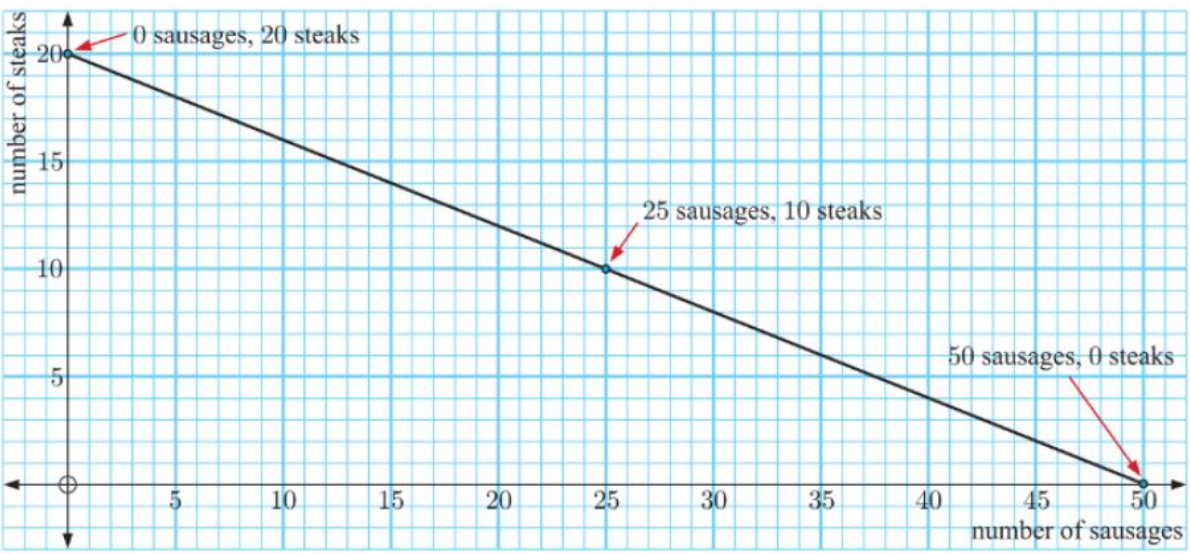
Jason and Kate are setting up a stall at the school fete. Their plan is to provide barbecued sausages and steaks for visitors at lunch time. Their butcher will sell them sausages for 40 cents each, and steaks for \$1.00 each. Jason and Kate have \$20 which they can spend on meat.

We can find several ways in which Jason and Kate can spend all of their \$20.

For example:

- If they buy 0 sausages and 20 steaks, the total cost $= 0 \times \$0.40 + 20 \times \$1 = \$20$.
- If they buy 50 sausages and 0 steaks, the total cost $= 50 \times \$0.40 + 0 \times \$1 = \$20$.
- If they buy 25 sausages and 10 steaks, the total cost $= 25 \times \$0.40 + 10 \times \$1 = \$20$.

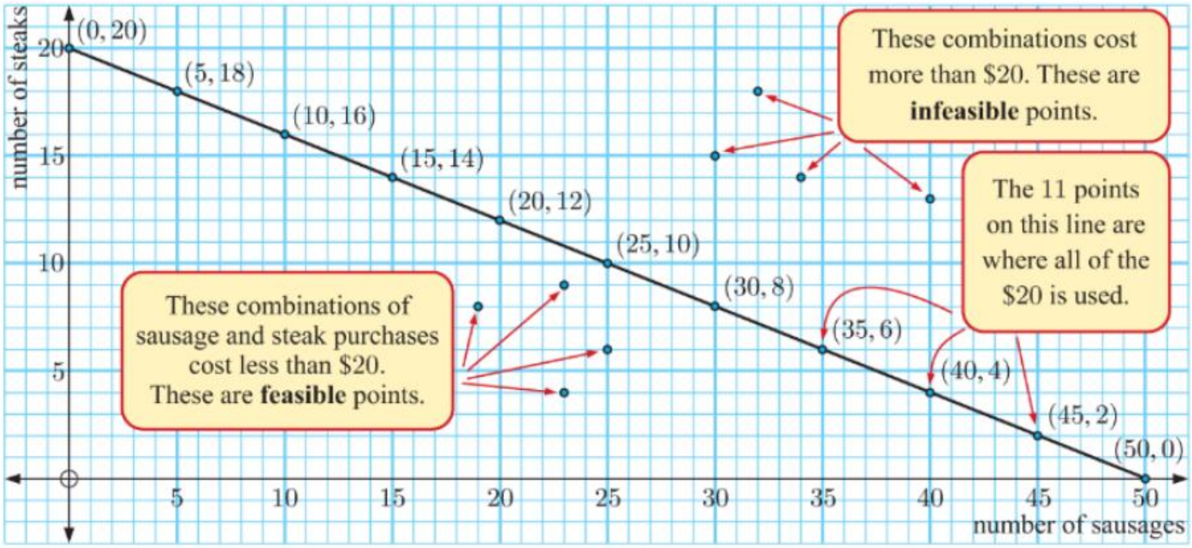
We can draw these solutions on a graph, and then use the graph to look for other solutions.



We notice that all solutions must be integer pairs, because we cannot buy a part of a sausage or steak. We therefore look along the line connecting our three points to see what other grid points it passes through. We can present these solutions in table form:

Sausages	0	5	10	15	20	25	30	35	40	45	50
Steaks	20	18	16	14	12	10	8	6	4	2	0

There are 11 different solutions for which exactly \$20 is spent. However, we still have not considered *all* the possible solutions, because there are many other feasible solutions for which Jason and Kate spend *less* than \$20. Some of those are shown on the graph below.



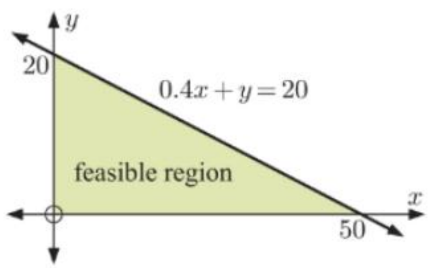
The information on the straight line can be written as an equation:

$$(\text{cost of a sausage}) \times (\text{number of sausages}) + (\text{cost of a steak}) \times (\text{number of steaks}) = \$20$$

If x represents the number of sausages and y represents the number of steaks, then this equation simplifies to

$$\$0.40 \times x + \$1 \times y = \$20 \qquad \textit{or} \qquad 0.4x + y = 20$$

The feasible points are **on or below** the line $0.4x + y = 20$. We write this as the inequality $0.4x + y \leqslant 20$, since Jason and Kate can spend less than or equal to \$20. Since we cannot have a negative number of sausages or steaks, the points must also satisfy $x \geqslant 0$ and $y \geqslant 0$.



The set of feasible points is shaded alongside, and is called the **feasible region**.

EXERCISE 23C.1

1 Suppose chops cost \$2 each, and sausages cost \$1 each. You have \$10 to spend on chops and sausages.

- **a** If you only bought chops, how many chops could you buy?
- **b** If you only bought sausages, how many sausages could you buy?
- **c** What other combinations could you buy using the whole \$10?
- **d** Graph the possible combinations found in **a**, **b**, and **c**.
- **e** Find the equation of the line through the points in **d**.
- **f** Shade the feasible region of possible purchases. Describe this region using *three* inequalities.

2 Suppose chocolates cost €2 each, and flowers cost €3 each. You have €24 available to spend on them.

a If you only bought chocolates, how many chocolates could you buy?

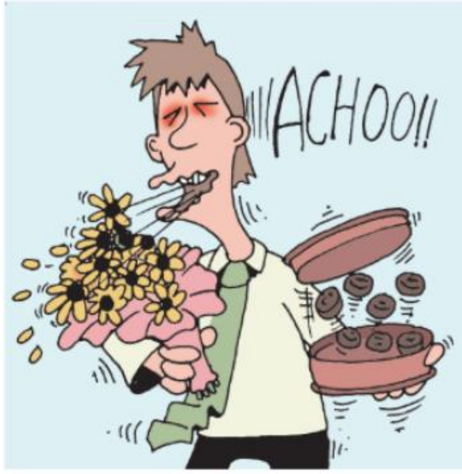
b If you only bought flowers, how many flowers could you buy?

c If the €24 is all used, what are the other combinations which could be purchased?

d Graph the possible combinations found in **a**, **b**, and **c**.

e Find the equation of the line through the points in **d**.

f Shade the feasible region of possible purchases. Describe this region using *three* inequalities.



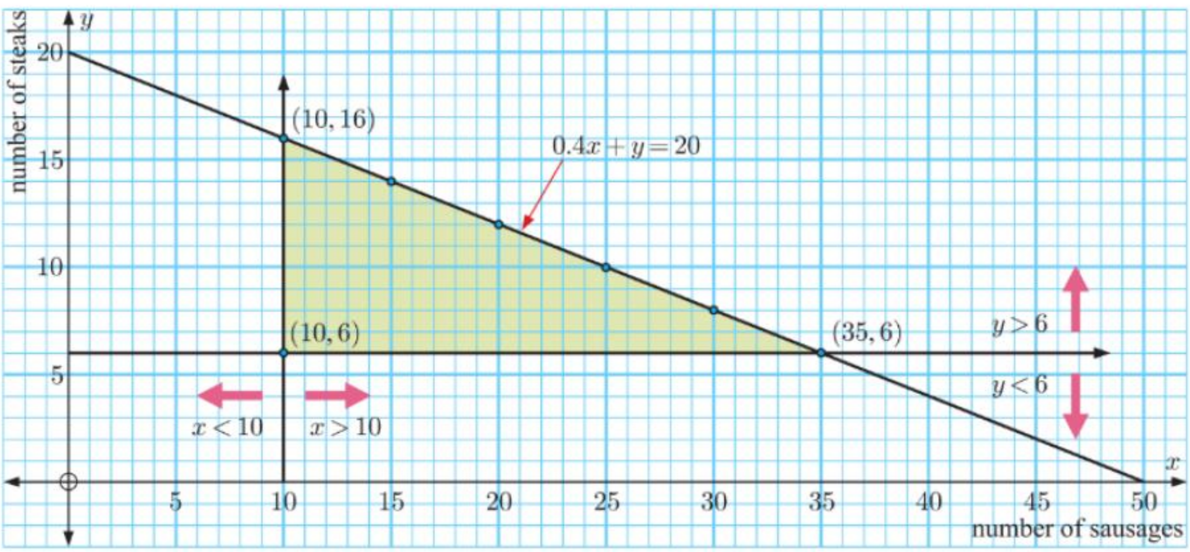
ADDITIONAL CONSTRAINTS

Consider again Jason and Kate's school barbecue. Suppose that when Jason and Kate go to the butcher to make their purchase, he says that he can only supply the sausages at \$0.40 each and the steaks at \$1 each if they buy *at least* 10 sausages and *at least* 6 steaks.

So, number of sausages $\geqslant 10$ and number of steaks $\geqslant 6$

$\therefore\ x \geqslant 10$ and $y \geqslant 6$

The three constraints are now $x \geqslant 10$, $y \geqslant 6$, and $0.4x + y \leqslant 20$.



To find which side of the boundary line $0.4x + y = 20$ the feasible region is, we can substitute a point.

For example, substituting $x = 0$, $y = 0$ into $0.4x + y \leqslant 20$ gives $0 \leqslant 20$, which is true. Since the inequality is satisfied at the origin, the feasible region is *below* the line $0.4x + y = 20$.

In the graph above, the feasible region is shaded green. All intersections of grid lines within or on the boundary of the region are possible combinations which satisfy the given constraints.

The feasible region is also called the **simplex**, and each corner point of the simplex is called a **vertex**.

Example 5 **Self Tutor**

Steve buys x pineapples for €2 each, and y watermelons for €3 each. He needs at least two of each, and can spend no more than €30. Graph the feasible region for what Steve can buy.

We first notice that $x \geqslant 2$ and $y \geqslant 2$. {Steve needs at least two of each}

The total cost of the pineapples is $€2 \times x = €2x$.

The total cost of the watermelons is $€3 \times y = €3y$.

$\therefore\ 2x + 3y \leqslant 30$ {Steve cannot spend more than €30}

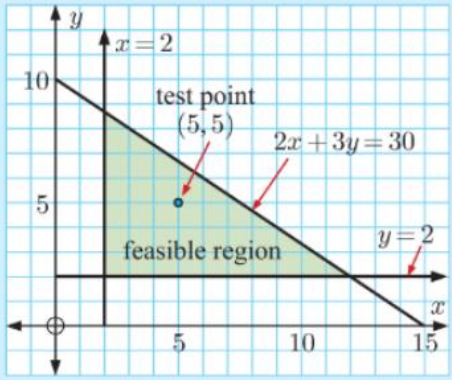
So, the feasible region is given by the constraints $x \geqslant 2$, $y \geqslant 2$, and $2x + 3y \leqslant 30$.

The boundary line of $2x + 3y \leqslant 30$ is $2x + 3y = 30$.

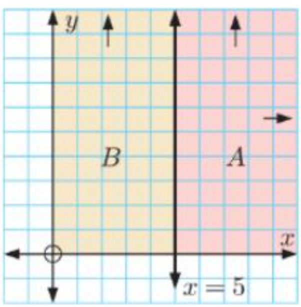
Using the test point $(5, 5)$,

$$\begin{aligned} 2x + 3y &= 2(5) + 3(5) \\ &= 25 \leqslant 30 \quad ✓ \end{aligned}$$

$(5, 5)$ satisfies $2x + 3y \leqslant 30$, so the feasible region is *below* the line $2x + 3y = 30$.



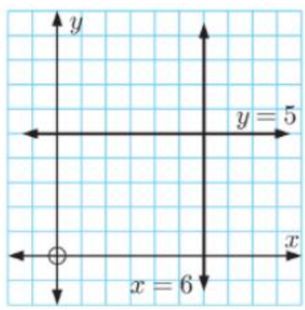
EXERCISE 23C.2

1 

Region A can be described using the constraints $x \geqslant 5$ and $y \geqslant 0$.
What constraints can be written to describe region B?

2 Display on a single grid, the following regions:

A: $x \geqslant 6$ and $y \geqslant 5$
B: $x \geqslant 6$ and $0 \leqslant y \leqslant 5$
C: $0 \leqslant x \leqslant 6$ and $0 \leqslant y \leqslant 5$



3 Sketch the simplex described by:

a $x \geqslant 0$
b $y \geqslant 0$
c $0 \leqslant x \leqslant 4$
d $0 \leqslant y \leqslant 7$
e $x \geqslant 2$ and $y \geqslant 3$
f $x \geqslant 2$ and $1 \leqslant y \leqslant 3$

4 At the supermarket I will buy x loaves of bread at \$2 a loaf, and y blocks of cheese at \$5 each. I need at least three of each item, and can spend at most \$40.

  a Find the constraints on the variables x and y.

  b Graph the feasible region described by the constraints.

5 Sarah will buy x watches and y clocks for her shop. The watches cost £25 each, and the clocks cost £40 each. She needs at least four watches and five clocks, and can spend at most £1000 on the purchase.

  a Find the constraints on the variables x and y.

  b Graph the simplex described by the constraints.

6 Sketch the region defined by:

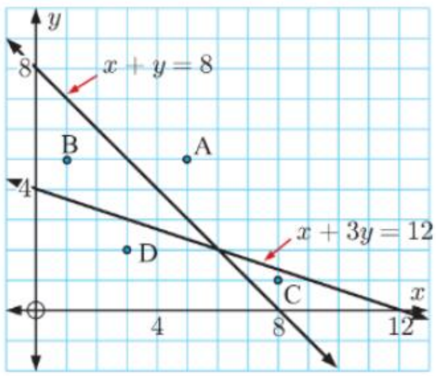
a $x+3y \leqslant 6$	**b** $x+y \geqslant 4$
c $2x+3y \geqslant 15$	**d** $5x+4y \leqslant 60$
e $4x+3y \leqslant 48$	**f** $8x+15y \geqslant 120$

Check your answers using the **region plotter** or your **graphics calculator**.

REGION PLOTTER

GRAPHICS CALCULATOR INSTRUCTIONS

7 The lines $x+y=8$ and $x+3y=12$ are illustrated.



  a Copy and complete this table:

Point	*Is* $x+y \leqslant 8$?	*Is* $x+3y \leqslant 12$?
A(5, 5)	No	
B(,)		
C(,)		
D(,)		

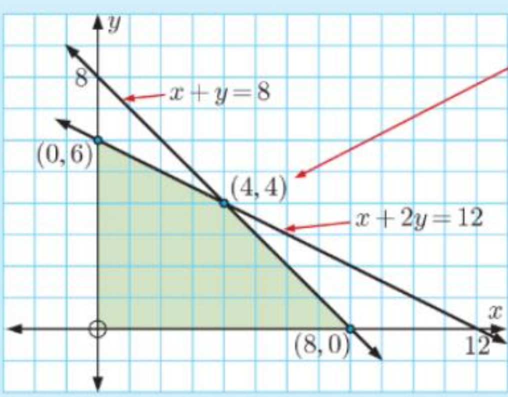
  b Draw the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 8$, $x+3y \leqslant 12$.

  c Using the same method, draw the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $x+y \geqslant 8$, $x+3y \geqslant 12$.

Example 6 **Self Tutor**

Graph the feasible region defined by: $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 8$, $x+2y \leqslant 12$

The boundary lines are: $x=0$, $y=0$, $x+y=8$, $x+2y=12$



The lines $x+y=8$ and $x+2y=12$ intersect at $(4, 4)$.

If a and b are positive then:
- $ax+by \leqslant c$ indicates on or **below** the line.
- $ax+by \geqslant c$ indicates on or **above** the line.

8 Graph the feasible region defined by:

a $x+4y \leqslant 12, \quad x+y \leqslant 6, \quad x \geqslant 0, \quad y \geqslant 0$

b $x+2y \leqslant 12, \quad 3x+2y \leqslant 24, \quad x \geqslant 0, \quad y \geqslant 0$

c $2x+y \leqslant 10, \quad x+y \leqslant 6, \quad x \geqslant 0, \quad y \geqslant 0$

d $x+2y \leqslant 22, \quad x+y \leqslant 12, \quad y \geqslant 2x, \quad x \geqslant 0, \quad y \geqslant 0.$

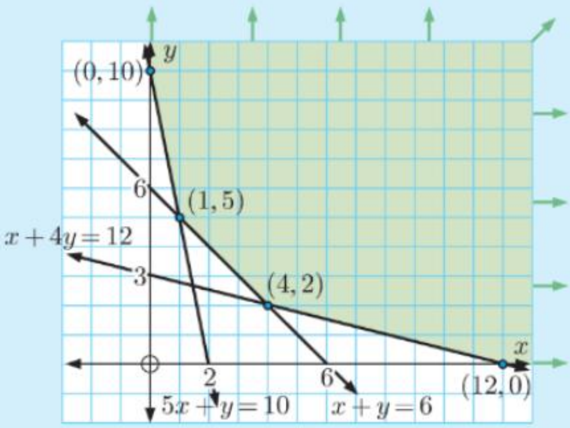
Check your answers using technology.

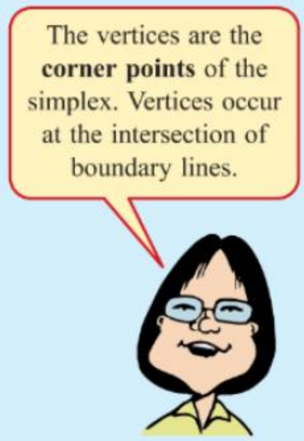
REGION PLOTTER

Example 7 Self Tutor

Graph the simplex defined by the following constraints, and find the coordinates of each vertex of the region: $x \geqslant 0, \; y \geqslant 0, \; x+4y \geqslant 12, \; 5x+y \geqslant 10, \; x+y \geqslant 6.$

The boundary lines are: $x=0, \; y=0, \; x+4y=12, \; 5x+y=10, \; x+y=6.$





The vertices of the region are $(0, 10)$, $(1, 5)$, $(4, 2)$, and $(12, 0)$.

9 Graph the simplex defined by the following constraints, and find the coordinates of each vertex of the region:

a $x+y \geqslant 8, \quad x+2y \geqslant 12, \quad x \geqslant 0, \quad y \geqslant 0$

b $2x+y \leqslant 8, \quad 4x+y \leqslant 12, \quad x \geqslant 0, \quad y \geqslant 0$

c $3x+y \geqslant 12, \quad 3x+2y \geqslant 18, \quad x+4y \geqslant 16, \quad x \geqslant 0, \quad y \geqslant 0.$

D CONSTRUCTING CONSTRAINTS

We have already constructed simple sets of constraints from given information. In more complicated situations, we can use a **table** to help us sort the information.

Example 8 **Self Tutor**

A cabinet manufacturer produces 2-drawer and 5-drawer filing cabinets. The manufacturer has 34 drawers, 8 locks, and 42 m^2 of sheet metal available. Each 2-drawer cabinet requires 1 lock and 2 m^2 of sheet metal. Each 5-drawer cabinet requires 1 lock and 4 m^2 of sheet metal.

Let x be the number of 2-drawer filing cabinets made and y be the number of 5-drawer filing cabinets made. Construct a set of constraints on x and y to represent this problem.

Type	*Number of drawers*	*Locks per cabinet*	*Metal per cabinet*	*Number made*
2-drawer	2	1	2 m^2	x
5-drawer	5	1	4 m^2	y

Now $x \geqslant 0,\ y \geqslant 0$ {neither x nor y can be negative}

Total number of drawers $= 2x + 5y$ $\quad \therefore \ 2x + 5y \leqslant 34$

Total number of locks $= x + y$ $\quad \therefore \ x + y \leqslant 8$

Total number of m^2 of metal $= 2x + 4y$ $\quad \therefore \ 2x + 4y \leqslant 42$

EXERCISE 23D

1 Base white paint can be made into lime green or pine green by adding yellow tint and blue tint in different proportions. For each litre can of lime green we add 5 units of yellow tint to one unit of blue tint. For each litre can of pine green we add 1 unit of yellow tint to 4 units of blue tint.
15 units of yellow tint and 12 units of blue tint are available.

- **a** State inequalities connecting x, the number of litre cans of lime green paint that can be made, and y, the number of litre cans of pine green paint that can be made.
- **b** Do x and y need to be integers in this case? Explain your answer.

2 A farmer has a week in which to plant lettuces and cauliflowers. Lettuces can be planted at a rate of 8 ha per day. Cauliflowers can be planted at a rate of 6 ha per day. 50 ha are available for planting.
Suppose the farmer plants lettuces for x days and cauliflowers for y days.

- **a** List, with reasons, the constraints involving x and y.
- **b** Do x and y need to be integers in this case?

3 Two varieties of dog food are sold in tins. Their nutritional information is shown in the table.

Food	*Carbohydrate*	*Protein*	*Vitamins*
Makemfast	10 units	30 units	500 units
Makemstrong	40 units	30 units	100 units

Each week a healthy dog must consume at least 120 units of carbohydrate, 180 units of protein, and 1000 units of vitamins.
Suppose x tins of Makemfast and y tins of Makemstrong are consumed by a healthy dog each week. List the inequalities connecting x and y.

E LINEAR PROGRAMMING

Linear programming is a method for finding the **optimal value** (maximum or minimum) of a linear expression whose variables are contained within a simplex.

CASE STUDY

Two varieties of special food, Fight-n-fit and Superlite, are used by athletes.

Fight-n-fit contains 30 units of carbohydrate, 30 units of protein, and 100 units of vitamins. Superlite contains 10 units of carbohydrate, 30 units of protein, and 200 units of vitamins.

Each week an athlete must consume at least 170 units of carbohydrate and at least 1400 units of vitamins, but no more than 330 units of protein.

Tins of Fight-n-fit cost \$5 each, and tins of Superlite cost \$3 each.

Athlete Enrique wants to satisfy his dietary needs while minimising his costs. We suppose he consumes x tins of Fight-n-Fit and y tins of Superlite each week.

We can now construct the table:

Food	*Carbohydrate*	*Protein*	*Vitamins*	*Tins bought*
Fight-n-fit	30	30	100	x
Superlite	10	30	200	y
Total needed	$\geqslant 170$	$\leqslant 330$	$\geqslant 1400$	

When buying x tins of Fight-n-fit and y tins of Superlite, $x \geqslant 0$ and $y \geqslant 0$.

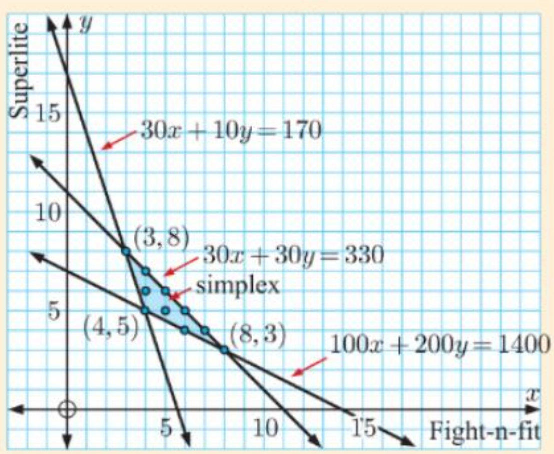
Total carbohydrate $= 30x + 10y$ $\quad \therefore \; 30x + 10y \geqslant 170$

Total protein $= 30x + 30y$ $\quad \therefore \; 30x + 30y \leqslant 330$

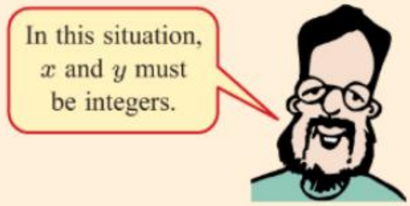
Total vitamins $= 100x + 200y$ $\quad \therefore \; 100x + 200y \geqslant 1400$

The boundary lines are:

$$x = 0, \;\; y = 0, \;\; 30x + 10y = 170, \;\; 30x + 30y = 330, \;\text{ and }\; 100x + 200y = 1400.$$



The ten illustrated points are the only possible feasible combinations to consider.



We now need to determine which of these feasible combinations will minimise Enrique's total cost. To do this we will find the cost of each feasible combination, and identify which is the lowest.

The cost to be considered is $5x + 3y$ dollars, and is called the **objective function**.

\$5 per tin x tins \$3 per tin y tins

Possible feasible points	*Cost* $= 5x + 3y$
(3, 8)	$5(3) + 3(8) = 39$
(4, 5)	$5(4) + 3(5) = 35$
(4, 6)	$5(4) + 3(6) = 38$
(4, 7)	$5(4) + 3(7) = 41$
(5, 5)	$5(5) + 3(5) = 40$
(5, 6)	$5(5) + 3(6) = 43$
(6, 4)	$5(6) + 3(4) = 42$
(6, 5)	$5(6) + 3(5) = 45$
(7, 4)	$5(7) + 3(4) = 47$
(8, 3)	$5(8) + 3(3) = 49$

The minimum cost is \$35 when we use 4 tins of Fight-n-fit and 5 tins of Superlite.

DISCUSSION

Consider what you have observed in the **Case Study**.

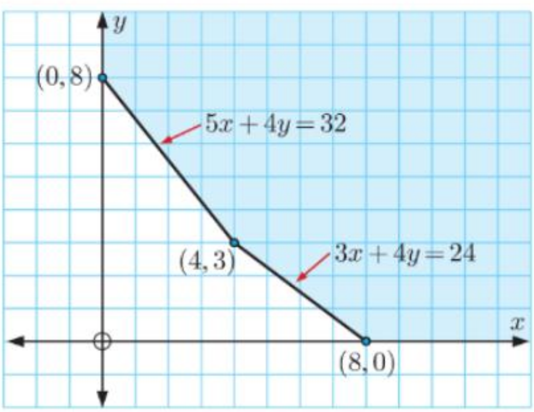
- The optimal solution occurred at (4, 5), which is a vertex of the feasible region. Will this always be the case?
- If the prices of the different tins were changed, would the optimal solution change?
- Imagine a simplex where there are hundreds of feasible points. Is the above method of optimisation still acceptable?

THE SIMPLEX ALGORITHM

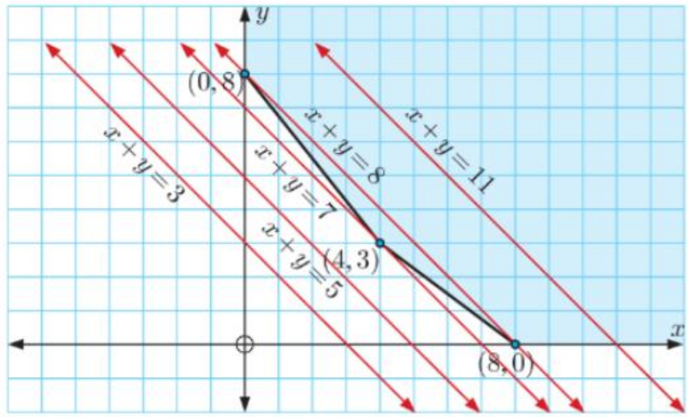
From experimenting with problems like the **Case Study**, we notice that:

The **optimum value** of a linear expression over a simplex occurs at a vertex of the simplex.

Suppose we wish to minimise the function $x + y$ over the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $5x + 4y \geqslant 32$, $3x + 4y \geqslant 24$.



On this graph we add five lines of the form $x + y = k$ where $k = 3, 5, 7, 8$, and 11.



We see that the lines $x + y = 3$ and $x + y = 5$ do not contain any points in the simplex. As the value of k increases, the lines get closer to the simplex.

The smallest value of k for which the objective function passes through the simplex, is $k = 7$. This occurs at the vertex $(4, 3)$. So, $(4, 3)$ is the point in the feasible region which gives the smallest value of $x + y$, and this value is 7.

All other points in the feasible region will give a higher value of $x + y$, as illustrated by the lines $x + y = 8$ and $x + y = 11$.

So, when we are trying to maximise or minimise the objective function, we do not have to evaluate the objective function at every feasible point. We need only evaluate the objective function at the *vertices* of the feasible region.

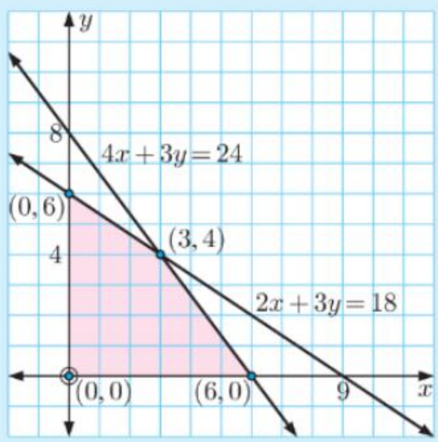
Example 9 **Self Tutor**

Find the maximum value of $5x + 4y$ over the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $2x + 3y \leqslant 18$, $4x + 3y \leqslant 24$. State the values of x and y where this maximum value occurs.

We first draw the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $2x + 3y \leqslant 18$, $4x + 3y \leqslant 24$.

The simplex has vertices $(0, 0)$, $(0, 6)$, $(3, 4)$, and $(6, 0)$.

Vertex	$5x + 4y$	
$(0, 0)$	$5(0) + 4(0) = 0$	
$(0, 6)$	$5(0) + 4(6) = 24$	
$(3, 4)$	$5(3) + 4(4) = 31$	← maximum
$(6, 0)$	$5(6) + 4(0) = 30$	



So, the maximum value of $5x + 4y$ over the simplex is 31, occurring when $x = 3$ and $y = 4$.

EXERCISE 23E.1

1 **a** Sketch the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $2x + y \leqslant 8$, $x + 2y \leqslant 10$.

b State the coordinates of the vertices of the simplex.

c Find the maximum value of $x + y$ over the simplex, and the values of x and y where the maximum value occurs.

2 **a** Sketch the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $x + 3y \leqslant 15$, $3x + y \leqslant 21$.

b Find the maximum value of each of the following linear expressions over the simplex. State the coordinates of the point where each maximum occurs.

i $x + y$ **ii** $2x + y$ **iii** $6y$ **iv** $4x + y$

3 **a** Sketch the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $2x + 3y \geqslant 12$, $4x + y \geqslant 14$.

b Find the minimum value of each of the following linear expressions over the simplex. State the coordinates of the point where each minimum occurs.

i $x + 5y$ **ii** $4x + 5y$ **iii** $6x + y$ **iv** $5x$

c Can you find the *maximum* value of any of the objective functions in **b**? Explain your answer.

d Add the additional constraint $2x + y \leqslant 12$ to redefine the feasible region. For the new feasible region, find the *maximum* value of each of the objective functions in **b**.

PROBLEM SOLVING USING LINEAR PROGRAMMING

We are now equipped to solve linear programming problems involving two variables x and y.

In most linear programming problems, the variables cannot be negative, and so $x \geqslant 0$, $y \geqslant 0$.

Following is a useful checklist of the steps required:

Step 1: In sentences, explain or define the **two variables** being considered.

Step 2: Prepare a **table** to display the given information.

Step 3: Write the **objective function** in algebraic form, and state whether you are trying to maximise or minimise it.

Step 4: State the **constraints** in algebraic form.

Step 5: Draw the **graph** of the simplex given by the constraints.

Step 6: Find the **optimum solution** and when it occurs. Give your answer in sentence form.

The following problems can all be solved by hand.

However, you are encouraged to use technology to help with their solution.

LINEAR PROGRAMMING

GRAPHICS CALCULATOR INSTRUCTIONS

Example 10 **Self Tutor**

Each week a cat needs at least 225 units of carbohydrate, 80 units of protein, and 90 units of fat.

Two tins of cat food, A and B, are analysed.

A contains 25 units of carbohydrate, 10 units of protein, and 15 units of fat.

B contains 50 units of carbohydrate, 10 units of protein, and 9 units of fat.

Tin A costs \$6, and tin B costs \$3.

a Find the cheapest combination of A and B which provides the necessary nutrients.

b Discuss the cat's diet for this optimal combination.

a *Step 1*: Let x and y be the numbers of tins of A and B used each week respectively.

Step 2:

Type	*Carbohydrate*	*Protein*	*Fat*	*Tins used*	*Cost*
A	25 units	10 units	15 units	x	\$6
B	50 units	10 units	9 units	y	\$3

Step 3: We need to minimise the total cost, $(6x+3y)$ dollars.

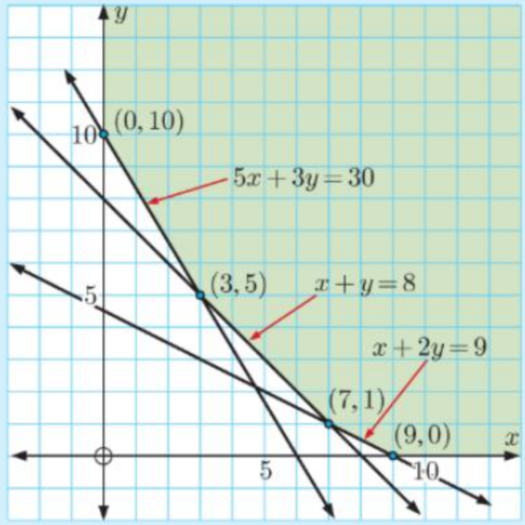
Step 4: $x \geqslant 0$ and $y \geqslant 0$

$25x + 50y \geqslant 225 \quad \therefore \quad x + 2y \geqslant 9$ {dividing each term by 25}

$10x + 10y \geqslant 80 \quad \therefore \quad x + y \geqslant 8$ {dividing each term by 10}

$15x + 9y \geqslant 90 \quad \therefore \quad 5x + 3y \geqslant 30$ {dividing each term by 3}

Step 5: The simplex is shown alongside.



Step 6:

Vertex	$6x+3y$	
(0, 10)	30	← minimum
(3, 5)	33	
(7, 1)	45	
(9, 0)	54	

The minimum cost of \$30 is obtained when we use 0 tins of A and 10 tins of B.

b If the cat is fed 10 tins of B, then it receives the minimum 90 units of fat, but receives 500 units of carbohydrate (more than double the minimum) and 100 units of protein (20 units more than the minimum).

EXERCISE 23E.2

1 A chemical factory makes two different chemicals A and B. It can sell all that it is able to produce. The profit is \$300 per kg of A, and \$400 per kg of B. The factory has standing orders for 200 kg of A and 100 kg of B which it must fill. Due to the limited size of the factory, it can produce at most 700 kg of chemicals in total.

Suppose the factory produces x kg of A and y kg of B. How many kilograms of each should be produced to maximise the profit?

2 A food analyst is supplied with two containers of pet food, one of Foodo and the other of Petmix. The composition of one scoop of food from each of the two containers is shown below:

Food	*Protein*	*Fat*	*Carbohydrate*	*Fibre*
1 scoop of Foodo	12 g	4 g	24 g	5 g
1 scoop of Petmix	4 g	8 g	16 g	5 g

The analyst knows that an animal requires at least 96 g of protein, 80 g of fat, and 288 g of carbohydrate, but should not have more than 100 g of fibre each day.

a Suppose the analyst mixes x scoops of Foodo with y scoops of Petmix. Write down the system of inequalities satisfied by x and y. Hence graph the corresponding feasible region.

b If a scoop of Foodo costs \$2 and a scoop of Petmix costs \$1, find the mixture which provides the cheapest food.

c Could the animal be fed a satisfactory diet using:
  i Foodo only ii Petmix only?

3 Stephen manufactures two models of wheelbarrow, Deluxe and Standard.
For the Deluxe model he requires the use of machine A for 2 minutes and machine B for 3 minutes.
For the Standard model he requires machine A for 4 minutes and machine B for 3 minutes.
Machine A is available for at most 44 minutes and machine B for 42 minutes every hour.

He must make at least as many of the Standard model as the Deluxe model. The Deluxe model earns him \$25 profit, and the Standard model earns him \$20 profit.

  a i How many of each wheelbarrow model should Stephen produce per hour in order to maximise his profit?
    ii Which machine is fully used in this case?
    iii For how long is the other machine idle each hour?
  b Due to wearing parts, machine B is now only available for 36 minutes every hour.
    i How many wheelbarrows of each model should Stephen produce now?
    ii Discuss the use of machines A and B in this case.

MULTIPLE OPTIMAL SOLUTIONS

If the optimal solution occurs on **two** of the vertices, the problem has multiple solutions. Both of these vertices, as well as any valid points on the line joining the vertices, will be optimal solutions to the problem.

Example 11 — Self Tutor

A store can sell every gift basket it is able to make. Small baskets contain 5 chocolates, 1 balloon, and 12 roses. Large baskets contain 10 chocolates, 6 balloons, and 12 roses.

The shop has access to 80 chocolates, 36 balloons, and 168 roses. The small baskets sell for \$20, and the large baskets sell for \$40.

What combination of baskets should the store make, to maximise their income?

Step 1: Let x be the number of small baskets made.
Let y be the number of large baskets made.

Step 2:

Type	*Chocolates*	*Balloons*	*Roses*	*Number of baskets*	*Price*
Small	5	1	12	x	\$20
Large	10	6	12	y	\$40

Step 3: We want to maximise the income of $(20x + 40y)$ dollars.

Step 4: $x \geqslant 0$ and $y \geqslant 0$
$5x + 10y \leqslant 80 \quad \therefore \; x + 2y \leqslant 16$ {dividing each term by 5}
$x + 6y \leqslant 36$
$12x + 12y \leqslant 168 \quad \therefore \; x + y \leqslant 14$ {dividing each term by 12}

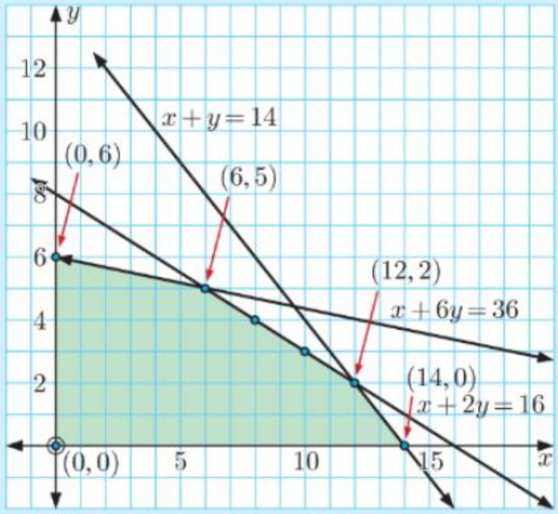
Step 5: The simplex is shown alongside.

Step 6:

Vertex	$20x + 40y$
(0, 0)	0
(0, 6)	240
(6, 5)	320
(12, 2)	320
(14, 0)	280

maximum

The maximum income of \$320 occurs at (6, 5), (8, 4), (10, 3), and (12, 2) since only integer solutions are valid.



The store should make 6 small and 5 large baskets *or* 8 small and 4 large baskets *or* 10 small and 3 large baskets *or* 12 small and 2 large baskets.

It does not matter to the store which of these optimal combinations it makes. All of the optimal combinations satisfy the constraints, and produce the maximum income of \$320.

Multiple solutions exist when the objective function is parallel to one of the constraint lines.

In the previous Example, the objective function $20x + 40y = k$ is parallel to the constraint line $x + 2y = 16$.

EXERCISE 23E.3

1 Three rare ingredients a, b, and c are used in two kinds of hair restorer X and Y.
Only 18 units of a, 21 units of b, and 18 units of c are available each week.
One bottle of X requires 2 units of a, 3 units of b, and 3 units of c.
One bottle of Y requires 2 units of a, 2 units of b, and 1 unit of c.

- **a** If X earns €20 profit per bottle and Y earns €12.50 profit, how many bottles should be made per week to maximise profits?
- **b** If the profit for Y rose to €20 per bottle, how much of each should now be produced to maximise profits?

2 A manufacturer produces two kinds of table tennis sets.
Set A contains 2 bats and 3 balls.
Set B contains 2 bats, 5 balls, and 1 net.
In one hour the factory can produce at most 56 bats, 108 balls, and 18 nets. Set A earns a profit of £3, and set B earns a profit of £5.

- **a** Find the optimal combinations of table tennis sets that the manufacturer should produce per hour to maximise his profit.
- **b** Of these optimal combinations, the manufacturer decides to produce the combination for which bat production is at full capacity.
  - **i** How many of each set is produced?
  - **ii** Which component is under-utilised in this case?

3 

Lenny owns a limousine hire company. He has two types of vehicle available for hire. The regular limousine holds 6 passengers, comes with a driver, and costs \$60 per night to run. The deluxe limousine holds 10 passengers, comes with a driver and a butler, and costs \$90 per night to run. Lenny has 20 staff, each of whom can work as a driver or a butler. His budget is \$1020 per night for running costs.

Each night Lenny must use a combination of limousines which can carry a total of at least 60 passengers.

- **a** On Sunday nights, Lenny earns \$400 per night from each regular limousine, and \$600 per night from each deluxe limousine. What combinations of limousines will maximise Lenny's earnings?
- **b** On Saturday nights, Lenny's earnings increase to \$600 and \$1200 respectively. What combinations of limousines will maximise Lenny's earnings?
- **c** What combination of limousines will maximise Lenny's earnings, regardless of which day it is?

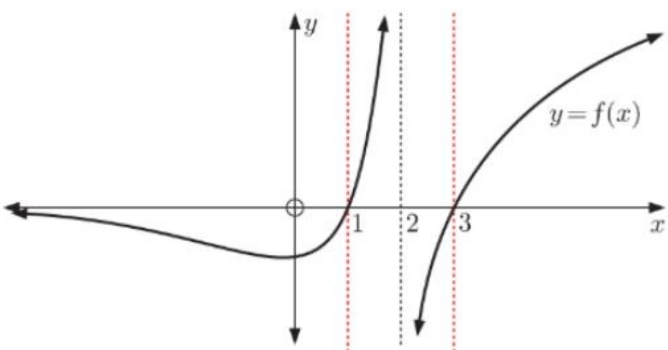
F SIGN DIAGRAMS

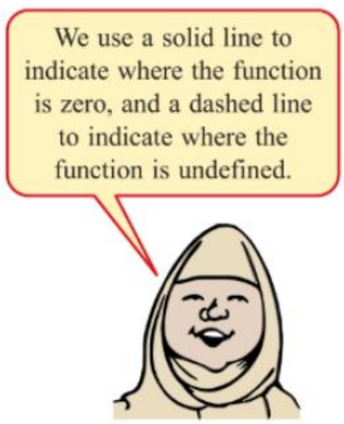
For some work involving functions, we do not need a complete graph of the function, but instead only need to know when the function is positive, negative, zero, or undefined. This situation corresponds to an inequality such as $f(x) \geqslant 0$ or $f(x) \leqslant 0$. A **sign diagram** enables us to do this.

A sign diagram consists of:

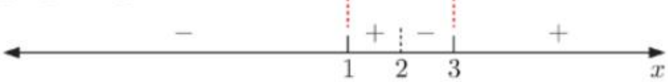
- a **horizontal line** which represents the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating where the graph is **above** and **below** the x-axis respectively
- **critical values**, which are the graph's x-intercepts, or where it is undefined.

Consider the graph:

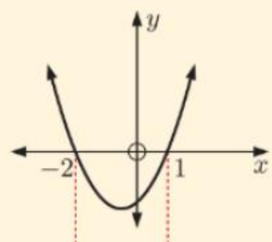
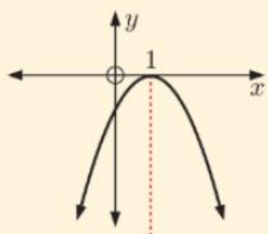
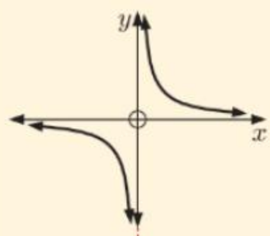
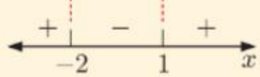
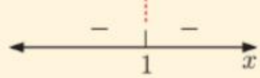
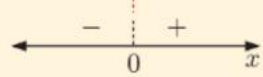




The corresponding sign diagram is:



Further examples are:

Function	$y = (x+2)(x-1)$	$y = -2(x-1)^2$	$y = \frac{4}{x}$
Graph			
Sign diagram			

Notice that:

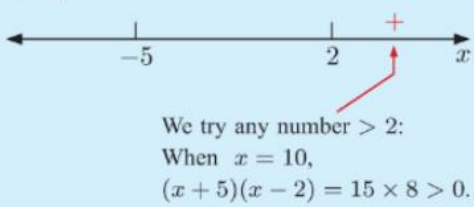
- A sign change occurs about the critical value for single factors such as $(x+2)$ and $(x-1)$, indicating **cutting** of the x-axis.
- No sign change occurs about the critical value for squared factors such as $(x-1)^2$, indicating **touching** of the x-axis.

Example 12 **Self Tutor**

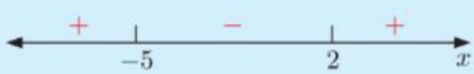
Draw a sign diagram for:

a $(x+5)(x-2)$ **b** $3(2x+1)(4-x)$

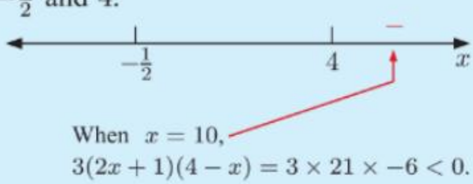
a $(x+5)(x-2)$ has critical values -5 and 2.



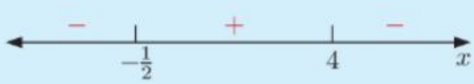
The factors are single so the signs alternate.
The sign diagram is



b $3(2x+1)(4-x)$ has critical values $-\frac{1}{2}$ and 4.



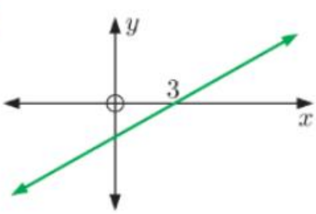
The factors are single so the signs alternate.
The sign diagram is



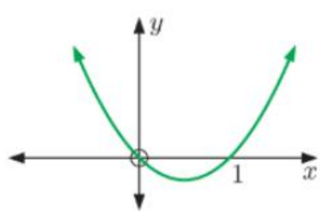
EXERCISE 23F

1 Draw a sign diagram for these graphs:

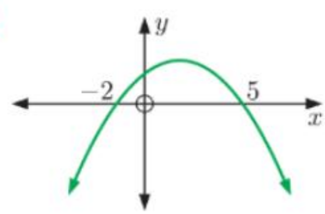
a



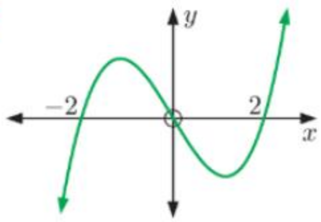
b



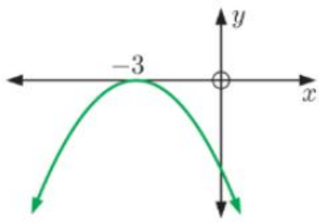
c



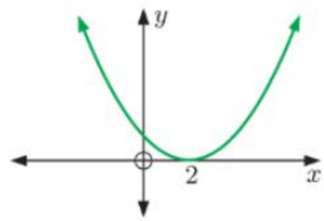
d



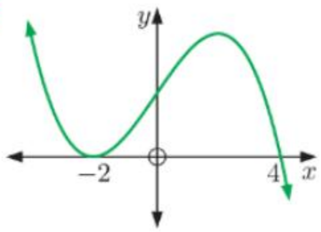
e



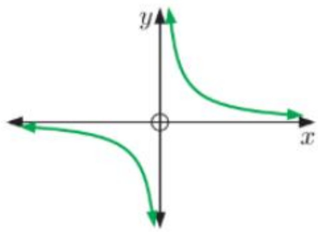
f



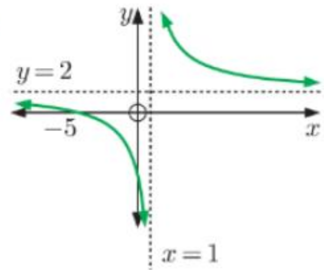
g



h



i



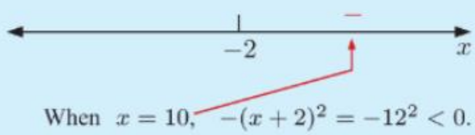
Example 13 **Self Tutor**

Draw a sign diagram for:

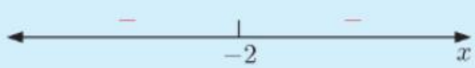
a $-(x+2)^2$

b $\dfrac{x-3}{x+1}$

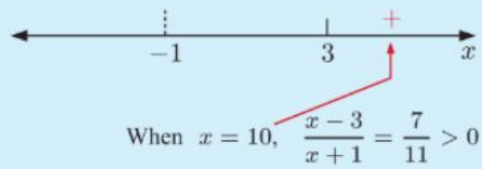
a $-(x+2)^2$ has critical value -2.



A squared factor indicates no change of sign about the critical value.
The sign diagram is



b $\dfrac{x-3}{x+1}$ is zero when $x = 3$ and undefined when $x = -1$.



Since $(x-3)$ and $(x+1)$ are single factors, the signs alternate.
The sign diagram is

2 Draw a sign diagram for:

a $(x+3)(x-1)$ **b** $x(x-4)$ **c** $x(x+5)$ **d** $-(x+2)(x-3)$

e $(3x-1)(4-x)$ **f** $(3-x)(1-2x)$ **g** x^2-16 **h** $1-x^2$

i $2x-x^2$ **j** x^2-4x+3 **k** $2-18x^2$ **l** $-x^2+2x+24$

3 Draw a sign diagram for:

a $(x-1)^2$ **b** $(x+4)^2$ **c** $-(x+3)^2$ **d** $-(x-2)^2$

e $(3x-1)^2$ **f** x^2-6x+9 **g** $-x^2-4x-4$ **h** $-x^2+2x-1$

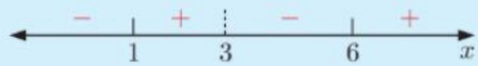
4 Draw a sign diagram for:

a $\dfrac{x+1}{x-2}$ **b** $\dfrac{x-1}{x}$ **c** $\dfrac{2x+5}{2-x}$ **d** $\dfrac{3x-1}{4-x}$

e $\dfrac{x+4}{3x-2}$ **f** $\dfrac{(x-2)^2}{x+1}$ **g** $\dfrac{3x}{(x+3)^2}$ **h** $\dfrac{x(x+2)}{4-x}$

Example 14 **Self Tutor**

Given the sign diagram alongside, use interval notation to describe where the function is:



a >0 **b** $\geqslant 0$

a The function is >0 where $1 < x < 3$ or $x > 6$.
Using interval notation, this is $\{x \mid 1 < x < 3 \text{ or } x > 6\}$.

b The function is $\geqslant 0$ where $1 \leqslant x < 3$ or $x \geqslant 6$.

We do not include 3 here because the function is undefined at $x = 3$.

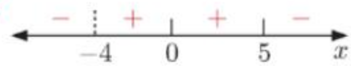
Using interval notation, this is $\{x \mid 1 \leqslant x < 3 \text{ or } x \geqslant 6\}$.

5 Use interval notation to describe where the functions with these sign diagrams are:

i >0 **ii** $\geqslant 0$ **iii** <0 **iv** $\leqslant 0$.

a


b


c


REVIEW SET 23A

1 Draw a number line graph to display:

a $\{x \mid x > -3\}$ **b** $\{x \mid 5 \leqslant x < 10\}$ **c** $]0, 9]$

2 Solve for x and graph the solution:

a $5x - 11 > -7$ **b** $9 \leqslant 4 - 2x$

3 Solve for x:

a $4x - 1 > x + 9$ **b** $5(1 - x) \leqslant 4(x + 3)$

4 Sketch the region described by:

a $x \geqslant 3$ and $y \geqslant 4$ **b** $x \geqslant 2$ and $2 \leqslant y \leqslant 5$

5 Nick is buying x cups and y bowls. The cups cost €3 each, and the bowls cost €5 each. He needs at least 4 cups and 2 bowls, and has at most €60 to spend.

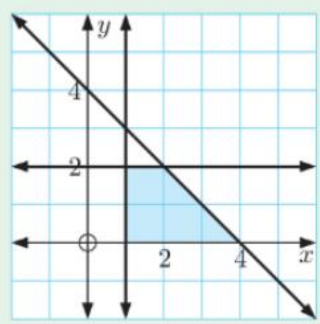
a Find the constraints on the variables x and y.

b Graph the simplex described by the constraints.

6 **a** State the four constraints that uniquely define the shaded region in the diagram.

b Maximise each of the following objective functions over the shaded region, and state the point(s) where the maximum occurs.

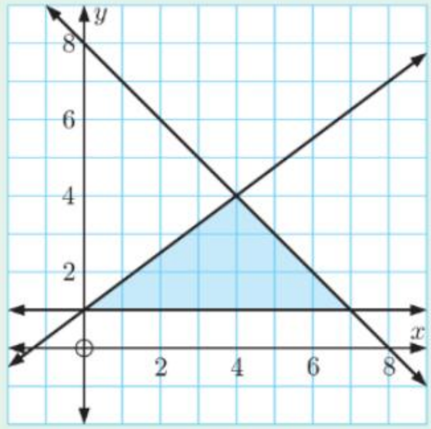
i $3x - y$ **ii** $x + 2y$



7 **a** Write the three inequalities that define the shaded region.

b Determine the coordinates of the vertices of the shaded region.

c Determine the maximum value of $6x + y$ over the shaded region.



8 **a** Sketch the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $x + 4y \leqslant 16$, $3x + 2y \leqslant 18$.

b State the coordinates of the vertices of the simplex.

c Find the maximum value of $x + 5y$ over the simplex, and the values of x and y where the maximum value occurs.

9 A factory makes gas meters and water meters. Gas meters need 4 gears and 1 dial, and are sold for a profit of £20. Water meters need 12 gears and 1 dial, and are sold for a profit of £31. There are 60 gears and 9 dials available for use in this production.

How many of each meter should be produced to maximise profit?

10 A civil engineer is designing a foundation for a bridge with round and square concrete pillars. Each round pillar gives 10 units of rigidity, 12 units of load strength, and 3 units of lateral support, at a cost of \$78 000. Each square pillar gives 4 units of rigidity, 8 units of load strength, and 6 units of lateral support, at a cost of \$63 000. The final structure needs 80 units of rigidity, 144 units of load strength, and 60 units of lateral support.

a Determine the number of each type of pillar that should be used to minimise cost.

b Due to an oversupply of certain materials, the cost of square pillars drops to \$52 000.

i What are the optimal combinations of pillars that can now be used to minimise cost?

ii Of these combinations, the engineer decides to use the combination that gives the most lateral support. How many of each type of pillar is used?

11 Draw a sign diagram for:

a $(x + 4)(x - 1)$ **b** $-x^2 + 2x + 15$ **c** $\dfrac{2x - 1}{x - 4}$

12 Describe where the function with sign diagram alongside is:

a > 0 **b** $\geqslant 0$ **c** < 0 **d** $\leqslant 0$

REVIEW SET 23B

1 Write these number sets using square bracket notation:

a $\{x \mid -2 \leqslant x < 8\}$

b $\{x \mid x \leqslant 1 \text{ or } 4 < x \leqslant 5\}$

c

d

2 Solve for x, and graph the solution:

a $3(2x+1) > 4$

b $40 \leqslant 7 - 3x$

c $7x - 2 \geqslant 2(x+3)$

3 Solve the inequality in the **Opening Problem** on page **432**.

4 Suppose pens cost \$3 each and pencils cost \$1 each. You have \$15 to spend on pens and pencils.

a If the \$15 is all used, what possible combinations could be purchased?

b Graph the possible combinations found in **a**.

c Shade the feasible region of possible purchases.

d What constraints specify the feasible region?

5 Sketch the region defined by:

a $2x + 5y \leqslant 20$

b $3x + 6y \geqslant 24$

c $6x + 5y \leqslant 120$

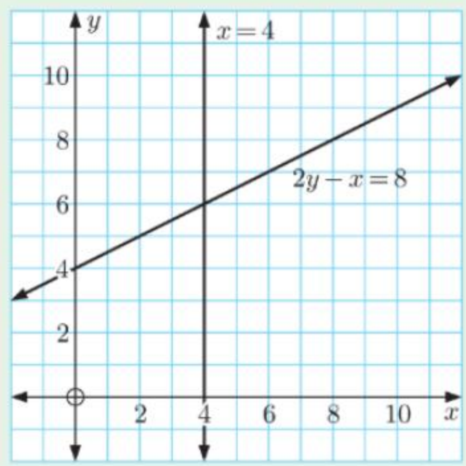
6 **a** Copy and complete the graph by shading the feasible region defined by the constraints:
$2y - x \leqslant 8$, $4 \leqslant x \leqslant 10$, and $x + y \geqslant 10$

b Show that the point:

i (10, 3) satisfies these constraints

ii (11, 3) does not satisfy the constraints.

c State the coordinates of the vertices of the feasible region.



7 **a** Sketch the simplex defined by $x \geqslant 0$, $y \geqslant 0$, $6x + 4y \geqslant 48$, $x + 3y \geqslant 15$.

b State the coordinates of the vertices of the simplex.

c Find the minimum value of $2x + 3y$ over the simplex.

8 A manufacturer makes 2-drawer filing cabinets, and desks with a single drawer. The 2-drawer cabinet uses 1 lock and 3 m^2 of metal, and yields €34 profit. The desk uses 1 lock and 9 m^2 of metal, and yields €47 profit. There are 14 drawers, 8 locks, and 54 m^2 of metal available.

a How much profit is made if the manufacturer produces:

i filing cabinets only

ii desks only?

b How many of each should be produced to earn the highest possible profit? What is the profit in this case?

9 A class is organising a student lunch. From the students' requests, the school needs to supply 50 pies and 60 sausage rolls for the lunch. Bev's Bakery offers a package containing 6 pies and 4 sausage rolls for \$15, while Dave's Bakery offers a package containing 4 pies and 8 sausage rolls for \$18. The school has a budget of \$189 for the lunch.

a Can the school fill their order within the budget by ordering exclusively from either bakery? Explain your answer.

b The school's accounts department suggests the packages should be bought such that the total cost is minimised.

i How many packages should be bought from each bakery in this case?

ii How much money will be left over?

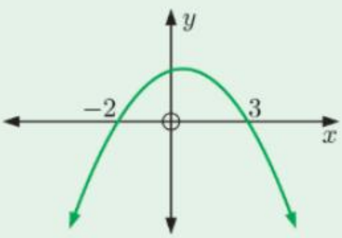
c The school's teachers think the packages should be bought such that the total number of food items is maximised.

i What are the two optimal combinations of packages in this case?

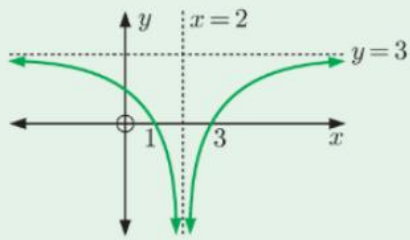
ii The teachers hope there will be some pies left over. Which of the two combinations will the teachers prefer, and how many pies will be left over in this case?

10 Draw a sign diagram for each graph:

a



b



11 Draw a sign diagram for:

a $-x(x+9)$

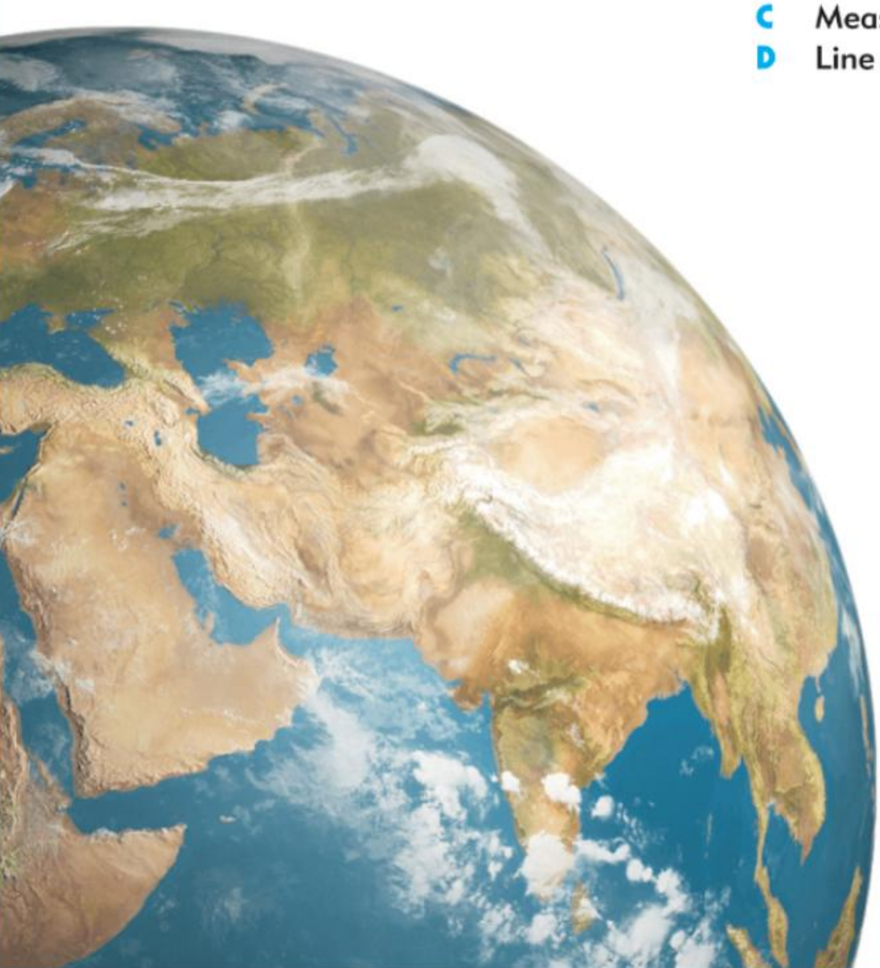
b $2x^2 - 16x + 32$

c $\dfrac{(x+4)(x-3)}{x-1}$

Chapter 24

Bivariate statistics

Contents:



OPENING PROBLEM

The relationship between the *height* and *weight* of members of a football team is to be investigated. The raw data for each player is given below, with heights in cm and weights in kg.

Player	*Height*	*Weight*
1	203	106
2	189	93
3	193	95
4	187	86
5	186	85
6	197	92

Player	*Height*	*Weight*
7	180	78
8	186	84
9	188	93
10	181	84
11	179	86
12	191	92

Player	*Height*	*Weight*
13	178	80
14	178	77
15	186	90
16	190	86
17	189	95
18	193	89

Things to think about:

a Which is the *dependent* variable?

b How could we display this data?

c Does an increase in the *height* of a player generally cause an increase or a decrease in their *weight*?

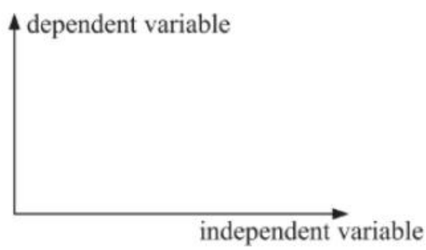
d How can we indicate the strength of the linear relationship between the variables?

e How can we use this data to estimate the weight of a player who is 200 cm tall? How reliable will this estimate be?

We often want to know how two variables are **associated** or **related**. We want to know whether an increase in one variable results in an increase or a decrease in the other. We call this **bivariate statistics** because we are dealing with *two* variables.

