IB Mathematics AA SL - Prediction Exams May 2025 - Paper 1

Paper 1 | 9 questions | 90 mins | 80 marks

Section A

Question 1

NO CALCULATOR

Easy

[Maximum mark: 4]

Consider the line L_1 which passes through the points $A(-4, -4)$ and $B(8, -1)$.

(a) Find the gradient of the line L_1. [1]

Line L_2 is perpendicular to L_1 and passes through the point $(-a, 2a)$ where $a \in \mathbb{R}$.

(b) Given that L_2 intersects the y-axis at $y = 2a - 3$, find the value of a. [3]

(a) Using the formula for the gradient of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

we can find the gradient of L_1

$$\begin{aligned} m_1 &= \frac{-1-(-4)}{8-(-4)} \\ &= \frac{3}{12} \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

A1

(b) As L_2 is perpendicular to L_1 we can find the gradient of L_2 using the formula

$$m_1 \times m_2 = -1$$

$$\frac{1}{4} \times m_2 = -1$$

$$m_2 = -4 \qquad \textbf{(A1)}$$

We can form an equation, using the gradient formula, and solve for a.

Substituting in m_2 and the points $(-a, 2a)$ and $(0, 2a-3)$ we get

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$-4 = \frac{(2a-3) - 2a}{0 - (-a)} \qquad \textbf{(M1)}$$

$$-4 = \frac{-3}{a}$$

$$-4a = -3$$

$$a = \boxed{\frac{3}{4}} \qquad \textbf{A1}$$

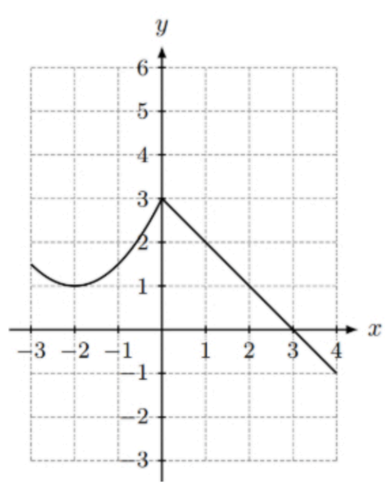
Question 2

NO CALCULATOR

Easy

[Maximum mark: 5]

The graph of $y = f(x)$ for $-3 \leq x \leq 4$ is shown in the following diagram.



(a) Write down the value of $f(2)$. [1]

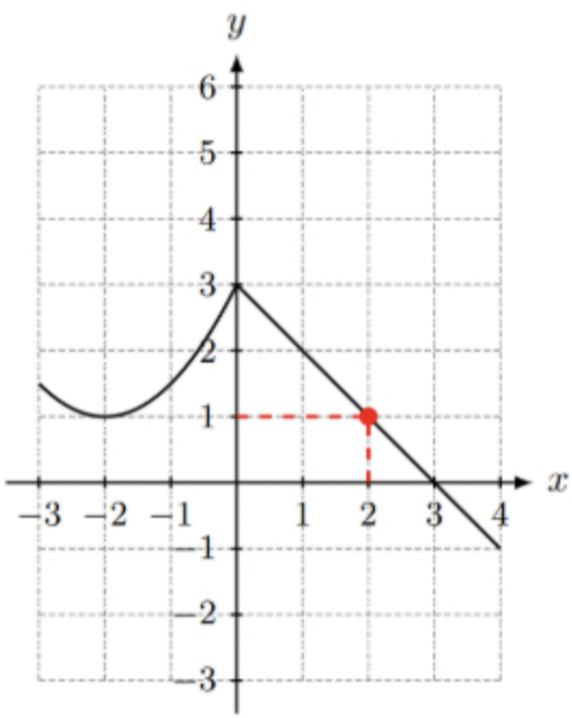
Let $g(x) = 2f(x) - 1$ for $-3 \le x \le 4$.

(b) On the axes above, sketch the graph of g. [2]

(c) Hence determine the value of $(g \circ f)(2)$. [1]

(d) Hence solve the equation $(f \circ g)(x) = 0$ when $x > 0$. [1]

(a) Evaluating $f(x)$ when $x = 2$



Hence $\boxed{f(2) = 1}$.

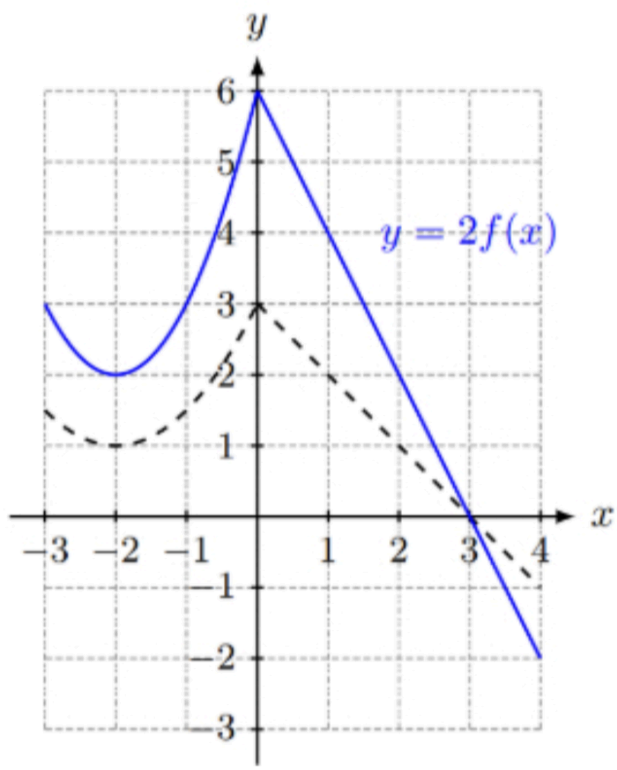
A1

(b) The function $f(x)$ is mapped to $g(x)$ by two transformations.

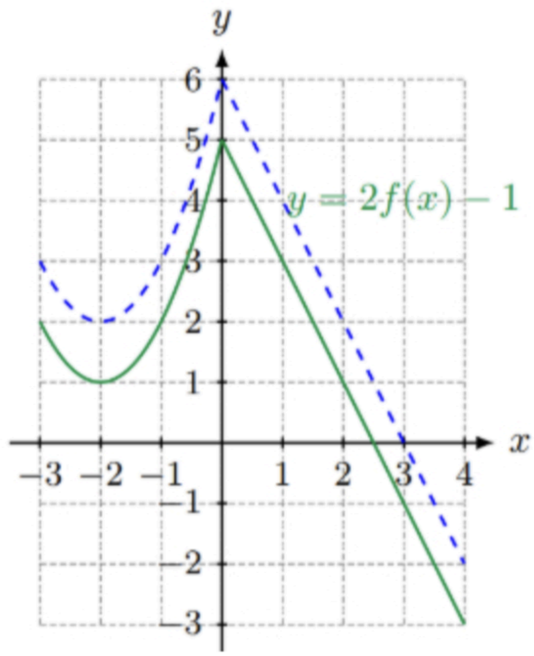
By considering $g(x) = 2f(x) - 1$, we can see there is a vertical stretch by a scale factor of 2 and a vertical shift down 1 unit.

The vertical stretch ($y = 2f(x)$) is shown in blue below.

Note that the vertical distance from the y-axis of every point on the curve is doubled.



Note that every point on the blue curve is shifted vertically down 1 unit.



Correct local minimum at $(-2, 1)$

A1

Correct y-intercept at $(0, 5)$

A1

(c) From part (a) we know that $f(2) = 1$.

$$(g \circ f)(2) = g(f(2))$$

Hence

$$= g(1)$$

$$= 3$$

Hence $\boxed{(g \circ f)(2) = 3}$.

A1

Here is the graph of $g(x)$ showing that $g(1) = 3$.

