# IB Mathematics AA HL - Prediction Exams May 2025 - Paper 3

Paper 3 | 2 questions | 75 mins | 55 marks

## Question 1

CALCULATOR | Medium

[Maximum mark: 20]

**In this question you will explore how the shadow cast by a building changes as the light source moves.**

**To begin with, two important terms used in the question, will be defined.**

**Cut scene**
In a theater production, a ”cut scene” typically refers to a brief, self-contained scene inserted between larger scenes or acts. It is often used to convey some information to the audience.

**Umbra**
This is the shadow region directly behind an object, where all light rays from a light source are completely blocked by the object.

A theatre director is planning the lighting for a particular cut scene.

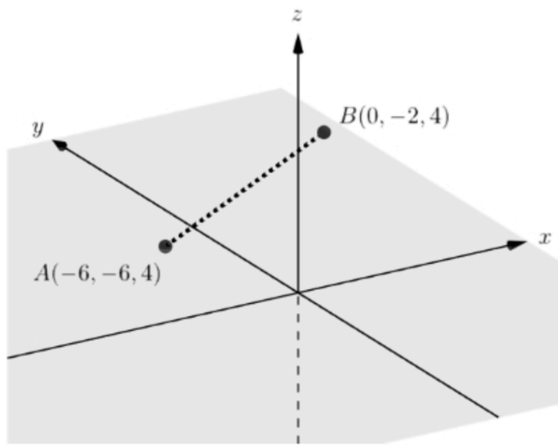
The scene before the cut scene is to be set early in the morning and the scene after the cut scene is set late in the afternoon.

During the cut scene the director wants to convey to the audience the passing of time. This will be done by showing how the shadow cast by a building changes as the Sun moves across the sky.

To do this he will use a spotlight mounted on a track to represent the Sun. The light from the spotlight will cast a shadow on the building on stage and as the spotlight moves along the track the shadows will change shape. Hence it will appear to the audience that the time in the day has changed.

All distances in this question are measured in metres and the time is in seconds.

Note: None of the diagrams given in this question are drawn to scale.




At time $t = 0$ the spotlight is at the point $A(-6, -6, 4)$ and begins moving in a straight line with a constant velocity. The path of the spotlight is shown as the dashed line above.

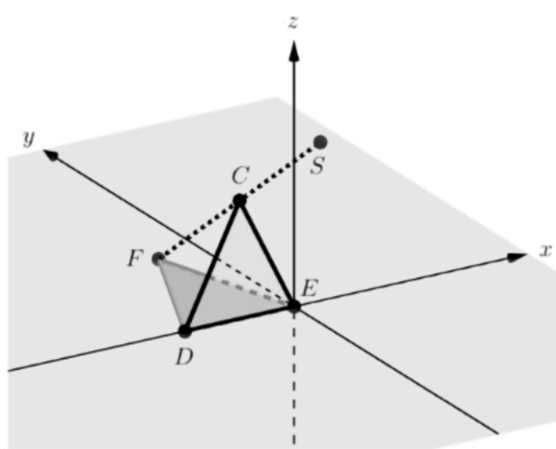
At time $t = 10$ the spotlight has finished moving and is at the point $B(0, -2, 4)$.

(a) (i) Show that the velocity vector, $\mathbf{v}$, of the spotlight is $\mathbf{v} = \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix}$ $\mathrm{m\,s^{-1}}$.

(ii) Hence write down a vector, $\boldsymbol{r_S}$ in terms of $t$, for the position of the spotlight after $t$ seconds. Where $0 \leq t \leq 10$. [3]

**For the rest of the question the building on stage will be represented by the triangle described below.**

When $t = 3$ the spotlight casts a shadow on the triangle with vertices $C(-1, 0, 2)$, $D(-2, 0, 0)$ and $E(0, 0, 0)$. The diagram below shows the umbra created.




The light ray from the spotlight, at point $S$, passes through $C$ and then intersects the $xy-$plane at point $F$. This path is represented by the dashed line above.

The path of the light ray can be expressed in the form $\boldsymbol{r_L} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

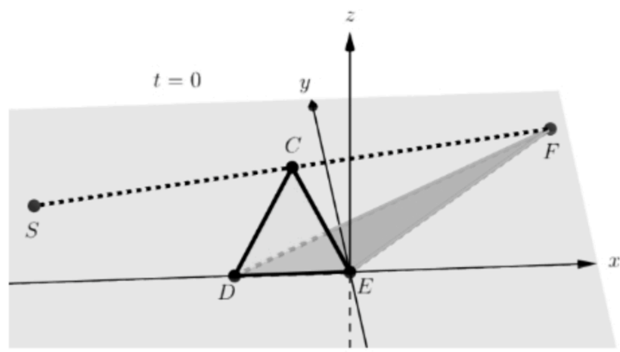
(b) For this part of the question $t = 3$.

