IB Mathematics AA HL - Prediction Exams May 2025 - Paper 3

Paper 3 | 2 questions | 75 mins | 55 marks

Question 1

CALCULATOR | Medium

[Maximum mark: 20]

In this question you will explore how the shadow cast by a building changes as the light source moves.

To begin with, two important terms used in the question, will be defined.

Cut scene
In a theater production, a "cut scene" typically refers to a brief, self-contained scene inserted between larger scenes or acts. It is often used to convey some information to the audience.

Umbra
This is the shadow region directly behind an object, where all light rays from a light source are completely blocked by the object.

A theatre director is planning the lighting for a particular cut scene.

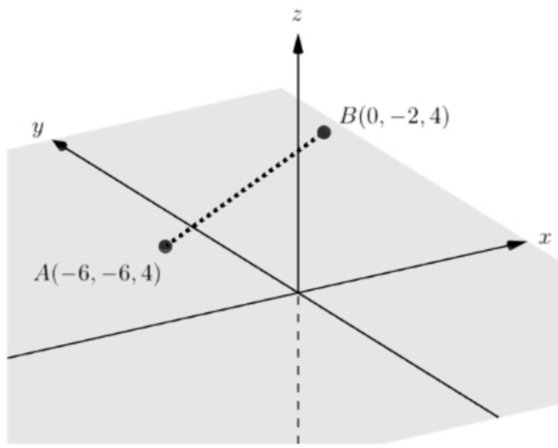
The scene before the cut scene is to be set early in the morning and the scene after the cut scene is set late in the afternoon.

During the cut scene the director wants to convey to the audience the passing of time. This will be done by showing how the shadow cast by a building changes as the Sun moves across the sky.

To do this he will use a spotlight mounted on a track to represent the Sun. The light from the spotlight will cast a shadow on the building on stage and as the spotlight moves along the track the shadows will change shape. Hence it will appear to the audience that the time in the day has changed.

All distances in this question are measured in metres and the time is in seconds.

Note: None of the diagrams given in this question are drawn to scale.



At time $t = 0$ the spotlight is at the point $A(-6, -6, 4)$ and begins moving in a straight line with a constant velocity. The path of the spotlight is shown as the dashed line above.

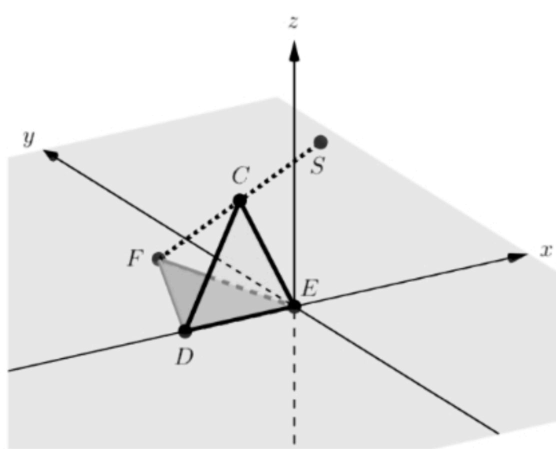
At time $t = 10$ the spotlight has finished moving and is at the point $B(0, -2, 4)$.

(a) (i) Show that the velocity vector, $\mathbf{v}$, of the spotlight is $\mathbf{v} = \begin{pmatrix} 0.6 \\ 0.4 \\ 0 \end{pmatrix}$ $\mathrm{m\,s^{-1}}$.

(ii) Hence write down a vector, $\boldsymbol{r_S}$ in terms of t, for the position of the spotlight after t seconds. Where $0 \le t \le 10$. [3]

For the rest of the question the building on stage will be represented by the triangle described below.

When $t = 3$ the spotlight casts a shadow on the triangle with vertices $C(-1, 0, 2)$, $D(-2, 0, 0)$ and $E(0, 0, 0)$. The diagram below shows the umbra created.



The light ray from the spotlight, at point S, passes through C and then intersects the $xy-$plane at point F. This path is represented by the dashed line above.

The path of the light ray can be expressed in the form $\boldsymbol{r_L} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

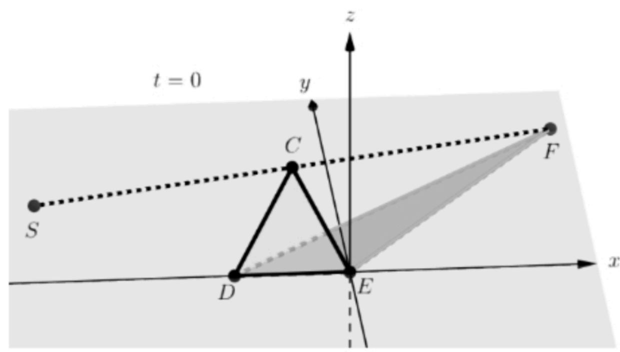
(b) For this part of the question $t = 3$.

