IB Mathematics AA HL - Prediction Exams May 2025 - Paper 1

Paper 1 | 12 questions | 120 mins | 110 marks

Section A

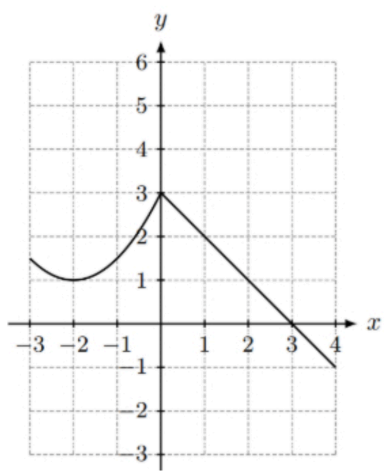
Question 1

NO CALCULATOR

Easy

[Maximum mark: 5]

The graph of $y = f(x)$ for $-3 \leq x \leq 4$ is shown in the following diagram.



(a) Write down the value of $f(2)$. [1]

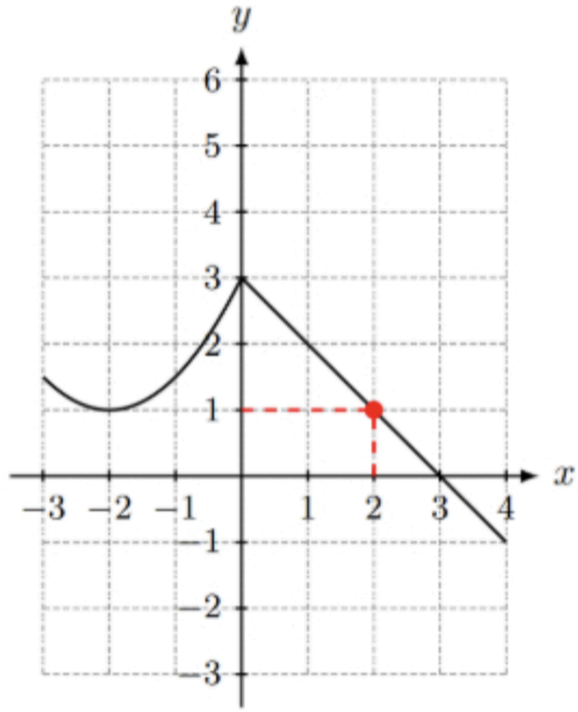
Let $g(x) = 2f(x) - 1$ for $-3 \leq x \leq 4$.

(b) On the axes above, sketch the graph of g. [2]

(c) Hence determine the value of $(g \circ f)(2)$. [1]

(d) Hence solve the equation $(f \circ g)(x) = 0$ when $x > 0$. [1]

(a) Evaluating $f(x)$ when $x = 2$



Hence $\boxed{f(2) = 1}$.

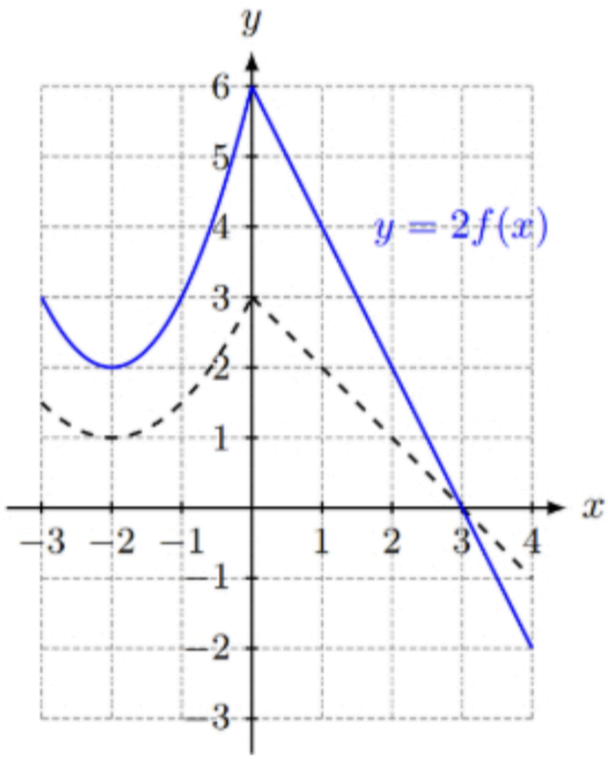
A1

(b) The function $f(x)$ is mapped to $g(x)$ by two transformations.

By considering $g(x) = 2f(x) - 1$, we can see there is a vertical stretch by a scale factor of 2 and a vertical shift down 1 unit.

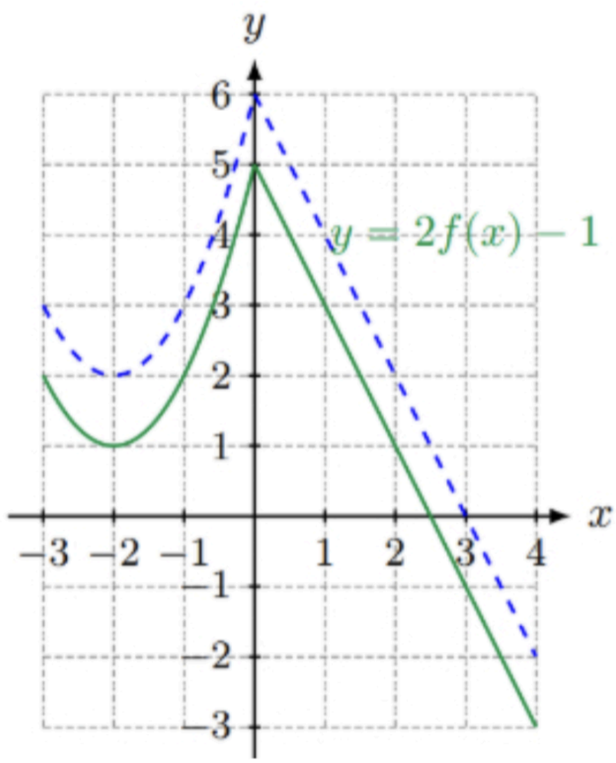
The vertical stretch ($y = 2f(x)$) is shown in blue below.

Note that the vertical distance from the y-axis of every point on the curve is doubled.



Then the vertical shift is shown in green ($y = 2f(x) - 1$) .

Note that every point on the blue curve is shifted vertically down 1 unit.



Correct local minimum at $(-2, 1)$

A1

Correct y-intercept at $(0, 5)$

A1

(c) From part (a) we know that $f(2) = 1$.

$$(g \circ f)(2) = g(f(2))$$

Hence

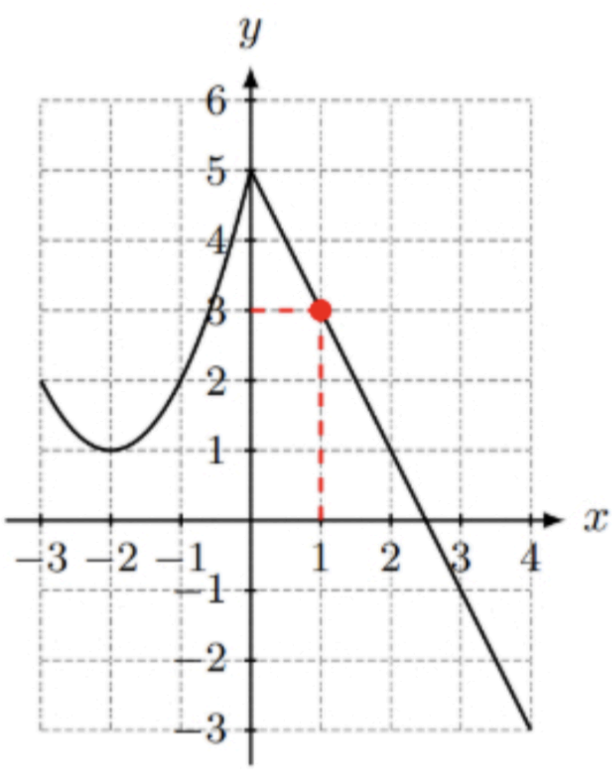
$$= g(1)$$

$$= 3$$

Hence $(g \circ f)(2) = 3$.

A1

Here is the graph of $g(x)$ showing that $g(1) = 3$.

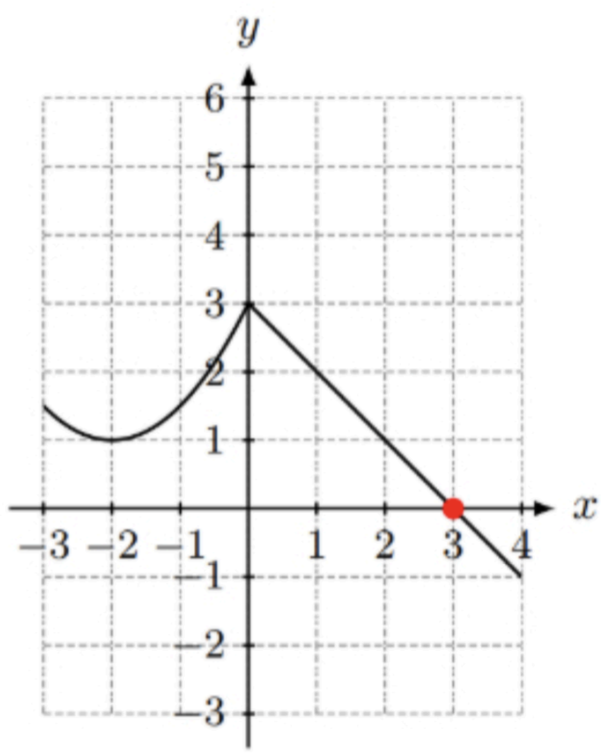


(d) We have the equation $(f \circ g)(x) = 0$. The L.H.S. is a composite function which can be rewritten

$$(f \circ g)(x) = f(g(x))$$

Here we can see that $g(x)$ is the inner function and $f(x)$ is the outer function, meaning that the output of g will be the input of f.

By considering the graph of $f(x)$ we see that when $x = 3$ then $f(x) = 0$.



This means in order to have an output of 0, the input must be 3.

$$f(g(x)) = 0$$

$$f(3) = 0$$

Therefore we need to find the value of x, where $x > 0$, such that $g(x) = 3$.

From part (c), we know that $g(1) = 3$, and we can see that this is the only possible solution when $x > 0$.

Hence if $x > 0$ and $(f \circ g)(x) = 0$ then $\boxed{x = 1}$.

A1

Question 2

NO CALCULATOR

Easy

[Maximum mark: 5]

(a) Show that $12 \log_x 2 = \dfrac{12}{\log_2 x}$. [1]

(b) Hence solve the equation $\log_2 x = 8 - 12 \log_x 2$. [4]

(a) Using the change of base formula we can write

$$\text{L.H.S.} = 12 \log_x 2$$

$$= 12 \left(\frac{\log_2 2}{\log_2 x} \right)$$

A1

As $\log_n n = 1$, we can replace $\log_2 2$ with 1 and simplify

$$= 12 \left(\frac{1}{\log_2 x} \right)$$

$$= \boxed{\frac{12}{\log_2 x}} \quad \text{...as required}$$

$$= \text{R.H.S.}$$

(b) We can replace $12\log_x 2$ with the R.H.S. of the identity from part (a)

$$\log_2 x = 8 - 12\log_x 2$$

$$\log_2 x = 8 - \frac{12}{\log_2 x} \qquad \textbf{(M1)}$$

We now multiply each side by $\log_2 x$ and then rearrange such that the R.H.S. is equal to 0

$$(\log_2 x)^2 = 8\log_2 x - 12$$

$$(\log_2 x)^2 - 8\log_2 x + 12 = 0 \qquad \textbf{(M1)}$$

This is a hidden quadratic equation. If we replace $\log_2$ with a variable, say a, we get

$$a^2 - 8a + 12 = 0$$

Let's solve this by factorising.

$$(a-2)(a-6) = 0$$

Therefore the solutions are

$$a - 2 = 0 \quad \text{and} \quad a - 6 = 0$$

$$a = 2 \qquad\qquad a = 6$$

Recall $a = \log_2 x$, therefore the solutions become

$$\log_2 x = 2 \qquad \log_2 x = 6 \qquad \textbf{A1}$$

Converting each to exponential form, we get

$$x = 2^2 \qquad x = 2^6$$

$$\boxed{x = 4} \qquad \boxed{x = 64} \qquad \textbf{A1}$$

Question 3

NO CALCULATOR Easy

[Maximum mark: 4]

When the resulting product of $3x^2 + 7x - 6$ multiplied by $ax + 1$ is divided by $x - 1$ the remainder is -4.