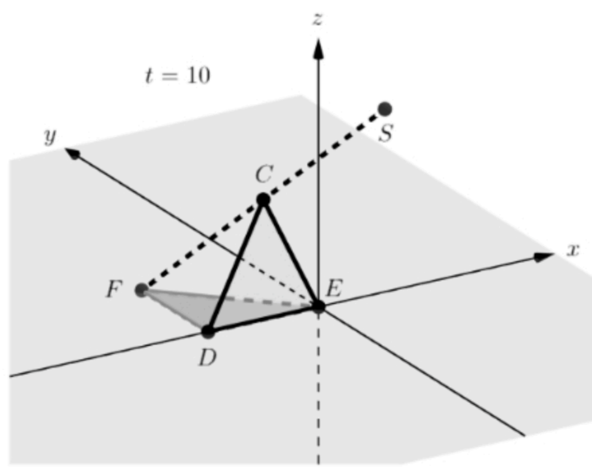
(i) Find the position vector of $S$.

(ii) Find a vector equation for $\boldsymbol{r_L}$.

(iii) Hence find the coordinates of $F$. [5]

As the value of $t$ increases from the 0 to 10 the shape of the umbra and the path of the light ray, $\boldsymbol{r_L}$, changes. This is shown below



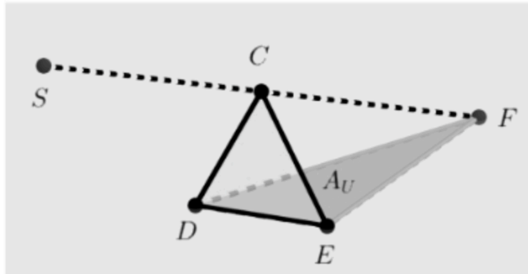




(c) (i) Write down vector $\boldsymbol{a}$ in terms of $t$.

(ii) Find vector $\boldsymbol{b}$ in terms of $t$. [3]

(d) Hence show that at time $t$ the coordinates of point $F$ are $(4 - 0.6t, 6 - 0.4t, 0)$. [3]

**In the this part of the question you will use previous results to find an expression for the area of the umbra.**




Let the area of the umbra be $A_U$.

(e) (i) Hence find an expression for $A_U$ in terms of $t$.

(ii) Hence show that the rate of change of $A_U$ with respect to $t$ is constant. [3]

Another part of the set also casts a shadow as the spotlight moves. The area of the umbra created by that is named $B_U$.

$B_U$, in $m^2$, is related to $A_U$ by the following formula

$$B_U = A_U - 3{A_U}^2$$

(f) Find the rate at which $B_U$ is changing when $t = 5$. [3]

(a) (i) First let's consider the entire journey, which takes 10 seconds

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -6 \\ -6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}\end{aligned}$$ **A1**

This vector represents the movement over 10 seconds. As the velocity is constant we can divide this by 10 to get the velocity vector

$$\begin{aligned}\boldsymbol{v} &= \tfrac{1}{10}\begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix}} \quad \mathrm{m\,s^{-1}} \text{ ...as required}\end{aligned}$$ **M1**

(ii) Using the vector equation of a line

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$$

Where $\boldsymbol{a}$ is the initial point, $A$, the direction $\boldsymbol{d}$ is the velocity vector and the parameter is $t$, this gives

$$\boldsymbol{r_S} = \boxed{\begin{pmatrix} -6 \\ -6 \\ 4 \end{pmatrix} + \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix} t}$$ **A1**

(b) (i) The coordinates of $S$, the position of the spotlight, can be found using the answer from part (a) when $t = 3$, hence

$$\begin{aligned} \boldsymbol{r_S} &= \begin{pmatrix} -6 \\ -6 \\ 4 \end{pmatrix} + \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix} \times 3 \\ &= \begin{pmatrix} -6 - (-1.8) \\ -6 - (-1.2) \\ 4 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} -4.2 \\ -4.8 \\ 4 \end{pmatrix}} \end{aligned}$$

**A1**

(ii) To find $\boldsymbol{r_L}$ we need the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\boldsymbol{a}$ is a point on the line of motion. For this we can use the position vector $\overrightarrow{OS}$.

$\boldsymbol{b}$ is the direction. As we know the light travels from $S$ through $C$ we can use the vector $\overrightarrow{SC}$ as $b$, hence we get

$$\begin{aligned} \boldsymbol{r_L} &= \overrightarrow{OS} + \lambda \overrightarrow{SC} \\ &= \begin{pmatrix} -4.2 \\ -4.8 \\ 4 \end{pmatrix} + \lambda \overrightarrow{SC} \end{aligned}$$

We can find $\overrightarrow{SC}$ using the coordinates of point $C$ to get

$$\begin{aligned} \overrightarrow{SC} &= \overrightarrow{OC} - \overrightarrow{OS} \\ &= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -4.2 \\ -4.8 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3.2 \\ 4.8 \\ -2 \end{pmatrix} \end{aligned}$$

Hence we get

$$\boldsymbol{r_L} = \boxed{\begin{pmatrix} -4.2 \\ -4.8 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3.2 \\ 4.8 \\ -2 \end{pmatrix}}$$

**A1A1**

(iii) Point $F$ is when $\boldsymbol{r_L}$ intersects the $xy-$plane.

