Mathematics: applications and interpretation

Higher level

Paper 2

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Answer all questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 15]

A hospital emergency department tracks the waiting time, W, for patients before seeing a doctor. Based on past data, the waiting time follows a normal distribution with a mean of $85\ minutes$ and a standard deviation of $20\ minutes$.

(a) State $P(W < 85)$ [1]

(b) Find $P(60 < W < 85)$. [2]

25% of patients are seen by a doctor in less than k minutes.

(c) (i) Sketch a diagram of this normal distribution, shading the region that represents $P(W < k)$
(ii) Find the value of k. [4]

For a study, hospital staff randomly select five patients arriving at the emergency department. These selections are independent.

(d) Find the probability that exactly 3 of these patients are seen in less than k minutes. [3]

The hospital parking lot charges a flat rate of \$50 for any car parked for up to 3 hours. For parking durations exceeding 3 hours, an additional \$20 is charged for each extra hour or portion thereof.

For example, a car parked for 3.5 hours will be charged \$50 plus an extra \$10.

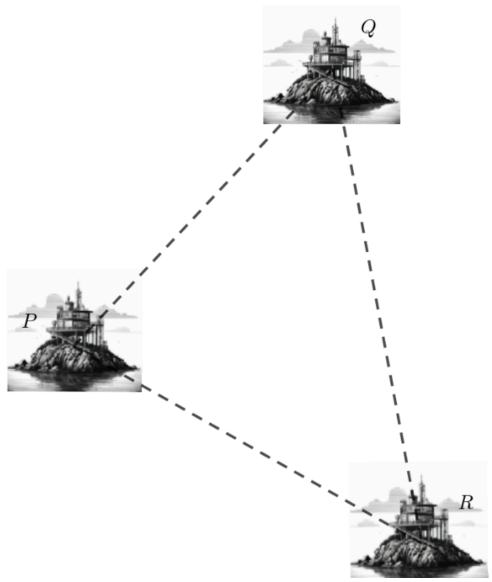
(e) Write an expression for the total parking charge, C, for a car parked for x hours, $x > 3$. [2]

(f) Calculate the total charge for a car parked for 6.2 hours. [1]

(g) Determine the total parking duration for a car that was charged \$110. [2]

2. [Maximum mark: 12]

A research station is set up on three islands in an ocean. The research posts are located at points P, Q, and R.



Using a coordinate system, the position of P is $(2.5, -1.3, 5.2)\ km$, and the position of Q is $(6.8, 3.4, 4.1) km$.

(a) (i) Find the vector $\vec{PQ}$

(ii) Hence, find PQ, the distance between P and Q. [3]

The vector $\vec{PR}$ is parallel to the vector

$$(2.2 \quad -1.63\ 0\ )$$

(b) Find the angle between $(2.2 \quad -1.63\ 0\ )$ and $\vec{PQ}$. [5]

The angle between $\vec{PQ}$ and $\vec{QR}$ is 47.5◦

(c) Find the distance PR. [4]

3. [Maximum mark: 13]

A water park tracks the number of visitors, V, per day in terms of x, the price of a single ticket (in dollars). The park uses the model:

$$V = 2400 - 30x^2$$

The maximum capacity of the water park is 600 visitors per day.

(a) Find the maximum price they could charge per ticket for the park to have 600 visitors in one day. [2]

(b) On a certain day the park sells tickets for $5 each, use the model to find:

(i) the number of visitors.

(ii) the total income from ticket sales. [2]

The cost of running the park per visitor is $4. The profit per day, P, is the total income from ticket sales minus the cost of accommodating visitors.

(c) (i) Show that, according to the model,

$$P = -30x^3 + 120x^2 + 1200x - 4800$$

(ii) Find $\frac{dP}{dx}$

(iii) Find the value of x for which $\frac{dP}{dx} = 0$.

(iv) Find the number of visitors when the profit is maximized. [9]

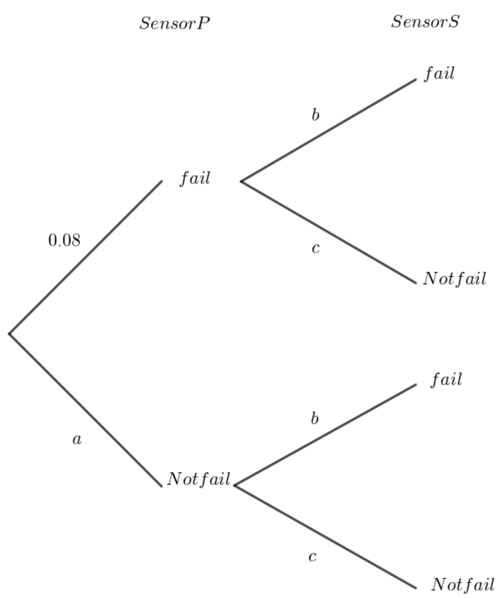
4. [Maximum mark: 12]

A new security system in a building requires at least one of two sensors to be functioning for the system to work. The system has a primary sensor (P) and a secondary sensor (S) for backup.

