IB Mathematics AA SL - Prediction Exams May 2025 - Paper 2

Paper 2 | 9 questions | 90 mins | 80 marks

Section A

Question 1

CALCULATOR | Easy

[Maximum mark: 4]

Consider the function $f(x) = \dfrac{3x+4}{2x+k}$.

(a) Write down the domain of f in terms of k. [1]

(b) Find $f^{-1}(x)$ in terms of k. [2]

(c) Hence write down the value of k such that $f(x)$ is a self-inverse function. [1]

(a) A rational function is undefined when the denominator is equal to zero.

$$2x + k = 0$$

$$x = -\frac{k}{2}$$

Any other value for x is valid.

Hence we get the domain

$$\boxed{x \in \mathbb{R}, x \neq -\frac{k}{2}}$$ **A1**

(b) We are trying to find the function $y = f^{-1}(x)$.

The definition of an inverse states that

$$(f \circ f^{-1})(x) = x$$

Hence we can rewrite this to get

$$f(f^{-1}(x)) = x$$
$$f(y) = x$$

Hence we get

$$f(y) = x$$
$$\frac{3y+4}{2y+k} = x$$

Recall, we want $y = f^{-1}(x)$ therefore we should rearrange this to make y the subject.

$$x(2y+k) = 3y+4$$
$$2xy + kx = 3y+4$$

To isolate y, we move all of the y terms to one side and then factorise out y.

$$2xy - 3y = -kx + 4$$

$$y(2x-3) = -kx + 4 \qquad \textbf{(M1)}$$

$$y = \frac{-kx+4}{2x-3}$$

$$\boxed{f^{-1}(x) = \frac{-kx+4}{2x-3}} \qquad \textbf{A1}$$

(c) A self-inverse function is such that $f(x) \equiv f^{-1}(x)$.

Therefore

$$\frac{3x+4}{2x+k} \equiv \frac{-kx+4}{2x-3}$$

Comparing either the numerators or denominators,

$$3x+4=-kx+4$$

Therefore

$$k=-3$$ **A1**

Question 2

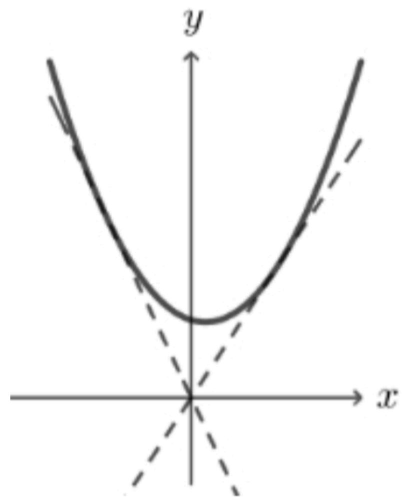
CALCULATOR

Easy

[Maximum mark: 5]

Find the equations of the two tangents to the curve $y=2x^2-x+\frac{9}{2}$ that pass through the origin.

To better understand the scenario we can make an approximate sketch.



The dashed lines are the two tangent lines whose equations we need to find.

The equation of a tangent line, through the origin, will be of the form

$$y = mx$$ **(M1)**

To find the intersection points of the curve and line we can solve both equations simultaneously, hence

$$mx = 2x^2 - x + \frac{9}{2} \qquad \textbf{(M1)}$$

$$0 = 2x^2 - x - mx + \frac{9}{2}$$

$$0 = 2x^2 - (1+m)x + \frac{9}{2} \qquad \textbf{(A1)}$$

If the lines are tangent to the curve then there is only one point of intersection i.e. only 1 solution.

As we have a quadratic equation then we can say that the discriminant must be equal to zero. Hence

$$\Delta = b^2 - 4ac$$

$$0 = [-(1+m)]^2 - 4(2)\left(\tfrac{9}{2}\right) \qquad \textbf{A1}$$

Solving for m we get

$$36 = [-(1+m)]^2$$

$$-(1+m) = \pm 6$$

$$m = -7 \text{ and } 5$$

Therefore the two equations are

$$\boxed{y = -7x} \text{ and } \boxed{y = 5x} \qquad \textbf{A1}$$

Question 3

CALCULATOR

Medium

[Maximum mark: 5]

Chun Li has a bag with five 6-sided dice.

Four of them are normal fair dice and one of them is biased with a 6 showing on each of its faces.

She draws two out at random and rolls them.

(a) Find the probability a six shows on both dice. [3]

(b) Given a six shows on both dice find the probability one of the dice is the biased dice. [2]

(a) We can begin by defining two events.

Event A is drawing two fair dice.

Event B is rolling two 6s.

Let's consider event A first.

As Chun Li is drawing out two dice, she can either have 2 fair dice or 1 biased dice and 1 fair dice.

The probability of her drawing (without replacement!) two fair dice is

$$\begin{aligned} \mathrm{P}(A) &= \frac{4}{5} \times \frac{3}{4} \\ &= \frac{12}{20} \\ &= \frac{3}{5} \end{aligned}$$

As she can only hold 2 fair dice or 1 fair and 1 biased this means that the probability she holds 1 fair dice and 1 biased is the compliment of the result we just found, hence

$$\begin{aligned} \mathrm{P}(A') &= 1 - \frac{3}{5} \\ &= \frac{2}{5} \end{aligned}$$

Now let's consider event B. Rolling two 6's.

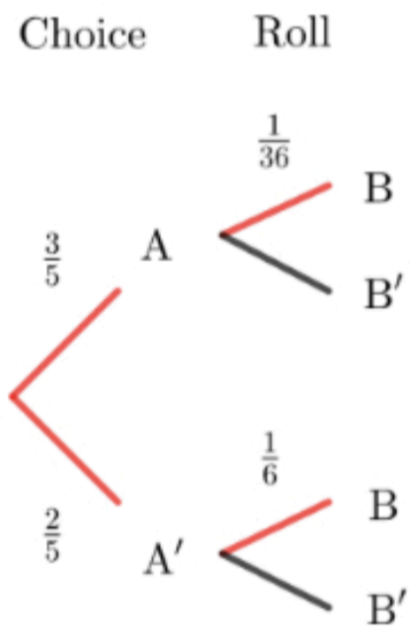
If she rolls 2 fair dice the probability of rolling two $6'$s is

$$\begin{aligned} &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

If she rolls 1 fair dice and 1 biased dice the probability of rolling two $6'$s is

$$\begin{aligned} &= \frac{1}{6} \times 1 \\ &= \frac{1}{6} \end{aligned}$$

We can now put this information into a tree diagram



Notice we don't need to calculate the all possible outcomes.