This occurs when the $z-$component is equal to 0.

Hence we can form an equation for the $z-$component of $\boldsymbol{r_L}$ and solve for $\lambda$

$$0 = 4 + (-2)\lambda$$
$$\lambda = 2$$

We can now find $\boldsymbol{r_L}$ when $\lambda = 2$

$$\boldsymbol{r_L} = \begin{pmatrix} -4.2 \\ -4.8 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 3.2 \\ 4.8 \\ -2 \end{pmatrix} \qquad \textbf{(M1)}$$
$$= \begin{pmatrix} -4.2 + 6.4 \\ -4.8 + 9.6 \\ 4 - 4 \end{pmatrix}$$
$$= \begin{pmatrix} 2.2 \\ 4.8 \\ 0 \end{pmatrix}$$

Therefore the coordinates of $F$ are $(2.2, 4.8, 0)$.

**A1**

(c) (i) $\boldsymbol{a}$ is the position vector of $S$ at time $t$, using our answer from (a)(ii) we get

$$\boldsymbol{a} = \boxed{\begin{pmatrix} -6+0.6t \\ -6+0.4t \\ 4 \end{pmatrix}} \qquad \textbf{A1}$$

(ii) Vector $\boldsymbol{b}$ is in the direction $\overrightarrow{SC}$, which in terms of $t$ is

$$\begin{aligned} \boldsymbol{b} &= \overrightarrow{SC} \\ &= \overrightarrow{OC} - \overrightarrow{OS} \\ &= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -6+0.6t \\ -6+0.4t \\ 4 \end{pmatrix} && \textbf{(M1)} \\ &= \boxed{\begin{pmatrix} 5-0.6t \\ 6-0.4t \\ -2 \end{pmatrix}} && \textbf{A1} \end{aligned}$$

(d) Using our answers to part (c) we can write down the vector equation of the path of the light, $\boldsymbol{r_L}$, in terms of $\lambda$ and $t$

$$\begin{aligned} \boldsymbol{r_L} &= \boldsymbol{a} + \lambda \boldsymbol{b} \\ &= \begin{pmatrix} -6+0.6t \\ -6+0.4t \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5-0.6t \\ 6-0.4t \\ -2 \end{pmatrix} \end{aligned}$$

The point $F$ is on the $xy-$plane which means the $z-$component is 0.

We can form an equation and find $\lambda$ for $F$

$$\begin{aligned} 0 &= 4 + (-2)\lambda \\ \lambda &= 2 && \textbf{A1} \end{aligned}$$

Note, the $z-$component of $F$ is zero when $\lambda = 2$ for all values of $t$.

Therefore the position vector $\overrightarrow{OF}$, in terms of $t$, is equal to $\boldsymbol{r_L}$ when $\lambda = 2$, hence we get

$$\overrightarrow{OF} = \begin{pmatrix} -6+0.6t \\ -6+0.4t \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5-0.6t \\ 6-0.4t \\ -2 \end{pmatrix} \qquad \textbf{(M1)}$$

$$= \begin{pmatrix} -6+0.6t+10-1.2t \\ -6+0.4t+12-0.8t \\ 0 \end{pmatrix} \qquad \textbf{A1}$$

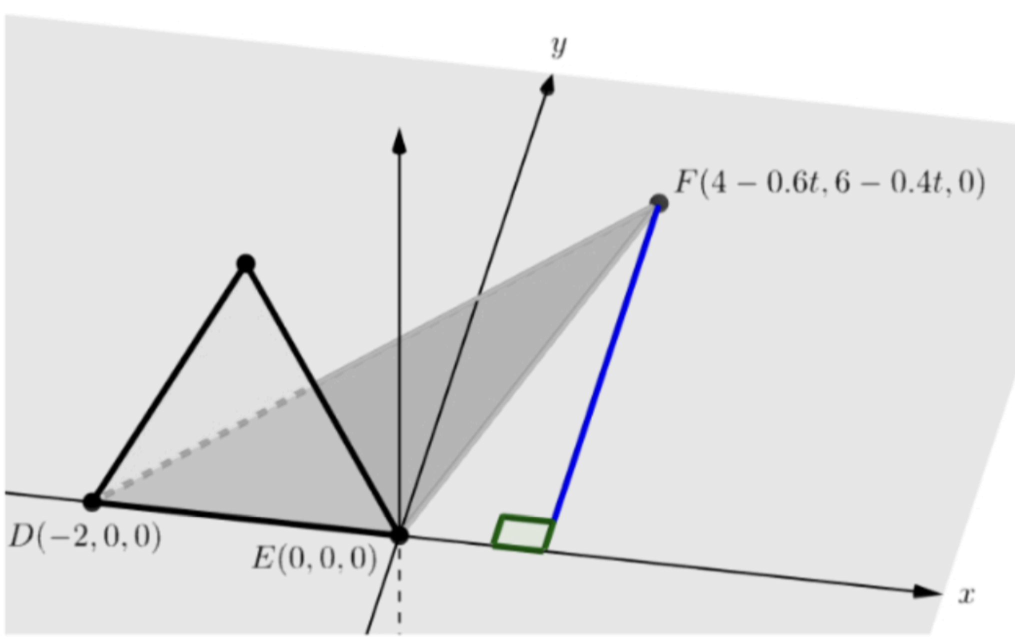
$$= \begin{pmatrix} 4-0.6t \\ 6-0.4t \\ 0 \end{pmatrix}$$