Find the integer a.

Although not essential, we may notice that we can factorise the quadratic expression

$$3x^2 + 7x - 6 = (3x - 2)(x + 3)$$

We will let the the product of the quadratic and linear expression be $p(x)$, hence

$$p(x) = (3x - 2)(x + 3)(ax + 1) \quad \textbf{(M1)}$$

Now let's consider dividing $p(x)$ by $x - 1$.

Using the remainder theorem we can form an equation

$$p(1) = -4 \quad \textbf{(M1)}$$

Hence substituting $x = 1$ into $p(x)$ the L.H.S. of above becomes

$$(3(1) - 2)(1 + 3)(a(1) + 1) = -4 \quad \textbf{A1}$$

We can now solve this for a

$$1 \times 4 \times (a + 1) = -4$$

$$a + 1 = -1$$

$$a = \boxed{-2} \quad \textbf{A1}$$

Question 4

NO CALCULATOR

Medium

[Maximum mark: 7]

(a) Show that $4 - 3\cos 2x = 6\sin^2 x + 1$. [1]

(b) Hence or otherwise solve $4 - 3\cos(4\theta + \frac{2\pi}{3}) - 9\sin(2\theta + \frac{\pi}{3}) = -2$ for $0 \leq \theta < \pi$. [6]

(a) In a 'show that' question we should work from the L.H.S. to the R.H.S.

$$\text{L.H.S.} = 4 - 3\cos 2x$$

The cosine double angle identity that contains only $\sin\theta$ is $\cos 2\theta = 1 - 2\sin^2\theta$.

Substituting this we obtain

$$= 4 - 3(1 - 2\sin^2 x)$$ **M1**

$$= 4 - 3 + 6\sin^2 x$$

$$= \boxed{6\sin^2 x + 1}$$ ...as required

$$= \text{R.H.S.}$$

(b) By considering a substitution $x = 2\theta + \frac{\pi}{3}$ we can write the equation in part (b) so that contains the expression from part (a).

$$4 - 3\cos(4\theta + \tfrac{2\pi}{3}) - 9\sin(2\theta + \tfrac{\pi}{3}) = -2$$

$$4 - 3\cos 2x - 9\sin x = -2$$

Hence we can substitute the R.H.S. of the equation from part (a) so that the equation is in terms of sine.

$$6\sin^2 x + 1 - 9\sin x = -2 \qquad \textbf{(M1)}$$

$$6\sin^2 x - 9\sin x + 3 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

Notice that this is a quadratic equation. We can factorise it using grouping which gives

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$2\sin^2 x - 2\sin x - \sin x + 1 = 0$$

$$2\sin x(\sin x - 1) - 1(\sin x - 1) = 0$$

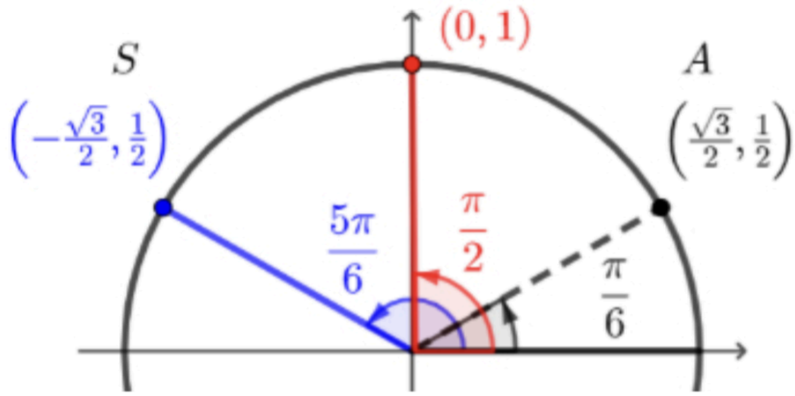
$$(2\sin x - 1)(\sin x - 1) = 0$$

Applying the null factor theorem we get

$$2\sin x - 1 = 0 \qquad \sin x - 1 = 0$$

$$\sin x = \tfrac{1}{2} \qquad \sin x = 1 \qquad \textbf{A1}$$

Using the unit circle, we know that $\sin x = 1$ when $x = \frac{\pi}{2}$ and $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. This is represented in the diagram below.



Hence the answers for x are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$

(A1)

Recall that we are solving for θ and we used a substitution.

Hence we can find values of θ that satisfy the equation

$$2\theta + \frac{\pi}{3} = \frac{\pi}{6}$$
$$\theta = -\frac{\pi}{12}$$

The first value we have found is outside of the given domain $(0 \leq \theta < \pi)$ hence we need to add (or subtract!) 2π to the value we found from the unit circle to obtain other values that could be in the domain

$$2\theta + \frac{\pi}{3} = \frac{\pi}{6} + 2\pi$$
$$= \frac{13\pi}{6}$$
$$2\theta = \frac{13\pi}{6} - \frac{2\pi}{6}$$
$$\theta = \boxed{\frac{11\pi}{12}}$$

A1

This value is now in the given domain.

Let's find the remaining values

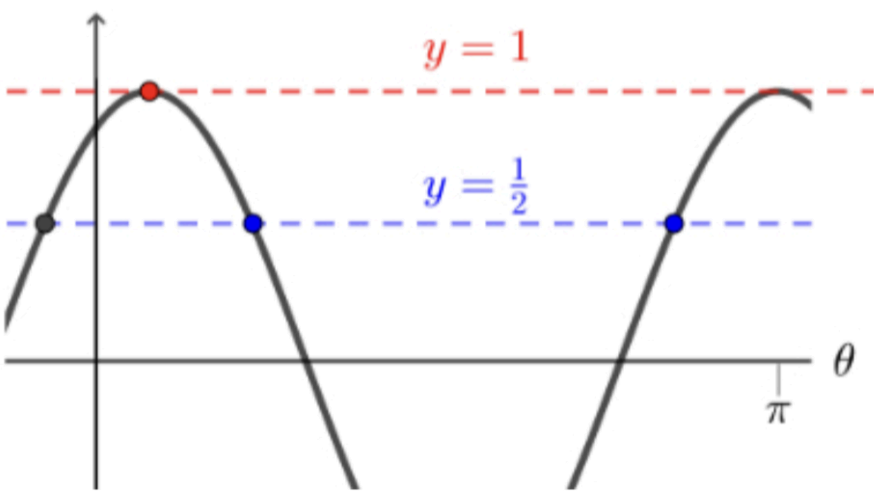
$$2\theta + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\theta = \boxed{\frac{\pi}{12}} \qquad \textbf{A1}$$

$$2\theta + \frac{\pi}{3} = \frac{5\pi}{6}$$

$$\theta = \boxed{\frac{\pi}{4}} \qquad \textbf{A1}$$

We have found three solutions. Although not required by the question we can view the values on a graph.



Notice three solutions, one in red and two in blue.

The solution identified with the black dot is the value we found which was not in the desired domain.

Question 5

NO CALCULATOR

Medium

[Maximum mark: 5]

Consider $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) + 3$ and $g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$.

The function f is mapped onto g by three transformations.

(a) Fully describe each of the transformations and the order in which they must be applied. [3]

A new function is such that $h(x) = g(x) + k$ where $k \in \mathbb{R}$.

(b) Find the minimum value of k such that $h(x) \geq 0$ for all x. [2]

(a) By considering the differences between the functions we can work out the transformations needed to map $f(x)$ to $g(x)$.

First we notice that the inner functions are different.

$$f(x) = 2\cos\left(x - \frac{\pi}{2}\right) + 3$$
$$g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$$

If we add π units to the inner function of $f(x)$, it will equal the inner function of $g(x)$.

A1

Hence

$$f(x+\pi) = 2\cos\left(x - \frac{\pi}{2} + \pi\right) + 3$$
$$f(x+\pi) = 2\cos\left(x + \frac{\pi}{2}\right) + 3$$

Comparing the function $f(x+\pi)$ to $g(x)$, we notice that the coefficient of cosine has been doubled.

$$f(x+\pi) = 2\cos\left(x + \frac{\pi}{2}\right) + 3$$
$$g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$$

Hence, if we multiply $f(x+\pi)$ by 2 we would have

A1

$$2f(x+\pi) = 2(2\cos\left(x+\frac{\pi}{2}\right)+3)$$
$$2f(x+\pi) = 4\cos\left(x+\frac{\pi}{2}\right)+6$$

If we continue comparing the function $2f(x+\pi)$ to $g(x)$, we see there is a difference of 4 units.

$$2f(x+\pi) = 4\cos\left(x+\frac{\pi}{2}\right)+6$$
$$g(x) = 4\cos\left(x+\frac{\pi}{2}\right)+2$$

Hence, if we apply a vertical shift of -4 to $2f(x+\pi)$, we will obtain $g(x)$.

A1

$$\begin{aligned} 2f(x+\pi)-4 &= 4\cos\left(x+\frac{\pi}{2}\right)+6-4 \\ &= 4\cos\left(x+\frac{\pi}{2}\right)+2 \\ &= g(x) \end{aligned}$$

Summarising the 3 transformations, there is

a horizontal shift **left** of π units, followed by a vertical stretch by a scale factor of 2 , followed by a vertical shift of -4 units.

Note: The horizontal shift could also come after the vertical transformations, however the two vertical transformations must be applied in the order given.

(b) In order to apply a vertical translation such that $g(x) > 0$ for all x, we need to know the minimum value of $g(x)$.

The minimum value of cosine is -1. Hence we can determine the minimum of $g(x)$

(M1)

$$\begin{aligned} g(x) &= 4\cos\left(x + \frac{\pi}{2}\right) + 2 \\ &= 4(-1) + 2 \\ &= -2 \end{aligned}$$

Hence, we need to translate the graph vertically upwards at least 2 units.

Therefore, the minimum value is $\boxed{k = 2}$.

A1

Question 6

NO CALCULATOR

Medium

[Maximum mark: 7]

(a) (i) Consider the following equation $2\binom{n}{r} = \binom{n}{r+1}$.

Show that it can be written as $3r + 2 = n$.

(ii) Now consider the following equation $7\binom{n}{r-1} = 2\binom{n}{r}$.

Show that it can be written as $9r - 2 = 2n$. [4]

Consider the expansion

$$(1 + x)^n = 1 + a_1x + ... + a_{k-1}x^{k-1} + a_kx^k + a_{k+1}x^{k+1} + ... + x^n$$

Where $a_i \in \mathbb{Q}$ and $k \in \mathbb{Z}$.