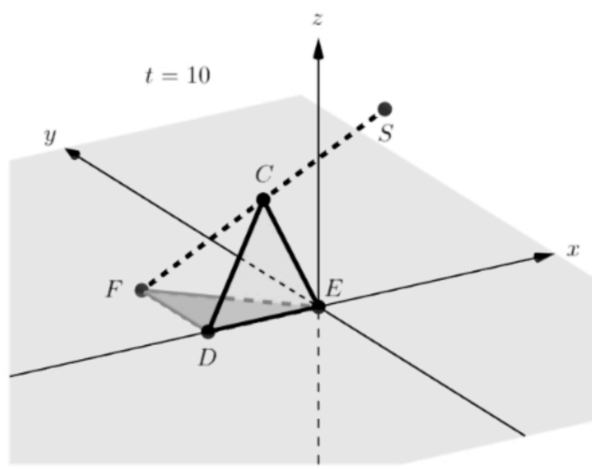
(i) Find the position vector of S.

(ii) Find a vector equation for $\boldsymbol{r_L}$.

(iii) Hence find the coordinates of F. [5]

As the value of t increases from the 0 to 10 the shape of the umbra and the path of the light ray, $\boldsymbol{r_L}$, changes. This is shown below



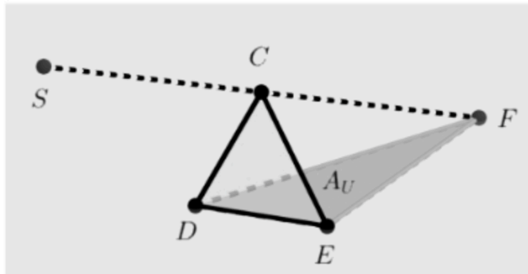


(c) (i) Write down vector $\boldsymbol{a}$ in terms of t.

(ii) Find vector $\boldsymbol{b}$ in terms of t. [3]

(d) Hence show that at time t the coordinates of point F are $(4 - 0.6t, 6 - 0.4t, 0)$. [3]

In the this part of the question you will use previous results to find an expression for the area of the umbra.



Let the area of the umbra be A_U.

(e) (i) Hence find an expression for A_U in terms of t.

(ii) Hence show that the rate of change of A_U with respect to t is constant. [3]

Another part of the set also casts a shadow as the spotlight moves. The area of the umbra created by that is named B_U.

B_U, in m^2, is related to A_U by the following formula

$$B_U = A_U - 3{A_U}^2$$

(f) Find the rate at which B_U is changing when $t = 5$. [3]

Question 2

CALCULATOR

Hard

[Maximum mark: 35]

(a) Consider the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 3x^2 y$$

(i) Find the integrating factor in the form $e^{f(x)}$.

(ii) Hence find the particular solution, in the form $y = g(x)$, when $g(0) = 0$. [8]

In this question you will explore how to use an integrating factor technique to solve Bernoulli Differential Equations.

Bernoulli Differential equations have the following general form

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} + \boldsymbol{P}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{n}}$$

Where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are functions of $\boldsymbol{x}$ and $\boldsymbol{n}$ is an integer.

In part (b) you will use a substitution to rewrite a Bernoulli Differential equation into the same form as the differential equation in part (a).

(b) The general form of Bernoulli Differential Equations is shown below.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n \qquad [1]$$

The solution to [1] is in the form $y = h(x)$,

(i) Rewrite equation [1] by dividing both sides by y^n.

Consider the substitution $u = y^{1-n}$ where $n \in \mathbb{Z}$.

(ii) Write down $\frac{\mathrm{d}u}{\mathrm{d}y}$ in terms of n.

(iii) Find a relationship between $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$

(iv) Hence show that [6]

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x)$$

The result in part (b) (iv) is now in the same form as the differential equation in part (a). Hence an integrating factor of $e^{\int (1-n)P(x)\,\mathrm{d}x}$ can be used to find the solution for u.

(c) Consider the following

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^3 - y \qquad [2]$$

(i) Show that the differential equation above can be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 2u = -2x$$

(ii) Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{-2x}u) = -2xe^{-2x}$$

(iii) Hence find the particular solution to [2], in the form $y = s(x)$, when $s(0) = 2$. [12]

In the final part you will use the methods from earlier in the question to find the time at which an airborne sky-diver lands on the ground.

(d) A sky-diver has jumped out of an aeroplane and is travelling towards the ground.

Her height H, above the ground, is measured in kilometres.

After falling some distance the sky-diver reaches a height of 0.2 km, opens her parachute and begins her final descent.

For the purposes of this question she begins her final descent at time $t = 0$, measured in seconds.

During her final descent the rate of change of H with respect to t is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{2H^2e^t}{1+9t^2} - H$$

Where $t \geq 0$.

She lands when the value of H is equal to 1.5 metres.

Find the time she lands after beginning her final descent. [9]