The sum of both red branches will give the probability she rolls two $6's$.

Therefore we get

$$= \frac{3}{5} \times \frac{1}{36} + \frac{2}{5} \times \frac{1}{6} \qquad \textbf{(M1A1)}$$

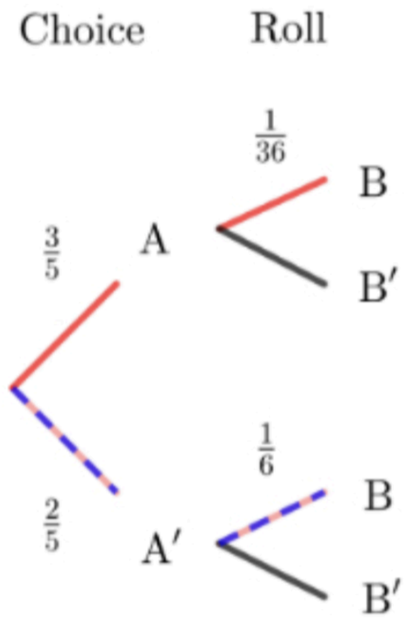
$$= \boxed{\frac{1}{12}} \qquad \textbf{A1}$$

$$= 0.08333...$$

(b) In this part some we have been given some information about the roll.

This is known as conditional probability.

Consider this tree diagram



We have been told that the roll is from either of the two red branches. This is now the universal set and is the denominator of the answer.

The blue dashed branches form the desired outcome.

Here is the conditional probability formula

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

The way we have defined the events in this solution means A' is the event that 1 fair and 1 biased dice is chosen.

Hence we can replace A with A' to get

$$\mathrm{P}(A'|B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)}$$

Our answer from part (a) is $\mathrm{P}(B)$ and the numerator is the blue dashed branch seen on the previous tree diagram.

$$\mathrm{P}(A'|B) = \frac{\frac{2}{5} \times \frac{1}{6}}{\frac{1}{12}} \qquad \textbf{(M1)}$$

$$= \boxed{\frac{4}{5}} \qquad \textbf{A1}$$

Question 4

CALCULATOR Medium ●●●●●

[Maximum mark: 7]

A cyclist leaves town A on a bearing of $240°$ and rides 11 kilometers to town B.

The cyclist then travels d km on a bearing of $090°$ until he is exactly 6 km from town A.

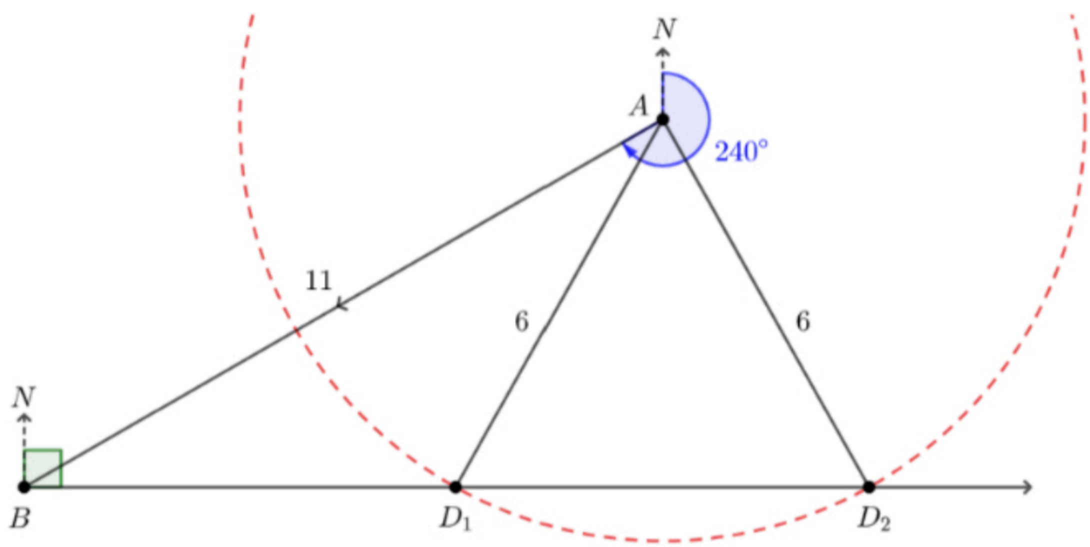
Find the possible values of d.

First let's sketch a diagram to better understand the given information.

We can sketch the approximate positions of towns A and B and also indicate the stopping places of the cyclist by D_1 and D_2.

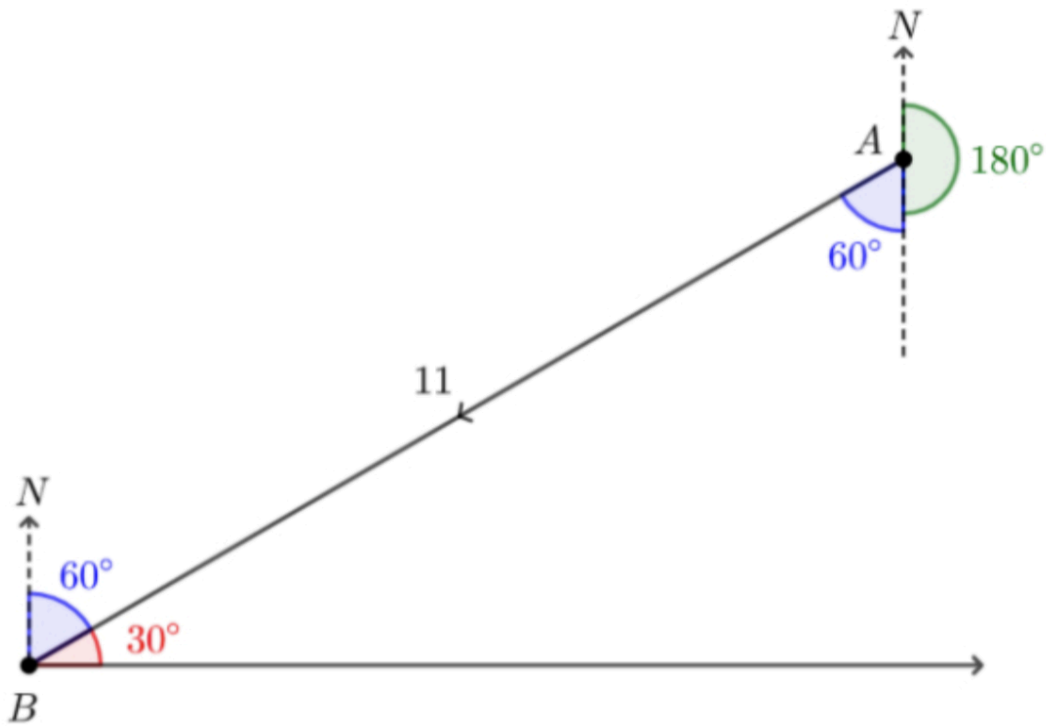
Any bearing measures start from the North and turn clockwise.