The coefficients of three consecutive terms of the expansion are such that

$$7 \times a_{k-1} = 2 \times a_k \quad \text{and} \quad 14 \times a_k = 7 \times a_{k+1}$$

(b) Find n. [3]

(a) (i) Using the combinations formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ we can rewrite both the LHS and the RHS of the equation

$$2\left(\frac{\cancel{n!}}{r!(n-r)!}\right) = \frac{\cancel{n!}}{(r+1)!(n-(r+1))!}$$

Now we can simplify and rearrange the equation

$$\frac{2(r+1)!}{r!} = \frac{(n-r)!}{(n-r-1))!}$$

Rewrite the numerators using the concept $n! = n(n-1)!$

$$\frac{2(r+1)r!}{r!} = \frac{(n-r)(n-r-1)!}{(n-r-1))!}$$ **M1**

We can cancel out the factorial terms

$$\frac{2(r+1)\cancel{r!}}{\cancel{r!}} = \frac{(n-r)\cancel{(n-r-1)!}}{\cancel{(n-r-1))!}}$$

$$2(r+1) = n-r$$

$$\boxed{3r+2=n}$$ **A1**

As required.

(ii) Rewrite the equation using the combinations formula and simplify in a similar way to part (a)

$$7\left(\frac{n!}{(r-1)!(n-(r-1))!}\right) = 2\left(\frac{n!}{r!(n-r)!}\right)$$

$$\frac{7}{(r-1)!(n-r+1)!} = \frac{2}{r!(n-r)!}$$

$$\frac{7r!}{(r-1)!} = \frac{2(n-r+1)!}{(n-r)!}$$

$$\frac{7r\cancel{(r-1)!}}{\cancel{(r-1)!}} = \frac{2(n-r+1)\cancel{(n-r)!}}{\cancel{(n-r)!}}$$ **M1**

$$7r = 2n-2r+2$$

$$\boxed{9r-2=2n}$$ **A1**

As required.

(b) We can begin by expanding $(1+x)^n$, using the binomial theorem, in terms of n and r.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots$$
$$+ \binom{n}{r-1}x^{r-1} + \binom{n}{r}x^r + \binom{n}{r+1}x^{r+1} + \ldots \qquad \textbf{(M1)}$$
$$+ x^n$$

The terms in red represent any three consecutive terms in the expansion.

We are told in the question the way in which the coefficients of three consecutive terms are related.

$$(1+x)^n = 1 + a_1x + \ldots + a_{k-1}x^{k-1} + a_kx^k + a_{k+1}x^{k+1} + \ldots + x^n$$

Hence we can use the information given to write the following two equations in terms of binomial coefficients

$$7 \times a_{k-1} = 2 \times a_k \qquad 14 \times a_k = 7 \times a_{k+1}$$
$$7\binom{n}{r-1} = 2\binom{n}{r} \qquad 14\binom{n}{r} = 7\binom{n}{r+1}$$
$$2\binom{n}{r} = \binom{n}{r+1}$$

In part (a) we already rewrote these equations without the combination notation, lets call them [1] and [2]. We can now solve them simultaneously and find n and r.

M1

$$3r + 2 = n \qquad [1]$$
$$9r - 2 = 2n \qquad [2]$$

Multiply [1] by 3 and subtract equation [2] from this result

$$9r - 9r + 6 - (-2) = 3n - 2n$$

$$\boxed{n = 8}$$ **A1**

Question 7

NO CALCULATOR

Medium

[Maximum mark: 8]

Consider the function $f(x) = \frac{\cos(mx) - \cos(nx)}{x^2}$ where $m, n \in \mathbb{R}$.

The function has a maximum value of $f_{\max}$ and it is known that $f_{\max} = \lim_{x \to 0} f(x)$.

(a) Show that $f_{\max} = \frac{n^2 - m^2}{2}$. [6]

It is now known that $m > 0$ and $n = 4\sqrt{m}$.

(b) Hence, using these conditions find the largest possible value of $f_{\max}$. [2]

(a) To find $f_{\max}$ we need to evaluate $f(x)$ as $x \to 0$.

Hence we can write

$$f_{\max} = \lim_{x \to 0} \left(\frac{\cos(mx) - \cos(nx)}{x^2} \right)$$

We can begin by attempting to evaluate the limit using direct substitution of $x = 0$, recall $\cos 0 = 1$.

$$f_{\max} = \frac{\cos(0) - \cos(0)}{(0)^2} \qquad \textbf{(M1)}$$

$$= \frac{1 - 1}{(0)^2}$$

$$= \frac{0}{0}$$

This is an indeterminate form. Hence we can use L'Hôpital's Rule to attempt to find the limit.

R1

Using the standard derivative for cosine and the power rule we can differentiate the numerator and denominator to get

$$\begin{aligned} f_{\max} &= \lim_{x\to 0}\left(\frac{-m\sin(mx)-(-n\sin(nx))}{2x}\right) \\ &= \lim_{x\to 0}\left(\frac{-m\sin(mx)+n\sin(nx)}{2x}\right) \end{aligned}$$

M1

Again we can attempt to evaluate the limit by direct substitution of $x = 0$

$$\begin{aligned} f_{\max} &= \lim_{x\to 0}\left(\frac{-m\sin(0)+n\sin(0))}{2(0)}\right) \\ &= \frac{0}{0} \end{aligned}$$

A1

Again we have an indeterminate form. Let's try L'Hôpital's Rule one more time.

$$f_{\max} = \lim_{x\to 0}\left(\frac{-m^2\cos(mx)+n^2\cos(nx)}{2}\right)$$

A1

As $\cos 0 = 1$ we get

M1

$$\begin{aligned} &= \frac{-m^2+n^2}{2} \\ &= \boxed{\frac{n^2-m^2}{2}} \end{aligned}$$

...as required

(b) We can take the result from part (b) and replace n with $4\sqrt{m}$, this gives $f_{\max}$ in terms of m

$$f_{\max} = \frac{(4\sqrt{m})^2 - m^2}{2}$$

This is a quadratic function in m.

By completing the square we can write $f_{\max}$ in vertex form and find the coordinates of the maximum point.

Note, although we are told to look for a maximum value we can also see that as the coefficient of m^2 is negative then the parabola will be concave down hence the vertex will be a maximum point.

When completing the square it is often easier to work with a positive squared term hence we will factorise by $-\frac{1}{2}$ which gives

$$f_{\max} = -\frac{1}{2}(m^2 - 16m)$$

Now we can complete the square

$$= -\frac{1}{2}\left[(m-8)^2 - 64\right] \quad \textbf{(M1)}$$

$$= -\frac{1}{2}(m-8)^2 + 32$$

Hence the largest value of $f_{\max}$ is $\boxed{32}$

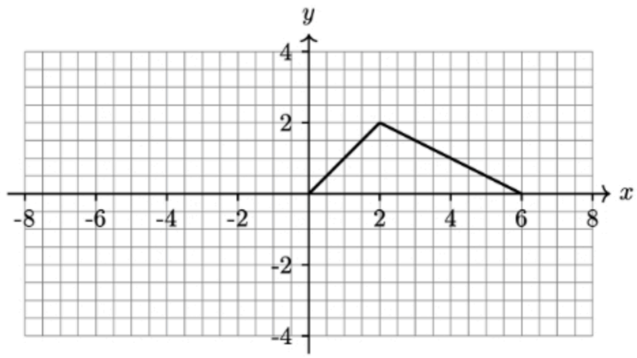
A1

Question 8

NO CALCULATOR   Medium

[Maximum mark: 8]

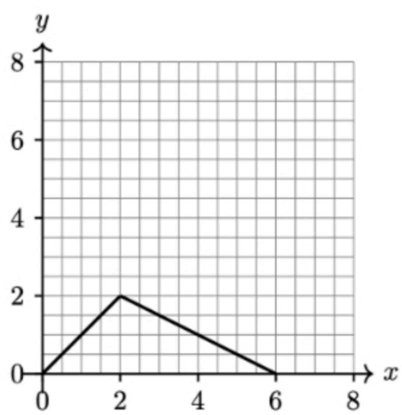
The graph of $y = f(x)$ for $0 \le x \le 6$ is shown below



The odd function $h(x)$ has the domain $-6 \le x \le 6$ and $h(x) = 2f(x)$ for $0 \le x \le 6$.

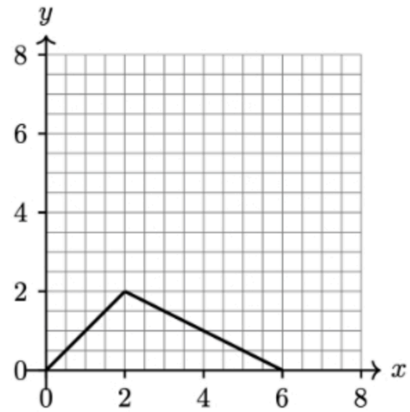
(a) Sketch $h(x)$ on the axes above. [2]

$f(x)$ is shown again below.



(b) Sketch the graph of $y = [f(x)]^2$ on the axes above. [3]

$f(x)$ is shown one more time below.



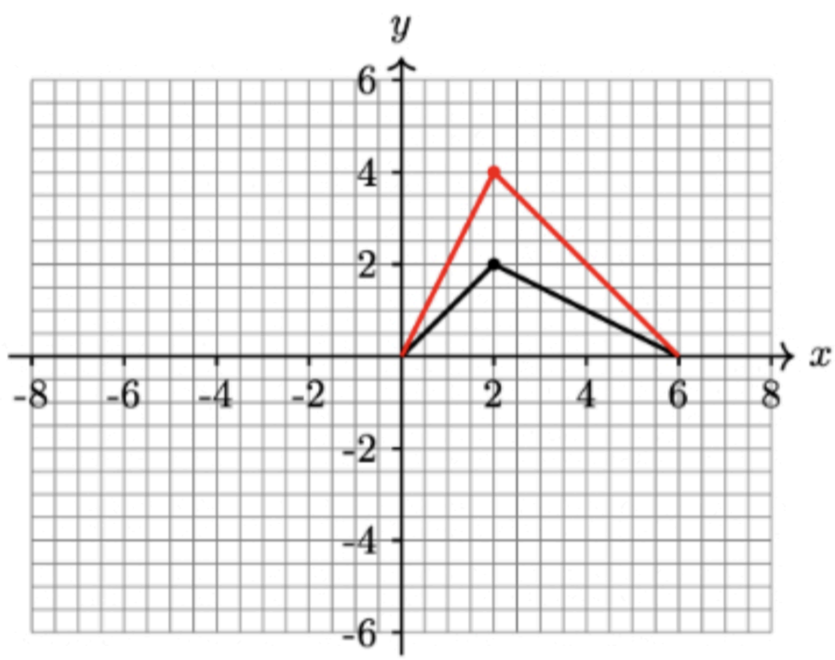
(c) Sketch the graph of $y = \frac{1}{f(x)}$ on the axes above. [3]

(a) Let's start with graphing $y = 2f(x)$.

This is a vertical stretch by a scale factor of 2.

The distance of all $y-$coordinates from the $x-$axis are multiplied by two.

The key point $(2, 2)$ will become $(2, 4)$. This is shown in the graph below in red.



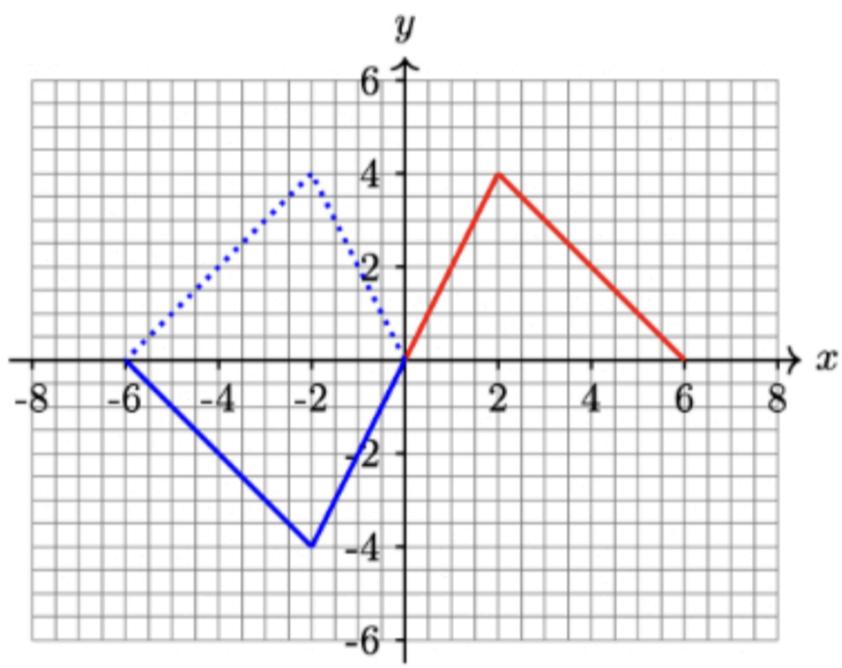
The definition of an odd function is $f(-x) = -f(x)$. This transformation represents rotational symmetry of order 2 about the origin which is equivalent to performing two reflections, one in each axis, in any order.

