IB Mathematics AA HL - Prediction Exams May 2025 - Paper 2

Paper 2 | 12 questions | 120 mins | 110 marks

Section A

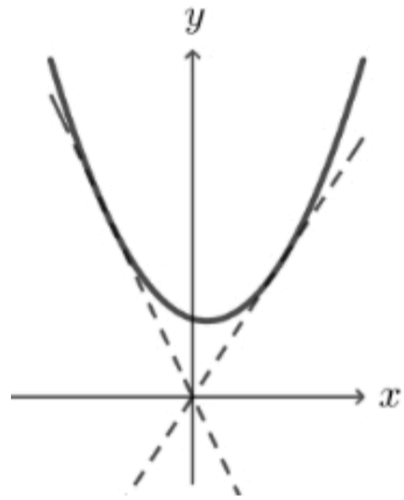
Question 1

CALCULATOR | Easy

[Maximum mark: 5]

Find the equations of the two tangents to the curve $y = 2x^2 - x + \frac{9}{2}$ that pass through the origin.

To better understand the scenario we can make an approximate sketch.



The dashed lines are the two tangent lines whose equations we need to find.

The equation of a tangent line, through the origin, will be of the form

$$y = mx \qquad \textbf{(M1)}$$

To find the intersection points of the curve and line we can solve both equations simultaneously, hence

$$mx = 2x^2 - x + \frac{9}{2} \qquad \textbf{(M1)}$$

$$0 = 2x^2 - x - mx + \frac{9}{2}$$

$$0 = 2x^2 - (1+m)x + \frac{9}{2} \qquad \textbf{(A1)}$$

If the lines are tangent to the curve then there is only one point of intersection i.e. only 1 solution.

As we have a quadratic equation then we can say that the discriminant must be equal to zero. Hence

$$\Delta = b^2 - 4ac$$

$$0 = [-(1+m)]^2 - 4(2)\left(\tfrac{9}{2}\right) \qquad \textbf{A1}$$

Solving for m we get

$$36 = [-(1+m)]^2$$

$$-(1+m) = \pm 6$$

$$m = -7 \text{ and } 5$$

Therefore the two equations are

$$\boxed{y = -7x} \quad \text{and} \quad \boxed{y = 5x} \qquad \textbf{A1}$$

Question 2

CALCULATOR Easy

[Maximum mark: 5]

Chun Li has a bag with five 6-sided dice.

Four of them are normal fair dice and one of them is biased with a 6 showing on each of its faces.

She draws two out at random and rolls them.

(a) Find the probability a six shows on both dice. [3]

(b) Given a six shows on both dice find the probability one of the dice is the biased dice. [2]

(a) We can begin by defining two events.

Event A is drawing two fair dice.

Event B is rolling two 6s.

Let's consider event A first.

As Chun Li is drawing out two dice, she can either have 2 fair dice or 1 biased dice and 1 fair dice.

The probability of her drawing (without replacement!) two fair dice is

$$\begin{aligned} \mathrm{P}(A) &= \frac{4}{5} \times \frac{3}{4} \\ &= \frac{12}{20} \\ &= \frac{3}{5} \end{aligned}$$

As she can only hold 2 fair dice or 1 fair and 1 biased this means that the probability she holds 1 fair dice and 1 biased is the compliment of the result we just found, hence

$$\begin{aligned} \mathrm{P}(A') &= 1 - \frac{3}{5} \\ &= \frac{2}{5} \end{aligned}$$

Now let's consider event B. Rolling two 6's.

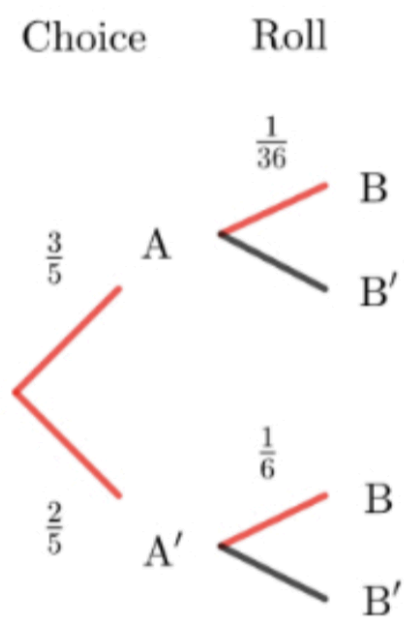
If she rolls 2 fair dice the probability of rolling two $6'$s is

$$= \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36}$$

If she rolls 1 fair dice and 1 biased dice the probability of rolling two $6'$s is

$$= \frac{1}{6} \times 1$$
$$= \frac{1}{6}$$

We can now put this information into a tree diagram



Notice we don't need to calculate the all possible outcomes.

The sum of both red branches will give the probability she rolls two $6's$.

Therefore we get

$$= \frac{3}{5} \times \frac{1}{36} + \frac{2}{5} \times \frac{1}{6} \qquad \textbf{(M1A1)}$$

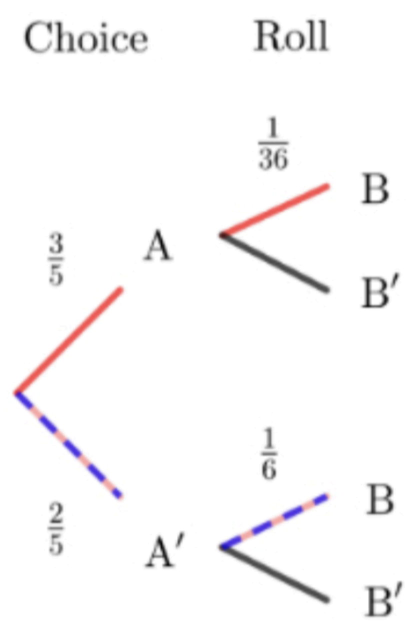
$$= \boxed{\frac{1}{12}} \qquad \textbf{A1}$$

$$= 0.08333...$$

(b) In this part some we have been given some information about the roll.

This is known as conditional probability.

Consider this tree diagram



We have been told that the roll is from either of the two red branches. This is now the universal set and is the denominator of the answer.

The blue dashed branches form the desired outcome.

Here is the conditional probability formula

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

The way we have defined the events in this solution means A' is the event that 1 fair and 1 biased dice is chosen.

Hence we can replace A with A' to get

$$\mathrm{P}(A'|B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)}$$

Our answer from part (a) is $\mathrm{P}(B)$ and the numerator is the blue dashed branch seen on the previous tree diagram.

$$\mathrm{P}(A'|B) = \frac{\frac{2}{5} \times \frac{1}{6}}{\frac{1}{12}} \qquad \textbf{(M1)}$$

$$= \boxed{\frac{4}{5}} \qquad \textbf{A1}$$

Question 3

CALCULATOR

Easy

[Maximum mark: 7]

A cyclist leaves town A on a bearing of $240°$ and rides 11 kilometers to town B.

The cyclist then travels d km on a bearing of $090°$ until he is exactly 6 km from town A.

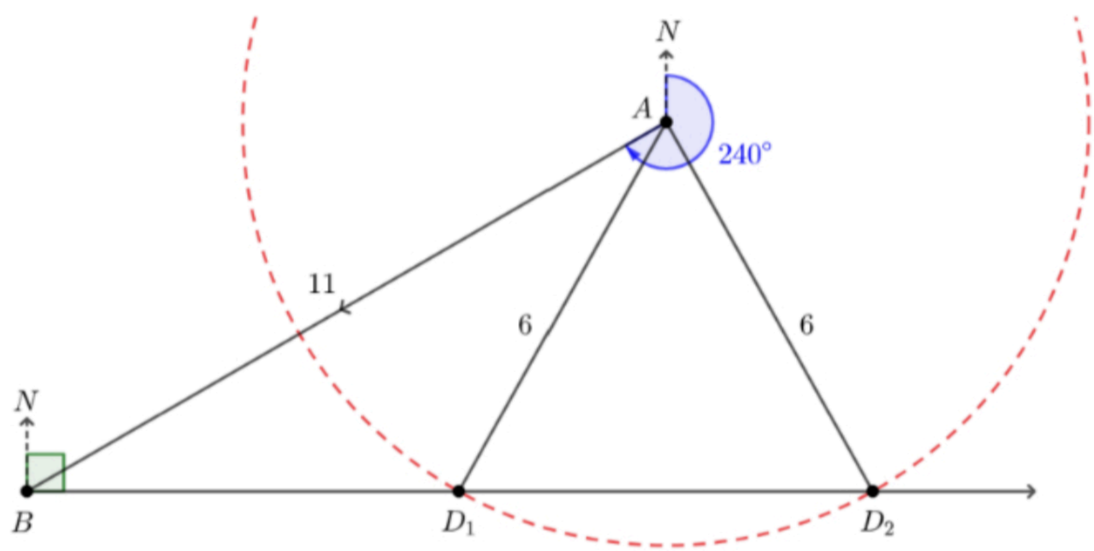
Find the possible values of d.

First let's sketch a diagram to better understand the given information.

We can sketch the approximate positions of towns A and B and also indicate the stopping places of the cyclist by D_1 and D_2.

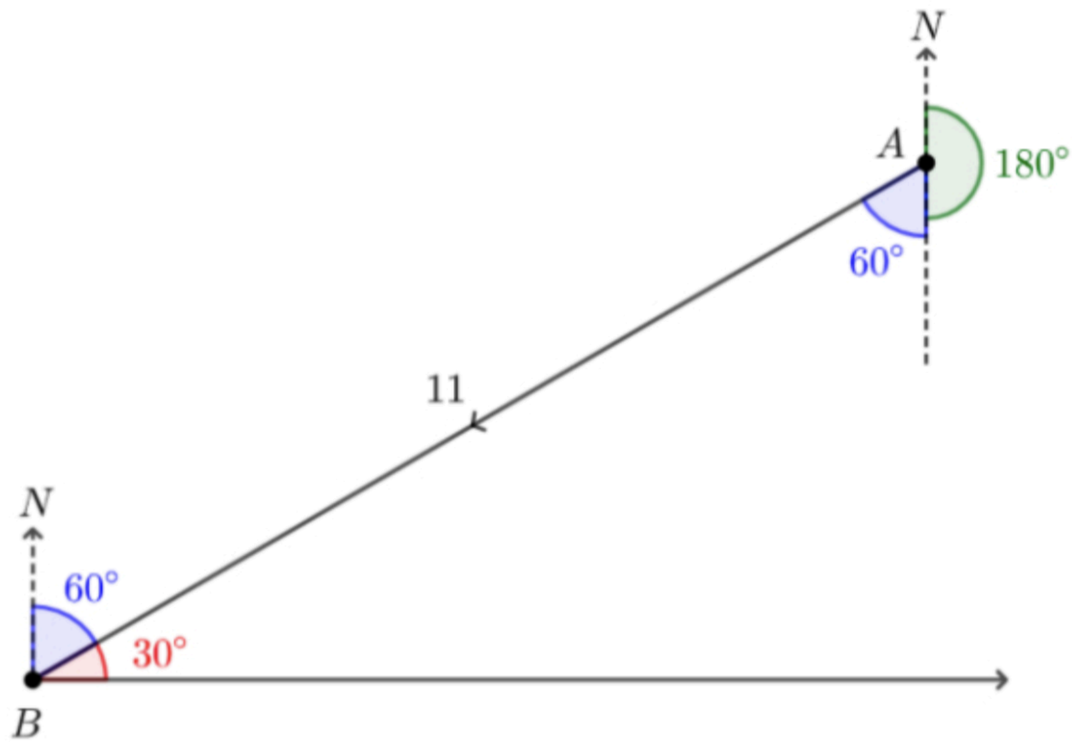
Any bearing measures start from the North and turn clockwise.