With all lengths in kilometres, our diagram will look like this.



Notice that the dashed red circle, with radius 6 centred on A, indicates that there are two places where the cyclist can stop that are 6 kilometers from town A.

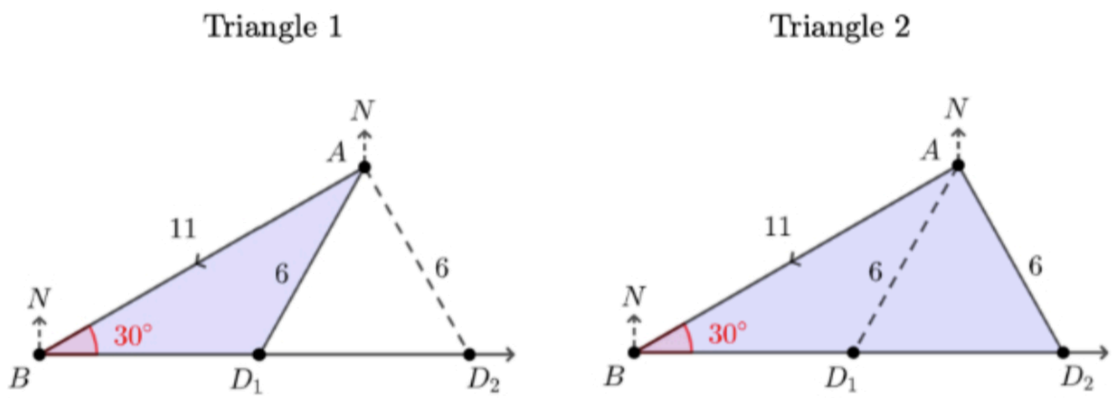
Using angle reasoning, we can determine $\angle ABD$.



$$240^\circ - 180^\circ = 60^\circ$$

$$90^\circ - 60^\circ = 30^\circ \quad \textbf{(A1)}$$

When we add this to the diagram we notice that there are 2 triangles which have sides of 11 and 6 and a non-included angle of 30°. This is an example of the ambiguous case of the sine rule.



Using the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Hence

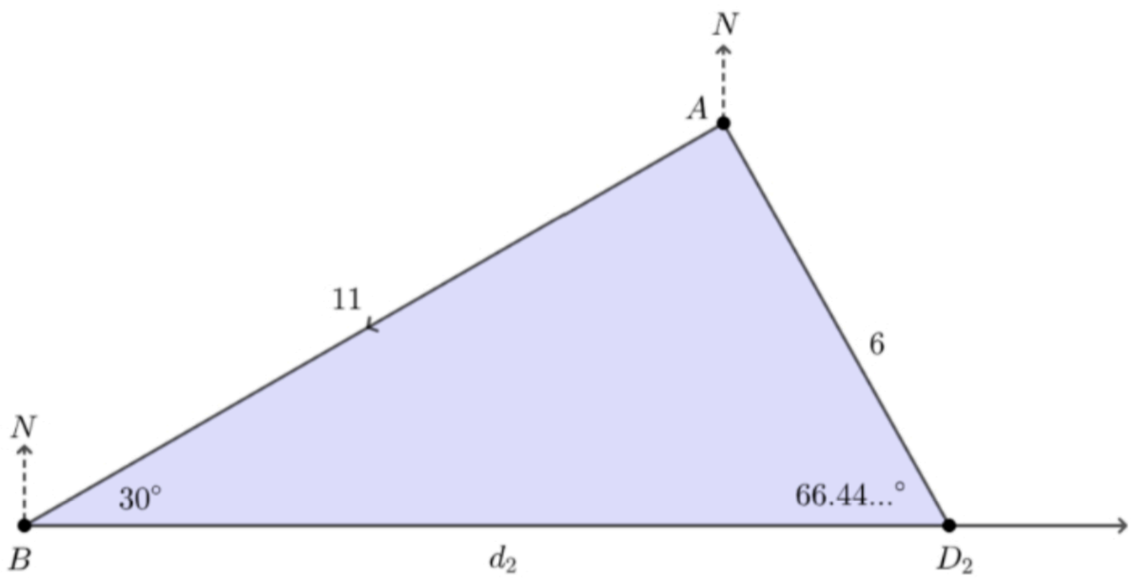
$$\frac{\sin D}{11} = \frac{\sin 30}{6} \quad \textbf{(M1)}$$

$$\sin D = 0.9166...$$

$$D = 66.44... \quad \textbf{A1}$$

As $D < 90^\circ$, this must be the angle associated with triangle 2.

Let's complete this triangle to find the distance d_2 associated with stopping at D_2.