Therefore, to complete the rest of the graph of $h(x)$, we can either take the graph of $y = 2f(x)$ and rotate it $180°$ about the origin or perform two reflections, one in each axis. Both result in the same final position.

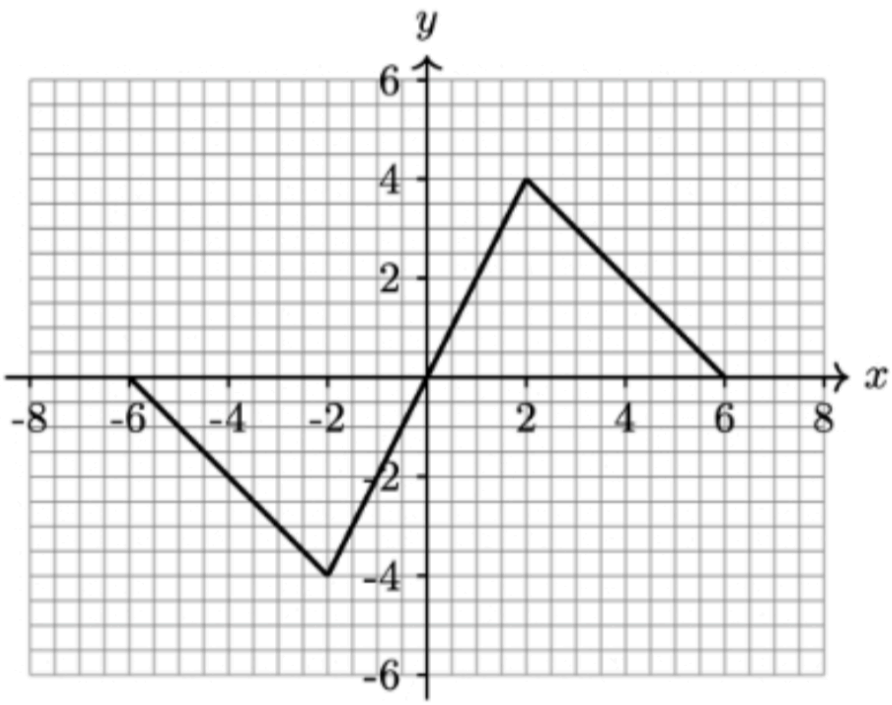
If we consider the reflection approach. First, we perform $f(x) \to f(-x)$, this is a reflection in the $y-$axis. This is the dotted blue line shown in the following graph.

Then we consider $f(-x) \to -f(-x)$, this is a reflection in the $x-$axis.

This fully transformed part is shown in solid blue in the following graph.



The final graph of $y = h(x)$ is shown below



Correct points at $(-2, -4)$ and $(-6, 0)$ **A1**

Correct transformation(s) **A1**

(b) First, let's look at the section of the graph $0 \leq x \leq 2$.

We can use the properties of the transformation $[f(x)]^2$ to identify some key points.

We know that if $f(x) = 0$ then $[f(x)]^2 = 0$.

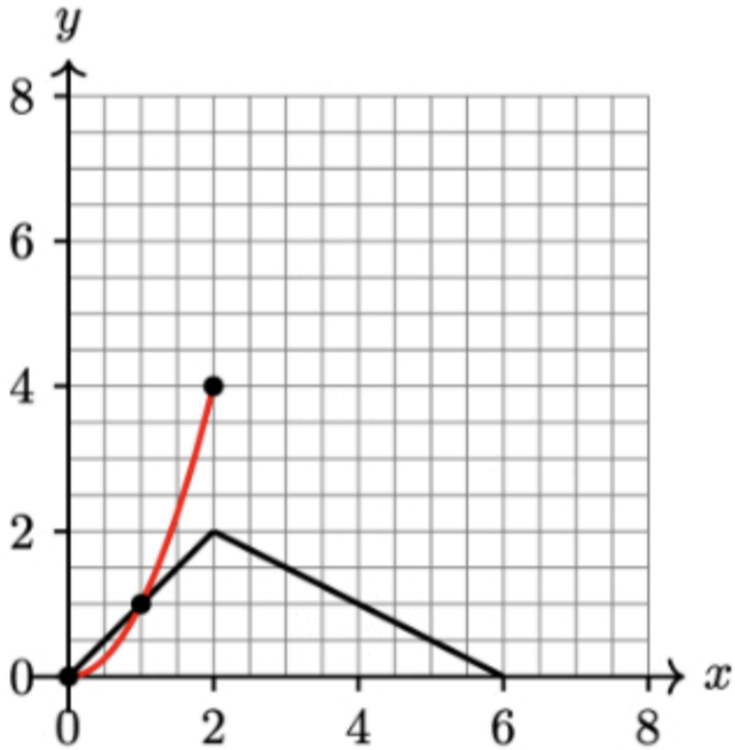
We also know that if $f(x) = 1$ then $[f(x)]^2 = 1$.

It is important to also realise that if $-1 < f(x) < 1$ then $[f(x)]^2 < f(x)$.

Therefore between $0 < x < 1$ then $[f(x)]^2 < f(x)$.

Using the end point of the first part of the curve we can see that $[f(2)]^2 = 4$.

Using the key points $(0, 0)$, $(1, 1)$ and $(2, 4)$ we can sketch the transformed curve, shown in red



Now, let's look at the section of the graph $2 \leq x \leq 6$.

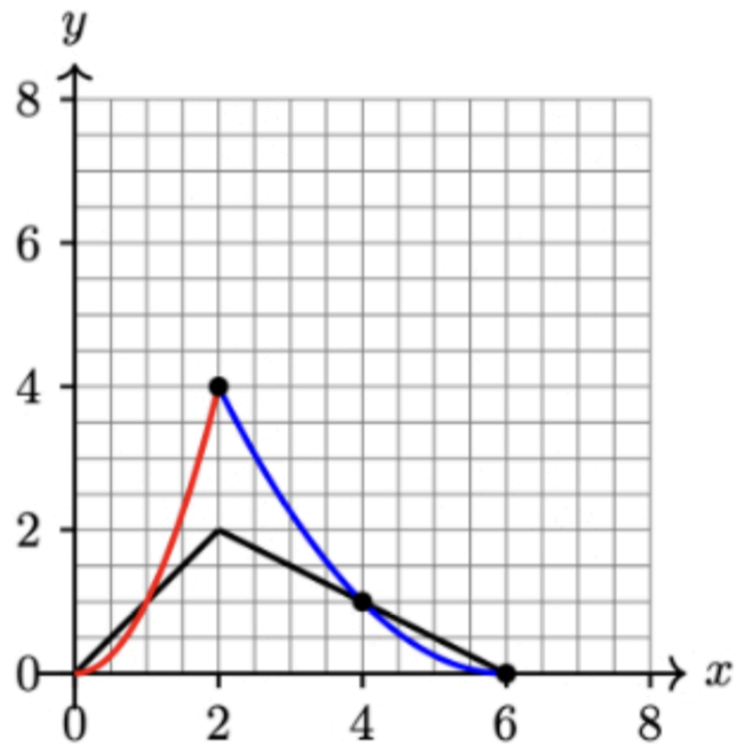
We know the second section will start at $(2, 4)$ and finish at $(6, 0)$.

Using the property if $f(x) = 1$ then $[f(x)]^2 = 1$ we know the curve will intersect the original function at $f(x) = 1$.

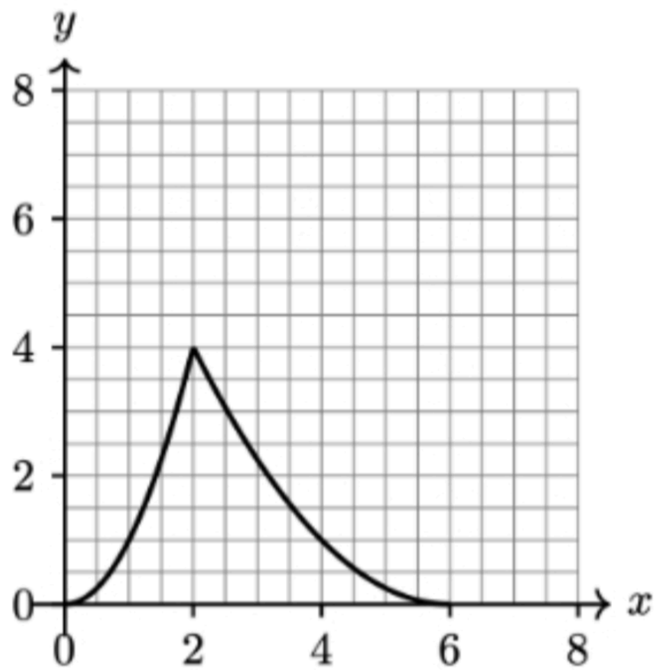
Which we can see is at $(4, 1)$.

It is important to clearly show on the graph that $[f(x)]^2 < f(x)$ when $0 < f(x) < 1$.

With this in mind, using the key points $(2, 4)$, $(1, 4)$ and $(6, 0)$ we can sketch the second part, shown in blue



The final graph of $y = [f(x)]^2$ is shown below



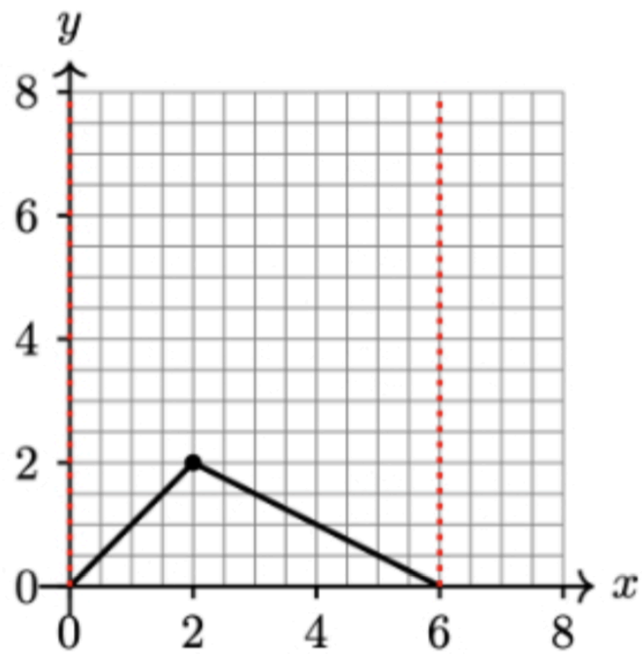
Correct shape **A1**

$f(x) = [f(x)]^2$ when $f(x) = 0, 1$ **A1**

$[f(x)]^2$ is less than $f(x)$ when $0 < f(x) < 1$ **A1**

(c) Let's start by drawing the vertical asymptotes.