Therefore the coordinates of $F$ are $(4-0.6t, 6-0.4t, 0)$ as required.

(e) (i) There are a number of approaches that could be used in this part of the question.

The simplest method is to realise that $A_U$ is in the shape of a triangle.

The base of the triangle is $DE$ and the perpendicular height of the triangle is the blue line in the diagram below.



The length of the blue line is the $y-$coordinate of point $F$, $F_y$.

Therefore

$$\begin{aligned} A_U &= \frac{DE \times F_y}{2} \\ &= \frac{2 \times (6 - 0.4t)}{2} & \textbf{M1} \\ &= \boxed{6 - 0.4t} & \textbf{A1} \end{aligned}$$

(ii) We can differentiate $A_U$ with respect to $t$ to get

$$\frac{\mathrm{d}A_U}{\mathrm{d}t} = -0.4\,\mathrm{m}^2\,\mathrm{s}^{-1} \qquad \textbf{A1}$$

Hence we can see the

$$\boxed{\text{rate of change of } A_U \text{ is not related to } t}$$

which means it is constant.

(f) This is a related rates question.

We have three variables, $A_U$, $B_U$ and $t$.

Which we can relate using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

We can replace $y$ with $B_U$, $x$ with $t$ and $u$ with $A_U$ giving

$$\frac{\mathrm{d}B_U}{\mathrm{d}t} = \frac{\mathrm{d}B_U}{\mathrm{d}A_U} \times \frac{\mathrm{d}A_U}{\mathrm{d}t}$$

We want to find $\frac{\mathrm{d}B_U}{\mathrm{d}t}$ when $t = 5$.

We know that from part (e) (ii) that $\frac{\mathrm{d}A_U}{\mathrm{d}t} = -0.4$ for all $t$.

Hence we differentiate the given relation and substitute in $t = 5$

$$\frac{\mathrm{d}B_U}{\mathrm{d}A_U} = 1 - 6A_U \qquad \textbf{(M1)}$$

$$= 1 - 6(6 - 0.4t)$$

$$= 1 - 6(6 - 0.4 \times 5)$$

$$= -23 \qquad \textbf{A1}$$

Finally, substituting these results back into our chain rule we get

$$\frac{\mathrm{d}B_U}{\mathrm{d}t} = -23 \times (-0.4)$$

$$= \boxed{9.2}\ \mathrm{m}^2\,\mathrm{s}^{-1} \qquad \textbf{A1}$$

## Question 2

CALCULATOR

Hard

[Maximum mark: 35]

(a) Consider the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 3x^2y$$

(i) Find the integrating factor in the form $e^{f(x)}$.

(ii) Hence find the particular solution, in the form $y = g(x)$, when $g(0) = 0$. [8]

**In this question you will explore how to use an integrating factor technique to solve Bernoulli Differential Equations.**

**Bernoulli Differential equations have the following general form**

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} + \boldsymbol{P}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{n}}$$

**Where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are functions of $\boldsymbol{x}$ and $\boldsymbol{n}$ is an integer.**

**In part (b) you will use a substitution to rewrite a Bernoulli Differential equation into the same form as the differential equation in part (a).**

(b) The general form of Bernoulli Differential Equations is shown below.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n \qquad [1]$$

The solution to [1] is in the form $y = h(x)$,

(i) Rewrite equation [1] by dividing both sides by $y^n$.

Consider the substitution $u = y^{1-n}$ where $n \in \mathbb{Z}$.

(ii) Write down $\frac{\mathrm{d}u}{\mathrm{d}y}$ in terms of $n$.