With all lengths in kilometres, our diagram will look like this.



Notice that the dashed red circle, with radius 6 centred on A, indicates that there are two places where the cyclist can stop that are 6 kilometers from town A.

Using angle reasoning, we can determine $\angle ABD$.

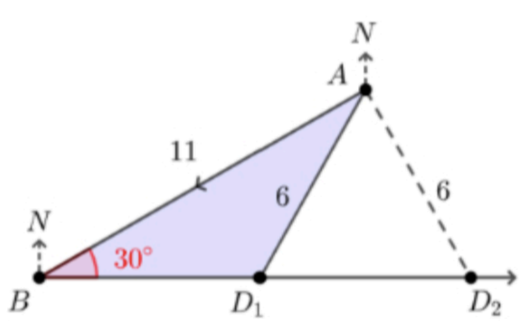


$$240^\circ - 180^\circ = 60^\circ$$

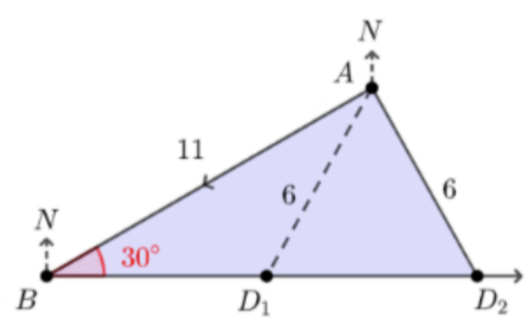
$$90^\circ - 60^\circ = 30^\circ \qquad \textbf{(A1)}$$

When we add this to the diagram we notice that there are 2 triangles which have sides of 11 and 6 and a non-included angle of 30°. This is an example of the ambiguous case of the sine rule.

Triangle 1



Triangle 2



Using the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Hence

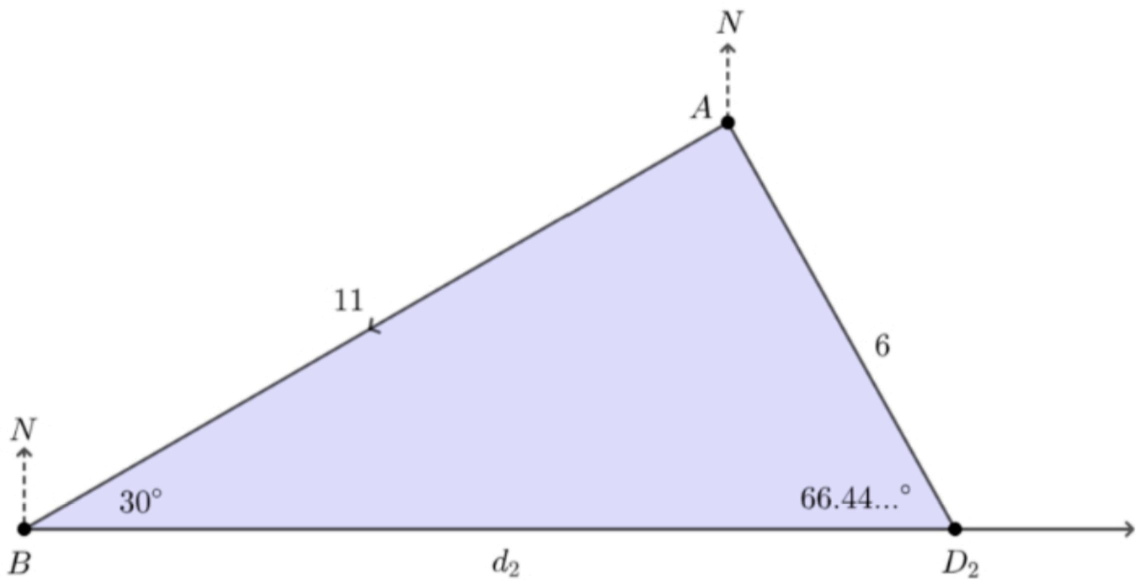
$$\frac{\sin D}{11} = \frac{\sin 30}{6} \qquad \textbf{(M1)}$$

$$\sin D = 0.9166...$$

$$D = 66.44... \qquad \textbf{A1}$$

As $D < 90^\circ$, this must be the angle associated with triangle 2.

Let's complete this triangle to find the distance d_2 associated with stopping at D_2.



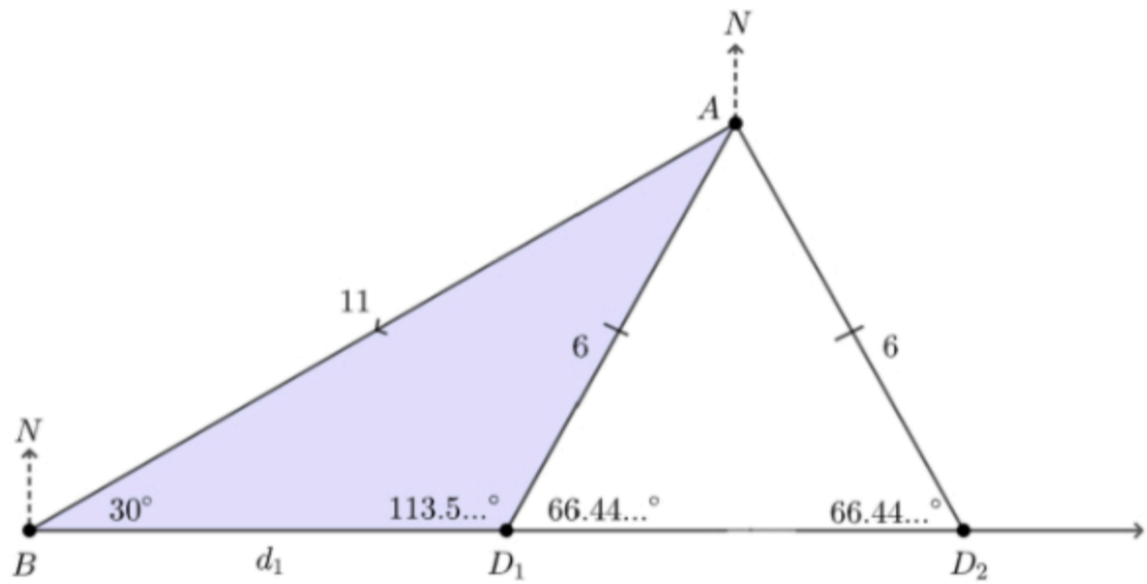
Finding angle BAD_2

$$\begin{aligned} BAD_2 &= 180° - 30° - 66.44...° \\ &= 83.55...° \qquad \textbf{(A1)} \end{aligned}$$

Using sine rule again

$$\begin{aligned} \frac{d_2}{\sin 83.55...°} &= \frac{6}{\sin 30°} \\ d_2 &= 11.92... \\ &= \boxed{11.9 \text{ km}} \qquad \textbf{A1} \end{aligned}$$

We can see that triangle AD_1D_2 is isosceles.



Hence we can find angle BD_1A

$$= 180° - 66.44...°$$

$$= 113.5...$$ **M1**

Finding angle BAD_1 in triangle 1

$$\angle BAD_1 = 180° - 30° - 113.55...°$$
$$= 36.44...°$$

Using sine rule again to find the distance associated with stopping at D_1

$$\frac{d_1}{\sin 36.44...°} = \frac{6}{\sin 30°}$$
$$d_1 = 7.128...$$
$$= \boxed{7.13 \text{ km}}$$ **A1**

Question 4

CALCULATOR

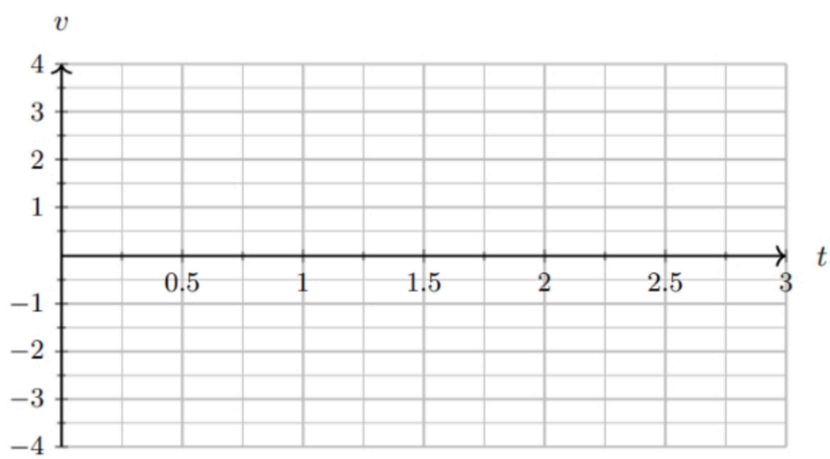
Medium

[Maximum mark: 6]

A particle P moves along a straight line such that its displacement, in metres, after t seconds, from a fixed point O is given by

$$s(t) = 3e^{-(t+1)}\sin(4t+4), \quad 0 \le t \le 2$$

(a) Sketch the graph of the velocity of P against t on the axes below. [2]



P reaches its maximum speed when $t = a$ seconds.

(b) Find a. [1]

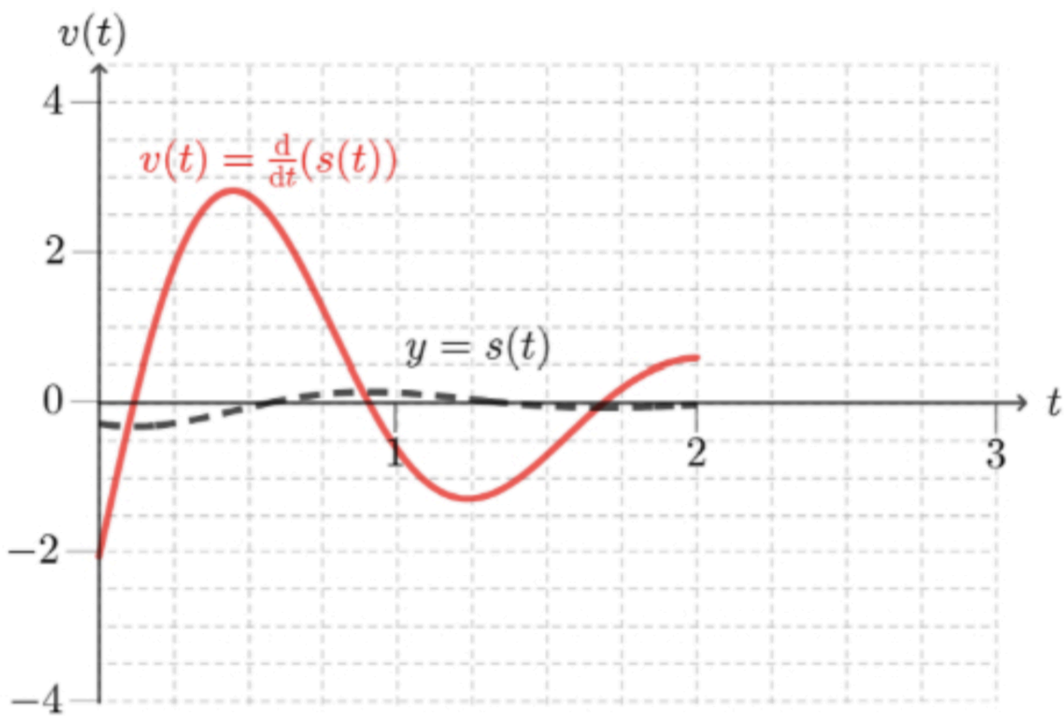
(c) Hence or otherwise, find the distance travelled whilst the acceleration of P is negative. [3]

(a) Recall

$$v(t) = s'(t)$$

$$= \frac{\mathrm{d}s}{\mathrm{d}t}$$

Hence, we can use the calculator to graph both $s(t)$ and $v(t)$.