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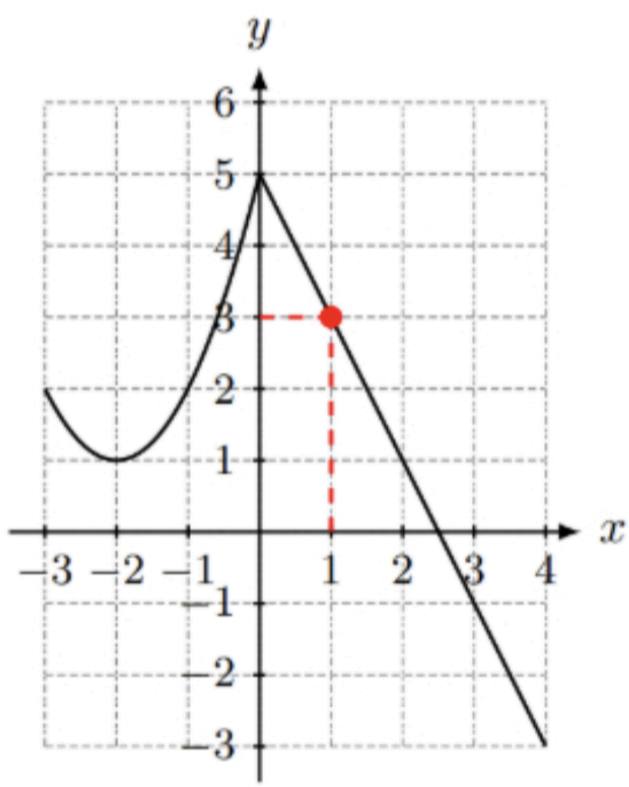
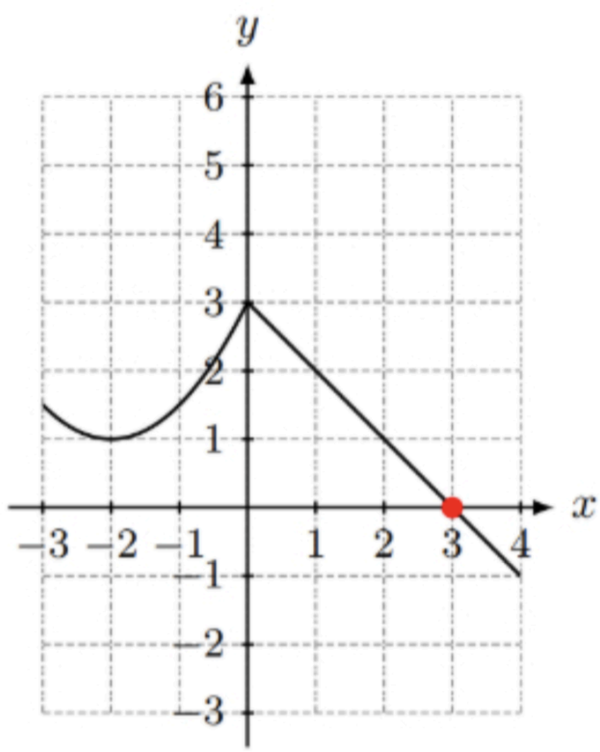


(d) We have the equation $(f \circ g)(x) = 0$. The L.H.S. is a composite function which can be rewritten

$$(f \circ g)(x) = f(g(x))$$

Here we can see that $g(x)$ is the inner function and $f(x)$ is the outer function, meaning that the output of g will be the input of f.

By considering the graph of $f(x)$ we see that when $x = 3$ then $f(x) = 0$.



This means in order to have an output of 0, the input must be 3.

$$f(g(x)) = 0$$

$$f(3) = 0$$

Therefore we need to find the value of x, where $x > 0$, such that $g(x) = 3$.

From part (c), we know that $g(1) = 3$, and we can see that this is the only possible solution when $x > 0$.

Hence if $x > 0$ and $(f \circ g)(x) = 0$ then $\boxed{x = 1}$.

A1

Question 3

NO CALCULATOR

Medium

[Maximum mark: 5]

(a) Show that $12 \log_x 2 = \dfrac{12}{\log_2 x}$. [1]

(b) Hence solve the equation $\log_2 x = 8 - 12 \log_x 2$. [4]

revisionvillage

(a) Using the change of base formula we can write

$$\text{L.H.S.} = 12\log_x 2$$

$$= 12\left(\frac{\log_2 2}{\log_2 x}\right) \quad \textbf{A1}$$

As $\log_n n = 1$, we can replace $\log_2 2$ with 1 and simplify

$$= 12\left(\frac{1}{\log_2 x}\right)$$

$$= \boxed{\frac{12}{\log_2 x}} \quad \text{...as required}$$

$$= \text{R.H.S.}$$

(b) We can replace $12\log_x 2$ with the R.H.S. of the identity from part (a)

$$\log_2 x = 8 - 12\log_x 2$$

$$\log_2 x = 8 - \frac{12}{\log_2 x} \quad \textbf{(M1)}$$

We now multiply each side by $\log_2 x$ and then rearrange such that the R.H.S. is equal to 0

$$(\log_2 x)^2 = 8\log_2 x - 12$$

$$(\log_2 x)^2 - 8\log_2 x + 12 = 0 \quad \textbf{(M1)}$$

This is a hidden quadratic equation. If we replace $\log_2$ with a variable, say a, we get

$$a^2 - 8a + 12 = 0$$

Let's solve this by factorising.

$$(a-2)(a-6) = 0$$

Therefore the solutions are

$$a - 2 = 0 \quad \text{and} \quad a - 6 = 0$$
$$a = 2 \qquad\qquad a = 6$$

Recall $a = \log_2 x$, therefore the solutions become

$$\log_2 x = 2 \qquad \log_2 x = 6$$ **A1**

Converting each to exponential form, we get

$$x = 2^2 \qquad x = 2^6$$
$$x = 4 \qquad x = 64$$ **A1**

Question 4

NO CALCULATOR

Medium

[Maximum mark: 7]

(a) Show that $4 - 3\cos 2x = 6\sin^2 x + 1$. [1]

(b) Hence or otherwise solve $4 - 3\cos(4\theta + \frac{2\pi}{3}) - 9\sin(2\theta + \frac{\pi}{3}) = -2$ for $0 \leq \theta < \pi$. [6]

(a) In a 'show that' question we should work from the L.H.S. to the R.H.S.

$$\text{L.H.S.} = 4 - 3\cos 2x$$

The cosine double angle identity that contains only $\sin\theta$ is $\cos 2\theta = 1 - 2\sin^2\theta$.

Substituting this we obtain

$$= 4 - 3(1 - 2\sin^2 x) \qquad \textbf{M1}$$

$$= 4 - 3 + 6\sin^2 x$$

$$= \boxed{6\sin^2 x + 1} \qquad \text{...as required}$$

$$= \text{R.H.S.}$$

(b) By considering a substitution $x = 2\theta + \frac{\pi}{3}$ we can write the equation in part (b) so that contains the expression from part (a).

$$4 - 3\cos(4\theta + \tfrac{2\pi}{3}) - 9\sin(2\theta + \tfrac{\pi}{3}) = -2$$

$$4 - 3\cos 2x - 9\sin x = -2$$

Hence we can substitute the R.H.S. of the equation from part (a) so that the equation is in terms of sine.

$$6\sin^2 x + 1 - 9\sin x = -2 \qquad \textbf{(M1)}$$

$$6\sin^2 x - 9\sin x + 3 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

Notice that this is a quadratic equation. We can factorise it using grouping which gives