The manufacturer claims that the probability that sensor P fails within a month is 0.08.The probability that sensor S fails within a month is 0.25.The failures of the two sensors are independent.

If both sensors fail, the security system will go offline and need repairs. All sensors are replaced at the end of the month or when the system goes offline.

The following tree diagram shows the probabilities of sensor failures within one month, assuming the manufacturer's claims are correct.



(a) Write down the values of
(i) a (ii) b (iii) c [2]

(b) Hence find the probability that the security system goes offline within one month. [1]

(This question continues on the following page)

(Question 4 continued)

The building manager is skeptical of the manufacturer's claims, so they analyze records from the last 200 months. They classify each month based on whether no sensors failed, one sensor failed, or both sensors failed.

The data collected are shown in the table:

No sensor failed	one sensor failed	both sensors failed
110	75	15

(c) Show that the expected number of months in which no sensors failed, assuming the manufacturer's claims are correct, is 138. [2]

(d) Perform a $\chi 2$ goodness-of-fit test at the 5% significance level to test whether the manufacturer's claims are accurate, using the following hypotheses:

H_0: *The manufacturer's claims are correct.*
H_1: *The manufacturer's claims are not both correct.* [7]

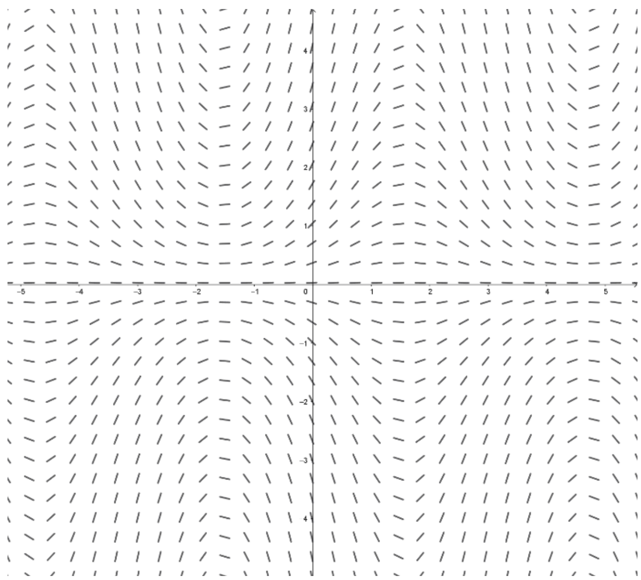
5. [Maximum mark: 13]

Consider the differential equation

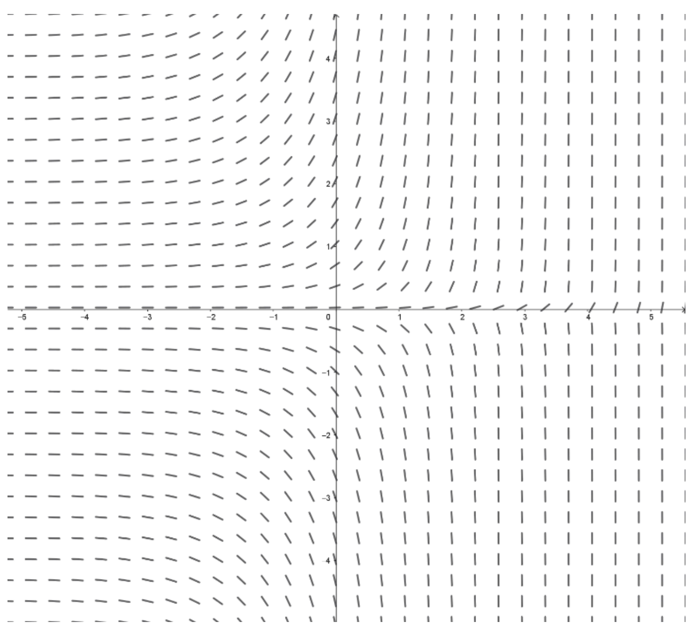
$$\frac{dy}{dx} = y\cos(x)$$

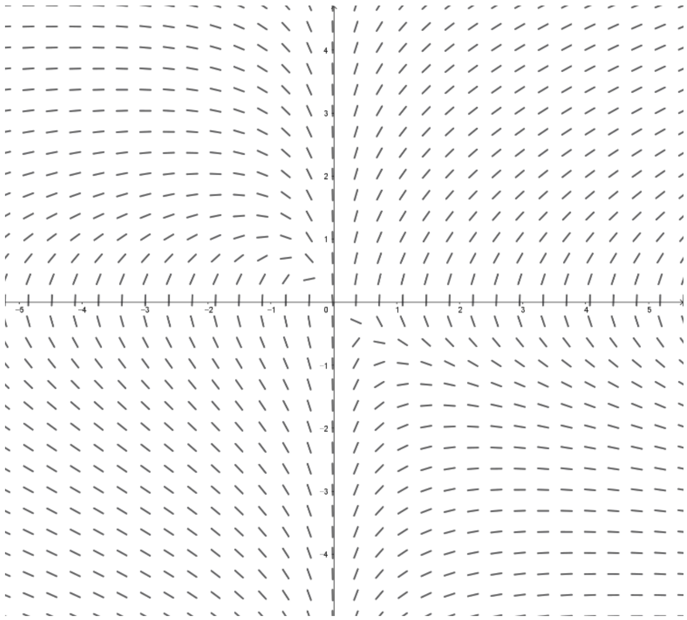
(a) Identify which of the following diagrams, A, B, or C, represents the slope field for the differential equation. Give a reason for your answer. [2]

A.