Correct endpoints

A1

Correct shape

A1

(b) Recall speed$= |v(t)|$.

The maximum speed occurs when $|v(t)|$ is at a maximum.

In this case, the maximum speed is also the maximum positive value of $v(t)$, which we find using a calculator to be $(0.4483..., 2.819...)$.

Hence, the maximum speed is $2.82\ \mathrm{m\,s^{-1}}$ and this occurs at $t = 0.448\ \mathrm{s}$

Therefore $\boxed{a = 0.448}$.

A1

(c) Recall

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

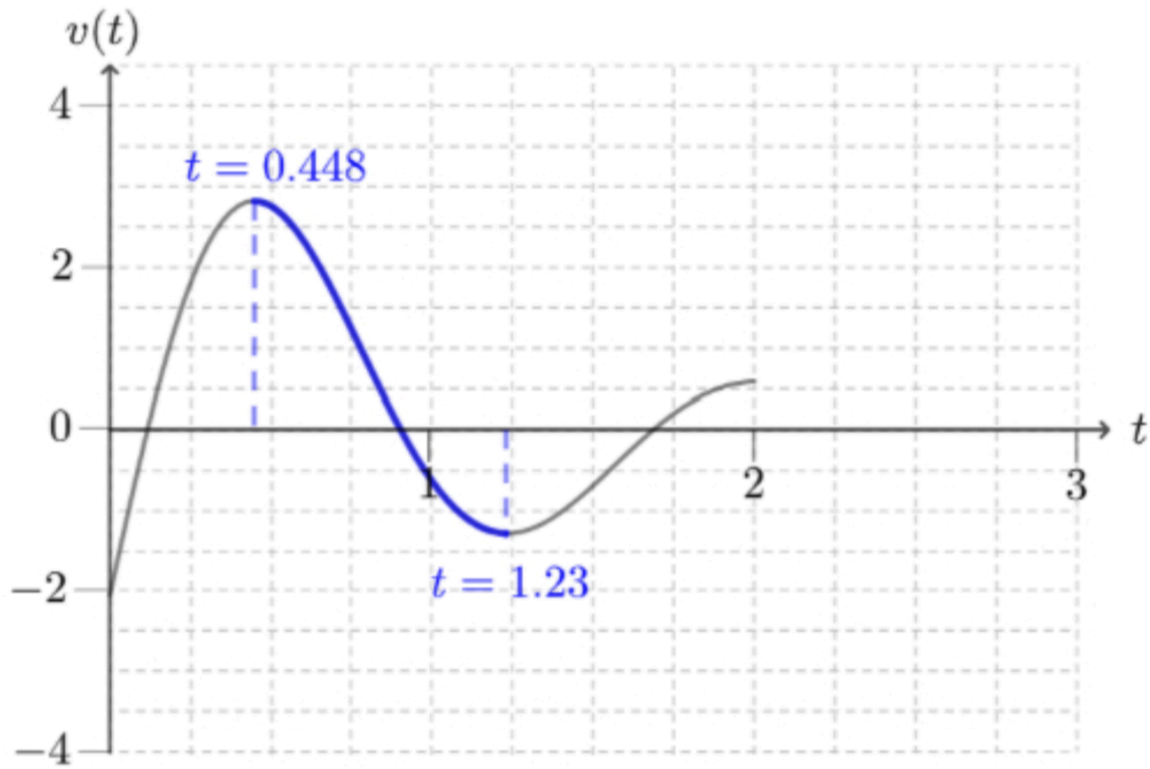
Therefore, the acceleration of P is negative when the gradient of the velocity function, $v(t)$, is negative.

From part (b) we know the local maximum occurs when $t = 0.448$ seconds.

Using technology we find the local minimum point of the velocity function to be $(1.233..., -1.285...)$. Hence, the minimum occurs when $t = 1.23$ seconds.

(A1)

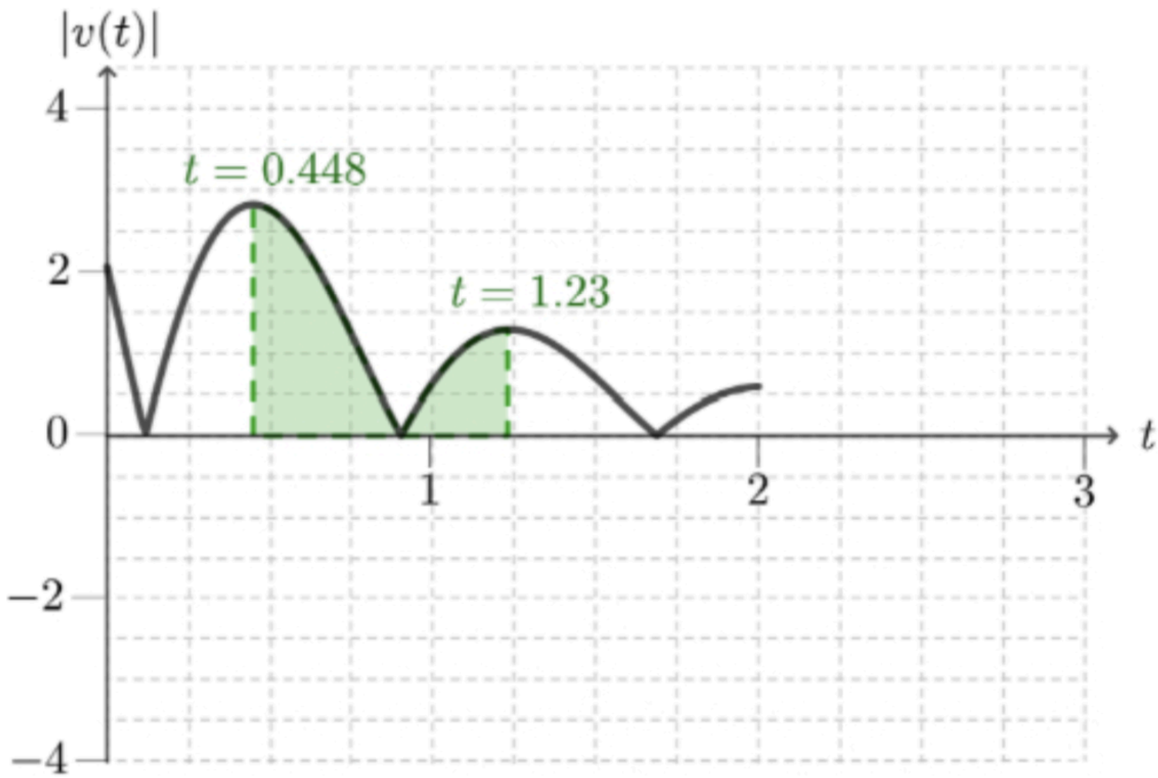
The period that the acceleration of P is negative is shown in blue below



The distance travelled is calculated using the formula

$$\text{distance travelled} = \int_{t_1}^{t_2} |v(t)|\, \mathrm{d}t$$

We can graph this function on the calculator



Using the calculator

$$\begin{aligned}\text{distance travelled} &= \int_{t_1}^{t_2} |v(t)|\, \mathrm{d}t \\ &= \int_{0.4483...}^{1.233...} |v(t)|\, \mathrm{d}t \qquad \textbf{(M1)} \\ &= 1.049... \\ &= \boxed{1.05\text{ m}} \qquad \textbf{A1}\end{aligned}$$

Question 5

CALCULATOR

Medium

[Maximum mark: 5]

The amount, in milligrams, of a medicinal drug in the body t hours after it is injected is given by

$$D(t) = 240e^{-kt}$$

Where $k > 0$ and $t \geq 0$. Before the injection, it is assumed the amount of drug in the body is zero.

A patient is to be injected with the drug and, for this patient, it is known that it takes 5 hours for the amount of drug remaining in the body to have decreased by 65% of the initial dose.

The patient is regularly checked and is allowed to go home when the amount of drug remaining in the body is 10%, or less, of the initial dose.

The initial dose is given to the patient at $9:00$ am.

Use this model to estimate, to the nearest hour, the earliest time the patient will be allowed to go home.

This model has a fixed value of k. In order to use the model we must find k.

First, we can find the initial amount of the drug injected into the patient, this occurs when $t = 0$

$$D(0) = 240e^{-k(0)}$$

$$D(0) = 240$$

Five hours later, $t = 5$, we are told that $100\% - 65\% = 35\%$ of the initial 240 mg remains.

We can use this to find k.

$$0.35 \times 240 = 240e^{-5k} \qquad \textbf{(M1)}$$

$$0.35 = e^{-5k}$$

$$\ln 0.35 = \ln e^{-5k}$$

Using the log law $\log_a x^m = m \log_a x$, we get

$$\ln 0.35 = -5k \ln e$$

Since $\ln e = 1$, we can simplify the equation and solve for k.

$$\ln 0.35 = -5k$$

$$k = \frac{\ln 0.35}{-5}$$

$$= 0.2099\ldots \qquad \textbf{A1}$$

We can now write out the model with out value of k

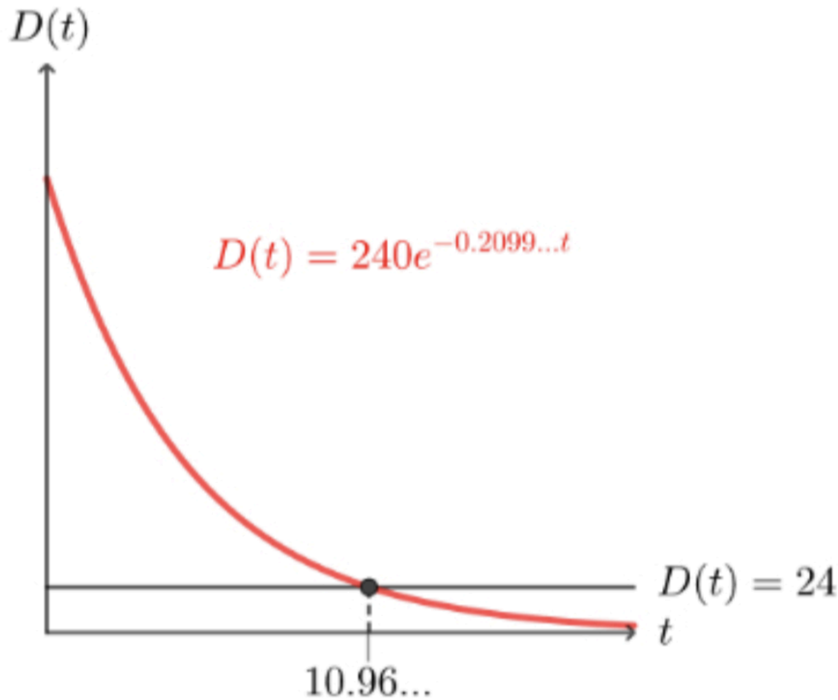
$$D(t) = 240e^{-0.2099\ldots t}$$

We need to find t when the drug remaining in the body is equal to $0.10 \times 240 = 24$ mg.

Hence we need to solve the following equation

$$24 = 240e^{-0.2099\ldots t} \qquad \textbf{(M1)}$$

We can do this graphically by sketching the function $y = D(t)$ and the horizontal line $y = 24$



The $x-$coordinate of the intersection point is the solution.