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$2\sin^2 x - 2\sin x - \sin x + 1 = 0$$

$$2\sin x(\sin x - 1) - 1(\sin x - 1) = 0$$

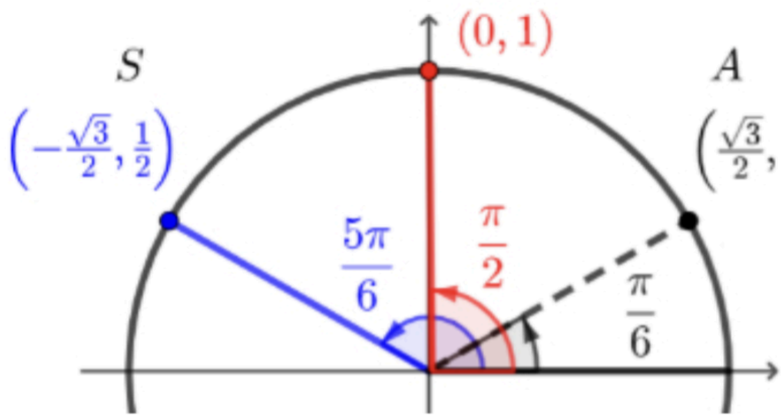
$$(2\sin x - 1)(\sin x - 1) = 0$$

Applying the null factor theorem we get

$$2\sin x - 1 = 0 \qquad \sin x - 1 = 0$$

$$\sin x = \tfrac{1}{2} \qquad \sin x = 1$$ **A1**

Using the unit circle, we know that $\sin x = 1$ when $x = \frac{\pi}{2}$ and $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. This is represented in the diagram below.



Hence the answers for x are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$

(A1)

Recall that we are solving for θ and we used a substitution.

Hence we can find values of θ that satisfy the equation

$$2\theta + \frac{\pi}{3} = \frac{\pi}{6}$$
$$\theta = -\frac{\pi}{12}$$

The first value we have found is outside of the given domain ($0 \leq \theta < \pi$) hence we need to add (or subtract!) 2π to the value we found from the unit circle to obtain other values that could be in the domain

$$2\theta + \frac{\pi}{3} = \frac{\pi}{6} + 2\pi$$
$$= \frac{13\pi}{6}$$
$$2\theta = \frac{13\pi}{6} - \frac{2\pi}{6}$$
$$\theta = \boxed{\frac{11\pi}{12}} \qquad \textbf{A1}$$

This value is now in the given domain.

Let's find the remaining values

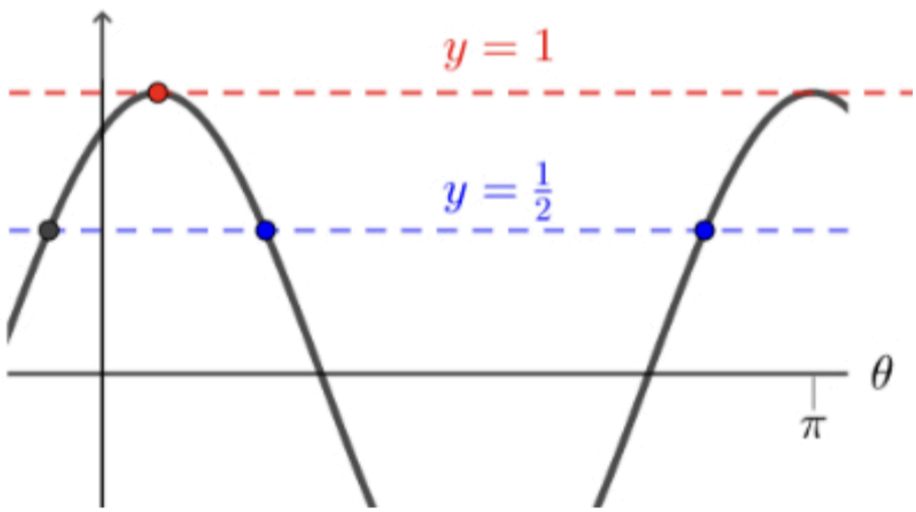
$$2\theta + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\theta = \boxed{\frac{\pi}{12}}$$ **A1**

$$2\theta + \frac{\pi}{3} = \frac{5\pi}{6}$$

$$\theta = \boxed{\frac{\pi}{4}}$$ **A1**

We have found three solutions. Although not required by the question we can view the values on a graph.



Notice three solutions, one in red and two in blue.

The solution identified with the black dot is the value we found which was not in the desired domain.

Question 5

NO CALCULATOR Medium

[Maximum mark: 5]

Consider $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) + 3$ and $g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$.

The function f is mapped onto g by three transformations.

(a) Fully describe each of the transformations and the order in which they must be applied. [3]

A new function is such that $h(x) = g(x) + k$ where $k \in \mathbb{R}$.

(b) Find the minimum value of k such that $h(x) \geq 0$ for all x. [2]

(a) By considering the differences between the functions we can work out the transformations needed to map $f(x)$ to $g(x)$.

First we notice that the inner functions are different.

$$f(x) = 2\cos\left(x - \frac{\pi}{2}\right) + 3$$
$$g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$$

If we add π units to the inner function of $f(x)$, it will equal the inner function of $g(x)$.

A1

Hence

$$f(x+\pi) = 2\cos\left(x - \frac{\pi}{2} + \pi\right) + 3$$
$$f(x+\pi) = 2\cos\left(x + \frac{\pi}{2}\right) + 3$$

Comparing the function $f(x+\pi)$ to $g(x)$, we notice that the coefficient of cosine has been doubled.

$$f(x+\pi) = 2\cos\left(x + \frac{\pi}{2}\right) + 3$$
$$g(x) = 4\cos\left(x + \frac{\pi}{2}\right) + 2$$

Hence, if we multiply $f(x+\pi)$ by 2 we would have

A1

$$\begin{aligned} 2f(x+\pi) &= 2\left(2\cos\left(x+\frac{\pi}{2}\right)+3\right) \\ 2f(x+\pi) &= 4\cos\left(x+\frac{\pi}{2}\right)+6 \end{aligned}$$

If we continue comparing the function $2f(x+\pi)$ to $g(x)$, we see there is a difference of 4 units.

$$\begin{aligned} 2f(x+\pi) &= 4\cos\left(x+\frac{\pi}{2}\right) + 6 \\ g(x) &= 4\cos\left(x+\frac{\pi}{2}\right) + 2 \end{aligned}$$

Hence, if we apply a vertical shift of -4 to $2f(x+\pi)$, we will obtain $g(x)$.

A1

$$\begin{aligned} 2f(x+\pi)-4 &= 4\cos\left(x+\frac{\pi}{2}\right)+6-4 \\ &= 4\cos\left(x+\frac{\pi}{2}\right)+2 \\ &= g(x) \end{aligned}$$

Summarising the 3 transformations, there is

a horizontal shift **left** of π units, followed by a vertical stretch by a scale factor of 2 , followed by a vertical shift of -4 units.

Note: The horizontal shift could also come after the vertical transformations, however the two vertical transformations must be applied in the order given.

(b) In order to apply a vertical translation such that $g(x) > 0$ for all x, we need to know the minimum value of $g(x)$.

The minimum value of cosine is -1. Hence we can determine the minimum of $g(x)$

(M1)

$$\begin{aligned} g(x) &= 4\cos\left(x + \frac{\pi}{2}\right) + 2 \\ &= 4(-1) + 2 \\ &= -2 \end{aligned}$$

Hence, we need to translate the graph vertically upwards at least 2 units.

Therefore, the minimum value is $\boxed{k = 2}$.

A1

Question 6

NO CALCULATOR   Hard

[Maximum mark: 7]

(a) (i) Consider the following equation $2\binom{n}{r} = \binom{n}{r+1}$.

Show that it can be written as $3r + 2 = n$.

(ii) Now consider the following equation $7\binom{n}{r-1} = 2\binom{n}{r}$.