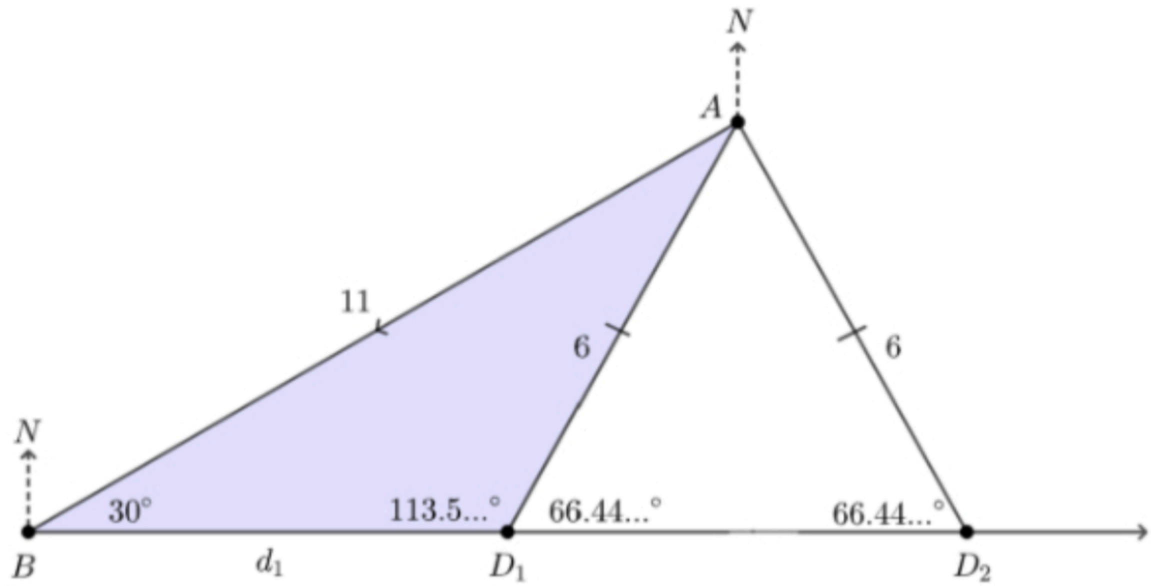
Finding angle BAD_2

$$\begin{aligned} BAD_2 &= 180^\circ - 30^\circ - 66.44...^\circ \\ &= 83.55...^\circ \end{aligned}$$ **(A1)**

Using sine rule again

$$\begin{aligned} \frac{d_2}{\sin 83.55...^\circ} &= \frac{6}{\sin 30^\circ} \\ d_2 &= 11.92... \\ &= \boxed{11.9 \text{ km}} \end{aligned}$$ **A1**

We can see that triangle AD_1D_2 is isosceles.



Hence we can find angle BD_1A

$$= 180^\circ - 66.44...^\circ$$

$$= 113.5...$$ **M1**

Finding angle BAD_1 in triangle 1

$$\angle BAD_1 = 180^\circ - 30^\circ - 113.55...^\circ$$

$$= 36.44...^\circ$$

Using sine rule again to find the distance associated with stopping at D_1

$$\frac{d_1}{\sin 36.44...^\circ} = \frac{6}{\sin 30^\circ}$$

$$d_1 = 7.128...$$

$$= \boxed{7.13 \text{ km}}$$ **A1**

Question 5

CALCULATOR

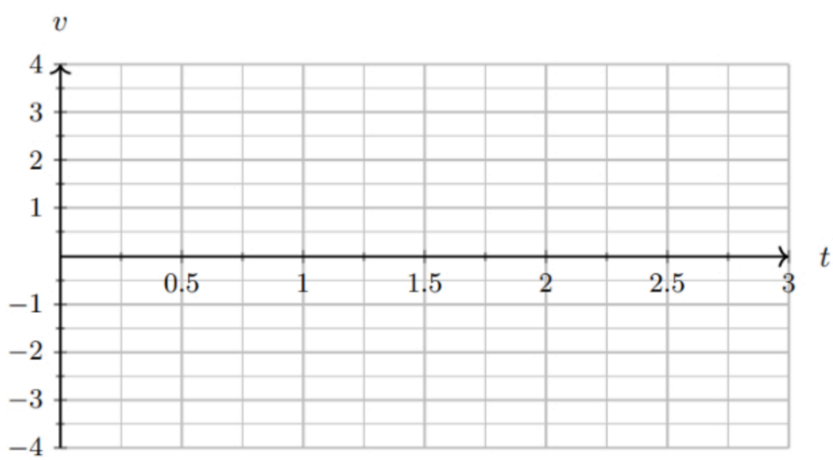
Medium

[Maximum mark: 6]

A particle P moves along a straight line such that its displacement, in metres, after t seconds, from a fixed point O is given by

$$s(t) = 3e^{-(t+1)} \sin(4t+4), \quad 0 \le t \le 2$$

(a) Sketch the graph of the velocity of P against t on the axes below. [2]



P reaches its maximum speed when $t = a$ seconds.

(b) Find a. [1]

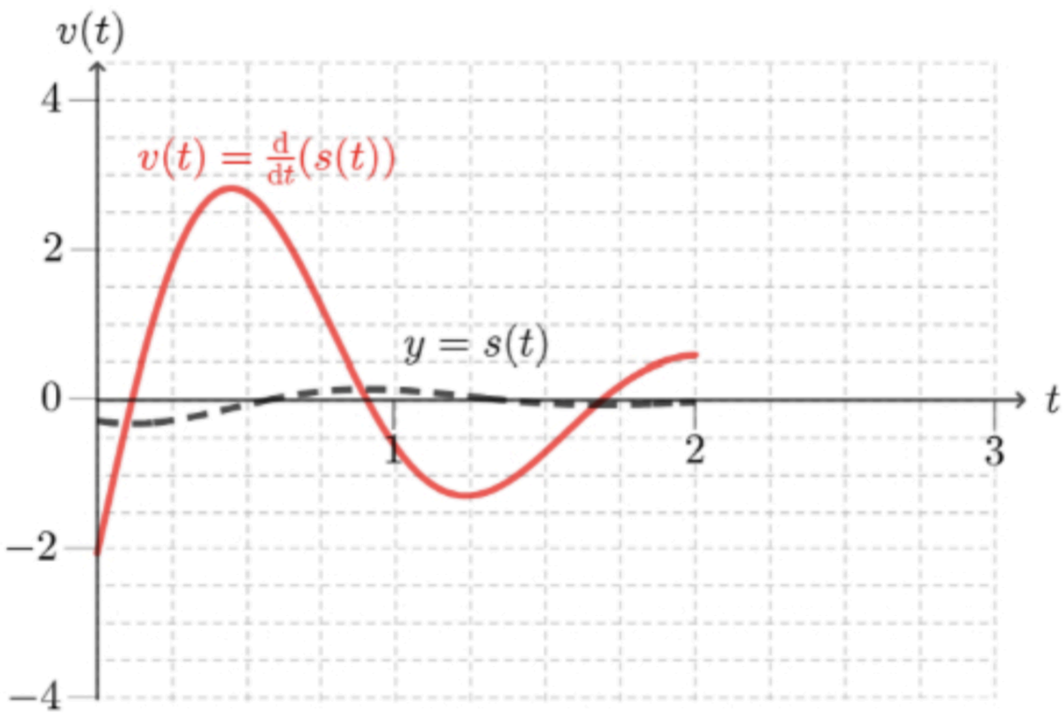
(c) Hence or otherwise, find the distance travelled whilst the acceleration of P is negative. [3]

(a) Recall

$$v(t) = s'(t)$$

$$= \frac{\mathrm{d}s}{\mathrm{d}t}$$

Hence, we can use the calculator to graph both $s(t)$ and $v(t)$.