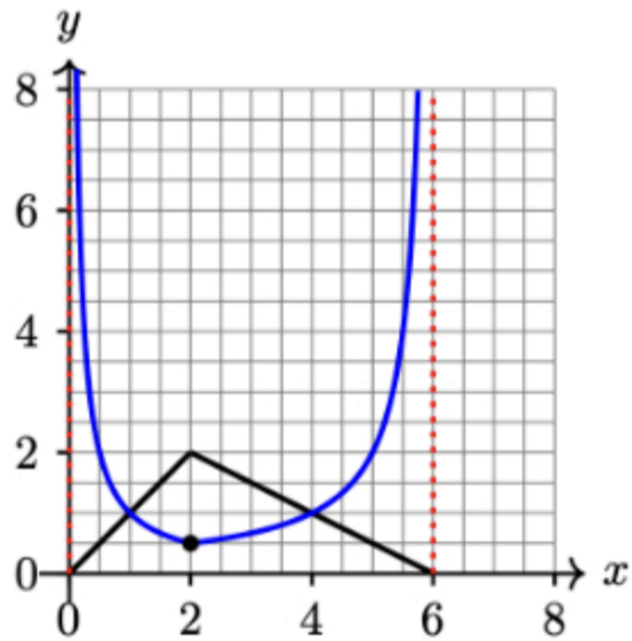
They will occur at the x-intercepts of $f(x)$ which are $x = 0$ and $x = 6$. Shown below as the dotted red vertical lines



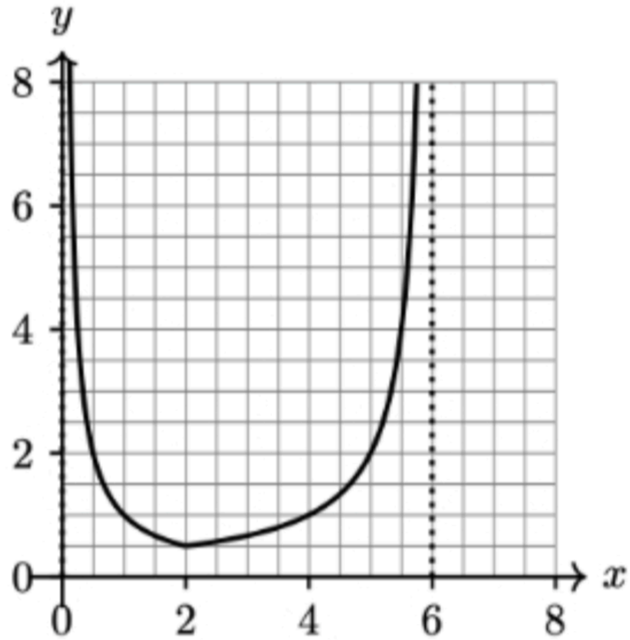
Let's consider the key point $(2, 2)$, after the transformation this point will become $(2, \frac{1}{2})$.

Considering the asymptotes, we can see that as $x \to 0$ and $x \to 6$ then $f(x) \to 0$ (from the positive direction) therefore $\frac{1}{f(x)} \to \infty^{+}$.

Hence we get the graph below in blue.



The final graph of $y = \frac{1}{f(x)}$ is shown below



Correct asymptotes **A1**

Correct shape **A1**

Correct minimum point at $(2, \frac{1}{2})$ **A1**

Question 9

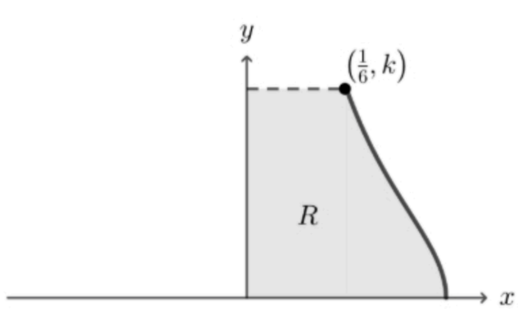
NO CALCULATOR

Hard

[Maximum mark: 8]

The function f is defined by $f(x) = \frac{\sqrt{1-9x^2}}{2x}$ for $x \geq 0$.

The region R is bounded by the curves $y = f(x)$ and the lines $x = 0$ and $y = 0$ as shown in the following diagram.



The shape of a solid clay sculpture can be modeled by rotating the region R though 2π radians about the $y-$axis.

The top edge of the sculpture has coordinates of $\left(\frac{1}{6}, k\right)$.

The volume of clay used to make the sculpture is $a\pi^2$ units^2. Where $a \in \mathbb{Q}$.

Find a.

In this question we are rotating region R about the $y-$axis.

According to the formula booklet rotation about the $y-$axis requires the following formula

$$V = \pi \int_a^b x^2 \, \mathrm{d}y$$

This means we need to find $x = f(y)$ hence we need to rewrite $y = f(x)$ to make x the subject.

M1

$$y = f(x)$$

$$y = \frac{\sqrt{1 - 9x^2}}{2x}$$

Let's begin by squaring both sides and then rewriting the expression

$$y^2 = \frac{1 - 9x^2}{4x^2}$$

$$4x^2y^2 = 1 - 9x^2$$

$$1 = 4x^2y^2 + 9x^2$$

Now we can factorise the R.H.S. and make x the subject

$$1 = x^2(4y^2 + 9)$$

$$x^2 = \frac{1}{4y^2 + 9}$$

$$\begin{aligned} x &= \sqrt{\frac{1}{4y^2+9}} \\ &= f(y) \end{aligned}$$ **A1**

Now substitute this expression into the formula mentioned to get

$$\begin{aligned} V &= \pi \int_a^b \left(\sqrt{\frac{1}{4y^2+9}}\right)^2 \mathrm{d}y \\ &= \pi \int_a^b \frac{1}{4y^2+9} \, \mathrm{d}y \end{aligned}$$

Next let's consider the limits.

We have been given the $x-$coordinate of the edge of the vase but, as we are integrating up the $y-$axis, we need to find k, the $y-$coordinate.

Hence, we find this by evaluating f at $x = \frac{1}{6}$

$$k = f(\tfrac{1}{6})$$ **M1**

$$\begin{aligned} &= \frac{\sqrt{1 - 9\left(\frac{1}{6}\right)^2}}{2\left(\frac{1}{6}\right)} \\ &= \frac{\sqrt{1 - 9 \times \frac{1}{36}}}{\frac{1}{3}} \\ &= 3\sqrt{1 - \tfrac{1}{4}} \\ &= 3\sqrt{\tfrac{3}{4}} \\ &= \tfrac{3}{2}\sqrt{3} \end{aligned}$$ **A1**

Hence our expression for the volume with the correct limits is

$$V = \pi \int_0^{\frac{3}{2}\sqrt{3}} \frac{1}{4y^2 + 9} \, \mathrm{d}y$$

We now have an expression, with limits in terms of y, which we are integrating with respect to y.

Notice that the integral looks similar to the following standard integral in the formula booklet

$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \arctan \frac{x}{a} + C$$

However, there are some differences, the first one is that our integral is a function of y but, as we are integrating with respect to y then we can ignore this difference.

The other difference is we have a composite function for y. As we need to be more careful here, we should use a substitution to make sure we perform the integral correctly.

$$V = \pi \int \frac{1}{3^2 + (2y)^2} \, \mathrm{d}y$$

Let $u = 2y$ therefore $\frac{\mathrm{d}u}{\mathrm{d}y} = 2$

Ignoring the limits for now and using $\mathrm{d}y = \frac{1}{2}\mathrm{d}u$ we can rewrite the integral in terms of u to get

$$V = \pi \int \frac{1}{3^2 + u^2} \times \frac{1}{2}\mathrm{d}u \qquad \textbf{M1}$$

$$= \frac{\pi}{2} \int \frac{1}{3^2 + u^2}\,\mathrm{d}u$$

Using the standard integral with $a = 3$ we get

$$= \frac{\pi}{2} \times \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C$$

Writing our answer back in terms of y we get

$$= \frac{\pi}{6}\arctan\left(\frac{2y}{3}\right) + C \qquad \textbf{A1}$$

Let's now find the volume by re-introducing the limits.

$$= \frac{\pi}{6}\left[\arctan\left(\frac{2y}{3}\right)\right]_0^{\frac{3}{2}\sqrt{3}}$$

Recall, the constant of C will cancel out and we can factorise out any common factors.

Substituting in the upper and lower limits we get

$$= \frac{\pi}{6}\left[\arctan\left(\frac{2(\frac{3}{2}\sqrt{3})}{3}\right) - \arctan\left(\frac{2(0)}{3}\right)\right] \qquad \textbf{M1}$$

$$= \frac{\pi}{6}\arctan\sqrt{3}$$

$$= \frac{\pi}{6} \times \frac{\pi}{3}$$

$$= \frac{\pi^2}{18}\ \text{units}^2$$

Therefore $\boxed{a = \frac{1}{18}}$

A1

Section B

Question 10

NO CALCULATOR   Medium

[Maximum mark: 23]

Consider the function $f(x) = \dfrac{\cos x}{2 + \sin x}$ for $-\pi \leq x \leq \pi$.

(a) Evaluate $f(0)$. [1]

(b) Find all possible values of a if $f(a) = 0$. [2]

(c) (i) Show that $f'(x) = -\dfrac{2\sin x + 1}{(2 + \sin x)^2}$.

(ii) Hence find the $x-$coordinates of any stationary points of f. [7]

(d) Given that $f''(x) = -\dfrac{2\cos x(1 - \sin x)}{(2 + \sin x)^3}$ find the nature of any stationary points of f. [5]

(e) Hence sketch the graph of f, clearly showing the values of the axes intercepts and the $x-$coordinates of any stationary points. [3]

The function f is positive and decreasing in the region $s < x < t$.

The area enclosed by f and the $x-$axis from $x = s$ to $x = t$ is $\ln c$ where $c \in \mathbb{Z}$.

(f) Find c. [5]

(a) To evaluate the function we substitute in $x = 0$ which gives

$$f(0) = \frac{\cos 0}{2 + \sin 0}$$

Using the fact that $\cos 0 = 1$ and $\sin 0 = 0$ we get

$$= \frac{1}{2 + 0}$$

$$= \boxed{\frac{1}{2}}$$

A1

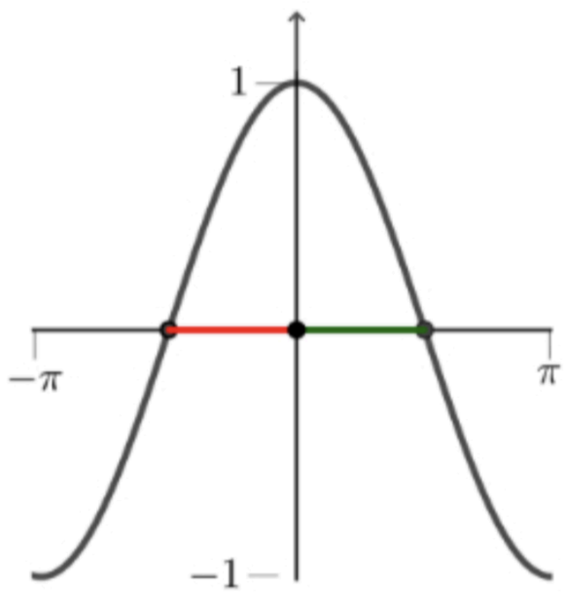
(b) Using $x = a$ and the function f we can form an equation

$$f(a) = \frac{\cos a}{2 + \sin a}$$
$$= 0$$

Only the numerator can provide solutions to the equation, hence we get

$$\cos a = 0$$ **(M1)**

Recall, the domain of f is $-\pi \leq x \leq \pi$, we can use a sketch of $\cos x$ to find all solutions of a in that domain



We know that the principal angle of $\cos \frac{\pi}{2}$ is 0, which is shown in green, and using the symmetries of the cosine curve we can see that $-\frac{\pi}{2}$, shown in red, is also a solution.