To analyse the relationship between two variables, we first need to decide which is the **dependent** variable and which is the **independent** variable. The value of the dependent variable *depends* on the value of the independent variable.

Having made this decision, we can then draw a **scatter plot** to display the data. The independent variable is placed on the horizontal axis, and the dependent variable is placed on the vertical axis.

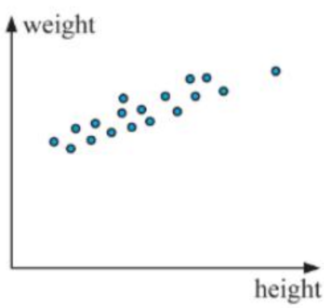


A SCATTER PLOTS

In the **Opening Problem**, we measured the *height* and *weight* of each football player.

We suspect that the weight of a footballer is dependent on his height, so we place height on the horizontal axis and weight on the vertical axis.

The data from each individual footballer is then displayed as a point on the scatter plot.



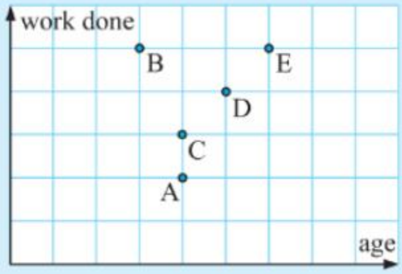
Example 1 **Self Tutor**

This scatter plot shows the ages and work done by 5 employees.

a Which person is the oldest?

b Who has done the most work?

c Who has done the least work?



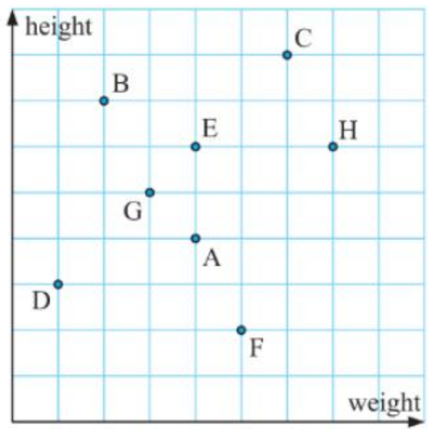
a E has the largest value on the *age* axis.
$\therefore$ E is the oldest.

b B and E have the highest values on the *work done* axis.
$\therefore$ B and E have done the most work.

c A has the lowest value on the *work done* axis.
$\therefore$ A has done the least work.

EXERCISE 24A

1 Liesl measured the height and weight of eight objects in her room. The results are shown in the scatter plot alongside.

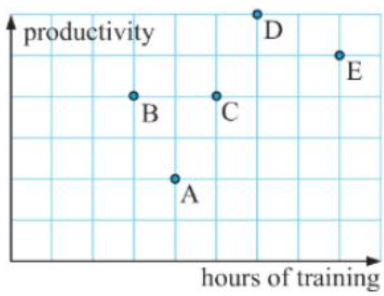


a Which object is the lightest?

b Which object is the tallest?

c Which two objects are the same height?

2 This scatter plot shows the productivity of five employees, and the number of hours of training they have received.



a Who has received the most hours of training?

b Who is the least productive?

c Are there employees with equal productivity?

d Do you think that the training given to the employees has been worthwhile? Explain your answer.

3 Peter's soccer team played seven games during an end-of-season carnival. The results of the games are displayed in the scatter plot.

a In which game did the team score the most goals?

b In which game did the team concede the least goals?

c How many of the games ended in a draw?

d How many games did Peter's team win?

4 Six students were asked how far they lived from school, and how long it took them to travel to school. The results are shown in the table.

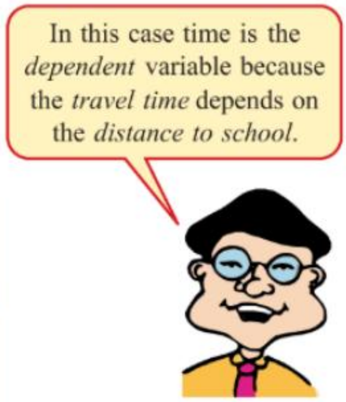
Student	P	Q	R	S	T	U
Distance to school (km)	8	5	2	12	6	4
Travel time (min)	10	5	20	15	15	7

a Draw a scatter plot to display the data, with *distance* on the horizontal axis.

b Which student lives furthest from the school?

c Which student took the least time to travel to school?

d One of the students walks to school. Which student do you think it is? Explain your answer.



B CORRELATION

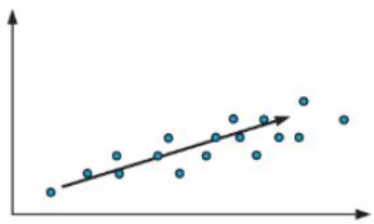
Correlation is a measure of the strength of the relationship or association between two variables.

We can use a scatter plot to describe the correlation between two variables.

Step 1: Look at the scatter plot for any **pattern**.

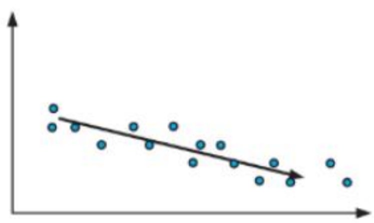
For a generally *upward* shape, we say that the correlation is **positive**.

As the independent variable increases, the dependent variable generally increases.

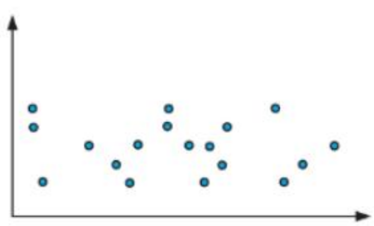


For a generally *downward* shape, we say that the correlation is **negative**.

As the independent variable increases, the dependent variable generally decreases.

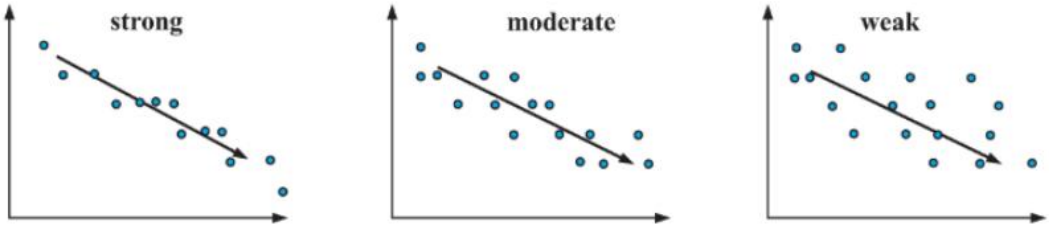


For *randomly scattered* points with no upward or downward trend, we say there is **no correlation**.



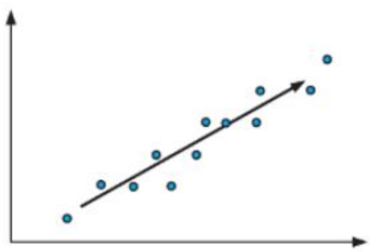
Step 2: Look at the spread of points to make a judgement about the **strength** of the correlation.
These scatter plots show strength classifications for positive relationships:

These scatter plots show strength classifications for negative relationships:

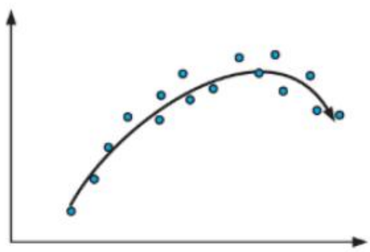


Step 3: Look at the pattern of points to see if the relationship is **linear**.

The relationship is approximately linear.



The relationship is not linear.



Example 2

Self Tutor

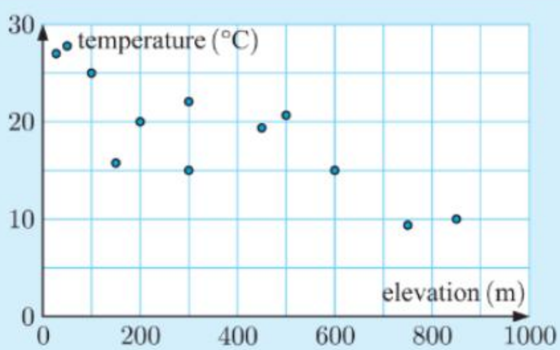
Alexander researched the elevation above sea level and mean annual temperature of 12 cities around the world. The results are given in this table.

Elevation (m)	600	850	150	300	100	200	500	450	750	30	300	50
Mean annual temperature (°C)	15	10	16	15	25	20	21	19	9	27	22	28

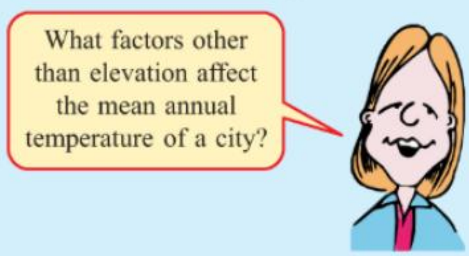
a Draw a scatter plot of the data.

b Describe the relationship between *elevation* and *mean temperature*.

a



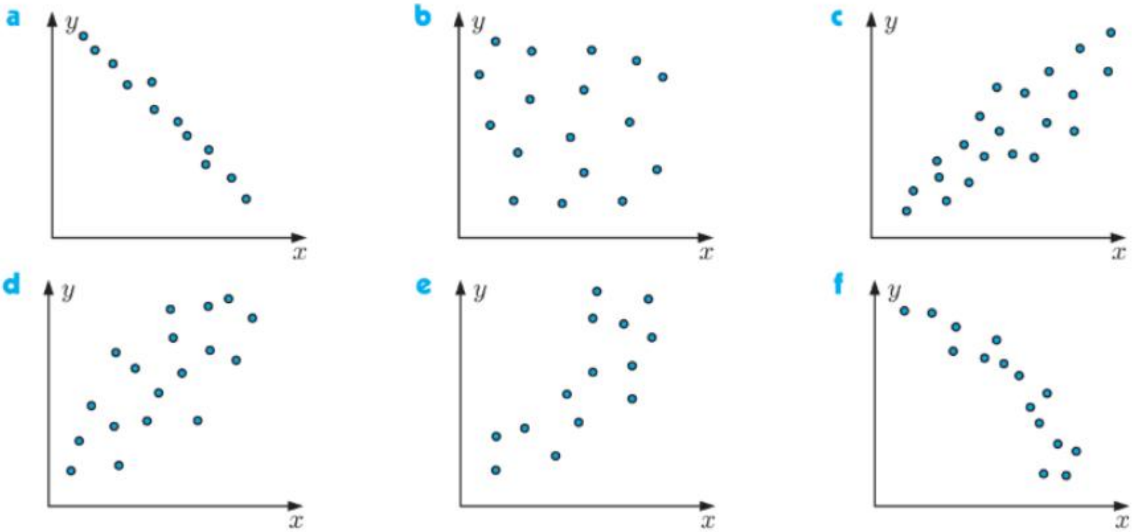
b There appears to be a moderate negative linear correlation between *elevation* and *mean temperature*.



EXERCISE 24B

1 For each of the scatter plots below:

- **i** state whether there is positive, negative, or no association between the variables
- **ii** decide whether the relationship between the variables appears to be linear
- **iii** describe the strength of the association (zero, weak, moderate, or strong).



2 Copy and complete the following:

- **a** If there is a positive association between x and y, then as x increases, y ......
- **b** If there is a negative correlation between T and d, then as T increases, d ......
- **c** If there is no association between two variables then the points on the scatter plot are ....

3 A class of 15 students was asked how many text messages they had sent and received in the last week. The results are shown below:

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Messages sent	5	0	12	9	17	15	10	4	8	18	25	17	0	6	13
Messages received	8	0	15	7	19	11	8	7	12	15	21	16	4	6	16

- **a** Draw a scatter plot of the data.
- **b** Describe the relationship between *messages sent* and *messages received.*

4 **a** 10 students were asked for their exam marks in Physics and Mathematics. Their percentages are given in the table below.

Student	A	B	C	D	E	F	G	H	I	J
Physics	75	83	45	90	70	78	88	50	55	95
Mathematics	68	70	50	65	60	72	75	40	45	80

- **i** Draw a scatter plot of the data, with the Physics marks on the horizontal axis.
- **ii** Comment on the relationship between the Physics and Mathematics marks.

b The same students were asked for their Art exam results. Their percentages were:

Student	A	B	C	D	E	F	G	H	I	J
Art	75	70	80	85	82	70	60	75	78	65

Draw a scatter plot to see if there is any relationship between the Physics marks and the Art marks of each student.

5 The following table shows the sales of hot drinks in a popular café each month, along with the average daily temperature for the month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (°C)	32	29	26	23	19	16	12	15	18	22	25	29
Sales ($\$ \times 1000$)	12	8	10	15	16	18	22	25	20	15	16	12

a Draw a scatter plot of the data, with the independent variable *temperature* along the horizontal axis.

b Comment on the relationship between the sales and the temperature.

C MEASURING CORRELATION

Simple observation of the points on a scatter plot is a fairly inaccurate way to describe the strength of correlation between two variables. Instead, we can calculate **Pearson's correlation coefficient**. This gives us a numerical value between -1 and 1 which measures the strength of correlation between two variables.

Consider a set of n data points given as ordered pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n), where $\overline{x}$ and $\overline{y}$ are the means of the x and y data respectively.

Pearson's correlation coefficient is $$r = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}}$$

where $\sum$ indicates the sum over all the data values.

You are not required to learn this formula, since we usually use technology to find the value of r.

GRAPHICS CALCULATOR INSTRUCTIONS

The values of r range from -1 to $+1$.

The **sign** of r indicates the **direction** of the correlation.

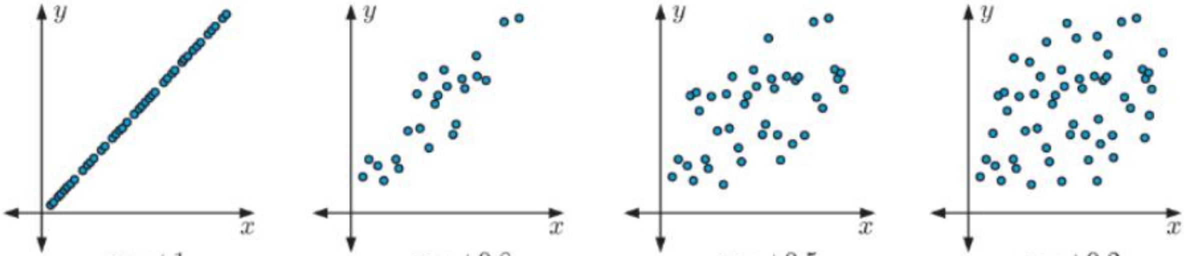
- A positive value for r indicates the variables are **positively correlated**.
  An increase in one of the variables will result in an increase in the other.
- A negative value for r indicates the variables are **negatively correlated**.
  An increase in one of the variables will result in a decrease in the other.

The **size** of r indicates the **strength** of the correlation.

- A value of r close to $+1$ or -1 indicates strong correlation between the variables.
- A value of r close to zero indicates weak correlation between the variables.

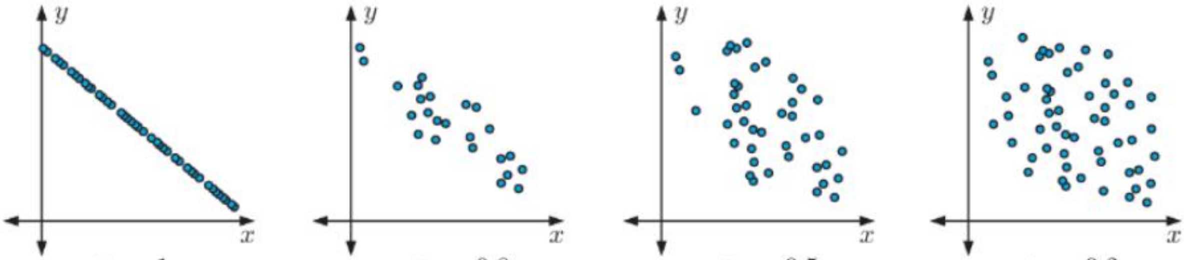
Some examples of scatter plots with their corresponding values of r are given below.

POSITIVE CORRELATION



$r=+1$ $\quad$ $r=+0.8$ $\quad$ $r=+0.5$ $\quad$ $r=+0.2$

NEGATIVE CORRELATION



$r=-1$ $\quad$ $r=-0.8$ $\quad$ $r=-0.5$ $\quad$ $r=-0.2$

Example 3

Self Tutor

Four students were asked how many children and pets were in their household. The results are shown in the table alongside.

Student	A	B	C	D
Number of children	1	4	2	1
Number of pets	1	6	3	2

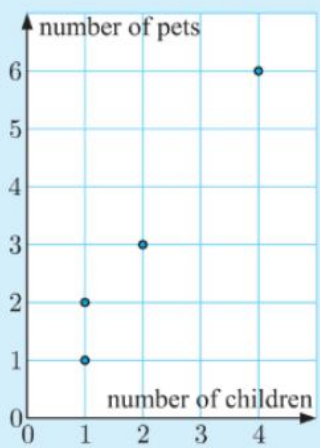
a Draw a scatter plot for the data.

b Find the correlation coefficient r *without* using technology.

c Is the data positively correlated or negatively correlated?

d Describe the strength of the correlation between the variables.

a



b $\overline{x} = \dfrac{1+4+2+1}{4} = 2$ $\qquad$ $\overline{y} = \dfrac{1+6+3+2}{4} = 3$

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})(y-\overline{y})$	$(x-\overline{x})^2$	$(y-\overline{y})^2$
1	1	-1	-2	2	1	4
4	6	2	3	6	4	9
2	3	0	0	0	0	0
1	2	-1	-1	1	1	1
			Total	9	6	14

$$r = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}} = \frac{9}{\sqrt{6 \times 14}}$$
$$\approx 0.982$$

c $r > 0$, so the data is positively correlated.

d r is close to 1, indicating a very strong correlation between the variables.

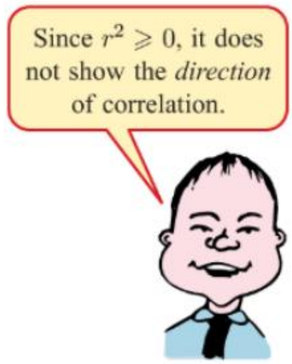
COEFFICIENT OF DETERMINATION r^2

Another commonly used statistic which describes the strength of the association between two variables is the square of Pearson's correlation coefficient r.

This value, r^2, is known as the **coefficient of determination**. Since $-1 \leqslant r \leqslant 1$, $0 \leqslant r^2 \leqslant 1$.

The following table is a guide for describing the strength of linear correlation using the coefficient of determination:

Value	*Strength of correlation*
$r^2 = 0$	no correlation
$0 < r^2 < 0.25$	very weak correlation
$0.25 \leqslant r^2 < 0.50$	weak correlation
$0.50 \leqslant r^2 < 0.75$	moderate correlation
$0.75 \leqslant r^2 < 0.90$	strong correlation
$0.90 \leqslant r^2 < 1$	very strong correlation
$r^2 = 1$	perfect correlation



Example 4 — Self Tutor

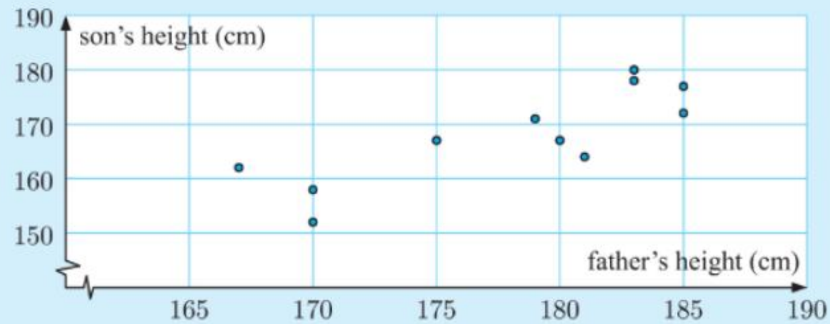
At a father-son camp, the heights of the fathers and their sons were measured.

Father's height (cm)	175	183	170	167	179	180	183	185	170	181	185
Son's height (cm)	167	178	158	162	171	167	180	177	152	164	172

a Draw a scatter plot of the data.

b Calculate the coefficient of determination r^2 for the data, and interpret its value.

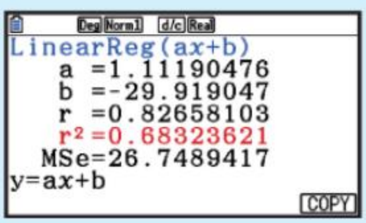
a



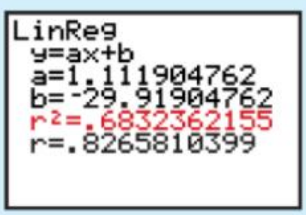
GRAPHICS CALCULATOR INSTRUCTIONS

b Using technology, $r^2 \approx 0.683$.

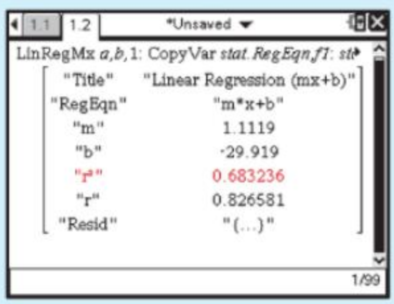
Casio fx-CG20



TI-84 Plus



TI-*n*spire



There is a moderate positive correlation between the *father's height* and the *son's height*.

EXERCISE 24C

1 The correlation coefficient for a set of bivariate data is $r \approx 0.812$.

a Are the variables positively correlated or negatively correlated? Explain your answer.

b Find the coefficient of determination r^2.

c Describe the strength of the correlation between the variables.

2 Five students each took 10 shots at a netball goal. The table alongside shows how many times each student scored a goal, and how many times they missed.

Student	A	B	C	D	E
Scored	7	5	3	9	6
Missed	3	5	7	1	4

a Draw a scatter plot of the data.

b Find r using the formula $r = \dfrac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2 \sum(y-\overline{y})^2}}$.

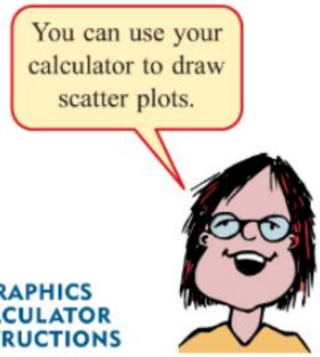
c Describe the correlation between the variables. Explain why this must be the case.

3 For each of the following data sets:

i Draw a scatter plot of the data.

ii Use technology to find r and r^2.

iii Describe the linear correlation between X and Y.

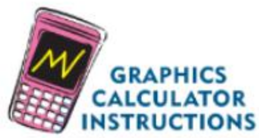


a

X	1	2	3	4	5	6
Y	3	2	5	5	9	6

b

X	3	8	5	14	19	10	16
Y	17	12	15	6	1	10	4



c

X	3	6	11	7	5	6	8	10	4
Y	2	8	8	4	7	9	11	1	5

4 Students were asked to measure their height in centimetres and their shoe size. The results are recorded in the table below:

Height (cm)	165	155	140	145	158	148	160	164	160	155	150	160
Shoe size	6.5	4.5	4	5.5	6	5.5	6	5.5	5.5	5	5	5.5

a Construct a scatter plot of the data.

b Calculate r and r^2.

c Describe the relationship between height and shoe size.

5 The scores awarded by two judges at a diving competition are shown in the table.

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

a Construct a scatter plot of the data, with Judge A's scores on the horizontal axis, and Judge B's scores on the vertical axis.

b Calculate r and r^2.

c Describe the correlation between the judges' scores.

6 A basketballer takes 20 shots from each of ten different positions marked on the court. The table below shows how far each position is from the goal, and how many shots were successful:

Position	A	B	C	D	E	F	G	H	I	J
Distance from goal (m)	2	5	3.5	6.2	4.5	1.5	7	4.1	3	5.6
Successful shots	17	6	10	5	8	18	6	8	13	9

a Draw a scatter plot of the data.
b Will r be positive or negative?
c Calculate r and r^2.
d Describe the correlation between these variables.

7 Jane wanted to see whether there was any correlation between the length of a movie, and its performance at the box office. She selected 10 movies, and recorded their lengths and box office takings.

Length (min)	107	122	92	103	96	161	121	178	95	135
Takings (€ × 1 m)	100	336	47	1063	363	190	164	871	543	313

a Draw a scatter plot of the data.
b Does there appear to be a strong correlation between the variables? Explain your answer.
c Calculate r and r^2.
d Describe the relationship between the variables.

ACTIVITY — WHAT'S IN A NAME?

Do people with long surnames generally have long first names as well?

In this Activity we will investigate whether there is a relationship between the length of a person's first name and surname.

What to do:

1. Count the number of letters in your first name and surname.
2. As a class, predict the nature of the correlation between the *lengths of first names* and the *lengths of surnames*.
3. Use the names of each student in your class to create a scatter plot. Which is the independent variable?
4. Calculate r and r^2.
5. Describe the correlation between the *lengths of first names* and the *lengths of surnames*.

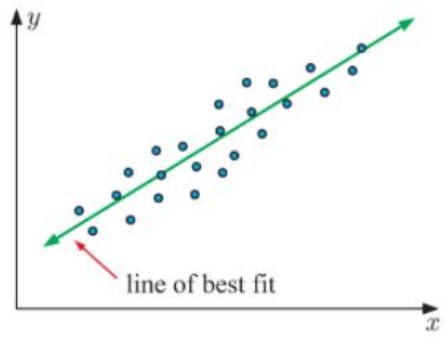
D LINE OF BEST FIT

If there is a strong linear correlation between two variables x and y, then it is reasonable to draw a **line of best fit** through the data.

We can find the line of best fit:

- by eye
- using linear regression.

The line of best fit can be used to estimate y for any x.



LINE OF BEST FIT BY EYE

For a bivariate data set involving x and y, we use these steps to find the line of best fit by eye:

Step 1: Find the means $\overline{x}$ and $\overline{y}$ of the x and y values respectively.

Step 2: Plot the **mean point** $(\overline{x}, \overline{y})$ on a scatter plot of the data.

Step 3: Draw a line through the mean point which fits the trend of the data and which has about as many points above the line as below it.

Example 5 **Self Tutor**

On a hot day, six cars were left in the sun in a car park. The length of time each car was left in the sun was recorded, as well as the temperature inside the car at the end of the period.

Car	A	B	C	D	E	F
Time x (min)	50	5	25	40	15	45
Temperature y (°C)	47	28	36	42	34	41

a Calculate $\overline{x}$ and $\overline{y}$.

b Draw a scatter plot of the data.

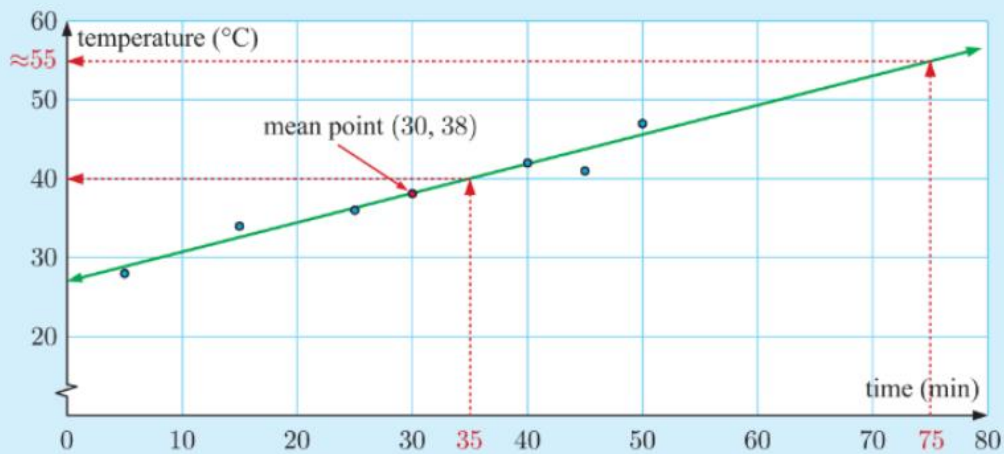
c Plot the mean point $(\overline{x}, \overline{y})$ on the scatter plot. Draw a line of best fit through this point.

d Predict the temperature of a car which has been left in the sun for:

i 35 minutes **ii** 75 minutes.

a $\overline{x} = \dfrac{50+5+25+40+15+45}{6} = 30, \quad \overline{y} = \dfrac{47+28+36+42+34+41}{6} = 38$

b, c



d **i** When $x = 35$, $y \approx 40$.
The temperature of a car left in the sun for 35 minutes will be approximately 40°C.

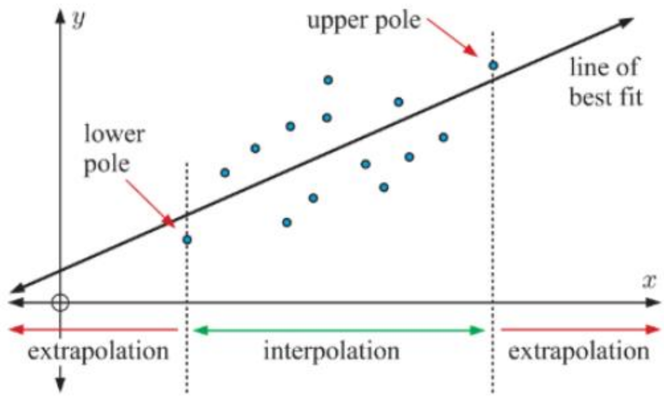
ii When $x = 75$, $y \approx 55$.
The temperature of a car left in the sun for 75 minutes will be approximately 55°C.

INTERPOLATION AND EXTRAPOLATION

Given a bivariate data set, the data values with the lowest and highest values of x are called the **poles**.

If we use values of x **in between** the poles to estimate y, we say we are **interpolating** between the poles.

If we use values of x **outside** the poles to estimate y, we say we are **extrapolating** outside the poles.



As a general rule, it is reasonable to interpolate between the poles, but unreliable to extrapolate outside them.

In **Example 5** above:

- The estimate in **d i** is an interpolation, so we would expect this estimate to be reliable.
- The estimate in **d ii** is an extrapolation, and therefore may not be reliable. We cannot assume that the linear trend shown in the data will continue up to a time of 75 minutes.

EXERCISE 24D.1

1 Consider the bivariate data alongside.

x	5	10	2	13	6
y	11	3	18	5	13

a Find $\overline{x}$ and $\overline{y}$.

b Draw a scatter plot of the data.

c Does the data appear to be positively correlated or negatively correlated?

d Plot the mean point $(\overline{x}, \overline{y})$ on the scatter plot, and draw a line of best fit through this point.

e Estimate the value of y when $x = 8$.

2 The table alongside shows the *percentage of unemployed adults* and the *number of major thefts per day* in eight large cities.

City	*Unemployment* (%)	*Thefts*
A	7	113
B	6	67
C	10	117
D	8	88
E	9	120
F	6	38
G	3	61
H	7	76

a Find the mean unemployment percentage $\overline{x}$ and the mean number of thefts $\overline{y}$.

b Calculate r for this data.

c Draw a scatter plot of the data.

d Plot $(\overline{x}, \overline{y})$ on the scatter plot.

e Draw the line of best fit on the scatter plot.

f Another city has 15% unemployment.

i Estimate the number of major thefts per day for that city.

ii Comment on the reliability of your estimate.

3 Each month, an opinion poll shows the approval rating of the Prime Minister and the Opposition leader. The approval ratings for the last 10 polls are shown below:

Prime Minister (x%)	55	59	68	61	46	42	38	45	42	44
Opposition (y%)	37	35	31	35	43	40	42	37	41	39

a Calculate $\overline{x}$ and $\overline{y}$.

b Draw a scatter plot of the data. Plot the mean point $(\overline{x}, \overline{y})$ on the scatter plot, and draw a line of best fit through this point.

c In a new opinion poll, the Prime Minister's approval rating is 47%. Estimate the approval rating of the Opposition leader.

4 A café manager believes that during April the *number of people wanting dinner* is related to the *temperature at noon*. Over a 13 day period, the number of diners and the noon temperature were recorded.

Temperature (x °C)	18	20	23	25	25	22	20	23	27	26	28	24	22
Number of diners (y)	63	70	74	81	77	65	75	87	91	75	96	82	88

a Find the mean point $(\overline{x}, \overline{y})$.

b Calculate r^2 for this data, and interpret this value.

c Draw a scatter plot of the data. Plot $(\overline{x}, \overline{y})$ on the scatter plot, and draw a line of best fit through this point.

d Estimate the number of diners at the café when the temperature is:

i 21°C **ii** 14°C.

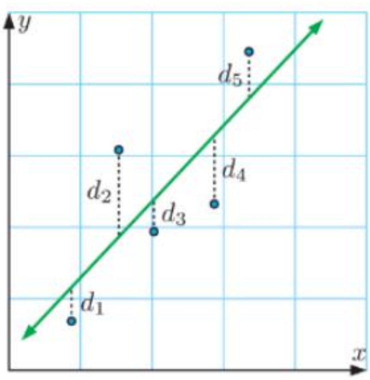
e Comment on the reliability of your estimates in **d**.

LINE OF BEST FIT USING LINEAR REGRESSION

The problem with drawing a line of best fit by eye is that the answer will vary from one person to another, and the line may not be very accurate. Instead, we can use a method called **linear regression**.

Linear regression is a formal method of fitting a line which best fits a set of data.

This line of best fit is called the **least squares regression line**. It is the line which makes the sum of the squares of the distances $d_1^2 + d_2^2 + d_3^2 + ...$ as small as possible.



We use a **graphics calculator** or the **statistics package** to find the equation of the least squares regression line.

Once we have found the equation of the least squares regression line, we can estimate y for any x by substituting the value of x into the equation.

Example 6 — Self Tutor

The annual income and average weekly grocery bill for a selection of families is shown below:

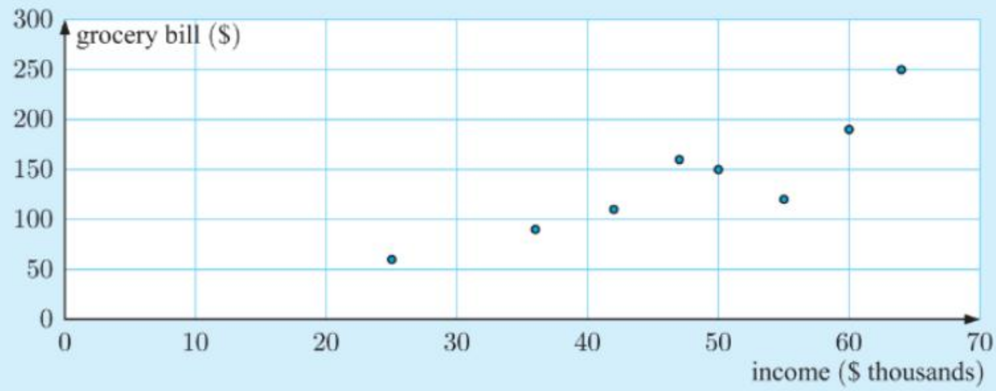
Income (x thousand dollars)	55	36	25	47	60	64	42	50
Grocery bill (y dollars)	120	90	60	160	190	250	110	150

a Construct a scatter plot to illustrate the data.

b Use technology to find the least squares regression line.

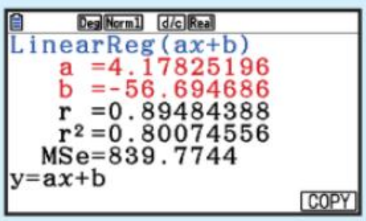
c Estimate the weekly grocery bill for a family with an annual income of \$95 000. Comment on whether this estimate is likely to be reliable.

a

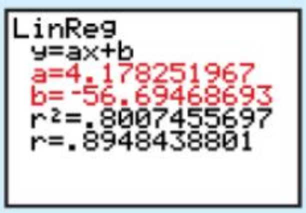


b

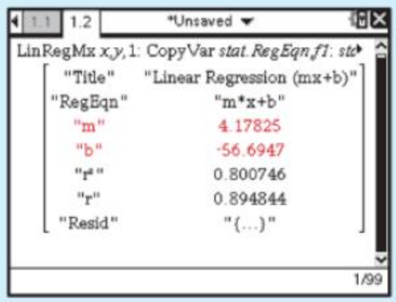
Casio fx-CG20



TI-84 Plus



TI-*n*spire



Using technology, the line of best fit is $y \approx 4.18x - 56.7$

c When $x = 95$, $y \approx 4.18(95) - 56.7 \approx 340$

So, we expect a family with an income of \$95 000 to have a weekly grocery bill of approximately \$340.

This is an extrapolation, however, so the estimate may not be reliable.

EXERCISE 24D.2

1 For each data set: **i** draw a scatter plot of the data
ii find the equation of the least squares regression line.

a

x	3	5	6	9	13	15	18	20
y	4	8	11	17	15	19	25	24

b

x	16	6	11	8	14	24	2	14	19	6
y	15	30	20	26	17	9	35	10	7	27

2 Tomatoes are sprayed with a pesticide-fertiliser mix. The table below gives the yield of tomatoes per row of bushes for various spray concentrations.

Spray concentration (x mL per 2 L)	2	4	6	8	10	12
Yield of tomatoes per row (y)	45	76	93	105	119	124

a Draw a scatter plot of the data.

b Find the equation of the least squares regression line.

c Interpret the y-intercept of this line.

d Use the equation of the line to predict the yield if the spray concentration was 7 mL.
Comment on whether this prediction is reasonable.

3 A group of friends competed in a fun-run. The table below shows how long each friend spent training, and the time they recorded for the fun-run.

Training time (x hours)	7	2	11	3	7	15	3	0	5	9	0
Fun-run time (y minutes)	60	75	47	70	52	37	72	75	60	62	80

a Draw a scatter plot of the data.

b Find r and r^2.

c Find the equation of the least squares regression line.

d Interpret the gradient of this line.

e Another friend of the group trained for 30 hours.

i Use the least squares regression line to estimate his fun-run time.

ii Comment on the reliability of your estimate.

4 The yield of cherries from an orchard each year depends on the number of frosty mornings. The following table shows the yield of cherries from an orchard over several years with different numbers of frosty mornings.

Frosty mornings (n)	18	29	23	38	35	27
Yield (Y tonnes)	29.4	34.6	32.1	36.9	36.1	32.5

a Produce a scatter plot of Y against n.

b Find the linear model which best fits the data.

c Estimate the yield from the orchard if there are 31 frosty mornings in the year.

d Copy and complete:

"The greater the number of frosty mornings, the ...... the yield of cherries."

5 Carbon dioxide (CO_2) is a chemical linked to acid rain and global warming. The concentration of CO_2 in the atmosphere has been recorded over a 40 year period. It is measured in parts per million or ppm found in Law Dome Ice Cores in Antarctica.

Year	1960	1970	1980	1990	2000
CO_2 *concentration* (ppm)	313	321	329	337	345

Let t be the number of years since 1960 and C be the CO_2 concentration.

a Draw a scatter plot of C against t.

b Describe the correlation between C and t.

c Obtain the linear model which best fits the data.

d Estimate the CO_2 concentration for 1987.

e If CO_2 emission continues at the same rate, estimate the concentration in 2020.

6 Paul researched the first 6 games of the 2012 Twenty20 Cricket World Cup. For the team which batted first, he recorded their score after 10 overs, and their final score.

Score after 10 *overs* (x)	76	46	68	51	75	74
Final score (y)	182	123	159	93	191	196

a Draw a scatter plot of the data.

b Describe the correlation between the variables.

c Find the equation of the least squares regression line.

d Predict the final score for a team which scores 60 runs in the first 10 overs. How reliable is your prediction?

Global context

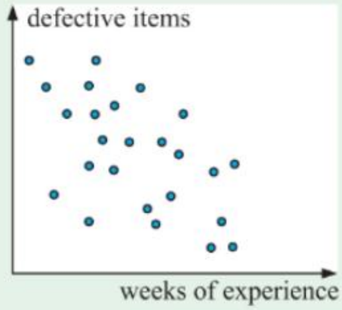
click here

What is a dollar worth to you?

Statement of inquiry:	Using a model to represent relationships can help us make predictions.
Global context:	Fairness and development
Key concept:	Relationships
Related concepts:	Model, Pattern
Objectives:	Communicating, Applying mathematics in real-life contexts
Approaches to learning:	Communication, Social, Research, Thinking

REVIEW SET 24A

1 The scatter plot shows the number of defective items made by each employee of a factory, plotted against the employee's number of weeks of experience.
Describe the correlation between the variables.

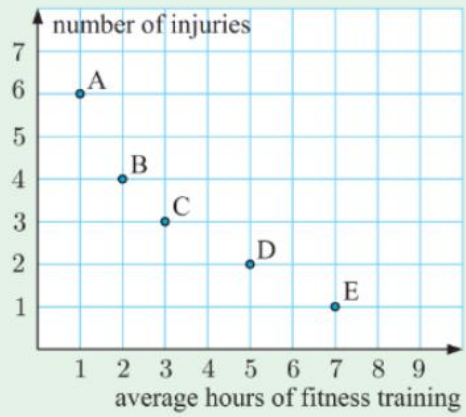


2 Five hockey clubs were surveyed on their players' average hours of general fitness training, and the number of injuries to players during matches.

a Which club had the lowest number of injuries?

b Which club's players had the lowest number of hours of fitness training?

c Write a sentence describing the general trend of the graph.



3 Sam and Richard competed in a series of 8 chess games. The table below shows how many pieces remained for each player at the end of each game.

Sam	4	2	6	2	4	1	5	7
Richard	2	1	3	7	4	8	3	5

a Construct a scatter plot of the data.

b Calculate r and r^2, and interpret these values.

4 Following an outbreak of a deadly virus, medical authorities begin taking records of the number of cases. Their records are shown below.

Days after outbreak (n)	2	3	4	5	6	7	8	9	10	11
Diagnosed cases (d)	8	14	33	47	80	97	118	123	139	153

a Produce a scatter plot of d against n.

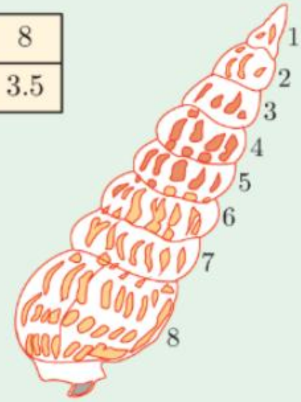
b Plot the point $(\overline{n}, \overline{d})$ on the scatter plot, and draw the line of best fit by eye.

c **i** Use the graph to predict the number of diagnosed cases on day 14.

ii Is this predicted value reliable? Give reasons for your answer.

5 The whorls on a cone shell get broader as you go from the top of the shell towards the bottom. Measurements from a shell are summarised in the following table:

Position of whorl (p)	1	2	3	4	5	6	7	8
Width of whorl (w cm)	0.7	1.2	1.4	2.0	2.0	2.7	2.9	3.5



a Obtain a scatter plot of the data.

b Find Pearson's correlation coefficient for this data.

c Find the linear regression model which best fits the data.

d **i** If a cone shell has 14 whorls, what width do you expect the 14th whorl to have?

ii How reliable is this prediction?

6 The table below shows the number of games won, and the final position on the ladder for the Liverpool football team from 2001-02 to 2012-13:

Games won (x)	24	18	16	17	25	20	21	25	18	17	14	16
Position (y)	2	5	4	5	3	3	4	2	7	6	8	7

a Would you expect x and y to be positively or negatively correlated? Explain your answer.

b Draw a scatter plot of the data.

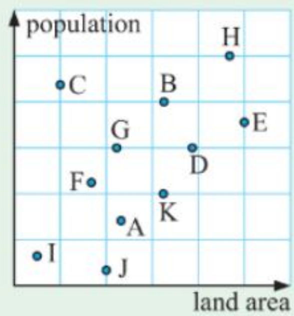
c Find r and r^2.

d Use technology to find the equation of the line of best fit.

e Suppose Liverpool wins 22 games next season. Predict their position on the ladder.

REVIEW SET 24B

1 This scatter plot displays the land area and population of 10 countries.



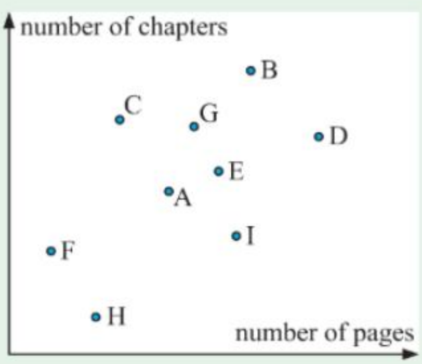
a Which country has the smallest population?

b Which country has the largest area?

c Which two countries have the same population?

d Which country is the most densely populated?

2



The scatter plot alongside shows the number of pages and chapters in the novels on Rashida's bookshelf.

a Which book has the:

i most pages

ii least chapters?

b Copy and complete: As the number of pages increases, the number of chapters generally ......

3 Consider the bivariate data alongside.

x	11	7	13	3	12	8
y	17	14	20	5	26	8

a Find $\overline{x}$ and $\overline{y}$.

b Draw a scatter plot of the data.

c Does the data appear to be positively correlated or negatively correlated?

d Plot the mean point $(\overline{x}, \overline{y})$ on the scatter plot, and draw a line of best fit through this point.

e Estimate the value of y when $x = 5$. How reliable is this estimate?

4 In a Los Angeles shopping mall, David asked 10 people how many coins they had in their wallet or purse, and the total value of those coins.

Number of coins	5	8	11	7	5	10	2	10	1	12
Value of coins	\$1.10	\$0.82	\$1.56	\$0.90	\$0.51	\$1.54	\$0.30	\$1.02	\$0.10	\$1.23

Let n be the number of coins, and v be the value of the coins.

a Draw a scatter plot of the data.

b Find the equation of the least squares regression line.

c Interpret the gradient of this line.

d Terese has 20 coins in her purse.

  i Estimate the total value of these coins.

  ii How reliable is your prediction?

5 Consider the relationship between a *number* and the *number of factors* it has.

a Would you expect the correlation between these variables to be:

  i positive or negative **ii** strong, moderate, or weak?

  Explain your answers.

b Copy and complete this table:

Number (x)	1	2	3	4	5	....	16	17	18	19	20
Number of factors (y)	1	2	2	3		....				2	6

c Draw a scatter plot of the data.

d Calculate the correlation coefficient r, and coefficient of determination r^2.

e Describe the correlation between the variables.

6 The following table gives peptic ulcer rates per 1000 people for differing family incomes.

Income (I £1000s)	20	25	30	35	40	50	60	80	100
Peptic ulcer rate (R)	8.3	7.7	6.9	7.3	5.9	4.7	3.6	2.6	1.2

a Draw a scatter plot of the data.

b Find the equation of the line of best fit for the data.

c Estimate the peptic ulcer rate in families with an income of £55 000.

d Explain why the model is inadequate for families with an income in excess of £120 000.

e Later it is realised that one of the figures was written incorrectly.

  i Which is it likely to be? Explain your answer.

  ii Repeat **b** and **c** with the incorrect data value removed.

Chapter 25

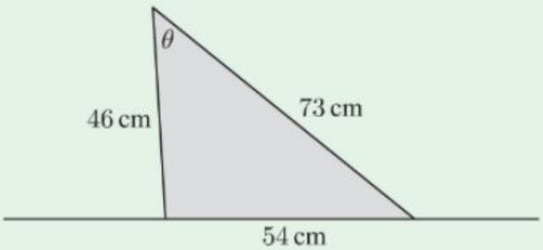
Non-right angled triangle trigonometry

Contents:

OPENING PROBLEM

Great white sharks can be recognised by their triangular dorsal fins.

The fin of a particular shark has the dimensions shown:



Things to think about:

a How can we use the dimensions given to find the angle θ at the top of the fin?

b How can we calculate the surface area of one side of the fin?

So far in this course we have studied the trigonometry of right angled triangles, and learnt how to calculate the trigonometric ratios for any angle size.

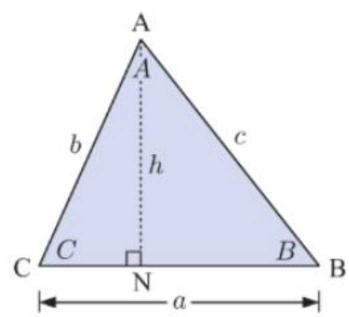
In this chapter we apply trigonometry to triangles which are not right angled, allowing us to solve a variety of real world problems.

A THE AREA OF A TRIANGLE

We can use trigonometry to find the area of a triangle if we are given the lengths of **two sides**, as well as the **included angle** between the sides.

Consider triangle ABC in which the sides opposite angles A, B, and C are labelled a, b, and c respectively.

Suppose N lies on [BC] such that [AN] is perpendicular to [BC].



In $\triangle$ANC, $\sin C = \dfrac{h}{b}$

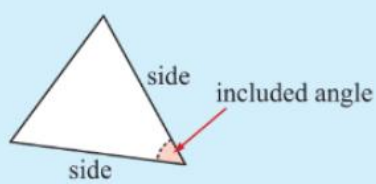
$\therefore \; h = b \sin C$

Since the area of $\triangle$ABC $= \frac{1}{2}ah$, we find: $\boxed{\textbf{Area} = \dfrac{1}{2}ab\sin C}$

If the altitudes from B and C were drawn, we could also show that the area is $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$.

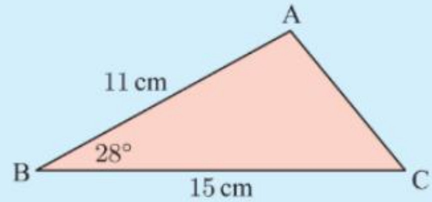
The **area of a triangle** is

a half of the product of two sides and the sine of the included angle.



Example 1 **Self Tutor**

Find the area of triangle ABC.



Area $= \frac{1}{2}ac\sin B$

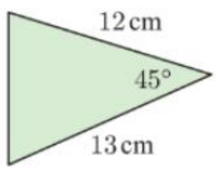
$= \frac{1}{2} \times 15 \times 11 \times \sin 28°$

$\approx 38.7 \text{ cm}^2$

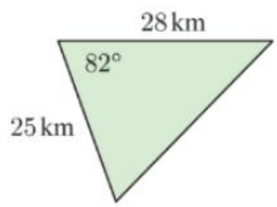
EXERCISE 25A

1 Find the area of:

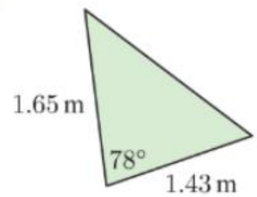
a



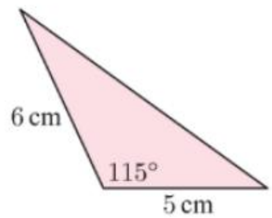
b



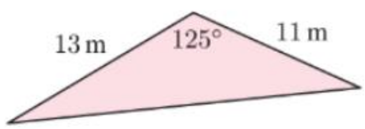
c



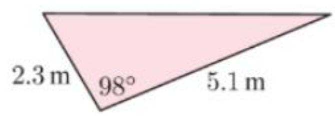
d



e

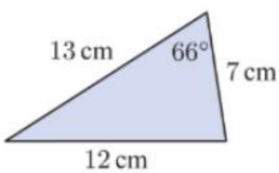


f

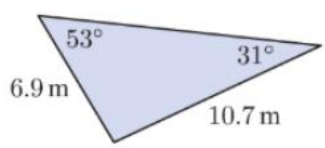


2 Find the area of:

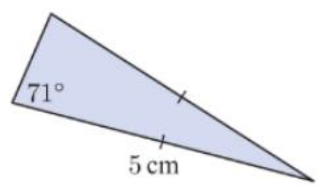
a



b

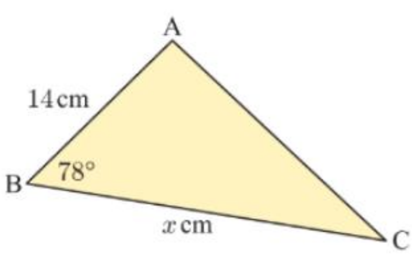


c

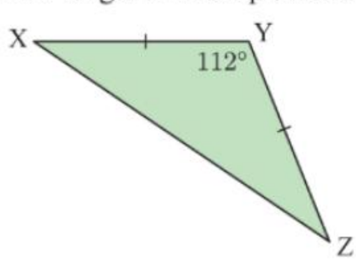


3 Find the area of a parallelogram with sides 6.4 cm and 8.7 cm, and one interior angle 64°.

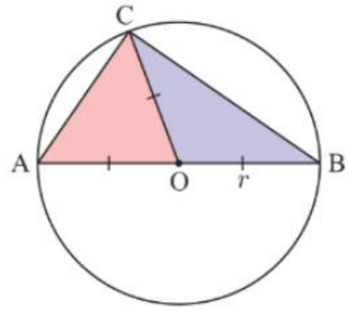
4 Triangle ABC has area 150 cm^2.
Find the value of x.



5 Triangle XYZ has area 80 cm^2.
Find the length of the equal sides.



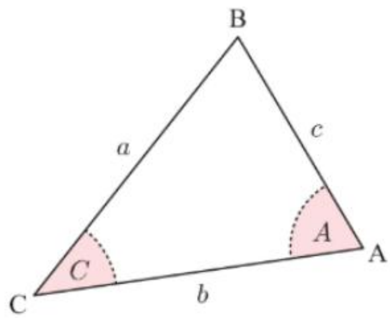
6 [AB] is the diameter of a circle with centre O and radius r. Show that the shaded triangles have equal area.



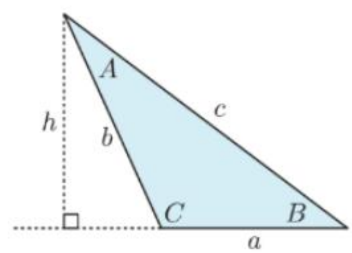
7 **a** Find the area of triangle ABC using:

i angle A **ii** angle C

b Hence, show that $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$.



8 The proof of the area formula given above assumes that the included angle C is acute. Use the diagram alongside to prove that the formula is also true in the case where C is obtuse.

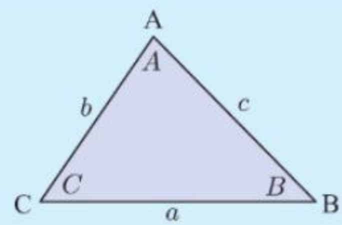


B THE SINE RULE

The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of the opposite angles.

In any triangle ABC with sides a, b, and c units, and opposite angles A, B, and C respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Proof:

The area of any triangle ABC is given by $\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

Dividing each expression by $\frac{1}{2}abc$ gives $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

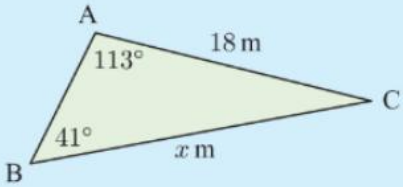
GEOMETRY PACKAGE

FINDING SIDES

If we are given two angles and one side of a triangle, we can use the sine rule to find another side length.

Example 2 — **Self Tutor**

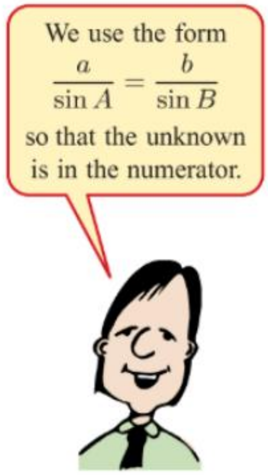
Find x, rounded to 2 decimal places:



Using the sine rule, $\dfrac{x}{\sin 113^\circ} = \dfrac{18}{\sin 41^\circ}$

$\therefore \quad x = \dfrac{18 \times \sin 113^\circ}{\sin 41^\circ}$

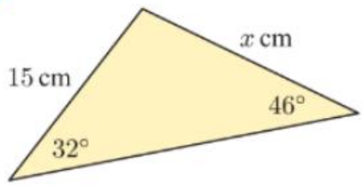
$\therefore \quad x \approx 25.26$



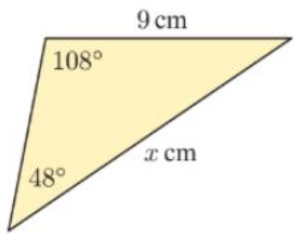
EXERCISE 25B.1

1 Find x, rounded to 2 decimal places:

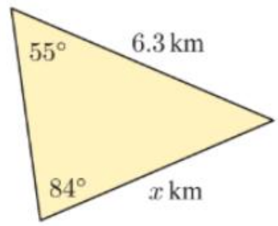
a



b

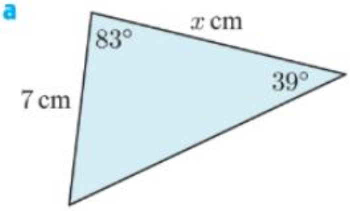


c

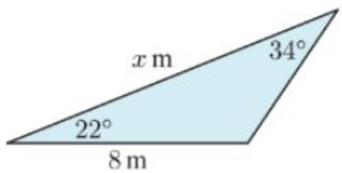


2 Find x:

a



b



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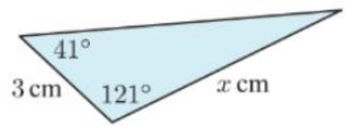
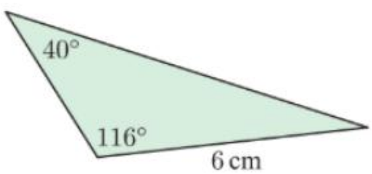
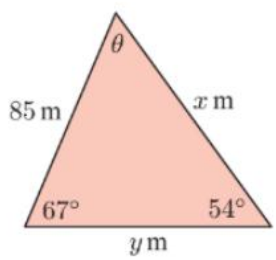


3 Find the area of this triangle.



4 Find *all* unknown sides and angles of:

a



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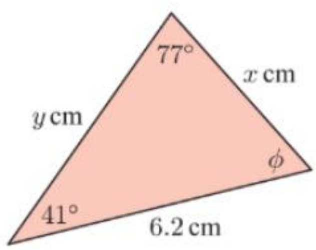

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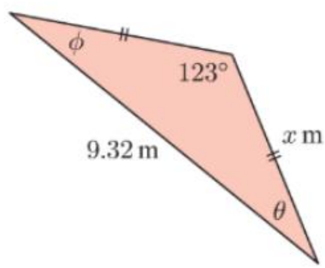


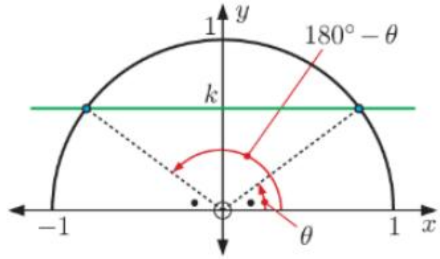
FINDING ANGLES

Finding angles using the sine rule is more complicated than finding sides because there may be two possible answers.

In **Chapter 21** we saw that $\sin(180° - \theta) = \sin\theta$.

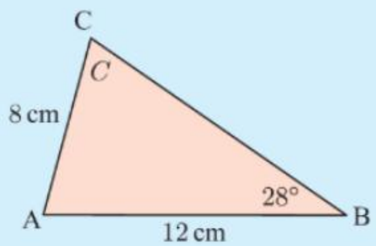
For these types of problems, when we solve $\sin\theta = k$, we need to test both θ and $180° - \theta$ to see whether each will work.

Sometimes there is information in the question which enables us to **reject** one of the solutions.



Example 3 — Self Tutor

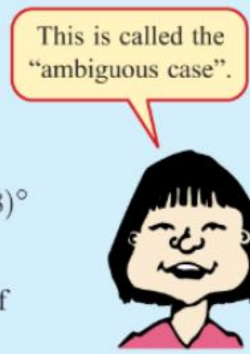
In triangle ABC, AB = 12 cm, AC = 8 cm, and angle B measures 28°.
Find, rounded to 1 decimal place, the measure of angle C.



$$\frac{\sin C}{12} = \frac{\sin 28°}{8} \quad \{\text{sine rule}\}$$

$$\therefore \sin C = \frac{12 \times \sin 28°}{8}$$

$$\text{Now} \quad \sin^{-1}\left(\frac{12 \times \sin 28°}{8}\right) \approx 44.8°$$



Since the angle at C could be acute or obtuse, $C \approx 44.8°$ or $(180 - 44.8)°$

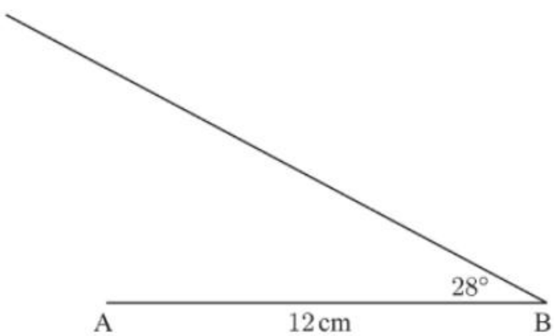
$\therefore C \approx 44.8°$ or $135.2°$

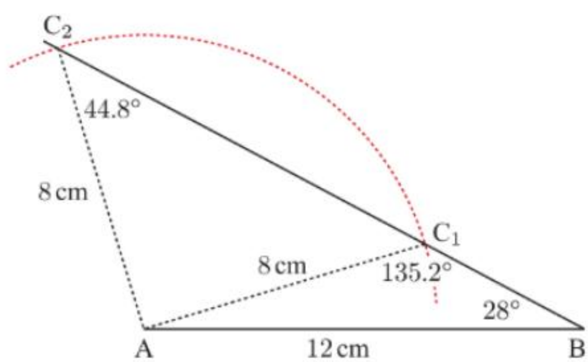
In this case there is insufficient information to determine the actual shape of the triangle.

The validity of the two answers in the above **Example** can be demonstrated by a simple construction.

Step 1: Draw [AB] of length 12 cm, and construct an angle of 28° at B.

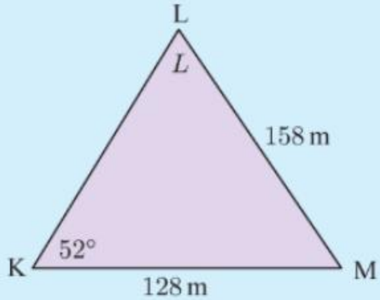
Step 2: From A, draw an arc of radius 8 cm.





Example 4 **Self Tutor**

In triangle KLM, $\mathrm{L\widehat{K}M}$ measures $52°$, $\mathrm{LM} = 158$ m, and $\mathrm{KM} = 128$ m.
Find the measure of angle L.



$$\frac{\sin L}{128} = \frac{\sin 52°}{158} \quad \{\text{sine rule}\}$$

$$\therefore \ \sin L = \frac{128 \times \sin 52°}{158}$$

$$\text{Now} \quad \sin^{-1}\left(\frac{128 \times \sin 52°}{158}\right) \approx 39.7°$$

$\therefore$ since L could be acute or obtuse,

$$L \approx 39.7° \quad \text{or} \quad (180 - 39.7)° \approx 140.3°$$

However, we can reject $L \approx 140.3°$ as $140.3° + 52° > 180°$ which is impossible.

$\therefore$ the angle $L \approx 39.7°$.

EXERCISE 25B.2

1 In triangle ABC, $\mathrm{AB} = 10$ cm, $\mathrm{BC} = 7$ cm, and $\mathrm{C\widehat{A}B}$ measures $42°$.

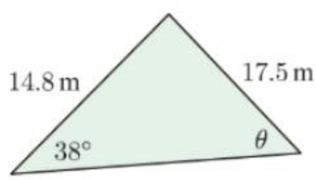
- **a** Find the two possible values for $\mathrm{A\widehat{C}B}$.
- **b** Draw a diagram to illustrate the two possible triangles.

2 In triangle PQR, $\mathrm{PQ} = 10$ cm, $\mathrm{QR} = 12$ cm, and $\mathrm{R\widehat{P}Q}$ measures $42°$.

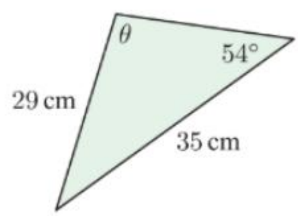
- **a** Show that there is only one possible value for $\mathrm{P\widehat{R}Q}$, and state its measure.
- **b** Draw a diagram to demonstrate that only one triangle can be drawn from the information given.

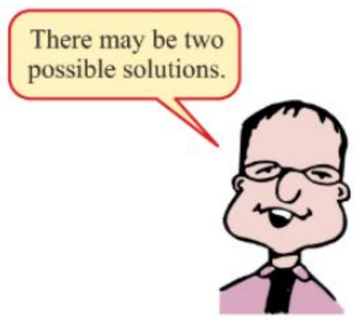
3 The following diagrams are not drawn to scale, but the information on them is correct.
Find the value of θ:

a

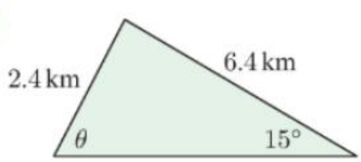


b

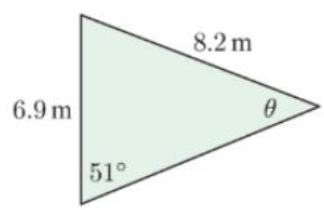




c



d



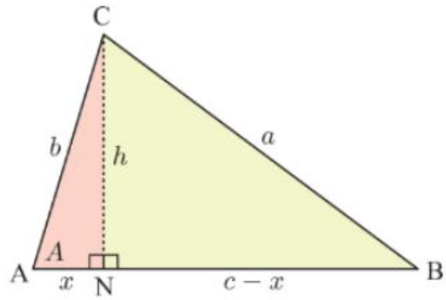
4 In triangle ABC, find the measure of:

- **a** angle A if $a = 12.6$ cm, $b = 15.1$ cm, and $\mathrm{A\widehat{B}C} = 65°$
- **b** angle B if $b = 38.4$ cm, $c = 27.6$ cm, and $\mathrm{A\widehat{C}B} = 43°$
- **c** angle C if $a = 5.5$ km, $c = 4.1$ km, and $\mathrm{B\widehat{A}C} = 71°$.