Correct endpoints

A1

Correct shape

A1

(b) Recall speed$= |v(t)|$.

The maximum speed occurs when $|v(t)|$ is at a maximum.

In this case, the maximum speed is also the maximum positive value of $v(t)$, which we find using a calculator to be $(0.4483..., 2.819...)$.

Hence, the maximum speed is 2.82 m s^{-1} and this occurs at $t = 0.448 \text{ s}$

Therefore $\boxed{a = 0.448}$.

A1

(c) Recall

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

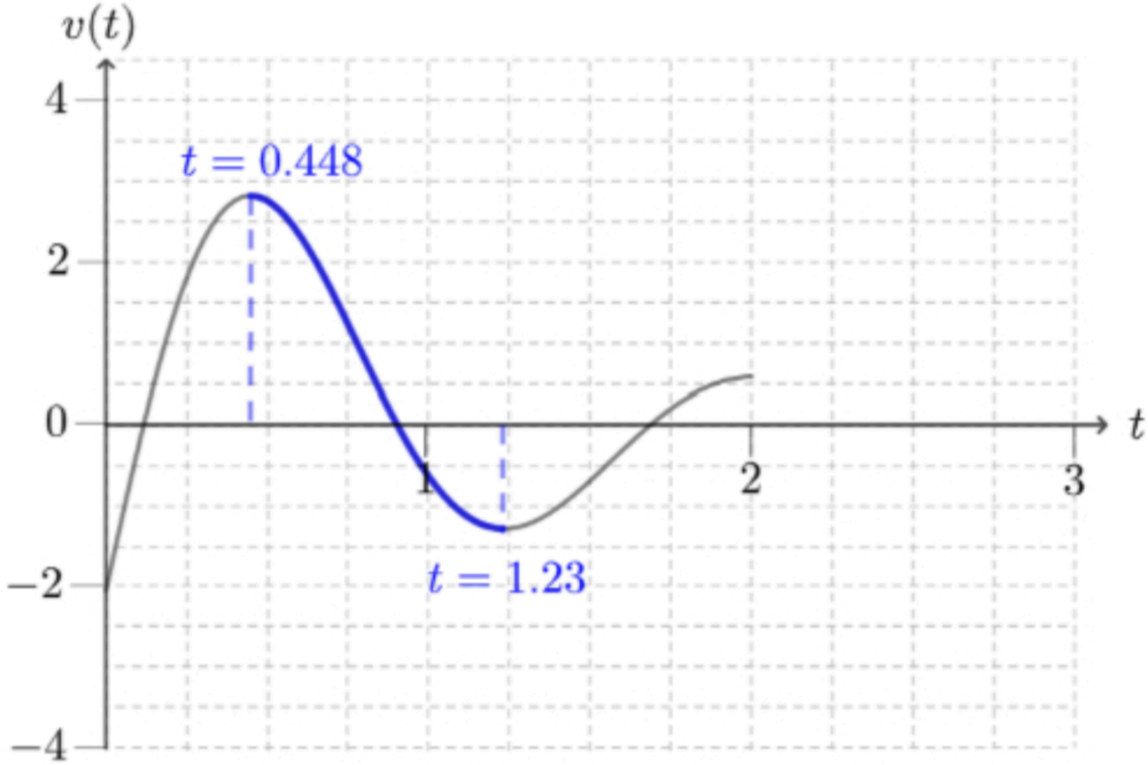
Therefore, the acceleration of P is negative when the gradient of the velocity function, $v(t)$, is negative.

From part (b) we know the local maximum occurs when $t = 0.448$ seconds.

Using technology we find the local minimum point of the velocity function to be $(1.233..., -1.285...)$. Hence, the minimum occurs when $t = 1.23$ seconds.

(A1)

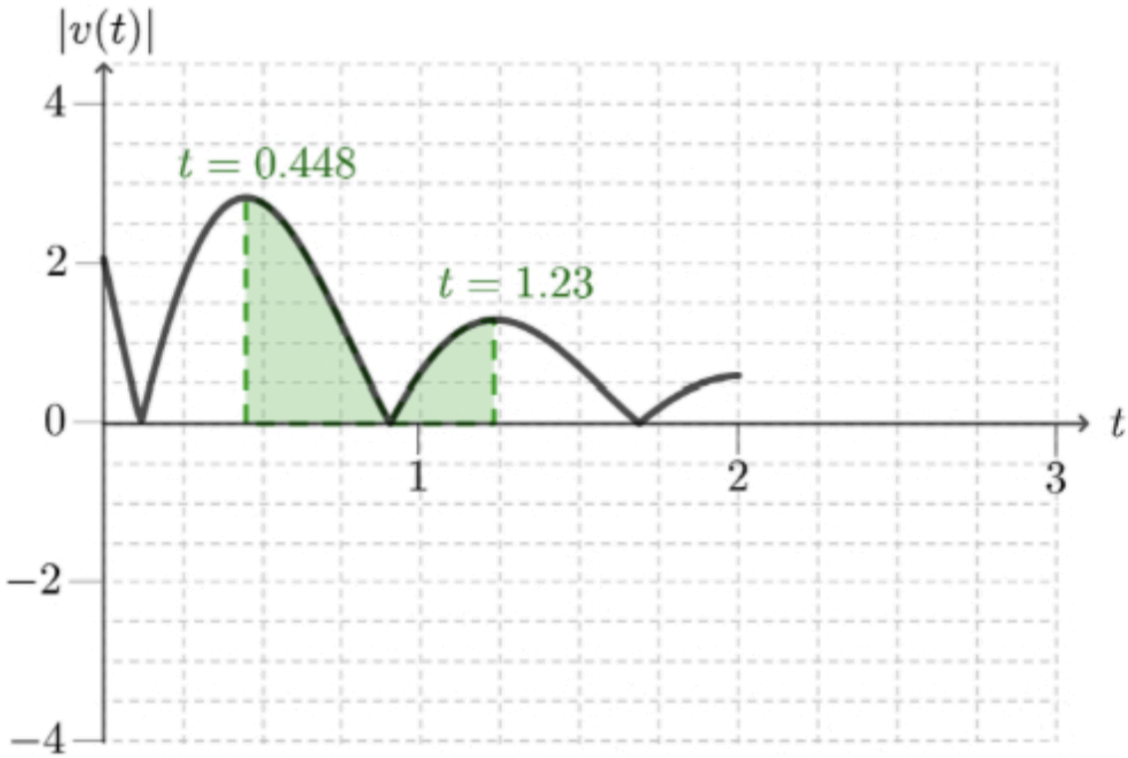
The period that the acceleration of P is negative is shown in blue below