B.



C. 

It is given that, for a particular solution, $x = 0$ and $y = 1$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{dy}{dx}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{dy}{dx} = y\, cos(x)$ [2]

6. [Maximum mark: 13]

Matías is playing *League of Legends (LoL)*, where he eliminates enemy champions and jungle monsters. The number of enemy champions he eliminates per minute follows a Poisson distribution with a mean of 2.1.The number of jungle monsters he eliminates per minute follows a Poisson distribution with a mean of 3.5.Each elimination occurs independently of the others.

(a) Find the probability that Matías eliminates

(i) at most 12 jungle monsters in 3 minutes.

(ii) a total of more than 8 enemy champions and jungle monsters in one minute. [5]

Every champion eliminated earns Matías 4 gold coins, and every jungle monster earns 2 gold coins. Let G be the total number of gold coins Matías earns in one minute.

(b) Find

(i) $E(G)$, the expected value of gold coins per minute.

(ii) $Var(G)$, the variance of the gold earned per minute. [3]

(c) State one reason why the distribution of G cannot be Poisson. [1]

Matías plans to play for one hour.

(d) Use the central limit theorem to find the probability that Matías' mean gold earned per minute is greater than 15. [4]

7. [Maximum mark: 14]

The interior of a wine glass is modeled by rotating the region bounded by the curve

$$y = \frac{x^2}{4} + 1$$

and the lines $x = 0,\ y = 0$, and $y = 13$, through 2π radians about the $y - axis$. The values of x and y are measured in $centimeters$.

The wine glass is filled with wine up to a height of h cm.

(a) Find an explicit expression for the volume of wine in terms of h. [5]

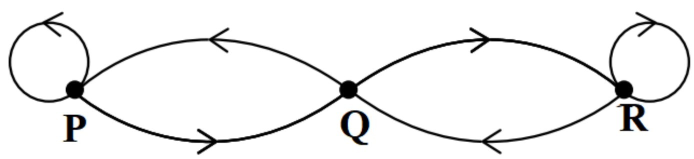
The glass is filled at a rate of $30\ cm^3 s^{-1}$

(b) Find the time taken to completely fill the glass. [2]

(c) Find the rate at which the height of the wine is increasing when $h = 12\ cm$. [7]

8. [Maximum mark: 18]

(a) Write down the adjacency matrix for the directed graph shown below. [2]



A drone patrols three charging stations, labelled P, Q, and R, with Q between P and R. The drone follows an automated movement pattern where:

- If it is at Q, it will move to P or R with a probability of 0.5 each.
- If it is at P or R, it will either stay at the same station with a probability of 0.5 or move to B with a probability of 0.5.

The movement of the drone can be represented by the graph in part (a).

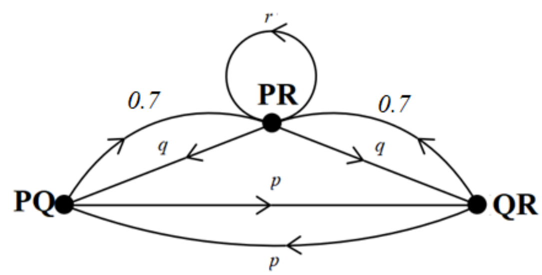
(b) Find the total number of sequences of 6 moves in which the drone starts at P and arrives at Q. [3]

(c) (i) Every possible sequence of 6 moves by the drone has the same probability of occurring. State this probability.
(ii) Use your answer to part (b) to find the probability that if the drone starts at station P , it will be at station Q after 6 moves. [3]

A second drone is added to the patrol, and their movements now follow a coordinated pattern:

- The drones will never occupy the same station.
- They always move at the same time.
- If they are on adjacent stations, the drone at Q will always move to the empty station, while the other drone will move to Q with a probability of 0.3 or stay in place with a probability of 0.7.
- If they are at opposite end stations, each will move to Q with a probability of 0.5 or stay in place with a probability of 0.5. However, if both attempt to move to Q at the same time, they will detect each other and immediately return to their previous stations.

The possible movements of the two drones can be represented by the following transition diagram, where the three vertices represent which stations are occupied.



(d) Write down the value of
(i) q.
(ii) p.
(iii) r. [3]

(e) Given that the drones start at stations P and Q, find the probability that they will be at stations Q and R after 6 moves. [4]

The drones continue moving in this pattern for a long time.

(f) Given that the time between moves is always the same, determine which station is occupied the least and the proportion of time it remains unoccupied. [3]