C THE COSINE RULE

The **cosine rule** relates the three sides of a triangle and one of its angles.

Consider triangle ABC with side lengths a, b, and c as shown.



Using Pythagoras' theorem in $\triangle$BCN,

$$a^2 = h^2 + (c-x)^2$$
$$\therefore\ a^2 = h^2 + c^2 - 2cx + x^2$$

In $\triangle$ACN, $b^2 = h^2 + x^2$ and so $h^2 = b^2 - x^2$

Thus, $a^2 = (b^2 - x^2) + c^2 - 2cx + x^2$

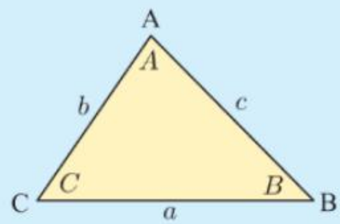
$$\therefore\ a^2 = b^2 + c^2 - 2cx$$

In $\triangle$ACN, $\cos A = \frac{x}{b}$ and so $x = b\cos A$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$

In any triangle ABC with sides a, b, and c units, and opposite angles A, B, and C respectively,

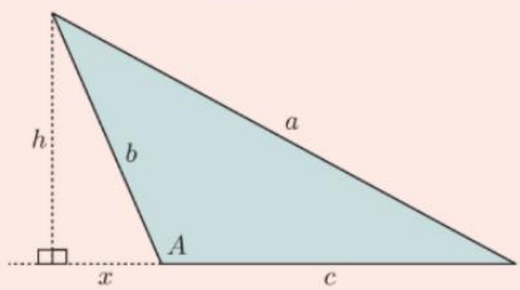
$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$
$$\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}.$$



ACTIVITY

Prove the **cosine rule** $a^2 = b^2 + c^2 - 2bc\cos A$ in the case where A is obtuse.

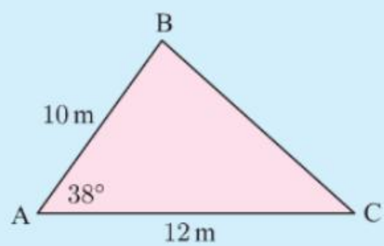
Hint: Use the supplementary angle formula $\cos(180° - A) = -\cos A$.

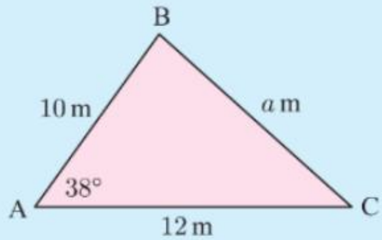


If we are given two sides of a triangle and the included angle, we can use the cosine rule to find the third side.

Example 5 **Self Tutor**

Find, correct to 2 decimal places, the length of [BC].





Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore \ a = \sqrt{12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 38^\circ}$$

$$\therefore \ a \approx 7.41$$

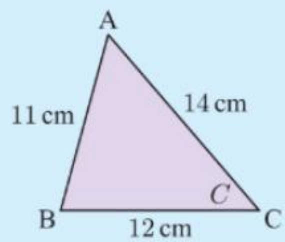
$\therefore$ [BC] is approximately 7.41 m long.

If we know all three side lengths of a triangle, we can use the cosine rule to find any of the angles. To do this, we rearrange the original cosine rule formulae:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

Example 6 **Self Tutor**

Find the measure of angle C in the given figure.



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \ \cos C = \frac{12^2 + 14^2 - 11^2}{2 \times 12 \times 14}$$

$$\therefore \ \cos C = \frac{219}{336}$$

$$\therefore \ C = \cos^{-1}\left(\frac{219}{336}\right)$$

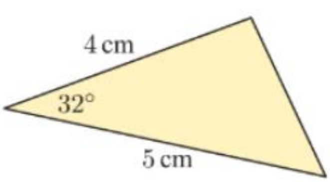
$$\therefore \ C \approx 49.3^\circ$$

GEOMETRY PACKAGE

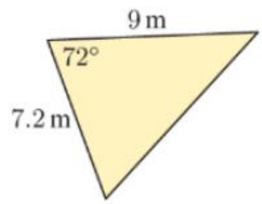
EXERCISE 25C

1 Find the length of the remaining side in the triangle:

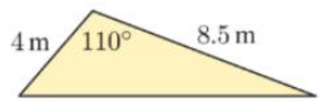
a



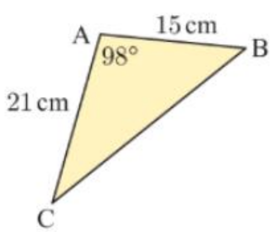
b



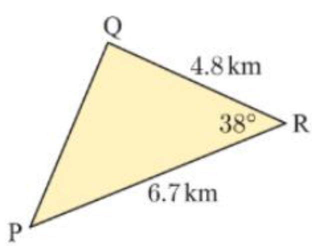
c



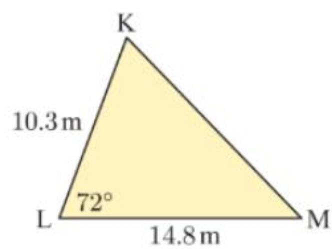
d



e

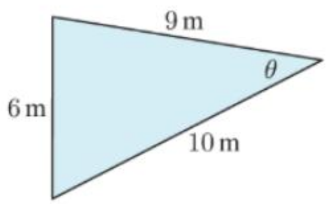


f



2 Find θ, rounded to 1 decimal place:

a



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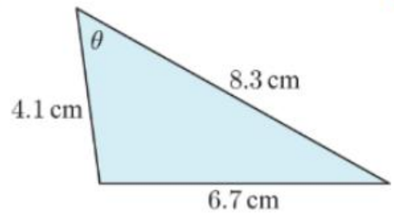

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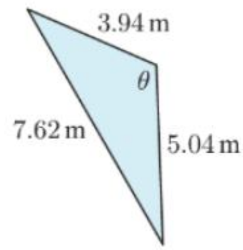
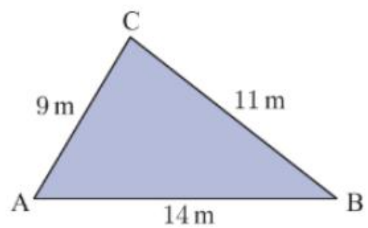


3 Find the measure of all angles of the triangle:

a



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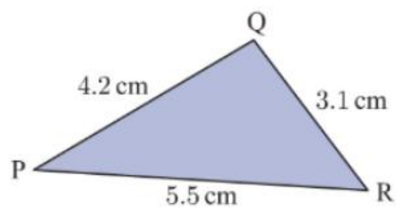

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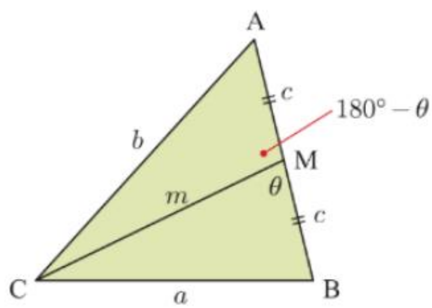


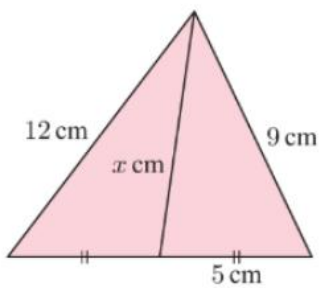
a Use the cosine rule in $\triangle$BCM to find $\cos\theta$ in terms of a, c, and m.

b Use the cosine rule in $\triangle$ACM to find $\cos(180° - \theta)$ in terms of b, c, and m.

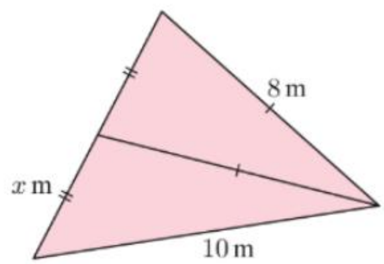
c Use the supplementary angle formula $\cos(180° - \theta) = -\cos\theta$ to prove **Apollonius' median theorem**: $a^2 + b^2 = 2m^2 + 2c^2$.

d Find x in the following:

i



ii



5 In triangle ABC, AB $= 10$ cm, AC $= 9$ cm, and $\mathrm{A\widehat{B}C} = 60°$. Let BC $= x$ cm.

a Use the cosine rule to show that x is a solution of $x^2 - 10x + 19 = 0$.

b Solve this equation for x.

c Use a scale diagram and a compass to explain why there are two possible values of x.

INVESTIGATION — ANGLE SIZES

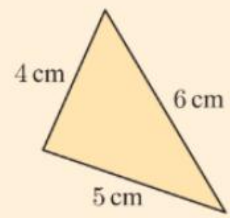
If we know the side lengths of a triangle, can we tell which angles are the largest and smallest?

What to do:

1 In the triangle alongside, calculate the size of the angle opposite:

a the 4 cm side **b** the 5 cm side

c the 6 cm side.

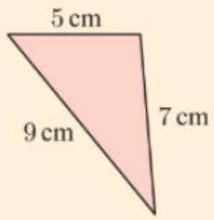


Record your results in a table like the one shown.

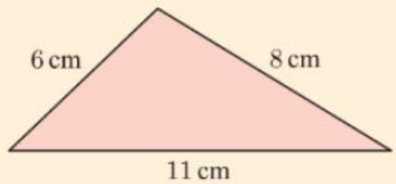
Side length	*Opposite angle*
4 cm	
5 cm	
6 cm	

2 Fill in a similar table for each of the following triangles:

a

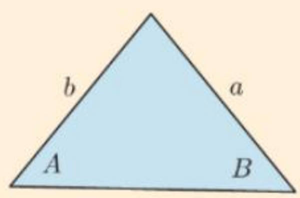


b



3 In each of your tables, compare the lengths of the sides with the sizes of their opposite angles. What do you notice?

4 Consider the sides a and b of a triangle, with opposite angles A and B respectively.
Copy and complete: If $a > b$, then A ...... B.



D PROBLEM SOLVING

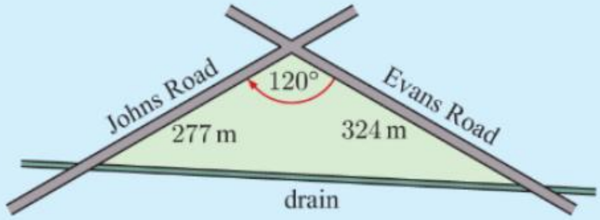
When using trigonometry to solve problems, you should draw a diagram of the situation. The diagram should be reasonably accurate, and all important information should be clearly marked on it.

Example 7 **Self Tutor**

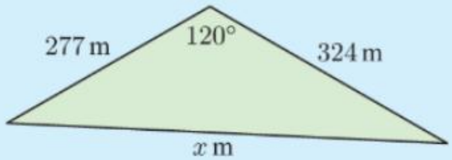
A triangular property is bounded by two roads and a long, straight drain. Find:

a the area of the property in hectares

b the length of the drain boundary.



a Area $= \frac{1}{2} \times 277 \times 324 \times \sin 120°$
$\approx 38\,862 \text{ m}^2$
≈ 3.89 ha $\{1 \text{ ha} = 10\,000 \text{ m}^2\}$



b Let the drain boundary be x m long.

$\therefore \; x^2 = 277^2 + 324^2 - 2 \times 277 \times 324 \times \cos 120°$ {cosine rule}

$\therefore \; x = \sqrt{277^2 + 324^2 - 2 \times 277 \times 324 \times \cos 120°}$

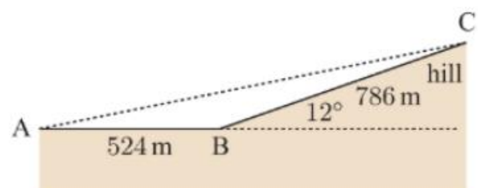
$\therefore \; x \approx 521$

The drain boundary is approximately 521 m long.

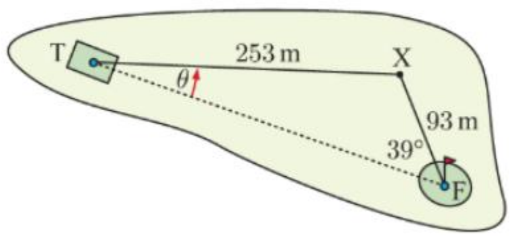
EXERCISE 25D

1 Two farm houses A and B are 10.3 km apart. A third farm house C is located such that $B\widehat{A}C = 83°$ and $A\widehat{B}C = 59°$. How far is C from A?

2 A roadway is horizontal from A to B, then rises up a 12° incline from B to C.
How far is it directly from A to C?



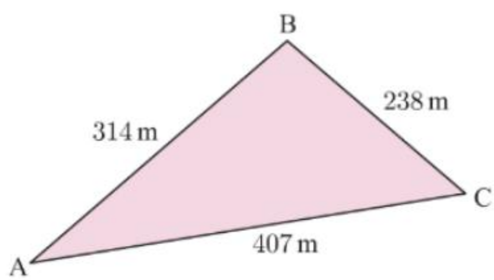
3 Sharon drives a golf ball 253 m from the tee T to point X on the fairway. X is 93 m from the flag, and $X\widehat{F}T$ is 39°.
Find the angle θ that Sharon's drive was off line.



4 Hazel's property is triangular with the dimensions shown.

a Find the measure of the angle at A, rounded to 2 decimal places.

b Find the area of Hazel's property, to the nearest hectare.



5 Answer the **Opening Problem** on page **478**.

Example 8 **Self Tutor**

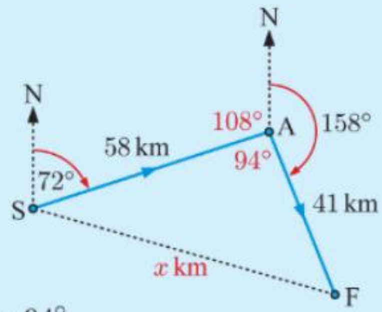
A ship sails 58 km on the bearing 072°. Once it has passed a reef, it turns and sails 41 km on the bearing 158°. How far is the ship from its starting point?

We suppose the ship starts at S, sails to A, then changes direction and sails to F.

$\widehat{SAN} = 180° - 72° = 108°$ {co-interior angles}

$\therefore\ \widehat{SAF} = 360° - 158° - 108°$
$= 94°$ {angles at a point}

Let SF $= x$ km.



Using the cosine rule, $x^2 = 58^2 + 41^2 - 2 \times 58 \times 41 \times \cos 94°$

$\therefore\ x = \sqrt{58^2 + 41^2 - 2 \times 58 \times 41 \times \cos 94°}$

$\therefore\ x \approx 73.3$

The ship is about 73.3 km from its starting point.

6 Fred walks 83 m in the direction 111°, and then 78 m in the direction 214°. How far is Fred from his starting point?

7 A boat travels 13 km in the direction 138°, and then a further 11 km in the direction 067°. Find the distance and bearing of the boat from its starting point.

8 Mount X is 9 km from Mount Y, on a bearing 146°. Mount Z is 14 km from Mount X, and on a bearing 072° from Mount Y. Find the bearing of X from Z.

9 X is 20 km north of Y. A mobile telephone mast M is to be placed 15 km from Y so the bearing of M from X is 140°.

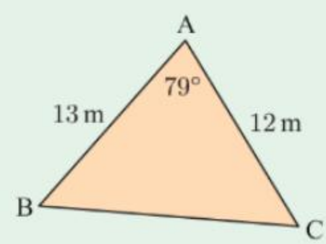
a Draw a sketch to show the two possible positions where the mast could be placed.

b Calculate the distance of each of these positions from X.

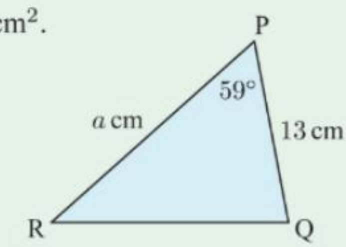
REVIEW SET 25A

1 **a** Find the area of triangle ABC, to the nearest m^2.

b Find the length of side [BC] rounded to 1 decimal place.

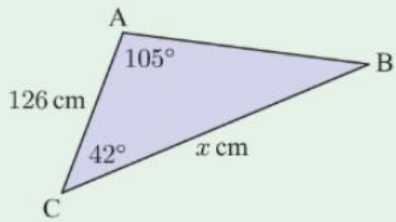


2 Triangle PQR has area 107 cm^2.
Find the value of a.

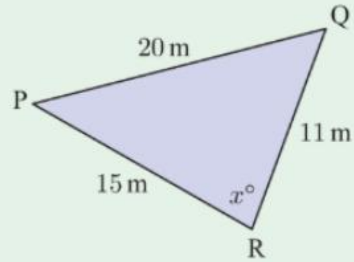


3 Find the value of x:

a

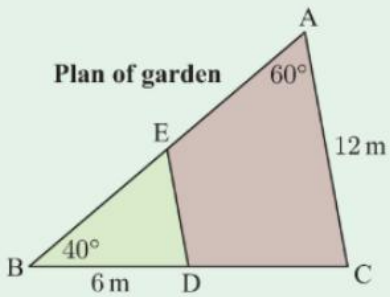


b



4 Stuart swam 200 m in the direction 124°, then 150 m in the direction 156°. Find the distance and bearing of Stuart from his starting point.

5

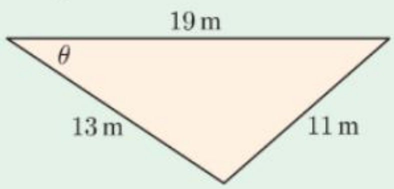


In the given plan view, $AC = 12$ m, $B\widehat{A}C = 60°$, and $A\widehat{B}C = 40°$.
D is a post 6 m from corner B, E is a tap, and triangle BDE is a lawn with area 13.5 m^2.

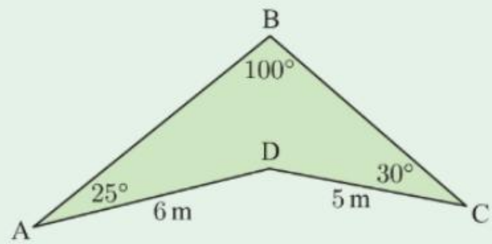
a Calculate the length DC.
b Calculate the length BE.
c Find the area of quadrilateral ACDE.

REVIEW SET 25B

1 **a** Find the angle θ.
b Hence find the area of this triangle.



2 Find the distance between A and C.



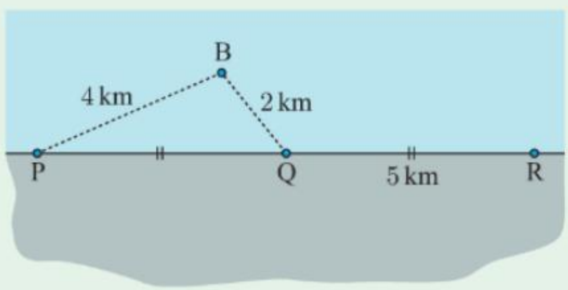
3 Triangle ABC has $AB = 12$ m, $BC = 10$ m, and $B\widehat{A}C = 40°$. Find the two possible values for $A\widehat{C}B$.

4 A surveyor at point A measures 200 m in the direction 138° to point B. From B the surveyor then measures 150 m in the direction 256° to point C.

a How far is C from A?
b What is the bearing of C from A?

5 Ports P, Q, and R are equally spaced along the coast. A boat B is 4 km from P, and 2 km from Q.

a Show that $\cos B\widehat{Q}P = \frac{13}{20}$.
b Hence, find $\cos B\widehat{Q}R$.
c Show that the boat is $\sqrt{42}$ km from port R.



Chapter 26

Vectors

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Introduction to calculus

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ANSWERS

EXERCISE 1A.1

1 a 125 b 27 c 32 d 343 e 540 f 1176 g 2925 h 4400

2 a $50 = 2 \times 5^2$ b $98 = 2 \times 7^2$ c $108 = 2^2 \times 3^3$ d $375 = 3 \times 5^3$ e $1128 = 2^3 \times 3 \times 47$ f $784 = 2^4 \times 7^2$ g $952 = 2^3 \times 7 \times 17$ h $6500 = 2^2 \times 5^3 \times 13$

3 a $n = 5$ b $n = 8$ c $n = 12$

4 a $n = 3$ b $n = 6$ c $n = 10$ **5** 3 **6** 7

EXERCISE 1A.2

1 a -1 b 1 c -1 d 1 e -1

2 a 9 b -27 c -27 d 27

3 a 25 b -25 c -125 d 125

EXERCISE 1A.3

1 a 256 b 625 c -243 d 2401 e 512 f 117 649 g $-117\,649$ h 1.795 856 326 i $-0.005\,487\,423\,935$ j $-325\,687.9871$

2 a i $0.\overline{1}$ ii $0.\overline{1}$ iii 0.0625 iv 0.0625 v $0.\overline{012\,345\,679}$ vi $0.\overline{012\,345\,679}$ vii 1 viii 1

b i It is the reciprocal of the number raised to the positive power.
ii Any non-zero number raised to the power zero is 1.

EXERCISE 1B

1 a 3^7 b x^9 c x^{5+n} d t^{12} e 7^4 f x^4 g t^{6-x} h t^{3m-1} i 5^6 j t^{12} k y^{3m} l a^{12m}

2 a 11^2 b 2^5 c 3^4 d 2^4 e 5^4 f 7^{t+2} g 3^{a-2} h 2^{3p-2} i 7^2 j 3^{2-x} k 5^{4t} l 2^{3k-12} m 2^{2a-b} n 2^{3x-4y} o 5^{2x+4} p 3^6

3 a x^2y^2 b a^3b^3 c $x^2y^2z^2$ d $27b^3$ e $625a^4$ f $100\,000x^5y^5$ g $\frac{p^2}{q^2}$ h $\frac{m^3}{n^3}$ i $\frac{x^4}{81}$ j $\frac{125}{z^3}$ k $\frac{16a^4}{b^4}$ l $\frac{27x^3}{64y^3}$

4 a $8b^5$ b a^2b^2 c $6a^4b^2$ d $\frac{x^2y}{3}$ e $\frac{a^6}{125b^3}$ f $\frac{8r^2}{5}$ g $16c^4d^3$ h $\frac{5k^2}{16}$

5 a 1 b $\frac{1}{6}$ c $\frac{1}{4}$ d 1 e 16 f $\frac{1}{16}$ g 125 h $\frac{1}{125}$ i 49 j $\frac{1}{49}$ k 1000 l $\frac{1}{1000}$

6 a 10^3 b 10^6 c 10^{-3} d 10^{-8}

7 a 2^3 b 2^{-3} c 3^3 d 3^{-3} e 5^3 f 5^{-3} g 2^5 h 2^{-5} i 3^4 j 3^{-4} k 5^{-2} l 2^0 or 3^0 or 5^0

8 a 1 b 1 c 2, $t \neq 0$ d 1, $t \neq 0$ e 3 f $\frac{1}{25}$ g $\frac{1}{x^5}$ h $\frac{8}{3}$ i $\frac{3}{2}$ j 3 k $\frac{4}{5}$ l $\frac{11}{3}$ m 9 n $\frac{27}{8}$ o $\frac{8}{27}$ p $\frac{4}{25}$

9 a $\frac{1}{3b}$ b $\frac{3}{b}$ c $\frac{7}{a}$ d $\frac{1}{7a}$ e t^2 f $\frac{y}{3x}$ g $\frac{1}{25t^2}$ h $\frac{t^2}{5}$ i $\frac{x}{y}$ j $\frac{1}{xy}$ k $\frac{x}{y^3}$ l $\frac{1}{x^3y^3}$ m $\frac{1}{3pq}$ n $\frac{3}{pq}$ o $\frac{3p}{q}$ p x^3y^5 q $\frac{125y^9}{x^6}$ r $\frac{4d^6}{c^2}$ s $\frac{r^6t^2}{9}$ t $\frac{125}{8p^3q^6}$

10 a
$$\begin{aligned}\frac{1}{a^{-n}} &= \frac{a^0}{a^{-n}}\\ &= a^{0-(-n)}\\ &= a^{0+n}\\ &= a^n\end{aligned}$$
b
$$\begin{aligned}\left(\frac{a}{b}\right)^{-n} &= \left[\left(\frac{a}{b}\right)^{-1}\right]^n\\ &= \left(\frac{b}{a}\right)^n\\ &= \frac{b^n}{a^n}\end{aligned}$$

11 a m s^{-1} b g s^{-1} c $\text{m}^3\,\text{h}^{-1}$ d $\text{cm}^2\,\text{s}^{-1}$ e $\text{cm}^3\,\text{min}^{-1}$ f m s^{-2}

EXERCISE 1C

1 a 4.816×10^3 b 4.816×10^6 c 4.816×10^0 d 4.816×10^{-2}

2 a 312 b 0.003 12 c 312 000 d 3.12

3 a 2.3×10^2 b 5.39×10^4 c 3.61×10^{-2} d 6.8×10^{-3} e 3.26×10^0 f 5.821×10^{-1} g 3.61×10^8 h 1.674×10^{-6}

4 a 2300 b 0.023 c 564 000 d 0.000 793 1 e 9.97 f 60 400 000 g 0.4215 h 0.000 000 036 21

5 a $\approx 4 \times 10^6$ red blood cells b 8×10^{-4} m c 6.38×10^6 m d 4.3252×10^{19} arrangements

6 a 6 990 000 m b 0.018 cm c 32 000 000 bacteria d 0.000 008 2 tonnes

7 a 6×10^{10} b 2.8×10^9 c 5.6×10^{-8} d 5.4×10^{-6} e 9×10^{10} f 1.6×10^{15} g 1.6×10^{-11} h 1.25×10^{-7}

8 a 2×10^3 b 3×10^{-2} c 2×10^8

9 a 1000 times b i 5×10^{-21} ii 100 000 times c 5×10^{10} times

10 a $\approx 4.01 \times 10^{13}$ b $\approx 2.59 \times 10^{12}$ c $\approx 7.08 \times 10^{-9}$ d $\approx 4.87 \times 10^{-11}$ e $\approx 8.01 \times 10^6$ f $\approx 3.55 \times 10^{-9}$ g $\approx 1.57 \times 10^{13}$ h $\approx 6.55 \times 10^{-22}$

11 a 5.84×10^7
b i 1.60×10^5 vehicles ii 5.84×10^8 vehicles

12 a i 2.88×10^8 km ii $\approx 7.01 \times 10^{10}$ km b 0.9 s c ≈ 26.6 times d i Microbe C ii ≈ 32.9 times

13 As land areas are rounded to 2 significant figures, the answers should be rounded to 2 significant figures.
a 9.1×10^6 km^2
b Quebec > British Columbia > Ontario > Alberta > Saskatchewan > Manitoba > Newfoundland and Labrador > New Brunswick > Nova Scotia > Prince Edward Island
c i ≈ 2.5 times ii ≈ 330 times d $\approx 0.58\%$

REVIEW SET 1A

1 **a** $96 = 2^5 \times 3$ **b** $180 = 2^2 \times 3^2 \times 5$
c $154 = 2 \times 7 \times 11$ **d** $2125 = 5^3 \times 17$

2 **a** $n = 3$ **b** $n = -2$ **c** $n = 5$ **d** $n = 0$

3 **a** k^8 **b** p^5 **c** m^{48}

4 **a** $9w^2$ **b** $8x^6y^3$ **c** $\dfrac{a^6}{b^6}$ **d** $\dfrac{1}{125n^3}$

5 **a** $\frac{1}{7}$ **b** $\frac{3}{4}$ **c** $\frac{10}{11}$ **d** $\frac{16}{49}$

6 **a** 5.9×10^4 **b** 9×10^{-3}
c 6.085×10^6 **d** 7.71×10^{-6}

7 **a** $623\,000$ **b** $0.000\,300\,8$ **c** 4.597

8 **a** $\dfrac{1}{m^2n^2}$ **b** $\dfrac{m}{n^2}$ **c** $\dfrac{125y^6}{x^3}$

9 **a** 1.8×10^{12} **b** 4×10^6 **c** 1.5×10^{-8}

10 **a** **i** $\approx 1.08 \times 10^{12}$ m **ii** $\approx 2.59 \times 10^{13}$ m
iii $\approx 9.46 \times 10^{15}$ m
b **i** $\approx 3.34 \times 10^{-9}$ s **ii** $\approx 3.34 \times 10^{-11}$ s
iii $\approx 3.34 \times 10^{-12}$ s
c $\approx 8.71 \times 10^5$ times faster

REVIEW SET 1B

1 **a** -343 **b** -343 **c** 343 **d** $-\frac{1}{343}$

2 **a** $\frac{9}{25}$ **b** $\frac{25}{9}$ **c** $\frac{4}{5}$ **d** $\frac{64}{27}$

3 **a** m^5 **b** $\dfrac{1}{m}$ **c** $\dfrac{1}{m}$ **d** m^5

4 **a** 2^6 **b** 5^{2x-3} **c** 7^{k+7}

5 **a** $\dfrac{9}{a^2}$ **b** $4p^2q^2$ **c** $\dfrac{y}{x}$ **d** x

6 **a** $15c^7$ **b** $7x^3y$ **c** $9p^2q^6$

7 **a** $\dfrac{b^2}{a^2}$ **b** $\dfrac{8a^3b^6}{27}$

8 **a** 2.1×10^{15} **b** 4×10^{10} **c** 6×10^{-12}

9 70 times

10 **a** **i** 1.21×10^5 km **ii** 1.21×10^{10} cm
b 6.05×10^6 m
c Mercury < Mars < Venus < Earth < Neptune < Uranus < Saturn < Jupiter
d **i** ≈ 10.5 times larger **ii** ≈ 20.6 times larger

EXERCISE 2A

1 **a** $8 \in P$ **b** $k \notin S$ **c** $14 \notin \{\text{odd numbers}\}$
d $n(Y) = 9$

2 **a** **i** $A = \{1, 2, 3, 6\}$ **ii** finite **iii** $n(A) = 4$
b **i** $B = \{6, 12, 18, 24, ...\}$ **ii** infinite
c **i** $C = \{1, 17\}$ **ii** finite **iii** $n(C) = 2$
d **i** $D = \{17, 34, 51, 68, ...\}$ **ii** infinite
e **i** $E = \{2, 3, 5, 7, 11, 13, 17, 19\}$ **ii** finite
iii $n(E) = 8$
f **i** $F = \{12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$
ii finite **iii** $n(F) = 13$

3 **a** $M_3 = \{3, 6, 9, 12, 15, 18, 21, 24, ...\}$
b $F_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
c 3, 6, 12, 15, 30, 60

4 $n(\varnothing) = 0$

EXERCISE 2B

1 **a** true **b** true **c** true **d** true
e false **f** false **g** true **h** true

2 **a** rational **b** rational **c** rational **d** rational
e irrational **f** neither **g** rational **h** rational
i neither **j** rational

3 **a** $0.\overline{7} = \frac{7}{9}$ **b** $0.\overline{41} = \frac{41}{99}$ **c** $0.\overline{324} = \frac{12}{37}$

4 0.527 can be written as $\frac{527}{1000}$, where 527 and 1000 are integers.

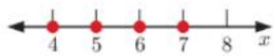
5 $0.\overline{9} = 1$ and $1 \in \mathbb{Z}$

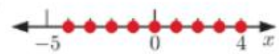
6 **Note:** There may be other answers.
a $\sqrt{2}, -\sqrt{2}$ are irrational, but $\sqrt{2} + (-\sqrt{2}) = 0$ which is rational.
b $\sqrt{2}, \sqrt{50}$ are irrational, but $\sqrt{2} \times \sqrt{50} = \sqrt{100} = 10$ which is rational.

EXERCISE 2C

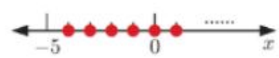
1 **a** The set of all real x such that x is greater than 4.
b The set of all integers x such that x is less than or equal to 5.
c The set of all real y such that y lies between 0 and 8.
d The set of all integers x such that x lies between 1 and 4, including 1 and 4.
e The set of all real t such that t lies between 2 and 7.
f The set of all real n such that n is less than or equal to 3, or n is greater than 6.

2 **a** $\{x \mid x > 3\}$ **b** $\{x \mid 2 < x \leqslant 5\}$
c $\{x \mid x \leqslant -1 \text{ or } x \geqslant 2\}$ **d** $\{x \mid -1 \leqslant x < 5, \ x \in \mathbb{Z}\}$
e $\{x \mid 0 \leqslant x \leqslant 6, \ x \in \mathbb{N}\}$ **f** $\{x \mid x < 0\}$

3 **a**


b


c


d


e


f


4 **a** $\{x \mid x > 7\}$ **b** $\{x \mid -8 < x < 15, \ x \in \mathbb{Z}\}$
c $\{x \mid 4 \leqslant x < 6, \ x \in \mathbb{Q}\}$

EXERCISE 2D

1 **a** $A \subseteq B$ **b** $A \not\subseteq B$ **c** $A \subseteq B$ **d** $A \not\subseteq B$
e $A \subseteq B$

2 **a** true **b** true **c** false **d** true **e** false **f** true

3 $E' = \{\text{odd integers}\}$ **4** $P' = \{\text{composites and } 1\}$

5 $B \not\subseteq A, \ C \subseteq A$

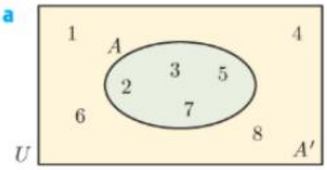
6 **a** false **b** true **c** false **d** true

7 **a** $A' = \{1, 3, 4, 7, 8, 9\}$ **b** $B' = \{1, 4, 6, 8, 9\}$
c $C' = \{2, 4, 6, 8\}$ **d** $D' = \{1, 2, 3, 5, 6, 7, 9\}$
e $E' = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (or $E' = U$)
f $F' = \{3, 4, 5, 6, 7, 8, 9\}$

8 **a** $P' = \{$A, B, D, E, G, H, I, K, L, N, O, Q, R, S, T, V, W, X$\}$
b $Q' = \{\text{vowels}\}$
c $R' = \{$B, C, D, E, F, G, J, K, M, N, Q, R, U, V, W, X, Y, Z$\}$
d $S' = \{$A, B, C, D, E, F, G, H, I, J$\}$

9 **a** $A' = \{1, 2, 3, 4, 13, 14, 15\}$
$B' = \{1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15\}$
b **i** false **ii** true **iii** true **iv** false

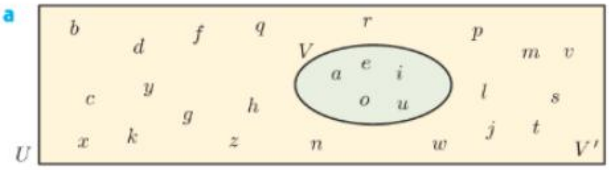
EXERCISE 2E

1 **a** 

$A = \{2, 3, 5, 7\}$

b $A' = \{1, 4, 6, 8\}$

c **i** $n(A) = 4$ **ii** $n(A') = 4$ **iii** $n(U) = 8$

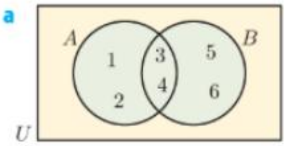
2 **a** 

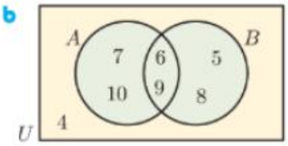
b $V' = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

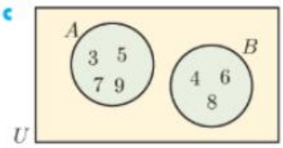
c **i** $n(V) = 5$ **ii** $n(V') = 21$ **iii** $n(U) = 26$

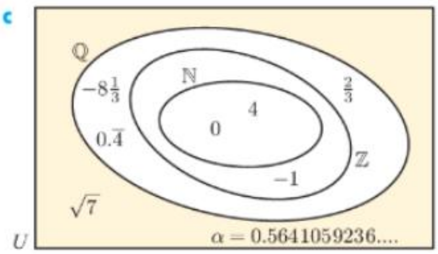
3 **a** **i** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
ii $N = \{3, 8\}$ **iii** $M = \{1, 3, 4, 7, 8\}$

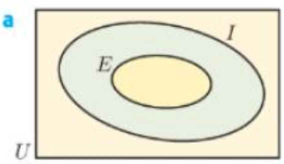
b $n(N) = 2$, $n(M) = 5$ **c** No, $N \subseteq M$.

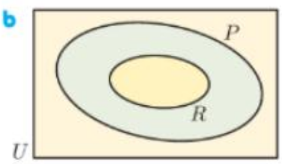
4 **a** 

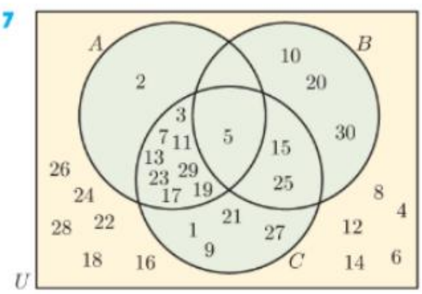
b 

c 

5 **a, b, c** 

6 **a** 

b 

7 

EXERCISE 2F.1

1 **a** **i** $C = \{1, 3, 7, 9\}$ **ii** $D = \{1, 2, 5\}$

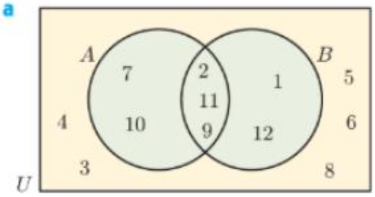
iii $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **iv** $C \cap D = \{1\}$

v $C \cup D = \{1, 2, 3, 5, 7, 9\}$

b **i** $n(C) = 4$ **ii** $n(D) = 3$ **iii** $n(U) = 9$
iv $n(C \cap D) = 1$ **v** $n(C \cup D) = 6$

2 **a** **i** $A = \{2, 7\}$ **ii** $B = \{1, 2, 4, 6, 7\}$
iii $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ **iv** $A \cap B = \{2, 7\}$
v $A \cup B = \{1, 2, 4, 6, 7\}$

b **i** $n(A) = 2$ **ii** $n(B) = 5$ **iii** $n(U) = 8$
iv $n(A \cap B) = 2$ **v** $n(A \cup B) = 5$

3 **a** 

b **i** $A \cap B = \{2, 9, 11\}$
ii $A \cup B = \{1, 2, 7, 9, 10, 11, 12\}$
iii $B' = \{3, 4, 5, 6, 7, 8, 10\}$

c **i** $n(A) = 5$ **ii** $n(B') = 7$
iii $n(A \cap B) = 3$ **iv** $n(A \cup B) = 7$

4 **a** $A \cap B = \{1, 3, 9\}$
b $A \cup B = \{1, 2, 3, 4, 6, 7, 9, 12, 18, 21, 36, 63\}$

5 **a** $X \cap Y = \{\text{B, M, T, Z}\}$
b $X \cup Y = \{\text{A, B, C, D, M, N, P, R, T, W, Z}\}$

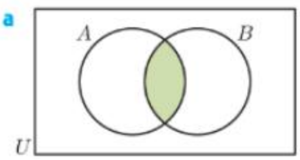
6 **a** **i** $n(A) = 8$ **ii** $n(B) = 10$
iii $n(A \cap B) = 3$ **iv** $n(A \cup B) = 15$

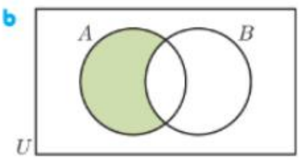
b $n(A) + n(B) - n(A \cap B) = 8 + 10 - 3$
$= 15 = n(A \cup B)$

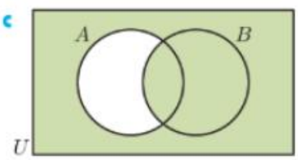
7 **a** $X \cap Y = \varnothing$ **b** X and Y are disjoint.

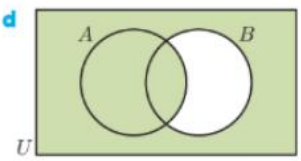
8 **a** $A \cup A' = U$ **b** $A \cap A' = \varnothing$

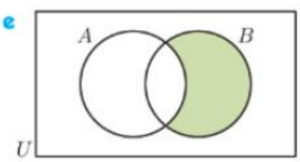
EXERCISE 2F.2

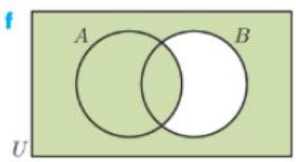
1 **a** 

b 

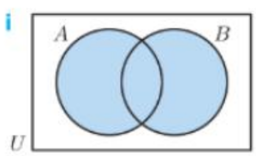
c 

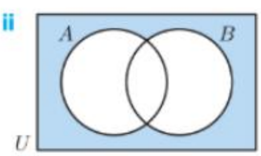
d 

e 

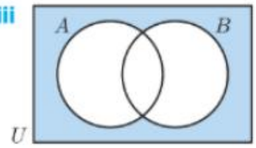
f 

2 **a** in A but not in B
b the complement of 'in exactly one of A or B'

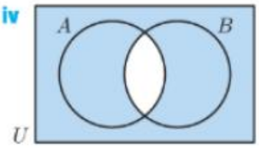
3 **a** **i** 

ii 

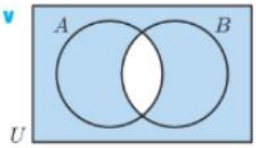
iii



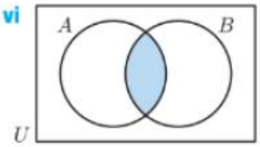
iv



v



vi



b **i** Since **a iv** and **a v** have the same regions shaded, $(A \cap B)' = A' \cup B'$.

ii Since **a ii** and **a iii** have the same regions shaded, $(A \cup B)' = A' \cap B'$.

EXERCISE 2G

1 **a** 10 **b** 9 **c** 14 **d** 4 **e** 7 **f** 2

2 **a** **i** $n(A) = a + b$ **ii** $n(B) = b + c$

iii $n(A \cap B) = b$ **iv** $n(A \cup B) = a + b + c$

b **i** $n(A) + n(B) - n(A \cap B) = a + b + b + c - b$
$= a + b + c$
$= n(A \cup B)$

ii $n(A) + n(B) - n(A \cup B)$
$= a + b + b + c - (a + b + c)$
$= a + 2b + c - a - b - c$
$= b$
$= n(A \cap B)$

iii If A and B are disjoint, then $n(A \cap B) = b = 0$.
So, $n(A \cup B) = a + b + c$
$= a + c$
and $n(A) + n(B) = a + b + b + c$
$= a + c$
$= n(A \cup B)$

3 **a** $n(A) - n(A \cap B) = (a + b) - b$
$= a$
$= n(A \cap B')$

b $n(U) - n(A' \cap B) = (a + b + c + d) - c$
$= a + b + d$
$= n(A \cup B')$

4 **a** $n(A \cup B) = 17$ **b** $n(B \cap A') = 5$

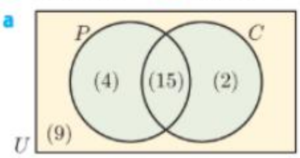
5 **a** $n(N) = 7$ **b** $n((M \cup N)') = 10$

EXERCISE 2H

1 **a** 18 students **b** 2 students **c** 17 students **d** 12 students

2 **a** 75 people **b** 9 people **c** 24 people **d** 42 people

3 **a**



b **i** 21 students **ii** 4 students **iii** 6 students **iv** 9 students

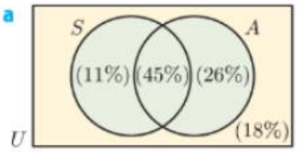
4 **a** 20 students **b** 32 students **c** 25 students **d** 13 students

5 **a** 10 students **b** 5 students

6 **a** 18 students **b** 38 students

7 **a** 15 students **b** 14 students **c** 8 students

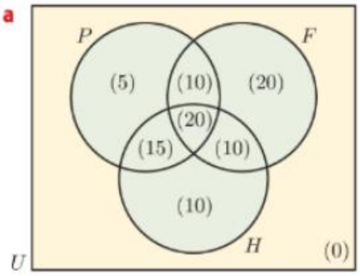
8 **a**



b **i** 45% **ii** 82% **iii** 11% **iv** 37%

9 11 violin players **10** 43% **11** 19 places

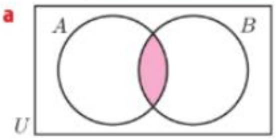
12 **a**



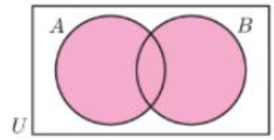
b 15 students **c** 55 students

EXERCISE 2I

1 **a**

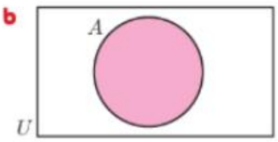


The shaded region could be either $A \cap B$ or $B \cap A$.
$\therefore\ A \cap B = B \cap A$



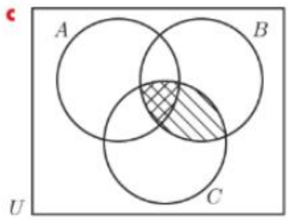
The shaded region could be either $A \cup B$ or $B \cup A$.
$\therefore\ A \cup B = B \cup A$

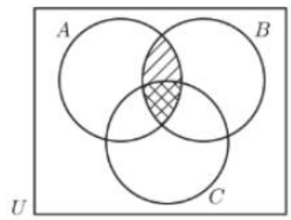
b



The shaded region is either $A \cap A$, A, or $A \cup A$.
$\therefore\ A \cap A = A$
and $A \cup A = A$

c



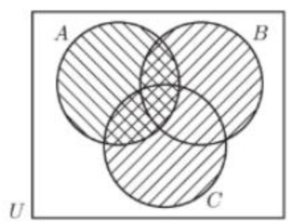


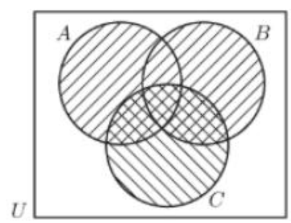
represents $B \cap C$
represents $A \cap (B \cap C)$

represents $A \cap B$
represents $(A \cap B) \cap C$

Region shaded is the same in each case.

$\therefore\ A \cap (B \cap C) = (A \cap B) \cap C$



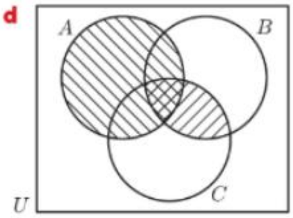


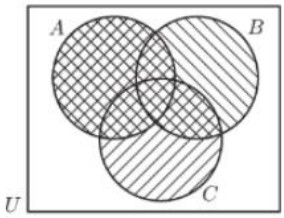
represents A
represents $B \cup C$
whole shaded region represents $A \cup (B \cup C)$

represents C
represents $A \cup B$
whole shaded region represents $(A \cup B) \cup C$

Total shaded region is the same in each case.

d

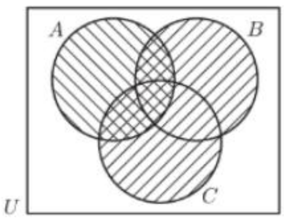


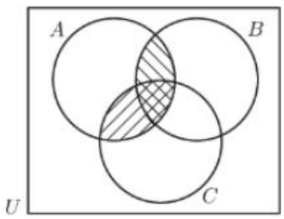


represents A represents $A \cup B$
represents $B \cap C$ represents $A \cup C$

The total shaded region in the first diagram is the same as the double shaded region in the second diagram.

$\therefore \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



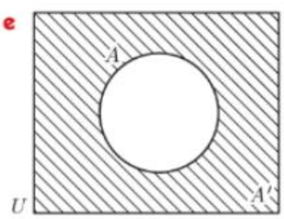


represents A represents $A \cap B$
represents $B \cup C$ represents $A \cap C$

The double shaded region in the first diagram is the same as the total shaded region in the second diagram.

$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e



represents A
represents A'

A' is the region outside A and is shaded.

$(A')'$ is the region not in A' and is unshaded.

$\therefore \quad (A')' = A$

2 **a** $A \cup (B \cup A')$
$= A \cup (A' \cup B)$ {commutative law}
$= (A \cup A') \cup B$ {associative law}
$= U \cup B$ {complement law}
$= U$ $\{A \cup U = U\}$

b $A \cap (B \cap A')$
$= A \cap (A' \cap B)$ {commutative law}
$= (A \cap A') \cap B$ {associative law}
$= \varnothing \cap B$ {complement law}
$= \varnothing$ $\{A \cap \varnothing = \varnothing\}$

c $A \cup (B \cap A')$
$= (A \cup B) \cap (A \cup A')$ {distributive law}
$= (A \cup B) \cap U$ {complement law}
$= A \cup B$ {identity law}

d $(A' \cup B')'$
$= ((A \cap B)')'$ $\{A' \cup B' = (A \cap B)'\}$
$= A \cap B$ $\{(A')' = A\}$

e $(A \cup B) \cap (A' \cap B')$
$= (A \cup B) \cap (A \cup B)'$ $\{A' \cap B' = (A \cup B)'\}$
$= \varnothing$ {complement law}

f $(A \cup B) \cap (C \cup D)$
$= ((A \cup B) \cap C) \cup ((A \cup B) \cap D)$ {distributive law}
$= (A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$ {distributive law}
$= (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$ {commutative law}

REVIEW SET 2A

1 1.3 can be written as $\frac{13}{10}$, and 13 and 10 are integers.

2 No, $\sqrt{4000} \notin \mathbb{Q}$.

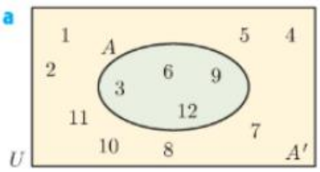
3 **a** yes **b** $P = \{23, 29, 31, 37\}$, $n(P) = 4$

4 S is the set of all real t such that t lies between -1 and 3, including -1.

5 $\{x \mid 0 < x \leqslant 5\}$ **6** **a** $P \not\subseteq Q$ **b** $P \subseteq Q$

7 **a** $A' = \{1, 2, 4, 5, 6, 8, 10\}$ **b** $B' = \{1, 2, 3, 5, 7\}$

8 **a**

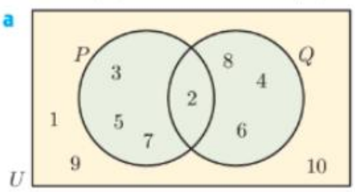


b $A' = \{1, 2, 4, 5, 7, 8, 10, 11\}$ **c** $n(A') = 8$

9 **a** False, as $0 \in \mathbb{N}$, but $0 \notin \mathbb{Z}^+$.
b False, as $\frac{1}{2} \in \mathbb{Q}$, but $\frac{1}{2} \notin \mathbb{Z}$.

10 **a** **i** $A = \{1, 2, 3, 4, 5\}$ **ii** $B = \{1, 2, 7\}$
iii $U = \{1, 2, 3, 4, 5, 6, 7\}$
iv $A \cup B = \{1, 2, 3, 4, 5, 7\}$ **v** $A \cap B = \{1, 2\}$
b **i** $n(A) = 5$ **ii** $n(B) = 3$ **iii** $n(A \cup B) = 6$

11 **a**



b **i** $P \cap Q = \{2\}$ **ii** $P \cup Q = \{2, 3, 4, 5, 6, 7, 8\}$
iii $Q' = \{1, 3, 5, 7, 9, 10\}$
c **i** $n(P') = 6$ **ii** $n(P \cap Q) = 1$ **iii** $n(P \cup Q) = 7$
d yes

12 **a** The shaded region is the complement of X, that is, everything that is not in X.
b The shaded region represents 'in exactly one of X or Y but not both'.
c The shaded region represents 'in X or not in Y'.

13 **a** 11 **b** 14 **c** 21 **d** 2 **14** 200 families

REVIEW SET 2B

1 no **2** $0.\overline{51} = \frac{17}{33}$, and 17 and 33 are integers.

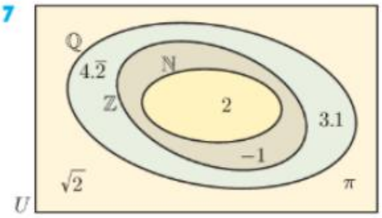
3

4 **a** **i** $A = \{1, 3, 5, 15\}$ **ii** finite **iii** $n(A) = 4$
b **i** $B = \{8, 16, 24, 32, ...\}$ **ii** infinite
c **i** $C = \{31, 33, 35, 37, 39, 41, 43, 45, 47, 49\}$
ii finite **iii** $n(C) = 10$

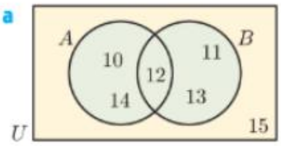
d **i** $D = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
ii finite **iii** $n(D) = 10$

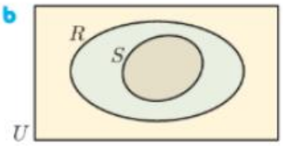
5 $Q \subseteq P,\quad R \nsubseteq P$

6 $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
a $A = \{2, 3, 5, 7, 11\}$ **b** $A' = \{1, 4, 6, 8, 9, 10, 12\}$
c $n(A) = 5$ **d** $n(A') = 7$ **e** $n(U) = 12$

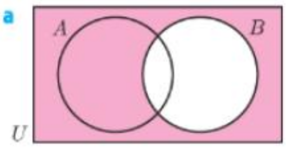
7


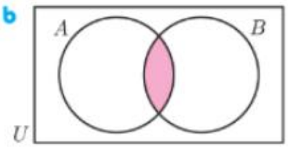
U is the set of all real numbers, $\mathbb{R}$.

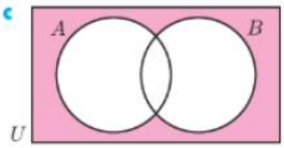
8 **a**


b


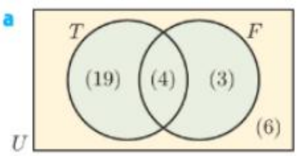
9 **a** $A \cap B = \{1, 2, 3, 6\}$
b $A \cup B = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

10 **a**


b


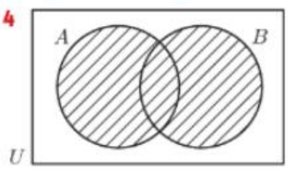
c


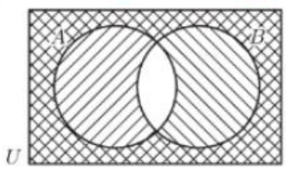
11 **a** $n(A \cup B) = 18$ **b** $n(B, \text{ but not } A) = 4$

12 **a**


b **i** 4 children
ii 19 children
iii 3 children

13 **a** 65% **b** 35% **c** 22% **d** 28% **e** 9%

14




represents $A \cup B$

represents A'

represents B'

The unshaded region in the first diagram is the same as the double shaded region in the second diagram.

$\therefore\ (A \cup B)' = A' \cap B'$

EXERCISE 3A.1

1 **a** $20 + 5x$ **b** $2x - 2y$ **c** $3t - 21$ **d** $21 + 7p$
e $9b + 9c$ **f** $4x - 20$ **g** $12 + 2j$ **h** $8q - 8p$

2 **a** $44x + 11y$ **b** $2m - 14n$ **c** $18g - 12h$ **d** $12 + 9x$
e $6x + 2z$ **f** $6c - 18d$ **g** $5p + 30q$ **h** $12a - 4bc$

3 **a** $10x - 2x^2$ **b** $15x - 3x^2$ **c** $4a^3 - 12a^2$
d $28n + 14n^2$ **e** $3x^2 - 2x$ **f** $4x + 2x^2$
g $pq^2 - p^2q$ **h** $ab^3 - ab^2$

4 **a** $-3x - 3$ **b** $-2x - 6$ **c** $-5x + 10$ **d** $-18 + 6x$
e $-a - 4$ **f** $-x + 2$ **g** $-15 + 5x$ **h** $-10 + 4c$

5 **a** $-2a - a^2$ **b** $-b^2 + 4b$ **c** $-2c - c^2$
d $x^2 - 7x$ **e** $-6n + 3n^2$ **f** $-4y^2 - 12y$
g $-18a + 6a^2$ **h** $-4b + 10b^2$

6 **a** $3a^2 + 9a + 3$ **b** $5x^2 - 15x + 10$
c $-8c^2 + 12c + 28$ **d** $6a^3 - 10a^2 + 2a$

7 **a** $6x + 8$ **b** $7x - 11$ **c** $m + 18$ **d** $-2x - 4$
e $-3x + 3$ **f** $-51x + 14$ **g** $-17n + 27$ **h** $11y - 23$

8 **a** $2x - 1$ **b** $1 - 8x$ **c** $33 - 12x$ **d** $11x - 6$
e $15x - 1$ **f** $9 - 4x$

9 **a** $2x^3$ **b** $a^2 + b^2$ **c** $-x^3 + 9x^2 - 12x$

EXERCISE 3A.2

1 **a** $x^2 + 7x + 10$ **b** $x^2 + x - 12$ **c** $x^2 + 2x - 15$
d $x^2 - 12x + 20$ **e** $2x^2 - 5x - 3$ **f** $6x^2 - 23x + 20$
g $2x^2 - xy - y^2$ **h** $-2x^2 - 7x - 3$
i $-x^2 - x - 2xy - 2y$

2 **a** $a^2 - 7a + 10$ **b** $wy + wz + xy + xz$
c $ap + bp + aq + bq$ **d** $3x^2 - x - 2$
e $-2x^2 - x + 3$ **f** $2x^2 - x - 15$ **g** $3x^2 + 10x - 8$
h $12x^2 - 29x + 15$ **i** $x^3 - x^2 + 5x - 5$

3 **a** $x^2 + 5x - 18$ **b** $2x^2 + 7x - 31$ **c** $x^2 - 8x - 12$
d $-7t + 2$ **e** $2x^2 + 3$ **f** $10x^2 + 53x - 96$

EXERCISE 3A.3

1 **a** $y^2 - 1$ **b** $b^2 - 4$ **c** $a^2 - 49$ **d** $x^2 - 9$
e $36 - b^2$ **f** $25 - x^2$ **g** $64 - a^2$ **h** $4 - 9y^2$
i $49 - 4a^2$ **j** $9x^2 - 1$ **k** $25 - 9y^2$ **l** $x^2 - 4$
m $4x^2 - 1$ **n** $16 - 9y^2$ **o** $9x^2 - 16z^2$

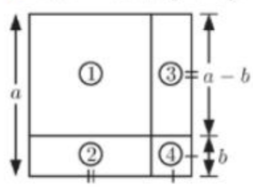
2 **a** 27 **b** $22p^2 + p - 4$ **c** $5y^2 + 8z^2$ **d** $8x^4$

EXERCISE 3A.4

1 **a** $x^2 + 2x + 1$ **b** $x^2 - 2x + 1$ **c** $x^2 + 16x + 64$
d $x^2 - 16x + 64$ **e** $y^2 + 8y + 16$ **f** $y^2 - 8y + 16$
g $9x^2 + 6x + 1$ **h** $9x^2 - 6x + 1$ **i** $4a^2 + 4a + 1$
j $4a^2 - 4a + 1$ **k** $a^2 + 2ab + b^2$ **l** $a^2 - 2ab + b^2$

2 **a** $x^2 + 10x + 25$ **b** $4x^2 + 12x + 9$ **c** $x^2 + 14x + 49$
d $9x^2 + 24x + 16$ **e** $x^4 + 10x^2 + 25$
f $9x^4 + 12x^2 + 4$ **g** $25x^2 + 30xy + 9y^2$
h $4x^4 + 28x^2y + 49y^2$ **i** $x^6 + 16x^4 + 64x^2$

3 **a** $x^2 - 6x + 9$ **b** $x^2 - 4x + 4$ **c** $9x^2 - 6x + 1$
d $25p^2 - 60p + 36$ **e** $4x^2 - 20xy + 25y^2$
f $a^2b^2 - 4ab + 4$ **g** $x^4 - 10x^2 + 25$
h $16x^4 - 24x^2y + 9y^2$ **i** $x^4 + 2x^2y^2 + y^4$

4


Hint:

Area 1
= total area − 2 × area 2
− area 4

5 **a** $2x^2+14x+85$ **b** $5x^2+18x-8$ **c** $19x+74$
d $p^4-7p^2-10p+41$ **e** $9x^4-10x^2+8x-3$
f $y^4-x^5+2x^3y+9xy^2+25x^2$

EXERCISE 3B

1 **a** x^3+3x^2+6x+8 **b** x^3+5x^2+3x-9
c x^3+5x^2+7x+3 **d** $2x^3+x^2-6x-5$
e $2x^3+7x^2+8x+3$ **f** $2x^3-9x^2+4x+15$
g $3x^3+14x^2-x+20$ **h** $8x^3-14x^2+7x-1$

2 **a** $x^3+9x^2+26x+24$ **b** $x^3-x^2-14x+24$
c $x^3-10x^2+31x-30$ **d** $2x^3+x^2-12x+9$
e $12x^3+11x^2-2x-1$ **f** $-3x^3+26x^2-33x-14$
g $-3x^3+16x^2-12x-16$ **h** x^3+x^2-5x+3
i $x^3+9x^2+27x+27$ **j** $x^3-6x^2+12x-8$

3 **a** $2\times 2=4$ terms **b** $3\times 2=6$ terms
c $2\times 3=6$ terms **d** $3\times 3=9$ terms
e $2\times 2\times 2=8$ terms **f** $3\times 2\times 2=12$ terms

4 **a** $x^4+2x^3+x^2+8x+3$
b $2x^4+7x^3-2x^2-5x+2$
c $6x^4-7x^3-8x^2+13x-4$
d $x^4-x^3-19x^2+49x-30$

EXERCISE 3C.1

1 **a** $3(a+b)$ **b** $8(x-2)$ **c** $3(p+6)$ **d** $7(x-2)$
e $6(2+x)$ **f** $c(a+b)$ **g** $a(5+b)$ **h** $c(b-6d)$
i $x(7-y)$ **j** $a(1+b)$ **k** $y(x-z)$ **l** $c(d-1)$

2 **a** $-6(a+b)$ **b** $-4(1+2x)$ **c** $-3(y+2z)$
d $-c(9+d)$ **e** $-x(1+y)$ **f** $-5x(x+4)$
g $9(b-a)$ **h** $3(2b-1)$ **i** $4(b-2a)$
j $c(d-7)$ **k** $a(b-1)$ **l** $5x(3x-1)$

3 **a** $x(x+2)$ **b** $x(5-2x)$ **c** $4x(x+2)$
d $7x(2-x)$ **e** $6x(x+2)$ **f** $x^2(x+9)$
g $xy(x+y)$ **h** $2x^2(2x-3)$ **i** $9x(x^2-2y)$
j $-3y(y+4)$ **k** $6x(2-x)$ **l** $-9a(2a+b)$

4 **a** $a(a^2+a+1)$ **b** $2(a^2+2a+4)$ **c** $3a(a^2-2a+3)$

5 **a** $(x+5)(3+x)$ **b** $(c-d)(a+b)$
c $(x-1)(ab+c)$ **d** $(b+3)(a-5)$
e $(x+2)(2x+5)$ **f** $(x+4)(x+1)$
g $(x+2)(a-1)$ **h** $(2+y)(y-1)$
i $(x-3)(x-2)$ **j** $(x+5)(x+8)$
k $2x(x-2)$ **l** $(a+b)(4-a)$
m $(x+4)(4x+1)$ **n** $5(x-1)(6-x)$

EXERCISE 3C.2

1 **a** $(x+y)(x-y)$ **b** $(p+q)(p-q)$ **c** $(x+5)(x-5)$
d $(x+9)(x-9)$ **e** $(2x+1)(2x-1)$
f $(3y+4)(3y-4)$ **g** $(8+x)(8-x)$
h $(4+3a)(4-3a)$ **i** $(1+t)(1-t)$

2 **a** $(7+2x)(7-2x)$ **b** $(y+2x)(y-2x)$
c $(2a+5b)(2a-5b)$ **d** $(9x+4y)(9x-4y)$
e $(2x^2+y)(2x^2-y)$ **f** $(3ab+4)(3ab-4)$
g $(x+5)(x+1)$ **h** $3(3x+2)(x-2)$
i $3(x-3)(x-1)$

3 **a** $2(x+2)(x-2)$ **b** $3(y+3)(y-3)$
c $2(1+3x)(1-3x)$ **d** $x(2+3x)(2-3x)$
e $ab(a+b)(a-b)$ **f** $2(5+xy)(5-xy)$
g $b(3b+2)(3b-2)$ **h** $x(x+y)(x-y)$

EXERCISE 3C.3

1 **a** $(x+1)^2$ **b** $(x-2)^2$ **c** $(x+3)^2$ **d** $(x+2)^2$
e $(x-5)^2$ **f** $(x-10)^2$ **g** $(x-4)^2$ **h** $(2x+7)^2$
i $(3x+5)^2$

2 **a** $(3x-1)^2$ **b** $-2(3x-1)^2$ **c** $3(x+3)^2$
d $-3(x+3)^2$ **e** $2(x-5)^2$ **f** $2(x-4)^2$

EXERCISE 3D

1 **a** $(a+1)(2+b)$ **b** $(c+d)(a+4)$
c $(a+2)(b+3)$ **d** $(n+3)(m+p)$
e $(x+5)(2y-1)$ **f** $(3-c)(2a+b)$

2 **a** $(x+4)(x+2)$ **b** $(x+3)(x+7)$
c $(x+5)(x+4)$ **d** $(2x+1)(x+3)$
e $(3x+2)(x+4)$ **f** $(5x+3)(4x+1)$

3 **a** $(x-4)(x+5)$ **b** $(x-7)(x+2)$
c $(x-3)(x-2)$ **d** $(x-5)(x-3)$
e $(x+7)(x-8)$ **f** $(2x+1)(x-3)$
g $(3x+2)(x-4)$ **h** $(4x-3)(x-2)$
i $(9x+2)(x-1)$

EXERCISE 3E

1 **a** $(x+1)(x+4)$ **b** $(x+2)(x+5)$
c $(x+3)(x+7)$ **d** $(x+6)(x+9)$
e $(x+2)(x+10)$ **f** $(x+3)(x+6)$
g $(x+2)(x+12)$ **h** $(x+3)(x+12)$
i $(x+3)(x+16)$

2 **a** $(x-1)(x-4)$ **b** $(x-1)(x-3)$
c $(x-2)(x-3)$ **d** $(x-2)(x-11)$
e $(x-7)(x-8)$ **f** $(x-4)(x-12)$
g $(x-2)(x-14)$ **h** $(x-1)(x-24)$
i $(x-3)(x-12)$

3 **a** $(x+1)(x-9)$ **b** $(x+7)(x-3)$
c $(x+2)(x-3)$ **d** $(x-6)(x+3)$
e $(x+8)(x-3)$ **f** $(x-12)(x+1)$
g $(x+9)(x-6)$ **h** $(x+8)(x-7)$
i $(x-7)(x+4)$ **j** $(x-5)(x+4)$
k $(x-9)(x+7)$ **l** $x+12)(x-5)$

4 **a** $2(x+1)(x+4)$ **b** $3(x-1)(x-6)$
c $2(x+3)(x+4)$ **d** $5(x+2)(x-8)$
e $4(x+1)(x-3)$ **f** $3(x-3)(x-11)$
g $2(x+9)(x-10)$ **h** $3(x+2)(x-4)$
i $2(x+4)(x+5)$

5 **a** $-(x-6)(x+9)$ **b** $-(x+2)(x+5)$
c $-(x+3)(x+7)$ **d** $-(x-8)(x+6)$
e $-(x-3)^2$ **f** $-3(x-3)(x-7)$

EXERCISE 3F

1 **a** **i** $3x^2+7x+2$
$=3x^2+6x+x+2$
$=3x(x+2)+1(x+2)$
$=(x+2)(3x+1)$

ii $3x^2+7x+2$
$=3x^2+x+6x+2$
$=x(3x+1)+2(3x+1)$
$=(3x+1)(x+2)$

b Yes, as $ab=ba$.