The distance travelled is calculated using the formula

$$\text{distance travelled} = \int_{t_1}^{t_2} |v(t)|\, \mathrm{d}t$$

We can graph this function on the calculator



Using the calculator

$$\begin{aligned} \text{distance travelled} &= \int_{t_1}^{t_2} |v(t)|\, \mathrm{d}t \\ &= \int_{0.4483...}^{1.233...} |v(t)|\, \mathrm{d}t \qquad \textbf{(M1)} \\ &= 1.049... \\ &= \boxed{1.05 \text{ m}} \qquad \textbf{A1} \end{aligned}$$

Question 6

CALCULATOR

Medium

[Maximum mark: 5]

The amount, in milligrams, of a medicinal drug in the body t hours after it is injected is given by

$$D(t) = 240e^{-kt}$$

Where $k > 0$ and $t \geq 0$. Before the injection, it is assumed the amount of drug in the body is zero.

A patient is to be injected with the drug and, for this patient, it is known that it takes 5 hours for the amount of drug remaining in the body to have decreased by 65% of the initial dose.

The patient is regularly checked and is allowed to go home when the amount of drug remaining in the body is 10%, or less, of the initial dose.

The initial dose is given to the patient at $9:00$ am.

Use this model to estimate, to the nearest hour, the earliest time the patient will be allowed to go home.

This model has a fixed value of k. In order to use the model we must find k.

First, we can find the initial amount of the drug injected into the patient, this occurs when $t = 0$

$$D(0) = 240e^{-k(0)}$$

$$D(0) = 240$$

Five hours later, $t = 5$, we are told that $100\% - 65\% = 35\%$ of the initial 240 mg remains.

We can use this to find k.

$$0.35 \times 240 = 240e^{-5k} \qquad \textbf{(M1)}$$

$$0.35 = e^{-5k}$$

$$\ln 0.35 = \ln e^{-5k}$$

Using the log law $\log_a x^m = m \log_a x$, we get

$$\ln 0.35 = -5k \ln e$$

Since $\ln e = 1$, we can simplify the equation and solve for k.

$$\ln 0.35 = -5k$$

$$k = \frac{\ln 0.35}{-5}$$

$$= 0.2099\ldots \qquad \textbf{A1}$$

We can now write out the model with out value of k

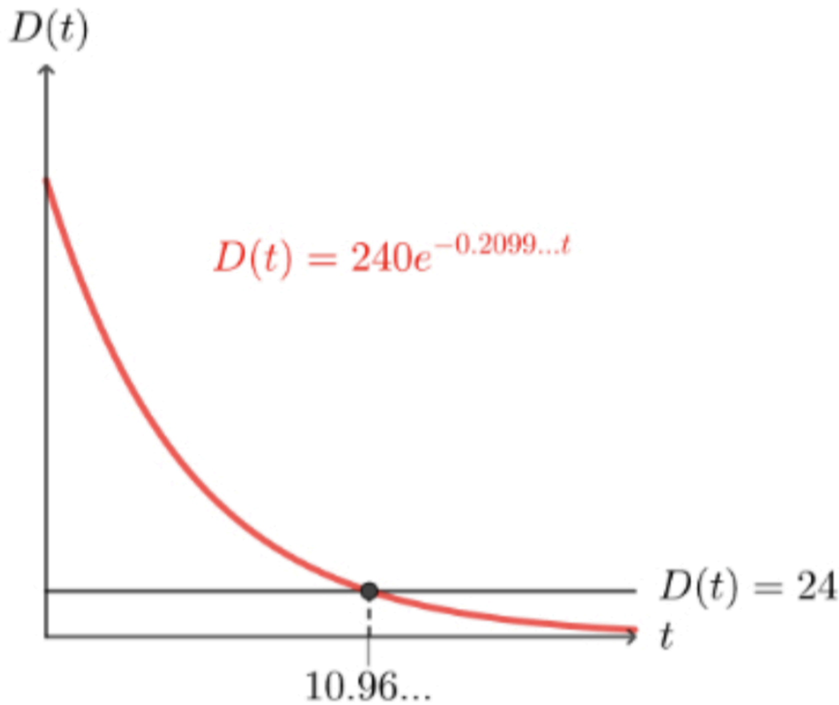
$$D(t) = 240e^{-0.2099\ldots t}$$

We need to find t when the drug remaining in the body is equal to $0.10 \times 240 = 24$ mg.

Hence we need to solve the following equation

$$24 = 240e^{-0.2099\ldots t} \qquad \textbf{(M1)}$$

We can do this graphically by sketching the function $y = D(t)$ and the horizontal line $y = 24$



The $x-$coordinate of the intersection point is the solution.

Although a calculator method is preferred, an algebraic approach is also included in case you attempted it that way

$$0.1 = e^{-0.2099...t}$$

$$\ln 0.1 = -0.2099\ldots t$$

$$t = \frac{\ln 0.1}{-0.2099\ldots}$$

$$t = 10.96\ldots \quad \textbf{A1}$$

$$t = 11 \quad \text{to the nearest hour}$$

Converting the answer to a time we get

$$9{:}00 + 11 \text{ hours} = 20{:}00$$

Therefore the earliest time, to the nearest hour, the patient will be allowed to go home is

$$\boxed{20{:}00 \text{ or } 8{:}00 \text{ pm}} \quad \textbf{A1}$$

Section B

Question 7

CALCULATOR

Medium

[Maximum mark: 13]

A geometric sequence, with common ratio r, has a first term of 81. The sum of the first four terms of the sequence is 195.

(a) (i) Find r.