Although a calculator method is preferred, an algebraic approach is also included in case you attempted it that way

$$0.1 = e^{-0.2099...t}$$

$$\ln 0.1 = -0.2099\ldots t$$

$$t = \frac{\ln 0.1}{-0.2099\ldots}$$

$$t = 10.96\ldots$$ **A1**

$$t = 11 \quad \text{to the nearest hour}$$

Converting the answer to a time we get

$$9{:}00 + 11 \text{ hours} = 20{:}00$$

Therefore the earliest time, to the nearest hour, the patient will be allowed to go home is

$$\boxed{20{:}00 \text{ or } 8{:}00 \text{ pm}}$$ **A1**

Question 6

CALCULATOR

Medium

[Maximum mark: 7]

(a) Write $\frac{3x-7}{3x^2+x-2}$ in form $\frac{A}{x+1}+\frac{B}{3x-2}$, where $x \neq -1, \frac{3}{2}$. [2]

(b) Hence determine the first three terms of the binomial expansion of $\frac{3x-7}{3x^2+x-2}$. [5]

(a) The process of rewriting a single fraction as two (or more!) fractions is known as partial fractions.

Let's setup an identity using the information given, this means the identity is true for all valid values of x.

$$\frac{3x-7}{3x^2+x-2} \equiv \frac{A}{x+1}+\frac{B}{3x-2}$$

We can then multiply throughout by $3x^2+x-2$ to get

$$3x-7=(3x-2)A+(x+1)B \qquad \textbf{(M1)}$$

Next expand the R.H.S. and then collect together the coefficients of x and the constant

$$3x-7=3Ax-2A+Bx+B$$

$$3x-7=(3A+B)x-2A+B$$

We can now equate the coefficients of x and the constant term, to form two equations

$$3A+B=3$$

$$-2A+B=-7$$

Subtracting the two equations gives

$$5A = 10$$
$$A = 2$$

And

$$3(2) + B = 3$$
$$B = -3$$

Hence we have

$$\frac{3x-7}{3x^2+x-2} = \boxed{\frac{2}{x+1} - \frac{3}{3x-2}} \qquad \textbf{A1}$$

(b) First we can rewrite both quotients in index form

$$2(x+1)^{-1} - 3(3x-2)^{-1}$$

Next consider the given formula for the extension of the binomial theorem

$$(a+b)^n = a^n\left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \ldots\right)$$

Let's begin with the first quotient

$$2(x+1)^{-1}$$

We can see that $a = 1$, $b = x$ and $n = -1$, hence the first three terms are

$$2(x+1)^{-1} = 2 \times 1^{-1}\left(1 + (-1)\left(\frac{x}{1}\right) + \frac{-1((-1)-1)}{2!}\left(\frac{x}{1}\right)^2\right)$$
$$= 2 - 2x + 2x^2 \qquad \textbf{(A1)}$$

Now lets consider the other quotient

$$-3(3x-2)^{-1}$$

This one is not as straight forward.

In order to use the formula effectively it is best to try and rewrite the quotient by factorising out the a^n term, this is the constant term inside the bracket

$$\begin{aligned} -3(3x-2)^{-1} &= -3(-2\left(-\tfrac{3}{2}x+1\right))^{-1} \\ &= -3 \times (-2)^{-1}\left(1-\tfrac{3}{2}x\right)^{-1} \\ &= \tfrac{3}{2}\left(1-\tfrac{3}{2}x\right)^{-1} \end{aligned}$$

In this form we can now see that $a=1$, $b=-\frac{3}{2}x$ and $n=-1$, hence we get

$$\begin{aligned} \tfrac{3}{2}\left(1-\tfrac{3}{2}x\right)^{-1} &= \tfrac{3}{2}\left(1+(-1)\left(-\tfrac{3}{2}x\right)+\frac{-1((-1)-1)}{2!}\left(-\tfrac{3}{2}x\right)^2\right) && \textbf{(M1)} \\ &= \tfrac{3}{2}\left(1+\tfrac{3}{2}x+\tfrac{9}{4}x^2\right) \\ &= \tfrac{3}{2}+\tfrac{9}{4}x+\tfrac{27}{8}x^2 && \textbf{(A1)} \end{aligned}$$

Hence we need to sum the two results which gives

$$\begin{aligned} 2(x+1)^{-1}-3(3x-2)^{-1} &= 2-2x+2x^2+\tfrac{3}{2}+\tfrac{9}{4}x+\tfrac{27}{8}x^2 && \textbf{M1} \\ &= \boxed{\tfrac{7}{2}+\tfrac{1}{4}x+\tfrac{43}{8}x^2} && \textbf{A1} \end{aligned}$$

Question 7

CALCULATOR Medium

[Maximum mark: 7]

Consider the real polynomial

$$p(z) = 2z^3 + az^2 + bz - 75$$

One of the roots of $p(z)$ is $3 + 4i$.

(a) Find the remaining roots of $p(z)$. [4]

(b) Hence determine the values of a and b. [3]

(a) We know, due to the Fundamental Theorem of Algebra, that a polynomial of order 3 must have 3 roots.

Also, $p(z)$ is a real polynomial, this means its coefficients are real numbers and that any complex roots **must** come in conjugate pairs.

Hence we can write down the second root as $\boxed{3 - 4i}$.

A1

We have one root remaining, it must be real.

Looking at the polynomial we can see we have been given the first and last coefficient.

Consider the formulae for the sum and products of the roots of a polynomial

$$\text{Sum of roots} = \frac{-a_{n-1}}{a_n} \qquad \text{Product of roots} = \frac{(-1)^n a_0}{a_n}$$

n is the order of the polynomial, which in this case is 3.

a_n, where $n = 3$ refers to the coefficient of the z^3 term and a_0 refers to the constant term.

We can write down the values

$$a_3 = 2$$

$$a_0 = -75 \qquad \textbf{(M1)}$$

This is enough information to find the value of the product of the roots.

Hence

$$\begin{aligned}\text{Product of roots} &= \frac{(-1)^n a_0}{a_n} \\ &= \frac{(-1)^3 \times (-75)}{2} \\ &= \frac{75}{2}\end{aligned}$$ **A1**

If we call the third root z_3 we can form an equation which we can solve on our G.D.C.

$$\begin{aligned}(3+4i)(3-4i)z_3 &= \frac{75}{2} \\ z_3 &= \boxed{\frac{3}{2}}\end{aligned}$$ **A1**

(b) We can use the roots from part (a) to write out $p(z)$ in factorised form

$$(z-(3+4i))(z-(3-4i)(z-\tfrac{3}{2})=0$$

$$(z-3-4i)(z-3+4i)(z-\tfrac{3}{2})=0 \qquad \textbf{(M1)}$$

There is a very useful identity, worth memorising, that is helpful when expanding brackets of this form

$$(z-w)(z-w^*)=z^2-2\text{Re}(w)(z)+|w|^2$$

In our case $w=3+4i$ therefore

$$\begin{aligned}|w|^2 &= (\sqrt{3^2+4^2})^2\\ &= 5^2\\ &= 25\end{aligned}$$

And

$$\begin{aligned}\text{Re}(w) &= \text{Re}(3+4i)\\ &= 3\end{aligned}$$

Putting these results together we can expand the red part of the factorised form to get

$$(z-3-4i)(z-3+4i)(z-\tfrac{3}{2})=0$$

$$(z^2-2\times 3z+25)(z-\tfrac{3}{2})=0 \qquad \textbf{A1}$$

Now we can expand the rest of the expression to get

$$z^3 - 6z^2 + 25z - \frac{3}{2}z^2 + 9z - \frac{75}{2} = 0$$

We can multiply throughout by 2 and simplify to get

$$2z^3 - 12z^2 + 50z - 3z^2 + 18z - 75 = 0$$

$$2x^3 - 15z^2 + 68z - 75 = 0$$

$$p(z) = 2x^3 - 15z^2 + 68z - 75$$

Therefore $\boxed{a = -15}$ and $\boxed{b = 68}$.

A1

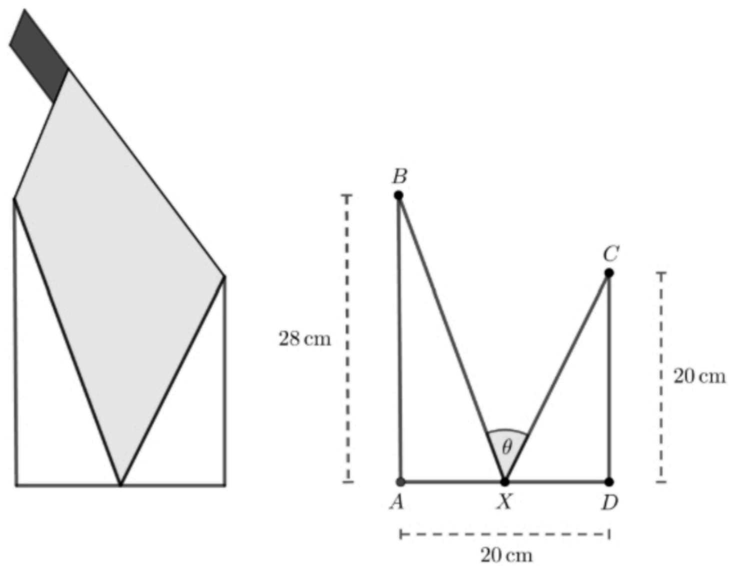
Question 8

CALCULATOR Hard

[Maximum mark: 6]

A group of designers are creating a knife holder.

The diagram on the left shows a knife inside the holder and the diagram on the right shows the empty holder.



The lengths of the holder, $AB = 28$, $AD = 20$ and $CD = 20$ are measured in centimeters and $\angle CXB = \theta$ radians.

The designers can move the vertex X anywhere along the line AD.

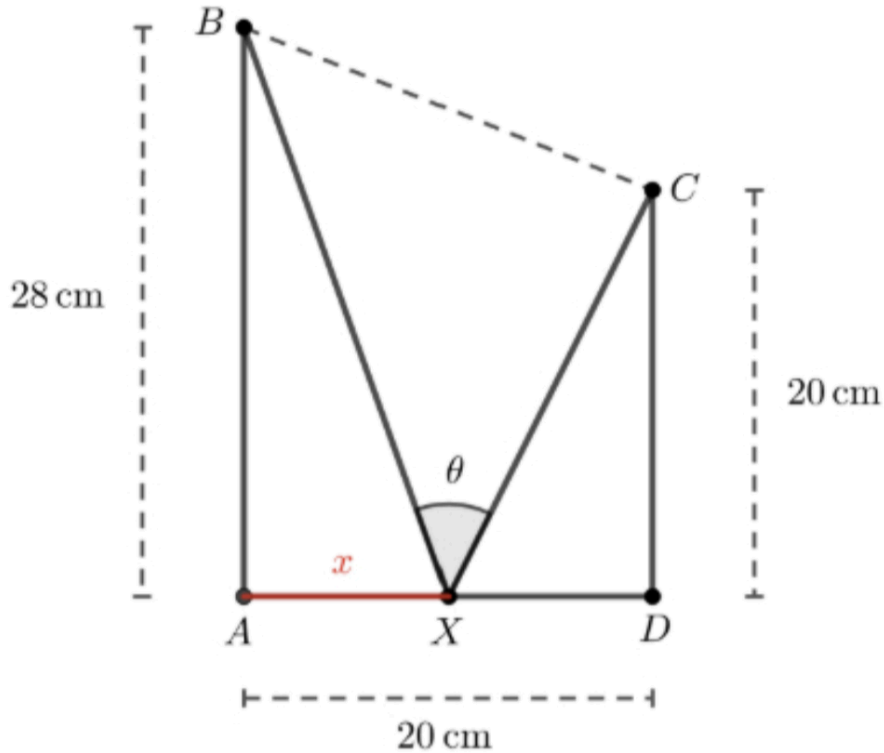
Market research suggests that the most appealing design is such that the value of θ is maximised.