Show that it can be written as $9r - 2 = 2n$. [4]

Consider the expansion

$$(1+x)^n = 1 + a_1 x + \ldots + a_{k-1}x^{k-1} + a_k x^k + a_{k+1}x^{k+1} + \ldots + x^n$$

Where $a_i \in \mathbb{Q}$ and $k \in \mathbb{Z}$.

The coefficients of three consecutive terms of the expansion are such that

$$7 \times a_{k-1} = 2 \times a_k \quad \text{and} \quad 14 \times a_k = 7 \times a_{k+1}$$

(b) Find n. [3]

(a) (i) Using the combinations formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ we can rewrite both the LHS and the RHS of the equation

$$2\left(\frac{\cancel{n!}}{r!(n-r)!}\right) = \frac{\cancel{n!}}{(r+1)!(n-(r+1))!}$$

Now we can simplify and rearrange the equation

$$\frac{2(r+1)!}{r!} = \frac{(n-r)!}{(n-r-1))!}$$

Rewrite the numerators using the concept $n! = n(n-1)!$

$$\frac{2(r+1)r!}{r!} = \frac{(n-r)(n-r-1)!}{(n-r-1))!}$$ **M1**

We can cancel out the factorial terms

$$\frac{2(r+1)\cancel{r!}}{\cancel{r!}} = \frac{(n-r)\cancel{(n-r-1)!}}{\cancel{(n-r-1))!}}$$

$$2(r+1) = n-r$$

$$\boxed{3r+2=n}$$ **A1**

As required.

(ii) Rewrite the equation using the combinations formula and simplify in a similar way to part (a)

$$7\left(\frac{n!}{(r-1)!(n-(r-1))!}\right) = 2\left(\frac{n!}{r!(n-r)!}\right)$$

$$\frac{7}{(r-1)!(n-r+1)!} = \frac{2}{r!(n-r)!}$$

$$\frac{7r!}{(r-1)!} = \frac{2(n-r+1)!}{(n-r)!}$$

$$\frac{7r\cancel{(r-1)!}}{\cancel{(r-1)!}} = \frac{2(n-r+1)\cancel{(n-r)!}}{\cancel{(n-r)!}}$$ **M1**

$$7r = 2n-2r+2$$

$$\boxed{9r-2=2n}$$ **A1**

As required.

(b) We can begin by expanding $(1+x)^n$, using the binomial theorem, in terms of n and r.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r-1}x^{r-1} + \binom{n}{r}x^r + \binom{n}{r+1}x^{r+1} + \ldots + x^n \qquad \textbf{(M1)}$$

The terms in red represent any three consecutive terms in the expansion.

We are told in the question the way in which the coefficients of three consecutive terms are related.

$$(1+x)^n = 1 + a_1x + \ldots + a_{k-1}x^{k-1} + a_kx^k + a_{k+1}x^{k+1} + \ldots + x^n$$

Hence we can use the information given to write the following two equations in terms of binomial coefficients

$$7 \times a_{k-1} = 2 \times a_k \qquad 14 \times a_k = 7 \times a_{k+1}$$

$$7\binom{n}{r-1} = 2\binom{n}{r} \qquad 14\binom{n}{r} = 7\binom{n}{r+1}$$

$$2\binom{n}{r} = \binom{n}{r+1}$$

In part (a) we already rewrote these equations without the combination notation, lets call them [1] and [2]. We can now solve them simultaneously and find n and r.

M1

$$3r + 2 = n \qquad [1]$$
$$9r - 2 = 2n \qquad [2]$$

Multiply [1] by 3 and subtract equation [2] from this result

$$9r - 9r + 6 - (-2) = 3n - 2n$$

$$n = 8 \qquad \textbf{A1}$$

Section B

Question 7

NO CALCULATOR

Medium

[Maximum mark: 11]

Consider the function

$$f(x) = \frac{2}{3}\sqrt{x}(9x^2 - 8x + 3)$$

(a) Show that $f'(x) = \frac{1}{\sqrt{x}}(15x^2 - 8x + 1)$. [4]

(b) Hence find the $x-$coordinates of the two stationary points of $f(x)$. [3]

A particle, P, is moving along the $x-$axis. Its position s, in metres, relative to the origin after time t, measured in seconds, is given by

$$s(t) = \frac{2}{3}\sqrt{t}(9t^2 - 8t + 3)$$

Where $t \geq 0$.

The particle is moving to the left for $t = k$ seconds.

(c) Hence find k. [4]

(a) There are a number of ways to differentiate this function.

We could expand the brackets using index laws then differentiate each term.

However, in this solution we will use the product rule.

(M1)

$$\text{If} \qquad y = uv$$

$$\text{then} \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

With $u = \frac{2}{3}x^{\frac{1}{2}}$ and $v = 9x^2 - 8x + 3$.

Using the power rule, we get $\frac{du}{dx} = \frac{1}{3}x^{-\frac{1}{2}}$ and $\frac{dv}{dx} = 18x - 8$.

By substituting these results into the formula we can write

$$f'(x) = (9x^2 - 8x + 3) \times \frac{1}{3}x^{-\frac{1}{2}} + \frac{2}{3}x^{\frac{1}{2}} \times (18x - 8) \qquad \textbf{A1A1}$$

If we look at the requested form of the answer we can see that $x^{-\frac{1}{2}}$ is a common factor.

Note that for the second term of the derivative we can use the following to help us factorise $x^{\frac{1}{2}} = x^{-\frac{1}{2}} \times x$.

Hence we get

$$= x^{-\frac{1}{2}}\left[\frac{1}{3}(9x^2 - 8x + 3) + \frac{2}{3} \times x(18x - 8)\right]$$

Now we can expand the brackets and simplify

\begin{align*}e}{ \tag*{\textbf{(M1)}} \end{align*}

$$= x^{-\frac{1}{2}}\left[3x^2 - \frac{8}{3}x + 1 + 12x^2 - \frac{16}{3}x\right]$$

$$= x^{-\frac{1}{2}}\left[15x^2 - \frac{24}{3}x + 1)\right]$$

$$= \boxed{\frac{1}{\sqrt{x}}(15x^2 - 8x + 1)} \qquad \text{... as required}$$

(b) Stationary points occur when $f'(x) = 0$.

Hence we can form an equation

$$\frac{1}{\sqrt{x}}(15x^2 - 8x + 1) = 0 \qquad \textbf{(M1)}$$

Note that $\frac{1}{\sqrt{x}} \neq 0$ for any value of x, therefore we only need to consider the term in the brackets when solving the equation above. Thus we need to solve

$$15x^2 - 8x + 1 = 0$$

We can attempt to factorise this. Note that $15x^2 = 3x \times 5x$ and the only combinations that give $+1$ are 1×1 or $(-1) \times (-1)$. Hence through inspection we get

$$(5x - 1)(3x - 1) = 0 \qquad \textbf{A1}$$

The $x-$coordinates of the stationary points are $x = \frac{1}{5}, \frac{1}{3}$

A1

(c) The particle is moving left when the velocity is negative. Hence we need to find the interval for t which satisfies

$$v(t) < 0 \qquad \textbf{(M1)}$$

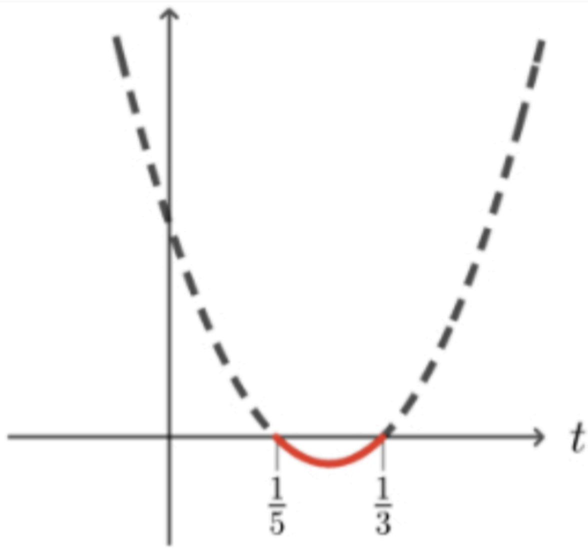
We can note that $v(t) = s'(t)$ and that the function given for $s(t)$ is of the same form as $f(x)$ from part (a).