(ii) Hence, find G_n, the sum of the first n terms of the geometric sequence. [3]

The first three terms of an arithmetic sequence, with a common difference of d, are $\log 96$, $\log 48$ and $\log 24$.

(b) Find d in the form $p \log q$ where $p, q \in \mathbb{Z}$. [2]

A_n is the sum of the first n terms of the arithmetic sequence.

(c) Show that $A_n = \log\left(3^n \times \left(\sqrt{2}\right)^{11n-n^2}\right)$. [5]

(d) Find the maximum value of n such that $|G_n| > |A_n|$. [3]

(a) (i) From the given information we know $u_1 = 81$ and $S_4 = 195$.

Using the formula for the sum of a geometric sequence we can form an equation and then solve for r.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$S_4 = \frac{u_1(1-r^4)}{1-r}$$

$$195 = \frac{81(1-r^4)}{1-r} \qquad \textbf{(M1)}$$

Using a G.D.C. we get

$$r = 0.6666...$$

$$r = \boxed{\frac{2}{3}} \qquad \textbf{A1}$$

(ii) Using the answer from (a) (i) and the formula for the sum of the first n terms of a geometric series we can write the general term for G_n as

$$\begin{aligned} G_n &= \frac{81(1-(\frac{2}{3})^n)}{1-\frac{2}{3}} \\ &= \frac{81(1-(\frac{2}{3})^n)}{\frac{1}{3}} \\ &= 81\left(1-\left(\frac{2}{3}\right)^n\right) \times 3 \\ &= \boxed{243\left(1-\left(\frac{2}{3}\right)^n\right)} \end{aligned} \qquad \textbf{A1}$$

(b) In an arithmetic sequence, there is a common difference between terms.

$$d = u_2 - u_1$$

Substituting the given values for u_1 and u_2 we get

$$d = \log 48 - \log 96 \qquad \textbf{(M1)}$$

$$d = \log\left(\frac{48}{96}\right)$$

$$d = \log\frac{1}{2}$$

$$d = \log 2^{-1}$$

$$d = \boxed{-\log 2} \qquad \textbf{A1}$$

(c) The formula for sum of an arithmetic sequence is

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

Substituting $u_1 = \log 96$ and $d = -\log 2$

$$\begin{aligned} A_n &= \frac{n}{2}(2\log 96 + (n-1)(-\log 2)) \\ &= \frac{n}{2}(\log 96^2 - (n-1)\log 2) \end{aligned}$$

(M1)

Applying logarithm rules we get

(M1)

$$= \frac{n}{2}(\log 96^2 - \log 2^{(n-1)})$$

$$= \frac{n}{2}\log\frac{96^2}{2^{(n-1)}}$$

$$= \log\left(\frac{96^2}{2^{(n-1)}}\right)^{\frac{n}{2}}$$

$$= \log\sqrt{\left(\frac{96^2}{2^{(n-1)}}\right)^n}$$

A1

$$= \log\left(\frac{\sqrt{96^2}}{\sqrt{2^{(n-1)}}}\right)^n$$

$$= \log\left(\frac{96^n}{(\sqrt{2})^{n^2-n}}\right)$$

Looking at the form of the answer we can attempt to rewrite the denominator using by negating its exponent

$$= \log(96^n \times (\sqrt{2})^{n-n^2})$$

Let's rewrite 96 as a product of prime factors

$$= \log((2^5 \times 3)^n \times (\sqrt{2})^{n-n^2}) \qquad \textbf{M1}$$

Then we'll rewrite 2^5 as a base of $\sqrt{2}$

$$= \log(((\sqrt{2})^{10} \times 3)^n \times (\sqrt{2})^{n-n^2})$$

Now we can distribute the power of n and then use exponent laws again to simplify to the required form.

$$= \log((\sqrt{2})^{10n} \times 3^n \times (\sqrt{2})^{n-n^2}) \qquad \textbf{A1}$$

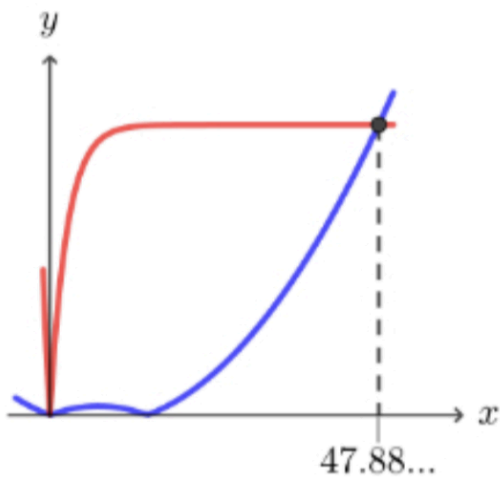
$$= \log(3^n \times (\sqrt{2})^{10n+n-n^2})$$

$$= \boxed{\log(3^n \times (\sqrt{2})^{11n-n^2})} \qquad \text{as required.}$$

(d) Using the functions from (a) (ii) and (c) we can form an inequality which we can solve for n.

$$\left|243\left(1-\left(\frac{2}{3}\right)^n\right)\right| > \left|\log\left(3^n \times (\sqrt{2})^{11n-n^2}\right)\right| \qquad \textbf{(M1)}$$

We can sketch the L.H.S. and the R.H.S. on a G.D.C. to get



Hence when

$$n < 47.88... \qquad \textbf{A1}$$

then $|G_n| > |A_n|$.

As n represents the term value, we therefore round down to the nearest integer value.

Hence $\boxed{n = 47}$.

A1

Question 8

CALCULATOR Hard

[Maximum mark: 18]

Lesta Laboratory conducts experiments on different metals.

A sample of a metal is taken, if the weight falls in a particular range then the sample is used in an experiment. Otherwise it is rejected.

One type of metal is *Alloy X*, for which the samples have a weight that is normally distributed with a mean of 19.6 grams and a standard deviation of 2.1 grams.

Lesta Laboratory will use a sample of *Alloy X* in an experiment if the weight is between 17 g and 23 g.

(a) Find the probability a randomly selected sample of *Alloy X* will be used in an experiment. [2]

Alloy Y is another type of metal used at *Lesta Laboratory*.

The weights, W_Y, of samples of *Alloy Y* are normally distributed with a mean of μ_Y and a standard deviation of σ_Y.

It is known that 3.061% of samples of *Alloy Y* weigh less than 35 g.