Hence $a = \pm\frac{\pi}{2}$

A1

(c) (i) To differentiate f we need to use the quotient rule.

(M1)

For this we need the derivative of both the numerator and the denominator

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(2+\sin x) = \cos x$$

Now we can use these results and the quotient rule to form an expression for $f'(x)$

$$f'(x) = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2}$$ **A1A1**

$$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - (\sin^2 x + \cos^2 x)}{(2+\sin x)^2}$$

Notice we can use a trigonometric identity to simplify the numerator

M1

$$= \frac{-2\sin x - 1}{(2+\sin x)^2}$$

$$= -\frac{2\sin x + 1}{(2+\sin x)^2}$$ ... as required.

(ii) To find any stationary points we must solve $f'(x) = 0$, hence we can form an equation using the result from part (c)(i)

$$-\frac{2\sin x + 1}{(2+\sin x)^2} = 0 \qquad \textbf{(M1)}$$

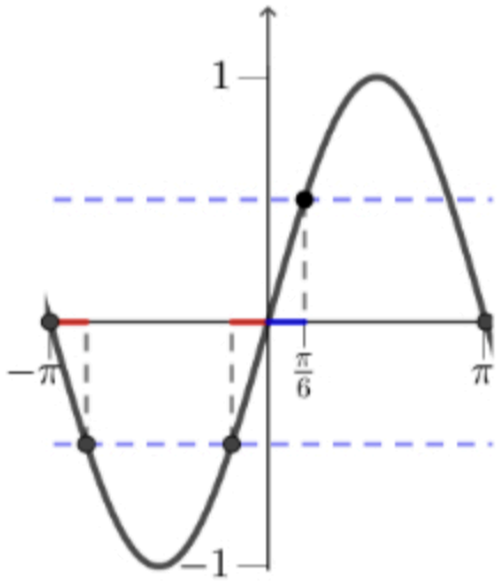
We only need to consider the numerator, hence we get

$$-2\sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \qquad \textbf{(A1)}$$

The principal solution (this means the solution in the first quadrant) is $x = \frac{\pi}{6}$.

However, if we make a sketch of $\sin x$ with the same domain as f we can see that the solutions to $\sin x = -\frac{1}{2}$ are both negative



Due to the symmetries of the graph we can see that the two angles, x_1 and x_2, marked in red are

$$x_1 = -\pi + \frac{\pi}{6} \qquad x_2 = 0 - \frac{\pi}{6}$$
$$= -\frac{5\pi}{6} \qquad x_2 = -\frac{\pi}{6}$$

Hence there are two stationary points with $x-$coordinates of $\boxed{x = -\frac{5\pi}{6}}$ and $\boxed{x = -\frac{\pi}{6}}$.

A1

(d) We can use the second derivative to determine the nature of the stationary points.

If $f''(x) > 0$ the point is a minimum and if $f''(x) < 0$ it is a maximum.

(M1)

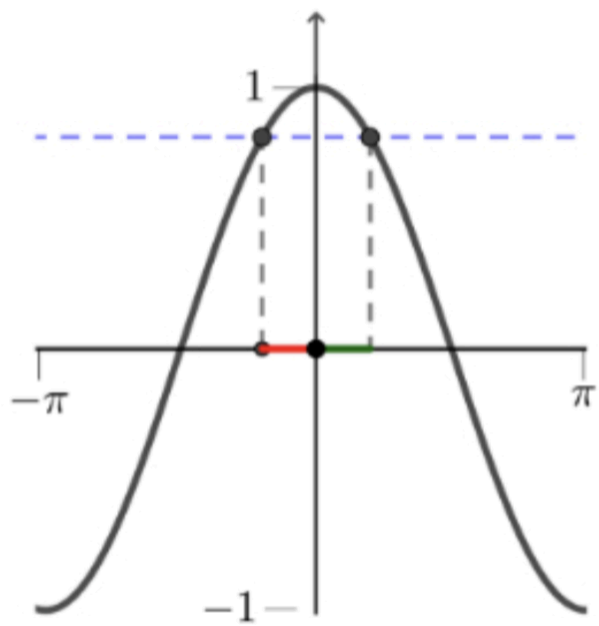
Let's first test the point $x = -\frac{\pi}{6}$

$$f''(-\tfrac{\pi}{6}) = -\frac{2\cos(-\frac{\pi}{6})(1-\sin(-\frac{\pi}{6}))}{(2+\sin(-\frac{\pi}{6}))^3}$$

To evaluate the expression above we need the exact values of $\sin(-\frac{\pi}{6})$ and $\cos(-\frac{\pi}{6})$.

From previous work we know that $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$.

We can use the symmetry of the cosine curve



to realise that $\cos(-\frac{\pi}{6}) = \cos(\frac{\pi}{6})$.

Hence $\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

Now let's substitute the two values into the earlier expression we had $f''(x)$ to get

$$f''(-\tfrac{\pi}{6}) = -\frac{2\times\frac{\sqrt{3}}{2}\times(1-(-\frac{1}{2}))}{(2+(-\frac{1}{2}))^3}$$ **M1**

We don't need to evaluate the expression exactly we just need to know if it is positive or negative, let's do a little simplification

$$= -\frac{2 \times \frac{\sqrt{3}}{2}(1+\frac{1}{2})}{(\frac{3}{2})^3}$$

We can now see that quotient will result in a positive value. Hence, as the entire quotient is being multiplied by -1, the result is negative.

Therefore

$$f''(-\tfrac{\pi}{6}) < 0$$

Hence $\boxed{x = -\frac{\pi}{6} \text{ is a maximum value}}$.

A1

We can use a similar process for the other stationary point $x = -\frac{5\pi}{6}$

$$f''(-\tfrac{5\pi}{6}) = -\frac{2\cos(-\frac{5\pi}{6})(1-\sin(-\frac{5\pi}{6}))}{(2+\sin(-\frac{5\pi}{6}))^3}$$ **M1**

From previous work we know that $\sin(-\frac{5\pi}{6}) = -\frac{1}{2}$.

And using the symmetries of the cosine curve we get

$$\begin{aligned}\cos(-\tfrac{5\pi}{6}) &= \cos(\tfrac{5\pi}{6}) \\ &= -\cos(\tfrac{\pi}{6}) \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

We can now substitute these two values into the second derivative to get

$$\begin{aligned}f''(-\tfrac{5\pi}{6}) &= -\frac{2\left(-\frac{\sqrt{3}}{2}\right)(1-(-\frac{1}{2}))}{(2+(-\frac{1}{2})^3} \\ &= -\frac{-2\sqrt{3}(1+\frac{1}{2})}{(\frac{3}{2})^3}\end{aligned}$$

We can see that the quotient will now be negative and hence the overall value will be positive.

Therefore

$$f''(-\tfrac{5\pi}{6}) > 0$$

Hence $x = -\frac{5\pi}{6}$ is a minimum value.

A1

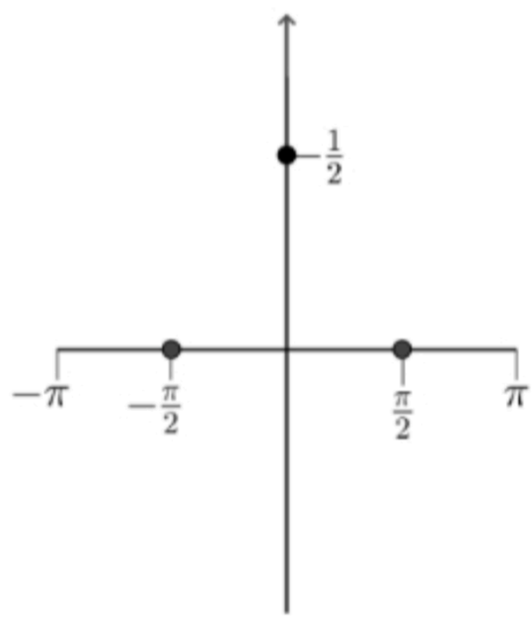
(e) To help us with the sketch we can summarise our findings so far.

From parts (a) and (b) we found the axes intercepts $(0, \frac{1}{2})$, $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$.

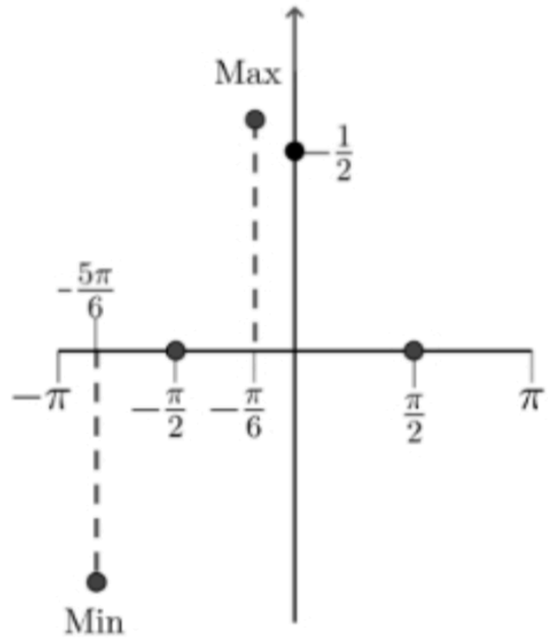
We have also found a maximum point at $x = -\frac{\pi}{6}$ and a minimum point at $x = -\frac{5\pi}{6}$.

We should also remind ourselves that the domain is $-\pi \leq x \leq \pi$.

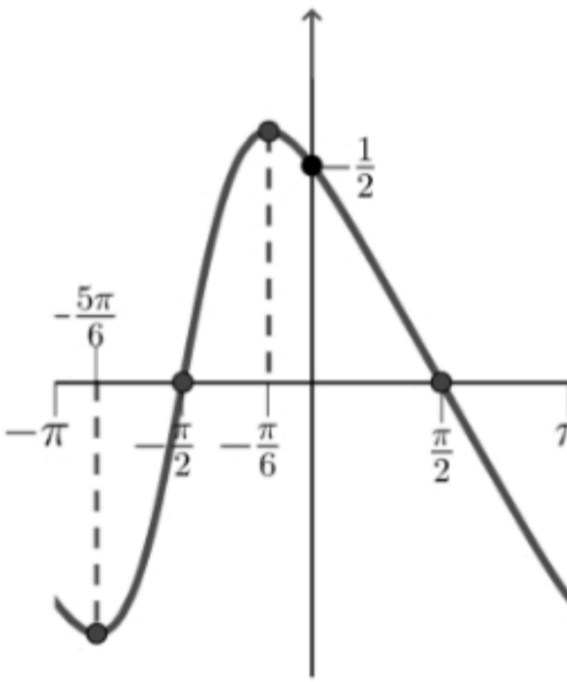
Let's first sketch the domain and the axes intercepts



We can now add the stationary points



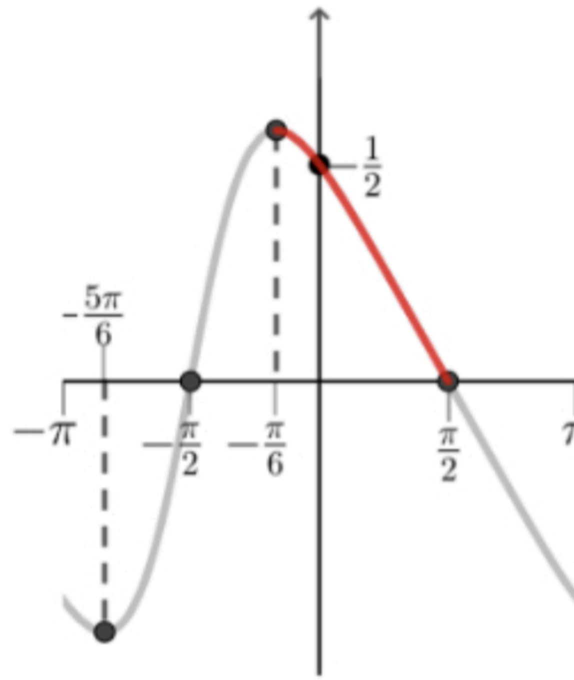
Finally we can use the plotted points to fit the function. Being careful to stop at the end-points!