Find the length AX which maximises the value of θ.

This is an optimisation question.

We need to use the information given to obtain the angle θ as a function of the length AX. We can then sketch the function and find the value of x for which θ is maximised.

We can begin by introducing a variable for the length AX, we'll call this x.



The trick to this question, and ones similar, is to use the diagram to spot a way to relate the two variables. In our case θ and x.

The triangle $\triangle BXC$ contains the angle θ. We can write the side lengths of this triangle in terms of in terms of x.

Hence we can use the cosine rule to relate the variables.

(M1)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Note that angle A and side length a are opposite each other.

In our example $a = BC$ and $A = \theta$. The other two sides, b and c, are BX and CX.

Hence we get

$$\cos\theta = \frac{BX^2 + CX^2 - BC^2}{2 \times BX \times CX}$$

$$\theta = \arccos\left(\frac{BX^2 + CX^2 - BC^2}{2 \times BX \times CX}\right)$$

Now we need to write the side lengths in terms of x.

Let's first consider triangle $\triangle ABX$. Using Pythagoras' theorem we can write

$$BX^2 = AX^2 + AB^2$$

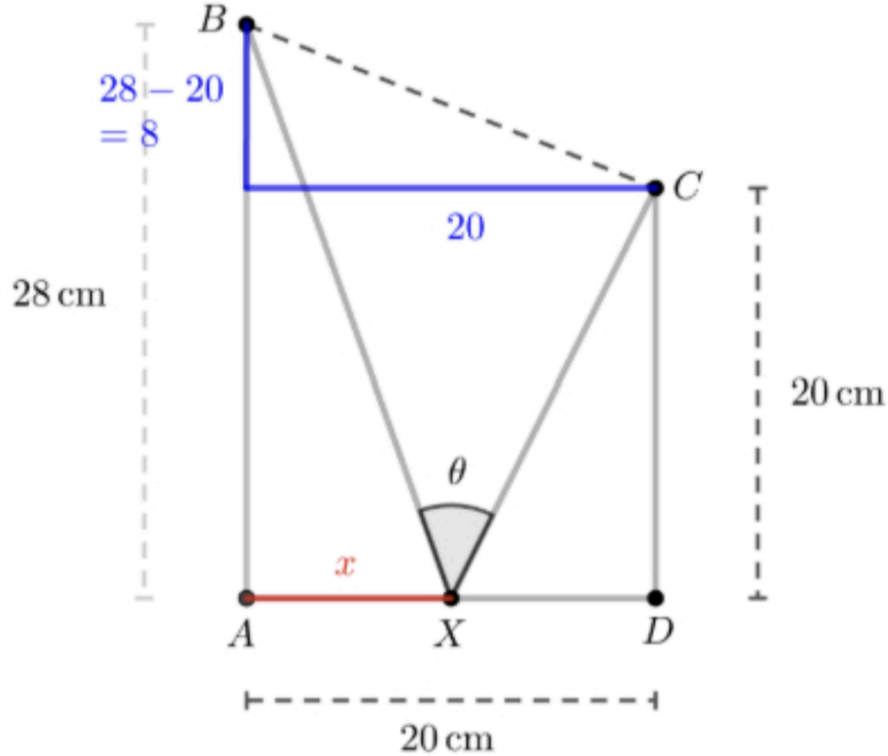
$$BX^2 = 28^2 + x^2 \qquad \textbf{(A1)}$$

Similarly, using the length $AX = 20 - x$, we get for triangle $\triangle CDX$

$$CX^2 = CD^2 + XD^2$$

$$CX^2 = 20^2 + (20 - x)^2 \qquad \textbf{(A1)}$$

And finally, to find length BC, using the diagram below



We can see that

$$BC^2 = 8^2 + 20^2$$

$$BC^2 = 464$$

Substituting all of these results into the cosine rule we get

$$\theta = \arccos\left(\frac{BX^2 + CX^2 - BC^2}{2 \times BX \times CX}\right)$$

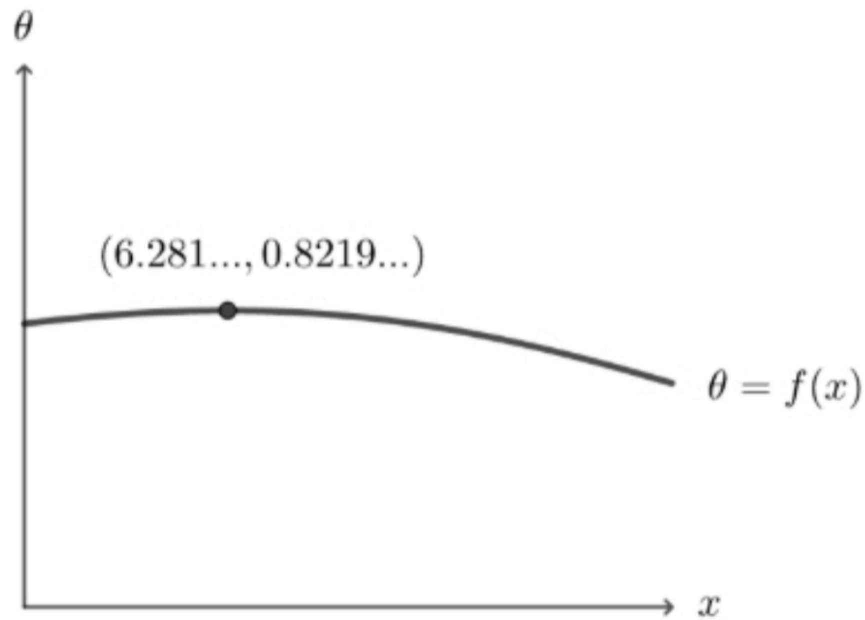
$$= \arccos\left(\frac{28^2 + x^2 + 20^2 + (20 - x)^2 - 464}{2\sqrt{28^2 + x^2}\sqrt{20^2 + (20 - x)^2}}\right) \qquad \textbf{A1}$$

Notice, we haven't simplified/expanded anything. We don't need to, we can simply let the calculator do the work.

This type of technique not only reduces the chance of making mistakes it also saves time! So it is an important exam technique.

We should now sketch the function on our G.D.C. and find the maximum point.

(M1)



Hence we can see that when $AX = 6.28\,\text{cm}$ the value of θ is maximised.

A1

Note, the maximum value of θ is 0.822 radians, however the question does not require us to state this.

Question 9

CALCULATOR

Hard ●●●●

[Maximum mark: 8]

Consider the differential equation

$$\frac{d^2y}{dx^2} = -10xe^{-x^2}$$

Where $y = -1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

(a) Find $\frac{dy}{dx}$ in terms of x. [2]

(b) Use Euler's method with a step length of 0.1 to estimate a value for y when $x = 0.5$. [4]

(c) Hence justify whether your answer to part (b) is an overestimate or an underestimate. [2]

(a) We can find the first derivative by integrating the second derivative.

$$\int \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \int -10xe^{-x^2}\,\mathrm{d}x$$

Using integration by substitution

$$\text{Let } u = -x^2$$

$$\text{Therefore } \mathrm{d}u = -2x\mathrm{d}x$$

$$\text{and } \mathrm{d}x = -\frac{1}{2x}\mathrm{d}u$$

Replacing $\mathrm{d}x$ we get

$$\int \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \int -10xe^{u} \times -\frac{1}{2x}\,\mathrm{d}u$$

$$= \int 5e^{u}\mathrm{d}u$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5e^{u} + c$$ **(A1)**

To find the value of c, we can use the given information of $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

$$2 = 5e^0 + c$$

$$2 = 5 + c$$

$$c = -3$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5e^{-x^2} - 3$$ **A1**

(b) We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5e^{-x^2} - 3$$
$$y(0) = -1$$
$$h = 0.1$$

Using the formulae we get

$$x\text{-iteration:} \quad x_{n+1} = x_n + 0.1$$
$$y\text{-iteration:} \quad y_{n+1} = y_n + 0.1 \times \left[\frac{\mathrm{d}y}{\mathrm{d}x} = 5e^{-(x_n)^2} - 3\right] \qquad [\text{for } n \in \mathbb{N}] \qquad \textbf{(A1)}$$

Using Euler's method, we obtain

(M1)

n	x_n	y_n
0	0	-1
1	0.1	-0.8
2	0.2	$-0.6049...$
3	0.3	$-0.4245...$
4	0.4	$-0.2676...$
5	0.5	$-0.1415...$

A1

Hence we find the estimate of y when $x = 0.5$ to be $\boxed{-0.142}$

A1

(c) We can use the second derivative to decide if our value is an over or underestimate.

We need to decide what sign the second derivative takes near to our estimated value.

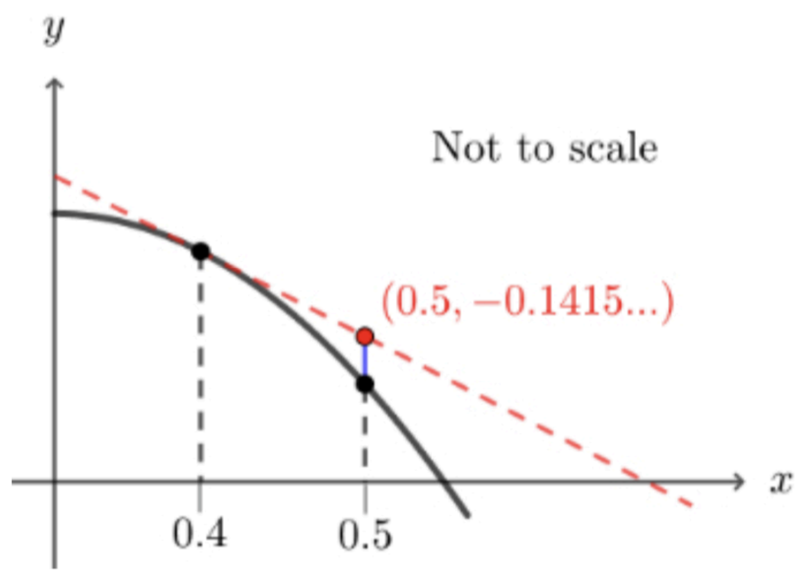
$$\frac{d^2y}{dx^2} = -10xe^{-x^2} \qquad \text{as } x > 0 \text{ and } e^{-x^2} > 0$$
$$< 0$$

Therefore we know that $f(x)$ is concave-down for all $x > 0$.

A1

Here is a visual of a function that is concave down. Note, this is not to scale.

Notice how the estimated point, in red, is vertically above the actual value on the curve.



The blue solid line is the difference between the estimated and the actual value.

Therefore Euler's method has produced an overestimate.

A1

Section B

Question 10

CALCULATOR   Hard ●●●●

[Maximum mark: 17]

Juanita wants to borrow some money to buy an apartment.

She finds an offer allowing her to borrow $480,000 over 10 years with an interest rate of $r\%$ P.A. compounded monthly. She repays the loan with a fixed amount p every month.

Juanita takes the loan out at the beginning of the month. At the end of the month, the interest is added **and then** she makes the monthly payment of p.

This continues until after 10 complete years, she has repaid the loan in its entirety.

Juanita wants to analyse three different scenarios in which she could repay the loan.

(a) In the first scenario her monthly payment is $p = \$5\,000$.

If $k = 1 + \frac{r}{1200}$

(i) Write down the number of payments that will be made over the entire 10 year term of the loan.