Hence, using the answer from part (a) we can say that

$$v(t) = \frac{1}{\sqrt{t}}(15t^2 - 8t + 1)$$

We can realise that if $t \geq 0$ then $\frac{1}{\sqrt{t}} > 0$. Therefore the expression in the bracket will determine the sign of $v(t)$.

To help us here we can make a sketch of $y = 15t^2 - 8t + 1$



(M1)

Notice the red part of the curve is when $y < 0$ and therefore $v(t) < 0$.

Using our answers from part (b) we can write that $v(t) < 0$ when $\frac{1}{5} < t < \frac{1}{3}$.

(A1)

However, the question doesn't ask for an interval it asks for a total time the particle is moving left, hence

$$\begin{aligned} k &= \tfrac{1}{3} - \tfrac{1}{5} \\ &= \tfrac{5}{15} - \tfrac{3}{15} \\ &= \boxed{\tfrac{2}{15}} \text{ seconds} \end{aligned}$$

A1

Question 8

NO CALCULATOR

Hard

[Maximum mark: 13]

Consider the function $f(x) = 4x - x^2 - 1$.

(a) Write $f(x)$ in the form $(x - h)^2 + k$. [2]

A line is drawn through the points $A(0, f(0))$ and $B(2, f(2))$.

(b) (i) Write down the coordinates of points A and B.

(ii) Find $g(x)$, the equation of the line passing through points A and B.

(iii) Hence, show that the area enclosed by $f(x)$ and $g(x)$ is $\frac{4}{3}$ units^2. [4]

A horizontal line is drawn through the points $C(1, f(1))$ and $D(3, f(3))$.

(c) Show that the area enclosed by $f(x)$ and line CD is $\frac{4}{3}$ units^2. [2]

Consider the two points $E(a, f(a))$ and $F(a + 2, f(a + 2))$.

(d) Show that the area enclosed by the function f and the line EF is $\frac{4}{3}$ units^2. [5]

(a) To write f in the desired form we must complete the square

(M1)

$$f(x) = 4x - x^2 - 1$$
$$= -(x^2 - 4x) - 1$$
$$= -[(x-2)^2 - (2)^2] - 1$$
$$= -(x-2)^2 + 4 - 1$$
$$= \boxed{3 - (x-2)^2}$$

A1

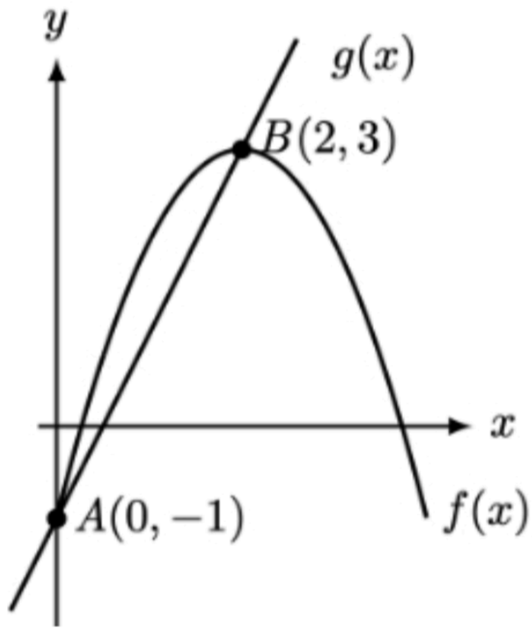
(b) (i) We can find the y−coordinate by evaluating the function at both $x = 0$ and $x = 2$.

$$f(0) = 3 - (0-2)^2 \qquad f(2) = 3 - (2-2)^2$$
$$= -1 \qquad\qquad = 3$$

Hence the coordinates are $\boxed{A(0, -1) \text{ and } B(2, 3)}$

A1

(ii) We can make a quick sketch of f



We have the $y-$intercept already.

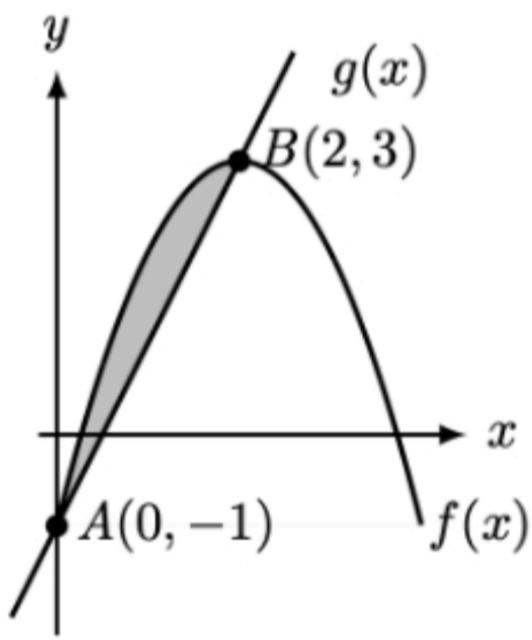
And the gradient of the line through A and B is

$$m = \frac{3-(-1)}{2-0}$$
$$m = 2$$

Therefore, the line through A and B is $\boxed{y = 2x - 1}$

A1

(iii) The shaded area, seen below, shows the area we need to find



To find the area we evaluate the following definite integral, where A_x and B_x are the $x-$coordinates of points A and B respectively

$$= \int_{A_x}^{B_x} (\mathrm{f}(x) - \mathrm{g}(x))\, dx$$

$$= \int_0^2 (4x - x^2 - 1 - (2x - 1))\, dx \qquad \textbf{M1}$$

$$= \int_0^2 (2x - x^2)\, dx$$

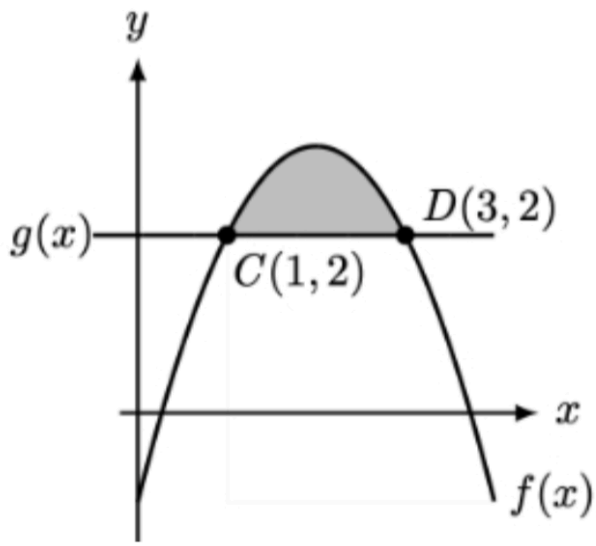
Using the inverse power rule and substituting in the limits we get

$$= \left[x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= (2)^2 - \frac{1}{3}(2)^3 - \left((0)^2 - \frac{1}{3}(0)^3\right) \qquad \textbf{A1}$$

$$= \boxed{\frac{4}{3}\text{ units}^2} \quad \text{...as required}$$

(c) We can make a quick sketch of the new region enclosed by $f(x)$ and the new (horizontal) $g(x)$.