2 **a** $(x+1)(2x+3)$ **b** $(x+2)(2x+9)$
c $(x+1)(7x+2)$ **d** $(3x+1)(x+4)$
e $(x+2)(3x+2)$ **f** $(x+3)(3x+7)$
g $(4x+1)(2x+3)$ **h** $(7x+1)(3x+2)$
i $(2x+1)(3x+1)$ **j** $(6x+1)(x+3)$
k $(5x+1)(2x+3)$ **l** $(7x+1)(2x+5)$

3 **a** **i** $(2x+3)(2x-1)$ **ii** $(2x-1)(2x+3)$ **b** yes

4 **a** $(2x+1)(x-5)$ **b** $(3x-1)(x+2)$
c $(3x+1)(x-2)$ **d** $(2x-1)(x+2)$
e $(x-1)(2x+5)$ **f** $(5x-3)(x-1)$
g $(11x+2)(x-1)$ **h** $(2x+3)(x-3)$
i $(3x-2)(x-5)$ **j** $(5x+2)(x-3)$
k $(3x-2)(x+4)$ **l** $(2x-1)(x+9)$
m $(2x-3)(x+6)$ **n** $(5x+2)(3x-1)$
o $(21x+1)(x-3)$

5 **a** $-(x-2)(3x+7)$ **b** $-(5x-1)(x-2)$
c $-(4x-3)(x+3)$ **d** $-(3x-2)^2$
e $-(4x+1)(2x+3)$ **f** $-(6x+1)(2x-3)$

EXERCISE 3G

1 **a** $x(3x+2)$ **b** $(x+9)(x-9)$
c $2(p^2+4)$ **d** $3(b+5)(b-5)$
e $2(x+4)(x-4)$ **f** $n^2(n+2)(n-2)$
g $(x-9)(x+1)$ **h** $(d+7)(d-1)$
i $(x+9)(x-1)$ **j** $4t(1+2t)$
k $(2x+5)(2x+1)$ **l** $2(g-11)(g+5)$
m $(2a+3d)(2a-3d)$ **n** $5(a-2)(a+1)$
o $2(c-3)(c-1)$ **p** $(2x+3)(x+7)$

2 **a** $7(x-5y)$ **b** $2(g+2)(g-2)$ **c** $-5x(x+2)$
d $m(m+3p)$ **e** $(a+3)(a+5)$ **f** $(m-3)^2$
g $5x(x+y-xy)$ **h** $(x+2)(y+2)$ **i** $(y-9)(y+5)$

3 **a** $(x+5)(2x+1)$ **b** $3(y+7)(y-7)$
c $(6x+1)(x-5)$ **d** $(2c+1)(2c-1)$
e $3(x+4)(x-3)$ **f** $2(x-3)(b+5)$
g $(3x+1)(4x+3)$ **h** $-2(x-1)(x-3)$
i $(4x+1)^2$ **j** $-2x(x-1)^2$
k $(a+b+3)(a+b-3)$ **l** $2(6x-1)(x-3)$

REVIEW SET 3A

1 Show that the total area equals the sum of the two smaller areas.

2 **a** x^2-x-30 **b** $6x^2+13x-5$ **c** $18x-x^2$

3 **a** $x(7x-4)$ **b** $x(x-1)(x+6)$ **c** $(x-8)(x+5)$

4 **a** $x^3+4x^2-7x-10$ **b** $2x^3+5x^2-8x-6$

5 **a** $(4+3m)(4-3m)$ **b** $x(x+9)(x-9)$
c $(x+12)(x+2)$

6 **a** t^2-49 **b** $4y^2-25$ **c** $4m^2-20mn+25n^2$

7 **a** $2(x+5)^2$ **b** $(b+d)(2-c)$

8 **a** $(x-2)(x+9)$ **b** $3(x+2)(x-5)$ **c** $-2(x+4)(x-8)$

9 **a** $(2x+1)(4x+3)$ **b** $(5x-3)(x-2)$
c $-(3x+1)(3x-2)$

10 **b** $3(x+2)(x+12)$
c $(2x+9+x-3)(2x+9-x+3)=3(x+2)(x+12)$

REVIEW SET 3B

1 **a** $20x-25$ **b** $12x-4x^2$ **c** $3x^2-5x+12$

2 **a** $x^3+2x-20$ **b** $3b^2$

3 **a** $2(x+7)(x-7)$ **b** $(4x-3)(2x+5)$

4 **a** $((x+5)+x+(x+5)+x)$ m or $2((x+5)+x)$ m
We can show that these expressions are equal by expanding and simplifying them.

b Area of a rectangle = length × width
$\therefore$ surface area of pool $=(x+5)\times x$
$=x(x+5)$ m^2

By expanding and simplifying this expression, we find that the surface area of the pool is (x^2+5x) m^2.

5 **a** $(x+9)(x-6)$ **b** $3(x+4)^2$

6 $4\times 2\times 2=16$ terms

7 **a** $(x+6)(x-11)$ **b** $2(x-3)(x+13)$
c $(2x+3)(2x-7)$

8 **a** $9x^4-30x^2+25$ **b** $x^4+x^3+5x^2+5x+12$

9 **a** $-(x-4)(x+3)$ **b** $-(2x-5)(3x+10)$

10 **a** We seek two numbers with product $6\times 12=72$ and sum 17. These are 9 and 8.
$\therefore$ to factorise $6x^2+17x+12$ we split the $17x$ into $9x+8x$.

b $(2x+3)(3x+4)$ **c** $(3x+4)(2x+3)$ ✓

EXERCISE 4A

1 **a** 7 **b** 13 **c** 15 **d** 24 **e** $2\sqrt{2}$ **f** 4
g $4\sqrt{2}$ **h** $\frac{1}{2}$ **i** $\frac{1}{3}$ **j** $\frac{4}{11}$ **k** $\frac{3}{17}$ **l** $\frac{2}{23}$

2 **a** 2 **b** -5 **c** $\frac{1}{5}$

3 **a** 12 **b** 6 **c** 30 **d** 24 **e** -30
f -30 **g** 12 **h** 18 **i** $54\sqrt{2}$ **j** 48
k $192\sqrt{3}$ **l** 64 **m** $45\sqrt{5}$ **n** $-150\sqrt{3}$ **o** -224

4 **a** $\sqrt{10}$ **b** $\sqrt{21}$ **c** $\sqrt{33}$ **d** 7
e 6 **f** $-2\sqrt{10}$ **g** $6\sqrt{6}$ **h** $6\sqrt{15}$
i $\sqrt{30}$ **j** $4\sqrt{3}$ **k** -12 **l** $162\sqrt{6}$

5 **a** 2 **b** $\frac{1}{2}$ **c** 3 **d** $\frac{1}{3}$ **e** 2 **f** $\frac{1}{2}$
g 3 **h** $\sqrt{6}$ **i** $\frac{1}{\sqrt{10}}$ **j** 5 **k** 1 **l** 25

6 **a** $\frac{1}{5}$ **b** $\frac{4}{3}$ **c** $\frac{9}{2}$ **d** $\frac{7}{4}$

7 **a** $\sqrt{9}+\sqrt{16}=3+4=7$, whereas $\sqrt{9+16}=\sqrt{25}=5$
$\therefore\ \sqrt{9}+\sqrt{16}\neq\sqrt{9+16}$
$\sqrt{25}-\sqrt{16}=5-4=1$, whereas $\sqrt{25-16}=\sqrt{9}=3$
$\therefore\ \sqrt{25}-\sqrt{16}\neq\sqrt{25-16}$

b No, as they are not true for all $a\geqslant 0,\ b\geqslant 0$.

8 **a** $(\sqrt{a}\sqrt{b})^2=\sqrt{a}\sqrt{b}\times\sqrt{a}\sqrt{b}=ab$
$(\sqrt{ab})^2=\sqrt{ab}\times\sqrt{ab}=ab$
As $a\geqslant 0,\ b\geqslant 0,\ (\sqrt{a}\sqrt{b})^2=(\sqrt{ab})^2$ and so
$\sqrt{a}\sqrt{b}=\sqrt{ab}$

b Using **a**, $\sqrt{b}\times\sqrt{\frac{a}{b}}=\sqrt{b\times\frac{a}{b}}=\sqrt{a}$
$\therefore\ \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

EXERCISE 4B

1 **a** $2\sqrt{6}$ **b** $5\sqrt{2}$ **c** $3\sqrt{6}$ **d** $2\sqrt{10}$ **e** $2\sqrt{14}$ **f** $3\sqrt{7}$ **g** $2\sqrt{13}$ **h** $2\sqrt{11}$ **i** $2\sqrt{15}$ **j** $3\sqrt{10}$ **k** $4\sqrt{6}$ **l** $2\sqrt{17}$ **m** $10\sqrt{7}$ **n** $5\sqrt{7}$ **o** $8\sqrt{2}$ **p** $9\sqrt{2}$

2 **a** $\frac{\sqrt{5}}{3}$ **b** $\frac{3\sqrt{2}}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{5\sqrt{3}}{6}$

3 **a** $2+\sqrt{2}$ **b** $3-\sqrt{3}$ **c** $1+\frac{1}{2}\sqrt{5}$ **d** $2-\sqrt{2}$ **e** $2+\sqrt{2}$ **f** $3+\frac{1}{2}\sqrt{3}$ **g** $\frac{7}{4}-\frac{1}{4}\sqrt{15}$ **h** $\frac{1}{2}-\sqrt{2}$

EXERCISE 4C

1 **a** $10\sqrt{2}$ **b** $3\sqrt{3}$ **c** $-\sqrt{5}$ **d** $\sqrt{10}$ **e** $5\sqrt{6}$ **f** $-6\sqrt{15}$

2 **a** $6\sqrt{2}-2\sqrt{3}$ **b** $7\sqrt{7}+3\sqrt{6}$ **c** $8\sqrt{10}+3\sqrt{5}$ **d** $-2\sqrt{3}-7\sqrt{13}$ **e** $6\sqrt{7}+5\sqrt{11}$ **f** $\sqrt{14}-2\sqrt{6}$ **g** $11-2\sqrt{17}$ **h** $-21+6\sqrt{15}$

3 **a** $\frac{7\sqrt{5}}{12}$ **b** $\frac{5\sqrt{6}}{14}$ **c** $\frac{23\sqrt{3}}{24}$ **d** $-\frac{14\sqrt{11}}{45}$ **e** $\frac{13\sqrt{10}}{12}$ **f** $-\frac{11\sqrt{2}}{28}$

4 **a** $\sqrt{20}+\sqrt{5}$
$= 2\sqrt{5}+\sqrt{5}$
$= 3\sqrt{5}$
$= \sqrt{9}\times\sqrt{5}$
$= \sqrt{45}$

b $\sqrt{147}-\sqrt{75}$
$= 7\sqrt{3}-5\sqrt{3}$
$= 2\sqrt{3}$
$= \sqrt{4}\times\sqrt{3}$
$= \sqrt{12}$

5 **a** **i** 9 units^2 **ii** 4 units^2 **b** 5 units^2
c $A=l^2=5$ units^2
Since $(\sqrt{5})^2=5$, the side length is $\sqrt{5}$ units.
d $4\sqrt{5}$ units

EXERCISE 4D

1 **a** $\sqrt{10}+2$ **b** $3\sqrt{2}-2$ **c** $3+\sqrt{3}$ **d** $\sqrt{3}-3$ **e** $7\sqrt{7}-7$ **f** $2\sqrt{5}-5$ **g** $22-\sqrt{11}$ **h** $\sqrt{6}-12$ **i** $3+\sqrt{6}-\sqrt{3}$ **j** $6-2\sqrt{15}$ **k** $6\sqrt{5}-10$ **l** $30+3\sqrt{10}$

2 **a** $2-3\sqrt{2}$ **b** $-2-\sqrt{6}$ **c** $2-4\sqrt{2}$ **d** $-3-\sqrt{3}$ **e** $-3-2\sqrt{3}$ **f** $-5-2\sqrt{5}$ **g** $-3-\sqrt{2}$ **h** $-5+4\sqrt{5}$ **i** $-3+\sqrt{7}$ **j** $11-2\sqrt{11}$ **k** $\sqrt{7}-\sqrt{3}$ **l** $4-2\sqrt{2}$ **m** $9-15\sqrt{3}$ **n** $-14-14\sqrt{3}$ **o** $4-6\sqrt{2}$

3 **a** $4+3\sqrt{2}$ **b** $7+4\sqrt{3}$ **c** $1+\sqrt{3}$ **d** $10+\sqrt{2}$ **e** -2 **f** $3-3\sqrt{7}$ **g** $-1-\sqrt{5}$ **h** $1-\sqrt{6}$ **i** $14-7\sqrt{2}$

4 **a** $3+2\sqrt{2}$ **b** $7-4\sqrt{3}$ **c** $7+4\sqrt{3}$ **d** $6+2\sqrt{5}$ **e** $5-2\sqrt{6}$ **f** $27-10\sqrt{2}$ **g** $9+2\sqrt{14}$ **h** $22-8\sqrt{6}$ **i** $8-4\sqrt{3}$ **j** $13+4\sqrt{10}$ **k** $13-4\sqrt{10}$ **l** $44+24\sqrt{2}$ **m** $51-10\sqrt{2}$ **n** $17-12\sqrt{2}$ **o** $19+6\sqrt{2}$

5 **a** 13 **b** 23 **c** 1 **d** -9 **e** 14 **f** 19 **g** -2 **h** -28 **i** -174

6 **a** 1 **b** -4 **c** $x-y$ **d** 15 **e** 20 **f** 2

EXERCISE 4E

1 **a** $\frac{\sqrt{2}}{2}$ **b** $\sqrt{2}$ **c** $2\sqrt{2}$ **d** $\frac{\sqrt{6}}{2}$ **e** $\frac{\sqrt{14}}{6}$ **f** $\frac{\sqrt{3}}{3}$ **g** $\sqrt{3}$ **h** $\frac{4\sqrt{3}}{3}$ **i** $\frac{\sqrt{21}}{3}$ **j** $\frac{\sqrt{33}}{12}$

2 **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{3\sqrt{5}}{5}$ **c** $3\sqrt{5}$ **d** $\frac{\sqrt{15}}{5}$ **e** $\frac{25\sqrt{5}}{2}$ **f** $\sqrt{5}$ **g** $\frac{\sqrt{3}}{6}$ **h** $\frac{2\sqrt{6}}{3}$ **i** $\frac{3\sqrt{5}}{2}$ **j** $\frac{\sqrt{2}}{4}$

3 **a** $\dfrac{3+\sqrt{5}}{4}$ **b** $2-\sqrt{3}$ **c** $\dfrac{4+\sqrt{11}}{5}$ **d** $\dfrac{5\sqrt{2}-2}{23}$ **e** $\dfrac{\sqrt{3}-1}{2}$ **f** $\dfrac{-10-15\sqrt{2}}{14}$ **g** $\dfrac{3\sqrt{5}-10}{11}$ **h** $\dfrac{-48+13\sqrt{7}}{59}$

4 **a** $4+2\sqrt{2}$ **b** $5-5\sqrt{2}$ **c** $-3+2\sqrt{2}$ **d** $-\frac{4}{7}+\frac{1}{7}\sqrt{2}$ **e** $1+\sqrt{2}$ **f** $3+2\sqrt{2}$ **g** $\frac{9}{7}+\frac{3}{7}\sqrt{2}$ **h** $\frac{10}{7}+\frac{6}{7}\sqrt{2}$

5 **a** -6 **b** $-14-2\sqrt{5}$ **c** $12+8\sqrt{2}-9\sqrt{3}+2\sqrt{6}$

6 **a** $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$
$= (\sqrt{a})^2-(\sqrt{b})^2$ {difference of two squares}
$= a-b$ which is an integer since $a, b\in\mathbb{Z}^+$

b **i** $\dfrac{4\sqrt{7}-4\sqrt{2}}{5}$ **ii** $\dfrac{10+2\sqrt{10}}{3}$ **iii** $\dfrac{-29-3\sqrt{39}}{10}$

REVIEW SET 4A

1 **a** 18 **b** -24 **c** $\sqrt{2}$ **d** -27 **e** $\frac{9}{16}$ **f** $1500\sqrt{3}$

2 **a** $4\sqrt{3}$ **b** $12\sqrt{6}$ **3** **a** $2\sqrt{5}-\sqrt{3}$ **b** $\frac{29\sqrt{6}}{60}$

4 **a** $8\sqrt{3}-6$ **b** $16-6\sqrt{7}$ **c** 1 **d** $\sqrt{5}-4$ **e** $8+5\sqrt{2}$ **f** $79+12\sqrt{14}$

5 **a** $4\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\dfrac{4\sqrt{3}-\sqrt{6}}{14}$ **d** $\dfrac{15+5\sqrt{3}}{12}$

6 $\frac{1}{7}\sqrt{7}$ **7** **a** $7+4\sqrt{3}$ **b** 13 **8** $\dfrac{-24+5\sqrt{35}}{23}$

REVIEW SET 4B

1 **a** $6\sqrt{15}$ **b** 20 **c** $-2\sqrt{2}$ **d** $2-2\sqrt{2}$ **e** 9 **f** 15

2 **a** $5\sqrt{3}$ **b** $3\sqrt{14}$ **3** **a** $\frac{3}{2}+\sqrt{6}$ **b** $2-\frac{3}{2}\sqrt{2}$

4 **a** 22 **b** $4\sqrt{5}-9$ **c** $6-4\sqrt{3}$ **d** $7\sqrt{2}-9$

5 **a** $7\sqrt{2}$ **b** $\dfrac{-\sqrt{2}-\sqrt{6}}{2}$ **c** $\dfrac{6-\sqrt{2}}{17}$ **d** $\dfrac{-25-10\sqrt{3}}{13}$

6 **a** $-\frac{1}{11}+\frac{1}{11}\sqrt{5}$ **b** $\frac{1}{2}-\frac{5}{2}\sqrt{5}$

7 **a** $-1+5\sqrt{2}$ **b** $\dfrac{16+5\sqrt{3}}{11}$

EXERCISE 5A

1 **a** $x=7$ **b** $x=8$ **c** $2x=10$ **d** $3x=9$

2 **a** $x=3$ **b** $x=-10$ **c** $2x=-4$ **d** $3x=-9$

3 **a** $x=12$ **b** $x=-9$ **c** $x=16$ **d** $x=-20$

4 **a** $x=2$ **b** $x=-10$ **c** $x+2=-4$ **d** $x-1=5$

EXERCISE 5B.1

1 **a** $x=-11$ **b** $x=-3$ **c** $x=-7$ **d** $x=-3$ **e** $x=5$ **f** $x=9$ **g** $x=1$ **h** $x=-5$ **i** $x=-2$ **j** $x=3$ **k** $x=-1\frac{1}{2}$ **l** $x=-6$

2 **a** $x=11$ **b** $x=-5\frac{1}{2}$ **c** $x=-4$ **d** $x=1$ **e** $x=3\frac{1}{2}$ **f** $x=11$ **g** $x=-6$ **h** $x=11$ **i** $x=-\frac{1}{2}$ **j** $x=-2$ **k** $x=4$ **l** $x=-9$

3 **a** $x=28$ **b** $x=-15$ **c** $x=-16$ **d** $x=-12$
e $x=19$ **f** $x=-11$ **g** $x=10$ **h** $x=24$

4 **a** $x=-5\frac{1}{2}$ **b** $x=-3$ **c** $x=17$ **d** $x=-7$
e $x=3$ **f** $x=8\frac{1}{2}$

EXERCISE 5B.2

1 **a** $x=9$ **b** $x=-12$ **c** $x=1$ **d** $x=-2$
e $x=2\frac{2}{3}$ **f** $x=-3$

2 **a** $x=-3$ **b** $x=6$ **c** $x=2$ **d** $x=3$
e $x=2$ **f** $x=1$

EXERCISE 5B.3

1 **a** $x=6$ **b** $x=-3$ **c** $x=1\frac{1}{5}$ **d** $x=-3$
e $x=-3\frac{1}{2}$ **f** $x=-4$ **g** $x=3$ **h** $x=2$
i $x=2$ **j** $x=6\frac{1}{2}$ **k** $x=1$ **l** $x=6$

2 **a** $x=2$ **b** $x=-1$ **c** $x=6$ **d** $x=3\frac{1}{2}$
e $x=\frac{2}{3}$ **f** $x=19$ **g** $x=12$

3 **a** $x=-18$ **b** $x=-5$ **c** $x=18$ **d** $x=-1$
e $x=-\frac{1}{5}$ **f** $x=-1$ **g** $x=-5$ **h** $x=-10\frac{3}{4}$

EXERCISE 5C

1 The number is 8. **2** The numbers are 86 and 87.
3 The smallest integer is 35. **4** The number is 91.
5 The number is 12. **6** The number is 4.
7 The number is $4\frac{1}{4}$. **8** The number is $\frac{1}{2}$.
9 The number is 9. **10** Isaac buys 6 shirts.
11 eight 5-cent coins **12** 6 apples **13** five 500 g cans

REVIEW SET 5A

1 **a** **i** **C** **ii** **B** **iii** **D** **iv** **A** **b** **A** and **C**
c Perform inverse operations to isolate the variables.

2 **a** $2x=7$ **b** $x+4=-3$ **c** $3x=2$ **d** $x=-4$

3 **a** $x=4$ **b** $x=-1$ **c** $x=-3$ **d** $x=-4$
e $x=-4$ **f** $x=-1$

4 **a** $x=-6$ **b** $x=2\frac{6}{7}$ **c** $x=-7$ **d** $x=19$
e $x=-3$ **f** $x=7$

5 **a** $x=1$ **b** $x=2$ **c** $x=2$ **d** $x=3\frac{3}{4}$

6 **a** $x=6$ **b** $x=-7$ **c** $x=-2$ **d** $x=1$

7 The number is 13. **8** The smallest integer is 20.
9 The number is 7. **10** twelve 5-cent coins

REVIEW SET 5B

1 **a** $6-x=-1$ **b** $5x=11$ **c** $2x-1=2$ **d** $9x=9$

2 **a** $x=10$ **b** $x=4$ **c** $x=-5$ **d** $x=-3$
e $x=-5$ **f** $x=-4\frac{2}{3}$

3 **a** $x=-20$ **b** $x=\frac{3}{4}$ **c** $x=7$ **d** $x=-3$
e $x=-6$ **f** $x=5$

4 **a** $x=-\frac{1}{2}$ **b** $x=-1$ **c** $x=-1$ **d** $x=\frac{4}{11}$

5 **a** $x=2$ **b** $x=-1$ **c** $x=2$ **d** $x=\frac{2}{5}$

6 The number is 5. **7** The largest integer is 19.
8 The number is 7. **9** 6 writing pads **10** 13 chogokin

EXERCISE 6A

1 **a** $x=\pm 2$ **b** $x=\pm 4$ **c** $x=\pm 8$ **d** $x=0$
e no real solutions **f** $x=\pm\sqrt{15}$ **g** $x=\pm 9$
h $x=\pm 2\sqrt{5}$ **i** $x=\pm 11$ **j** no real solutions
k $x=\pm 13$ **l** no real solutions

2 **a** $x=\pm 4$ **b** $x=\pm 4$ **c** $x=\pm 3$
d $x=\pm 6$ **e** $x=\pm 5$ **f** $x=\pm 12$
g $x=\pm\sqrt{11}$ **h** $x=\pm\sqrt{26}$ **i** $x=\pm 2\sqrt{2}$

3 **a** $x=\pm 4$ **b** $x=\pm 3$ **c** $x=\pm 4$ **d** $x=\pm 8$
e $x=\pm 5$ **f** $x=\pm 2$ **g** $x=\pm 3$ **h** $x=\pm\sqrt{2}$
i $x=\pm\sqrt{3}$

4 **a** $x=\pm 3$ **b** $x=\pm 2$ **c** $x=\pm 4$ **d** $x=\pm\sqrt{3}$

EXERCISE 6B

1 **a** $\sqrt{65}$ cm **b** 10 cm **c** $\sqrt{233}$ km **d** $5\sqrt{2}$ cm
e $2\sqrt{61}$ m **f** $\sqrt{481}$ mm

2 **a** $\sqrt{85}$ cm **b** 8 m **c** $6\sqrt{2}$ cm **d** $3\sqrt{13}$ cm
e $\sqrt{26}$ m **f** 180 cm or 1.80 m

3 **a** ≈ 9.4 cm **b** ≈ 7.1 m **c** ≈ 8.8 cm **d** ≈ 5.9 cm
e ≈ 5.2 m **f** ≈ 16.9 cm

4 **a** $x=\sqrt{11}$ **b** $x=\sqrt{2}$ **c** $x=\sqrt{5}$ **d** $x=\sqrt{19}$
e $x=\frac{1}{\sqrt{2}}$ **f** $x=\frac{1}{2}$

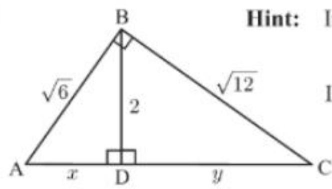
5 **a** $x=3\sqrt{3}$ **b** $x=2\sqrt{13}$ **c** $x=2$

6 **a** $x=2\sqrt{2},\ \ y=\sqrt{17}$ **b** $x=3\sqrt{5},\ \ y=\sqrt{29}$
c $x=\sqrt{5},\ \ y=\sqrt{6}$

7 **a** $x=12$ **b** $x=\sqrt{2}$ **c** $x=12$

8 $AC=\sqrt{39}$ m ≈ 6.24 m

9



Hint: In $\triangle ABD$, $x^2=6-2^2$
$\therefore\ x=\sqrt{2},\ \ x>0$
In $\triangle BCD$,
$y^2+2^2=12$
$\therefore\ y=\sqrt{8},\ \ y>0$

In $\triangle ABC$, $AC^2=(\sqrt{6})^2+(\sqrt{12})^2=18\ \ \therefore\ AC=\sqrt{18}$

10 **a** $\sqrt{17}$ cm **b** $\sqrt{29}$ m **c** $\sqrt{41}$ m

EXERCISE 6C

1 **a** No {as $4^2+5^2=41$ and $7^2=49$}
b Yes {as $9^2+12^2=225$ and $15^2=225$}
c No {as $5^2+8^2=89$ and $9^2=81$}
d No {as $3^2+(\sqrt{7})^2=16$ and $(\sqrt{12})^2=12$}
e Yes {as $(\sqrt{27})^2+(\sqrt{48})^2=75$ and $(\sqrt{75})^2=75$}
f Yes {as $8^2+15^2=289$ and $17^2=289$}

2 **a** As $(\sqrt{5})^2=1^2+2^2$, it is right angled at A.
b As $8^2+12^2=208\neq(\sqrt{218})^2$, it is not right angled.
c As $7^2=49=(\sqrt{24})^2+5^2$, it is right angled at C.
d As $4^2+(3\sqrt{2})^2\neq(\sqrt{19})^2$, it is not right angled.
e $5^2+(\sqrt{47})^2=72=(6\sqrt{2})^2$
$\therefore$ it is right angled at A.
f $(\sqrt{13})^2+(5\sqrt{2})^2=13+50=63$ and $(3\sqrt{7})^2=63$
$\therefore$ it is right angled at B.

EXERCISE 6D

1 **a** yes **b** no **c** yes
d no {numbers must be integers} **e** yes
f no {numbers must be positive}

2 **a** $k = 20$ **b** $k = 10$ **c** $k = 48$ **d** $k = 15$
e $k = 21$ **f** $k = 11$

3 **a** As $\{a, b, c\}$ is a Pythagorean triple, $a^2 + b^2 = c^2$
Hint: $(ka)^2 + (kb)^2 = k^2a^2 + k^2b^2 = k^2(a^2 + b^2)$
b **i** $\{6, 8, 10\}$, $\{9, 12, 15\}$, $\{12, 16, 20\}$, $\{15, 20, 25\}$
ii $\{10, 24, 26\}$, $\{15, 36, 39\}$, $\{20, 48, 52\}$, $\{25, 60, 65\}$

4 **a** $\{3, 4, 5\}$, $3^2 + 4^2 = 5^2$
b $\{5, 12, 13\}$, $5^2 + 12^2 = 13^2$
c $\{7, 24, 25\}$, $7^2 + 24^2 = 25^2$
d $\{9, 40, 41\}$, $9^2 + 40^2 = 41^2$

5 **a** Show $(be - ad)^2 + (bd + ae)^2 = (cf)^2$
b $\{36, 77, 85\}$

EXERCISE 6E

1 $\sqrt{73}$ cm ≈ 8.54 cm
2 $3\sqrt{10}$ cm by $\sqrt{10}$ cm, or, ≈ 9.49 cm by 3.16 cm
3 **a** $24\sqrt{5}$ cm ≈ 53.7 cm **b** 160 cm^2
4 $2\sqrt{11}$ cm ≈ 6.63 cm **5** $5\sqrt{2}$ cm ≈ 7.07 cm
6 $4\sqrt{41}$ cm ≈ 25.6 cm
7 $8^2 + 11.55^2 = 197.4025$ and $14.05^2 = 197.4025$
$\therefore$ is right angled.
8 $6\sqrt{3}$ cm ≈ 10.4 cm **9** **a** 0.86 m **b** ≈ 1.81 m
10 $80\sqrt{2}$ m ≈ 113 m **11** 7.75 m
12 **a** $\sqrt{7.2}$ m ≈ 2.68 m **b** ≈ 9.88 m **13** ≈ 7.52 m
14 ≈ 2.72 m **15** ≈ 35.8 m
16 **a** ≈ 7.07 cm **b** ≈ 148 cm

EXERCISE 6F

1 ≈ 11.4 km
2 **a** ≈ 10.8 km **b** ≈ 1.08 hours (1 hour 5 minutes)
3 Yes, since $240^2 + 100^2 = 260^2$.
4 $2\sqrt{445}$ km ≈ 42.2 km
5 **a** X: 45 km, Y: 60 km **b** 75 km
6 B: ≈ 11.2 km h^{-1} and C: ≈ 22.4 km h^{-1}
7 **a** ≈ 115.4 m **b** 100 m **c** ≈ 15.4 m

EXERCISE 6G

1 BC = 6 cm **2** ≈ 6.26 cm **3** ≈ 3.71 cm
4 $3\sqrt{2}$ cm ≈ 4.24 cm **5** $2\sqrt{39}$ cm ≈ 12.5 cm
6 $6\sqrt{2}$ cm ≈ 8.49 cm **7** $4\sqrt{7}$ cm ≈ 10.6 cm
8 $(\sqrt{2} - 1)$ m ≈ 41.4 cm **9** ≈ 717 km
10 AB $= 4\sqrt{6}$ cm ≈ 9.80 cm **11** $\sqrt{101}$ m ≈ 10.050 m
12 AB $= 2\sqrt{7}$ cm ≈ 5.29 cm **13** 5 cm

EXERCISE 6H

1 15 cm **2** 10 cm **3** $\sqrt{10}$ cm ≈ 3.16 cm
4 $3\sqrt{3}$ cm ≈ 5.20 cm **5** ≈ 6.80 m **6** ≈ 4.12 cm
7 No, the maximum length would be about 8.06 m.
8 height $= \frac{100}{\sqrt{2}} \approx 71$ m
9 $5\sqrt{10}$ cm $\times$ $5\sqrt{10}$ cm ≈ 15.8 cm $\times$ 15.8 cm
10 ≈ 29.87 m

REVIEW SET 6A

1 **a** $\sqrt{29}$ cm ≈ 5.39 cm **b** $\sqrt{33}$ cm ≈ 5.74 cm
c $\sqrt{69}$ cm ≈ 8.31 cm
2 No, $5^2 + 11^2 = 146$ but $13^2 = 169$.
3 Yes, as $4^2 + 1^2 = (\sqrt{17})^2$. The right angle is at A.
4 **a** $x = \sqrt{113}$, $y = 2\sqrt{14}$ **b** $x = 4\sqrt{5}$, $y = 6$
5 **Hint:** Find the two lengths that make up the base of the triangle, then use Pythagoras in the largest triangle.
6 1.6 m **7** 7.54 m **8** $30\sqrt{2}$ m ≈ 42.4 m
9 **a** Mia: 7.5 km, Yvette: 6 km
b $7.5^2 + 6^2 \approx 9.6^2$, that is, Mia and Yvette travelled at right angles to each other.
c either north or south
10 ≈ 41.4 cm **11** $5\sqrt{2}$ cm ≈ 7.07 cm
12 **a** $x = 2\sqrt{5}$ **b** $x = 2\sqrt{14}$
13 **a** $\sqrt{61}$ m ≈ 7.81 m **b** $\sqrt{70}$ m ≈ 8.37 m

REVIEW SET 6B

1 **a** $x = 3\sqrt{2}$ **b** $x = \sqrt{2}$
2 $2^2 + 5^2 = 29$ and AC$^2 = 29$ $\therefore$ right angled at B.
3 $k = 35$ **4** 6 cm **5** 35.4 m **6** 17 km
7 **a** AB $= \sqrt{85}$ m ≈ 9.22 m **b** AB $= \sqrt{33}$ cm ≈ 5.74 cm
8 ≈ 1.431 m **9** ≈ 14.2 m
10 Max. distance $= \sqrt{122}$ m ≈ 11.05 m which is > 10.5 m.
$\therefore$ the beam will fit.
11 ≈ 21.5 m **12** $\sqrt{99}$ cm ≈ 9.95 cm **13** AB ≈ 6.55 m

EXERCISE 7A

1 **a** 38.8 cm **b** 17.8 cm **c** 11.9 km **d** 29 m
e 11.6 cm **f** 30 m **g** 17.6 m **h** 99.6 km
i 19.1 m
2 **a** ≈ 22.0 cm **b** ≈ 25.1 m **c** ≈ 25.7 cm
d ≈ 10.7 m **e** ≈ 441 m **f** ≈ 14.3 cm
g ≈ 85.7 cm **h** ≈ 22.8 cm **i** ≈ 75.4 cm
3 **a** ≈ 8.09 cm **b** ≈ 19.6 mm **c** ≈ 8.09 cm
4 **a** 1800 m **b** 5400 m **c** \$1512
5 **a** 69.38 m **b** 146.08 m
6 **a** 226.2 km **b** 26.1 km h^{-1}
7 Yes, Al has ≈ 31.9 m of rope.
8 **a** ≈ 39.0 m **b** 14 posts **c** $\approx$ £936.16 **9** 155 cm

EXERCISE 7B

1 **a** 49 cm^2 **b** 7.5 mm^2 **c** 13 cm^2
d 186.34 mm^2 **e** 5.46 cm^2 **f** 1.02 m^2
2 **a** 30 cm^2 **b** ≈ 31.2 cm^2 **c** ≈ 26.9 cm^2
3 ≈ 22.2 cm^2
4 **a** ≈ 28.3 m^2 **b** ≈ 38.5 cm^2 **c** ≈ 56.5 m^2
d ≈ 12.6 cm^2 **e** ≈ 26.2 m^2 **f** ≈ 51.8 cm^2
g ≈ 30.2 cm^2 **h** ≈ 15.7 m^2
5 **a** 112 m^2 **b** 39 m^2 **c** 74 m^2
d 84 cm^2 **e** 31.5 cm^2 **f** 189 cm^2
g ≈ 6.85 cm^2 **h** ≈ 30.6 cm^2 **i** ≈ 30.9 cm^2
j ≈ 6430 m^2 **k** ≈ 113 cm^2 **l** ≈ 36.9 cm^2
6 2475 kg **7** **a** 160 tiles **b** €1504
8 **a** ≈ 9.42 m^2 **b** $\approx$ £74.50 **9** $\approx 21.5\%$
10 The square has the larger area, by 6.25 cm^2.
11 **a** 4 m **b** \$73.60 **c** ≈ 5.85 m^2 **d** $\approx 8.58\%$

EXERCISE 7C.1

1 **a** 150 cm^2 **b** 6.88 m^2 **c** ≈ 684 cm^2
d 306 304 mm^2 **e** 600 cm^2
2 **a** 336 cm^2 **b** ≈ 1.73 m^2

3 **a** $13\,056$ cm^2 **b** $\approx 63\,500$ cm^2 **4** \$3717

5 **a** ≈ 1010 m^2 **b** $\approx$ €25 300

EXERCISE 7C.2

1 **a** ≈ 88.0 cm^2 **b** ≈ 2210 cm^2 **c** ≈ 5.16 m^2
d ≈ 302 m^2 **e** ≈ 855 mm^2 **f** ≈ 1420 cm^2

2 **a** ≈ 201 cm^2 **b** ≈ 5670 mm^2 **c** ≈ 271 cm^2

3 **a** ≈ 1020 cm^2 **b** ≈ 6.16 km^2 **c** ≈ 1430 mm^2

4 **a** ≈ 251 m^2 **b** eleven 5 L cans **c** €577.50

5 **a** $\approx 5.15\times10^8$ km^2 **b** $\approx 3.65\times10^8$ km^2
c **i** $\approx 1.89\%$ **ii** $\approx 6.50\%$

EXERCISE 7D

1 **a** 385 m^3 **b** ≈ 339 cm^3 **c** 320 cm^3
d 45 cm^3 **e** 704 cm^3 **f** 432 cm^3
g ≈ 254 cm^3 **h** 288 cm^3 **i** ≈ 2320 cm^3

2 **a** ≈ 670 cm^3 **b** 32 cm^3 **c** ≈ 288 cm^3
d ≈ 58.3 cm^3 **e** ≈ 85.3 cm^3 **f** 24 cm^3

3 **a** ≈ 125 m^3 **b** ≈ 1440 cm^3 **c** ≈ 56.5 cm^3

4 $\approx 15\,100$ cm^3 **5** **a** 32.5 m^2 **b** 195 m^3

6 **a** ≈ 5.94 m^2 **b** ≈ 3.56 m^3

7 8820 m^3 **8** ≈ 15.0 cm^3

9 **a** ≈ 13.5 m^3 **b** $\approx 10\,400$ kg **c** $\approx$ €8800

10 **a** ≈ 2.54 cm^2 **b** ≈ 6360 cm^3 **c** ≈ 9.54 kg

11 **a** 3 sugar cubes **b** 184 mm^3

12 **a** 30.912 cm^3 **b** ≈ 905 mm^3 **c** 34 spheres
d $\approx 0.484\%$

EXERCISE 7E

1 **a** 22.05 kL **b** ≈ 23.6 mL **c** ≈ 9.11 mL

2 ≈ 1.53 L **3** 368 bottles **4** ≈ 22.6 kL

5 ≈ 8.33 cm **6** ≈ 24.4 L

7 **a** ≈ 358 m^2 **b** €210 **c** ≈ 536 m^3 **d** ≈ 536 kL

8 ≈ 8.35 kL

9 **a** 13.824 cm^3 **b** 15.2064 mL **c** ≈ 0.113 mL
d 134 drops

REVIEW SET 7A

1 **a** perimeter $= 9.2$ m, area ≈ 4.51 m^2
b perimeter $= 42$ m, area $= 60$ m^2
c perimeter ≈ 256 mm, area ≈ 3320 mm^2
d perimeter $= 28$ cm, area $= 40$ cm^2
e perimeter ≈ 24.6 cm, area ≈ 45.9 cm^2

2 **a** 9.1 m **b** \$305 **3** ≈ 21.9 km h^{-1}

4 **a** ≈ 183 cm^2 **b** ≈ 2630 mm^2 **c** ≈ 102 cm^2

5 **a** ≈ 729 cm^3 **b** ≈ 303 cm^3 **c** ≈ 65.4 cm^3

6 ≈ 2.04 m^3

7 **a** ≈ 212 m^2 **b** $\approx 46\,700$ m^3 **c** $\approx$ €11 900 000

8 **a** 375 kL **b** 350 kL

9 **a** $\approx 2\,460\,000$ m^3 **b** $\approx 6.56\times10^6$ tonnes **c** ≈ 327 kL

REVIEW SET 7B

1 **a** perimeter ≈ 20.8 cm, area $= 20$ cm^2
b perimeter ≈ 17.9 m, area ≈ 19.6 m^2
c perimeter ≈ 19.8 m, area $= 22.5$ m^2
d perimeter ≈ 41.7 cm, area ≈ 37.7 cm^2

2 **a** 105 m **b** \$3150 **3** ≈ 18.8 cm^2

4 **a** ≈ 138 mm^2 **b** ≈ 692 cm^2 **c** 3024 m^2
d ≈ 37.7 m^2

5 **a** $\approx 10\,900$ km **b** $\approx 3.79\times10^7$ km^2 **6** 2842 m^3

7 **a** ≈ 339 cm^3 **b** ≈ 1270 cm^3 **c** $\approx 14\,100$ cm^3

8 ≈ 12.3 km **9** **a** 19.8 kL **b** 5 hours 30 minutes

10 $\approx 53\,100$ kL

EXERCISE 8A

1 **a** 3 **b** 2 **c** $\frac{1}{2}$ **d** -6 **e** $\frac{9}{2}$ **f** 1
g $\frac{9}{2}$ **h** 12 **i** 2 **j** 6

2 **a** -2 **b** $-\frac{1}{2}$ **c** $\frac{1}{3}$ **d** 8 **e** 4 **f** -1
g $\frac{1}{2}$ **h** -3 **i** $-\frac{2}{3}$ **j** -4

EXERCISE 8B.1

1 **a** $\dfrac{a}{2}$ **b** $2m$ **c** 6 **d** 3 **e** $2a$ **f** x^2
g $2x$ **h** 2 **i** $\dfrac{1}{2a}$ **j** $2m$ **k** 4 **l** $\dfrac{2}{t}$
m $2d$ **n** $\dfrac{b}{2}$ **o** $\dfrac{2b}{3a}$

2 **a** t **b** cannot be simplified **c** y
d cannot be simplified **e** $\dfrac{a}{b}$ **f** cannot be simplified
g $\dfrac{a}{2}$ **h** cannot be simplified **i** $\dfrac{7c}{4d}$

3 **a** 4 **b** $2n$ **c** a **d** 1
e a^2 **f** $\dfrac{3n}{2}$ **g** $\dfrac{1}{2b^3}$ **h** $\dfrac{k^4}{2a^3}$

4 **a** $2(x+5)$ **b** $\dfrac{n+5}{6}$ **c** $\dfrac{b+2}{2}$ **d** $\dfrac{3(k-2)}{4}$
e $\dfrac{5}{t-1}$ **f** $\dfrac{2}{5(k+4)}$ **g** $\dfrac{1}{3(x-3)}$ **h** $\dfrac{5(p+4)}{3}$

5 **a** $\dfrac{x+2}{9}$ **b** $\dfrac{12}{a+1}$ **c** $\dfrac{x+y}{3}$ **d** $x+y$
e $\dfrac{2}{x+2}$ **f** $\dfrac{a+5}{3}$ **g** $\dfrac{y}{3}$ **h** $\dfrac{y-3}{3}$
i $\dfrac{x+1}{3}$ **j** $\dfrac{1}{2(b-4)}$ **k** $\dfrac{2(p+q)}{3}$ **l** $\dfrac{8}{5(r-2)}$

EXERCISE 8B.2

1 **a** $x+2$ **b** $x-2$ **c** $\dfrac{x+2}{2}$ **d** $\dfrac{x-5}{2}$
e $\dfrac{y+3}{3}$ **f** $\dfrac{3(x-5)}{2}$ **g** $a+b$ **h** $\dfrac{a+b}{c+d}$

2 **a** $\dfrac{2x+3}{3}$ **b** cannot be simplified
c cannot be simplified **d** $2a-1$ **e** $\dfrac{3a+1}{2}$
f cannot be simplified **g** $\dfrac{b+3}{2}$ **h** $\dfrac{4b-6}{3}$

3 **a** $\frac{3}{4}$ **b** $\frac{5}{3}$ **c** x **d** $\frac{4}{5}$ **e** $\dfrac{1}{y}$ **f** $\dfrac{x}{y}$ **g** $4x$
h $3x$ **i** $\dfrac{5x}{7}$ **j** $\dfrac{3b}{4}$ **k** $\dfrac{2x}{3}$ **l** $\dfrac{3(a+b)}{2a^2(2a+b)}$

4 **a** -2 **b** $-\frac{3}{2}$ **c** -1 **d** $-\frac{1}{2}$ **e** -3
f $-\dfrac{2}{x}$ **g** $-\dfrac{ab}{2}$ **h** $-2x$

EXERCISE 8C

1 **a** $\frac{xy}{10}$ **b** $\frac{3}{2}$ **c** $\frac{a^2}{2}$ **d** $\frac{1}{6}$ **e** $\frac{1}{5}$ **f** $\frac{c^2}{10}$
g $\frac{ac}{bd}$ **h** 1 **i** $\frac{1}{2m}$ **j** 2 **k** $\frac{a}{b}$ **l** 4
m $\frac{3}{m}$ **n** $\frac{a^2}{b^2}$ **o** $\frac{4}{x^2}$ **p** $\frac{1}{c}$

2 **a** $\frac{3}{2}$ **b** $\frac{3}{a}$ **c** $\frac{3x}{16}$ **d** $\frac{3}{4}$ **e** 2 **f** $\frac{c}{25}$
g $\frac{1}{5}$ **h** $\frac{m^2}{2}$ **i** 2 **j** $\frac{n}{m}$ **k** $\frac{3}{4g}$ **l** $\frac{g}{3}$
m $\frac{8}{x^3}$ **n** $\frac{x^2}{3}$ **o** $\frac{1}{a}$ **p** $\frac{3a}{5}$

3 **a** $\frac{5x}{2(x-2)}$ **b** $\frac{4}{3t}$ **c** $\frac{20}{a}$ **d** $\frac{4k}{3}$
e $-\frac{x}{2}$ **f** $\frac{3m}{2(5m-1)}$

EXERCISE 8D

1 **a** $\frac{5a}{6}$ **b** $\frac{b}{10}$ **c** $\frac{7c}{4}$ **d** $\frac{5d-6}{10}$
e $\frac{2x+15}{24}$ **f** $-\frac{5x}{14}$ **g** $\frac{4a+3b}{12}$ **h** $-\frac{2t}{9}$
i $\frac{5m}{21}$ **j** $\frac{d}{2}$ **k** $\frac{11p}{35}$ **l** $\frac{19k}{72}$
m m **n** $\frac{5a}{12}$ **o** $\frac{x}{12}$ **p** $\frac{41q}{21}$

2 **a** $\frac{7b+3a}{ab}$ **b** $\frac{3c+2a}{ac}$ **c** $\frac{4d+5a}{ad}$ **d** $\frac{2an-am}{mn}$
e $\frac{2a+b}{2x}$ **f** $\frac{5}{2a}$ **g** $\frac{4y-1}{xy}$ **h** $\frac{31}{5x}$
i $\frac{35}{12z}$ **j** $\frac{ad+bc}{bd}$ **k** $\frac{6+a^2}{2a}$ **l** $\frac{3x+2y}{3y}$
m $\frac{40-2p}{5p}$ **n** $\frac{7x}{18y}$ **o** $-\frac{19}{40t}$ **p** $\frac{5x+6}{2x^2}$

3 **a** $\frac{x+2}{2}$ **b** $\frac{y-3}{3}$ **c** $\frac{3a}{2}$ **d** $\frac{b-12}{4}$
e $\frac{x-8}{2}$ **f** $\frac{6+a}{3}$ **g** $\frac{4x}{5}$ **h** $\frac{2x+1}{x}$
i $\frac{5x-2}{x}$ **j** $\frac{a^2+2}{a}$ **k** $\frac{3+b^2}{b}$ **l** $\frac{1-2x^3}{x^2}$

4 **a** $\frac{5x+2}{6}$ **b** $\frac{-x-1}{4}$ **c** $\frac{11x+9}{12}$ **d** $\frac{5x+1}{6}$
e $\frac{-2x-1}{12}$ **f** $\frac{10x+3}{6}$ **g** $\frac{7x+3}{12}$ **h** $\frac{5x+2}{4}$
i $\frac{23x-6}{30}$ **j** $\frac{14x-1}{20}$ **k** $\frac{3x-4}{20}$ **l** $\frac{3x-2}{8}$
m $\frac{3x-2}{10}$ **n** $\frac{32-7x}{15}$ **o** $\frac{-17x-1}{12}$

5 **a** $\frac{7x+3}{x(x+1)}$ **b** $\frac{2x-6}{x(x+2)}$ **c** $\frac{x-7}{(x+1)(x-1)}$
d $\frac{3x+7}{x+2}$ **e** $\frac{5x-4}{x(x-4)}$ **f** $\frac{-4x-10}{x+3}$
g $\frac{2x^2+x+1}{(x+1)(x-1)}$ **h** $\frac{11x-10}{x(x-2)}$ **i** $\frac{-2x-6}{(x+1)(x+2)}$
j $\frac{2x^2+5x-4}{(x-1)(2x+1)}$ **k** $\frac{x^2+2x+17}{(x-1)(x+3)}$
l $\frac{-8x}{(x+5)(x-3)}$

6 **a** $\$\left(\frac{210x+150}{x(x+1)}\right)$ **b** **i** \$49.50 **ii** \$40

7 **a** $\frac{2+x}{x(x+1)}$ **b** $\frac{2(x^2+2x+2)}{(x+2)(x-3)}$
c $\frac{x^2-2x+3}{(x-2)(x+3)}$ **d** $\frac{x+14}{x+7}$

EXERCISE 8E

1 **a** $x=15$ **b** $x=\frac{9}{2}$ **c** $x=\frac{35}{2}$ **d** $x=\frac{9}{4}$
e $x=\frac{3}{8}$ **f** $x=-\frac{35}{12}$ **g** $x=-\frac{2}{15}$ **h** $x=\frac{15}{2}$
i $x=-\frac{25}{21}$

2 **a** $x=2$ **b** $x=-\frac{4}{3}$ **c** $x=-\frac{7}{2}$

REVIEW SET 8A

1 **a** -2 **b** 4 **c** $\frac{1}{3}$ **d** $-\frac{13}{11}$

2 **a** $\frac{2t}{3}$ **b** $\frac{x}{3}$ **c** $\frac{8}{3}$ **d** $\frac{2}{x+2}$

3 **a** $\frac{3(x+2)}{x}$ **b** $\frac{3x}{3x+1}$ **c** $\frac{2}{x-3}$

4 **a** -2 **b** $-\frac{5}{x}$ **c** $-\frac{x}{2}$

5 **a** $\frac{a}{3}$ **b** $\frac{3a}{b^2}$ **c** $\frac{3a+b^2}{3b}$ **d** $\frac{3a-b^2}{3b}$

6 **a** $\frac{21}{x}$ **b** $\frac{t}{24}$ **7** **a** $\frac{3}{2n}$ **b** $\frac{7x}{3(x+5)}$

8 **a** $\frac{11x}{12}$ **b** $\frac{14+x}{7}$ **c** $\frac{x-4}{4}$

9 **a** $\frac{7x-3}{12}$ **b** $\frac{3x+2}{6}$ **c** $\frac{3x+3}{10}$

10 **a** $\frac{3x}{(x+1)(x-2)}$ **b** $\frac{x+9}{(x-1)(x+1)}$ **c** $\frac{x^2+x+1}{x^2(x+1)}$

11 **a** $x=-\frac{4}{5}$ **b** $x=6$ **c** $x=-27$

REVIEW SET 8B

1 **a** -6 **b** 0 **c** 2

2 **a** $\frac{3}{2x}$ **b** $a+2b$ **c** $\frac{x+2}{x}$

3 **a** $\frac{1}{3}$ **b** cannot be simplified **c** $\frac{3-x}{4}$

4 **a** $\frac{2m}{n^2}$ **b** $\frac{m}{2}$ **c** $\frac{m^3}{n}$

5 **a** $\frac{11}{2x}$ **b** $\frac{6b-ay}{by}$ **c** $\frac{35}{12x}$

6 **a** $\frac{5x}{14}$ **b** $\frac{4x+9}{3x^2}$

7 **a** $\frac{10+x}{2}$ **b** $\frac{3x-y}{x}$ **c** $\frac{2a^2+1}{a}$

8 **a** $\frac{3y}{2(y+2)}$ **b** $-\frac{3}{4x}$

9 **a** $\frac{3x-2}{8}$ **b** $\frac{9x+27}{10}$ **c** $\frac{9-7x}{18}$

10 **a** $\frac{7-x}{(x-1)(x+2)}$ **b** $\frac{x^2-2x+2}{x^2(x-1)}$ **c** $\frac{x^2-2x-4}{x(x+2)}$

11 **a** $x=\frac{10}{3}$ **b** $x=-9$ **c** $x=-1$

EXERCISE 9A

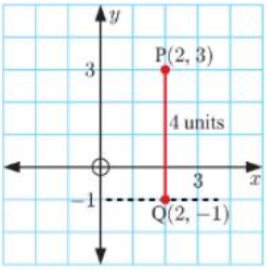
1 **a** $2\sqrt{2}$ units **b** 7 units **c** $2\sqrt{5}$ units **d** $3\sqrt{5}$ units
e 7 units **f** $\sqrt{5}$ units **g** $\sqrt{10}$ units **h** $\frac{\sqrt{5}}{2}$ units

2 **a** $10\sqrt{2} \approx 14.1$ km **b** $20\sqrt{5} \approx 44.7$ km
c $10\sqrt{26} \approx 51.0$ km

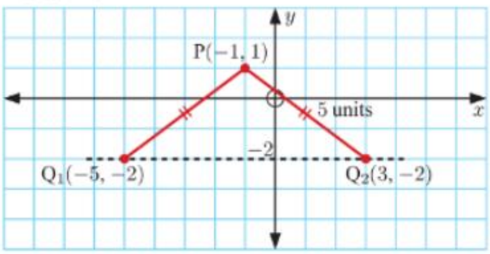
3 **a** isosceles with $\text{AB} = \text{AC} = \sqrt{85}$ units **b** scalene
c isosceles (and right angled at B) with $\text{AB} = \text{BC}$
d isosceles with $\text{BC} = \text{AC} = \sqrt{7}$ units
e equilateral, all sides $2\sqrt{3}$ units
f isosceles (AC = BC) if $b \neq 2 \pm a\sqrt{3}$
equilateral if $b = 2 \pm a\sqrt{3}$

4 **a** right angled at B **b** not right angled
c right angled at A **d** not right angled

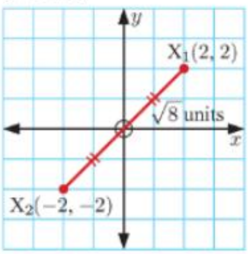
5 **a** $a = 2$



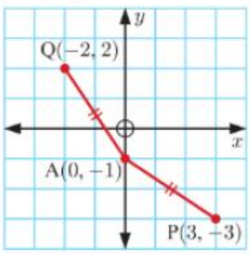
b $a = 3$ or -5



c $a = \pm 2$



d $a = -1$



6 **a** $x^2 + y^2 = 9$ **b** $(x-1)^2 + (y-3)^2 = 4$

EXERCISE 9B

1 **a** $(5, 3)$ **b** $(1, -1)$ **c** $(\frac{3}{2}, 3)$ **d** $(0, 4)$
e $(2, -\frac{3}{2})$ **f** $(1, \frac{5}{2})$

2 **a** B$(0, -6)$ **b** B$(5, -2)$ **c** B$(0, 6)$ **d** B$(0, 7)$
e B$(-7, 3)$ **f** B$(-3, 0)$

3 C$(1, -3)$ **4** P$(7, -3)$ **5** X$(\frac{3}{2}, \frac{3}{2})$, S$(-2, 0)$

6 $\frac{\sqrt{89}}{2}$ units ≈ 4.72 units

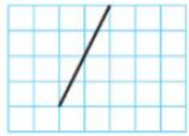
EXERCISE 9C.1

1 **a** $\frac{1}{3}$ **b** 0 **c** -3 **d** $\frac{2}{3}$ **e** $-\frac{3}{4}$
f undefined **g** -4 **h** $-\frac{2}{5}$

2 **a**

b

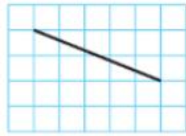
c



d

e

f



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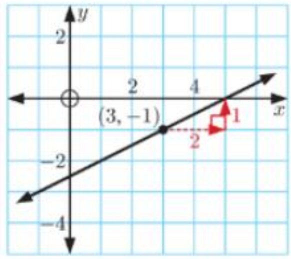

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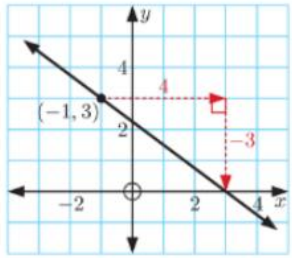

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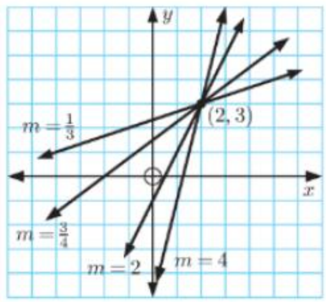

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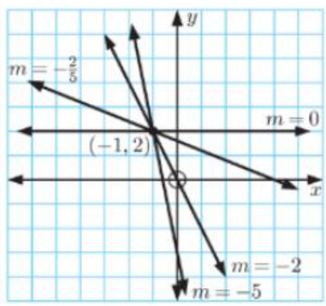
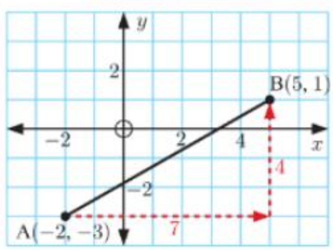


EXERCISE 9C.2

1 $\frac{4}{7}$



2 **a** $\frac{1}{5}$ **b** $\frac{1}{4}$ **c** 4 **d** 0 **e** undefined
f $\frac{2}{7}$ **g** $-\frac{2}{7}$ **h** 1 **i** $-\frac{4}{3}$

3 **a** $t = 19$ **b** $t = \frac{29}{2}$ **c** $t = \frac{13}{3}$ **d** $t = \frac{2}{3}$
e $t = 9$ **f** $t = 5$

4 Q$(0, 7)$, R$(\frac{28}{3}, 0)$

EXERCISE 9D.1

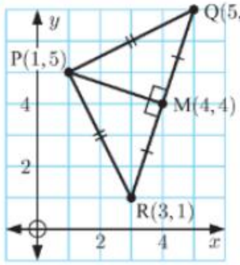
1 **a** -2 **b** $-\frac{5}{2}$ **c** $-\frac{1}{3}$ **d** $-\frac{1}{7}$ **e** $\frac{5}{2}$ **f** $\frac{2}{7}$
g $\frac{3}{4}$ **h** 1

2 The line pairs in **c**, **d**, **f**, and **h** are perpendicular.

3 **a** [AB]: $\frac{3}{5}$, [BC]: -2, [CD]: $\frac{9}{2}$, [DE]: $-\frac{1}{2}$, [EF]: -2, [FA]: 2
b **i** [BC] $\parallel$ [EF] **ii** [DE] $\perp$ [FA]

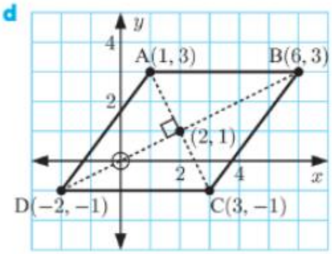
4 $k=-2$ **5** **a** $t=4$ **b** $t=4$ **c** $t=14$ **d** $t=\frac{22}{7}$

6 **a** $PQ = PR = 2\sqrt{5}$ units **b** M(4, 4)
c gradient of [PM] $= -\frac{1}{3}$,
gradient of [QR] $= 3$,
and their product is -1
$\therefore$ [PM] $\perp$ [QR]
d



7 M(0, 3) and N(3, 3)
a gradient of [MN] $= 0$, gradient of [AC] $= 0$
$\therefore$ [MN] $\parallel$ [AC]
b MN $= 3$ units and AC $= 6$ units $\therefore$ MN $= \frac{1}{2}$AC

8 **a** All sides have length 5 units. $\therefore$ ABCD is a rhombus.
b (2, 1) and (2, 1)
c gradient of [AC] $= -2$,
gradient of [BD] $= \frac{1}{2}$
and their product $= -1$
$\therefore$ [AC] $\perp$ [BD]
d



9 **a** **i** P(0, 5) **ii** Q($\frac{9}{2}$, 2) **iii** R($\frac{1}{2}$, $-\frac{5}{2}$) **iv** S(-4, $\frac{1}{2}$)
b **i** $-\frac{2}{3}$ **ii** $\frac{9}{8}$ **iii** $-\frac{2}{3}$ **iv** $\frac{9}{8}$
c PQRS is a parallelogram.