(ii) Show that

$$96k^{120} = \frac{k^{120} - 1}{k - 1}$$

(iii) Hence, or otherwise, find r. [6]

(b) In the second scenario Juanita uses the same values for p and r as part (a). She makes the monthly payments of p for 7 years and 4 months.

She then makes a final payment to clear the remaining balance of the loan.

(i) Find the number of payments she makes **before** the final payment.

(ii) Hence, find the final payment required to clear the remaining balance to 4 significant figures. [3]

(c) In the third scenario Juanita pays p per month for 5 complete years and then she increases her monthly loan repayment to $2p$ for the remaining 5 years.

Find the value of p, to the nearest dollar, for the third scenario. [8]

(a) (i) There are 12 payments per year for 10 years

$$= 10 \times 12$$

$$= \boxed{120}$$ **A1**

(ii) Note: Multiplying by k represents an increase of $r\%$.

We can build up an expression which represents the remaining value, FV, of the loan. After the first month, this remaining value is increased by $r\%$ and then the payment \$5 000 is subtracted

$$FV = 480\,000 \times k - 5\,000$$

$$= 480\,000k - 5\,000$$

We then continue into the second month, this process is repeated. The remaining value is increased by $r\%$ and the payment \$5 000 is subtracted

$$FV = (480\,000k - 5\,000) \times k - 5\,000$$

$$= 480\,000k^2 - 5\,000k - 5\,000$$

Similarly for the third month

(M1)

$$FV = (480\,000k^2 - 5\,000k - 5\,000) \times k - 5\,000$$

$$= 480\,000k^3 - 5\,000k^2 - 5\,000k - 5\,000$$

This continues for all 120 payments and can be shown using the following series

$$FV = 480\,000k^{120} - 5\,000k^{119} - 5\,000k^{118} - 5\,000k^{117} - \dots - 5\,000k - 5\,000$$

(A1)

As the loan has been repaid in full we can replace FV with 0.

(M1)

$$0 = 480\,000k^{120} - 5\,000k^{119} - 5\,000k^{118} - 5\,000k^{117} - \dots - 5\,000k - 5\,000$$

We can then rearrange the equation above to get

$$480\,000k^{120} = 5\,000k^{119} + 5\,000k^{118} + 5\,000k^{117} + \dots + 5\,000k + 5\,000$$

Recognise that the R.H.S. of the above is a geometric series with a first term of 5 000 and a common ratio of k. Hence we can use the sum of a geometric sequence formula to write

$$480\,000k^{120} = 5\,000\left(\frac{k^{120}-1}{k-1}\right) \qquad \textbf{A1}$$

$$\boxed{96k^{120} = \frac{k^{120}-1}{k-1}} \qquad \text{... as required.}$$

(iii) Using a G.D.C. we can solve the equation found above.

$$96k^{120} = \frac{k^{120}-1}{k-1}$$

$$k = 1.0038412\ldots \qquad \text{[by using G.D.C.]}$$

Hence

$$1.0038412\ldots = 1 + \tfrac{r}{1200}$$

$$r = 4.609419\ldots \qquad \textbf{A1}$$

$$= \boxed{4.61\%} \qquad \text{[by using G.D.C.]}$$

(b) (i) She makes 7 full years of payments plus an additional 4 months, which gives

$$= 7 \times 12 + 4$$

$$= 88$$

She makes 88 payments before the final payment.

A1

(ii) You could use the Financial Solver on the G.D.C. with the following values

N	I	PV	Pmt	FV	PpY	CpY
88	4.609...%	\$480 000	−\$5 000		12	12

(A1)

Solving for FV gives a remaining value of 150 286.301....

Hence the final payment is \$150 300.

A1

(c) Juanita pays p per month for 60 months, she then pays $2p$ per month for the remaining 60 months.

(A1)

Lets consider the first 60 payments of p using a similar method to part (a) where $k = 1 + \frac{4.61}{1200} = 1.0038...$

$$FV = ((480\,000 \times k - p) \times k - p)...$$

$$FV = 480\,000k^{60} - pk^{59} - pk^{58}... - pk - p \qquad \textbf{A1}$$

Now let's consider the next 60 payments of $2p$ which will clear the remaining balance, which means the future value is \$0

(M1)

$$0 = ((480\,000k^{60} - pk^{59} - pk^{58}... - p) \times k - 2p) \times k - 2p)...$$

$$0 = 480\,000k^{120} - pk^{119}... - pk^{60} - 2pk^{59} - 2pk^{58} - ... - 2pk - 2p$$

$$480\,000k^{120} = pk^{119}... + pk^{60} + 2pk^{59} + 2pk^{58} + ... + 2pk + 2p \qquad \textbf{A1}$$

This equation has a single unknown (p). However in its current form we cannot enter it into the calculator as there are too many terms!

We need to find a way to rewrite the sum in a simpler form.

We can do this by summing the RHS of the equation in two parts. We will let the first 60 terms be part A and the last 60 terms be part B

To find the sum of part A we can use the formula directly, pasrt A has parameters $u_1 = 2p$, $r = 1.0038...$ and $n = 60$, therefore

$$S_A = 2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1} \qquad \textbf{A1}$$

Let's reconsider the original equation and replace part A with the simpler form found above

$$480\,000k^{120} = \underbrace{pk^{119}... + pk^{60}}_{\text{B}} + \underbrace{2pk^{59} + 2pk^{58} + ... + 2pk + 2p}_{\text{A}}$$
$$480\,000k^{120} = \underbrace{pk^{119}... + pk^{60}}_{\text{B}} + \underbrace{2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{A}}$$

Part B is more complicated. We can't simply sum 120 terms as we only want to sum the last 60 terms of the sequence.

Lets find the sum of the full 120 terms and then subtract the sum of the first 60 terms.

Part B has parameters $u_1 = p$, $r = 1.0038...$ and $n = 120$, therefore

$$S_B = p\frac{(1.0038...)^{120} - 1}{(1.0038...) - 1} - p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}$$ **M1A1**

Let's now rewrite the original equation and replace S_B with the formula above

$$480\,000(1.0038...)^{120} = \underbrace{p\frac{(1.0038...)^{120} - 1}{(1.0038...) - 1} - p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{B}} + \underbrace{2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{A}}$$

We can now solve this on the G.D.C.

$$p = 3465.61...$$

$$p = \boxed{\$3466}$$ **A1**

Question 11

CALCULATOR

Medium

[Maximum mark: 21]

A continuous random variable, X, has a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{4}x, & 0 \leq x \leq a \\ \frac{2}{7} - \frac{1}{28}x, & a \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases}$$

(a) It is known that $\mathrm{P}(X \leq a) = 0.125$.

(i) Show that $a = 1$.

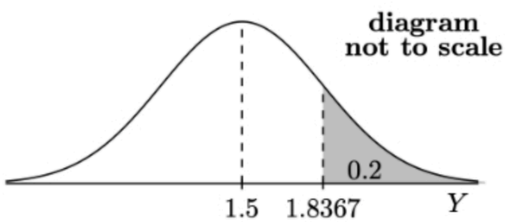
(ii) Hence show that $\mathrm{E}(X) = 3$. [4]

(b) Find $\mathrm{P}(0.5 \leq X \leq 2)$. [3]

(c) Given that $\mathrm{E}(X^2) = \frac{73}{6}$ find $\mathrm{Var}(X)$. [1]

Another continuous random variable, Y, is normally distributed with a mean of 1.5 and a standard deviation of σ_Y.

For this distribution it is known that $\mathrm{P}(Y \geq 1.8367) = 0.2$. This information is shown below



(d) Find σ_Y. [3]

(e) Hence find $\mathrm{P}(0.5 \leq Y \leq 2)$. [1]

A water utility company serves a large number of households. 90% of their households are classed as **regular** households. The rest are classed as **premium** households.

The water usage per day, in m^3, of the regular households is modeled by the random variable X.

The water usage per day, in m^3, of the premium households is modeled by the random variable Y.

If a household uses between $0.5\,m^3$ and $2\,m^3$ of water per day they are eligible for a special deal.

(f) A household is chosen at random. It is found that they are eligible for a special deal. What is the probability they are a premium household? [3]

The water company charges the households a daily fee based on how much water is used in that day.

The charge per day, C_X, in \$US, for the regular households is calculated using the following formula

$$C_X = 1.5X + 0.5$$

(g) Find
(i) $\mathrm{E}(C_X)$.

(ii) $\mathrm{Var}(C_X)$. [3]

Premium households are charged in a different way to regular households.

The charge per day, C_Y, in \$US, for the premium households is calculated using the following formula

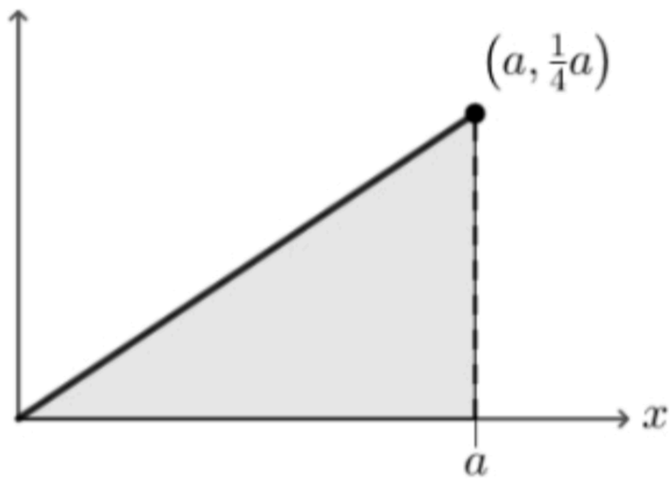
$$C_Y = sY + t \qquad \text{where } s, t \in \mathbb{R}.$$

If the value of $\mathrm{E}(C_Y)$ is \$1 less than the value of $\mathrm{E}(C_X)$ and $\mathrm{Var}(C_Y) = \frac{32}{57}\mathrm{Var}(C_X)$ then

(h) Given that $s > 0$, find s and t. [3]

(a) (i) The probability density function (P.D.F.) of X is a piecewise function consisting of two straight lines.

We can sketch the first part of the P.D.F., from $x = 0$ to $x = a$, noting that the end-point has coordinates $(a, \frac{1}{4}a)$



From the information given we know the area under the graph for this part is 0.125.

Hence we can form an equation for the area of the triangle

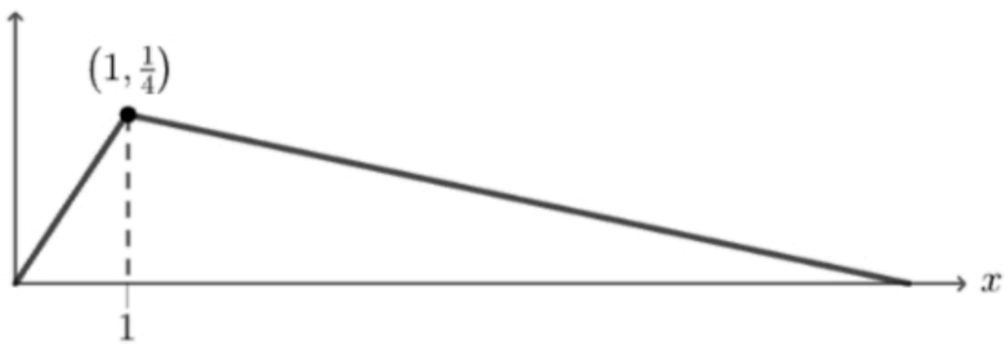
$$0.125 = \frac{a \times \frac{1}{4}a}{2}$$

$$a^2 = 1$$

$$a = \boxed{1}$$ **A1**

From the information given we can see that $a > 0$.