We now evaluate the definite integral, where C_x and D_x are the $x-$coordinates of points C and D respectively

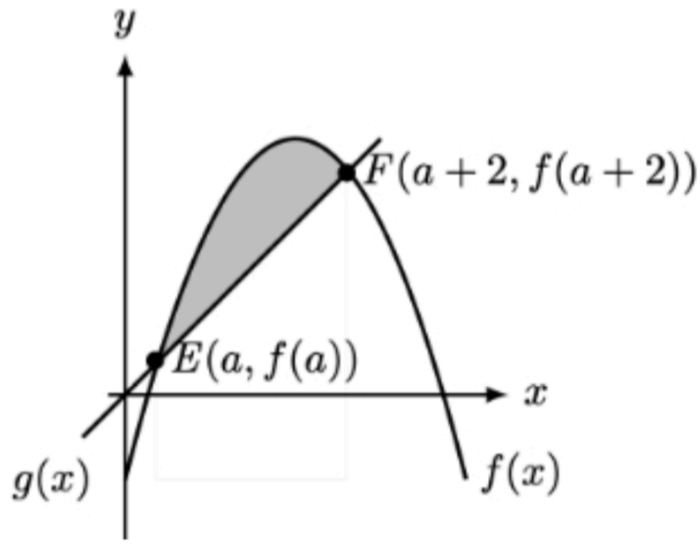
$$= \int_{C_x}^{D_x} (\text{f}(x) - \text{g}(x))\, dx$$
$$= \int_1^3 (4x - x^2 - 1 - (2))\, dx$$
$$= \int_1^3 (4x - x^2 - 3)\, dx \qquad \textbf{M1}$$

Using the inverse power rule, to integrate each term and substitute in the limits, we get

$$= \left[2x^2 - \frac{1}{3}x^3 - 3x\right]_1^3$$
$$= 2(3)^2 - \frac{1}{3}(3)^3 - 3(3) - \left((2(1)^2 - \frac{1}{3}(1)^3 - 3(1)\right)$$
$$= 18 - 9 - 9 - (2 - \frac{1}{3} - 3) \qquad \textbf{A1}$$
$$= \boxed{\frac{4}{3}\text{ units}^2}$$ ...as required

(d) The diagram below shows a region bound by $f(x)$ and $g(x)$ and the points $E(a, f(a))$ and $F(a+2, f(a+2)))$.

The function f needs to be used to calculate the $y-$coordinate values of E and F.



To find the area we need to find the linear function $g(x)$ in terms of a.

(M1)

g(x) is of the form $g(x) = m_g x + c_g$.

Where the gradient, m_g, is

$$\begin{aligned} m_g &= \frac{y_2 + y_1}{x_2 - x_1} \\ &= \frac{\mathrm{f}(a+2) - \mathrm{f}(a)}{a + 2 - a} \\ &= \frac{(3 - a^2) - (3 - (a-2)^2))}{2} \end{aligned}$$

Simplifying we get

$$\begin{aligned} &= \frac{-a^2 + (a-2)^2}{2} \\ &= \frac{-a^2 + a^2 - 4a + 4}{2} \\ m_g &= 2 - 2a \end{aligned} \qquad \textbf{A1}$$

Using the point $A(a, f(a))$ we can find the $y-$intercept, c_g, of $g(x)$

$$\begin{aligned} y &= (2 - 2a)x + c_g \\ \mathrm{f}(a) &= (2 - 2a)a + c_g \\ 3 - (a-2)^2 &= (2 - 2a)a + c_g \\ c_g &= 3 - (a-2)^2 - 2a + 2a^2 \\ c_g &= a^2 + 2a - 1 \end{aligned} \qquad \textbf{A1}$$

The function $g(x)$ in terms of a is

$$\mathrm{g}(x) = (2 - 2a)x + a^2 + 2a - 1$$

The shaded area is given by the following definite integral

$$= \int_{E_x}^{F_x} (f(x) - g(x))\, dx$$

$$= \int_{a}^{a+2} (4x - x^2 - 1 - ((2 - 2a)x + a^2 + 2a - 1))\, dx$$ **M1**

Simplifying we get

$$= \int_{a}^{a+2} (4x - x^2 - 1 - (2x - 2ax + a^2 + 2a - 1))\, dx$$

$$= \int_{a}^{a+2} (2x - x^2 + 2ax - a^2 - 2a)\, dx$$

$$= \int_{a}^{a+2} (-x^2 + (2a + 2)x - a^2 - 2a)\, dx$$

Now we can integrate each term, using the inverse power rule, and substitute in the limits.

$$= \left[-\frac{1}{3}x^3 + \frac{2a+2}{2}x^2 - x(a^2+2a)\right]_a^{a+2} \qquad \textbf{A1}$$

$$= -\frac{1}{3}(a+2)^3 + (a+1)(a+2)^2 - (a+2)(a^2+2a) - \ldots$$
$$\ldots\left(-\frac{1}{3}a^3 + (a+1)a^2 - a(a^2+2a)\right)$$

Expanding and simplifying we gradient

$$= -\frac{1}{3}(a^3+6a^2+12a+8) + a^3 + 5a^2 + 8a + 4 - (a^3+4a^2+4a) - \ldots$$
$$\ldots\left(-\frac{1}{3}a^3 + a^3 + a^2 - a^3 - 2a^2\right)$$

We continue to collect any like terms and simplify to get

$$= \cancel{-\frac{1}{3}a^3} - 2a^2 - 4a - \frac{8}{3} + a^2 + 4a + 4 - \ldots$$
$$\ldots\left(\cancel{-\frac{1}{3}a^3} - a^2\right)$$

$$= \cancel{-2a^2} \cancel{-4a} - \frac{8}{3} + \cancel{a^2} + \cancel{4a} + 4 + \cancel{a^2} \qquad \textbf{A1}$$
$$= -\frac{8}{3} + 4$$
$$= \boxed{\frac{4}{3}\text{ units}^2} \qquad \text{...as required}$$

Question 9

NO CALCULATOR

Hard ●●●●

[Maximum mark: 23]

Consider the function $f(x) = \dfrac{\cos x}{2+\sin x}$ for $-\pi \leq x \leq \pi$.

(a) Evaluate $f(0)$. [1]

(b) Find all possible values of a if $f(a) = 0$. [2]

(c) (i) Show that $f'(x) = -\dfrac{2\sin x + 1}{(2+\sin x)^2}$.

(ii) Hence find the $x-$coordinates of any stationary points of f. [7]

(d) Given that $f''(x) = -\dfrac{2\cos x(1-\sin x)}{(2+\sin x)^3}$ find the nature of any stationary points of f. [5]

(e) Hence sketch the graph of f, clearly showing the values of the axes intercepts and the $x-$coordinates of any stationary points. [3]

The function f is positive and decreasing in the region $s < x < t$.

The area enclosed by f and the $x-$axis from $x = s$ to $x = t$ is $\ln c$ where $c \in \mathbb{Z}$.

(f) Find c. [5]

(a) To evaluate the function we substitute in $x = 0$ which gives

$$f(0) = \frac{\cos 0}{2+\sin 0}$$

Using the fact that $\cos 0 = 1$ and $\sin 0 = 0$ we get

$$= \frac{1}{2+0}$$

$$= \boxed{\frac{1}{2}}$$ **A1**

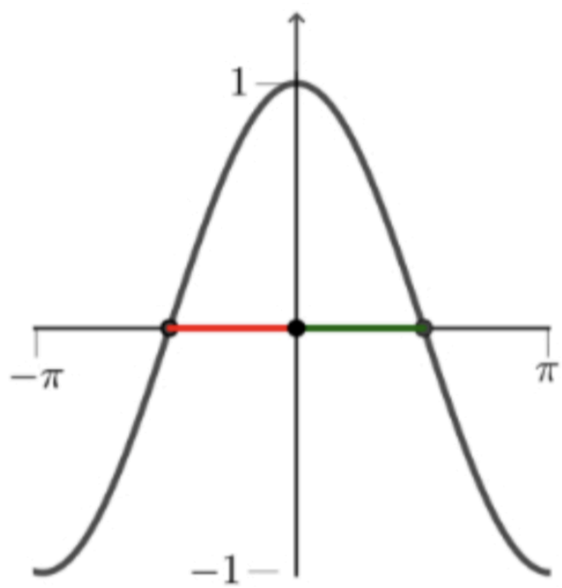
(b) Using $x = a$ and the function f we can form an equation

$$f(a) = \frac{\cos a}{2 + \sin a}$$
$$= 0$$

Only the numerator can provide solutions to the equation, hence we get

$$\cos a = 0$$ **(M1)**

Recall, the domain of f is $-\pi \leq x \leq \pi$, we can use a sketch of $\cos x$ to find all solutions of a in that domain



We know that the principal angle of $\cos \frac{\pi}{2}$ is 0, which is shown in green, and using the symmetries of the cosine curve we can see that $-\frac{\pi}{2}$, shown in red, is also a solution.