(iii) Find a relationship between $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$

(iv) Hence show that [6]

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x)$$

**The result in part (b) (iv) is now in the same form as the differential equation in part (a). Hence an integrating factor of $e^{\int (1-n)P(x)\,dx}$ can be used to find the solution for $u$.**

(c) Consider the following

$$\frac{dy}{dx} = xy^3 - y \qquad [2]$$

(i) Show that the differential equation above can be written in the form

$$\frac{du}{dx} - 2u = -2x$$

(ii) Hence show that

$$\frac{d}{dx}(e^{-2x}u) = -2xe^{-2x}$$

(iii) Hence find the particular solution to [2], in the form $y = s(x)$, when $s(0) = 2$. [12]

**In the final part you will use the methods from earlier in the question to find the time at which an airborne sky-diver lands on the ground.**

(d) A sky-diver has jumped out of an aeroplane and is travelling towards the ground.

Her height $H$, above the ground, is measured in kilometres.

After falling some distance the sky-diver reaches a height of 0.2 km, opens her parachute and begins her final descent.

For the purposes of this question she begins her final descent at time $t = 0$, measured in seconds.

During her final descent the rate of change of $H$ with respect to $t$ is given by

$$\frac{dH}{dt} = \frac{2H^2e^t}{1+9t^2} - H$$

Where $t \geq 0$.

She lands when the value of $H$ is equal to 1.5 metres.

Find the time she lands after beginning her final descent. [9]

(a) (i) To find the integrating factor we first need to rearrange the differential equation into the form provided in the formula book

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

Hence we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y = 6x^2$$

We know that the integrating factor is $e^{\int P(x)\,\mathrm{d}x}$.

We can see from our differential equation that $P(x) = 3x^2$.

**(A1)**

Hence the integrating factor is

$$= e^{\int 3x^2\,\mathrm{d}x}$$

$$= \boxed{e^{x^3}}$$ **A1**

We don't need a constant of integration when finding the integrating factor as we introduce one when solving the differential equation.

(ii) To solve a differential equation means we need to find $y$ as a function of $x$.

Doing this with an integrating factor means we first multiply the entire equation by the factor, this gives

$$e^{x^3} \times \frac{\mathrm{d}y}{\mathrm{d}x} + e^{x^3} \times 3x^2y = e^{x^3} \times 6x^2 \qquad [3] \qquad \textbf{(M1)}$$

The next step is a very crucial step to understanding why using an integrating factor works.

To better understand this process let's first use the product rule to differentiate the following product of functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right) = e^{x^3} \times \frac{\mathrm{d}y}{\mathrm{d}x} + e^{x^3} \times 3x^2 \times y$$

Notice that the R.H.S. of the above is the same as the L.H.S. of equation [3], hence we can equate the L.H.S. of [3] with the R.H.S. of the equation above, which gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right) = e^{x^3} \times 6x^2 \qquad \textbf{(M1)}$$

Recall, we are trying to find $y = g(x)$ hence we should integrate both sides to get

$$\int \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)\right] \mathrm{d}x = \int 6x^2e^{x^3}\,\mathrm{d}x$$

$$e^{x^3}y = \int 6x^2e^{x^3}\,\mathrm{d}x \qquad \textbf{A1}$$

Notice that the coefficient of $e^{f(x)}$ is a multiple of $f'(x)$ hence we can integrate the expression using a substitution of $u = x^3$.

If $u = x^3$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$, therefore we can write $\mathrm{d}x = \frac{1}{3x^2}\mathrm{d}u$.

Hence we can rewrite the integral entirely in terms of $u$, which gives

$$\int 6x^2 e^{x^3}\,\mathrm{d}x = \int 6x^2 e^u \times \frac{1}{3x^2}\mathrm{d}u$$ **M1**

Notice the $x^2$ cancel out which means we can now integrate with respect to $u$

$$= 2\int e^u\,\mathrm{d}u$$
$$= 2e^u + C$$ **A1**

We can now rewrite integral in terms of $x$

$$e^{x^3} y = 2e^{x^3} + C$$
$$y = 2 + \frac{C}{e^{x^3}}$$

Using the condition $g(0) = 0$ we can find $C$

$$0 = 2 + \frac{C}{e^0}$$
$$C = -2$$

Hence the particular solution to the differential equation is

$$y = 2 - \frac{2}{e^{x^3}}$$ **A1**

(b) (i) First we divide [1] by $y^n$ and then attempt to simplify

$$\frac{1}{y^n} \times \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{P(x)y}{y^n} = \frac{Q(x)y^n}{y^n}$$

$$\boxed{y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)}$$ **A1**

(ii) We can use the power rule to differentiate $u$ with respect to $y$ to get

$$u = y^{1-n}$$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \boxed{(1-n)y^{-n}}$$ **A1**

(iii) We have three variables, $x$, $y$ and $u$ and we can relate them using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$ **(M1)**

Using our answer from part (b)(ii) we can write an expression for $\frac{\mathrm{d}y}{\mathrm{d}u}$ as

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}u} &= \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}u}} \\ &= \frac{1}{(1-n)y^{-n}} \\ &= \frac{y^n}{(1-n)} \end{aligned}$$

Substituting this into the chain rule we get

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^n}{(1-n)} \times \frac{\mathrm{d}u}{\mathrm{d}x}}$$ **A1**

(iv) Notice the equation we need to show is in terms of $u$ and $x$.