10 **a** $s=6$ **b** **i** $\frac{1}{2}$ **ii** -2
c gradient of [PS] $\times$ gradient of [SQ] $= -1$
$\therefore$ $\widehat{PSQ} = 90°$

EXERCISE 9D.2

1 **a** gradient of [AB] $= \frac{4}{3}$, gradient of [BC] $= \frac{5}{4}$
$\therefore$ not collinear
b gradient of [PQ] = gradient of [QR] $= \frac{6}{5}$
$\therefore$ P, Q, R are collinear
c gradient of [RS] $= -\frac{3}{11}$, gradient of [ST] $= -\frac{3}{2}$
$\therefore$ not collinear
d gradient of [AB] = gradient of [BC] $= 3$
$\therefore$ A, B, C are collinear

2 **a** $c=3$ **b** $c=-5$

3 **a** M(1, -4)
b gradient of [AM] = gradient of [MC] $= 1$, $\therefore$ collinear
c gradient of [AC] $= 1$, gradient of [BD] $= -1$,
$\therefore$ perpendicular

EXERCISE 9E.1

1 **a** $y=2x+1$ **b** $y=-x+1$ **c** $y=\frac{2}{3}x+3$
d $y=-\frac{4}{5}x+\frac{6}{5}$ **e** $y=-\frac{3}{4}x-\frac{1}{2}$

2 **a** $4x-y=7$ **b** $3x+5y=-1$ **c** $x-3y=-11$
d $3x+4y=24$ **e** $2x-7y=25$

3 **a** $y=x-4$ **b** $y=-x+4$ **c** $y=6x+16$
d $y=\frac{1}{2}x+8$ **e** $y=-\frac{1}{3}x-\frac{7}{3}$ **f** $y=-\frac{3}{5}x-\frac{14}{5}$

4 **a** $3x-y=-1$ **b** $2x+y=3$ **c** $x+4y=1$
d $5x-3y=10$ **e** $y=-3$ **f** $x=2$

5 **a** $4x+5y=13$ **b** $4(12)+5(-7)=48-35=13$ ✓

6 **a** $2x-5y=10$ **b** $y=-2x-2$ **c** $y=2x+6$
d $4x+3y=20$ **e** $x-2y=-8$

7 **a** $-\frac{3}{4}$ **b** $4x-3y=10$

8 **a** $y=\frac{2}{3}x-\frac{5}{3}$ **b** $-\frac{5}{3}$

9 $7x-5y=-42$ **10** $y=\frac{3}{2}x-3$

11 **a** **i** $y=-\frac{1}{2}x+2$ **ii** $y=2x-3$ **iii** $y=-\frac{1}{2}x+\frac{19}{2}$
b **Hint:** Solve $2x-3=-\frac{1}{2}x+\frac{19}{2}$.

EXERCISE 9E.2

1 **a** $x-2y=2$ **b** $2x-3y=-19$ **c** $3x-4y=15$
d $3x-y=11$ **e** $x+3y=13$ **f** $3x+4y=-6$
g $2x+y=4$ **h** $3x+y=4$

2 **a** $-\frac{2}{3}$ **b** $\frac{3}{7}$ **c** $\frac{6}{11}$ **d** $-\frac{5}{6}$ **e** $-\frac{1}{2}$ **f** 3

3 **a** Parallel lines have the same gradient of $-\frac{3}{5}$.
$\therefore$ equations have the form $3x+5y=$ a constant.
b $3x+5y=2$ has gradient $-\frac{3}{5}$
$\therefore$ perpendicular lines have gradient $\frac{5}{3}$.
$\therefore$ equations have the form $5x-3y=$ a constant.

4 **a** $3x+4y=10$ **b** $2x-5y=3$ **c** $3x+y=-12$
d $x-3y=0$

5 **a** $\frac{2}{3}$ and $-\frac{6}{k}$ **b** $k=-9$ **c** $k=4$

6 **a** $2\sqrt{34}$ km ≈ 11.7 km **b** $(6, -\frac{1}{2})$ **c** no
d **i** $11x+8y=31$
ii No, as $11(2)+8(1)=30 \neq 31$.

7 **a** $x-3y=-16$ **b** $3x-2y=13$ **c** $2x-y=-3$
d $x=5$

8 **a** **i** $x-7y=-12$ **ii** $x+y=8$
b $(\frac{11}{2})-7(\frac{5}{2})=-\frac{24}{2}=-12$ ✓ $(\frac{11}{2})+(\frac{5}{2})=8$ ✓
c $PR=QR=\frac{5\sqrt{2}}{2}$ units

EXERCISE 9F

1 **a** $x-y=4$ **b** $2x-y=-6$
c $12x-10y=-35$ **d** $y=1$

2 **a** $2x+3y=-1$ **b** $2(-5)+3(3)=-1$ ✓
c $AC=BC=\sqrt{65}$ units

3 $2x-3y=-5$

4 **a** $x+2y=5$, $3x+y=10$, $x-3y=0$ **b** (3, 1)

REVIEW SET 9A

1 $(-3, 3)$ **2** $\sqrt{58}$ units **3** $-\frac{3}{2}$ **4** $m=6$ or -2

5 $t=-11$

6 Gradient of [AB] and [BC] $= 2$.
$\therefore$ [AB] $\parallel$ [BC] with B common.

7 $2x+3y=10$ **8** $a=4$

9 **a** $AB = BC = 5$ units, $m_{AB} = \frac{3}{4}$, $m_{BC} = -\frac{4}{3}$
$\therefore$ right angle at B.
b $X(\frac{1}{2}, \frac{1}{2})$
c gradient of [BX] $= 7$, gradient of [AC] $= -\frac{1}{7}$
and $7 \times -\frac{1}{7} = -1$, $\therefore$ [BX] $\perp$ [AC].

10 **a** $x - 5y = -19$ **b** P(3, 7) **c** $5x + y = 22$
d **i** $M(\frac{7}{2}, \frac{9}{2})$ **ii** $5(\frac{7}{2}) + \frac{9}{2} = \frac{44}{2} = 22$ ✓

11 **a** $y = 4x - 11$ **b** $6x - 16y = -11$

REVIEW SET 9B

1 **a** ST $= \sqrt{73}$ units **b** $(3, -\frac{1}{2})$ **2** $x + 2y = 1$

3 **a** 2 **b** No, as $2x + y = 3$ has gradient -2.

4 $a = -1$ or -3 **5** $b = -3$

6 **a** $y = -2x + 7$ **b** $2x + 3y = 7$ **c** $3x - 2y = 15$

7 $x + 2y = 8$ **8** $5x + 6y = 29$ **9** $y = 3x - 5$

10 **a** **i** [AB]: $\frac{1}{5}$, [BC]: -2, [CD]: $\frac{1}{5}$, [AD]: -2
ii ABCD is a parallelogram.
b **i** $(\frac{1}{2}, \frac{1}{2})$ for both diagonals.
ii The diagonals of a parallelogram bisect each other.
c **i** [AC]: $-\frac{3}{7}$, [BD]: $\frac{5}{3}$
ii product $= -\frac{5}{7}$ $\therefore$ not a rhombus

11 **a** 5 units **b** $4x - 3y = -18$ **c** B(5, −4)
d $4x - 3y = 32$

EXERCISE 10A

1 **A** and **D**; **B** and **E** **2** **A** and **O**; **E**, **I**, and **M**; **F** and **H**

3 **a** **i** [FG] **ii** $\widehat{FGH}$ **b** **i** [ON] **ii** $\widehat{ONM}$
c **i** [QR] **ii** $\widehat{QRP}$ **d** **i** [TS] **ii** $\widehat{TSR}$
e **i** [ON] **ii** $\widehat{ONM}$ **f** **i** [FE] **ii** $\widehat{FED}$

EXERCISE 10B

1 **a** $\triangle ABC \cong \triangle FED$ {AAcorS}
b $\triangle PQR \cong \triangle ZYX$ {SAS}
c $\triangle ABC \cong \triangle EDF$ {AAcorS}
d $\triangle ABC \cong \triangle LKM$ {SSS}
e $\triangle XYZ \cong \triangle FED$ {RHS} **f** not congruent
g not enough information **h** not enough information
i $\triangle ABC \cong \triangle PQR$ {SSS} **j** $\triangle ABC \cong \triangle FED$ {AAcorS}

2 **a** **B** and **D** {SAS} **b** **A** and **D** {RHS}
c **B** and **C** {AAcorS} **d** **A** and **D** {SSS}

EXERCISE 10C

1 $\triangle ABC \cong \triangle EDC$ {AAcorS}

2 **a** $\triangle ABD \cong \triangle CBD$ {SSS} **b** **i** 47° **ii** 51°

3 **a** Join [AC].
Hint: Show $\triangle ABC \cong \triangle CDA$ {AAcorS}
b Now join [DB] and let the diagonals meet at M.
Hint: Show $\triangle AMD \cong \triangle CMB$ {AAcorS}

4 **a** **Hint:** Show $\triangle XYZ \cong \triangle ZWX$ {SSS}
b $\widehat{WZX} = \widehat{YXZ}$ so [WZ] $\parallel$ [XY]
and $\widehat{WXZ} = \widehat{YZX}$ so [WX] $\parallel$ [ZY]

5 **Hint:** Join [OA], [OB], and [OP], and
show $\triangle OAP \cong \triangle OBP$. {RHS}

6 **Hint:** Join [AC], [BC], and [CX], and
show $\triangle ACX \cong \triangle BCX$. {RHS}

7 **Hint:** Show $\triangle APB \cong \triangle AQC$ {AAcorS}

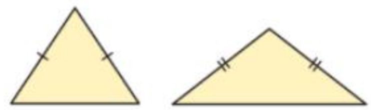
8 **Hint:** Join [AX], [BX], and [CX].
Show $\triangle AQX \cong \triangle BQX$ {SAS}
and $\triangle APX \cong \triangle CPX$ {SAS}

9 **Hint:** Join [WZ] and [ZY].
Show $\triangle WAZ \cong \triangle YDZ$ {SAS}

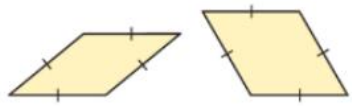
EXERCISE 10D

1 **a** $x = 18$ **b** $x = 12.6$ **c** $x = 48$ **d** $x \approx 4.71$

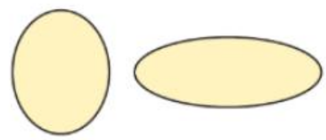
2 **a** true **b** false



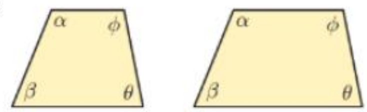
c false **d** true



e true **f** false



g true **h** false



EXERCISE 10E

1 **e** **Hint:** Let $\widehat{NJM}$ be α; find all other angles in terms of α.
f **Hint:** Let $\widehat{ACB} = \alpha$, then $\widehat{ABC} = \alpha$ and $\widehat{BDC} = \alpha$.

2 **a** $x = 1.2$ **b** $x = \frac{10}{3}$ **c** $x = \frac{20}{7}$ **d** $x = 7.5$
e $x = 10.8$ **f** $x = \frac{8}{3}$ **g** $x = \frac{4}{3}$ **h** $x = 4.8$
i $x = \frac{35}{3}$

3 7 m **4** ≈ 1.62 m **5** 10.625 km

6 ≈ 651 m **7** 8 cm

8 She is correct as the viewing region is always 20 m long no matter where the camera is located on [AB].

9 1.52 m **10** ≈ 35.7 m **11** ≈ 5.56 cm

EXERCISE 10F

1 **a** $x \approx 14.2$ **b** $x \approx 30.7$ **c** $x \approx 41.5$ **d** $x \approx 6.26$
e $x = 8$ **f** $x \approx 10.7$

2 **a** $x = 96$ **b** $x = 432$ **c** $x \approx 0.150$ **d** $x = 7.5$
e $x = 13.2$ **f** $x \approx 6.94$

3 **a** 12.544 cm^2 **b** 6.144 cm^2

4 **a** $x = 1.5$ **b** 17.6 m^2 **5** 30 cm

6 **a** **i** It is multiplied by 8. **ii** It is increased by 72.8%.
b **i** It is divided by 8. **ii** It is increased by 237.5%.
c **i** ≈ 48.8 cm^3 **ii** 1310.72 grams

7 The length of [RS] is the length of [PM] enlarged with scale factor $k = 2$.
$\therefore$ the area of the similar triangle will be enlarged by a factor of $k^2 = 4$.

8 **a** similar **b** not similar

9 **a** 10 cm **b** 30 cm^2 **c** 6 mm, 2.25 mm **d** 1.25 mL

10 **B** and **F**

Hint: Create a table for each pair of glasses.

	k	k^3	*Calc. V*	*Actual V*	
A, B	$\frac{10}{8.5}$	≈ 1.628	≈ 204	160	✗

REVIEW SET 10A

1 **a** **B** and **C** {AAcorS} **b** **A** and **C** {AAcorS}

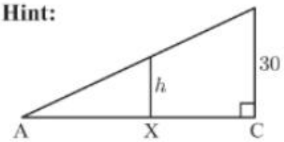
4 **a** $x = \frac{13}{8}$ **b** $x = \frac{23}{3}$ **c** $x = \frac{40}{3}$

5 **a** 38.4 cm^2 **b** 28.8 cm^2

6 **Hint:** Show $\triangle$s APQ and ABC are similar, then $\triangle$s PBC and QCB congruent, etc.

7 ≈ 117 m **8** 2 cm

9 **a** **Hint:**



Explain why $\frac{h}{30} = \frac{\text{AX}}{\text{AC}}$.

Similarly find $\frac{h}{50}$.

b 18.75 m

10 **a** $5:4$ **b** $25:16$ **c** $125:64$

REVIEW SET 10B

1 **a** yes {RHS} **b** no **c** no

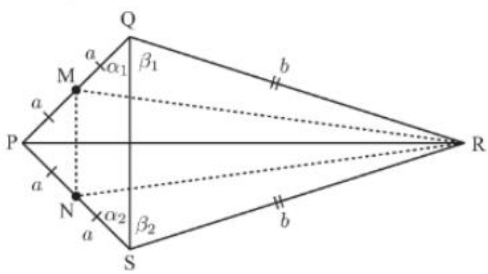
2 **a** $x = 2.8$ **b** $x = 1.92$

3 **a** **Hint:** Use 'angle in semi-circle' theorem.

b $\text{BP} = \text{BQ}$, $\widehat{\text{PAB}} = \widehat{\text{QAB}}$, $\widehat{\text{PBA}} = \widehat{\text{QBA}}$
The triangles have equal area.

4 $\text{AE} \approx 4.74$ m **5** **a** $x = 3$ **b** $x = 4$

6 **Hint:** Explain equally marked sides and angles, then consider $\triangle$s MQR and NSR.



7 $x = 10$ **8** no **10** **a** ≈ 7.56 cm **b** ≈ 5.24 cm

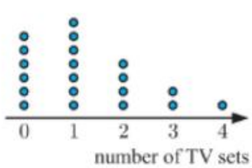
EXERCISE 11A

1 **a** 45 shoppers **b** 18 shoppers **c** $\approx 15.6\%$
d positively skewed

2 **a** 10 employees **b** $\approx 4.44\%$
c positively skewed **d** It is an outlier.

3 **a**

Number of TV sets in students' households



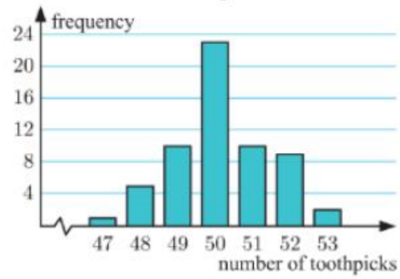
b positively skewed, no outliers
c 6 households
d 15%

4 **a**

No. of toothpicks	*Tally*	*Frequency*
47	|	1
48	𝍸	5
49	𝍸 𝍸	10
50	𝍸 𝍸 𝍸 𝍸 |||	23
51	𝍸 𝍸	10
52	𝍸 ||||	9
53	||	2
	Total	60

b

Number of toothpicks in boxes



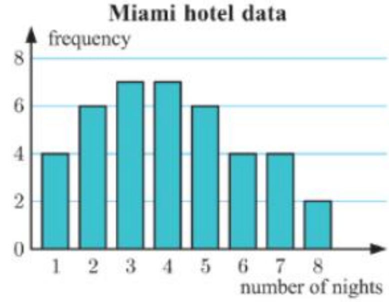
c approximately symmetrical **d** $\approx 38.3\%$

5 **a**

Number of nights	*Tally*	*Frequency*
1	||||	4
2	𝍸 |	6
3	𝍸 ||	7
4	𝍸 ||	7
5	𝍸 |	6
6	||||	4
7	||||	4
8	||	2
	Total	40

b

Miami hotel data



c no **d** slightly positively skewed **e** the Miami hotel

6 **a**

Test score	*Tally*	*Frequency*
20 - 29	|	1
30 - 39	||	2
40 - 49	|||	3
50 - 59	𝍸 ||||	9
60 - 69	𝍸 𝍸 |||	13
70 - 79	𝍸 |||	8
80 - 89	𝍸 𝍸	10
90 - 100	||||	4
	Total	50

b 28%
c 12%

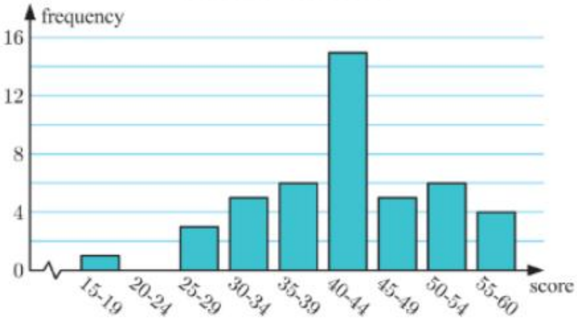
d More students had a test score in the interval 60 - 69 than in any other interval.

e negatively skewed

7 **a**

Test score	*Tally*	*Frequency*
15 - 19	|	1
20 - 24		0
25 - 29	|||	3
30 - 34	𝍸	5
35 - 39	𝍸 |	6
40 - 44	𝍸 𝍸 𝍸	15
45 - 49	𝍸	5
50 - 54	𝍸 |	6
55 - 60	||||	4
	Total	45

b **Students' test scores**



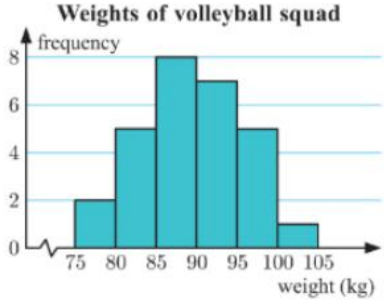
c slightly negatively skewed with no outliers

d $\approx 22.2\%$

EXERCISE 11B

1 **a** *Weight* can take any value in the given range, and is measured, not counted.

b **Weights of volleyball squad**

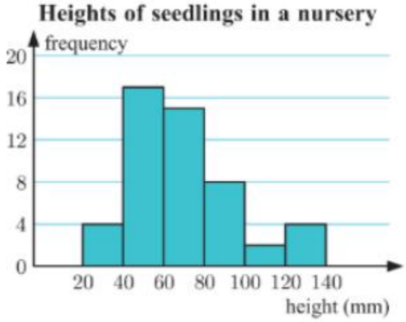


c $85 \leqslant w < 90$ kg. More people in the volleyball squad have weights between 85 and 90 kg than in any other interval.

d approximately symmetrical

2 **a** 6 seedlings **b** 30%

c **Heights of seedlings in a nursery**



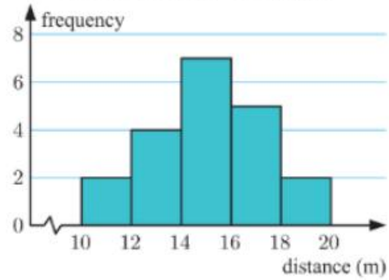
d $40 \leqslant h < 60$ mm **e** positively skewed

f **i** ≈ 754 seedlings **ii** ≈ 686 seedlings

3 **a**

Distance d (m)	*Frequency*
$10 \leqslant d < 12$	2
$12 \leqslant d < 14$	4
$14 \leqslant d < 16$	7
$16 \leqslant d < 18$	5
$18 \leqslant d < 20$	2

b **Shotput distances thrown**



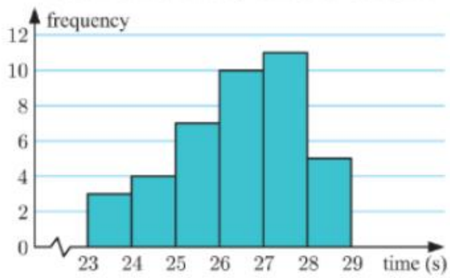
c $14 \leqslant d < 16$ m **d** approximately symmetrical

4 **a**

Time t (s)	*Frequency*
$23 \leqslant t < 24$	3
$24 \leqslant t < 25$	4
$25 \leqslant t < 26$	7
$26 \leqslant t < 27$	10
$27 \leqslant t < 28$	11
$28 \leqslant t < 29$	5

b 17.5%

c **200 m times run by athletics students**



d $27 \leqslant t < 28$ s **e** negatively skewed

EXERCISE 11C.1

1 **a** **i** 25 **ii** 24 **iii** 30

b **i** ≈ 13.3 **ii** 11.5 **iii** 8

c **i** ≈ 10.3 **ii** 10 **iii** 11.2

d **i** ≈ 429 **ii** 428 **iii** 415, 427

2 **a** *Data set A*: $\overline{x} \approx 7.73$ *Data set B*: $\overline{x} \approx 8.45$

b *Data set A*: 7 *Data set B*: 7

c The data sets are the same except for the last value, and the last value of A is less than the last value of B, so the mean of A is less than the mean of B.

d The middle value of both data sets is the same, so the median is the same.

3 **a** mean: \$582 000, median: \$420 000, mode: \$290 000

b The mode is the second lowest value, so does not take the higher values into account.

c No, since the data is unevenly distributed, the median is not in the centre.

4 **a** mean: ≈ 3.11 mm, median: 0 mm, mode: 0 mm

b **i** The data is very positively skewed so the median is not in the centre.

ii The mode is the lowest value so does not take the higher values into account.

Both median and mode indicate no rain for February if daily values are unknown.

c 15 and 27; the outliers should not be removed unless they are a result of a recording error.

5 a 44 points b 44 points c ≈ 40.6 points

d i increase ii 40.75 points

6 a 1 head b 1 head c 1.4 heads

7 a

Donation (\$)	*Frequency*
1	7
2	9
5	2
10	4
20	8
Total	30

b 30 donations

c i $\approx \$7.83$ ii \$2 iii \$2

d the mode

8 a i 4.25 ducklings ii 5 ducklings iii 5 ducklings

b Yes, it is negatively skewed.

c The mean is lower than the mode and median.

9 a i New York: 3.475 nights, Miami: 4.075 nights

ii New York: 2 nights, Miami: 3 nights and 4 nights

iii New York: 3 nights, Miami: 4 nights

b the Miami hotel

10 26 goals **11** 7.875 **12** 48 trees

13 a $x = 12$ b $x = 8$ **14** 15 **15** 6

EXERCISE 11C.2

1 $\approx 32.1°C$

2 a $165 \leqslant s < 170$ km h^{-1}

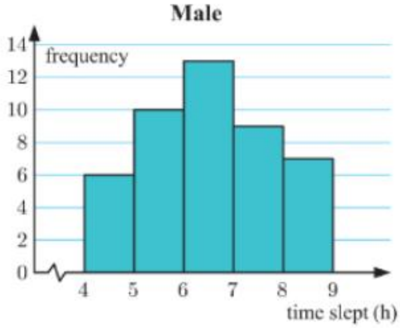
b i 76 serves ii not possible iii 73.5%

c ≈ 167 km h^{-1}

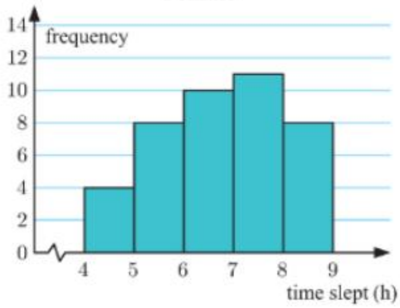
3 a 23 times b 16 times c ≈ 24.1 runs

4 a

Male



Female



b Male: approximately symmetrical,
Female: negatively skewed

c Male: ≈ 6.52 hours, Female: ≈ 6.77 hours

d The female students. However the means are only estimates, and the difference in means is small, so the answer may not be reliable.

5 a $\overline{p} \approx 42.28$ litres

b i

Petrol bought p (L)	*Frequency*
$20 \leqslant p < 30$	8
$30 \leqslant p < 40$	13
$40 \leqslant p < 50$	17
$50 \leqslant p < 60$	12

ii $\overline{p} \approx 41.6$ litres

c i

Petrol bought p (L)	*Frequency*
$20 \leqslant p < 25$	2
$25 \leqslant p < 30$	6
$30 \leqslant p < 35$	6
$35 \leqslant p < 40$	7
$40 \leqslant p < 45$	9
$45 \leqslant p < 50$	8
$50 \leqslant p < 55$	3
$55 \leqslant p < 60$	9

ii $\overline{p} \approx 42.1$ litres

d The estimate found in c ii is closer to the actual mean found in a. However, both estimates are reasonable approximations.

e Using smaller intervals should produce a more accurate estimate of the mean. This is because as the intervals get smaller, the values in those intervals get closer to the interval midpoint used in estimating the mean.

EXERCISE 11D

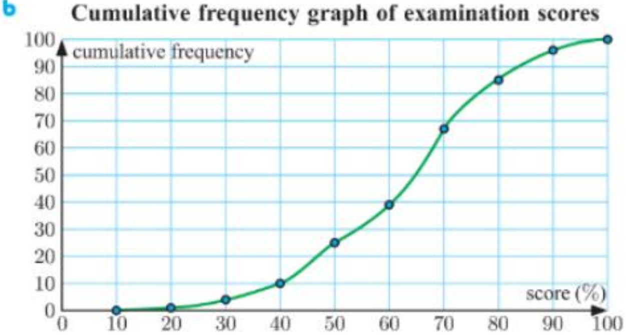
1 a 50 trout b i ≈ 5 trout ii ≈ 15 trout

c ≈ 26.8 cm

2 a

Score x (%)	*Frequency*	*Cumulative frequency*
$10 \leqslant x < 20$	1	1
$20 \leqslant x < 30$	3	4
$30 \leqslant x < 40$	6	10
$40 \leqslant x < 50$	15	25
$50 \leqslant x < 60$	14	39
$60 \leqslant x < 70$	28	67
$70 \leqslant x < 80$	18	85
$80 \leqslant x < 90$	11	96
$90 \leqslant x < 100$	4	100

b **Cumulative frequency graph of examination scores**

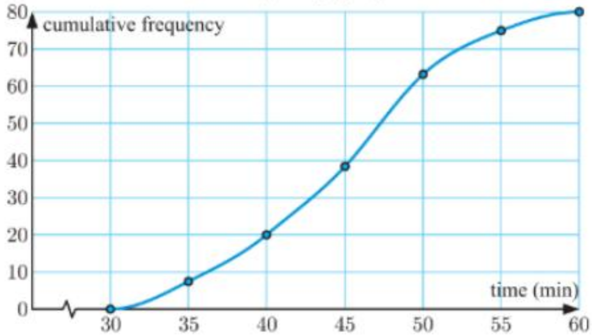


c i $\approx 64\%$ ii ≈ 52 students iii $\approx 74\%$ or more

3 a

Time t (min)	*Frequency*	*Cumulative frequency*
$30 \leqslant t < 35$	7	7
$35 \leqslant t < 40$	13	20
$40 \leqslant t < 45$	18	38
$45 \leqslant t < 50$	25	63
$50 \leqslant t < 55$	12	75
$55 \leqslant t < 60$	5	80

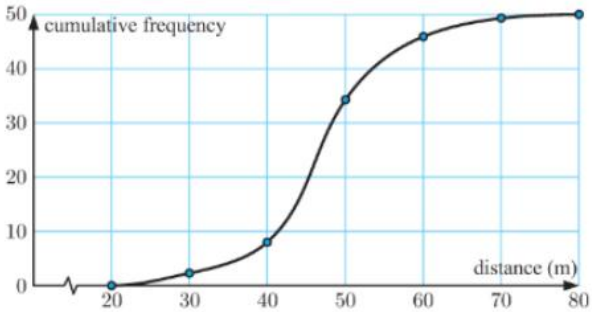
b Cumulative frequency graph of times



c **i** ≈ 45.4 min **ii** ≈ 14 runners **iii** ≈ 43 minutes

4 **a** ≈ 47.2 m

b Baseball distances thrown



c **i** ≈ 46 m **ii** ≈ 4 students **iii** ≈ 53 m

EXERCISE 11E

1 **a** **i** 7 **ii** 9 **iii** $Q_1 = 7$, $Q_3 = 10$ **iv** 3

b **i** 14 **ii** 18.5 **iii** $Q_1 = 16$, $Q_3 = 20$ **iv** 4

c **i** 7.7 **ii** 26.9 **iii** $Q_1 = 25.5$, $Q_3 = 28.1$ **iv** 2.6

2 range = 12 seeds, IQR = 7 seeds

3 **a** ≈ 6.33 tows **b** 7 tows **c** 6 tows **d** 2.5 tows

4 **a** **i** Kylie: 11, Chris: 12.5 **ii** Kylie: 17, Chris: 9

iii Kylie: 8, Chris: 4.5

b Chris **c** Kylie

5 **a** **i** Year 6: 4 visits, Year 10: 2 visits

ii Year 6: 4 visits, Year 10: 8 visits

iii Year 6: 2 visits, Year 10: 3 visits

b **i** the Year 6 class **ii** the Year 10 class

6 **a** ≈ 9.8 min **b** ≈ 6.3 min **c** ≈ 13 min **d** ≈ 6.7 min

EXERCISE 11F.1

1 **a** **i** 31 **ii** 54 **iii** 16 **iv** 40 **v** 26

b **i** 38 **ii** 14

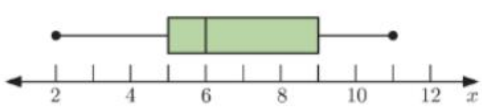
2 **a** **i** ...... 89 points **ii** ...... 25 points **iii** ...... 62 points

iv ...... 73 points **v** ...... between 45 and 73 points

b 64 points **c** 28 points

3 **a** **i** min = 2, $Q_1 = 5$, median = 6, $Q_3 = 9$, max = 11

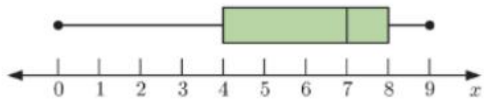
ii



iii range = 9 **iv** IQR = 4

b **i** min = 0, $Q_1 = 4$, median = 7, $Q_3 = 8$, max = 9

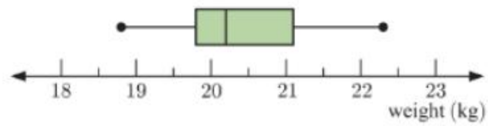
ii



iii range = 9 **iv** IQR = 4

4 **a** median = 20.2 kg, $Q_1 = 19.8$ kg, $Q_3 = 21.1$ kg, max. weight = 22.3 kg, min. weight = 18.8 kg

b



c **i** IQR = 1.3 kg **ii** range = 3.5 kg

d **i** ...... 20.2 kg **ii** 31.8% of the bags ......

iii ...... 1.3 kg **iv** ...... 19.8 kg or less

e slightly positively skewed

EXERCISE 11F.2

1 **a** Year 10: min = 4 hours, $Q_1 = 6.5$ hours, med = 9 hours, $Q_3 = 11$ hours, max = 15 hours

Year 12: min = 8 hours, $Q_1 = 10$ hours, med = 12 hours, $Q_3 = 16$ hours, max = 17 hours

b **i** Year 10: 11 hours, Year 12: 9 hours

ii Year 10: 4.5 hours, Year 12: 6 hours

2 **a** **i** Indonesia: 88 cm, Australia: 93 cm

ii Indonesia: 43 cm, Australia: 41 cm

iii Indonesia: 45 cm, Australia: 52 cm

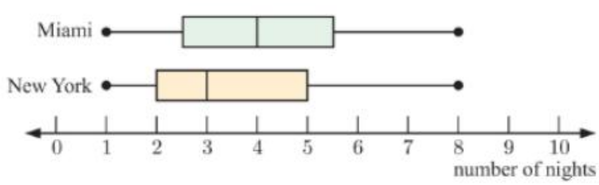
iv Indonesia: 15 cm, Australia: 20 cm

b **i** 75% **ii** 50% **c** **i** Indonesia **ii** Australia

3 **a** New York: min = 1 night, $Q_1 = 2$ nights, med = 3 nights, $Q_3 = 5$ nights, max = 8 nights

Miami: min = 1 night, $Q_1 = 2.5$ nights, med = 4 nights, $Q_3 = 5.5$ nights, max = 8 nights

b



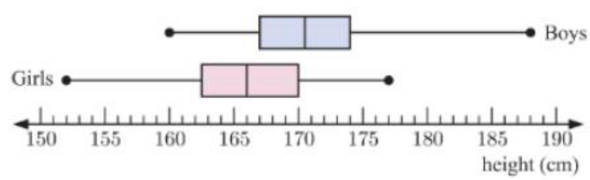
c New York: positively skewed, Miami: slightly positively skewed

d the Miami hotel

4 **a** Boys: min = 160 cm, $Q_1 = 167$ cm, med = 170.5 cm, $Q_3 = 174$ cm, max = 188 cm

Girls: min = 152 cm, $Q_1 = 162.5$ cm, med = 166 cm, $Q_3 = 170$ cm, max = 177 cm

b

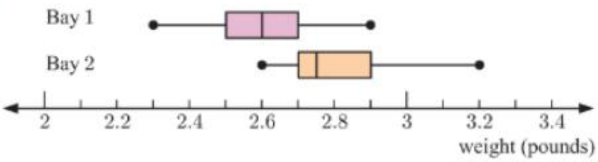


c In general, the boys are taller than the girls and are more varied in their heights.

5 **a** Bay 1: min = 2.3 pounds, Q_1 = 2.5 pounds, med = 2.6 pounds, Q_3 = 2.7 pounds, max = 2.9 pounds

Bay 2: min = 2.6 pounds, Q_1 = 2.7 pounds, med = 2.75 pounds, Q_3 = 2.9 pounds, max = 3.2 pounds

b



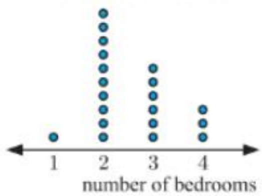
c Clearly Bay 2 catches weigh more than those from Bay 1. Median weight 2.75 pounds compared with 2.6 pounds. The range (0.6 pounds) and IQR (0.2 pounds) are identical for each bay indicating consistency of spread.

REVIEW SET 11A

1 positively skewed

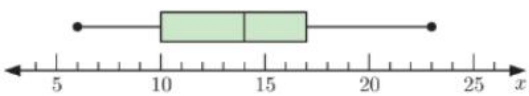
2 **a** discrete **b** no **c**

Number of bedrooms in students' houses

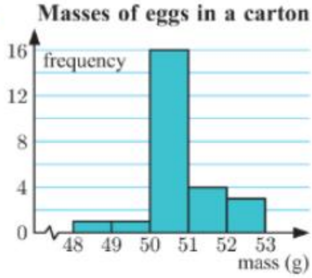


3 **a** **i** 14 **ii** 13.8 **iii** 14 **iv** 17 **v** $Q_1 = 10$, $Q_3 = 17$ **vi** 7

b



4 **a** **Masses of eggs in a carton**



b $50 \leqslant m < 51$ g This is the most common weight interval for the eggs.

c approximately symmetrical

d ≈ 50.8 g

5 **a** **i** 48 **ii** 98 **iii** 15 **iv** 66 **v** 42

b **i** 83 **ii** 24

6 $x = 13$

7 **a** Comparing the median swim times for girls and boys shows that, in general, the boys swim 2.5 seconds faster than the girls.

b The range of the girls' swim times is 10 seconds compared to the range of 7 seconds for the boys.

c The fastest 25% of the boys swim as fast or faster than 100% of the girls.

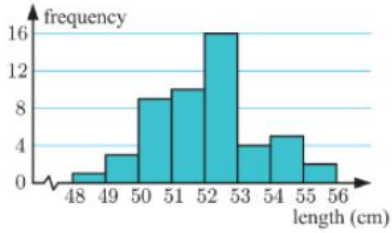
d 100% of the boys swim faster than 60 seconds whereas 75% of the girls swim faster than 60 seconds.

8 **a** 60 competitors **b** 25% **c** ≈ 207 points

9 **a** Davis: mean = \$115.28, IQR = \$14.55
Douglas: mean = \$102.37, IQR = \$34.30

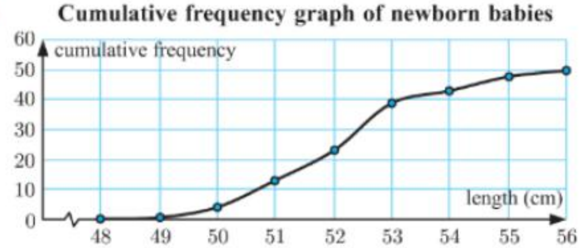
b the Davis family **c** the Douglas family

10 **a** **Histogram of lengths of newborn babies**



b 27 babies **c** 70%

d **Cumulative frequency graph of newborn babies**



e **i** ≈ 52.1 cm **ii** ≈ 18 babies

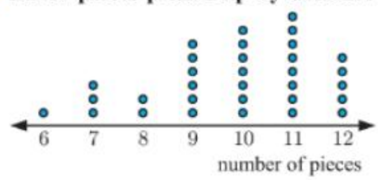
REVIEW SET 11B

1 **a** 14.55 **b** 14.5 **c** 14, 15 **2** 13.56

3 **a** 32 students

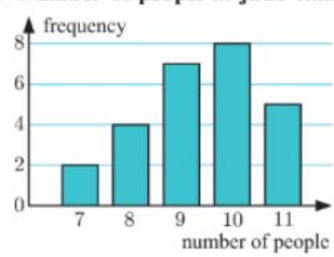
b **i** ≈ 9.84 pieces **ii** 11 pieces **iii** 10 pieces

c **Litter pieces picked up by students**



d negatively skewed

4 **a** **Number of people at judo class**

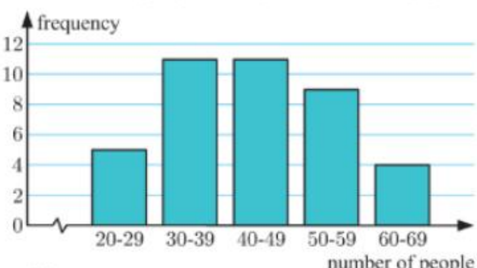


b negatively skewed

5 **a**

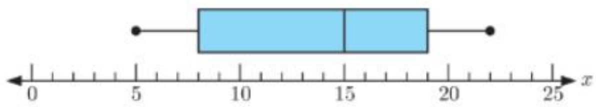
Number of people	*Tally*	*Frequency*
20 - 29	~~||||~~	5
30 - 39	~~||||~~ ~~||||~~ |	11
40 - 49	~~||||~~ ~~||||~~ |	11
50 - 59	~~||||~~ ||||	9
60 - 69	||||	4
	Total	40

b **Number of people using the swimming pool**



c 32.5%

6

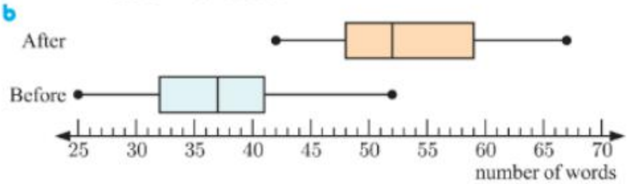


7 18 males **8** 19.5 **9** **a** 32 houses **b** $\approx \$383\,000$

10 **a** Before: min = 25 words, Q_1 = 32 words, med = 37 words, Q_3 = 41 words, max = 52 words

After: min = 42 words, Q_1 = 48 words, med = 52 words, Q_3 = 59 words, max = 67 words

b



c Yes. Each value in the 5-number summary for the After data is greater than the corresponding value for the Before data.

EXERCISE 12A

1 **a** $x = 0$ or 7 **b** $x = 0$ or 5 **c** $x = 0$ or 8 **d** $x = 0$ or 4 **e** $x = 0$ or -2 **f** $x = 0$ or $-\frac{5}{2}$ **g** $x = 0$ or $\frac{3}{4}$ **h** $x = 0$ or $\frac{5}{4}$ **i** $x = 0$ or 3

2 **a** $x = -1$ or -2 **b** $x = 1$ or 2 **c** $x = 5$ **d** $x = -2$ or -3 **e** $x = 2$ or 3 **f** $x = -1$ or -6 **g** $x = -2$ or -7 **h** $x = -5$ or -6 **i** $x = -5$ or 3 **j** $x = -6$ or 2 **k** $x = 3$ or 8 **l** $x = 7$

3 **a** $x = -3$ or -6 **b** $x = -4$ or -7 **c** $x = -4$ or 2 **d** $x = -4$ or 3 **e** $x = 3$ or 2 **f** $x = 2$ **g** $x = 3$ or -2 **h** $x = -12$ or 5 **i** $x = 10$ or -7 **j** $x = \frac{1}{2}$ or 2 **k** $x = -3$ or $\frac{1}{3}$ **l** $x = -4$ or $-\frac{5}{3}$ **m** $x = \frac{1}{2}$ or -3 **n** $x = \frac{1}{2}$ or 5 **o** $x = -1$ or $-\frac{5}{2}$

4 **a** $x = -\frac{1}{3}$ or -4 **b** $x = -\frac{2}{5}$ or 3 **c** $x = \frac{1}{2}$ or -9 **d** $x = -1$ or $\frac{5}{2}$ **e** $x = \frac{4}{3}$ or -2 **f** $x = \frac{3}{2}$ or -6 **g** $x = -\frac{1}{6}$ or 3 **h** $x = \frac{3}{2}$ or -4 **i** $x = \frac{3}{2}$ or $\frac{1}{3}$ **j** $x = -\frac{8}{3}$ or 2 **k** $x = \frac{4}{7}$ or $-\frac{1}{2}$ **l** $x = \frac{1}{4}$ or $-\frac{4}{3}$

5 **a** $x = -4$ or -3 **b** $x = -3$ or 1 **c** $x = \pm 3$ **d** $x = -1$ or $\frac{2}{3}$ **e** $x = -\frac{1}{2}$ **f** $x = \frac{5}{2}$ or 4 **g** $x = 11$ or -3 **h** $x = -\frac{3}{4}$ or $\frac{5}{2}$

6 **a** $x = \pm 2$ **b** $x = \pm\sqrt{10}$ **c** $x = \pm 4$ **d** $x = -2$ or 1 **e** $x = -6$ or 2 **f** $x = 2$ or -1 **g** $x = 4$ or -1 **h** $x = 1$ or $-\frac{1}{3}$ **i** $x = \frac{1}{2}$ or -1

7 **a** $x = 4$ or -1 **b** $x = 15$ or 2 **c** $x = -9$ or -2 **d** $x = 3$ or $-\frac{7}{2}$ **e** $x = 4$ or $\frac{4}{3}$ **f** $x = 1$ or $\frac{6}{5}$ **g** $x = 0$ or $\pm\sqrt{11}$ **h** $x = 0$ or -3

EXERCISE 12B.1

1 **a** $x = \pm 8$ **b** no real solution **c** $x = 0$

2 **a** $x = 4$ or -2 **b** $x = 0$ or -8 **c** no real solution **d** $x = 4 \pm \sqrt{5}$ **e** no real solution **f** $x = -2$ **g** $x = 2\frac{1}{2}$ **h** $x = 0$ or $-\frac{4}{3}$ **i** $x = \frac{8}{3}$ or $-\frac{10}{3}$ **j** $x = -\frac{1}{2} \pm 2\sqrt{3}$ **k** $x = \dfrac{3 \pm \sqrt{7}}{2}$ **l** $x = \dfrac{\pm\sqrt{6} - 3}{2}$

EXERCISE 12B.2

1 **a** **i** 1^2 **ii** $(x+1)^2 = 6$
b **i** 1^2 **ii** $(x-1)^2 = -6$
c **i** 3^2 **ii** $(x+3)^2 = 11$
d **i** 3^2 **ii** $(x-3)^2 = 6$
e **i** 5^2 **ii** $(x+5)^2 = 26$
f **i** 4^2 **ii** $(x-4)^2 = 21$
g **i** 6^2 **ii** $(x+6)^2 = 49$
h **i** $(\frac{5}{2})^2$ **ii** $(x+\frac{5}{2})^2 = \frac{17}{4}$
i **i** $(\frac{7}{2})^2$ **ii** $(x-\frac{7}{2})^2 = \frac{65}{4}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = 1 \pm \sqrt{3}$ **c** $x = 2 \pm \sqrt{7}$ **d** $x = -1 \pm \sqrt{2}$ **e** $x = -2 \pm \sqrt{3}$ **f** $x = -3 \pm \sqrt{6}$ **g** $x = -1$ or -2 **h** $x = -4 \pm \sqrt{2}$ **i** $x = \dfrac{3 \pm \sqrt{13}}{2}$

3 **a** no real solution **b** $x = 3$ or 2 **c** no real solution **d** $x = \dfrac{-1 \pm \sqrt{5}}{2}$ **e** $x = \dfrac{-5 \pm \sqrt{33}}{2}$ **f** no real solution

4 **a** $x = \dfrac{-2 \pm \sqrt{6}}{2}$ **b** $x = \dfrac{6 \pm \sqrt{15}}{3}$ **c** $x = \dfrac{5 \pm \sqrt{10}}{5}$

EXERCISE 12C

1 **a** **i, ii** $x = -2$ or -4 **b** **i, ii** $x = 5$ **c** **i, ii** $x = -\frac{2}{3}$ or 3

2 **a** $x = \dfrac{-1 \pm \sqrt{21}}{2}$ **b** $x = \dfrac{5 \pm \sqrt{5}}{2}$ **c** $x = 2 \pm \sqrt{5}$ **d** $x = \dfrac{-5 \pm \sqrt{37}}{6}$ **e** $x = \dfrac{1 \pm \sqrt{57}}{4}$ **f** $x = \dfrac{4 \pm \sqrt{11}}{5}$ **g** $x = \dfrac{3 \pm \sqrt{5}}{2}$ **h** $x = \dfrac{1 \pm \sqrt{7}}{2}$ **i** $x = \dfrac{1 \pm \sqrt{2}}{3}$ **j** $x = \dfrac{5 \pm \sqrt{53}}{14}$ **k** $x = \dfrac{-1 \pm \sqrt{7}}{3}$ **l** $x = \dfrac{2 \pm \sqrt{3}}{5}$

3 **a** $x = \dfrac{-1 \pm \sqrt{29}}{2}$ **b** $x = \dfrac{-1 \pm \sqrt{5}}{2}$ **c** no real solution **d** $x = 1 \pm 2\sqrt{2}$ **e** $x = \dfrac{7 \pm \sqrt{217}}{6}$ **f** $x = \dfrac{3 \pm \sqrt{13}}{2}$

EXERCISE 12D

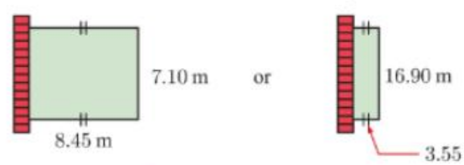
1 The number is -11 or 10. **2** The number is -3 or 8.

3 The numbers are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

4 The numbers are -2 and 5, or 2 and -5.

5 8 cm **6** 10 m **7** 18 m by 12 m

8



9 **a** $x = 5$ **b** $x = 8$ **c** $x = 20$

10 17.9 cm **11** BC = 16 cm or 5 cm

12 **a** $x = 2$ **b** $x = 5$ **c** $x = 6$ **d** $x = \sqrt{31} - 1$ **e** $x = \dfrac{3 + \sqrt{5}}{2}$ as $x > \frac{1}{2}$ **f** $x = 3 + \sqrt{34}$

13 BE = 6 cm **14** CD = $(1 + \sqrt{41})$ m **15** $n = 6$

16 $\frac{2}{5}$ or $\frac{-9}{-6}$ **17** 40 oranges **18** $\frac{4}{3}$ or $\frac{3}{4}$

19 The numbers are $\dfrac{4+\sqrt{14}}{2}$ and $\dfrac{4-\sqrt{14}}{2}$.

20 Cut out squares with sides 2 cm.

21 **a** $x + 2r = 10 \quad \therefore \quad r = 5 - \dfrac{x}{2}$

b **Hint:** Lawn area $= 4 \times$ total of flower bed areas

$\therefore \quad 2(\pi \times 5^2) = 4 \times \pi r^2$

$\therefore \quad 50\pi = 4\pi\left(5 - \dfrac{x}{2}\right)^2$ etc.

c ≈ 2.93 m

22 **a** **Hint:** Draw a diagram and show that area of path $= 2\left[x(12+2x)+6x\right]$.

b $4x^2 + 36x = \frac{7}{8}(6 \times 12) = 63$ **c** 1.5 m

23 $x = 5$ **24** 3.2 cm

REVIEW SET 12A

1 **a** $x = \pm\sqrt{11}$ **b** $x = \pm 4$

2 **a** $x = 9$ or -1 **b** $x = 0$ or -2

3 **a** $x = 7$ or -3 **b** $x = \pm\frac{5}{2}$ **c** $x = \frac{2}{3}$ or $-\frac{1}{2}$

4 **a** $x = 6$ or -4 **b** $x = -3$ **c** $x = \frac{1}{2}$ or 4

5 $x = -3 \pm \sqrt{5}$

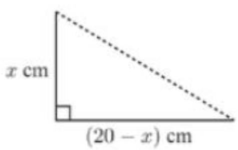
6 **a** $x = -12 \pm \sqrt{155}$ **b** $x = 9$ or -2 **c** $x = -\frac{2}{5}$ or $\frac{3}{2}$

7 $\frac{2}{3}$ or $\frac{3}{2}$

8 **a** $x = 2$ or $-\frac{1}{2}$ **b** $x = \dfrac{-2 \pm \sqrt{19}}{3}$ **c** $x = \dfrac{13 \pm \sqrt{105}}{8}$

9 $x = 7$

10



a $(10 + 2\sqrt{10})$ cm, $(10 - 2\sqrt{10})$ cm, and hypotenuse $2\sqrt{70}$ cm

b $(20 + 2\sqrt{70})$ cm

REVIEW SET 12B

1 **a** $x = \pm\sqrt{6}$ **b** $x = \pm 1$

2 **a** $x = -3 \pm \sqrt{19}$ **b** $x = \dfrac{1 \pm \sqrt{17}}{3}$

3 **a** $x = 11$ or -3 **b** $x = \frac{1}{2}$ or $-\frac{3}{4}$ **4** $x = 1 \pm \sqrt{101}$

5 **a** $x = 9$ or -5 **b** $x = \frac{2}{3}$ or -5

6 **a** $x = \pm\sqrt{35}$ **b** $x = -3$ or 1

7 8 cm, 15 cm, and 17 cm

8 **a** $x = \dfrac{-1 \pm \sqrt{3}}{2}$ **b** $x = \dfrac{2 \pm \sqrt{2}}{2}$

9 12 people, \$40 each **10** $x = 3$ or 12 **11** 10 cm

EXERCISE 13A.1

1 **a** [BC] **b** [AB] **c** [AC]

2 **a** **i** z **ii** y **iii** x

b **i** $\sin\theta = \dfrac{y}{z}$ **ii** $\cos\theta = \dfrac{x}{z}$ **iii** $\tan\theta = \dfrac{y}{x}$

c **i**
$$\begin{aligned}\frac{\sin\theta}{\cos\theta} &= \frac{\frac{y}{z}}{\frac{x}{z}} && \{\text{using } \mathbf{b\ i}, \mathbf{b\ ii}\}\\ &= \frac{y}{x} && \{\text{multiplying RHS by } \tfrac{z}{z}\}\\ &= \tan\theta && \{\text{using } \mathbf{b\ iii}\}\end{aligned}$$

ii
$$\begin{aligned}\cos^2\theta + \sin^2\theta &= \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2\\ &= \frac{x^2}{z^2} + \frac{y^2}{z^2}\\ &= \frac{x^2+y^2}{z^2}\end{aligned}$$

But $x^2 + y^2 = z^2$ {Pythagoras}

So, $\cos^2\theta + \sin^2\theta = \dfrac{z^2}{z^2} = 1$

3 **a** **i** c **ii** b **iii** a

b **i** $\sin\theta = \dfrac{b}{c}$ **ii** $\cos\theta = \dfrac{a}{c}$ **iii** $\tan\theta = \dfrac{b}{a}$

c $\theta + 90° +$ third angle $= 180°$ {angles in a triangle}

$\therefore$ third angle $= 180° - 90° - \theta = 90° - \theta$

d **i** $\sin(90° - \theta) = \dfrac{a}{c}$ **ii** $\cos(90° - \theta) = \dfrac{b}{c}$

iii $\tan(90° - \theta) = \dfrac{a}{b}$

e **i** $\sin(90° - \theta) = \dfrac{a}{c} = \cos\theta$ {using **b ii**}

ii $\cos(90° - \theta) = \dfrac{b}{c} = \sin\theta$ {using **b i**}

iii
$$\begin{aligned}\tan(90° - \theta) = \frac{a}{b} &= \frac{1}{\left(\frac{b}{a}\right)}\\ &= \frac{1}{\tan\theta} && \{\text{from } \mathbf{b\ iii}\}\end{aligned}$$

EXERCISE 13A.2

1 **a** $\tan 64° = \dfrac{c}{x}$ **b** $\sin 70° = \dfrac{a}{x}$ **c** $\cos 43° = \dfrac{x}{e}$

d $\sin 35° = \dfrac{x}{b}$ **e** $\cos 61° = \dfrac{g}{x}$ **f** $\tan 30° = \dfrac{h}{x}$

2 **a** $x \approx 12.99$ **b** $x \approx 15.52$ **c** $x \approx 9.84$

d $x \approx 6.73$ **e** $x \approx 3.31$ **f** $x \approx 20.01$

g $x \approx 16.86$ **h** $x \approx 4.45$ **i** $x \approx 1.65$

j $x \approx 26.75$ **k** $x \approx 7.53$ **l** $x \approx 9.61$

3 **a** $\theta = 22°$, $a \approx 8.09$, $b \approx 3.03$

b $\theta = 59°$, $a \approx 13.0$, $b \approx 7.83$

c $\theta = 65°$, $a \approx 5.42$, $b \approx 2.29$

4 ≈ 28.8 cm

EXERCISE 13A.3

1 **a** $\theta \approx 48.2°$ **b** $\theta \approx 45.6°$ **c** $\theta \approx 56.3°$

d $\theta \approx 37.4°$ **e** $\theta \approx 42.2°$ **f** $\theta = 45°$

g $\theta \approx 40.2°$ **h** $\theta \approx 35.3°$ **i** $\theta \approx 35.9°$

2 **a** $\phi \approx 45.6°$, $\theta \approx 44.4°$, $x \approx 7.14$

b $\alpha \approx 53.9°$, $\beta \approx 36.1°$, $x \approx 5.94$

c $a \approx 50.3°$, $b \approx 39.7°$, $x \approx 8.65$

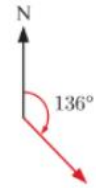
3 **a** **i** $\sin\theta = \frac{3}{8}$ **ii** $\cos\theta = \frac{\sqrt{55}}{8}$ **iii** $\tan\theta = \frac{3}{\sqrt{55}}$

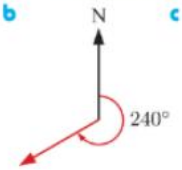
b In each case, $\theta \approx 22.0°$.

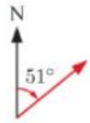
EXERCISE 13B

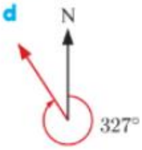
1 ≈ 110 m **2** $\approx 32.9°$ **3** ≈ 238 m **4** ≈ 765 m
5 ≈ 280 m **6** $\theta \approx 6.89°$ **7** ≈ 23.5 m **8** $\approx 21.8°$
9 ≈ 15.8 cm **10** $\approx 106°$ **11** No, ≈ 0.721 cm.
12 $\approx 41.4°$ **13** $\approx 53.2°$ **14** **a** ≈ 248 m **b** ≈ 128 m
15 $\approx 14.3°$ **16** ≈ 729 m
17 **a** $d \approx 1.66$ units **b** $d \approx 1.66$ units

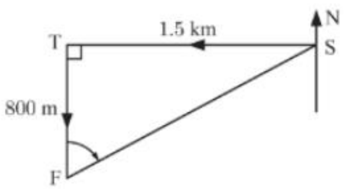
EXERCISE 13C

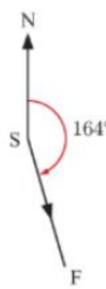
1 **a**


b


c


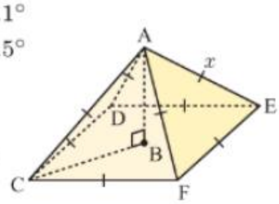
d


2 **a** 040° **b** 235° **c** 297° **d** 132° **e** 225°
f 337°
3 **a** 041° **b** 142° **c** 279° **d** 322° **e** 221°
f 099°
4 **a** 055° **b** 235° **c** 095° **d** 275° **e** 130°
f 310°
5 $\approx 057.3°$ **6** $\approx 308°$
7 **a**


b 1.7 km
c 061.9°T
8 ≈ 7.81 km, $\approx 130°$ **9** ≈ 46.0 km
10 **a**


b 35 km
c **i** ≈ 9.65 km
ii ≈ 33.6 km
11 ≈ 2.44 km **12** ≈ 221 km

EXERCISE 13D

1 **a** EG ≈ 21.2 cm **b** $\approx 35.3°$
2 **a** HX ≈ 9.43 cm **b** $\approx 32.5°$
c HY ≈ 10.8 cm **d** $\approx 29.1°$
3 **a** DF ≈ 8.94 cm **b** $\approx 18.5°$
4 ≈ 69.2 cm
5 **a** 45°
b **Hint:** Show that $BC = \frac{\sqrt{2}}{2}x$.


6 $\theta \approx 61.9°$

REVIEW SET 13A

1 **a** $x \approx 14.0$ **b** $x \approx 35.2$ **c** $x \approx 64.6$
2 **a** $\theta = 36°$, $x \approx 12.4$, $y \approx 21.0$
b $\theta = 64°$, $x \approx 40.1$, $y \approx 17.6$
3 ≈ 55.2 m **4** ≈ 31.5 cm^2 **5** **a** 157° **b** 281°
6 $\approx 201°$ **7** ≈ 2.28 m **8** **a** $\approx 56.3°$ **b** $\approx 33.9°$

REVIEW SET 13B

1 **a** $\theta \approx 38.7°$ **b** $\theta \approx 37.1°$
2 **a** $x \approx 3.18$ **b** $x \approx 9.40$
3 $\alpha \approx 36.4°$, $\theta \approx 53.6°$, $x \approx 25.7$ **4** $\approx 32.2°$
5 ≈ 187 m **6** **a** 230° **b** 165° **c** 140°
7 **a** 45° **b** 60°
8 **a** ≈ 39.4 km **b** at about 12:50 pm

EXERCISE 14A

1 $\frac{168}{200} = 0.84$ **2** $\frac{11}{123} \approx 0.0894$ **3** $\frac{169}{227} \approx 0.744$
4 $\frac{172}{417} \approx 0.412$
5 **a** $\frac{137}{200} = 0.685$ **b** $\frac{145}{200} = 0.725$
c We could combine Sam and Karla's results and estimate the chance of an end occurring using the combined data.
d yes

EXERCISE 14B

1 **a** 407 people **b** **i** $\frac{93}{407} \approx 0.229$ **ii** $\frac{207}{407} \approx 0.509$
2 **a** 57 ice creams **b** **i** $\frac{17}{57} \approx 0.298$ **ii** $\frac{36}{57} \approx 0.632$
3 **a**

Councillor	*Frequency*
Mr Tony Trimboli	216
Mrs Andrea Sims	72
Mrs Sara Chong	238
Mr John Henry	74
Total	600

b **i** $\frac{74}{600} \approx 0.123$
ii $\frac{310}{600} \approx 0.517$

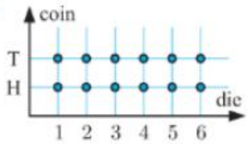
4 **a**

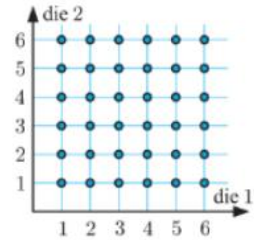
	Like	Dislike	*Total*
Junior students	87	38	125
Senior students	129	56	185
Total	216	94	310

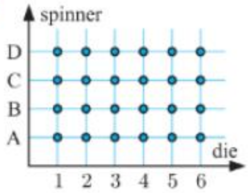
b **i** $\frac{87}{310} \approx 0.281$ **ii** $\frac{129}{310} \approx 0.416$ **iii** $\frac{94}{310} \approx 0.303$
c The total is 1. This is because the three probabilities in **b** cover all possible outcomes that could occur.
5 **a** 100 students
b **i** $\frac{29}{100} = 0.29$ **ii** $\frac{8}{100} = 0.08$ **iii** $\frac{26}{100} = 0.26$
iv $\frac{68}{100} = 0.68$
6 **a** $\frac{69}{147} \approx 0.469$ **b** $\frac{26}{147} \approx 0.177$ **c** $\frac{43}{147} \approx 0.293$
7 **a** $\frac{3}{82} \approx 0.0366$ **b** $\frac{52}{82} \approx 0.634$ **c** $\frac{32}{82} \approx 0.390$
d $\frac{19}{82} \approx 0.232$
8 **a** $\frac{4822}{5038} \approx 0.957$ **b** $\frac{448}{5038} \approx 0.0889$
c $\frac{31}{5038} \approx 0.006\,15$ **d** $\frac{1864}{5038} \approx 0.370$

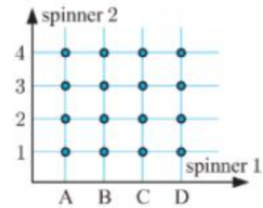
EXERCISE 14C

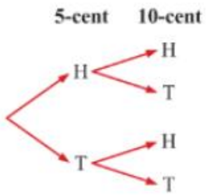
1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
e **i** {HH, HT, TH, TT}
ii {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
iii {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}

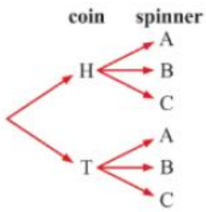
2 **a** 

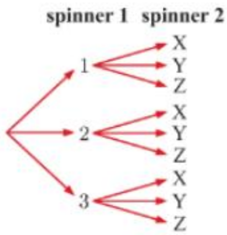
b 

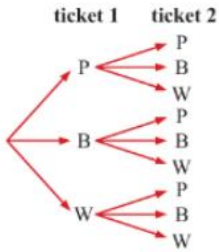
c 

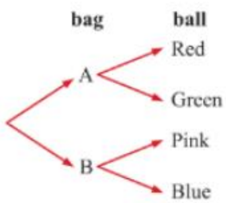
d 

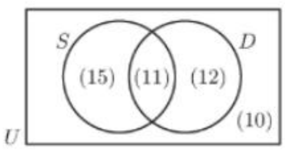
3 **a** 

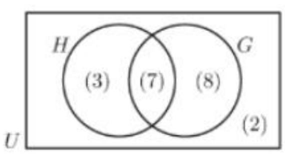
b 

c 

d 

e 

4 

5 

EXERCISE 14D

1 {ODG, OGD, DOG, DGO, GOD, GDO}

a $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{1}{3}$

2 {BBB, GBB, BGB, BBG, BGG, GBG, GGB, GGG}

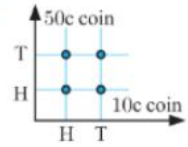
a $\frac{1}{8}$ **b** $\frac{1}{8}$ **c** $\frac{1}{8}$ **d** $\frac{3}{8}$ **e** $\frac{1}{2}$ **f** $\frac{7}{8}$

3 **a** E' is the event that the token is not red.

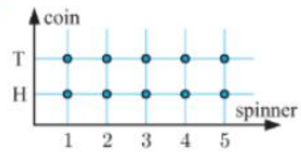
b $P(E) = \frac{1}{4}$ **c** $P(E') = \frac{3}{4}$

4 {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

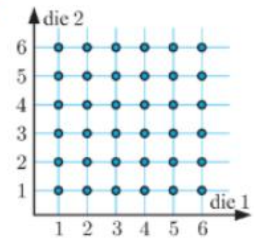
a $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{1}{6}$ **d** $\frac{1}{2}$

5 **a** 

b **i** $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{2}$ **iv** $\frac{3}{4}$

6 **a** 

b **i** $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{5}$ **iv** $\frac{3}{5}$

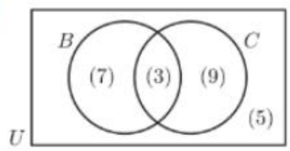
7 **a** 

b **i** $\frac{1}{18}$ **ii** $\frac{1}{6}$ **iii** $\frac{11}{36}$ **iv** $\frac{5}{9}$ **v** $\frac{1}{4}$ **vi** $\frac{1}{6}$

8 **a** **i** $\frac{1}{15}$ **ii** $\frac{2}{15}$ **iii** $\frac{7}{15}$ **iv** $\frac{5}{15} = \frac{1}{3}$

b $\text{P(odd)} = \frac{6}{15}$, $\text{P(even)} = \frac{9}{15}$ $\therefore$ even

9 **a** 33 students **b** **i** $\frac{17}{33}$ **ii** $\frac{5}{33}$ **iii** $\frac{8}{11}$ **iv** $\frac{19}{33}$

10 **a** 

b **i** $\frac{3}{8}$ **ii** $\frac{1}{8}$

11 **a** $\frac{11}{50}$ **b** $\frac{29}{50}$

EXERCISE 14E.1

1 **a** $\frac{1}{48}$ **b** $\frac{5}{16}$ **2** **a** $\frac{2}{9}$ **b** $\frac{4}{15}$

3 **a** $\frac{6}{625} = 0.0096 = 0.96\%$ **b** $\frac{506}{625} = 0.8096 \approx 81.0\%$

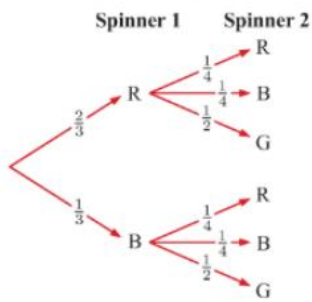
4 **a** $\frac{1}{49}$ **b** $\frac{10}{49}$ **c** $\frac{25}{49}$

5 **a** ≈ 0.0545 **b** ≈ 0.0584 **c** ≈ 0.441 **d** ≈ 0.0840

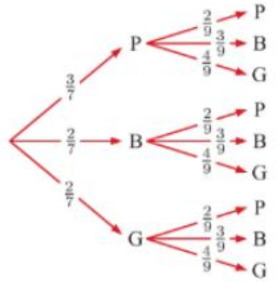
6 **a** **i** ≈ 0.405 **ii** ≈ 0.595

b **i** ≈ 0.164 **ii** ≈ 0.354

7 **a** ≈ 0.366 **b** ≈ 0.0231

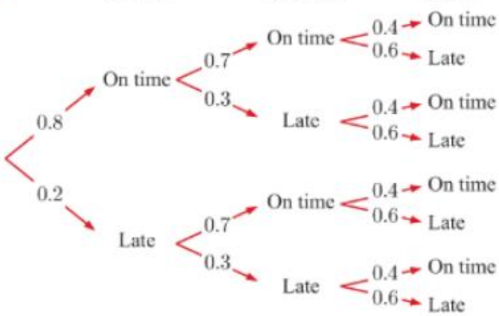
8 **a** 

b **i** $\frac{1}{4}$ **ii** $\frac{3}{4}$ **iii** $\frac{5}{12}$

9 **a** 

b **i** $\frac{2}{9}$ **ii** $\frac{20}{63}$ **iii** $\frac{29}{63}$

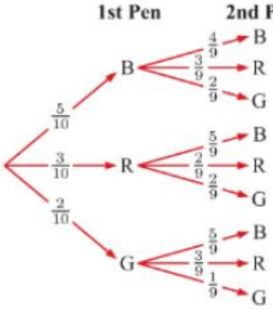
c 1; this is because every possible outcome is covered by these three events.