(ii) We can make a rough sketch of the rest of the P.D.F., noting that the two lines do intersect when $x = 1$.



We can use the formula for the expected value of a continuous random variable but we must be careful to evaluate the integral over both parts and sum the results

$$\mathrm{E}(X) = \int_0^1 x \times \tfrac{1}{4}x \, \mathrm{d}x + \int_1^8 x \times \left[\tfrac{2}{7} - \tfrac{1}{28}x\right] \, \mathrm{d}x$$ **M1**

We can integrate each term to get

$$= \left[\tfrac{1}{12}x^3\right]_0^1 + \left[\tfrac{1}{7}x^2 - \tfrac{1}{84}x^3\right]_1^8$$ **A1**

$$= \tfrac{1}{12} + \left(\tfrac{64}{7} - \tfrac{512}{84}\right) - \left(\tfrac{1}{7} - \tfrac{1}{84}\right)$$

$$= \frac{1}{12} + \frac{35}{12}$$ **A1**

$$= \boxed{3}$$ ...as required.

(b) We can identify on our sketch the region whose area we need to find

In a similar fashion to the previous part we need to evaluate the different parts of the function and then sum them together.

Hence

$$\begin{aligned} \mathrm{P}(0.5 \leq X \leq 2) &= \int_{0.5}^{1} \tfrac{1}{4}x \, \mathrm{d}x + \int_{1}^{2} \left[\tfrac{2}{7} - \tfrac{1}{28}x\right] \mathrm{d}x \quad \textbf{(M1A1)} \\ &= 0.3258... \\ &= \boxed{0.326} \quad \textbf{A1} \end{aligned}$$

(c) Using the formula for variance and the value for $\mathrm{E}(X)$ from part (a) we get

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\ &= \tfrac{73}{6} - 3^2 \\ &= \boxed{\frac{19}{6}} \quad \textbf{A1} \\ &= 3.166.... \end{aligned}$$

(d) To find an unknown standard deviation we need to use the $Z-$distribution along with the standardisation formula.

This distribution is defined as $Z \sim N(0, 1^2)$.

Recall, $\mathrm{P}(Y \geq 1.8367) = 0.2$, we want to find the corresponding $Z-$score.

$$\text{inverseNorm}(0.8,\ 0,\ 1) = 0.8416... \qquad \textbf{(A1)}$$

Notice we used 0.8, this is because the 0.2 region is in the upper tail of the distribution.

We can now use the standardised normal variable formulae, with Y replacing X

$$Z = \frac{Y - \mu}{\sigma}$$

We know that $\mu = 1.5$ and when $Y = 1.8367$ then $Z = 0.8416...$, hence we can form an equation and solve for σ_Y

$$0.8416... = \frac{1.8367 - 1.5}{\sigma_Y} \qquad \textbf{(M1)}$$

$$\sigma_Y = 0.4000...$$

$$= \boxed{0.400} \qquad \textbf{A1}$$

(e) We can now fully define the distribution for Y

$$Y \sim N(1.5, 0.400^2)$$

Hence, using a G.D.C. we find

$$\mathrm{P}(0.5 \leq Y \leq 2) = 0.8878...$$

$$= \boxed{0.888} \qquad \textbf{A1}$$

(f) In this question we have been given a condition. We are told that the chosen customer is eligible for the special deal.

Consider the conditional probability formula

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

If we assign event A as 'the household is a premium customer' and event B as 'the household is eligible for the special deal', then $\mathrm{P}(A|B)$ is the probability that the household is premium **given** they are eligible.

First consider $\mathrm{P}(B)$. This represents the probability that the chosen customer is eligible for a special deal.

The calculation for this includes **both** the regular households who are eligible and the premium households who are eligible.

We can use our answers from part (b) and (e) and the given information about the proportion of household type to get

$$\begin{aligned}\mathrm{P}(B) &= 0.9 \times 0.3258... + 0.1 \times 0.8878...\\ &= 0.3820... \qquad \textbf{(A1)}\end{aligned}$$

To find $\mathrm{P}(A \cap B)$, the household is premium **and** eligible, we calculate

$$\begin{aligned}\mathrm{P}(A \cap B) &= 0.1 \times 0.8878...\\ &= 0.08878...\end{aligned}$$

Therefore putting these results together we get

$$\begin{aligned}\mathrm{P}(A|B) &= \frac{0.08878...}{0.3820...} \qquad \textbf{(M1)}\\ &= 0.2323...\\ &= \boxed{0.232} \qquad \textbf{A1}\end{aligned}$$

(g) This part of the questions concerns the linear transform of a random variable.

We will be using the following formulae

(M1)

$$\mathrm{E}(aX+b) = a\mathrm{E}(X) + b$$
$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

(i) Using the formula above we get

$$\begin{aligned}\mathrm{E}(C_X) &= \mathrm{E}(1.5X+0.5)\\ &= 1.5\,\mathrm{E}(X) + 0.5\end{aligned}$$

From part (a) we know that $\mathrm{E}(X) = 3$, hence

$$\begin{aligned}&= 1.5 \times 3 + 0.5\\ &= \boxed{\$5} \quad \text{per day}\end{aligned}$$

A1

(ii) Similarly we get

$$\begin{aligned}\mathrm{Var}(C_X) &= \mathrm{Var}(1.5X+0.5)\\ &= 1.5^2\,\mathrm{Var}(X)\end{aligned}$$

Using our answer from part (c) we get

$$\begin{aligned}\mathrm{Var} &= 1.5^2 \times \frac{19}{6}\\ &= \boxed{7.125} \quad \text{dollars}^2 \text{ per day}\\ &= \tfrac{57}{8}\end{aligned}$$

A1

(h) We can begin by forming two equations from the information provided

$$\mathrm{E}(C_Y) = \mathrm{E}(C_X) - 1 \qquad \mathrm{Var}(C_Y) = \tfrac{32}{57}\mathrm{Var}(C_X)$$

Using the answers from part (g) we get

$$\begin{aligned} \mathrm{E}(C_Y) &= 5 - 1 & \mathrm{Var}(C_Y) &= \tfrac{32}{57} \times \tfrac{57}{8} \\ &= 4 & &= 4 \end{aligned} \qquad \textbf{(A1)}$$

Recall $C_Y = sY + t$, hence, using the results above, we get

$$\begin{aligned} \mathrm{E}(C_Y) &= \mathrm{E}(sY + t) & \mathrm{Var}(C_Y) &= \mathrm{Var}(sY + t) \\ 4 &= s\,\mathrm{E}(Y) + t & 4 &= s^2\,\mathrm{Var}(Y) \end{aligned}$$

In part (d) we found $\sigma_Y = 0.4000...$, therefore

$$\begin{aligned} \mathrm{Var}(Y) &= \sigma_Y^2 \\ &= 0.4000...^2 \\ &= 0.1600... \end{aligned}$$

Recall, we know that $\mu_Y = 1.5$, hence we get

$$4 = s \times 1.5 + t \qquad 4 = s^2 \times 0.1600...$$

Solving the equation on the right we get

$$\begin{aligned} s^2 &= \frac{4}{0.1600...} \\ s &= 4.999... \\ &= \boxed{5.00} \end{aligned} \qquad \textbf{A1}$$

As $s > 0$ we take only the positive value.

Using the other equation we can find t

$$\begin{aligned} 4 &= 1.5 \times 4.999... + t \\ t &= -3.498... \\ &= \boxed{-3.50} \end{aligned} \qquad \textbf{A1}$$

Question 12

CALCULATOR Hard

[Maximum mark: 16]

Consider the planes Π_1, Π_2 and Π_3 with the following equations.

$$\begin{aligned} \Pi_1 : &\quad x - y + z = -4 \\ \Pi_2 : &\quad 2x + y - z = -1 \\ \Pi_3 : &\quad -x + y + kz = -3 \end{aligned}$$

Where $k \in \mathbb{R}$.

The system of equations that represents the three planes is inconsistent.

(a) (i) Find k.

(ii) Describe the geometrical relationship of the three planes. [3]

L is the line of intersection between Π_1 and Π_2 and it crosses the $xy-$plane at point D.

(b) (i) Verify that the vector equation of L can be written as

$$\boldsymbol{r} = \begin{pmatrix} -\frac{5}{3} \\ 0 \\ -\frac{7}{3} \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Hence find the coordinates of point D. [4]

A fourth plane, Π_4, is created by reflecting Π_1 in Π_3.

(c) Find the Cartesian equation of Π_4. [9]

(a) (i) A system is inconsistent when there is no solution of any type - either a single solution or infinite solutions.

We should use Guassian Elimination to attempt to solve the system in terms of k

Let's begin by adding the equations for Π_1 and Π_3 this gives

$$x - y + z + (-x + y + kz) = -4 + (-3)$$ **(M1)**

$$z + kz = -7$$

$$z(k + 1) = -7$$

We can see here that if $k = -1$ then we have $z \times 0 = -7$ which leads to an impossible (inconsistent) equation to solve.

Therefore when $\boxed{k = -1}$ the system is inconsistent.

A1

(ii) There are a number of different geometrical arrangements of planes that could lead to an inconsistent solution.

Let's check for parallelism first.

If the L.H.S. of one plane is a multiple of the L.H.S. of another plane and the constants on the R.H.S.'s are different then the planes are parallel.

Using $k = -1$ and rewriting Π_3, we get

$$\begin{aligned} \Pi_1 : \quad & x - y + z = -4 \\ \Pi_3 : \quad & -(x - y + z) = -3 \end{aligned}$$

We can't do a similar manipulation with Π_2.

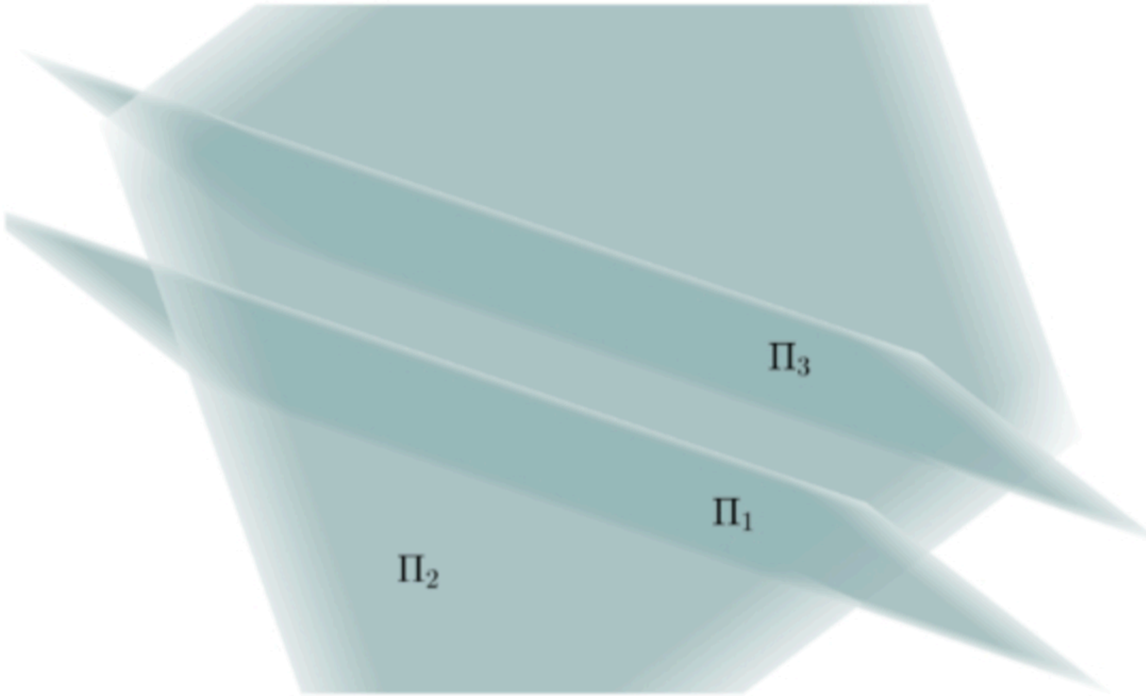
Hence plane Π_1 and Π_3 are parallel

and

Π_2 is oblique as it is distinct to Π_1 and Π_3 and parallel to neither.