Recall our answer from part (b)(i)

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)$$

Let's replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ with our answer from part (b)(iii)

$$y^{-n} \times \frac{y^n}{(1-n)} \times \frac{\mathrm{d}u}{\mathrm{d}x} + P(x)y^{1-n} = Q(x) \qquad \textbf{M1}$$

Now using the substitution $u = y^{1-n}$ and simplifying we can remove $y$ from the equation

$$\cancel{y^{-n}} \times \frac{\cancel{y^n}}{(1-n)} \times \frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x)$$

$$\frac{1}{(1-n)} \times \frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x) \qquad \textbf{A1}$$

Finally multiply throughout by $1-n$ to get

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x)} \qquad \text{...as required}$$

(c) (i) This appears to be a Bernoulli equation, let's rearrange it into the form of equation [1].

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n \qquad [1]$$

Hence we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^3 \qquad \textbf{(A1)}$$

By comparing to [1] we notice that the value of $n = 3$, $P(x) = 1$ and $Q(x) = x$.

**A1**

We can now rewrite the differential equation using the result from part (b)(iv) to get

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-3)u = (1-3)x$$

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}x} - 2u = -2x} \qquad \text{...as required}$$

(ii) We have been told that the integrating factor for this type of differential equation is given by $e^{\int (1-n)P(x)\,\mathrm{d}x}$.

Recall, $n = 3$ and $P(x) = 1$ hence the integrating factor is

$$= e^{\int (1-3)\times 1\,\mathrm{d}x} \qquad \textbf{(M1)}$$

$$= e^{\int -2\,\mathrm{d}x}$$

$$= e^{-2x}$$

Now we multiply the answer from the previous part by the integrating factor we have just found to get

$$e^{-2x} \times \frac{\mathrm{d}u}{\mathrm{d}x} - e^{-2x} \times 2u = e^{-2x} \times (-2x) \qquad \textbf{A1}$$

Hence using the techniques discussed in part (a) we can rewrite the L.H.S. of above equation to get

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(e^{-2x}u) = -2xe^{-2x}} \qquad \text{...as required}$$

(iii) To find $y = s(x)$ we first should solve to find $u$ as a function of $x$.

Hence let's integrate both sides of the differential equation from the previous part.

$$\int \left[\frac{\mathrm{d}}{\mathrm{d}x}(e^{-2x}u)\right] \mathrm{d}x = \int -2xe^{-2x}\mathrm{d}x$$

$$e^{-2x}u = -2\int xe^{-2x}\mathrm{d}x \qquad \textbf{(M1)}$$

We need to integrate the expression on the R.H.S. which is a product of two functions, for this we can use integration by parts.

We begin by finding the different elements we need

$$u = x \qquad v' = e^{-2x} \qquad \textbf{(A1)}$$

$$u' = 1 \qquad v = -\tfrac{1}{2}e^{-2x}$$

and then we can substitute them into the formula for integration by parts getting

$$-2\int xe^{-2x}\mathrm{d}x = -2\left(x \times -\tfrac{1}{2}e^{-2x} - \int 1 \times \left(-\tfrac{1}{2}e^{-2x}\right)\, \mathrm{d}x\right) \qquad \textbf{M1}$$

$$= -2\left(-\frac{x}{2}e^{-2x} - \left(\frac{1}{4}e^{-2x}\right)\right)$$

$$= xe^{-2x} + \tfrac{1}{2}e^{-2x} + C \qquad \textbf{A1}$$

We can now replace the R.H.S. of our differential equation with the result above and make $u$ the subject to get

$$
\begin{aligned}
e^{-2x}u &= xe^{-2x} + \tfrac{1}{2}e^{-2x} + C \\
u &= x + \tfrac{1}{2} + \frac{C}{e^{-2x}}
\end{aligned}
$$

To convert the equation into the form $y = s(x)$ we need to recall the substitution $u = y^{1-n}$ with $n = 3$, therefore we get

**(M1)**

$$
\begin{aligned}
y^{-2} &= x + \tfrac{1}{2} + \frac{C}{e^{-2x}} \\
&= x + \tfrac{1}{2} + Ce^{2x}
\end{aligned}
$$

In order to find $y = s(x)$ we need to raise both sides of the equation to the power of $-\frac{1}{2}$.