10 **a** 

b 0.712

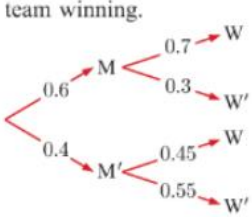
EXERCISE 14E.2

1 **a** $\frac{1}{45}$ **b** $\frac{1}{6}$ **2** **a** $\frac{25}{102}$ **b** $\frac{13}{204}$ **c** $\frac{1}{221}$

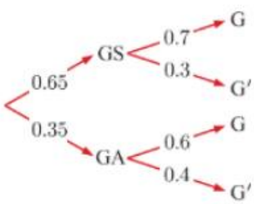
3 **a** 

b **i** $\frac{14}{45}$ **ii** $\frac{16}{45}$

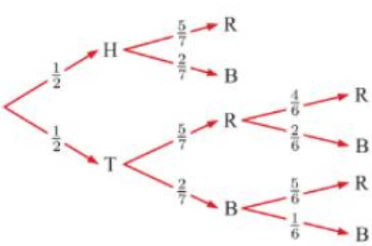
4 **a** M is the event of Matt playing in a game.
W is the event of his team winning.



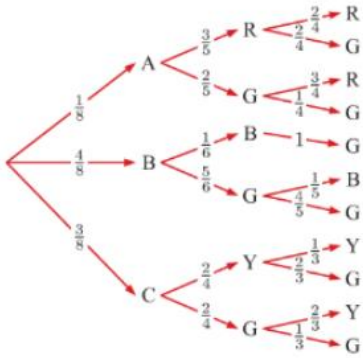
b 0.6

5 

P(G) = 0.665

6 

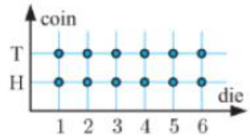
P(at least one red) $= \frac{5}{6}$

7 

P(exactly one G) $= \frac{59}{120}$

EXERCISE 14F

1 A and B, A and D, A and E, A and F, B and D, B and F, C and D

2 **a** 

b **i** No. It is possible to 'toss a head' *and* 'roll a 5'.
ii $\text{P}(A \text{ or } B) = \frac{7}{12}$, $\text{P}(A \text{ and } B) = \frac{1}{12}$
iii $\text{P}(A) + \text{P}(B) - \text{P}(A \text{ and } B) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12}$
$= \frac{7}{12} = \text{P}(A \text{ or } B)$ ✓

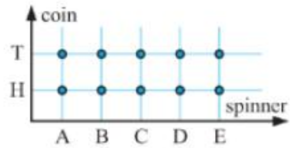
3 **a** No, since $\text{P}(A \text{ and } B) \neq 0$. **b** 0.75 **4** $\text{P}(Y) = 0.4$

5 If C and D were mutually exclusive, then $\text{P}(C \text{ or } D) = 0.6 + 0.7 = 1.3$, which is not possible.

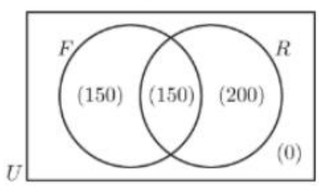
6 **a** 0 **b** 0.1 **7** 0.65 **8** $\text{P}(A) = 0.8$, $\text{P}(B) = 0.5$

REVIEW SET 14A

1 **a** 38 days **b** **i** $\frac{1}{38}$ **ii** $\frac{11}{38}$ **iii** $\frac{19}{38} = \frac{1}{2}$

2 

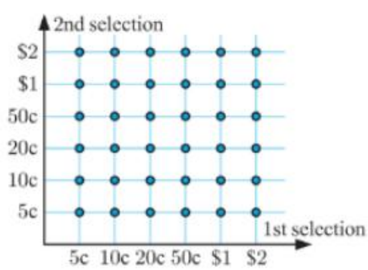
3 **a** $\frac{49}{81} \approx 0.60$ **b** $\frac{32}{81} \approx 0.40$ **4** **a** $\frac{1}{20}$ **b** $\frac{3}{10}$ **c** $\frac{9}{20}$

5 **a** 

b **i** $\frac{2}{5}$ **ii** $\frac{3}{10}$

6 **a** $\frac{2}{7}$ **b** $\frac{4}{7}$ **7** **a** $\frac{1}{9}$ **b** $\frac{2}{9}$

8 0.66 **9** **a** 47 books **b** **i** $\frac{31}{47}$ **ii** $\frac{4}{47}$

10 **a** 

b **i** $\frac{4}{9}$ **ii** $\frac{11}{36}$ **iii** $\frac{23}{36}$

11

Injuries	0	1	2	3
Probability	0.336	0.452 ✓	0.188	0.024

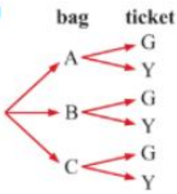
$\therefore$ one injury is most likely.

12 **a** 0 **b** 0.14

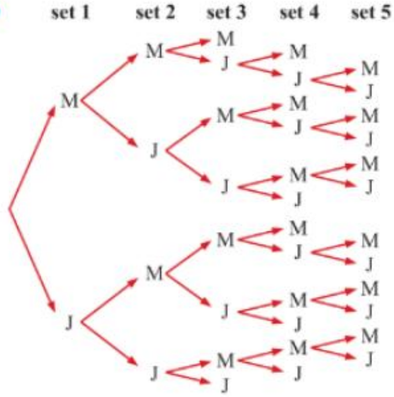
REVIEW SET 14B

1 **a** ≈ 0.364 **b** ≈ 0.551 **c** ≈ 0.814

2 **a**



b

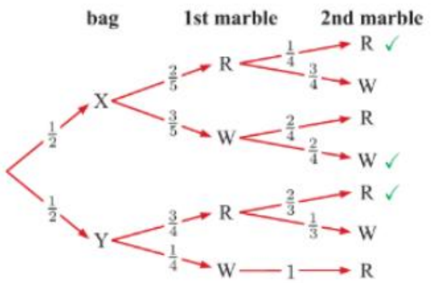


3 **a** $\frac{318}{450} \approx 0.707$ **b** $\frac{132}{450} \approx 0.293$ **c** $\frac{21}{450} \approx 0.0467$

4 $\frac{1}{2}$ **5** **a** $\frac{1}{9}$ **b** $\frac{4}{9}$ **c** $\frac{4}{9}$

6 **a** $\frac{37}{50} = 0.74$ **b** ≈ 0.0176

7 **a**



b $\frac{9}{20}$

8 0.697 **9** $\frac{7}{24}$ **10** $P(X \text{ or } Y) = 0.8$

11 **a** $\frac{13}{55}$

b **i** Increases, as the number of tickets of the desired colour does not decrease with each draw.

ii $\frac{37}{121}$

12 **a** **i** 0.21 **ii** 0.28 **iii** 0.3

b P(Ihor wins) $= 0.238$ which is > 0.21

$\therefore$ the strategy increases his chance of winning.

EXERCISE 15A

1 **a** $A = 5 \times 15$ **b** $A = 5p$ **c** $A = tp$

2 **a** $A = 2000 + 150 \times 8$ **b** $A = 2000 + 150w$
c $A = 2000 + dw$ **d** $A = P + dw$

3 **a** $C = 40 + 60 \times 5$ **b** $C = 40 + 60t$
c $C = 40 + xt$ **d** $C = F + xt$

4 **a** $P = 10 \times 3 - 1(15 - 10)$

b $P = c \times 3 - 1(20 - c)$
$\therefore\ P = 4c - 20$

c $P = c \times 3 - 1(a - c)$
$\therefore\ P = 4c - a$

5 **a** $D = 4 \times 6 + 2(4 - 1)$ **b** $D = 5m + 3 \times (5 - 1)$
c $D = 8m + b(8 - 1)$ **d** $D = mp + b(p - 1)$

6 **a** $G = 2 \times (3 - 1) + 3 \times (2 - 1)$
b $G = 3 \times (5 - 1) + 5 \times (3 - 1)$
c $G = 4 \times (4 - 1) + 4 \times (4 - 1)$
d $G = m(n - 1) + n(m - 1)$
$\therefore\ G = 2mn - m - n$

7 **a** $P = (2x + y)$ cm **b** $P = (4x + 6)$ m
c $P = (2\pi r + 2d)$ m

8 **a** $A = 2ab + \dfrac{\pi a^2}{2}$ **b** $A = ar + \dfrac{\pi r^2}{2}$
c $A = aw + \dfrac{\pi w^2}{4}$ **d** $A = 2ar - \pi r^2$
e $A = \dfrac{\pi(b^2 - a^2)}{8} + \dfrac{a\sqrt{b^2 - a^2}}{2}$ **f** $A = \dfrac{\theta}{360}\pi r^2 + dr$

9 **a** $V = Al$ **b** $V = \dfrac{\pi d^2 h}{4}$ **c** $V = \dfrac{abc}{2}$
d $V = (\frac{2}{3}\pi r^3)$ cm^3 **e** $V = (\frac{2}{3}\pi r^3 + \pi r^2 h)$ m^3

10 **a** $A = 2ab + 2bc + 2ac$ **b** $A = 6a^2 + 8ab$
c $A = \pi r^2 + 2\pi rh + \pi rs$ **d** $A = 10rh + 6r^2 + 4\pi r^2$

11 **Hint:** Area of end section $= \pi R^2 - \pi r^2$
$= \pi(R + r)(R - r)$

12 $V = \frac{\sqrt{2}}{12}x^3$

EXERCISE 15B

1 **a** ≈ 26.4 cm **b** ≈ 17.8 cm **c** ≈ 127 m

2 **a** 19.6 m **b** ≈ 4.52 s

3 **a** ≈ 129 cm^2 **b** ≈ 7.14 m

4 **a** ≈ 4260 cm^3 **b** ≈ 1.06 cm **c** ≈ 4.99 mm

5 **a** ≈ 707 cm^2 **b** ≈ 39.9 cm

6 **a** **i** $P = 4x$ cm **ii** $x = 3$
b **i** $P = 2\pi x$ cm **ii** $x \approx 1.91$

7 **a** $A = \frac{\sqrt{3}}{4}x^2$ m^2 **b** 8 cm

8 **a** ≈ 1.34 s **b** 81 cm

EXERCISE 15C

1 **a** $y = \dfrac{10 - 2x}{5}$ **b** $y = \dfrac{20 - 3x}{4}$ **c** $y = 2x - 8$
d $y = \dfrac{14 - 2x}{7}$ **e** $y = \dfrac{20 - 5x}{2}$ **f** $y = \dfrac{2x + 12}{3}$

2 **a** $x = r - p$ **b** $x = \dfrac{z}{y}$ **c** $x = \dfrac{d - a}{3}$
d $x = \dfrac{d - 2y}{5}$ **e** $x = \dfrac{p - by}{a}$ **f** $x = \dfrac{y - c}{m}$
g $x = \dfrac{s - 2}{t}$ **h** $x = \dfrac{m - p}{q}$ **i** $x = \dfrac{6 - a}{b}$

3 **a** $y = \dfrac{t - z}{5}$ **b** $y = \dfrac{c - p}{2}$ **c** $y = \dfrac{a - t}{3}$
d $y = \dfrac{n - 5}{k}$ **e** $y = \dfrac{a - n}{b}$ **f** $y = \dfrac{a - p}{n}$

4 **a** $z = \dfrac{b}{ac}$ **b** $z = \dfrac{a}{d}$ **c** $z = \dfrac{2d}{3}$
d $z = \pm\sqrt{2a}$ **e** $z = \pm\sqrt{bn}$ **f** $z = \pm\sqrt{m(a - b)}$

5 **a** $a = \dfrac{F}{m}$ **b** $r = \dfrac{C}{2\pi}$ **c** $d = \dfrac{V}{lh}$
d $K = \dfrac{b}{A}$ **e** $h = \dfrac{2A}{b}$ **f** $T = \dfrac{100I}{PR}$

6 $h = \dfrac{A - 2\pi r^2}{2\pi r}$ or $h = \dfrac{A}{2\pi r} - r$

7 **a** $r = \sqrt{\dfrac{A}{\pi}}$ **b** $x = \pm\sqrt{aN}$ **c** $k = \pm\sqrt{\dfrac{M}{5}}$

d $x = \sqrt[3]{\dfrac{n}{D}}$ **e** $x = -\sqrt{\dfrac{y+7}{4}}$ **f** $Q = \pm\sqrt{P^2 - R^2}$

8 **a** $a = d^2n^2$ **b** $l = 25T^2$ **c** $a = \pm\sqrt{b^2 + c^2}$

d $d = \dfrac{25a^2}{k^2}$ **e** $l = \dfrac{gT^2}{4\pi^2}$ **f** $b = \dfrac{16a}{A^2}$

9 **a** $x = \dfrac{c-a}{3-b}$ **b** $x = \dfrac{c}{a+b}$ **c** $x = \dfrac{a+2}{n-m}$

d $x = -\dfrac{a}{b+8}$ **e** $x = \dfrac{a-b}{1-c}$ **f** $x = \dfrac{e-d}{r+s}$

10 **a** $a = \dfrac{2-bP}{P}$ **b** $r = \dfrac{8-qT}{T}$ **c** $q = \dfrac{Ap-B}{A}$

d $x = \dfrac{3-Ay}{2A}$

11 **a** $x = \dfrac{y}{1-y}$ **b** $x = \dfrac{2y+3}{1-y}$ **c** $x = \dfrac{3y+1}{3-y}$

d $x = \dfrac{2y+1}{y+4}$ **e** $x = \dfrac{3y-1}{y-1}$ **f** $x = \dfrac{-4y-3}{y+2}$

EXERCISE 15D

1 **a** $\theta = \dfrac{360A}{\pi r^2}$

b **i** $\theta \approx 63.7°$ **ii** $\theta \approx 105°$ **iii** $\theta \approx 214°$

2 **a** $a = \dfrac{d^2}{2bK}$ **b** **i** $a \approx 1.29$ **ii** $a = 16.2$

3 **a** $t = (H-1)^2$

b **i** 1 year **ii** 4 years **iii** $6\frac{1}{4}$ years

4 **a** $r = \sqrt[3]{\dfrac{3V}{4\pi}}$

b **i** ≈ 2.12 cm **ii** ≈ 5.76 cm **iii** ≈ 62.0 cm

5 **a** $v = \sqrt{u^2 + 2as}$ **b** **i** ≈ 20.6 m/s **ii** ≈ 52.9 m/s

6 **a** $\approx 58.8\%$ **b** $w = \dfrac{Pl}{100-P}$ **c** 9 matches

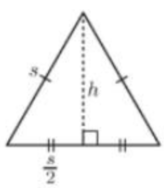
d 7 consecutive matches

7 **a** $F \approx 2.01 \times 10^{20}$ Newtons **b** $d = \sqrt{\dfrac{Gm_1m_2}{F}}$

c **i** $\approx 1.50 \times 10^{11}$ m **ii** $\approx 1.43 \times 10^{14}$ m

8 **b** $p = \dfrac{3g}{g-1}$ **c** $0g$ $0p$, $2g$ $6p$, $4g$ $4p$

9 **a** **Hint:**



Use the diagram to show that the area of one triangle face $= \dfrac{s}{2} \times \dfrac{\sqrt{3}s}{2} = \dfrac{\sqrt{3}s^2}{4}$

b $s^2 = \dfrac{A}{1+\sqrt{3}}\left(\dfrac{1-\sqrt{3}}{1-\sqrt{3}}\right)$ etc.

c **i** ≈ 4.28 cm **ii** ≈ 7.41 cm **iii** ≈ 14.8 cm

EXERCISE 15E

1 **a** 6, 11, 16, 21, 26 **b** 51 **c** $5n+1$

2 **a** 7, 10, 13, 16, 19 **b** 34 **c** $3n+4$

3 **a** **i** 4 **ii** 9 **iii** 16 **iv** 25 **b** $S_n = n^2$

4 **a** **i** 3 **ii** 7 **iii** 15 **iv** 31

b As $3 = 2^2 - 1$, $7 = 2^3 - 1$, $15 = 2^4 - 1$, etc., $S_n = 2^n - 1$

5 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{2}{3}$, $S_3 = \frac{3}{4}$, $S_4 = \frac{4}{5}$, ...

b **i** $S_{10} = \frac{10}{11}$ **ii** $S_n = \dfrac{n}{n+1}$

6 **a** $S_1 = \dfrac{1 \times 2 \times 3}{6} = 1$ and $1^2 = 1$ ✓

$S_2 = \dfrac{2 \times 3 \times 5}{6} = 5$ and $1^2 + 2^2 = 5$ ✓

$S_3 = \dfrac{3 \times 4 \times 7}{6} = 14$ and $1^2 + 2^2 + 3^2 = 14$ ✓

$S_4 = \dfrac{4 \times 5 \times 9}{6} = 30$ and $1^2 + 2^2 + 3^2 + 4^2 = 30$ ✓

b $S_{100} = \dfrac{100 \times 101 \times 201}{6} = 338\,350$

REVIEW SET 15A

1 **a** **i** $V = 6 \times 8$ litres **ii** $V = 8n$ litres **iii** $V = ln$ litres

b $V = 25 + ln$ litres

2 **a** 90 km/h **b** 3900 km **3** $A = \frac{\sqrt{3}}{2}a^2 + 3ab$

4 **a** $x = \dfrac{3p-n}{m}$ **b** $x = \dfrac{5y}{7}$

5 **a** $k = T^2 + l^2$ **b** $k = -\sqrt{\dfrac{P+r}{2}}$

6 **a** $V = \frac{1}{12}\pi x^3$ **b** ≈ 1.79 m

7 **a** 8 amperes **b** $r = \dfrac{E - IR}{I}$ **c** 0.15 ohms

8 **a** 4, 7, 10 **b** $M = 3n + 1$

9 **a** **i** ≈ 283.2 K **ii** ≈ 183.2 K **iii** ≈ 338.7 K

b $F = \frac{9}{5}(K - 273.15) + 32$

c **i** 104°F **ii** −459.67°F **iii** −99.67°F

REVIEW SET 15B

1 **a** $B = 15 + 25 \times 5$ **b** $B = c + 25 \times p$ **c** $B = c + mp$

2 **a** $E = 2 \times (3-2) + 2 \times (5-2) = 8$

b $E = 2 \times (4-2) + 2 \times (8-2) = 16$

c $E = 2(m-2) + 2(n-2)$

3 **a** $M = 37$ **b** $r = 8$ **4** $V = 4x^2y$

5 **a** $a = \dfrac{B+f}{d}$ **b** $a = \dfrac{9Q^2}{t^2}$ **c** $a = \dfrac{5-G^2}{G^2}$

6 **a** $b = \dfrac{a}{a-1}$ **b** $b = \frac{3}{2}$; $3 \times \frac{3}{2} = 3 + \frac{3}{2} = 4\frac{1}{2}$ ✓

7 $x = \dfrac{2y+3}{4-3y}$ **8** 8, 13, 18, ... $\therefore$ $M = 5n + 3$

9 **a** **i** 6 **ii** 12 **iii** 20 **iv** 30

b $S_1 = 2 = 1 \times 2$

$S_2 = 6 = 2 \times 3$

$S_3 = 12 = 3 \times 4$

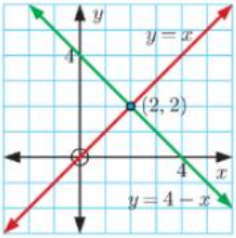
$S_4 = 20 = 4 \times 5$

$S_5 = 30 = 5 \times 6$ $\therefore$ $S_n = n(n+1)$

10 **a** $E = 1000$ joules **b** $v = \sqrt{\dfrac{2E}{m}}$ **c** 8 m/s

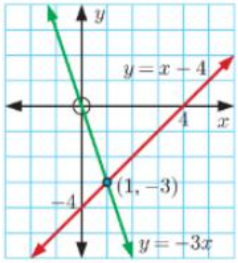
EXERCISE 16A.1

1 **a**



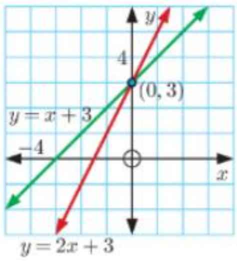
$x = 2,\ y = 2$

b



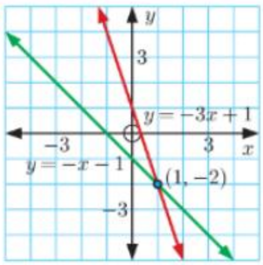
$x = 1,\ y = -3$

c



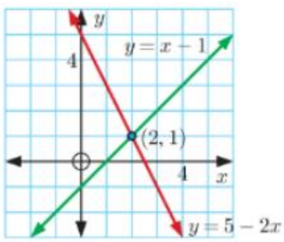
$x = 0,\ y = 3$

d



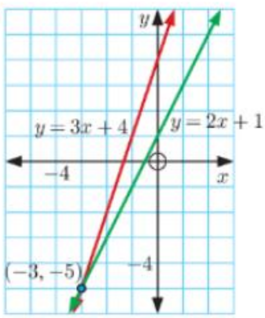
$x = 1,\ y = -2$

e



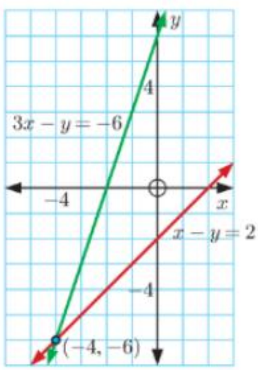
$x = 2,\ y = 1$

f



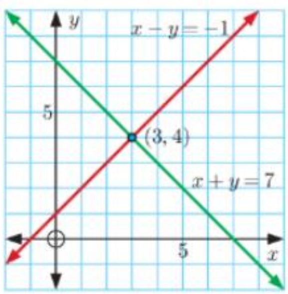
$x = -3,\ y = -5$

2 **a**



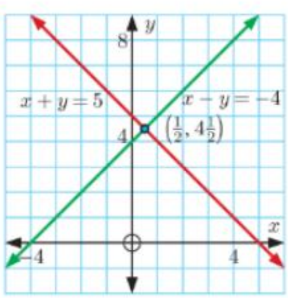
$x = -4,\ y = -6$

b



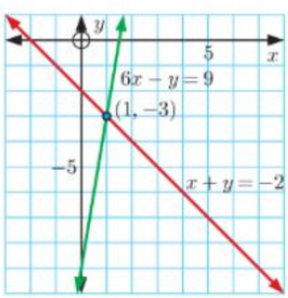
$x = 3,\ y = 4$

c



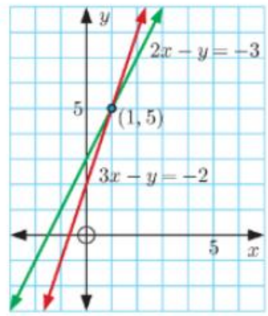
$x = \frac{1}{2},\ y = 4\frac{1}{2}$

d



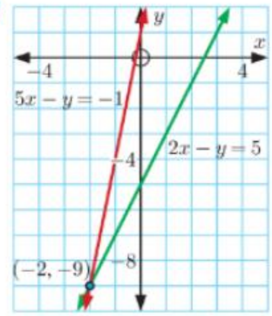
$x = 1,\ y = -3$

e



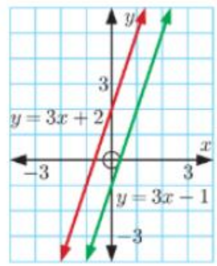
$x = 1,\ y = 5$

f



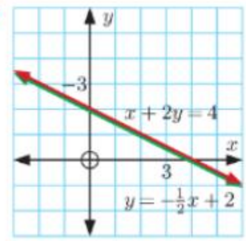
$x = -2,\ y = -9$

3 **a**



The lines are parallel.
$\therefore$ no solution exists.

b



The lines are coincident.
$\therefore$ infinitely many solutions exist.

EXERCISE 16A.2

1 **a** $x = 4,\ y = 5$ **b** $x = 0.5,\ y = 3.5$ **c** $x = 1,\ y = 5$

2 **a** $x = 16,\ y = -2$ **b** $x = -14,\ y = 5$
c $x = 21,\ y = 9$ **d** $x = -5,\ y = 15$
e $x = 2.25,\ y = 1.25$ **f** $x = 15.2,\ y = 13.2$

EXERCISE 16B

1 **a** $x = 1,\ y = 5$ **b** $x = 5,\ y = 7$
c $x = 2,\ y = 6$ **d** $x = -4,\ y = -7$
e $x = -2,\ y = -8$ **f** $x = -4,\ y = -6$

2 **a** $(3, 4)$ **b** $(\frac{1}{2}, 4\frac{1}{2})$ **c** $(2, -1)$ **d** $(0, -4)$
e $(-1, -1)$ **f** $(0, 6)$

EXERCISE 16C

1 **a** $x = 2,\ y = 5$ **b** $x = 3,\ y = 1$ **c** $x = 0,\ y = 5$
d $x = 5,\ y = 9$ **e** $x = 0,\ y = 4$ **f** $x = 2,\ y = 1$

2 **a** $x = 5,\ y = 3$ **b** $x = 1,\ y = \frac{2}{5}$
c $x = 2,\ y = \frac{4}{3}$ **d** $x = 5,\ y = -2$
e $x = -3,\ y = -\frac{1}{2}$ **f** $x = -\frac{9}{2},\ y = -\frac{7}{2}$

3 **a** reduces to $-10 = -14$ which is never true
$\therefore$ no solutions exist.
b reduces to $-6 = -6$ which is always true
$\therefore$ infinitely many solutions exist.

EXERCISE 16D

1 **a** $4x = 20$ **b** $3x = 3$ **c** $y = 5$

2 **a** $x = 4,\ y = -2$ **b** $x = -14,\ y = -10$
c $x = -2,\ y = 3$ **d** $x = -1,\ y = 4$
e $x = -1,\ y = 1$ **f** $x = 3,\ y = \frac{4}{3}$

3 **a** $4x + 6y = 2$ **b** $2x - 4y = -14$
c $12x - 3y = -6$ **d** $-15x + 20y = 15$

4 **a** $x = 2,\ y = 1$ **b** $x = -3,\ y = 1$
c $x = -1,\ y = 2$ **d** $x = 3,\ y = 1$
e $x = -4,\ y = -2$ **f** $x = 3,\ y = 2$

5 **a** $x=4,\ y=-1$ **b** $x=3,\ y=0$ **c** $x=-2,\ y=2$
d $x=-1,\ y=-3$ **e** $x=5,\ y=7$ **f** $x=-1,\ y=6$

EXERCISE 16E

1 24 and 9 **2** 21 and 10 **3** $40\frac{1}{2}$ and $23\frac{1}{2}$

4 7 £1 coins, 5 £2 coins

5 **a** **i** The number of hen heads and the number of goat heads must total 48. Hence, $x+y=48$.
ii Hens have 2 legs, so there are $2x$ hen legs.
Goats have 4 legs, so there are $4y$ goat legs.
There are 122 legs in total, so $2x+4y=122$.
b **i** infinitely many solutions **ii** infinitely many solutions
c $x=35,\ y=13$ (35 hens, 13 goats)

6 CDs cost RM12, DVDs cost RM17

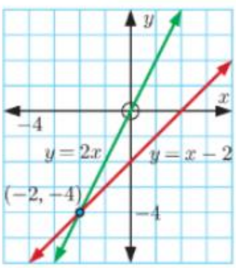
7 17 chairs, 6 stools **8** small bags: 2 kg, large bags: 5 kg

9 **a** $x=2\frac{1}{2},\ y=3$ **b** perimeter = 17 cm

10 13 years, 4 months

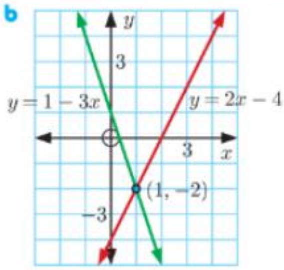
REVIEW SET 16A

1 **a**



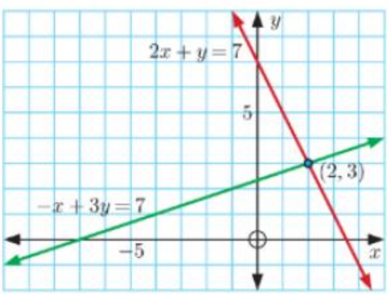
$x=-2,\ y=-4$

b



$x=1,\ y=-2$

c



$x=2,\ y=3$

2 **a** $x=11,\ y=-3$ **b** $x\approx 0.949,\ y\approx -1.18$

3 **a** $x=2,\ y=7$ **b** $x=-2,\ y=-2$

4 **a** $x=-4,\ y=-16$ **b** $x=-1,\ y=3$

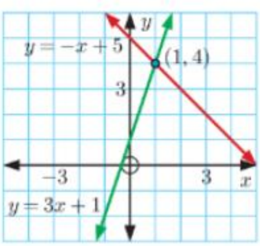
5 reduces to $0=-2$ which is never true
$\therefore$ no solutions exist.

6 **a** $x=2,\ y=-1$ **b** $x=1,\ y=0$

7 17 and 5 **8** \$1.50

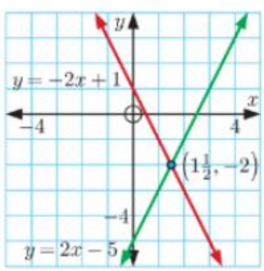
REVIEW SET 16B

1 **a**



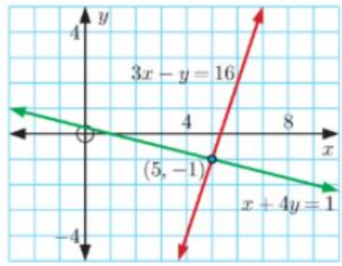
$x=1,\ y=4$

b



$x=1\frac{1}{2},\ y=-2$

c



$x=5,\ y=-1$

2 **a** $x=-5,\ y=18$ **b** $x\approx 0.194,\ y\approx 0.771$

3 **a** $(4, 18)$ **b** $(-1, 5)$

4 **a** $x=-4,\ y=1$ **b** $x=3,\ y=1$

5 reduces to $0=0$ which is always true
$\therefore$ infinitely many solutions exist.

6 **a** $x=9,\ y=-11$ **b** $x=\frac{7}{11},\ y=-\frac{32}{11}$

7 18 giraffes, 21 ostriches

8 small bottle: €1.75, large bottle: €3.15

EXERCISE 17A

1 **a** The *hours of training* an employee has completed, and their *productivity*.
b A **c** E

2 **a** A **b** A **c** yes; B and C
d The less sleep an employee has, the more likely they are to have an accident.

3 **a** The *time* of day, and the *number of shoppers in store*.
b **i** 11 am **ii** 8 pm
c The number of customers increased from 13 to 39.
d 28 people

4 **a** **i** CHF5 **ii** CHF10 **b** 2 kg

c

Package weight w (kg)	*Cost* (CHF)
$0 < w \leqslant 0.5$	5
$0.5 < w \leqslant 1$	8
$1 < w \leqslant 2$	10
$2 < w \leqslant 3$	11

EXERCISE 17B

1 **a** Domain is $\{x \mid x > -4\}$. Range is $\{y \mid y > -2\}$.
b Domain is $\{x \mid -3 \leqslant x \leqslant 4\}$. Range is $\{y \mid -5 \leqslant y \leqslant 2\}$.
c Domain is $\{x \mid -3 < x < 4\}$. Range is $\{y \mid -5 < y < 6\}$.
d Domain is $\{x \mid x = 2\}$. Range is $\{y \mid y \in \mathbb{R}\}$.
e Domain is $\{x \mid -3 \leqslant x \leqslant 3\}$. Range is $\{y \mid -3 \leqslant y \leqslant 3\}$.
f Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \leqslant 0\}$.
g Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = -5\}$.
h Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant 1\}$.
i Domain is $\{x \mid x \geqslant -5\}$. Range is $\{y \mid y \leqslant 7\}$.
j Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \leqslant 4\}$.
k Domain is $\{x \mid x \geqslant -5\}$. Range is $\{y \mid y \in \mathbb{R}\}$.
l Domain is $\{x \mid x \in \mathbb{R},\ x \neq 1\}$.
Range is $\{y \mid y \in \mathbb{R},\ y \neq 0\}$.

2 **a** Domain is $\{x \mid 0 \leqslant x \leqslant 2\}$. Range is $\{y \mid -3 \leqslant y \leqslant 2\}$.
b Domain is $\{x \mid -2 < x < 2\}$. Range is $\{y \mid -1 < y < 3\}$.
c Domain is $\{x \mid -4 \leqslant x \leqslant 4\}$. Range is $\{y \mid -2 \leqslant y \leqslant 2\}$.

EXERCISE 17C

1 **a**, **b**, and **e** are functions as no two ordered pairs have the same first member.

2 **a**, **b**, **d**, **e**, **g**, **h**, and **i** are functions.

3 No, a vertical line is not a function as it does not satisfy the vertical line test.

EXERCISE 17D.1

1 a 3 b 7 c 1 d -7 e 2

2 a -2 b -17 c 13 d $5x+3$ e $-5x-17$

3 a 2 b 11 c 46 d $2x^2+3x+2$ e $2x^2+x+1$

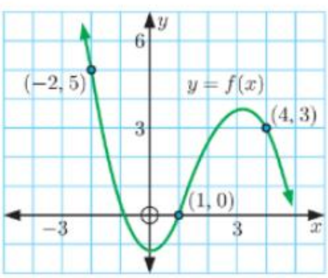
4 a 45 b -3 c 0 d $4x^2-20x+21$ e $16x^2+8x-3$

5 a i $-\frac{3}{2}$ ii $-\frac{1}{3}$ iii $-\frac{8}{3}$ b $x=-2$ c $2-\frac{7}{x}$ d $x=-1$

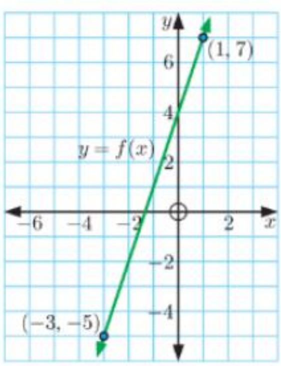
6 a $V(4) = 12\,000$. The value of the car after 4 years is \$12 000.
b $V(t) = 8000$ when $t = 5$. It takes 5 years for the value of the car to fall to \$8000.
c \$28 000

7 a i $f(2)=1$ ii $f(3)=-1$ b $x=-4$

8 a i $f(4)=2$ ii $g(0)=-6$ iii $g(5)=-1$
b $x=0$ and $x=3$ c $x=2$
d g has gradient 1 and y-intercept -6.

9


Other graphs are possible.

10 a


b $f(-3)=-5$, $f(1)=7$
c $f(x)=3x+4$

EXERCISE 17D.2

1 a $\{x \mid x\in\mathbb{R}\}$ b $\{x \mid x\neq 0\}$
c $\{x \mid x\neq 3\}$ d $\{x \mid x\neq -2 \text{ or } 1\}$
e $\{x \mid x\neq -3 \text{ or } 3\}$ f $\{x \mid x\neq 1 \text{ or } 4\}$

2 a $\{x \mid x\geqslant 2\}$ b $\{x \mid x\leqslant 3\}$
c $\{x \mid x>0\}$ d $\{x \mid x<4\}$

EXERCISE 17E

1 a 2 b 10 c 5 d 11 e 11 f 40 g 40 h -2

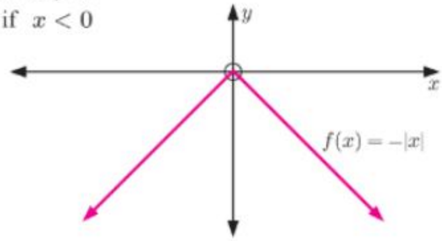
2 a 3 b 10 c 7 d 3 e 13 f 2 g $\frac{5}{2}$ h $\frac{4}{5}$

3 a

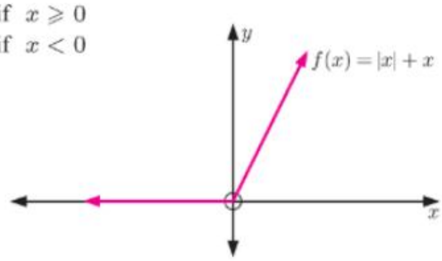
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$\lvert x\rvert^2$	9	4	1	0	1	4	9

b $x^2 = |x|^2$

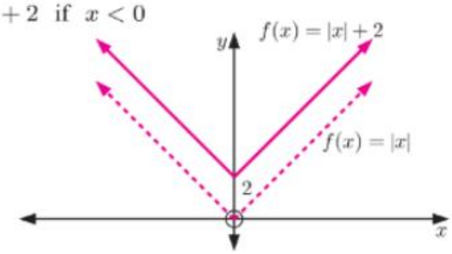
4 a $f(x)=\begin{cases}-x & \text{if } x\geqslant 0\\ x & \text{if } x<0\end{cases}$



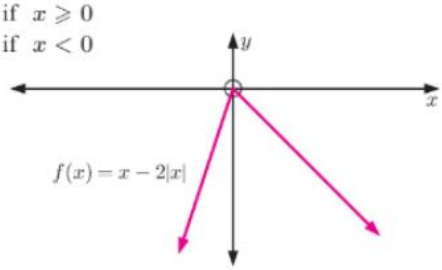
b $f(x)=\begin{cases}2x & \text{if } x\geqslant 0\\ 0 & \text{if } x<0\end{cases}$



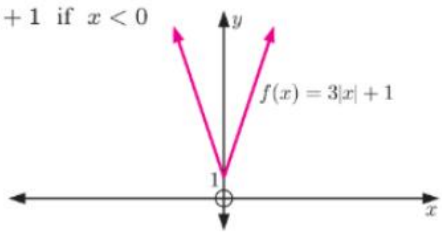
c $f(x)=\begin{cases}x+2 & \text{if } x\geqslant 0\\ -x+2 & \text{if } x<0\end{cases}$



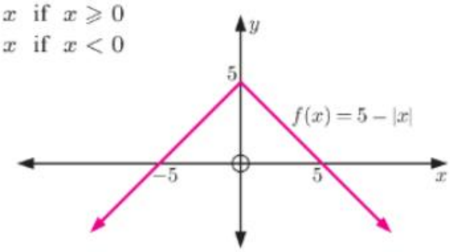
d $f(x)=\begin{cases}-x & \text{if } x\geqslant 0\\ 3x & \text{if } x<0\end{cases}$



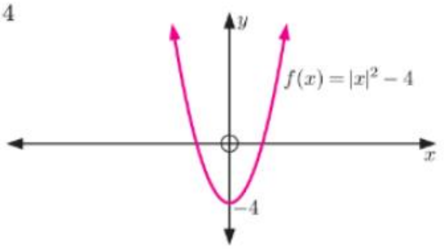
e $f(x)=\begin{cases}3x+1 & \text{if } x\geqslant 0\\ -3x+1 & \text{if } x<0\end{cases}$



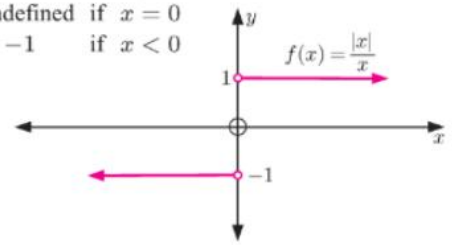
f $f(x)=\begin{cases}5-x & \text{if } x\geqslant 0\\ 5+x & \text{if } x<0\end{cases}$



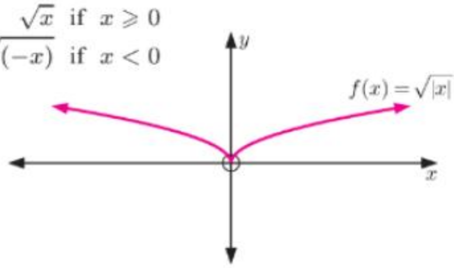
g $f(x) = x^2 - 4$



h $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



i $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ \sqrt{(-x)} & \text{if } x < 0 \end{cases}$



EXERCISE 17F

1 **a** $(2, 9)$ **b** $(\frac{5}{4}, \frac{43}{4})$ **c** $(5, \frac{1}{5})$ **d** $(1, 3)$

2 **a** at $(-3, 2)$ and $(2, 7)$ **b** at $(2, 1)$ and $(-1, -2)$
c at $(-\frac{1}{2}, -\frac{9}{4})$ and $(-3, 14)$ **d** at $(1, 1)$ and $(-\frac{1}{5}, -5)$

3 **a** at $(-1.62, -1.24)$ and $(0.62, 3.24)$ **b** at $(4.71, 0.64)$
c They do not intersect, $\therefore$ no solutions exist.
d at $(-0.75, -0.43)$

REVIEW SET 17A

1 **a** The *average hours of fitness training* completed by the players at each hockey club, and the *number of injuries* at each hockey club.
b E **c** A
d The more hours of fitness training done at a hockey club, the less injuries sustained by its players.

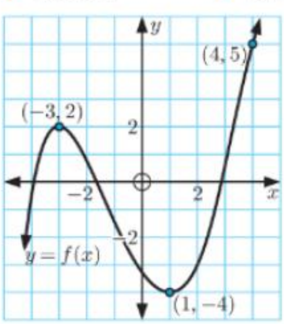
2 **a** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant -2\}$.
b Domain is $\{x \mid x \geqslant 0\}$. Range is $\{y \mid y \in \mathbb{R}\}$.

3 **a** 2 **b** -4 **c** $-x^2 + 9x - 18$

4 **a** 8 **b** $9x^2 + 6x$ **c** $x = -5$ or 3

5 **a** function **b** not a function

6

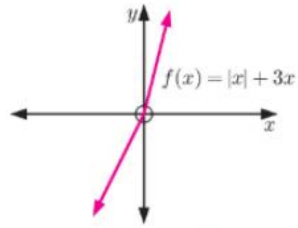


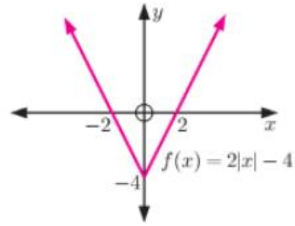
Note: There may be other answers.

7 **a** Domain is $\{x \mid x \neq -1\}$ **b** Domain is $\{x \mid x \neq \pm 2\}$
c Domain is $\{x \mid x > 1\}$

8 **a** 7 **b** -1 **c** 0

9 **a** $f(x) = \begin{cases} 4x & \text{if } x \geqslant 0 \\ 2x & \text{if } x < 0 \end{cases}$ **b** $f(x) = \begin{cases} 2x - 4 & \text{if } x \geqslant 0 \\ -2x - 4 & \text{if } x < 0 \end{cases}$





10 **a** $(2, 4)$ **b** $(-3, -5)$ and $(1, 3)$

REVIEW SET 17B

1 **a** The time elapsed in the race in minutes (x), and how far Fernando is ahead of Gerard in metres (y).
b There are corresponding values of y for all $0 \leqslant x \leqslant 15$. This tells us that the race was 15 minutes long.
c Fernando, as the final point on the graph is above the x-axis.
d Fernando, 15 m

2 **a** Domain is $\{x \mid x > -6\}$. Range is $\{y \mid y > -2\}$.
b Domain is $\{x \mid x \leqslant 1\}$. Range is $\{y \mid y \in \mathbb{R}\}$.

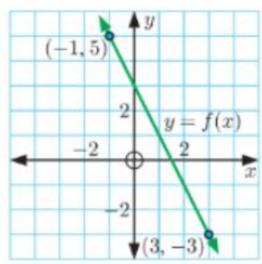
3 **a** function **b** not a function

4 **a** function **b** not a function

5 **a** -24 **b** $-5x - x^2$ **c** $-x^2 + 3x + 4$

6 **a** Domain is $\{x \mid x \neq -4\}$.
b Domain is $\{x \mid x \neq -5 \text{ or } 1\}$.
c Domain is $\{x \mid x \leqslant 6\}$.

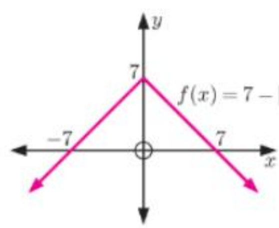
7 **a**

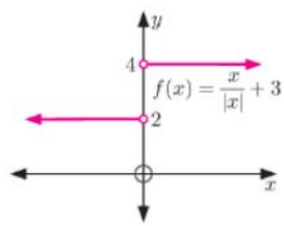


b $f(-1) = 5$, $f(3) = -3$
c $f(x) = -2x + 3$

8 **a** 36 **b** 13 **c** $\frac{7}{4}$

9 **a** $f(x) = \begin{cases} 7 - x & \text{if } x \geqslant 0 \\ 7 + x & \text{if } x < 0 \end{cases}$ **b** $f(x) = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \\ \text{undef.} & \text{if } x = 0 \end{cases}$





10 **a** $(-1, -1)$ **b** $(-1, -3)$ and $(\frac{3}{2}, 2)$

EXERCISE 18A

1 **a**, **c**, **d**, and **f** are exponential functions.

2 **a** $\frac{1}{8}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** 1 **e** 2
f 4 **g** 8 **h** 2^{2x} or 4^x

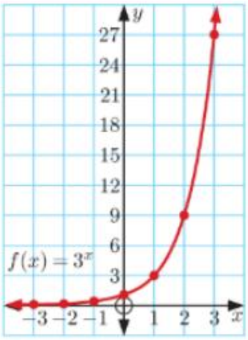
3 **a** 3 **b** 11 **c** $2\frac{1}{3}$ **d** $3^{2x} + 2$

4 **a** -2 **b** $-2\frac{4}{5}$ **c** 22 **d** $5^x - 3$

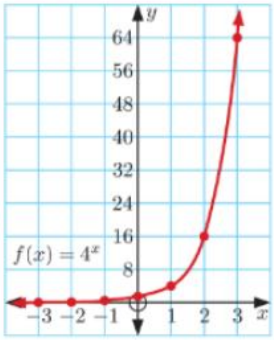
5 **a** $\frac{1}{9}$ **b** 9 **c** $\frac{1}{27}$ **d** 3^{x+3}

EXERCISE 18B.1

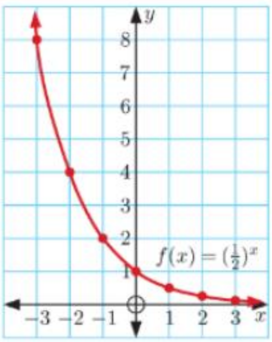
1 **a**



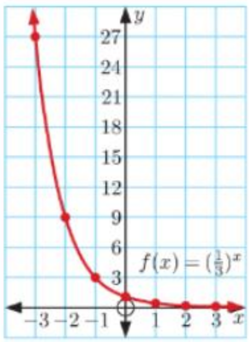
b



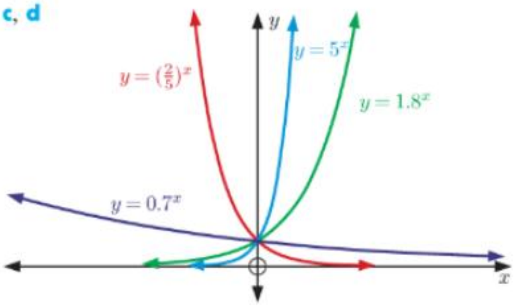
c



d



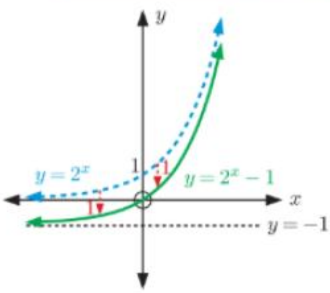
2 **a, b, c, d**



3 **a** D **b** A **c** F **d** C **e** B **f** E

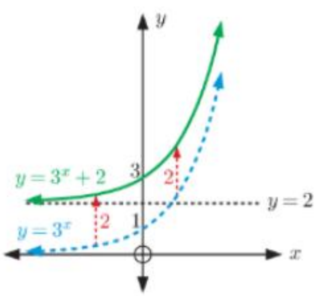
EXERCISE 18B.2

1 **a** **i**



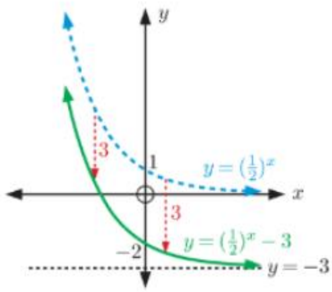
ii $y = -1$
iii $\{y \mid y > -1\}$
iv 0

b **i**



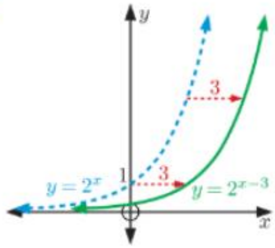
ii $y = 2$
iii $\{y \mid y > 2\}$
iv 3

c **i**

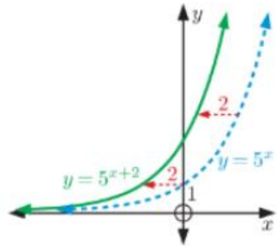


ii $y = -3$
iii $\{y \mid y > -3\}$
iv -2

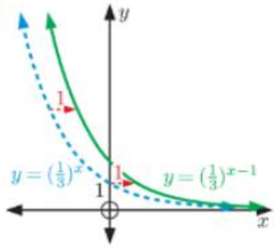
2 **a**



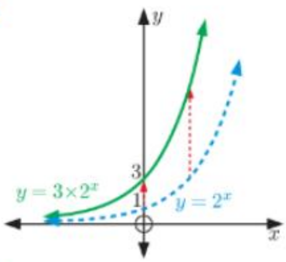
b



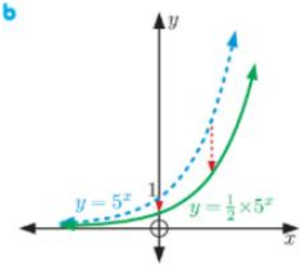
c



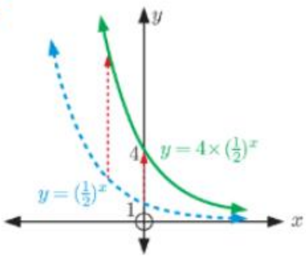
3 **a**



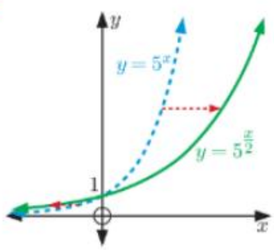
b



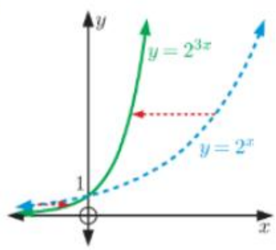
c



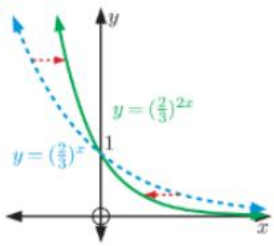
4 **a**



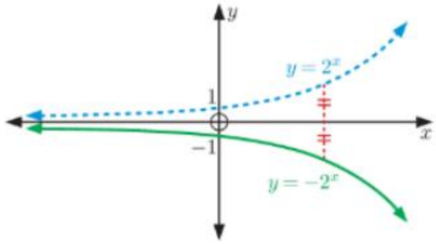
b



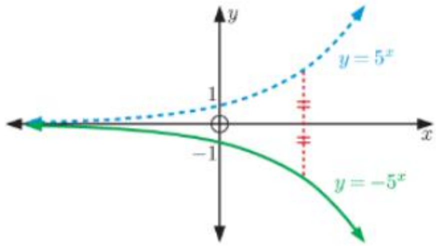
c



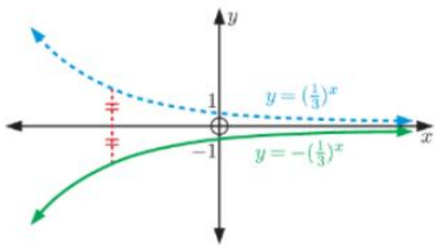
5 **a**



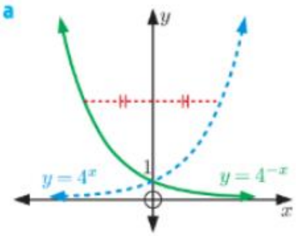
b



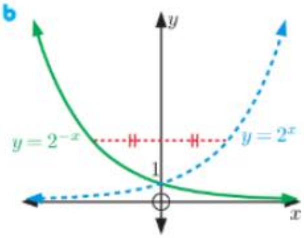
c



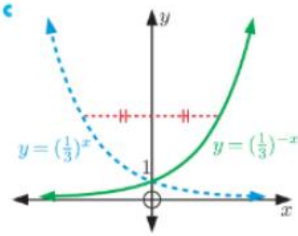
6 **a**



b



c



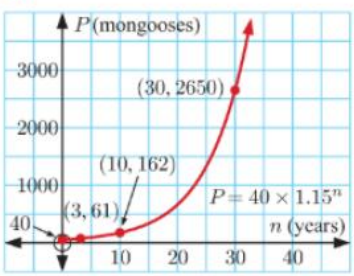
EXERCISE 18C.1

1 **a** 40 mongooses

b **i** ≈ 61 mongooses **ii** ≈ 162 mongooses

iii ≈ 2650 mongooses

c

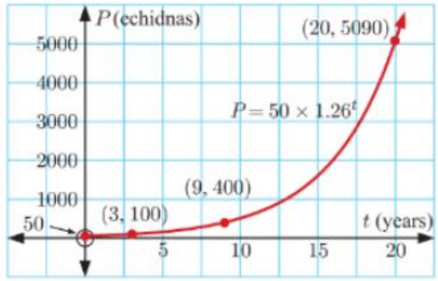


2 **a** 50 echidnas

b **i** ≈ 100 echidnas **ii** ≈ 400 echidnas

iii ≈ 5090 echidnas

c



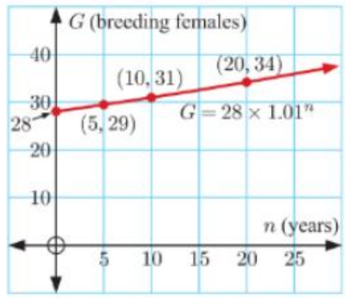
3 **a** $G_0 = 28$

b **i** ≈ 29

ii ≈ 31

iii ≈ 34

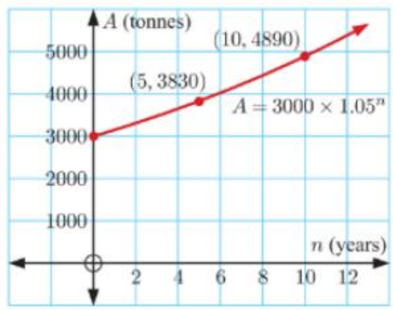
c



4 **a** The initial amount is 3000 tonnes, and the amount each year is 1.05 times the amount from the previous year.

b **i** ≈ 3830 tonnes **ii** ≈ 4890 tonnes

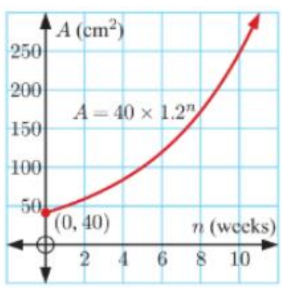
c



5 **a** No, for example the area increases by 8 cm^2 in the first week, and 9.6 cm^2 in the second week.

b The initial area is 40 cm^2, and the area each week is 1.2 times the amount from the previous week.

c



d 69.12 cm^2

EXERCISE 18C.2

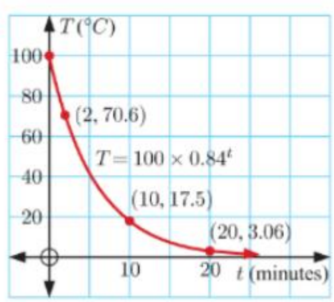
1 **a** 100°C

b **i** ≈ 70.6°C

ii ≈ 17.5°C

iii ≈ 3.06°C

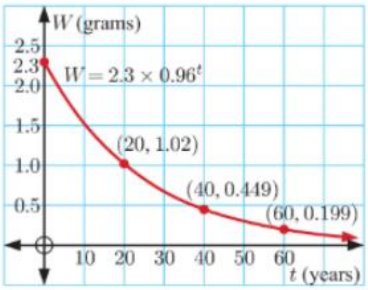
c



2 **a** 2.3 g

b **i** ≈ 1.02 g **ii** ≈ 0.449 g **iii** ≈ 0.199 g

c



d $\approx 55.8\%$ loss

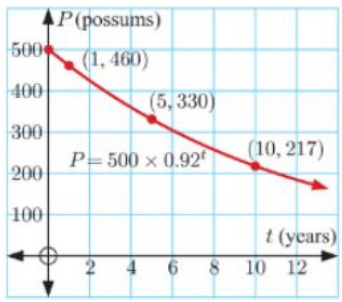
3 **a** $P = 500 \times 0.92^t$

b **i** 460

ii ≈ 330

iii ≈ 217

c



EXERCISE 18D

1 **a** $x = 1$ **b** $x = 2$ **c** $x = 3$ **d** $x = 0$
e $x = -1$ **f** $x = -1$ **g** $x = -4$ **h** $x = 0$

2 **a** $x = -4$ **b** $x = -2$ **c** $x = 6$ **d** $x = 2$
e $x = -\frac{3}{4}$ **f** $x = \frac{7}{2}$ **g** $x = 0$ **h** $x = 4$

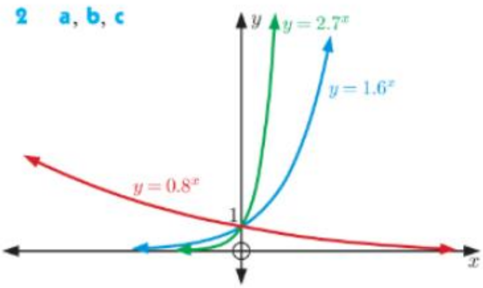
3 **a** $x = 3$ **b** $x = 0$ **c** $x = -2$ **d** $x = 3$
e $x = -5$ **f** $x = -2$ **g** $x = \frac{2}{7}$ **h** $x = \frac{1}{3}$
i $x = 1$

4 5 months **5** **a** $x = 1$ **b** $x = 2$ **c** $x = 1$

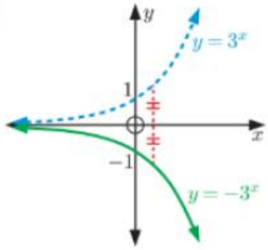
REVIEW SET 18A

1 **a** 0 **b** 26 **c** $-\frac{2}{3}$ **d** $3^{2x} - 1$

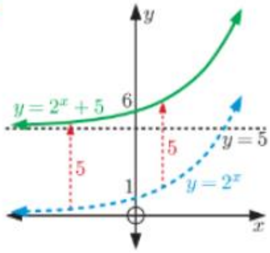
2 **a, b, c**



3 **a**



b

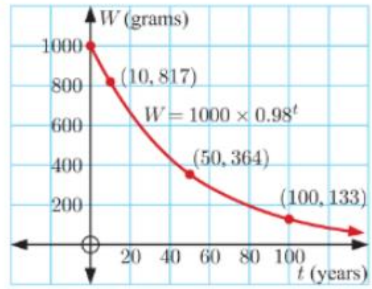


4 $y = -\frac{3}{2}$

5 **a** 1000 g

b **i** ≈ 817 g **ii** ≈ 364 g **iii** ≈ 133 g

c



6 **a** $x = \frac{1}{3}$ **b** $x = \frac{13}{7}$ **c** $x = -\frac{2}{5}$ **d** $x = -2$
e $x = \frac{5}{2}$ **f** $x = \frac{7}{11}$

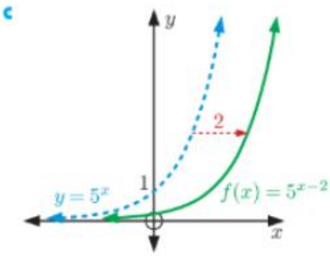
7 **a** 450 seals **b** 4 years

REVIEW SET 18B

1 **a** The variable x appears in the exponent.

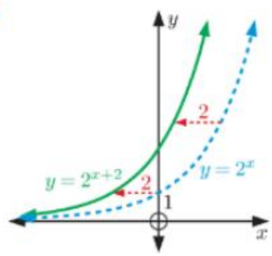
b Translate 2 units to the right.

c

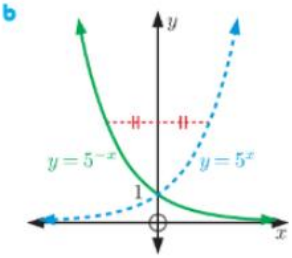


2 **a** 2 **b** $\frac{2}{9}$ **c** $2 \times 3^{-x-4}$

3 **a**



b



4 **a** $\{y \mid y < 0\}$ **b** $\{y \mid y > 0\}$ **c** $\{y \mid y > -2\}$
d $\{y \mid y > 3\}$

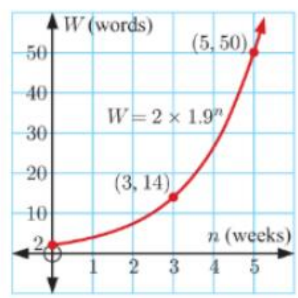
5 **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = \frac{2}{3}$

6 **a** **i** 2 words

ii ≈ 14 words

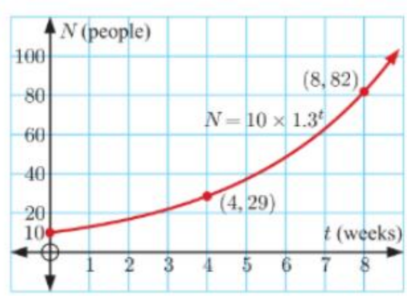
iii ≈ 50 words

b



7 **a** **i** 10 people **ii** ≈ 29 people **iii** ≈ 82 people

b



EXERCISE 19A

1 **a** $x = 37$ {angle in a semi-circle, angles in a triangle}
b $x = 30$ {angle in a semi-circle, angles in a triangle}
c $x = 18$ {angle in a semi-circle, angles in a triangle}

2 **a** $a = b = 55$ {tangents from an external point, base angles of isosceles triangle}
b $a = 3$ {chords of a circle}, $b \approx 36.9$
c $a = 60$ {angle in a semi-circle}
$b = 40$ {angles in a triangle, radius-tangent theorem}

3 **a** $x = 40$ {angles in a triangle}
b If $x = 40$, then $2x + 10 = 90$
$\therefore$ [AB] is a diameter {angle in a semi-circle}
c ≈ 7.83 cm

4 **Hint:** $AB + CD = (AP + PB) + (DR + CR)$
$= AS + QB + DS + CQ$, etc.