Hence $a = \pm \frac{\pi}{2}$

A1

(c) (i) To differentiate f we need to use the quotient rule.

(M1)

For this we need the derivative of both the numerator and the denominator

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(2 + \sin x) = \cos x$$

Now we can use these results and the quotient rule to form an expression for $f'(x)$

$$\begin{aligned} f'(x) &= \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2} \qquad \textbf{A1A1} \\ &= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} \\ &= \frac{-2\sin x - (\sin^2 x + \cos^2 x)}{(2+\sin x)^2} \end{aligned}$$

Notice we can use a trigonometric identity to simplify the numerator

M1

$$\begin{aligned} &= \frac{-2\sin x - 1}{(2+\sin x)^2} \\ &= \boxed{-\frac{2\sin x + 1}{(2+\sin x)^2}} \end{aligned}$$

... as required.

(ii) To find any stationary points we must solve $f'(x) = 0$, hence we can form an equation using the result from part (c)(i)

$$-\frac{2\sin x + 1}{(2 + \sin x)^2} = 0 \qquad \textbf{(M1)}$$

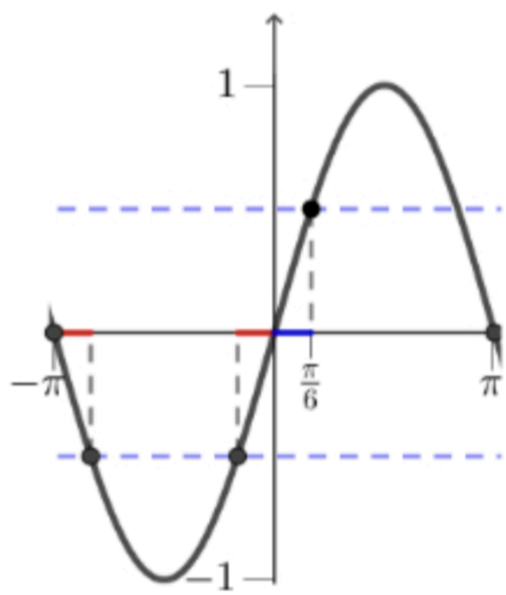
We only need to consider the numerator, hence we get

$$-2\sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \qquad \textbf{(A1)}$$

The principal solution (this means the solution in the first quadrant) is $x = \frac{\pi}{6}$.

However, if we make a sketch of $\sin x$ with the same domain as f we can see that the solutions to $\sin x = -\frac{1}{2}$ are both negative



Due to the symmetries of the graph we can see that the two angles, x_1 and x_2, marked in red are

$$x_1 = -\pi + \frac{\pi}{6} \qquad x_2 = 0 - \frac{\pi}{6}$$
$$= -\frac{5\pi}{6} \qquad x_2 = -\frac{\pi}{6}$$

Hence there are two stationary points with $x-$coordinates of $\boxed{x = -\frac{5\pi}{6}}$ and $\boxed{x = -\frac{\pi}{6}}$.

A1

(d) We can use the second derivative to determine the nature of the stationary points.

If $f''(x) > 0$ the point is a minimum and if $f''(x) < 0$ it is a maximum.

(M1)

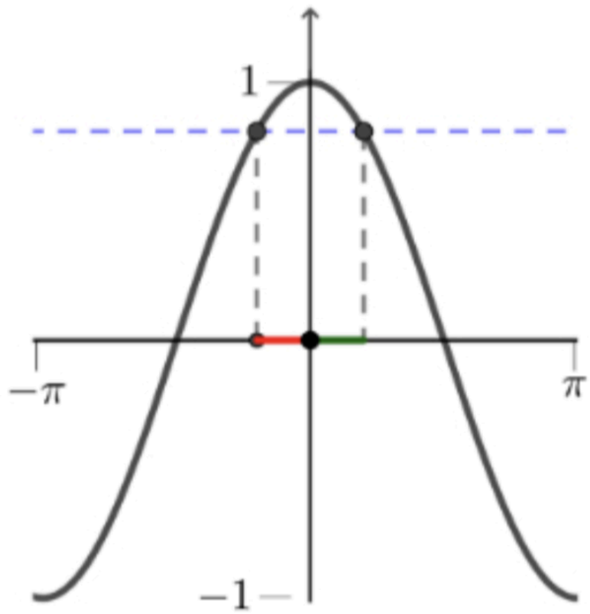
Let's first test the point $x = -\frac{\pi}{6}$

$$f''(-\tfrac{\pi}{6}) = -\frac{2\cos(-\frac{\pi}{6})(1 - \sin(-\frac{\pi}{6}))}{(2 + \sin(-\frac{\pi}{6}))^3}$$

To evaluate the expression above we need the exact values of $\sin(-\frac{\pi}{6})$ and $\cos(-\frac{\pi}{6})$.

From previous work we know that $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$.

We can use the symmetry of the cosine curve



to realise that $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$.

Hence $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Now let's substitute the two values into the earlier expression we had $f''(x)$ to get

$$f''(-\tfrac{\pi}{6}) = -\frac{2 \times \frac{\sqrt{3}}{2} \times (1-(-\frac{1}{2}))}{(2+(-\frac{1}{2}))^3} \qquad \textbf{M1}$$

We don't need to evaluate the expression exactly we just need to know if it is positive or negative, let's do a little simplification

$$= -\frac{2 \times \frac{\sqrt{3}}{2}(1+\frac{1}{2})}{(\frac{3}{2})^3}$$

We can now see that quotient will result in a positive value. Hence, as the entire quotient is being multiplied by -1, the result is negative.

Therefore

$$f''(-\tfrac{\pi}{6}) < 0$$

Hence $\boxed{x = -\frac{\pi}{6} \text{ is a maximum value}}$.

A1

We can use a similar process for the other stationary point $x = -\frac{5\pi}{6}$

$$f''(-\tfrac{5\pi}{6}) = -\frac{2\cos(-\frac{5\pi}{6})(1-\sin(-\frac{5\pi}{6}))}{(2+\sin(-\frac{5\pi}{6}))^3}$$ **M1**

From previous work we know that $\sin(-\frac{5\pi}{6}) = -\frac{1}{2}$.

And using the symmetries of the cosine curve we get

$$\begin{aligned}\cos(-\tfrac{5\pi}{6}) &= \cos(\tfrac{5\pi}{6}) \\ &= -\cos(\tfrac{\pi}{6}) \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

We can now substitute these two values into the second derivative to get

$$\begin{aligned}f''(-\tfrac{5\pi}{6}) &= -\frac{2\left(-\frac{\sqrt{3}}{2}\right)(1-(-\frac{1}{2}))}{(2+(-\frac{1}{2})^3} \\ &= -\frac{-2\sqrt{3}(1+\frac{1}{2})}{(\frac{3}{2})^3}\end{aligned}$$

We can see that the quotient will now be negative and hence the overall value will be positive.