Correct axes intercepts	**A1**
Correct position of two stationary points	**A1**
Correct shape and end-points	**A1**

(f) We need to identity the region in which f is positive (above the $x-$axis) and is decreasing (which means the gradient is negative).

This region is the highlighted red part of f shown below



Hence we can see that the value of $s = -\frac{\pi}{6}$ and $t = \frac{\pi}{2}$.

To find the area under the curve between these two values we need to evaluate the definite integral of f from $-\frac{\pi}{6}$ to $\frac{\pi}{2}$

$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,\mathrm{d}x \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{2+\sin x}\,\mathrm{d}x \end{aligned} \qquad \textbf{(M1)}$$

This integral is a quotient. Often a good strategy to integrate a quotient is to use a substitution for the denominator.

Hence let $u = 2 + \sin x$.

To rewrite the integral we will need an expression for $\mathrm{d}x$ in terms of $\mathrm{d}u$, hence

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

$$\mathrm{d}x = \frac{1}{\cos x}\mathrm{d}u$$

Let's now replace the original integral so that it is in terms of u. We will omit the limits for now.

$$\begin{aligned} \int \frac{\cos x}{2+\sin x}\,\mathrm{d}x &= \int \frac{\cancel{\cos x}}{u} \times \frac{1}{\cancel{\cos x}}\mathrm{d}u \\ &= \int \frac{1}{u}\mathrm{d}u \\ &= \ln|u| + C \end{aligned}$$

A1

Let's now substitute $u = 2 + \sin x$ and reintroduce the limits.

As the integral is now definite we can omit the constant.

This gives

$$\text{Area} = \Big[\ln |2 + \sin x| \Big]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \qquad \textbf{A1}$$

As $2 + \sin x > 0$ for all x we can remove the absolute value signs.

Substituting in the upper and lower limit we get

$$= \ln\left(2 + \sin\left(\tfrac{\pi}{2}\right)\right) - \ln\left(2 + \sin\left(-\tfrac{\pi}{6}\right)\right) \qquad \textbf{M1}$$

Recall $\sin\left(\frac{\pi}{2}\right) = 1$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, hence we get

$$= \ln(2+1) - \ln\left(2 + -\left(\tfrac{1}{2}\right)\right)$$

$$= \ln 3 - \ln \tfrac{3}{2}$$

$$= \ln \frac{3}{\frac{3}{2}}$$

$$= \ln 2$$

Hence $\boxed{c = 2}$

A1

Question 11

NO CALCULATOR | Hard ●●●●○

[Maximum mark: 16]

(a) Use mathematical induction to prove that

$$2^n \times \cos x \times \cos 2x \times \cos 4x \times \ldots \times \cos 2^{n-1}x = \frac{\sin 2^n x}{\sin x}$$

where $n \in \mathbb{Z}^+$. [4]

(b) (i) Find the first two non-zero terms of the Maclaurin series for $\sin 8x$.

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\frac{\sin 8x}{\sin x}$.

(iii) Hence find an estimate for $\int_0^{0.1} \cos x \cos 2x \cos 4x \, dx$. [12]

(a) We start by showing the base case is true.

This is when $n = 1$.

Let's first consider the value of the L.H.S. of the proposition

$$2^1 \times \cos 2^{1-1}x = 2\cos x$$

And the value of the R.H.S. is

$$\begin{aligned}\frac{\sin 2^1 x}{\sin x} &= \frac{2\sin x\cos x}{\sin x}\\ &= 2\cos x\\ &= \text{L.H.S.}\end{aligned}$$ **R1**

Hence we have shown the proposition is true when $n = 1$.

The next step is to assume the proposition is true when $n = k$ where $k \in \mathbb{Z}^+$

$$2^k \times \cos x \times \cos 2x \times \cos 4x \times \ldots \times \cos 2^{k-1}x = \frac{\sin 2^k x}{\sin x}$$

Next we need to use the assumption to show that the proposition is true when $n = k + 1$.

Hence we need to prove

$$2^{k+1} \times \cos x \times \cos 2x \times \cos 4x \times \ldots \times \cos 2^k x = \frac{\sin 2^{k+1} x}{\sin x}$$ **M1**

Let's start by rewriting the L.H.S. so it looks similar to the L.H.S. of the assumption

$$\text{L.H.S.} = 2 \times 2^k \times \cos x \times \cos 2x \times \cos 4x \times \ldots \times \cos 2^{k-1}x \times \cos 2^k x$$

Notice the blue part is now the same as the L.H.S. of the assumption, hence we can replace it with the R.H.S. of the assumption

$$= 2\left(\frac{\sin 2^k x}{\sin x}\right) \times \cos 2^k x \qquad \textbf{A1}$$

This is known as the *inductive* step.

We can rearrange the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ to

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$

and use it to rewrite our L.H.S. giving

$$= 2 \times \frac{\sin 2^k x}{\sin x} \times \cos 2^k x$$

$$= 2 \times \frac{\frac{1}{2}\sin\left(2 \times 2^k x\right)}{\sin x} \qquad \textbf{A1}$$

$$= \frac{\sin\left(2 \times 2^k x\right)}{\sin x}$$

$$= \boxed{\frac{\sin 2^{k+1}x}{\sin x}} \text{ ...as required.}$$

$$= \text{R.H.S.}$$

We have shown the proposition is true when $n = 1$ and we have shown it to be true when $n = k + 1$, assuming $n = k$ is true. Therefore by the principle of mathematical induction the proposition is true for all positive integer values of n.

(b) (i) We begin by noting that $\sin 8x$ is a composite function.

Hence we can use the formula for the Macluarin series for sine but replace x with $8x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\sin 8x = 8x - \frac{(8x)^3}{3!} \qquad \textbf{(M1)}$$

We only need the first two non-zero terms hence we get

$$= 8x - \frac{512x^3}{6}$$

$$= \boxed{8x - \frac{256}{3}x^3} \qquad \textbf{A1}$$

(ii) Let's begin by using a general power series to represent the result of the quotient

$$\frac{\sin 8x}{\sin x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Where a_i are real numbers we must determine.

We can use the result from part (a) and the general formula for sine to write the numerator and denominator as a Maclaurin series up to the x^3 term

$$\frac{8x - \frac{256}{3}x^3}{x - \frac{x^3}{3!}} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Recall, we only need the first two non-zero terms which will mean up to and including the x^3 term.

It can be challenging to attempt to use long division to rewrite the L.H.S. Often a better strategy, when facing problems like this, is to rewrite the division as a multiplication

$$8x - \frac{256}{3}x^3 = \left(x - \frac{x^3}{3!}\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) \qquad \textbf{(M1)}$$

We can now expand the R.H.S. until we have all of the terms required to give the x and x^3 terms.

$$\begin{aligned} 8x - \frac{256}{3}x^3 &= a_0 x - \frac{a_0 x^3}{6} + \cancel{a_1 x^2} - \cancel{\frac{a_1 x^4}{6}} + a_2 x^3 - \cancel{\frac{a_2 x^5}{6}} \\ &= a_0 x - \frac{a_0 x^3}{6} + + a_2 x^3 \qquad \textbf{A1} \end{aligned}$$

There is no x^2 term therefore a_1 must be zero. Also we don't need any power of x higher than 3. So we can cancel out another two terms.

We can now equate the coefficients of the $x-$terms on the left and right side giving

$$8 = a_0 \qquad \text{for } x$$

$$-\frac{256}{3} = -\frac{a_0}{6} + a_2 \qquad \text{for } x^3 \qquad \textbf{A1}$$

We can see that $a_0 = 8$ hence we can use this to find a_2

$$\begin{aligned} -\frac{256}{3} &= -\frac{8}{6} + a_2 && \textbf{(M1)} \\ &= -\frac{8}{6} + \frac{6a_2}{6} \\ -512 &= -8 + 6a_2 \\ 6a_2 &= -504 \\ a_2 &= -84 && \textbf{A1} \end{aligned}$$

Therefore, referring back to the series expansion at the start of the solution, we can see that the first two non-zero terms are

$$\begin{aligned} \frac{\sin 8x}{\sin x} &= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \\ &= \boxed{8 - 84x^2} && \textbf{A1} \end{aligned}$$

(iii) The integral given is a very difficult integral to perform directly.

However, notice that it is of the form of the proposition from part (a).

If we take the proposition and let $n = 3$ we get

$$\begin{aligned} 2^3 \times \cos x \times \cos 2x \times \cos 4x &= \frac{\sin 2^3 x}{\sin x} \\ \cos x \times \cos 2x \times \cos 4x &= \frac{\sin 8x}{8 \sin x} \qquad \textbf{(A1)} \end{aligned}$$

Hence we can rewrite the integral as

$$\begin{aligned} \int_0^{0.1} \cos x \cos 2x \cos 4x \, \mathrm{d}x &= \int_0^{0.1} \frac{\sin 8x}{8 \sin x} \, \mathrm{d}x \\ &= \tfrac{1}{8} \int_0^{0.1} \frac{\sin 8x}{\sin x} \, \mathrm{d}x \qquad \textbf{(A1)} \end{aligned}$$

The result is still a difficult integral to perform but the key word in the question is *estimate*. This suggests we should use the series expansion from part (b)(ii).

Hence we get

$$\int_0^{0.1} \cos x \cos 2x \cos 4x \, \mathrm{d}x \approx \tfrac{1}{8} \int_0^{0.1} \left(8 - 84x^2\right) \, \mathrm{d}x$$

Integrating each term we get

$$= \tfrac{1}{8} \left[8x - 28x^3 \right]_0^{0.1} \qquad \textbf{A1}$$

Substituting in the limits we get

$$\begin{aligned} &= \tfrac{1}{8} \left(8(0.1) - 28(0.1)^3 \right) \\ &= 0.1 - 3.5 \times 0.001 \\ &= 0.1 - 0.0035 \\ &= \boxed{0.0965} \qquad \textbf{A1} \end{aligned}$$

Question 12

NO CALCULATOR Hard

[Maximum mark: 14]

The complex number z is a root of the equation $|z+4i|=|z-10i|$.

(a) Show that the imaginary part of z is 3. [2]

(b) Let ω_1 and ω_2 be two possible values of z such that $|z|=6$.

(i) If ω_1 is in the first quadrant sketch both solutions on an Argand diagram.

(ii) Hence find the arguments of ω_1 and ω_2. [4]

A different complex number, v, is defined such that

$$v=\frac{\omega_1^k\,\omega_2}{-i}$$

Where k is a real number that can take any value in the interval $-10 \leq k \leq 10$.

(c) (i) Find $\arg(v)$ in terms of k and π.

(ii) Hence find all possible values of k such that v is a real number. [8]

(a) We are told z is a complex number, therefore z is of the form $a + bi$ where $a, b \in \mathbb{R}$.