5 **a** 4 cm **b** $4\sqrt{3}$ cm ≈ 6.93 cm
c $(8 - 4\sqrt{3})$ cm ≈ 1.07 cm

6 **a** **Hint:** Consider Pythagoras' theorem in $\triangle$OXD.
b 10.25 cm

7 1 cm

EXERCISE 19B

1 **a** $x = 64$ {angle at the centre}
b $x = 94$ {angle at the centre}
c $x = 70$ {angle at the centre, angles in an isosceles triangle}
d $x = 45$ {angles in an isosceles triangle, angle at the centre}
e $x = 66$ {angle at the centre}
f $x = 25$ {angle at the centre, base angles of isosceles triangle}

2 **a** $x = 46$ {angles subtended by the same arc}
b $x = y = 55$ {angles subtended by the same arc}
c $a = 50$, $b = 40$ {angles subtended by the same arc}
d $a = 55$, $c = 70$ {angles subtended by the same arc}
e $a = 80$ {angles subtended by the same arc}
$b = 200$ {angle at the centre, angles at a point}
f $x = 75$ {angles subtended by the same arc}
$y = 118$ {exterior angle of a triangle}
g $x = 42$ {angles subtended by the same arc, exterior angle of a triangle}
h $x = 25$ {angle in a semi-circle, angles in a triangle, angles subtended by the same arc}
i $x = 25$ {angles in a semi-circle, base angles of isosceles triangle, angles subtended by the same arc}

3 **a** $x = 70$ {angle between a tangent and a chord, angles in an isosceles triangle}
b $x = 40$ {angle between a tangent and a chord, angles in an isosceles triangle}
c $x = 35$ {angle at the centre, angle between a tangent and a chord}

EXERCISE 19C

1 **a** **Hint:** Use RHS. **b** AM = BM {equating sides}

2 **a** [OA], [OP], [OB] are radii of the circle, $\therefore$ are equal.
b **i** $\widehat{APO} = a$ **ii** $\widehat{BPO} = b$ **iii** $\widehat{AOX} = 2a$
iv $\widehat{BOX} = 2b$ **v** $\widehat{APB} = (a + b)$
vi $\widehat{AOB} = (2a + 2b) = 2(a + b)$
c The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.

3 **a** $\widehat{AOB} = 2\alpha$ **b** $\widehat{ACB} = \alpha$ **c** $\widehat{ADB} = \widehat{ACB}$

4 **a** **i** $\widehat{TAX} = 90°$ **ii** $\widehat{ACX} = 90°$
b **i** $90° - \alpha$ **ii** α **iii** α **c** $\widehat{TAC} = \widehat{CBA}$

5 **Hint:** Use equal alternate angles, 'angle at the centre' theorem, and isosceles triangle.

6 **a** **i** $\widehat{YOX} = \alpha$ **ii** $\widehat{AXO} = 2\alpha$ **iii** $\widehat{XAO} = 2\alpha$
iv $\widehat{XOB} = 4\alpha$ **v** $\widehat{BOY} = 3\alpha$
b $\widehat{BOY} = 3 \times \widehat{YOX}$

7 **a** $\widehat{BXA} = \widehat{BXC} = 90°$ {angles in a semi-circle}
b A, X, and C are collinear. **c** yes
d yes (Repeat **a** to **c** using semi-circles with diameter [AD] and [CD].)

8 **Hint:** Use the 'angle between a tangent and a chord' theorem.

9 **Hint:** Use the 'angle in a semi-circle' theorem.

10 **Hint:** Use vertically opposite angles and the 'angles subtended by the same arc' theorem.

11 **Hint:** Use the 'angle in a semi-circle' theorem.

EXERCISE 19D

1 **a** **i** $\widehat{DOB} = 2a°$ **ii** reflex $\widehat{DOB} = 2b°$
b $2a + 2b = 360$ $\therefore$ $a + b = 180$

3 **a** $x = 107$ {opposite angles of a cyclic quadrilateral}
b $x = 60$ {opposite angles of a cyclic quadrilateral}
c $x = 70$ {co-interior angles, opposite angles of a cyclic quadrilateral}
d $x = 90$ {angles on a line, opposite angles of a cyclic quadrilateral}
e $x = 62$ {opposite angles of a cyclic quadrilateral, angle between a tangent and a chord, angles on a line}

4 **a** Yes, one pair of opposite angles are supplementary.
b Yes, [AD] subtends equal angles at B and C.
c no
d Yes, opposite angles are supplementary.
e Yes, one pair of opposite angles are supplementary.
f Yes, [AD] subtends equal angles at B and C.

5 **Hint:** Use co-interior angles and 'opposite angles of a cyclic quadrilateral'.

6 **Hint:** Construct [OD]. Use opposite angles of a parallelogram, equal corresponding angles, base angles of an isosceles triangle.

7 **Hint:** Use 'chords of a circle theorem'.

8 **Hint:** Use equal corresponding angles.

9 **Hint:** Use 'one side subtends equal angles at the other two vertices'.

10 **Hint:** Use congruent triangles.

11 **Hint:** Use angle at the centre, 'one side subtends equal angles at the other two vertices'.

12 **Hint:** Join [PX] and use 'angles subtended by the same arc'.

13 **Hint:** Use 'opposite angles of a cyclic quadrilateral'.

REVIEW SET 19A

1 **a** $a = 54$ {angle at the centre, angles in an isosceles triangle}

b $a = 62$ {angle in a semi-circle, angles in a triangle}

c $a = 61$ {angles subtended by the same arc, exterior angle of a triangle}

d $a = 80$ {angles on a line, opposite angles of a cyclic quadrilateral}

e $a = 63$ {equal co-interior angles, opposite angles of a cyclic quadrilateral}

f $a = 45$ {opposite angles of a cyclic quadrilateral}

2 $\alpha + \beta + \gamma = 180°$ {angle between a tangent and a chord}

3 **a** $x = 38$ {angle in a semi-circle, angles in a triangle}

b $x = 140$ {angle at the centre, exterior angle of a triangle}

c $x = 104$ {angle between a tangent and a chord, angles in an isosceles triangle}

4 **a** **Hint:** Use the 'tangents from an external point' theorem.

b **Hint:** Let $\widehat{ACM} = \alpha$ and $\widehat{BCM}$ be β. Use the isosceles triangle theorem to show that $\alpha + \beta = 90°$.

5 **Hint:** Construct [PB] and [XB]. Use 'angle between a tangent and a chord' and 'angles subtended by the same arc' to show that alternate angles are equal.

6 **a** **i** $\frac{1}{2}\beta$ **ii** $\frac{1}{2}\alpha$

b **Hint:** Use the 'angles in a triangle' theorem.

7 **a** The angle between a tangent and a chord through the point of contact is equal to the angle subtended by the chord in the alternate segment.

b **i** $\widehat{PQB} = \alpha$, $\widehat{PQA} = \beta$, $\widehat{AQB} = (\alpha + \beta)$

ii **Hint:** Use the angles in a triangle theorem.

8 **Hint:** Use 'angle between a tangent and a chord', and equal alternate angles.

REVIEW SET 19B

1 **a** $x = 86$ {angles subtended by the same arc, exterior angle of a triangle}

b $x = \sqrt{34} \approx 5.83$ {chords of a circle, Pythagoras}

c $x = 9$ {radius-tangent, Pythagoras}

2 **a** $x = 55$ {angles at a point, angle at the centre}

b $x = 55$ {angles in an isosceles triangle, radius-tangent, angles in a triangle}

c $x \approx 7.68$ {tangents from an external point, angles in a triangle, radius-tangent}

3 ≈ 6.63 m

4 **Hint:** Use the isosceles triangle theorem and the 'angles subtended by the same arc' theorem.

5 **Hint:** Use the 'angles subtended by the same arc' theorem and the 'exterior angle of a triangle' theorem.

6 **a** **i** $\widehat{PSR} = 2\alpha$ **ii** $\widehat{PQR} = 2\alpha$ **iii** $\widehat{PRQ} = 2\alpha$

iv $\widehat{QPR} = 2\alpha$

b **Hint:** Use the angles in a triangle theorem.

c $\widehat{QRS} = 90°$ **d** [RS] is a diameter of the circle.

7 **Hint:** Use angle in a semi-circle, and 'one side subtends equal angles at the other two vertices'.

8 **a** **Hint:** Use 'angle between a tangent and a chord', and angles in a triangle.

b **Hint:** Use the 'angles subtended by the same arc' theorem.

c **Hint:** ABCT is a cyclic quadrilateral.

EXERCISE 20A

1 **b** and **c** are quadratic functions.

2 **a** $y = 19$ **b** $y = 82$ **c** $y = 34$

3 **a** $y = 0$ **b** $y = 5$ **c** $y = -15$ **d** $y = 12$

4 **a** no **b** yes **c** yes **d** no **e** no **f** no

5 **a** $x = -1$ or 3 **b** $x = 1$ **c** $x = -2$ or 4

6 **a** $x = -3$ **b** $x = -2$ or -3 **c** $x = 1$ or 4

d no real solutions

7 **a** $x = 0$ or 1 **b** $x = -1$ or 3 **c** $x = -7$ or $\frac{1}{2}$

d $x = 2$ or 3

8 **a** **i** 75 m **ii** 195 m **iii** 275 m

b **i** At $t = 2$ s and $t = 14$ s.

ii At $t = 0$ s and $t = 16$ s.

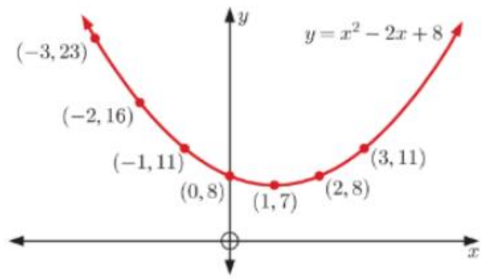
c The object leaves the ground at $t = 0$ s. At $t = 2$ it is rising and at $t = 14$ it is falling. Height 0 m is ground level and the time of flight is 16 seconds.

9 **a** **i** $-\$40$, a loss of $40 **ii** $480 profit

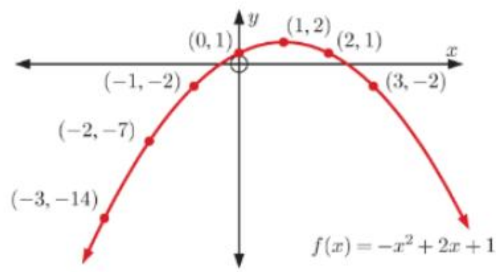
b 10 cakes or 62 cakes

EXERCISE 20B.1

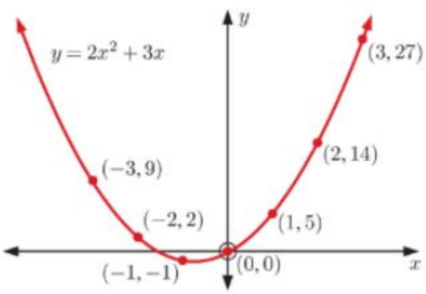
1 **a**



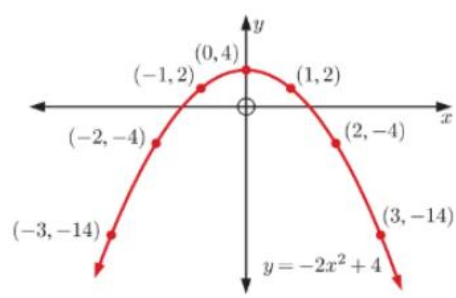
b



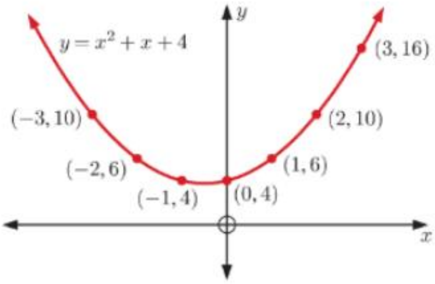
c



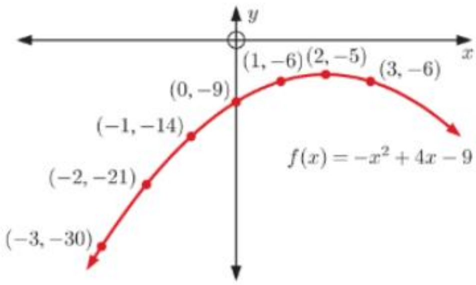
d



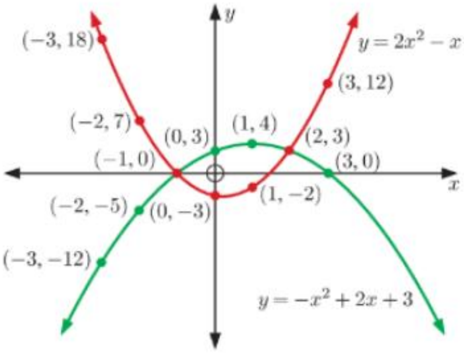
e



f



2 **a**

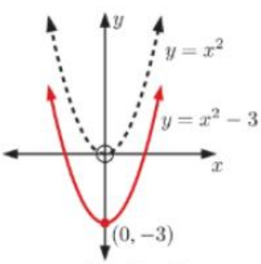


$x = -1$ or 2 (where they meet)

b $x = -1$ or 2

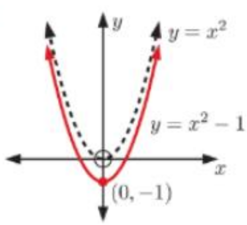
EXERCISE 20B.2

1 **a**



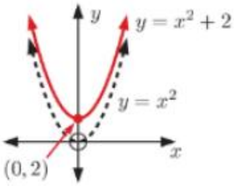
vertex is at $(0, -3)$

b



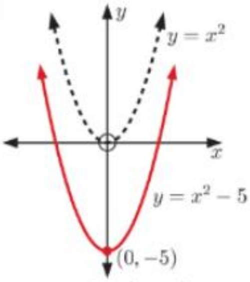
vertex is at $(0, -1)$

c



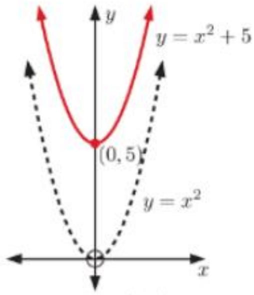
vertex is at $(0, 2)$

d



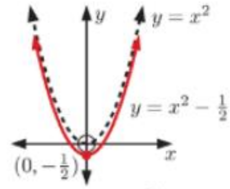
vertex is at $(0, -5)$

e



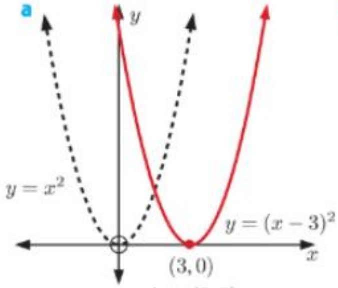
vertex is at $(0, 5)$

f



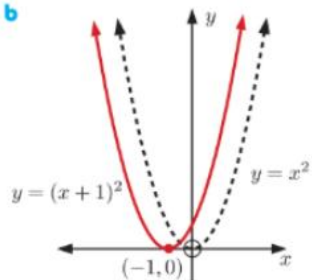
vertex is at $(0, -\frac{1}{2})$

2 **a**



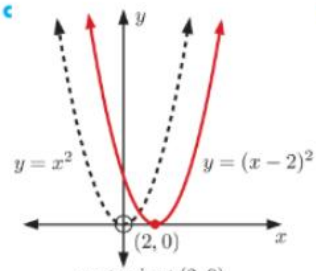
vertex is at $(3, 0)$

b



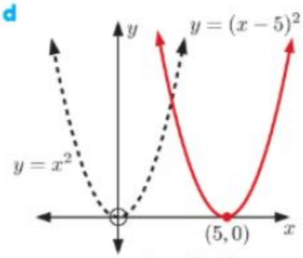
vertex is at $(-1, 0)$

c



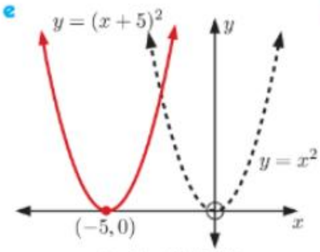
vertex is at $(2, 0)$

d



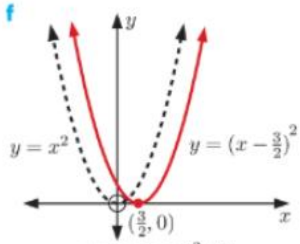
vertex is at $(5, 0)$

e



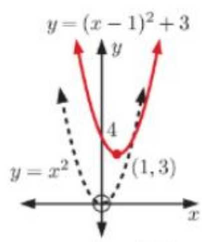
vertex is at $(-5, 0)$

f



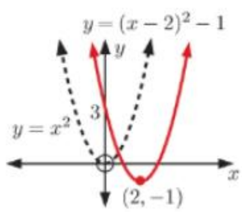
vertex is at $(\frac{3}{2}, 0)$

3 **a**



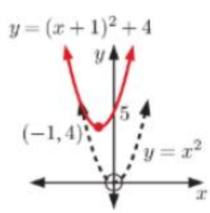
vertex is at $(1, 3)$

b



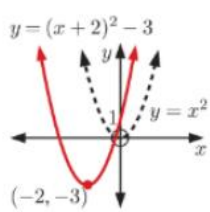
vertex is at $(2, -1)$

c



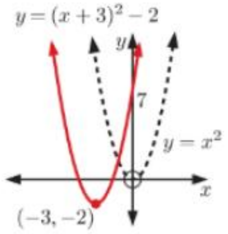
vertex is at $(-1, 4)$

d



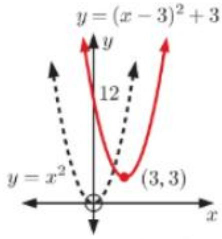
vertex is at $(-2, -3)$

e



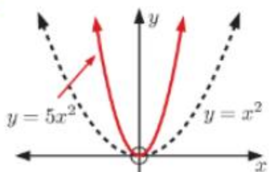
vertex is at $(-3,-2)$

f



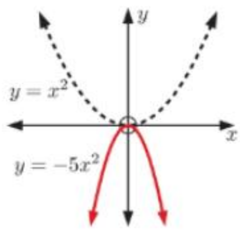
vertex is at $(3,3)$

4 **a**



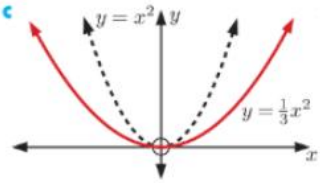
$y=5x^2$ is 'thinner' than $y=x^2$ and the graph opens upwards.

b



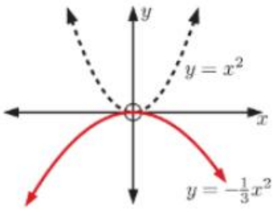
$y=-5x^2$ is 'thinner' than $y=x^2$ and the graph opens downwards.

c



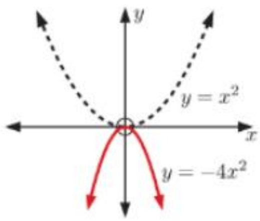
$y=\frac{1}{3}x^2$ is 'wider' than $y=x^2$ and the graph opens upwards.

d



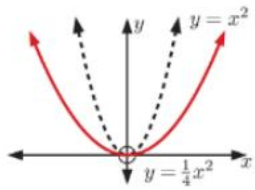
$y=-\frac{1}{3}x^2$ is 'wider' than $y=x^2$ and the graph opens downwards.

e



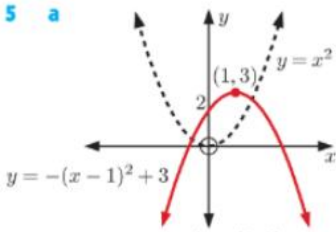
$y=-4x^2$ is 'thinner' than $y=x^2$ and the graph opens downwards.

f



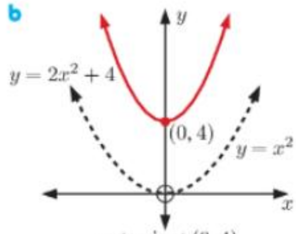
$y=\frac{1}{4}x^2$ is 'wider' than $y=x^2$ and the graph opens upwards.

5 **a**



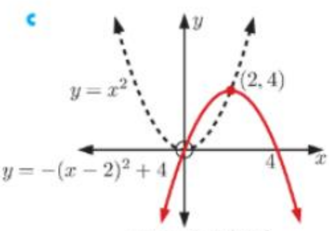
vertex is at $(1,3)$

b



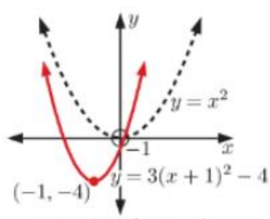
vertex is at $(0,4)$

c



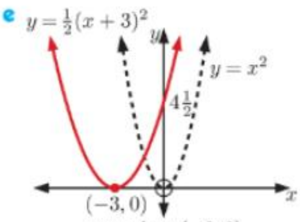
vertex is at $(2,4)$

d



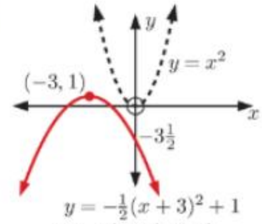
vertex is at $(-1,-4)$

e



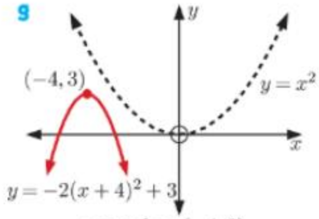
vertex is at $(-3,0)$

f



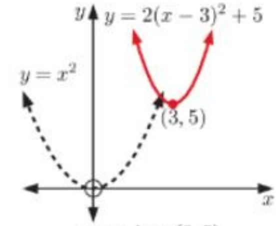
vertex is at $(-3,1)$

g



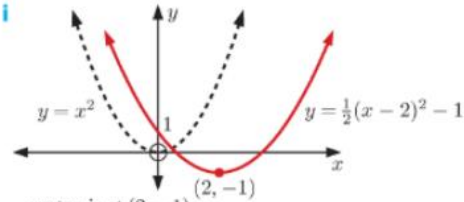
vertex is at $(-4,3)$

h



vertex is at $(3,5)$

i

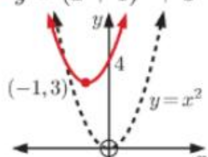


vertex is at $(2,-1)$

6 **a** F **b** C **c** B **d** D **e** E **f** A

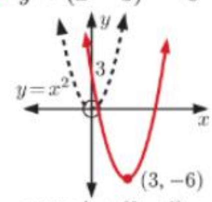
EXERCISE 20B.3

1 **a** $y=(x+1)^2+3$



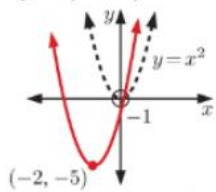
vertex is at $(-1,3)$

b $y=(x-3)^2-6$



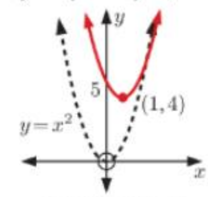
vertex is at $(3,-6)$

c $y=(x+2)^2-5$



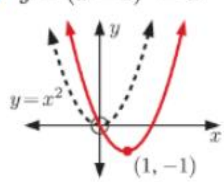
vertex is at $(-2,-5)$

d $y=(x-1)^2+4$



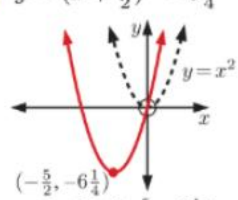
vertex is at $(1,4)$

e $y=(x-1)^2-1$



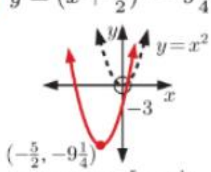
vertex is at $(1,-1)$

f $y=(x+\frac{5}{2})^2-6\frac{1}{4}$



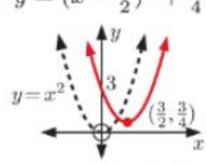
vertex is at $(-\frac{5}{2},-6\frac{1}{4})$

g $y=(x+\frac{5}{2})^2-9\frac{1}{4}$



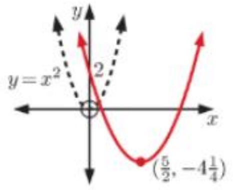
vertex is at $(-\frac{5}{2},-9\frac{1}{4})$

h $y=(x-\frac{3}{2})^2+\frac{3}{4}$



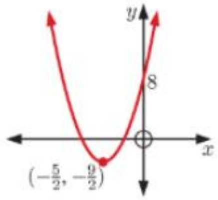
vertex is at $(\frac{3}{2},\frac{3}{4})$

i $y = (x - \frac{5}{2})^2 - \frac{17}{4}$



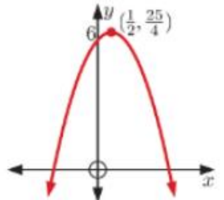
vertex is at $(\frac{5}{2}, -4\frac{1}{4})$

2 **a** $y = 2(x + \frac{5}{2})^2 - \frac{9}{2}$



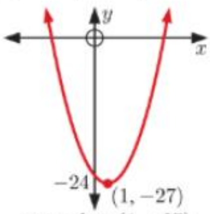
vertex is at $(-\frac{5}{2}, -\frac{9}{2})$

b $y = -(x - \frac{1}{2})^2 + \frac{25}{4}$



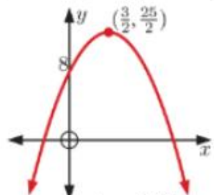
vertex is at $(\frac{1}{2}, \frac{25}{4})$

c $y = 3(x - 1)^2 - 27$



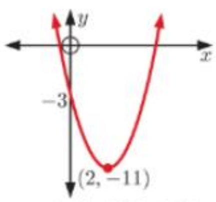
vertex is at $(1, -27)$

d $y = -2(x - \frac{3}{2})^2 + \frac{25}{2}$



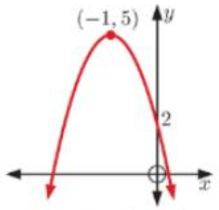
vertex is at $(\frac{3}{2}, \frac{25}{2})$

e $y = 2(x - 2)^2 - 11$



vertex is at $(2, -11)$

f $y = -3(x + 1)^2 + 5$



vertex is at $(-1, 5)$

EXERCISE 20C

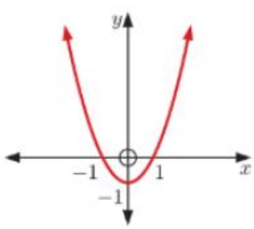
1 **a** 3 **b** 2 **c** -8 **d** 1 **e** 6 **f** 5
g 6 **h** 8 **i** -2

2 **a** 3 and -1 **b** 2 and 4 **c** -3 and -2
d 4 and 5 **e** -3 (touching) **f** 1 (touching)

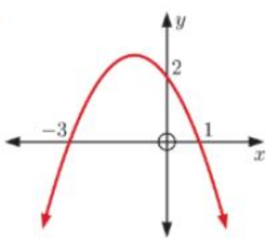
3 **a** -3 and 3 **b** -5 and 5 **c** 0 and 6
d -5 and -2 **e** -4 and 3 **f** 0 and 4
g -2 and -4 **h** -1 (touching) **i** 3 (touching)

4 **a** $2 + \sqrt{3}$ and $2 - \sqrt{3}$ **b** $-2 + \sqrt{7}$ and $-2 - \sqrt{7}$
c $3 + \sqrt{5}$ and $3 - \sqrt{5}$ **d** $\frac{7 + \sqrt{73}}{6}$ and $\frac{7 - \sqrt{73}}{6}$
e $\frac{1 + \sqrt{41}}{4}$ and $\frac{1 - \sqrt{41}}{4}$ **f** $\frac{9 + \sqrt{33}}{8}$ and $\frac{9 - \sqrt{33}}{8}$

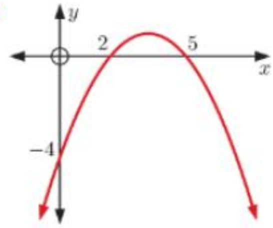
5 **a**



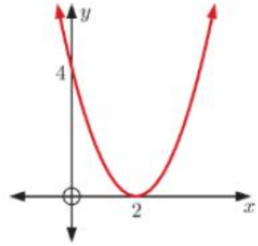
b



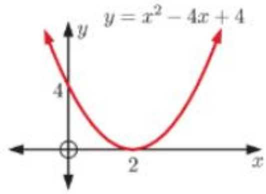
c



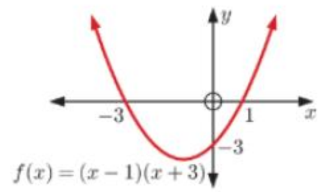
d



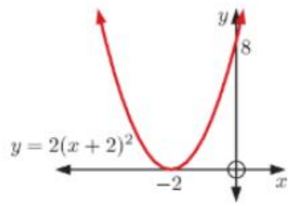
6 **a** **i** $a = 1$ **ii** 4
iii 2 (touching)



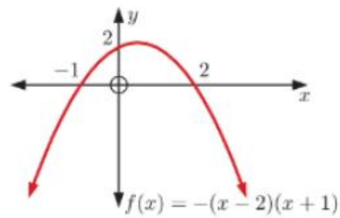
b **i** $a = 1$ **ii** -3
iii 1 and -3



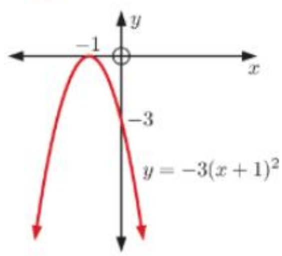
c **i** $a = 2$ **ii** 8
iii -2 (touching)



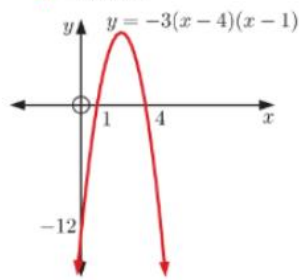
d **i** $a = -1$ **ii** 2
iii 2 and -1



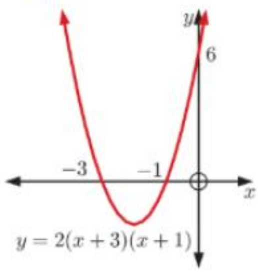
e **i** $a = -3$ **ii** -3
iii -1 (touching)



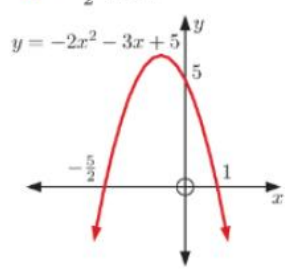
f **i** $a = -3$ **ii** -12
iii 4 and 1



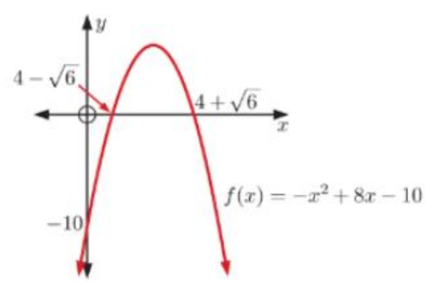
g **i** $a = 2$ **ii** 6
iii -3 and -1



h **i** $a = -2$ **ii** 5
iii $-\frac{5}{2}$ and 1



i **i** $a = -1$
ii -10
iii $4 + \sqrt{6}$
and
$4 - \sqrt{6}$



EXERCISE 20D

1 **a** $x = 2$ **b** $x = 1$ **c** $x = -2$ **d** $x = \frac{3}{2}$
e $x = -\frac{5}{2}$ **f** $x = 2$

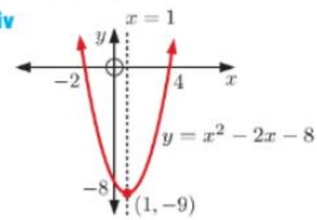
2 **a** $x = 3$ **b** $x = 2$ **c** $x = 0$ **d** $x = -\frac{5}{2}$
e $x = -4$ **f** $x = \frac{3}{2}$

3 **a** $x = -2$ **b** $x = \frac{3}{2}$ **c** $x = -\frac{2}{3}$ **d** $x = -2$
e $x = \frac{5}{4}$ **f** $x = 10$ **g** $x = -6$ **h** $x = 12\frac{1}{2}$
i $x = 150$

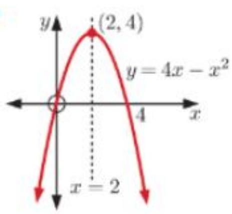
EXERCISE 20E

1 **a** **i** $V(2, -2)$ **ii** minimum **iii** $\{y \mid y \geqslant -2\}$
b **i** $V(-1, -4)$ **ii** minimum **iii** $\{y \mid y \geqslant -4\}$
c **i** $V(0, 4)$ **ii** minimum **iii** $\{y \mid y \geqslant 4\}$
d **i** $V(0, 1)$ **ii** maximum **iii** $\{y \mid y \leqslant 1\}$
e **i** $V(-2, 0)$ **ii** maximum **iii** $\{y \mid y \leqslant 0\}$
f **i** $V(\frac{5}{2}, -\frac{19}{2})$ **ii** minimum **iii** $\{y \mid y \geqslant -9\frac{1}{2}\}$

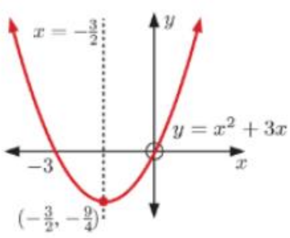
2 **a** **i** x-intercepts -2 and 4, y-intercept -8
ii $x = 1$
iii $V(1, -9)$
iv



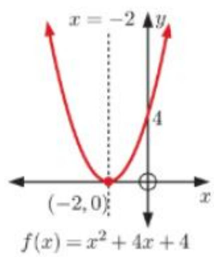
b **i** x-intercepts 0 and 4, y-intercept 0
ii $x = 2$
iii $V(2, 4)$
iv



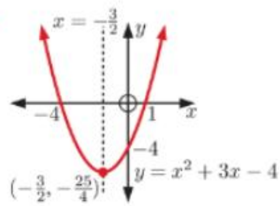
c **i** x-intercepts 0 and -3, y-intercept 0
ii $x = -\frac{3}{2}$
iii $V(-\frac{3}{2}, -\frac{9}{4})$
iv



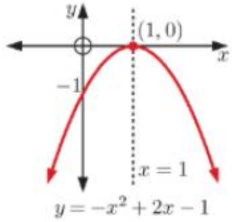
d **i** x-intercept -2, y-intercept 4
ii $x = -2$
iii $V(-2, 0)$
iv



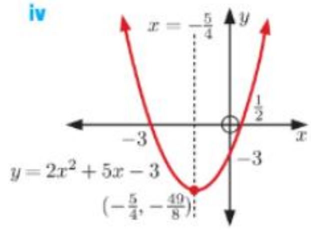
e **i** x-intercepts -4 and 1, y-intercept -4
ii $x = -\frac{3}{2}$
iii $V(-\frac{3}{2}, -6\frac{1}{4})$
iv



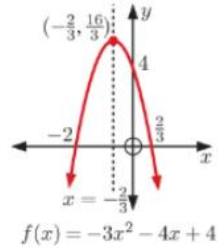
f **i** x-intercept 1, y-intercept -1
ii $x = 1$
iii $V(1, 0)$
iv



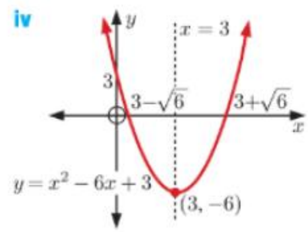
g **i** x-intercepts -3 and $\frac{1}{2}$, y-intercept -3
ii $x = -\frac{5}{4}$
iii $V(-\frac{5}{4}, -\frac{49}{8})$
iv



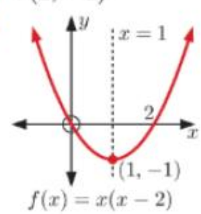
h **i** x-intercepts -2 and $\frac{2}{3}$, y-intercept 4
ii $x = -\frac{2}{3}$
iii $V(-\frac{2}{3}, \frac{16}{3})$
iv



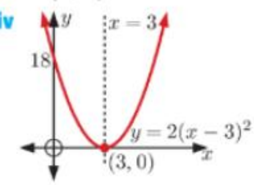
i **i** x-intercepts $3 + \sqrt{6}$ and $3 - \sqrt{6}$, y-intercept 3
ii $x = 3$
iii $V(3, -6)$
iv



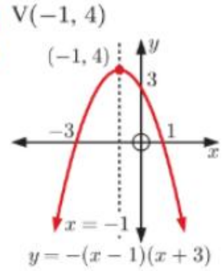
3 **a** **i** x-intercepts 0 and 2, y-intercept 0
ii $x = 1$
iii $V(1, -1)$
iv



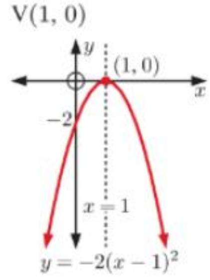
b **i** x-intercept 3, y-intercept 18
ii $x = 3$
iii $V(3, 0)$
iv



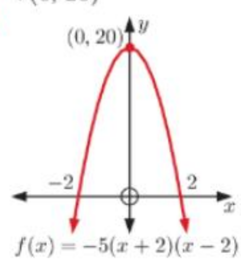
c **i** x-intercepts -3 and 1, y-intercept 3
ii $x = -1$
iii $V(-1, 4)$
iv



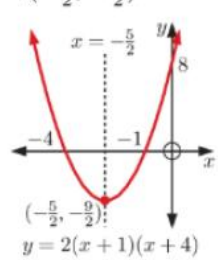
d **i** x-intercept 1, y-intercept -2
ii $x = 1$
iii $V(1, 0)$
iv



e **i** x-intercepts -2 and 2, y-intercept 20
ii $x = 0$
iii $V(0, 20)$
iv



f **i** x-intercepts -1 and -4, y-intercept 8
ii $x = -\frac{5}{2}$
iii $V(-\frac{5}{2}, -\frac{9}{2})$
iv



EXERCISE 20F

1 **a** 2 seconds **b** 20 m **c** 4 seconds

2 **a** 25 bicycles **b** €425
c €200 (Due to fixed daily costs such as wages and electricity.)

3 **a** 60 km h^{-1} (when $t = 0$) **b** $t = 1$ s **c** 66 km h^{-1}

4 **a** 30 taxis **b** \$1600/hour **c** \$200

5 **a** 25°C **b** 7:00 am the next day **c** −11°C

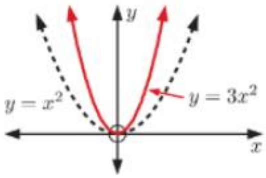
6 **a** 2 m **b** 3.8 m **c** 1 s

7 **a** **Hint:** [FB] ∥ [EC] **b** $\frac{\text{BF}}{2} = \frac{1-x}{1}$ etc.
c Area $= x(2(1-x))$ etc. **d** **i** $x = \frac{1}{2}$ **ii** $\frac{1}{2}$ cm^2

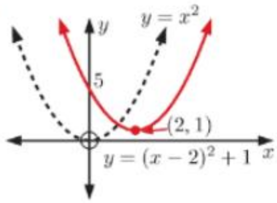
REVIEW SET 20A

1 **a** $y = -11$ **b** $x = 6$ or -3

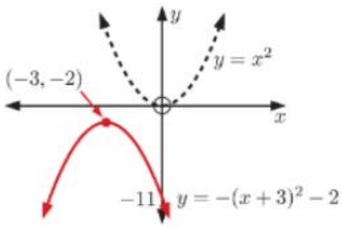
2 **a**



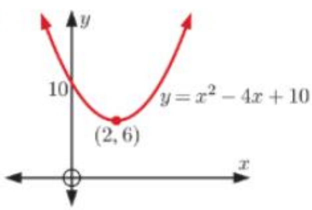
b



c



3 **a** $y = (x-2)^2 + 6$ **b**



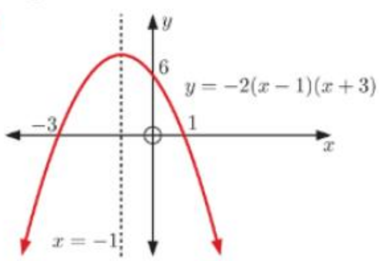
vertex is at (2, 6)

4 **a** 0 and −4 **b** −7 and 4 **5** **a** $x = -\frac{3}{2}$ **b** $x = 2$

6 **a** V(4, −19) **b** V($\frac{1}{2}$, −2)

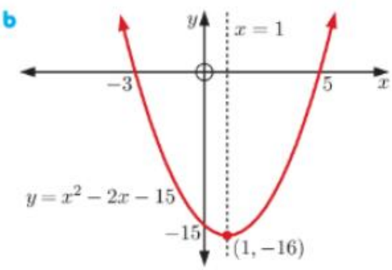
7 **a** **i** downwards
ii 6
iii −3 and 1
iv $x = -1$

b



8 **a** **i** −15
ii −3 and 5
iii $x = 1$
iv V(1, −16)

b

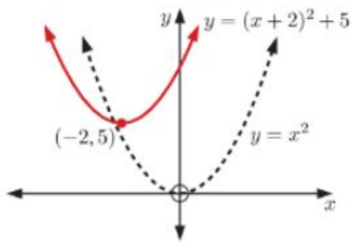


9 **a** unmarked side $= (40 - 2x)$ m
$\therefore\ A = x(40-2x)$ m^2, etc.
b $x = 10$ **c** 200 m^2

REVIEW SET 20B

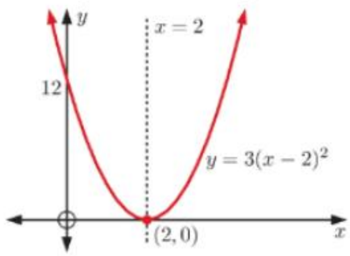
1 $x = -7$ or 6 **2** No, as $f(2) = 4 - 6 + 8 = 6 \neq 5$.

3



4 **a** $\frac{1+\sqrt{61}}{6}$ and $\frac{1-\sqrt{61}}{6}$ **b** $1 + \sqrt{7}$ and $1 - \sqrt{7}$

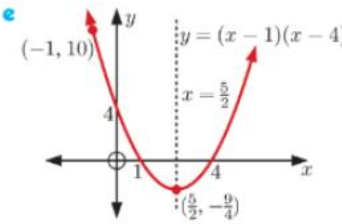
5



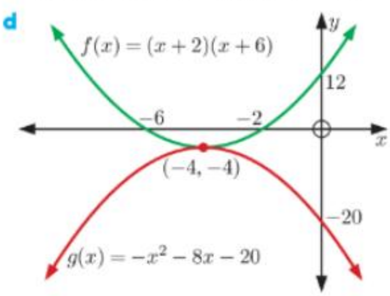
6 **a** $x = 1$
b $x = \frac{5}{6}$

7 **a** V(2, −3)
b $\{y \mid y \leqslant -3\}$

8 **a** $f(-1) = 10$ **b** x-intercepts 1 and 4, y-intercept 4
c $x = \frac{5}{2}$ **d** V($\frac{5}{2}$, $-\frac{9}{4}$)
e



9 **a** $f(x)$ has x-intercepts −6 and −2 and its y-intercept is 12.
$g(x)$ does not cut the x-axis and its y-intercept is −20.
b Both $f(x)$ and $g(x)$ have axis of symmetry $x = -4$.
Both $f(x)$ and $g(x)$ have vertex (−4, −4).
c Range of $f(x)$ is $\{y \mid y \geqslant -4\}$.
Range of $g(x)$ is $\{y \mid y \leqslant -4\}$.
d

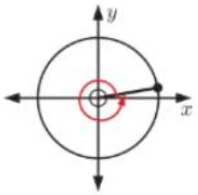


10 **a** 1.6 m **b** ≈ 22.6 m **c** ≈ 4.22 seconds

EXERCISE 21A

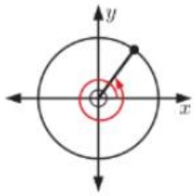
1 **a** A(cos 67°, sin 67°), B(cos 148°, sin 148°),
C(cos 281°, sin 281°), D(cos(−24°), sin(−24°))
b A(0.391, 0.921), B(−0.848, 0.530),
C(0.191, −0.982), D(0.914, −0.407)

2 **a**



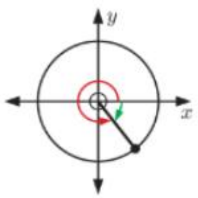
same point on unit circle
$\therefore$ same x-coordinate
$\therefore$ $\cos 380° = \cos 20°$

b



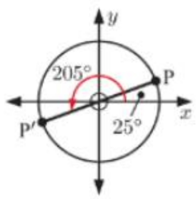
same point on unit circle
$\therefore$ same y-coordinate
$\therefore$ $\sin 413° = \sin 53°$

c



same point on unit circle
$\therefore$ same x-coordinate
$\therefore$ $\cos(-52°) = \cos 308°$

d



[OP] and [OP′] have the same gradient
$\therefore$ $\tan 25° = \tan 205°$

3 **a** $\cos 0° = 1,\ \sin 0° = 0$ **b** $\cos 90° = 0,\ \sin 90° = 1$
c $\cos 360° = 1,\ \sin 360° = 0$
d $\cos 450° = 0,\ \sin 450° = 1$
e $\cos(-90°) = 0,\ \sin(-90°) = -1$
f $\cos(-180°) = -1,\ \sin(-180°) = 0$

4 **a** **i** ≈ -0.454 **ii** ≈ -0.891 **iii** ≈ 0.809
iv ≈ -0.588
b **i** ≈ 1.96 **ii** ≈ -0.727

5 **a** $\cos 135°$ is negative, $\sin 135°$ is positive, $\tan 135°$ is negative
b $\cos 30°$ is positive, $\sin 30°$ is positive, $\tan 30°$ is positive
c $\cos 300°$ is positive, $\sin 300°$ is negative, $\tan 300°$ is negative
d $\cos 210°$ is negative, $\sin 210°$ is negative, $\tan 210°$ is positive
e $\cos 49°$ is positive, $\sin 49°$ is positive, $\tan 49°$ is positive
f $\cos 158°$ is negative, $\sin 158°$ is positive, $\tan 158°$ is negative
g $\cos 207°$ is negative, $\sin 207°$ is negative, $\tan 207°$ is positive
h $\cos 296°$ is positive, $\sin 296°$ is negative, $\tan 296°$ is negative

6 **a**

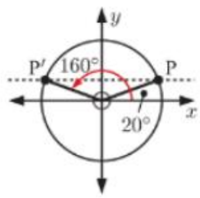
Quadrant	*Degree*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	+ ve	+ ve	+ ve
2	$90° < \theta < 180°$	− ve	+ ve	− ve
3	$180° < \theta < 270°$	− ve	− ve	+ ve
4	$270° < \theta < 360°$	+ ve	− ve	− ve

b **i** 3 and 4 **ii** 1 and 4 **iii** 3 **iv** 4

EXERCISE 21B

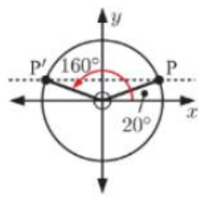
1 **a** $-42°$ **b** $-61°$ **c** $18°$ **d** $-117°$
2 **a** $-24°$ **b** $-70°$ **c** $5°$ **d** $-157°$
3 **a** $50°$ **b** $32°$ **c** $74°$ **d** $5°$
4 **a** $40°$ **b** $55°$ **c** $11°$ **d** $69°$
5 **a** $154°$ **b** $135°$ **c** $111°$ **d** $94°$
6 **a** $82°$ **b** $53°$ **c** $24°$ **d** $12°$
7 **a** $154°$ **b** $135°$ **c** $111°$ **d** $94°$
8 **a** $82°$ **b** $53°$ **c** $24°$ **d** $12°$

9 **a**



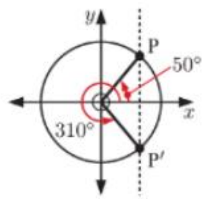
P′ is the reflection of P in the y-axis
$\therefore$ same y-coordinate
$\therefore$ $\sin 20° = \sin 160°$

b



P′ is the reflection of P in the y-axis
$\therefore$ x-coordinates have same magnitude, but opposite sign
$\therefore$ $\cos 160° = -\cos 20°$

c



P′ is the reflection of P in the x-axis
$\therefore$ same x-coordinate
$\therefore$ $\cos 310° = \cos 50°$

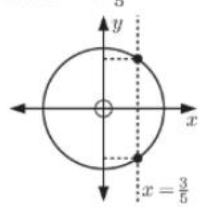
10 **a** Q is $(\cos(-\theta), \sin(-\theta))$ **b** **i** $\cos\theta$ **ii** $-\sin\theta$
c $\cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta$
d $\tan(-\theta) = -\tan\theta$

11 **a** Q is $(\cos(90° - \theta), \sin(90° - \theta))$
b $\cos(90° - \theta) = \sin\theta,\ \sin(90° - \theta) = \cos\theta$
c $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$

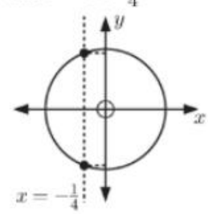
12 **a** Q is $(\cos(180° - \theta), \sin(180° - \theta))$
b **i** $-\cos\theta$ **ii** $\sin\theta$
c $\cos(180° - \theta) = -\cos\theta,\ \sin(180° - \theta) = \sin\theta$
d $\tan(180° - \theta) = -\tan\theta$

EXERCISE 21C

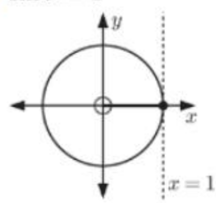
1 **a** $\sin\theta = \pm\frac{4}{5}$



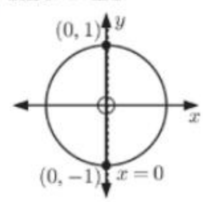
b $\sin\theta = \pm\frac{\sqrt{15}}{4}$



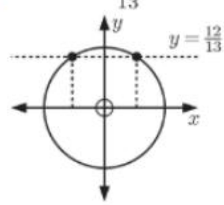
c $\sin\theta = 0$



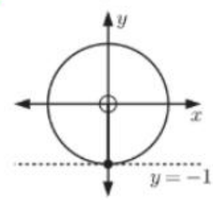
d $\sin\theta = \pm 1$



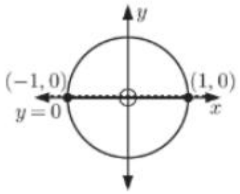
2 **a** $\cos\theta = \pm\frac{5}{13}$

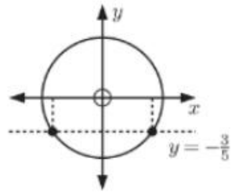


b $\cos\theta = 0$



c $\cos\theta = \pm 1$ **d** $\cos\theta = \pm\frac{4}{5}$





3 **a** $\cos\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{3}{5}$ **c** $\cos\theta = -\frac{2\sqrt{2}}{3}$
d $\cos\theta = \frac{12}{13}$

4 **a** $\sin\theta = \frac{4}{5}$ **b** $\sin\theta = -\frac{\sqrt{15}}{4}$ **c** $\sin\theta = \frac{\sqrt{7}}{4}$
d $\sin\theta = -\frac{12}{13}$

5 **a** $\cos\theta = -\frac{\sqrt{55}}{8}$ **b** $\tan\theta = \frac{3}{\sqrt{55}}$

EXERCISE 21D

1 **a** $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$ **c** $0, 1, 0$
d $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$ **e** $1, 0$, undefined **f** $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$
g $-1, 0$, undefined **h** $0, -1, 0$
i $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ **j** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}$ **k** $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$
l $0, 1, 0$ **m** $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$ **n** $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1$
o $1, 0$, undefined **p** $0, -1, 0$

2 **a** $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{3}$ **d** $\frac{3\sqrt{3}}{8}$ **e** $\frac{1}{4}$ **f** 3

3 **a** $150°$ **b** $225°$ **c** $30°$ **d** $315°$

EXERCISE 21E.1

1 **a** 0
b **i** $\theta = 0°, 180°, 360°, 540°, 720°$
ii $\theta = 270°, 630°$ **iii** $\theta = 30°, 150°, 390°, 510°$
iv $\theta = 60°, 120°, 420°, 480°$
c $\theta \approx 15°, 165°, 375°, 525°$
d **i** $\sin\theta$ is positive for $0° < \theta < 180°,\ 360° < \theta < 540°$
ii $\sin\theta$ is negative for $180° < \theta < 360°$, $540° < \theta < 720°$
e $\{y \mid -1 \leqslant y \leqslant 1\}$

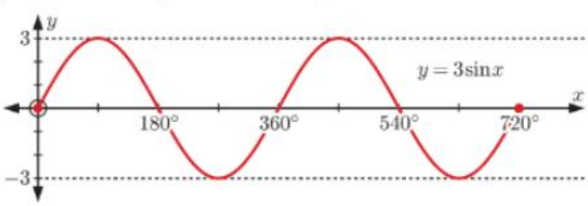
2 **a** 1
b **i** $\theta = 90°, 270°, 450°, 630°$
ii $\theta = 0°, 360°, 720°$
iii $\theta = 120°, 240°, 480°, 600°$
iv $\theta = 135°, 225°, 495°, 585°$
c $\theta \approx 75°, 285°, 435°, 645°$
d **i** $\cos\theta$ is positive for $0° \leqslant \theta < 90°,\ 270° < \theta < 450°$, $630° < \theta \leqslant 720°$
ii $\cos\theta$ is negative for $90° < \theta < 270°$, $450° < \theta < 630°$
e $\{y \mid -1 \leqslant y \leqslant 1\}$

EXERCISE 21E.2

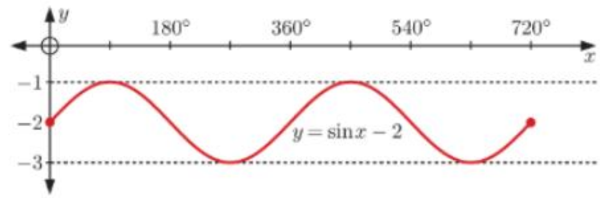
1 **a** 4 **b** 2 **c** 3 **2** **a** $120°$ **b** $90°$ **c** $720°$

3 **a** $y = -3$ **b** $y = 5$ **c** $y = 0$

4 **a**



b



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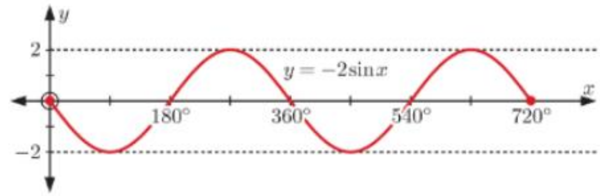
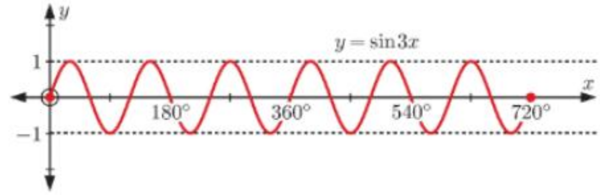
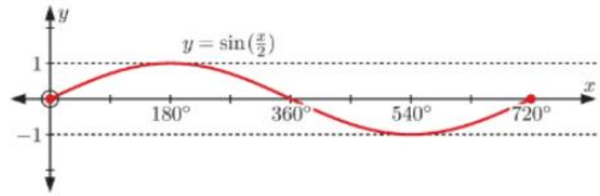


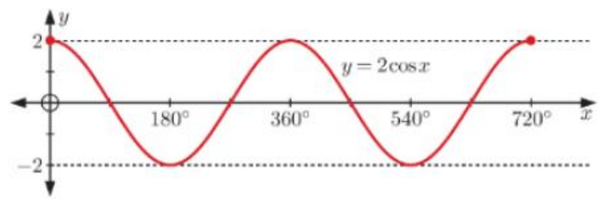
d



e



5 **a**



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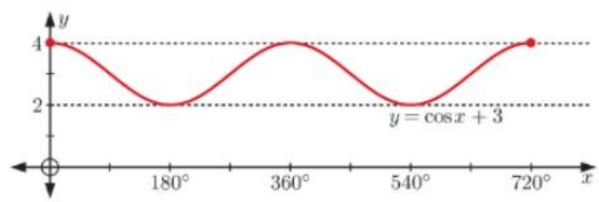

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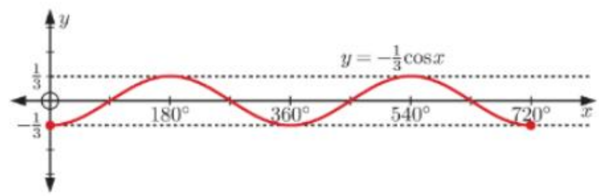

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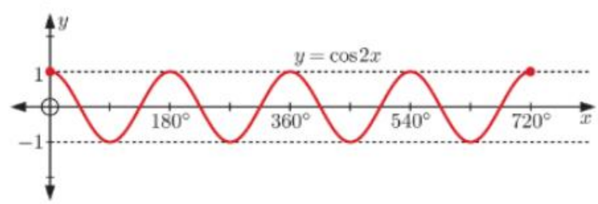
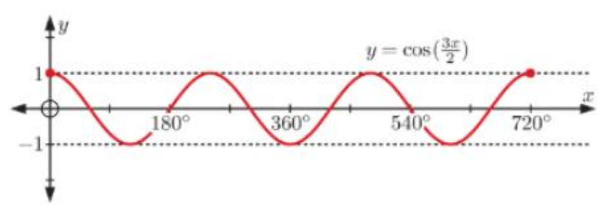
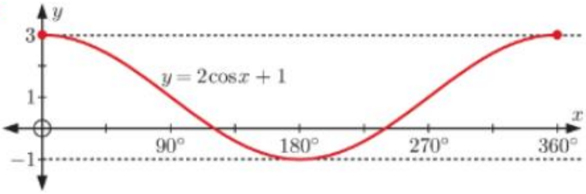


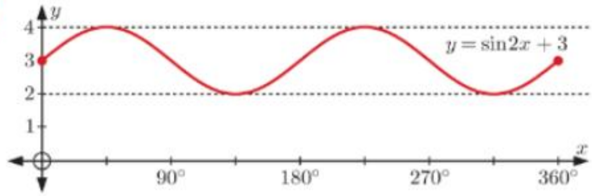
e



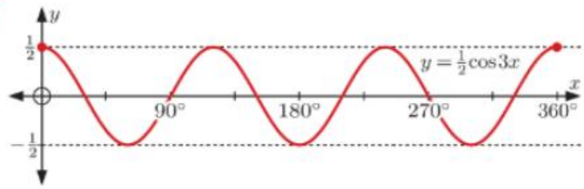
6 **a**



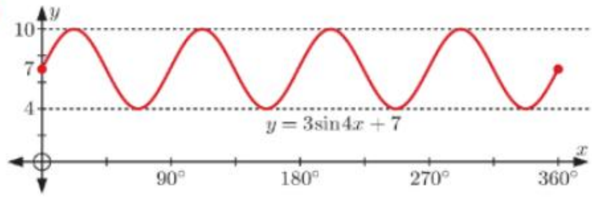
b



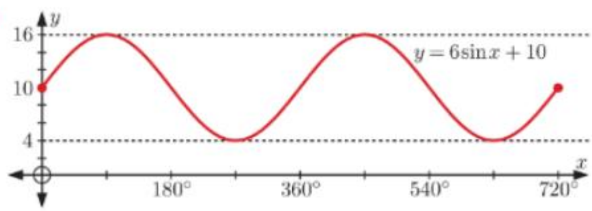
c



d



7 **a**



b $y = 13$

c maximum value $= 16$, when $x = 90°$ and $450°$

d minimum value $= 4$, when $x = 270°$ and $630°$

8 **a** $y = -\sin x$ with domain $0° \leqslant x \leqslant 720°$

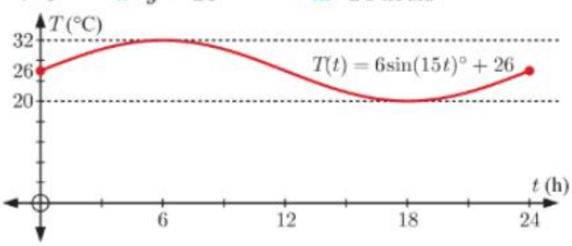
b $y = 3\cos 3x + 3$ with domain $0° \leqslant x \leqslant 360°$

c $y = 4\sin x + 3$ with domain $0° \leqslant x \leqslant 720°$

d $y = -4\cos(\frac{3}{2}x) + 1$ with domain $0° \leqslant x \leqslant 360°$

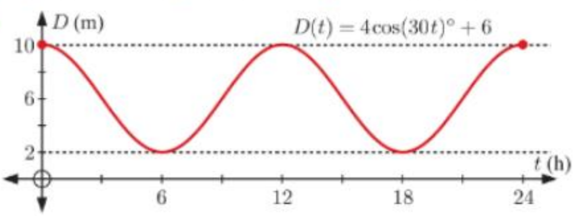
9 **a** **i** 6 **ii** $y = 26$ **iii** 24 hours

b



c **i** 26°C **ii** 29°C **d** 32°C, at 6 pm

10 **a**



b highest $= 10$ m, at midnight, midday, and midnight the next day

lowest $= 2$ m, at 6 am and 6 pm

c no (water height is 4 m)

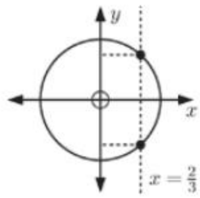
REVIEW SET 21A

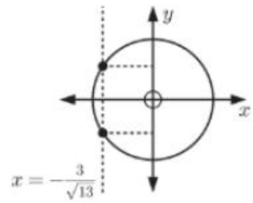
1 **a** **i** $\cos 167° \approx -0.974$ **ii** $\sin 167° \approx 0.225$

b $\tan 167° \approx -0.231$

2 **a** $-23°$ **b** $-62°$ **3** **a** $16°$ **b** $83°$

4 **a** $\sin\theta = \pm\frac{\sqrt{5}}{3}$ **b** $\sin\theta = \pm\frac{2}{\sqrt{13}}$



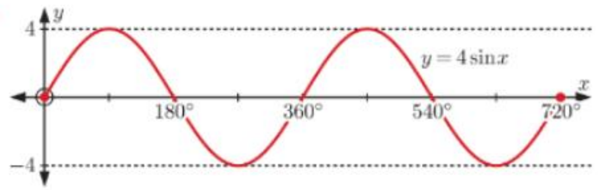


5 $\cos 315°$ is positive, $\sin 315°$ is negative, $\tan 315°$ is negative

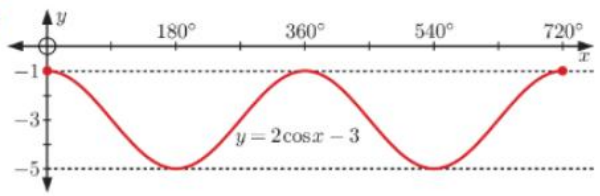
6 **a** $\sin\theta = \frac{2}{\sqrt{5}}$ **b** $\cos\theta = \frac{\sqrt{40}}{7}$

7 $150°$ **8** **a** $\frac{3}{4}$ **b** 1

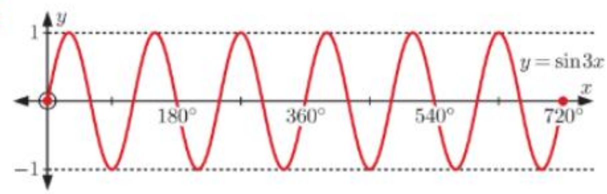
9 **a**



b

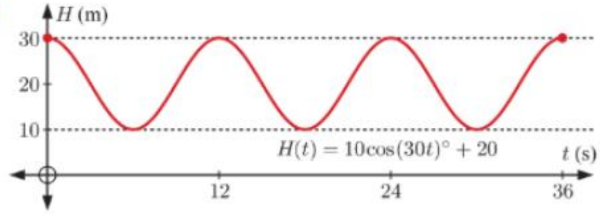


c



10 $y = 2\cos(\frac{3}{2}x)$ with domain $0° \leqslant x \leqslant 720°$

11 **a**



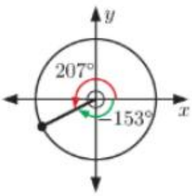
b 20 m **c** 10 m **d** 12 seconds

REVIEW SET 21B

1 **a** $(\cos 237°, \sin 237°)$ **b** $(-0.545, -0.839)$

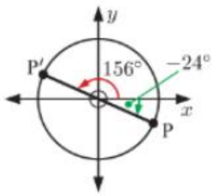
2 **a** $65°$ **b** $136°$ **c** $167°$ **d** $50°$

3 **a**



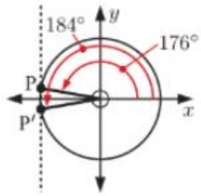
same point on unit circle
$\therefore$ same y-coordinate
$\therefore$ $\sin(-153°) = \sin 207°$

b



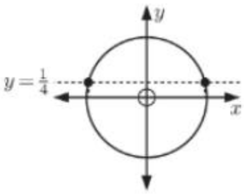
[OP] and [OP′] have the same gradient
$\therefore$ $\tan(-24°) = \tan 156°$

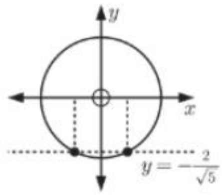
c



P′ is the reflection of P in the x-axis
$\therefore$ same x-coordinate
$\therefore$ $\cos 184° = \cos 176°$

4 **a** $\cos\theta = \pm\frac{\sqrt{15}}{4}$ **b** $\cos\theta = \pm\frac{1}{\sqrt{5}}$



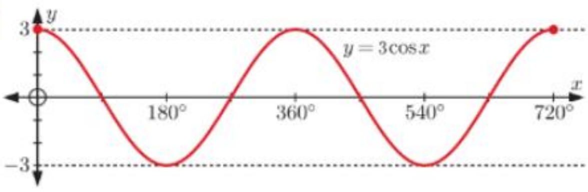


5 **a** $\frac{\sqrt{40}}{7}$ **b** $-\frac{\sqrt{40}}{3}$ **c** $\frac{3}{7}$ **6** **a** $\frac{1}{2}$ **b** 3