R1

Although not required by the question, here is a visual showing the two parallel planes and the oblique plane.



(b) (i) There are a number of ways to answer this question.

One way to check that the line is the intersection of both planes is to verify that the line lies in both planes for any value of λ.

Hence we will substitute each component of line L into the L.H.S. of Π_1 and Π_2 and ensure the result equals the R.H.S.

M1

First let's try Π_1

$$\begin{aligned}\Pi_1: \text{ L.H.S.} &= x - y + z \\ &= \left(-\tfrac{5}{3}\right) - (\lambda) + \left(-\tfrac{7}{3} + \lambda\right) \\ &= -\tfrac{12}{4} \\ &= -4 \\ &= \text{R.H.S.}\end{aligned}$$

As the L.H.S. equals the R.H.S. for any value of λ this means L lies in Π_1.

Let's do the same again for Π_2

$$\begin{aligned}\Pi_2: \text{ L.H.S.} &= 2x + y - z \\ &= 2\left(-\tfrac{5}{3}\right) + (\lambda) - \left(-\tfrac{7}{3} + \lambda\right) \\ &= -\tfrac{3}{3} \\ &= -1 \\ &= \text{R.H.S.}\end{aligned}$$

Hence L also lies in Π_2.

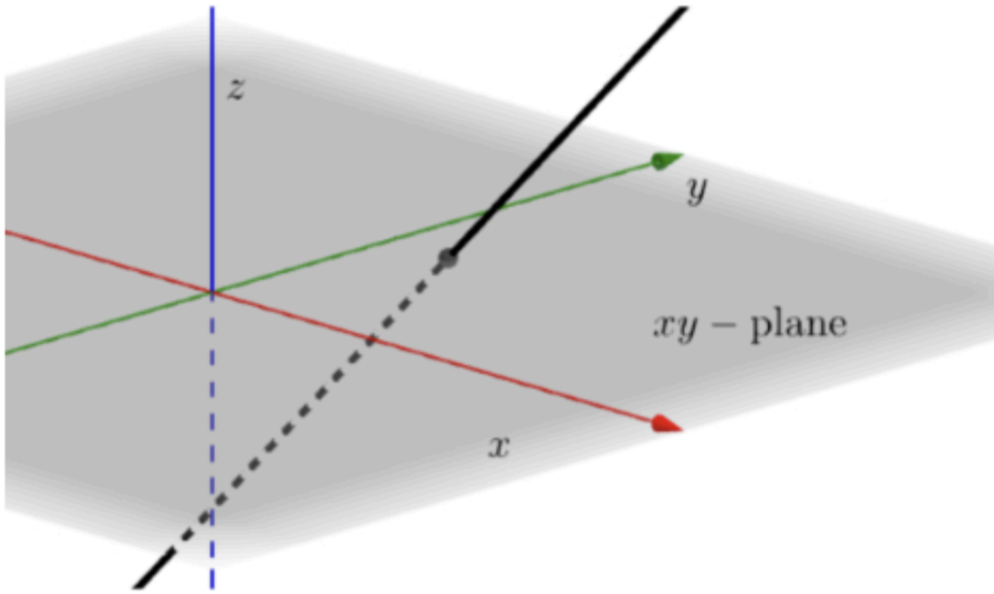
As L lies in both Π_1 and Π_2 and as

Π_1 and Π_2 are not parallel it must be the line of intersection of both planes.

A1

(ii) The value of the $z-$component of a line that intersects the $xy-$plane is zero.

This diagram can help you visualise it



We need to find the value of λ that makes the $z-$component of L equal zero.

$$r_z: -\tfrac{7}{3} + \lambda = 0 \qquad \textbf{(M1)}$$
$$\lambda = \tfrac{7}{3}$$

Hence the coordinates of D when $\lambda = \frac{7}{3}$ are

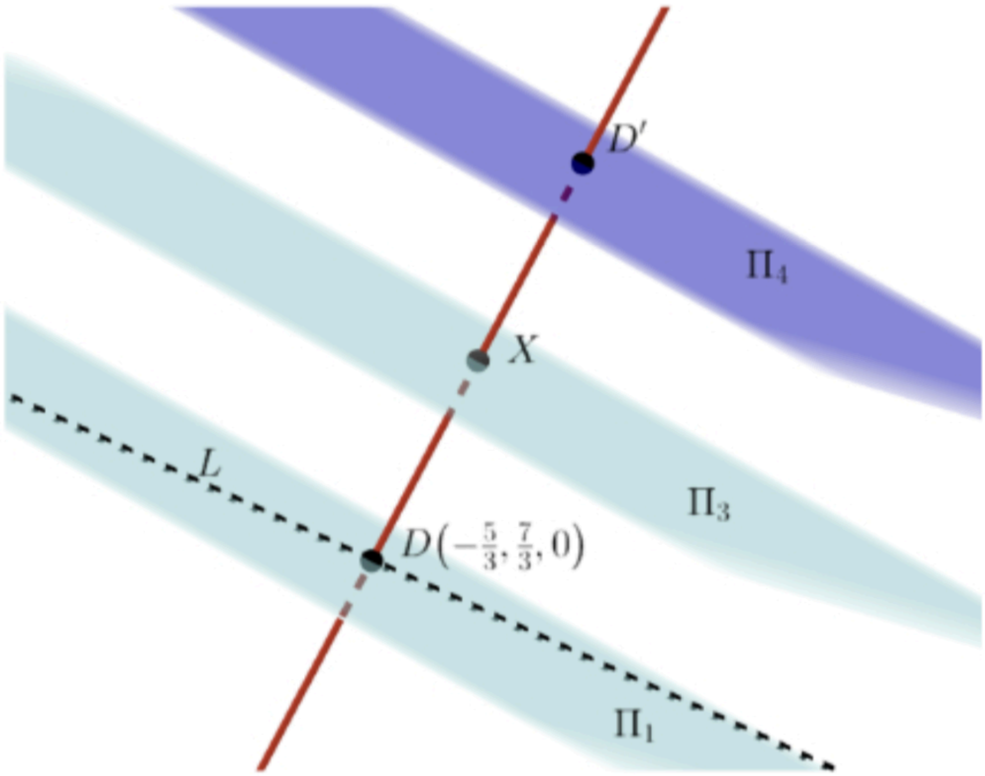
$$D\left(-\tfrac{5}{3} + 0, 0 + \tfrac{7}{3} \times 1, -\tfrac{7}{3} + \tfrac{7}{3}\right)$$
$$\boxed{D\left(-\tfrac{5}{3}, \tfrac{7}{3}, 0\right)} \qquad \textbf{A1}$$

(c) Before we begin answering the question let's outline a method.

We can make a rough sketch of the planes to help us.

The diagram below shows Π_1, Π_3 and Π_4.

Π_2 is not shown.



In order to find Π_4 we need to find a point in Π_4.

It makes sense to use point D, from part (b) (ii), to help us - however we don't need to pick D, we could find a different point in Π_4!

Let the point D' be the reflection of point D in Π_3.

Due to the symmetry of reflection, the point D' will lie on the line which is normal to plane Π_1 and goes through D.

This is shown as the red line.

The point X is the point on Π_3 which lies on the red line.

Π_4, shown in blue, is the plane whose equation we want to find.

Note that the vector from D to X is equal to the vector from X to D'.

Hence, we can form an equation using displacement vectors such that

$$\overrightarrow{DX} = \overrightarrow{XD'}$$

The will allow us to find the coordinates of D'.

We can summarise this information into four steps.

- Find the equation of the red line.
- Find the position vector of $\overrightarrow{OX}$.
- Solve the displacement vector equation above to find the coordinates of D'.
- Find the equation of Π_4.

Let's begin by finding the equation of the red line.

Recall the Cartesian form of plane 1

$$\Pi_1: \qquad x - y + z = -4$$

Written in this form, the coefficients of the L.H.S. give the normal, $\boldsymbol{n_1}$, to Π_1. Hence

$$\boldsymbol{n_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore using the coordinates of D and $\boldsymbol{n_1}$ as the direction we can write down the vector equation of the red line, $\boldsymbol{r_R}$, as

$$\boldsymbol{r_R} = \begin{pmatrix} -\frac{5}{3} \\ \frac{7}{3} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \textbf{(A1)}$$

where $\mu \in \mathbb{R}$.

The next step is to find the coordinates of X.

For this we substitute the components of $\boldsymbol{r_R}$ into the plane equation for Π_3 and solve for μ. Hence we get

$$-x+y-z=-3$$

$$-\left(-\tfrac{5}{3}+\mu\right)+\left(\tfrac{7}{3}-\mu\right)-(\mu)=-3 \qquad \textbf{(M1)}$$

$$\mu=\tfrac{7}{3} \qquad \text{[solved on G.D.C.]} \qquad \textbf{A1}$$

Hence the position vector of $\overrightarrow{OX}$ is

$$\overrightarrow{OX}=\begin{pmatrix}-\frac{5}{3}\\ \frac{7}{3}\\ 0\end{pmatrix}+\left(\tfrac{7}{3}\right)\times\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{5}{3}+\frac{7}{3}\\ \frac{7}{3}-\frac{7}{3}\\ \frac{7}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3}\\ 0\\ \frac{7}{3}\end{pmatrix} \qquad \textbf{(A1)}$$

Using the coordinates of $D\left(-\frac{5}{3},\frac{7}{3},0\right)$ we can write down the position vector $\overrightarrow{OD}$ but first we can rewrite the equation mentioned earlier in terms of position vectors

$$\overrightarrow{DX}=\overrightarrow{XD'} \qquad \textbf{M1}$$

$$\overrightarrow{OX}-\overrightarrow{OD}=\overrightarrow{OD'}-\overrightarrow{OX}$$

Substituting in what we have found we get

$$\begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{5}{3} \\ \frac{7}{3} \\ 0 \end{pmatrix} = \overrightarrow{OD'} - \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{7}{3} \end{pmatrix} \qquad \textbf{A1}$$

We can simplify and rearrange to find $\overrightarrow{OD'}$, giving

$$\begin{pmatrix} \frac{7}{3} \\ -\frac{7}{3} \\ \frac{7}{3} \end{pmatrix} = \overrightarrow{OD'} - \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{7}{3} \end{pmatrix}$$

$$\overrightarrow{OD'} = \begin{pmatrix} \frac{7}{3} \\ -\frac{7}{3} \\ \frac{7}{3} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{7}{3} \end{pmatrix} \qquad \textbf{A1}$$

$$= \begin{pmatrix} 3 \\ -\frac{7}{3} \\ \frac{14}{3} \end{pmatrix}$$

Hence the coordinates of point D' are $\left(3, -\frac{7}{3}, \frac{14}{3}\right)$.

Finally we need to find the plane which is parallel to $-x + y - z = c$, where c is a constant.

The plane must contain D', therefore we can find c by substituting in point D', this gives

$$-(3) + \left(-\frac{7}{3}\right) - \left(\frac{14}{3}\right) = c \qquad \textbf{(M1)}$$

$$c = -10$$

Hence Π_4 has equation

$$\boxed{-x + y - z = -10} \qquad \textbf{A1}$$