Which gives

$$
\begin{aligned}
(y^{-2})^{-\frac{1}{2}} &= (x + \tfrac{1}{2} + Ce^{2x})^{-\frac{1}{2}} \\
y &= \frac{1}{\sqrt{x + \frac{1}{2} + Ce^{2x}}}
\end{aligned}
$$

**A1**

Using the initial condition of $s(0) = 2$ we can find $C$

$$2 = \frac{1}{\sqrt{0 + \frac{1}{2} + Ce^0}}$$ **M1**

$$2 = \frac{1}{\sqrt{\frac{1}{2} + C}}$$

Solving for $C$ we get

$$\sqrt{\tfrac{1}{2} + C} = \frac{1}{2}$$

$$\tfrac{1}{2} + C = \tfrac{1}{4}$$

$$C = -\tfrac{1}{4}$$

Hence the particular solution is

$$\boxed{y = \frac{1}{\sqrt{x + \frac{1}{2} - \frac{1}{4}e^{2x}}}}$$ **A1**

(d) In order to find out the time at which she lands we need to find the particular solution, when $H = 0.2\,\text{km}$ and $t = 0\,\text{s}$, to the differential equation in the form $H = f(t)$.

We can then use this solution and the given value of $H$ to form the equation $0.015 = f(t)$ which can be solved to find $t$, her landing time.

Notice that this is a Bernoulli Equation and can be written in the form previously described.

Replacing $y$ with $H$ and $x$ with $t$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} + P(t)H = Q(t)H^n$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} + H = \frac{2H^2e^t}{1+9t^2}$$

Hence $n = 2$, $P(t) = 1$ and $Q(t) = \dfrac{2e^t}{1+9t^2}$.

**(A1)**

To begin with we will rewrite the differential equation using the substitution $u = H^{1-n} = H^{-1}$ and the result from part (b)(iv).

$$\frac{\mathrm{d}u}{\mathrm{d}t} + (1-2) \times 1 \times u = (1-2) \times \frac{2e^t}{1+9t^2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} - u = \frac{-2e^t}{1+9t^2}$$ **A1**

Recall the integrating factor is

$$
\begin{aligned}
&= e^{\int (1-n)P(t)\,\mathrm{d}t} \\
&= e^{\int -1\,\mathrm{d}t} \\
&= e^{-t} \qquad \textbf{(A1)}
\end{aligned}
$$

Next we multiply the differential equation, in $u$, by the integrating factor.

$$e^{-t} \times \frac{\mathrm{d}u}{\mathrm{d}t} - e^{-t} \times u = e^{-t} \times \frac{-2e^{t}}{1+9t^2}$$

Using the technique discussed in part (a)(ii) we can rewrite the L.H.S. and do some canceling on the R.H.S. to get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}u\right) &= \cancel{e^{-t}} \times \frac{-2\cancel{e^{t}}}{1+9t^2} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}u\right) &= \frac{-2}{1+9t^2} \qquad \textbf{A1}
\end{aligned}
$$

Integrating both sides we get

$$
\begin{aligned}
\int \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}u\right)\right]\,\mathrm{d}t &= \int \frac{-2}{1+9t^2}\mathrm{d}t \\
e^{-t}u &= \int \frac{-2}{1+9t^2}\mathrm{d}t
\end{aligned}
$$

The next step is to integrate the R.H.S.

Notice that the R.H.S. is of the form of the standard integral $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan x + C$.

Next we can rewrite the integral, so it is in the appropriate form

$$e^{-t}u = -2\int \frac{1}{1+(3t)^2}\mathrm{d}t \qquad \textbf{(A1)}$$

We need to use a substitution here as we have a function of $x$.

Let $v = 3x$ therefore $\frac{\mathrm{d}v}{\mathrm{d}x} = 3$.

Hence we have $\mathrm{d}x = \frac{1}{3}\mathrm{d}v$.

The R.H.S. becomes

$$= -2\int \frac{1}{1+v^2} \times \frac{1}{3}\mathrm{d}v$$

$$= -\frac{2}{3}\int \frac{1}{1+v^2}\,\mathrm{d}v$$

Now we apply the standard integral with $a = 1$ to get

$$= -\frac{2}{3}\arctan v + C$$

$$= -\frac{2}{3}\arctan 3t + C \qquad \textbf{A1}$$

Let's make $u$ the subject and then use the substitution, $u = H^{-1}$, to rewrite the solution in terms of $H$

$$\begin{aligned}
\frac{1}{e^t}u &= -\frac{2}{3}\arctan 3t + C \\
u &= -\frac{2e^t}{3}\arctan 3t + e^t C \\
\frac{1}{H} &= -\frac{2e^t}{3}\arctan 3t + e^t C \\
H &= \frac{1}{-\frac{2e^t}{3}\arctan 3t + e^t C} \\
&= -\frac{3}{2e^t\arctan 3t - 3e^t C}
\end{aligned}$$