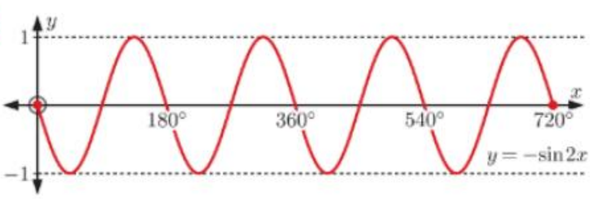
7 **a** amplitude $= 5$, period $= 180°$
b amplitude $= 4$, period $= 720°$

8 $\frac{\sqrt{11}}{4}$

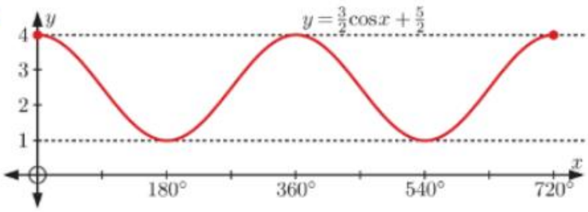
9 **a**



b

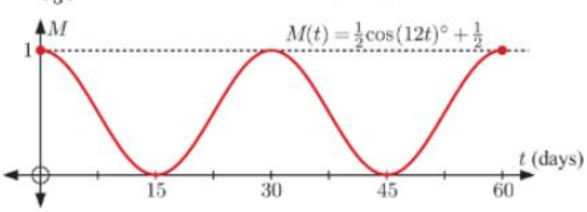


c



10 $y = \sin(\frac{x}{3}) - 1$ with domain $0° \leqslant x \leqslant 1080°$

11 **a**



b **i** 0.75 **ii** 0.25 **iii** ≈ 0.835 **iv** ≈ 0.165
c once every 30 days **d** January 16, February 15

EXERCISE 22A

1 **a** 4 **b** 16 **c** 13

2 **a** 3, 7, 11, 15, 19 **b** 40, 35, 30, 25, 20
c 2, 3, 5, 7, 11

3 **a** The sequence starts at 5 and increases by 3 each time.
b $u_1 = 5$, $u_n = u_{n-1} + 3$ for $n \geqslant 2$

4 **a** The nth term of the sequence is n squared. **b** $u_n = n^2$

5 **a** 19 **b** 19 **c** 96 **d** 27

6 **a** 14 **b** 136 **c** $u_8 = -14$

7 **a** 7, 10, 13, 16 **b** 25, 21, 17, 13 **c** 5, 15, 45, 135
d 100, 50, 25, 12.5 **e** 3, 5, 9, 17
f 4, 6, 4, 6 **g** 3, 4, 12, 48

EXERCISE 22B

1 **a** arithmetic **b** not arithmetic **c** not arithmetic
d arithmetic

2 **a** $\square = 16$ **b** $\square = 34$ **c** $\square = 15$, $\triangle = 3$
d $\square = 13$, $\triangle = -5$

3 **a** $u_1 = 41$, $d = 1$ **b** $u_1 = 1$, $d = 11$
c $u_1 = 98$, $d = -10$ **d** $u_1 = 91$, $d = -9$

4 **a** $11 - 4 = 7$, $18 - 11 = 7$, $25 - 18 = 7$, $32 - 25 = 7$
Consecutive terms have a common difference of 7.
$\therefore$ the sequence is arithmetic with $u_1 = 4$ and $d = 7$.
b $u_n = 7n - 3$ **c** 207 **d** yes, the 49th term **e** no

5 **a** $63 - 67 = -4$, $59 - 63 = -4$, $55 - 59 = -4$
Consecutive terms have a common difference of -4.
$\therefore$ the sequence is arithmetic with $u_1 = 67$ and $d = -4$.
b $u_n = 71 - 4n$ **c** -169 **d** no **e** no

6 **a** $u_1 = 4$ and $d = 11$ **b** 400 **c** $u_{24} = 257$

7 **a** $k = 22$ **b** $k = 2\frac{1}{2}$ **c** $k = 4\frac{1}{3}$

8 **a** $k = 6$ **b** 29, 37

9 **a** $u_n = 5n + 17$ **b** $u_n = 10 - 4n$
c $u_n = 3n - 16$ **d** $u_n = -\frac{17}{2} - \frac{3}{4}n$

10 **a** $u_n = 6n - 8$ **b** 172

EXERCISE 22C

1 **a** geometric **b** not geometric **c** geometric
d not geometric

2 **a** $r = 5$ **b** $r = \frac{1}{2}$ **c** $r = -2$ **d** $r = -\frac{1}{10}$

3 **a** $b = 12$, $c = 24$ **b** $b = \frac{1}{2}$, $c = \frac{1}{8}$ **c** $b = \frac{5}{3}$, $c = -\frac{5}{9}$

4 **a** $\frac{3}{1} = 3$, $\frac{9}{3} = 3$, $\frac{27}{9} = 3$
Consecutive terms have a common ratio of 3.
$\therefore$ the sequence is geometric with $u_1 = 1$ and $r = 3$.
b $u_n = 3^{n-1}$ **c** 19 683

5 **a** $\frac{-20}{40} = -\frac{1}{2}$, $\frac{10}{-20} = -\frac{1}{2}$, $\frac{-5}{10} = -\frac{1}{2}$
Consecutive terms have a common ratio of $-\frac{1}{2}$.
$\therefore$ the sequence is geometric with $u_1 = 40$ and $r = -\frac{1}{2}$.
b $u_n = 40 \times (-\frac{1}{2})^{n-1}$ **c** $-\frac{5}{256}$

6 **a** $\frac{-4}{16} = -\frac{1}{4}$, $\frac{1}{-4} = -\frac{1}{4}$, $\frac{-0.25}{1} = -\frac{1}{4}$
Consecutive terms have a common ratio of $-\frac{1}{4}$.
$\therefore$ the sequence is geometric with $u_1 = 16$ and $r = -\frac{1}{4}$.
b $-0.000\,976\,562\,5$

7 $u_n = 3 \times 2^{\frac{n-1}{2}}$ or $3\left(\sqrt{2}\right)^{n-1}$

8 **a** $k = \frac{2}{3}$ **b** $k = 9$ **c** $k = 2$ **d** $k = \pm 9$
e $k = -7$ **f** $k = -\frac{8}{7}$ or 2

9 **a** $u_n = \frac{16}{9} \times 3^{n-1}$ **b** $u_n = 128 \times (-\frac{1}{2})^{n-1}$
c $u_n = 2 \times 5^{n-1}$ or $u_n = (-2) \times (-5)^{n-1}$
d $u_n = 6 \times (\pm\frac{1}{\sqrt{2}})^{n-1}$

10 $\pm\frac{3}{32}$

EXERCISE 22D

1 **a** 8 **b** 16 **c** 19 **d** 46

2 **a** **i** 5, 7, 9, 11 **ii** 32
b **i** $-5, -2, 3, 10$ **ii** 6
c **i** 7, 14, 28, 56 **ii** 105
d **i** 3, 5, 11, 29 **ii** 48

3 12

EXERCISE 22E

1 **a, b, c** 75 **2** 450 **3** 220

4 **a** 210 **b** 75 **c** 4075 **d** 6780
e -2280 **f** -2400 **g** 275 **h** 387.5

5 2500 **6** 2835 **7** **a** 24 terms **b** 1476

8 **a** 775 **b** 1705 **c** 969 **d** 1040 **e** -345 **f** 306

9 **a** $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ where $u_1 = 4$, $d = 4$
$= \frac{n}{2}(8 + 4(n-1))$ etc.
b 840

10 **a** €65 **b** €1350 **11** $n = 10$

EXERCISE 22F.1

1 **a, b** 315

2 **a** 3280 **b** 4 882 812 **c** $\frac{1533}{32}$ **d** $63 + 63\sqrt{2}$

3 **a** -1364 **b** $\frac{1640}{27}$ **c** ≈ 52.2 **d** ≈ 12.8

4 **a** $u_1 = 2$, $r = 3$ **b** 59 048

5 $S_n = \dfrac{u_1(1-(-1)^n)}{2}$
If n is odd, then $S_n = \dfrac{u_1(1--1)}{2} = u_1$
If n is even, then $S_n = \dfrac{u_1(1-1)}{2} = 0$

6 **a** **i** ≈ 158 mL **ii** ≈ 37.5 mL **iii** ≈ 8.91 mL
b Each day Doug drinks $\frac{3}{4}$ of the amount he drank the previous day. So, the amount of water he drinks is a geometric sequence with $r = \frac{3}{4}$.
c **i** ≈ 1887.4 mL **ii** ≈ 1993.7 mL **iii** ≈ 1999.6 mL

EXERCISE 22F.2

1 **a** diverge **b** converge **c** diverge **d** converge

2 **a** **i** ≈ 23.4 **ii** ≈ 26.5 **iii** ≈ 27.0
b 27 **c** $S = \dfrac{9}{1-\frac{2}{3}} = 27$

3 **a** 32 **b** $\frac{5}{4}$ **c** 27 **d** 128 **e** $\frac{432}{7}$ **f** $\frac{2}{3}$

4 **a** Start with $\frac{1}{3}$ unit^2 shaded blue. So $u_1 = \frac{1}{3}$.
Each step shades $\frac{1}{3}$ of the previous area blue, so $r = \frac{1}{3}$.
$\therefore$ $S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$
b At each step, the same amount is coloured blue as is coloured red.
c If the process continues indefinitely, half the 1 unit^2 rectangle will be shaded blue and half will be shaded red. And from **a**, total blue area $= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$
$\therefore$ $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... = \frac{1}{2}$
d $S = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

REVIEW SET 22A

1 **a** 8, 13, 18, 23, 28 **b** 19, 12, 5, -2, -9
c 1, 8, 27, 64, 125

2 2, 8, 20, 44 **3** **a** $\square = 25$ **b** $\square = 21$, $\triangle = 9$

4 $k = 6$ **5** **a** $r = \frac{4}{5}$ **b** $r = -3$

6 $u_n = -3 \times (-2)^{n-1}$ **7** **a** 3, 8, 15, 24 **b** 50

8 **a** 1180 **b** -1410

9 **a** 1 747 625 **b** ≈ 11.1 **10** **a** $\frac{125}{4}$ **b** 16

REVIEW SET 22B

1 **a** 29 **b** 54

2 **a** $-2-(-6) = 4$, $2-(-2) = 4$, $6-2 = 4$
Consecutive terms have a common difference of 4.
$\therefore$ the sequence is arithmetic with $u_1 = -6$ and $d = 4$.
b $u_n = 4n - 10$ **c** 390 **d** $u_{127} = 498$

3 **a** $\frac{-32}{64} = -\frac{1}{2}$, $\frac{16}{-32} = -\frac{1}{2}$, $\frac{-8}{16} = -\frac{1}{2}$, $\frac{4}{-8} = -\frac{1}{2}$
Consecutive terms have a common ratio of $-\frac{1}{2}$.
$\therefore$ the sequence is geometric with $u_1 = 64$ and $r = -\frac{1}{2}$.
b $-\frac{1}{512}$

4 **a** 5, 7, 5, 7, 5 **b** 7 **5** 165

6 $u_n = 39 - \frac{15}{2}(n-1)$

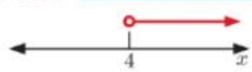
7 **a** 1070 **b** 270 **c** -119 **d** $\dfrac{58\,025}{32} \approx 1813.3$

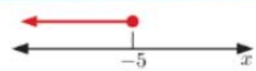
8 **a** $k = -5$ or 10
b If $k = -5$, $r = -2$. If $k = 10$, $r = -\frac{1}{3}$.

9 **a** $u_1 = 6$, $r = -\frac{1}{2}$ **b** $\frac{1023}{256}$

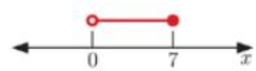
10 **a** **i** 42 km **ii** 58 km **iii** 78 km
b **i** 82 km **ii** 490 km **iii** 1180 km
c Yes, using $u_n = 2n + 38$ and $S_n = \frac{n}{2}(u_1 + u_n)$.
d Yes, $S_{30} = 2070$, so Jasmine will have cycled 2070 km in total.

EXERCISE 23A

1 **a**


b


c

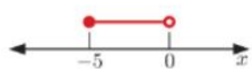

d


e

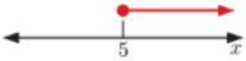

f

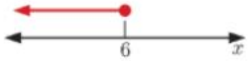

g

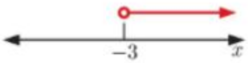

h

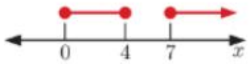

i


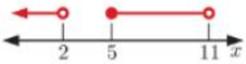
j


k 

l 

m 

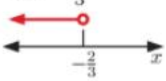
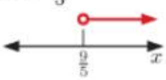
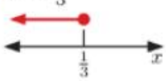
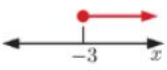
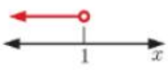
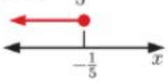
n 

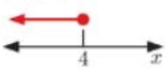
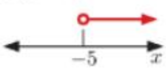
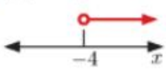
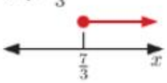
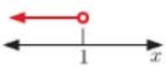
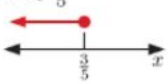
o 

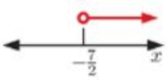
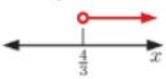
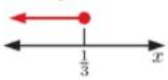
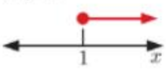
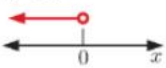
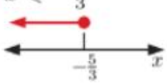
2 **a** $\{x \mid x \geqslant 6\}$ **b** $\{x \mid x < 3\}$ **c** $\{x \mid -2 \leqslant x < 1\}$
d $\{x \mid x \leqslant 0 \text{ or } x > 1\}$ **e** $\{x \mid x < -1 \text{ or } x > 1\}$
f $\{x \mid 0 < x \leqslant 4\}$ **g** $\{x \mid x < -2 \text{ or } 1 < x \leqslant 4\}$
h $\{x \mid -2 \leqslant x \leqslant 1 \text{ or } x > 3\}$

3 **a** $[-1, 6]$ **b** $]0, 5[$ **c** $]-4, 7]$ **d** $[4, 8[$
e $]-7, \infty[$ **f** $]-\infty, 0]$ **g** $]-\infty, 2] \cup [5, \infty[$
h $]-\infty, -3[\cup]4, \infty[$ **i** $]-1, 1] \cup [2, \infty[$
j $]-\infty, -4[\cup [2, 7[$ **k** $[3, 8[$ **l** $[-2, 7[$
m $[5, \infty[$ **n** $]-\infty, -2[$ **o** $]-\infty, 2] \cup]3, 5[$
p $[-4, 1] \cup]4, \infty[$ **q** $]-\infty, -2] \cup]2, \infty[$
r $]-\infty, -3[\cup]10, \infty[$

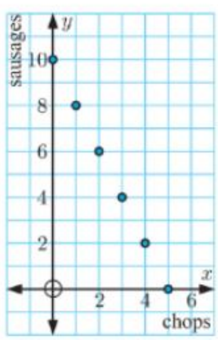
EXERCISE 23B

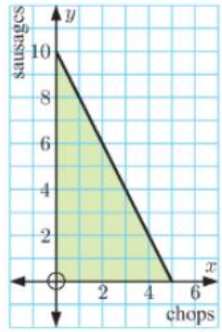
1 **a** $x < -\frac{2}{3}$ 
b $x > \frac{9}{5}$ 
c $x \leqslant \frac{1}{3}$ 
d $x \geqslant -3$ 
e $x < 1$ 
f $x \leqslant -\frac{1}{5}$ 

2 **a** $x \leqslant 4$ 
b $x > -5$ 
c $x > -4$ 
d $x \geqslant \frac{7}{3}$ 
e $x < 1$ 
f $x \leqslant \frac{3}{5}$ 

3 **a** $x > -\frac{7}{2}$ 
b $x > \frac{4}{3}$ 
c $x \leqslant \frac{1}{3}$ 
d $x \geqslant 1$ 
e $x < 0$ 
f $x \leqslant -\frac{5}{3}$ 

EXERCISE 23C.1

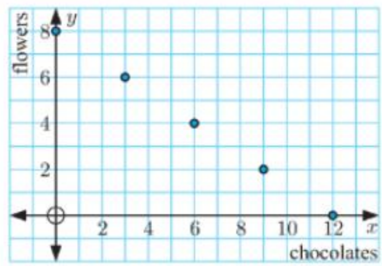
1 **a** 5 chops **b** 10 sausages
c 4 chops, 2 sausages
3 chops, 4 sausages
2 chops, 6 sausages
1 chop, 8 sausages
d 

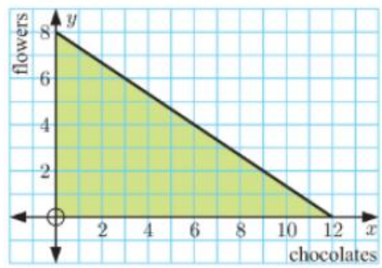
e $2x + y = 10$
f 

$x \geqslant 0, \quad y \geqslant 0, \quad 2x + y \leqslant 10$

2 **a** 12 chocolates
b 8 flowers
c

Chocolates	*Flowers*
3	6
6	4
9	2

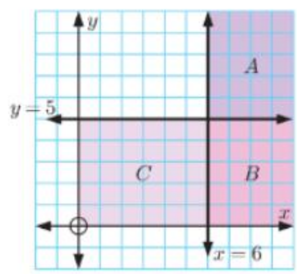
d 

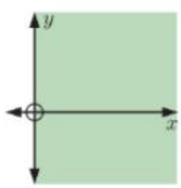
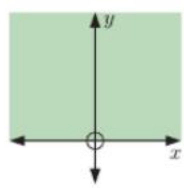
e $2x + 3y = 24$
f 

$x \geqslant 0, \quad y \geqslant 0, \quad 2x + 3y \leqslant 24$

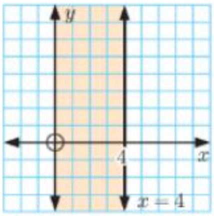
EXERCISE 23C.2

1 $0 \leqslant x \leqslant 5, \quad y \geqslant 0$

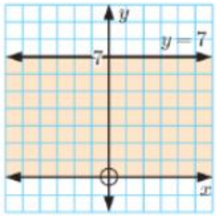
2 

3 **a** 
b 

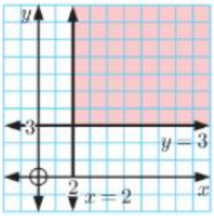
c



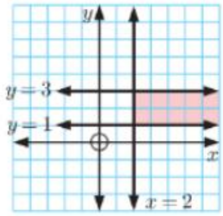
d



e

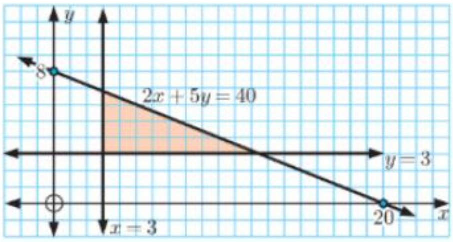


f



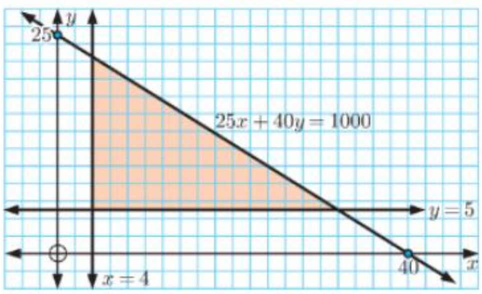
4 **a** $x \geqslant 3, \quad y \geqslant 3, \quad 2x + 5y \leqslant 40$

b

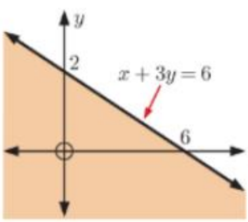


5 **a** $x \geqslant 4, \quad y \geqslant 5, \quad 25x + 40y \leqslant 1000$

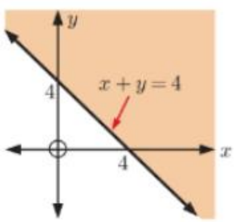
b



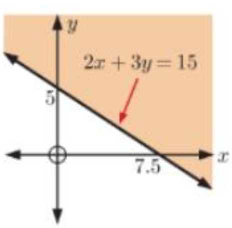
6 **a**



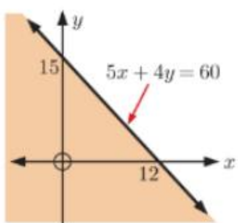
b



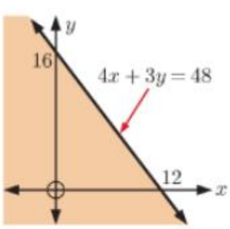
c



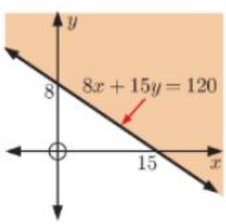
d



e



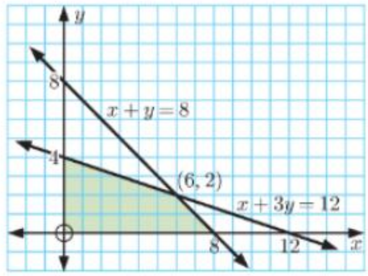
f



7 **a**

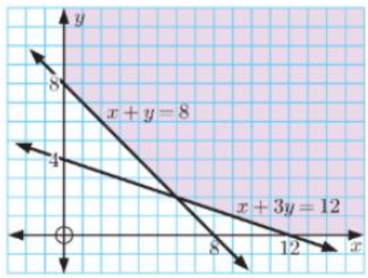
Point	Is $x + y \leqslant 8$?	Is $x + 3y \leqslant 12$?
A(5, 5)	No	No
B(1, 5)	Yes	No
C(8, 1)	No	Yes
D(3, 2)	Yes	Yes

b

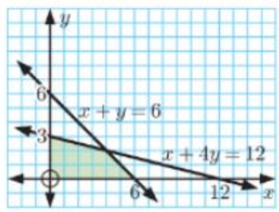


c

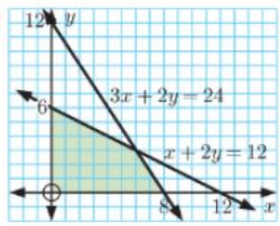
Point	Is $x + y \geqslant 8$?	Is $x + 3y \geqslant 12$?
A(5, 5)	Yes	Yes
B(1, 5)	No	Yes
C(8, 1)	Yes	No
D(3, 2)	No	No



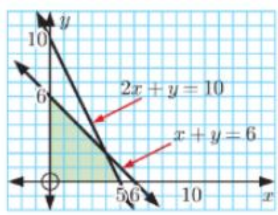
8 **a**



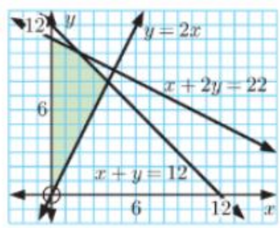
b



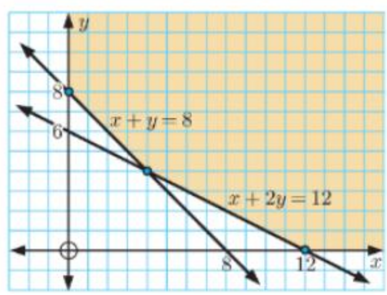
c



d

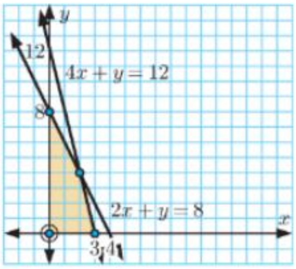


9 **a**



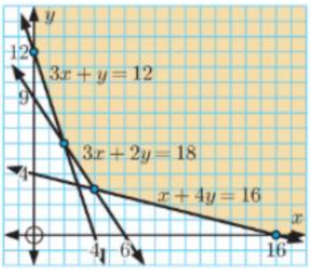
Vertices are $(0, 8)$, $(4, 4)$, and $(12, 0)$.

b



Vertices are $(0, 0)$, $(0, 8)$, $(2, 4)$, and $(3, 0)$.

c



Vertices are $(0, 12)$, $(2, 6)$, $(4, 3)$, and $(16, 0)$.

EXERCISE 23D

1 **a** $5x+y\leqslant 15$, $x+4y\leqslant 12$, $x\geqslant 0$, $y\geqslant 0$

b Yes, as we want to know the number of whole litre cans which can be made.

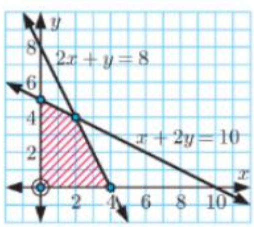
2 **a** $8x+6y\leqslant 50$ {50 ha planting space available}
$x+y\leqslant 7$ {7 days available to plant}
$x\geqslant 0$, $y\geqslant 0$ {neither x nor y can be negative}

b No, as part days are possible.

3 $10x+40y\geqslant 120$, $30x+30y\geqslant 180$, $500x+100y\geqslant 1000$, $x\geqslant 0$, $y\geqslant 0$

EXERCISE 23E.1

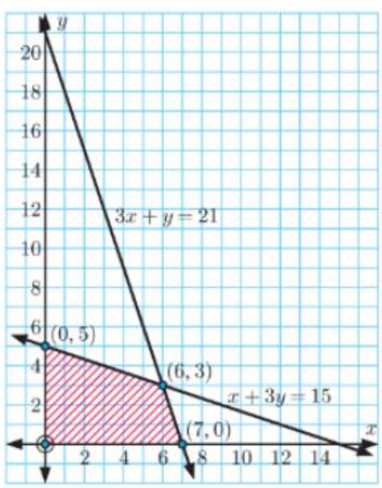
1 **a**



b Vertices are $(0, 0)$, $(0, 5)$, $(2, 4)$, and $(4, 0)$.

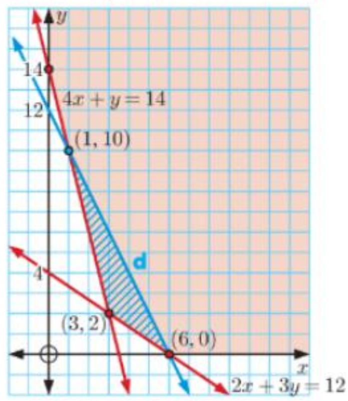
c Maximum value is 6, when $x=2$ and $y=4$.

2 **a**



b **i** 9 at $(6, 3)$ **ii** 15 at $(6, 3)$
iii 30 at $(0, 5)$ **iv** 28 at $(7, 0)$

3 **a**



b **i** 6 at $(6, 0)$ **ii** 22 at $(3, 2)$
iii 14 at $(0, 14)$ **iv** 0 at $(0, 14)$

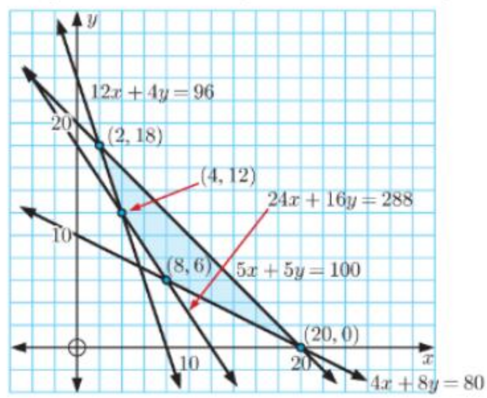
c No, because the simplex does not have any upper bounds.

d **i** 51 at $(1, 10)$ **ii** 54 at $(1, 10)$
iii 36 at $(6, 0)$ **iv** 30 at $(6, 0)$

EXERCISE 23E.2

1 200 kg of A, 500 kg of B

2 **a** $12x+4y\geqslant 96$, $4x+8y\geqslant 80$,
$24x+16y\geqslant 288$, $5x+5y\leqslant 100$, $x\geqslant 0$, $y\geqslant 0$



b 4 scoops of Foodo, 12 of Petmix

c **i** Yes - 20 scoops/day **ii** No

3 **a** **i** 7 Deluxe, 7 Standard **ii** Machine B
iii 2 minutes

b **i** 6 Deluxe, 6 Standard
ii Machine B is still fully used, and Machine A is now idle for 8 minutes each hour.

EXERCISE 23E.3

1 **a** 5 bottles of X, 3 bottles of Y

b 9 bottles of Y, 1 bottle of X and 8 bottles of Y, 2 bottles of X and 7 bottles of Y, or 3 bottles of X and 6 bottles of Y

2 **a** 6A and 18B, or 11A and 15B, or 16A and 12B

b **i** 16A and 12B **ii** nets

3 **a** 8 regular and 6 deluxe, or 11 regular and 4 deluxe, or 14 regular and 2 deluxe, or 17 regular and 0 deluxe

b 8 regular and 6 deluxe, or 6 regular and 7 deluxe, or 4 regular and 8 deluxe, or 2 regular and 9 deluxe, or 0 regular and 10 deluxe

c 8 regular and 6 deluxe

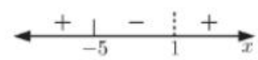
EXERCISE 23F

1 **a** **b**


c **d**


e **f**


g **h**


i


2 **a** **b**


c **d**


e **f**


g **h**


i **j**


k **l**


3 **a** **b**


c **d**


e **f**


g **h**


4 **a** **b**


c **d**


e **f**

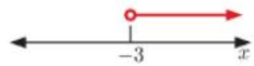

g **h**

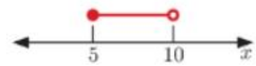

5 **a** **i** $\{x \mid -3 < x < 6\}$ **ii** $\{x \mid -3 \leqslant x \leqslant 6\}$
iii $\{x \mid x < -3 \text{ or } x > 6\}$ **iv** $\{x \mid x \leqslant -3 \text{ or } x \geqslant 6\}$

b **i** $\{x \mid x < -1 \text{ or } x > 2\}$ **ii** $\{x \mid x \leqslant -1 \text{ or } x > 2\}$
iii $\{x \mid -1 < x < 2\}$ **iv** $\{x \mid -1 \leqslant x < 2\}$

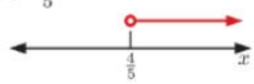
c **i** $\{x \mid -4 < x < 0 \text{ or } 0 < x < 5\}$
ii $\{x \mid -4 < x \leqslant 5\}$ **iii** $\{x \mid x < -4 \text{ or } x > 5\}$
iv $\{x \mid x < -4 \text{ or } x = 0 \text{ or } x \geqslant 5\}$

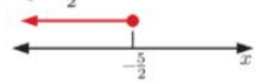
REVIEW SET 23A

1 **a**


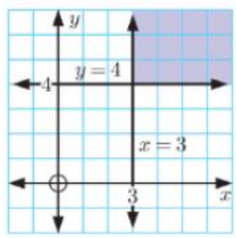
b


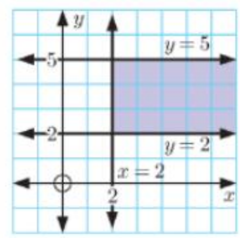
c


2 **a** $x > \frac{4}{5}$


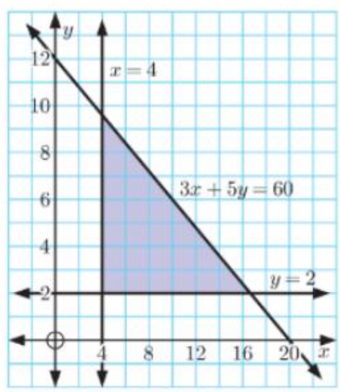
b $x \leqslant -\frac{5}{2}$


3 **a** $x > \frac{10}{3}$ **b** $x \geqslant -\frac{7}{9}$

4 **a**


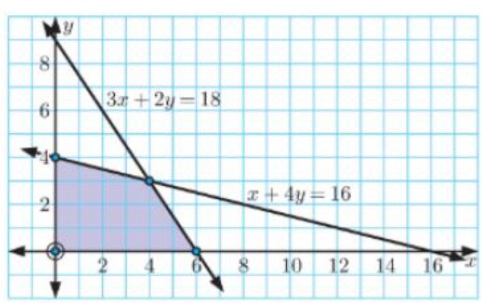
b


5 **a** $x \geqslant 4, \quad y \geqslant 2, \quad 3x + 5y \leqslant 60$

b


6 **a** $x \geqslant 1, \quad y \geqslant 0, \quad y \leqslant 2, \quad x + y \leqslant 4$
b **i** 12 at $(4, 0)$ **ii** 6 at $(2, 2)$

7 **a** $y \geqslant 1, \quad 3x - 4y \geqslant -4, \quad x + y \leqslant 8$
b Vertices are $(0, 1)$, $(4, 4)$, and $(7, 1)$.
c 43 at $(7, 1)$

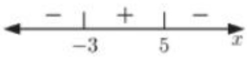
8 **a**


b Vertices are $(0, 0)$, $(0, 4)$, $(4, 3)$, and $(6, 0)$.
c 20, when $x = 0$ and $y = 4$.

9 6 gas meters, 3 water meters

10 **a** 8 round pillars and 6 square pillars
b **i** 2 round and 15 square, 4 round and 12 square, 6 round and 9 square, or 8 round and 6 square
ii 2 round pillars and 15 square pillars

11 **a** 

b 

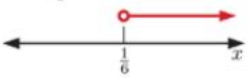
c 

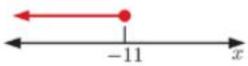
12 **a** $\{x \mid x < -1 \text{ or } x > 6\}$ **b** $\{x \mid x \leqslant -1 \text{ or } x > 6\}$
c $\{x \mid -1 < x < 6\}$ **d** $\{x \mid -1 \leqslant x < 6\}$

REVIEW SET 23B

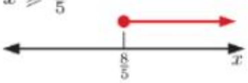
1 **a** $[-2, 8[$ **b** $]-\infty, 1] \cup]4, 5]$ **c** $]-3, 1[$
d $]-\infty, -7] \cup]3, \infty[$

2 **a** $x > \frac{1}{6}$ **b** $x \leqslant -11$





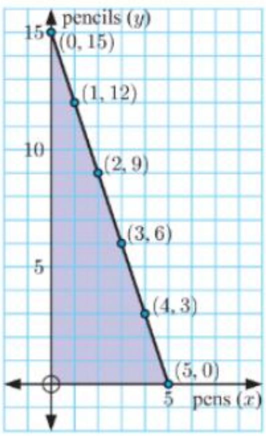
c $x \geqslant \frac{8}{5}$



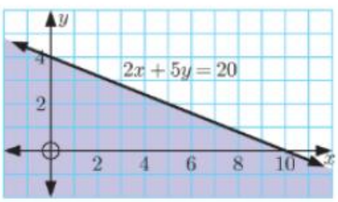
3 $x < 0$

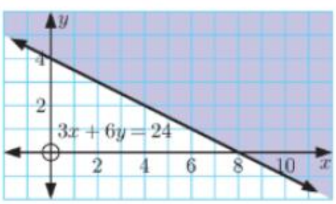
4 **a**

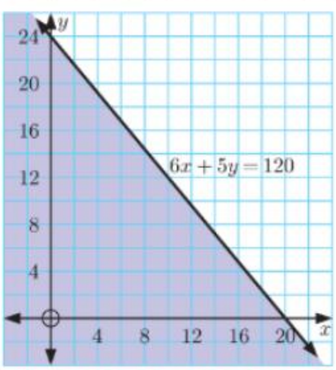
Pens	*Pencils*
5	0
4	3
3	6
2	9
1	12
0	15

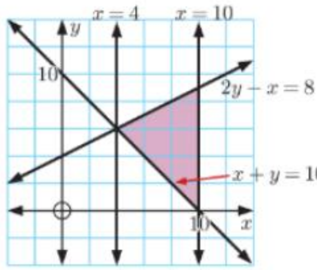
b, c 

d $x \geqslant 0,\ y \geqslant 0,$
$3x + y \leqslant 15$

5 **a** 

b 

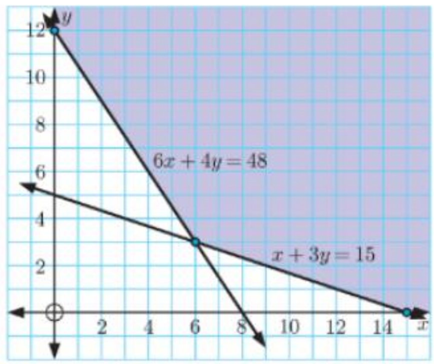
c 

6 **a** 

b **i** $2(3) - 10 = -4 \leqslant 8$ ✓
$4 \leqslant 10 \leqslant 10$ ✓
$10 + 3 = 13 \geqslant 10$ ✓
$\therefore$ (10, 3) satisfies the constraints.

ii $4 \leqslant 11 \not\leqslant 10$ ✗
$\therefore$ (11, 3) does not satisfy the constraints.

c Vertices are (4, 6), (10, 9), and (10, 0).

7 **a** 

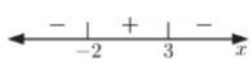
b Vertices are (0, 12), (6, 3), and (15, 0). **c** 21

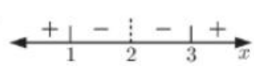
8 **a** **i** €238 **ii** €282
b 3 filing cabinets and 5 desks, profit = €337

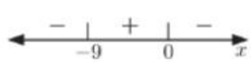
9 **a** No, as it would cost \$225 to purchase enough of each item from Bev's Bakery exclusively and \$234 to purchase enough of each item from Dave's Bakery. Both of these amounts are over the budget of \$189.

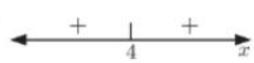
b **i** 5 packages from Bev's, 5 packages from Dave's
ii \$24

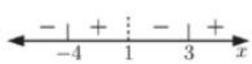
c **i** 3 packages from Bev's and 8 packages from Dave's, or 9 packages from Bev's and 3 packages from Dave's
ii 9 packages from Bev's, 3 packages from Dave's
There will be 16 pies left over.

10 **a** 

b 

11 **a** 

b 

c 

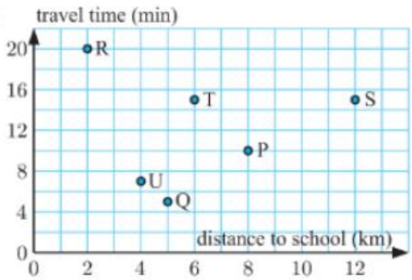
EXERCISE 24A

1 **a** D **b** C **c** E and H

2 **a** E **b** A **c** yes, B and C
d Yes, generally more hours of training results in higher productivity (from the graph).

3 **a** D **b** F **c** 2 **d** 3

4 **a**

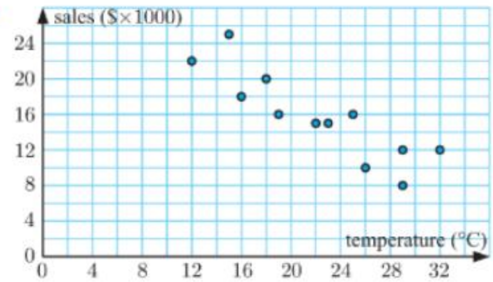


b S

c Q

d Student R. They live the shortest distance from school, yet take the most time to get there.

5 **a**



b A moderate, linear, negative correlation.

EXERCISE 24B

1 **a** **i** negative association **ii** linear **iii** strong

b **i** no association **ii** not linear **iii** zero

c **i** positive association **ii** linear **iii** moderate

d **i** positive association **ii** linear **iii** weak

e **i** positive association **ii** not linear **iii** moderate

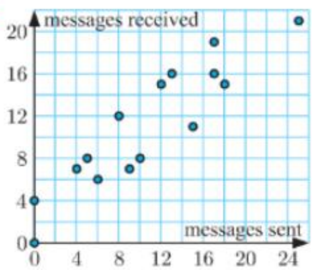
f **i** negative association **ii** not linear **iii** strong

2 **a** "...... as x increases, y increases"

b "...... as T increases, d decreases"

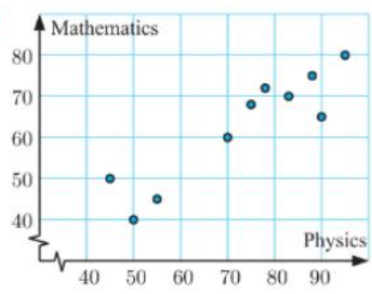
c "...... randomly scattered"

3 **a**



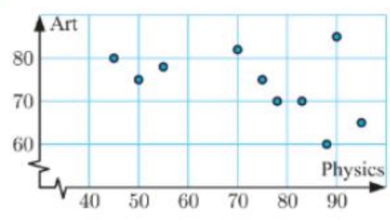
b A moderate, positive, linear correlation.

4 **a** **i**



ii A moderate, positive, linear correlation.

b



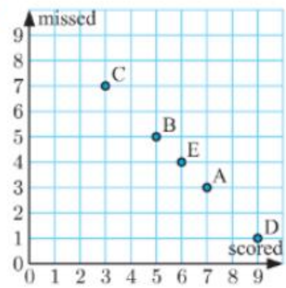
A very weak, negative, linear correlation (virtually zero correlation).

EXERCISE 24C

1 **a** positive correlation as $r > 0$ **b** $r^2 \approx 0.659$

c moderate positive linear correlation, $0.50 \leqslant r^2 < 0.75$

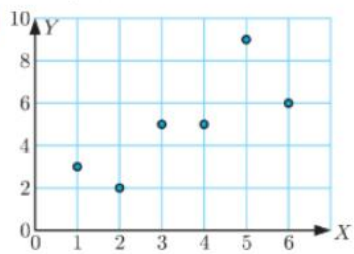
2 **a**



b $r = -1$

c perfect negative linear correlation
The number of shots scored is directly related to those missed. Out of 10 shots, each time the number of goals scored increases by 1, the number of misses decreases by 1.

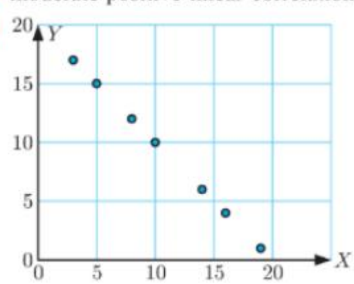
3 **a** **i**



ii $r \approx 0.786$, $r^2 \approx 0.617$

iii moderate positive linear correlation

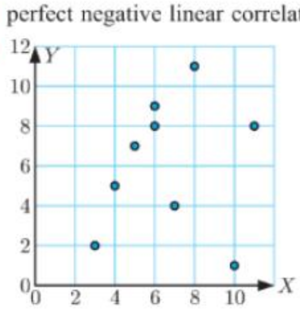
b **i**



ii $r = -1$, $r^2 = 1$

iii perfect negative linear correlation

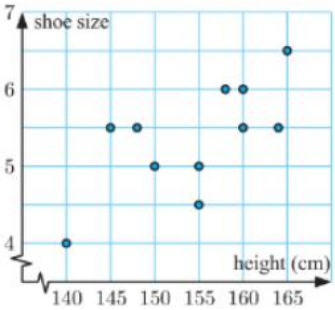
c **i**



ii $r \approx 0.146$, $r^2 \approx 0.0215$

iii very weak positive linear correlation

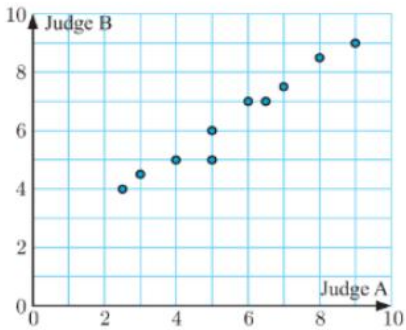
4 **a**



b $r \approx 0.673, \quad r^2 \approx 0.452$

c A weak, positive, linear correlation exists between *shoe size* and *height*.

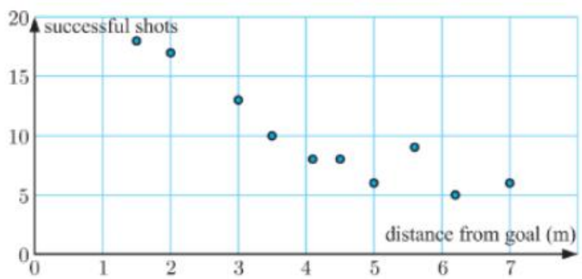
5 **a**



b $r \approx 0.981, \quad r^2 \approx 0.962$

c A very strong, positive, linear correlation exists between Judge B's scores and Judge A's scores.

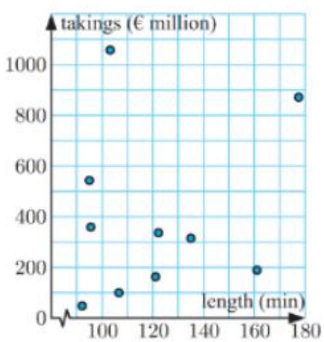
6 **a**



b negative **c** $r \approx -0.911, \quad r^2 \approx 0.830$

d There is a strong, negative, linear correlation between the number of successful shots and the distance from goal.

7 **a**



b No, there appears to be almost no correlation between the variables as the points look randomly scattered.

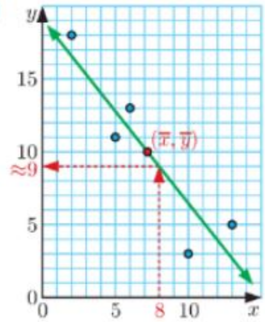
c $r \approx 0.191, \quad r^2 \approx 0.0366$

d There is a very weak, positive, linear correlation (almost no correlation) between movie length and box office performance.

EXERCISE 24D.1

1 **a** $\overline{x} = 7.2, \quad \overline{y} = 10$

b, d

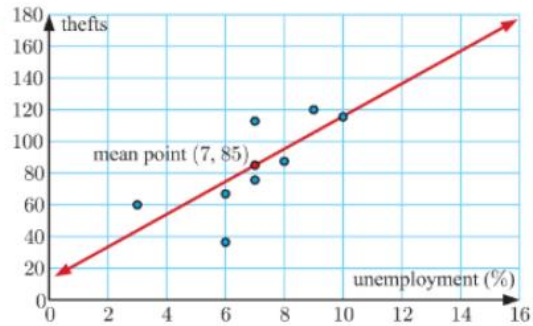


c negatively correlated

e when $x = 8, \; y \approx 9$

2 **a** $\overline{x} = 7, \quad \overline{y} = 85$ **b** $r \approx 0.739$

c, d, e

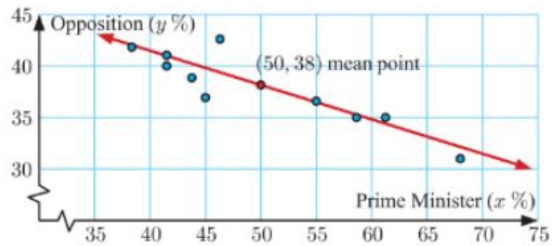


f **i** ≈ 170 thefts

ii Unreliable as it is an extrapolation well beyond the upper pole.

3 **a** $\overline{x} = 50, \quad \overline{y} = 38$

b

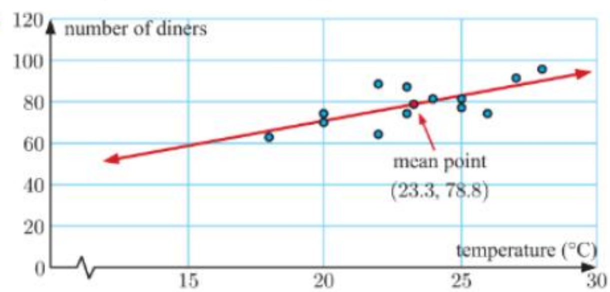


c 39%

4 **a** $\overline{x} \approx 23.3, \quad \overline{y} \approx 78.8$

b $r^2 \approx 0.509$ There is a moderate linear correlation between *number of diners* and *noon temperature*.

c

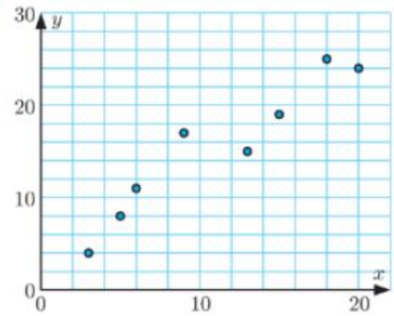


d **i** 73 diners **ii** 56 diners

e We expect the first estimate to be reliable as it is an interpolation, but the second may not be reliable as it is an extrapolation.

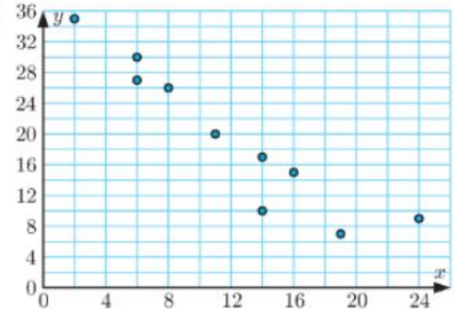
EXERCISE 24D.2

1 **a** **i**



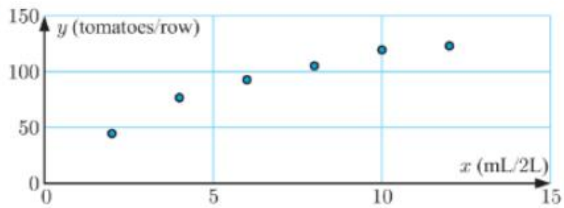
ii $y \approx 1.12x + 2.90$

b **i**



ii $y \approx -1.34x + 35.7$

2 **a**

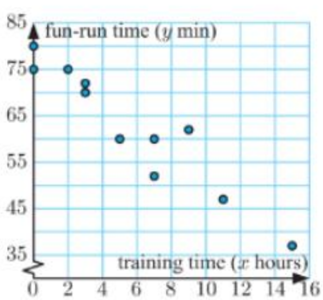


b $y \approx 7.66x + 40.1$

c The y-intercept indicates that a row can be expected to produce on average 40.1 tomatoes when no spray is applied.

d 94 tomatoes/row. Even though this is an interpolation, this seems low compared with the yield at 6 and 8 mL/2 L. Looking at the graph it would appear that the relationship is not linear.

3 **a**



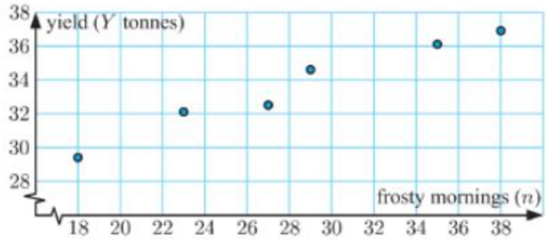
b $r \approx -0.952$, $r^2 \approx 0.907$ **c** $y \approx -2.69x + 77.9$

d The gradient indicates that an increase in training time of 1 hour will decrease the fun-run time by ≈ 2.7 minutes.

e **i** -2.8 minutes

ii This value is clearly absurd as one cannot record a negative time for a fun-run. It is very unreliable as it is an extrapolation well beyond the upper pole.

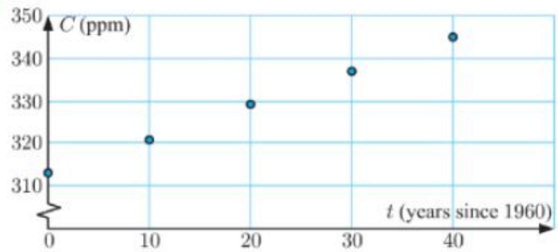
4 **a**



b $Y \approx 0.371n + 23.1$ **c** ≈ 34.6 tonnes

d "...... , the *greater* the yield of cherries."

5 **a**

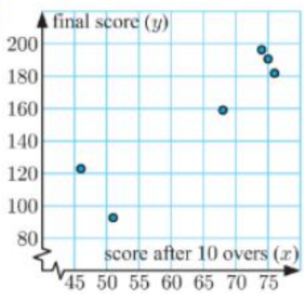


b $r = 1$, perfect positive linear correlation

c $C = 0.8t + 313$ **d** 335 parts per million

e 361 parts per million

6 **a**



b $r \approx 0.922$, $r^2 \approx 0.851$, strong positive linear correlation

c $y \approx 2.90x - 30.9$

d ≈ 143 runs. This should be fairly reliable as it is interpolated.

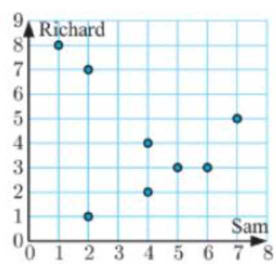
REVIEW SET 24A

1 There is a weak, negative, linear correlation between the variables.

2 **a** E **b** A

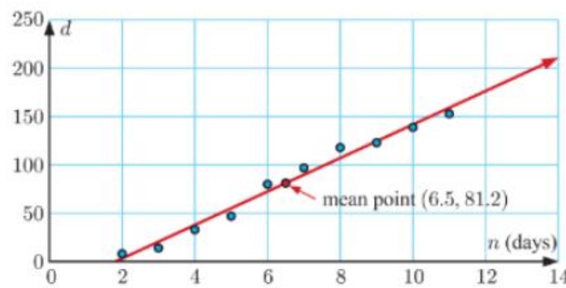
c The greater the time spent on player fitness, the smaller the chance of player injury.

3 **a**

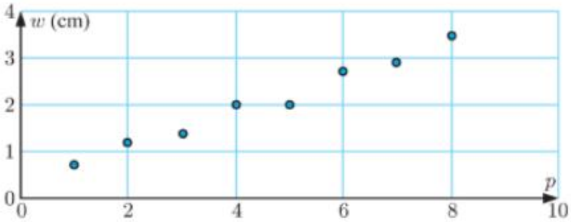


b $r \approx -0.334$
$r^2 \approx 0.112$
There is a very weak, negative, linear correlation between the variables.

4 **a, b**



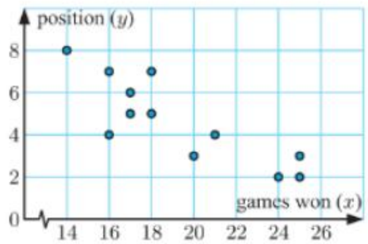
c **i** about 210 diagnosed cases
ii Very unreliable as 14 is outside the poles. The medical team have probably isolated those infected at this stage and there could be a downturn which may be very significant.

5 **a**


b $r \approx 0.990$ **c** $w \approx 0.381p + 0.336$

d **i** ≈ 5.7 cm
ii As $p = 14$ is outside the poles, this prediction could be unreliable.

6 **a** Negatively correlated. The more games won, the higher the team's position on the ladder (and so the value for *Position* is smaller).

b


c $r \approx -0.843$, $r^2 \approx 0.711$

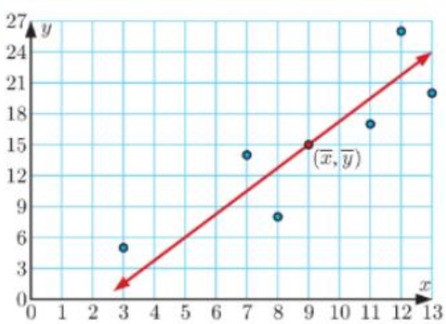
d $y \approx -0.454x + 13.4$ **e** 3rd

REVIEW SET 24B

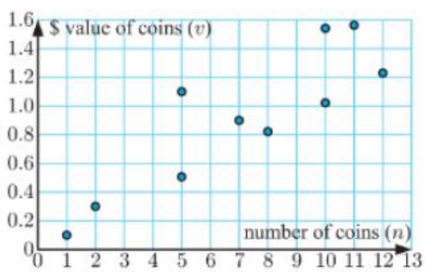
1 **a** J **b** E **c** D and G **d** C

2 **a** **i** D **ii** H
b As the number of pages increases, the number of chapters generally increases.

3 **a** $\overline{x} = 9$, $\overline{y} = 15$

b, d


c positively correlated
e $y \approx 6$. Reasonably accurate by interpolation.

4 **a**


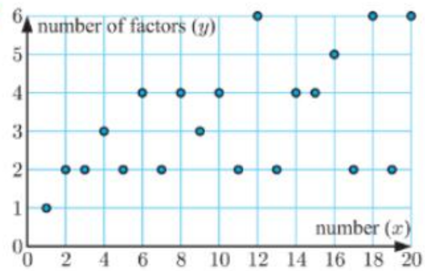
b $v \approx 0.114n + 0.100$
c Each coin added to a wallet or purse increases the value by 11.4 cents on average.
d **i** $\approx$ \$2.38
ii 20 coins is too far beyond the upper pole to extrapolate with any reliable accuracy.

5 **a** **i** Positive, as numbers get larger they have more possible factors.
ii Weak, even amongst large numbers there exist primes, squares, cubes, and so on which have a small number of factors.

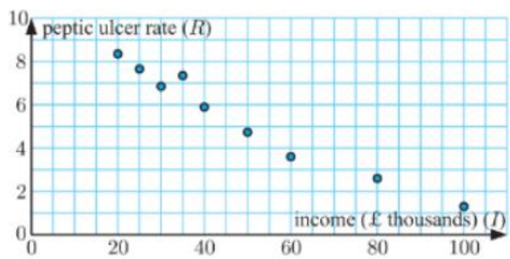
b

Number	1	2	3	4	5	6	7	8	9	10
Number of factors	1	2	2	3	2	4	2	4	3	4

Number	11	12	13	14	15	16	17	18	19	20
Number of factors	2	6	2	4	4	5	2	6	2	6

c


d $r \approx 0.531$, $r^2 \approx 0.282$
e weak, positive, linear correlation

6 **a**


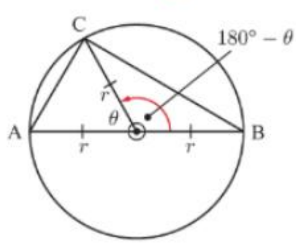
b $R \approx -0.0907I + 9.79$ **c** when $I = 55$, $R \approx 4.80$
d £120 000 gives a rate of -1.09 which is meaningless. So, £120 000 is outside the data range of this model.
e **i** The point $(35, 7.3)$, as it does not follow the general trend of the data in that area.
ii $R \approx -0.0887I + 9.61$; when $I = 55$, $R \approx 4.73$

EXERCISE 25A

1 **a** ≈ 55.2 cm^2 **b** ≈ 347 km^2 **c** ≈ 1.15 m^2
d ≈ 13.6 cm^2 **e** ≈ 58.6 m^2 **f** ≈ 5.81 m^2

2 **a** ≈ 41.6 cm^2 **b** ≈ 36.7 m^2 **c** ≈ 7.70 cm^2

3 ≈ 50.0 cm^2 **4** $x \approx 21.9$ **5** ≈ 13.1 cm

6


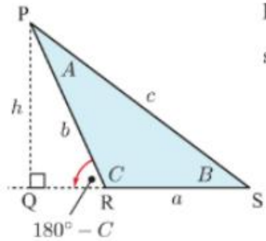
Area $\triangle$OBC
$= \frac{1}{2}r^2 \sin(180° - \theta)$
$= \frac{1}{2}r^2 \sin\theta$
$=$ area $\triangle$OAC

7 **a** **i** Area $= \frac{1}{2}bc\sin A$ **ii** Area $= \frac{1}{2}ab\sin C$

b From **a**, $\frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C$

Divide both sides by $\frac{1}{2}abc$, etc.

8



In $\triangle$PQR,

$\sin(180° - C) = \dfrac{h}{b}$

$\therefore\ h = b\sin(180° - C)$

$\therefore\ h = b\sin C$

But, area $\triangle$PRS $= \frac{1}{2}ah$

$= \frac{1}{2}ab\sin C$

EXERCISE 25B.1

1 **a** $x \approx 11.05$ **b** $x \approx 11.52$ **c** $x \approx 5.19$

2 **a** $x \approx 9.43$ **b** $x \approx 11.9$ **c** $x \approx 6.37$

3 ≈ 10.2 cm^2

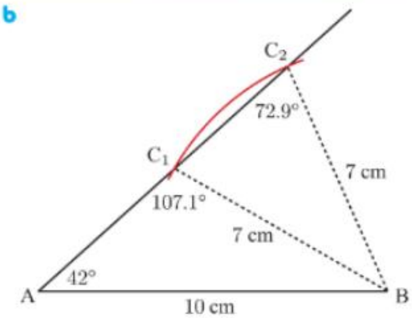
4 **a** $\theta = 59°$, $x \approx 96.7$, $y \approx 90.1$

b $\phi = 62°$, $x \approx 4.17$, $y \approx 5.62$

c $\theta = \phi = 28.5°$, $x \approx 5.30$

EXERCISE 25B.2

1 **a** $\widehat{\text{ACB}} \approx 72.9°$ or $\approx 107.1°$ **b**

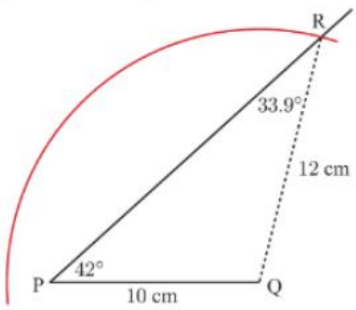


2 **a** Using the sine rule, $\widehat{\text{PRQ}} \approx 33.9°$ or $146.1°$.

In the case of $146.1°$, $42° + 146.1°$ is already $> 180°$.

$\therefore$ $\widehat{\text{PRQ}} \approx 33.9°$ only.

b



3 **a** $\theta \approx 31.4°$ **b** $\theta \approx 77.5°$ or $102.5°$

c $\theta \approx 43.6°$ or $136.4°$ **d** $\theta \approx 40.8°$

4 **a** $\widehat{\text{A}} \approx 49.1°$ **b** $\widehat{\text{B}} \approx 71.6°$ or $108.4°$ **c** $\widehat{\text{C}} \approx 44.8°$

EXERCISE 25C

1 **a** ≈ 2.66 cm **b** ≈ 9.63 m **c** ≈ 10.6 m

d ≈ 27.5 cm **e** ≈ 4.15 km **f** ≈ 15.2 m

2 **a** $\theta \approx 36.3°$ **b** $\theta \approx 53.2°$ **c** $\theta \approx 115.6°$

3 **a** $\widehat{\text{A}} \approx 51.8°$, $\widehat{\text{B}} \approx 40.0°$, $\widehat{\text{C}} \approx 88.3°$

b $\widehat{\text{P}} \approx 34.0°$, $\widehat{\text{Q}} \approx 96.6°$, $\widehat{\text{R}} \approx 49.3°$

4 **a** $\cos\theta = \dfrac{m^2 + c^2 - a^2}{2cm}$

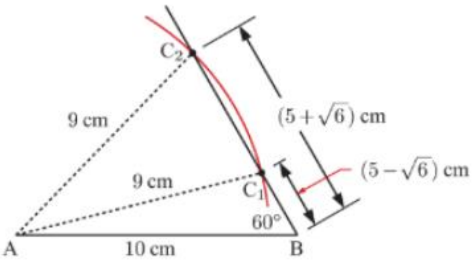
b $\cos(180° - \theta) = \dfrac{m^2 + c^2 - b^2}{2cm}$

c **Hint:** $\cos(180° - \theta) = -\cos\theta$

d **i** $x \approx 9.35$ **ii** $x \approx 4.24$

5 **b** $x = 5 \pm \sqrt{6}$

c



EXERCISE 25D

1 AC ≈ 14.3 km **2** AC ≈ 1300 m **3** $\theta \approx 13.4°$

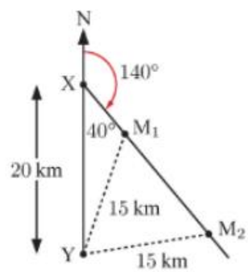
4 **a** $\widehat{\text{A}} \approx 35.69°$ **b** ≈ 4 ha

5 **a** Use the (rearranged) cosine rule to find θ.

b Once θ is known, we can use Area $= \frac{1}{2}ab\sin\theta$ to find the surface area of one side of the fin.

6 ≈ 100 m **7** ≈ 19.6 km in direction $106°$ **8** $\approx 214°$

9 **a**



b XM$_1 \approx 7.59$ km

XM$_2 \approx 23.0$ km

REVIEW SET 25A

1 **a** ≈ 77 m^2 **b** ≈ 15.9 m **2** $a \approx 19.2$

3 **a** $x \approx 223$ **b** $x \approx 99.4$

4 ≈ 337 m in the direction $\approx 138°$

5 **a** DC ≈ 10.2 m **b** BE ≈ 7.00 m **c** ≈ 82.0 m^2

REVIEW SET 25B

1 **a** $\approx 34.1°$ **b** ≈ 69.3 m^2

2 AC ≈ 10.7 m **3** $\widehat{\text{ACB}} \approx 50.5°$ or $129.5°$

4 **a** ≈ 185 m **b** $\approx 184°$

5 **a** $\cos \widehat{\text{BQP}} = \dfrac{2^2 + 5^2 - 4^2}{2 \times 2 \times 5} = \frac{13}{20}$

b $\cos(\widehat{\text{BQR}}) = \cos(180° - \widehat{\text{BQP}})$

$= -\cos\widehat{\text{BQP}}$

$= -\frac{13}{20}$

c BR$^2 = 2^2 + 5^2 - 2 \times 2 \times 5(-\frac{13}{20})$

$= 2^2 + 5^2 + 13$

etc.

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