Hence we can rewrite the equation in terms of a and b to get

$$|a + bi + 4i| = |a + bi - 10i|$$

If we collect together the real and imaginary parts it is easier to consider the magnitude of the number

$$|a + i(b + 4)| = |a + i(b - 10)|$$

Recall the magnitude of a complex number $|a + bi| = \sqrt{a^2 + b^2}$

$$\sqrt{a^2 + (b + 4)^2} = \sqrt{a^2 + (b - 10)^2}$$

$$a^2 + (b + 4)^2 = a^2 + (b - 10)^2 \quad \textbf{M1}$$

$$\cancel{a^2} + \cancel{b^2} + 8b + 16 = \cancel{a^2} + \cancel{b^2} - 20b + 100$$

We can find b

$$28b = 84 \quad \textbf{A1}$$

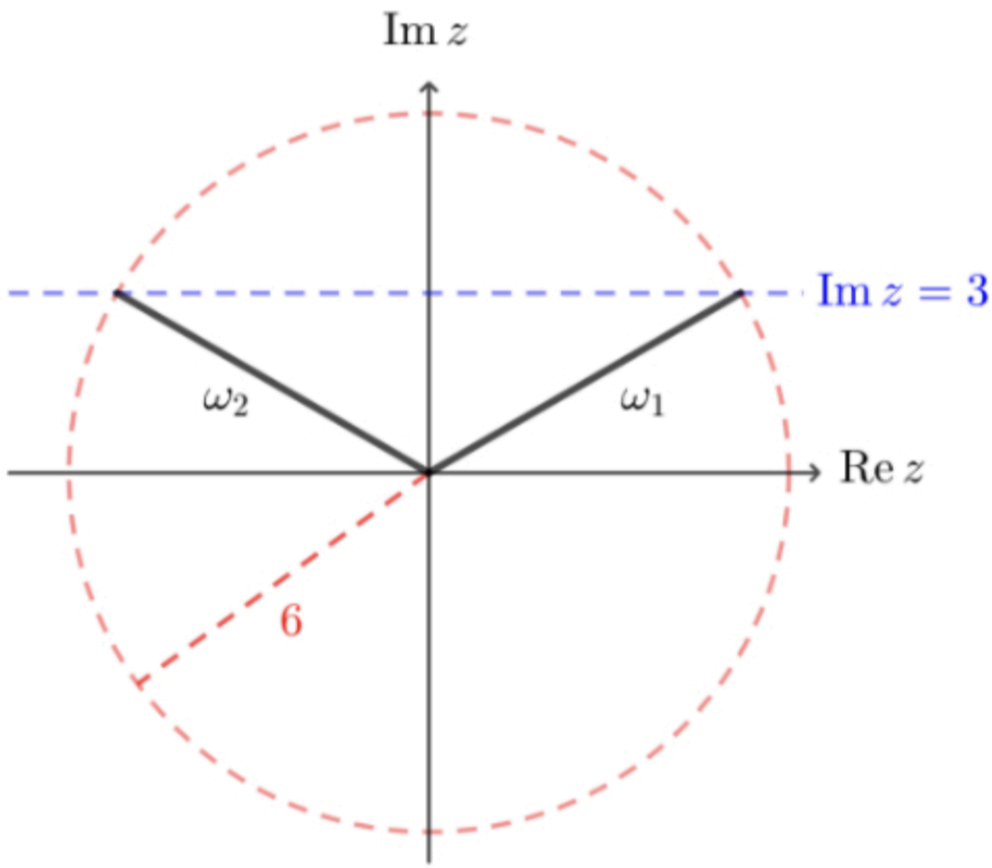
$$b = 3$$

Therefore $\boxed{\text{Im}(z) = 3}$ as required.

(b) (i) The red dashed circle of radius 6 represents the fact that $|z| = 6$. Therefore z must lie somewhere on this circle.

The blue horizontal line shows the possible values of z when $\text{Im}(z) = 3$, with ω_1 being in the first quadrant.

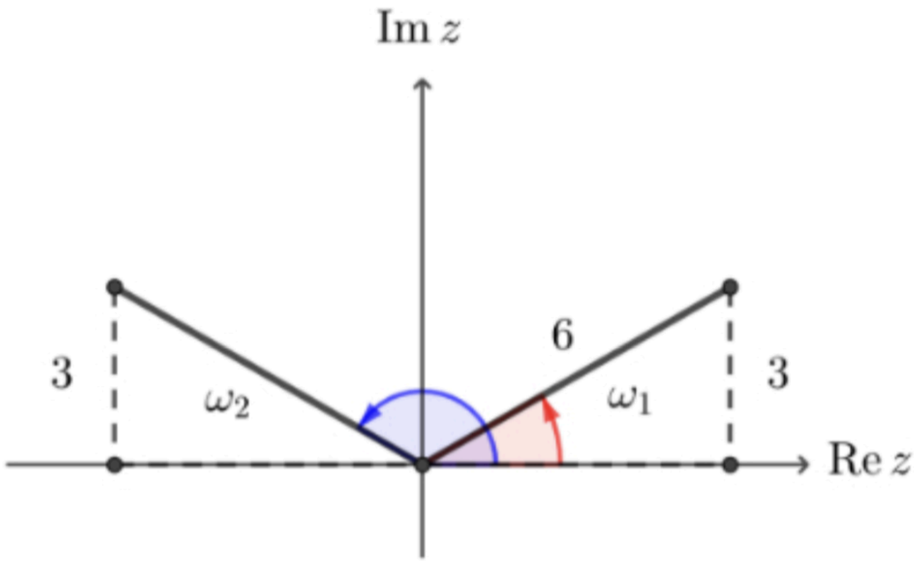
z must lie on both the red circle and blue line



ω_1 and ω_2 both have imaginary values of 3. **A1**

Correct quadrants for ω_1 and ω_2. **A1**

(ii) Using trigonometric ratios we can find $\arg(\omega_1)$



$$\sin[\arg(\omega_1)] = \frac{3}{6}$$

$$[\arg(\omega_1)] = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\arg(\omega_1) = \boxed{\frac{\pi}{6}}$$ **A1**

Using the symmetries of the diagram

$$\sin[\arg(\omega_2)] = \pi - \sin[\arg(\omega_1)]$$

$$= \boxed{\frac{5\pi}{6}}$$ **A1**

(c) (i) In this question we are combining complex numbers and working with their arguments.

For questions of this nature it is most efficient to use Euler's form of a complex number.

Recall

$$z = re^{i\theta}$$

Where r is the magnitude of the number of θ the argument.

Hence, using our answers from part (b) we can write

$$\omega_1 = 6e^{i\frac{\pi}{6}}$$

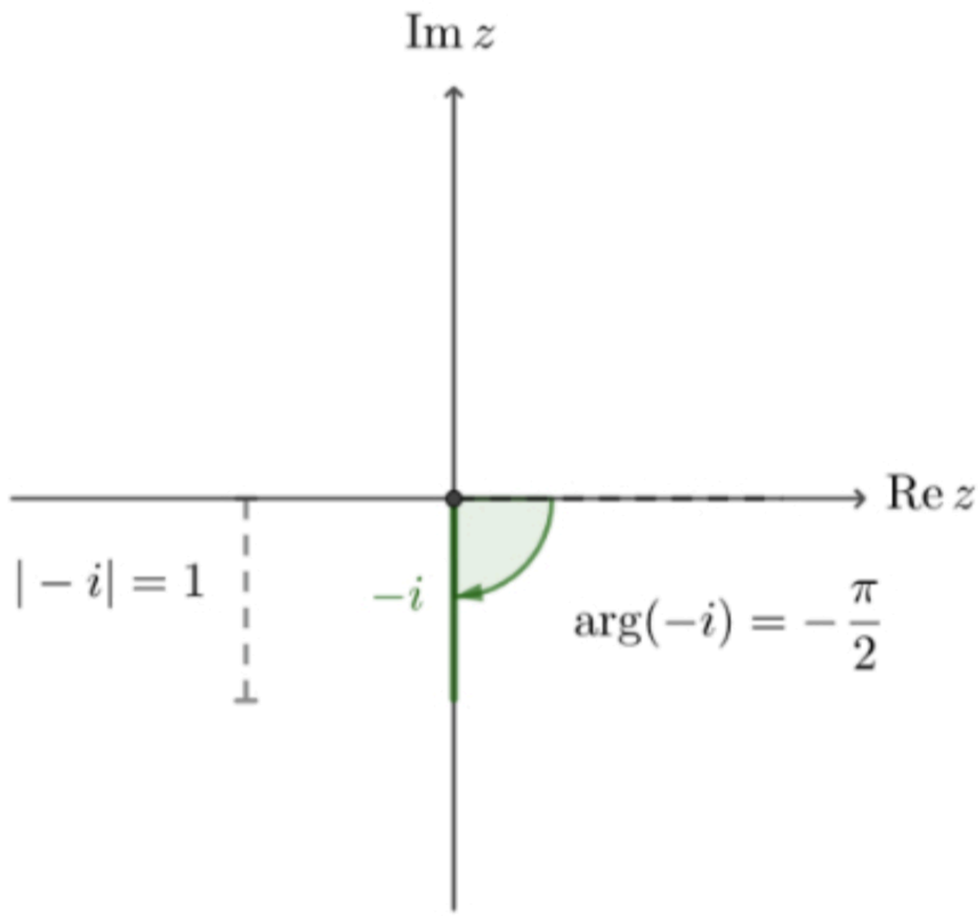
$$\omega_2 = 6e^{i\frac{5\pi}{6}}$$

If we raise ω_1 to the power k, using De Moivre's theorem, we get

$$\omega_1 = 6^k e^{i\frac{k\pi}{6}} \qquad \textbf{(M1)}$$

Next we need to consider how to write $-i$ in Euler's form.

Here is a sketch of $-i$



We can see that the magnitude is 1 and the argument is $-\frac{\pi}{2}$, angles measured in the clockwise direction are negative.

Hence we get

$$-i = e^{-i\frac{\pi}{2}} \qquad \textbf{(A1)}$$

Putting these results together for v we get

$$\begin{aligned} v &= \frac{\omega_1^k \omega_2}{-i} \\ &= \frac{6^k e^{i\frac{k\pi}{6}} \times 6e^{i\frac{5\pi}{6}}}{e^{-i\frac{\pi}{2}}} \end{aligned} \qquad \textbf{M1}$$

Recall we want to find the argument of v, hence we need to simplify our expression so that it is in standard Euler's form $\rightarrow z = re^{i\theta}$.

We can now see the advantage of using Euler's form. To combine the angles we need to resolve the powers of e using basic index laws

$$= 6 \times 6^k e^{i\frac{k\pi}{6} + \frac{5\pi}{6} - \left(-\frac{\pi}{2}\right)} \qquad \textbf{M1}$$

$$= 6^{k+1} e^{i\frac{k\pi}{6} + \frac{5\pi}{6} + \frac{\pi}{2}}$$

$$= 6^{k+1} e^{i\frac{k\pi}{6} + \frac{5\pi}{6} + \frac{3\pi}{6}}$$

$$= 6^{k+1} e^{i\frac{k\pi}{6} + \frac{8\pi}{6}}$$

$$= 6^{k+1} e^{i\frac{\pi}{6}(k+8)}$$

By looking at the exponent of e we can find the argument of v in terms of k and π, hence

$$\arg(v) = \boxed{\frac{\pi}{6}(k+8)} \qquad \textbf{A1}$$

(ii) v is a real number when it lies entirely on the real axis (Re z).

Therefore its argument must be a multiple of π. Note 0 is a multiple of π.

We can form an equation for the argument. We aren't sure how many multiples of π we'll need so we can pick a few to get us started

$$\frac{\pi}{6}(k+8) = \dots - 2\pi, \quad -\pi, \quad 0, \quad \pi, \quad 2\pi\dots \qquad \textbf{(M1)}$$

We can now multiply by 6 and divide by π to get

$$k+8 = \dots - 12, \quad -6, \quad 0, \quad 6, \quad 12\dots \qquad \textbf{A1}$$

Subtracting 8 from each answer we get

$$k = \dots - 20, \quad -14, \quad -8, \quad -2, \quad 4\dots$$

Recall that $-10 \leq k \leq 10$.

Hence we reject the first two answers. We may have missed one on the positive side.

$$\frac{\pi}{6}(k+8) = 3\pi$$
$$k+8 = 18$$
$$k = 10$$

Hence all answers in the interval for k are

$$k = \boxed{-8, -2, 4, 10} \qquad \textbf{A1}$$