**A1**

We have been given initial conditions in the question, when $t = 0\,\mathrm{s}$ then $H = 0.2\,\mathrm{km}$, hence we can form an equation and solve it to find $C$

$$\begin{aligned}
0.2 &= -\frac{3}{2e^0\arctan 0 - 3e^0 C} \\
0.2 &= \frac{-3}{-3C} \\
C &= \frac{1}{0.2} \\
&= 5 \qquad \text{[by using G.D.C.]}
\end{aligned}$$

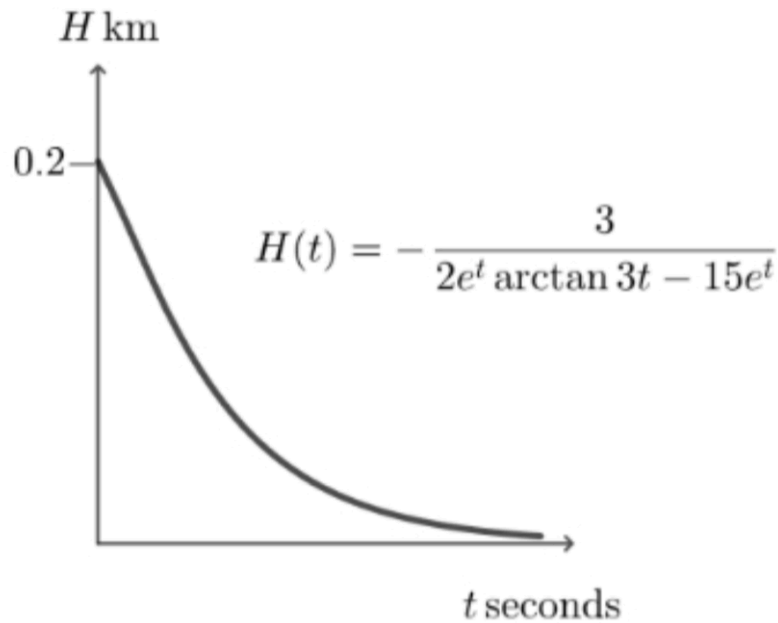
**A1**

We have now found $H = f(t)$

$$H = -\frac{3}{2e^t \arctan 3t - 15e^t}$$

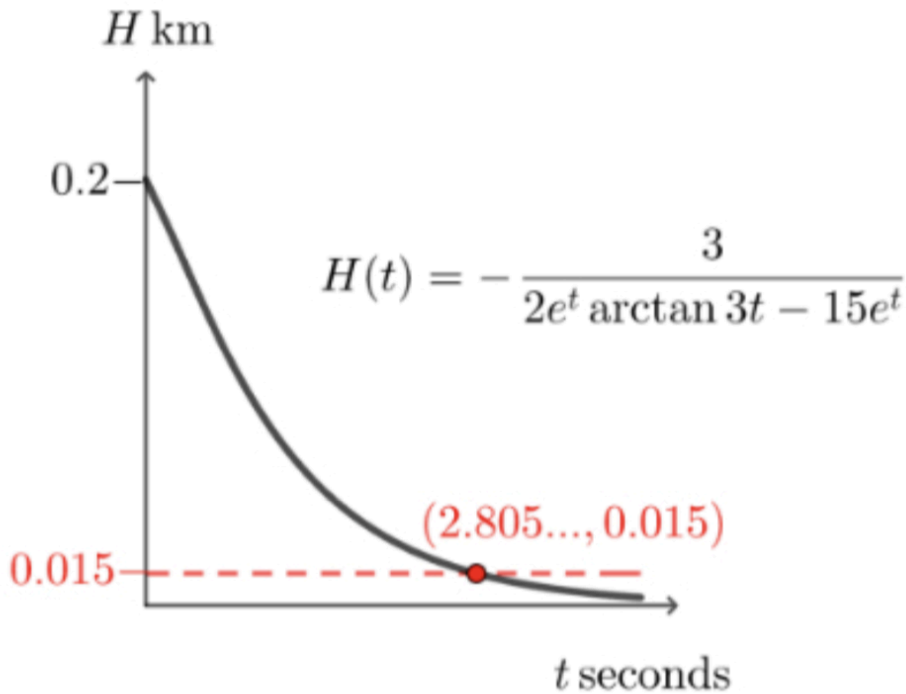
Where $t \geq 0$.

At this stage it is a good idea to check your solution by sketching it on your G.D.C.



Notice that the curve begins at $(0, 0.2)$ which is what we would expect to see. Also, as this is a contextual problem, the shape of the curve appears to model a sky-diver in the final stages of landing.

Now, as mentioned, we need to solve $0.015 = f(t)$. We can do this by drawing the line $H = 0.015$ and finding the point of intersection with the curve.




Hence the sky-diver lands 2.81 seconds after beginning her final descent.

**A1**