Therefore

$$f''(-\tfrac{5\pi}{6}) > 0$$

Hence $x = -\frac{5\pi}{6}$ is a minimum value.

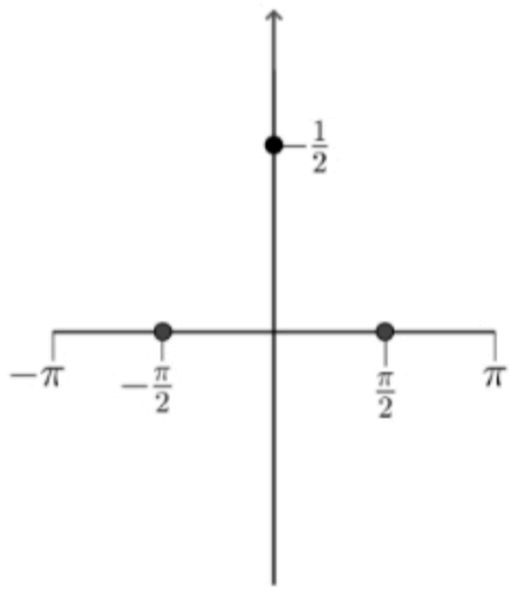
(e) To help us with the sketch we can summarise our findings so far.

From parts (a) and (b) we found the axes intercepts $(0, \frac{1}{2})$, $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$.

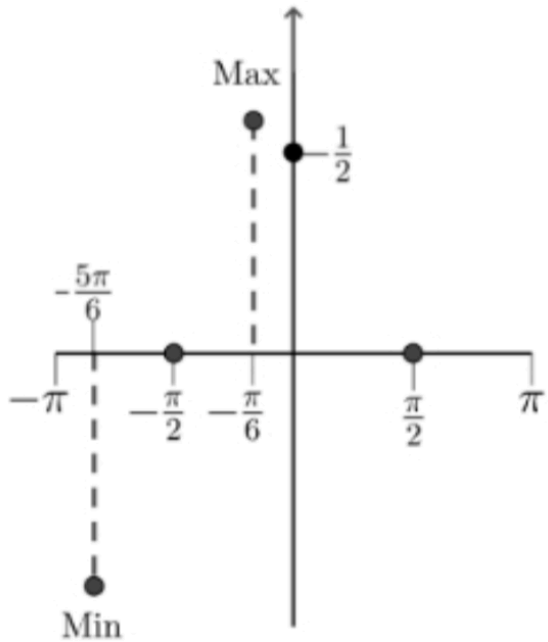
We have also found a maximum point at $x = -\frac{\pi}{6}$ and a minimum point at $x = -\frac{5\pi}{6}$.

We should also remind ourselves that the domain is $-\pi \leq x \leq \pi$.

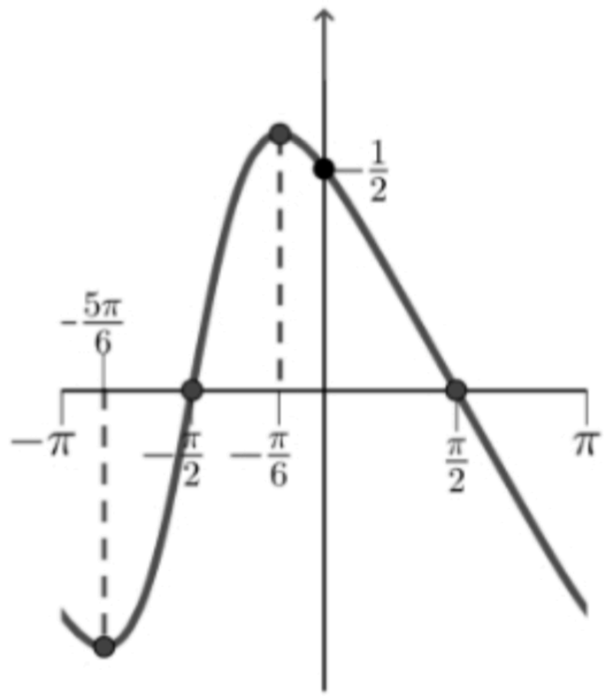
Let's first sketch the domain and the axes intercepts



We can now add the stationary points



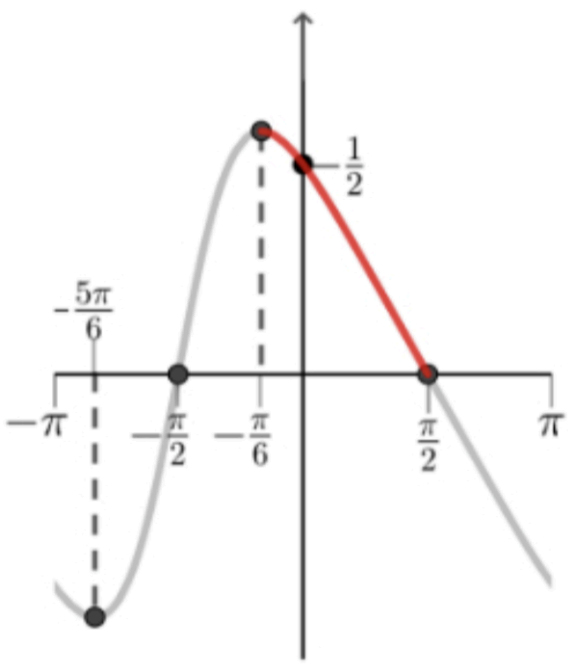
Finally we can use the plotted points to fit the function. Being careful to stop at the end-points!



Correct axes intercepts	**A1**
Correct position of two stationary points	**A1**
Correct shape and end-points	**A1**

(f) We need to identity the region in which f is positive (above the $x-$axis) and is decreasing (which means the gradient is negative).

This region is the highlighted red part of f shown below



Hence we can see that the value of $s = -\frac{\pi}{6}$ and $t = \frac{\pi}{2}$.

To find the area under the curve between these two values we need to evaluate the definite integral of f from $-\frac{\pi}{6}$ to $\frac{\pi}{2}$

$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,\mathrm{d}x \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{2+\sin x}\,\mathrm{d}x \qquad \textbf{(M1)} \end{aligned}$$

This integral is a quotient. Often a good strategy to integrate a quotient is to use a substitution for the denominator.

Hence let $u = 2 + \sin x$.

To rewrite the integral we will need an expression for $\mathrm{d}x$ in terms of $\mathrm{d}u$, hence

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

$$\mathrm{d}x = \frac{1}{\cos x}\mathrm{d}u$$

Let's now replace the original integral so that it is in terms of u. We will omit the limits for now.

$$\int \frac{\cos x}{2+\sin x}\,\mathrm{d}x = \int \frac{\cancel{\cos x}}{u} \times \frac{1}{\cancel{\cos x}}\mathrm{d}u$$

$$= \int \frac{1}{u}\mathrm{d}u$$

$$= \ln|u| + C$$ **A1**

Let's now substitute $u = 2 + \sin x$ and reintroduce the limits.

As the integral is now definite we can omit the constant.

This gives

$$\text{Area} = \Big[\ln|2+\sin x|\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$ **A1**

As $2 + \sin x > 0$ for all x we can remove the absolute value signs.

Substituting in the upper and lower limit we get

$$= \ln\left(2+\sin\left(\tfrac{\pi}{2}\right)\right) - \ln\left(2+\sin\left(-\tfrac{\pi}{6}\right)\right)$$ **M1**

Recall $\sin\left(\frac{\pi}{2}\right) = 1$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, hence we get

$$= \ln(2+1) - \ln\left(2+-\left(\tfrac{1}{2}\right)\right)$$

$$= \ln 3 - \ln\tfrac{3}{2}$$

$$= \ln\frac{3}{\frac{3}{2}}$$

$$= \ln 2$$

Hence $\boxed{c = 2}$

A1