(b) Show that $\mu_Y = 35 + 1.872\sigma_Y$ [2]

It is also known that $\mathrm{P}(W_Y > 46) = 0.1714$.

(c) (i) Show that $\mu_Y = 46 - 0.9486\sigma_Y$

(ii) Hence write down the values of μ_Y and σ_Y. [3]

A sample of *Alloy Y* is used in an experiment if the weight of the sample lies within k standard deviations, σ_Y, of its mean, μ_Y. The probability of this occurring is 54.67%.

(d) (i) Find k.

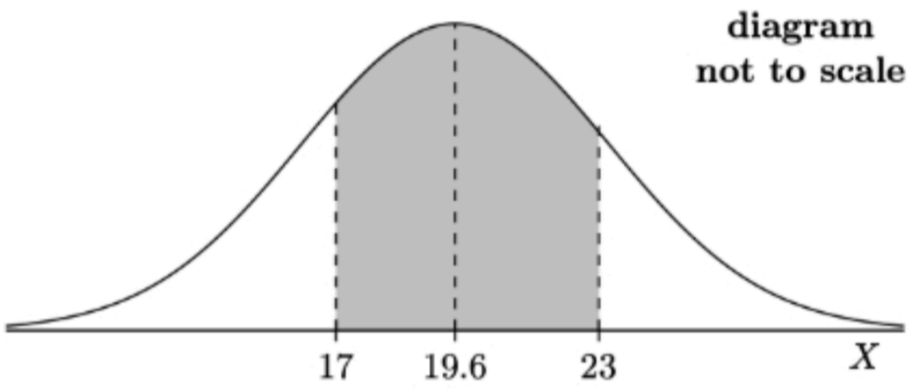
(ii) If *Lesta Laboratory* takes 8 independent samples of *Alloy Y* what is the probability more than 5 are used in an experiment? [6]

Lesta Laboratory is testing some samples of metals for an experiment. They select 2 samples of *Alloy X* and 2 samples of *Alloy Y*.

(e) What is the probability there will be more samples of *Alloy X* than *Alloy Y* that can be used in the experiment? [5]

(a) When working with normal distribution questions it is helpful to sketch the information on a normal distribution curve.

From the given information we have



where $X \sim \mathrm{N}(19.6, 2.1^2)$.

(M1)

The area of the shaded region represents the probability of X being between 17 g and 23 g.

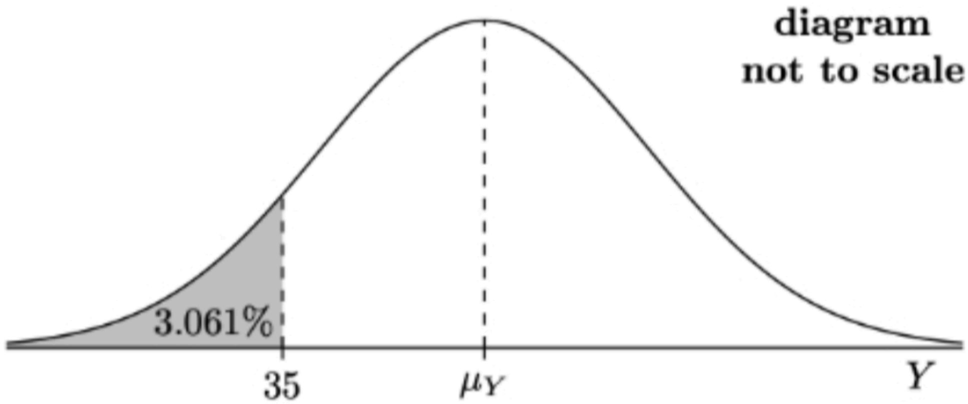
Using the calculator we can find this area

$$P(17 \leq X \leq 23) = \text{normCdf}(17, 23, 19.6, 2.1)$$

$$= 0.8394...$$

$$= \boxed{0.839}$$ **A1**

(b) The sketch below represents the region $P(Y < 35) = 0.03061$

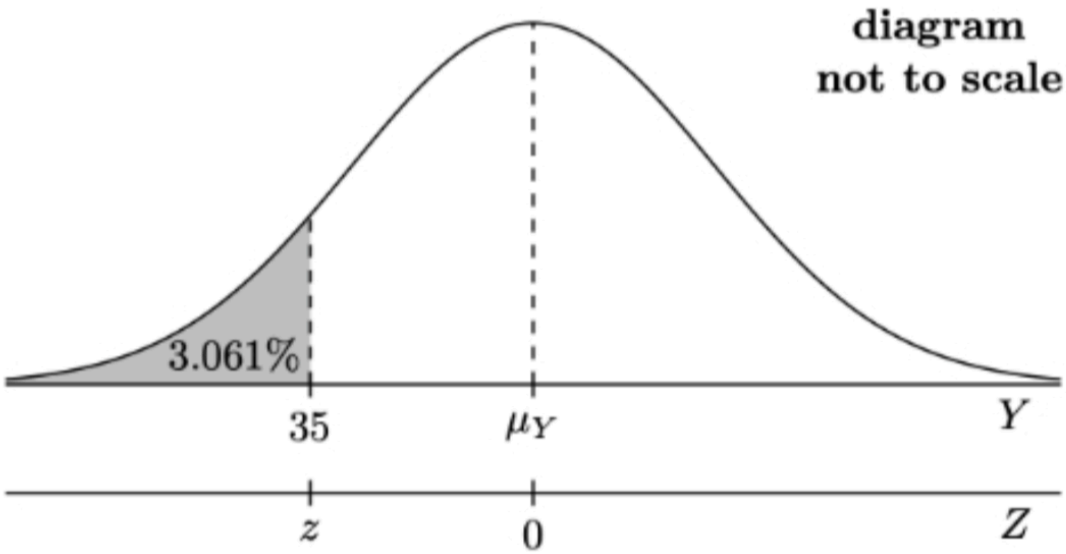


When μ and σ are unknown we will need to standardise the distribution. For this we will use the formula

$$z = \frac{x - \mu}{\sigma}$$

Recall that the C.R.V. Z follows the standard normal distribution such that $Z \sim \mathrm{N}(0, 1^2)$.

In the diagram below, the axis representing the $Z-$distribution is drawn below the distribution for Y. The $Z-$value is aligned with $Y = 35$, which is the boundary of the region with a probability of 0.03601.



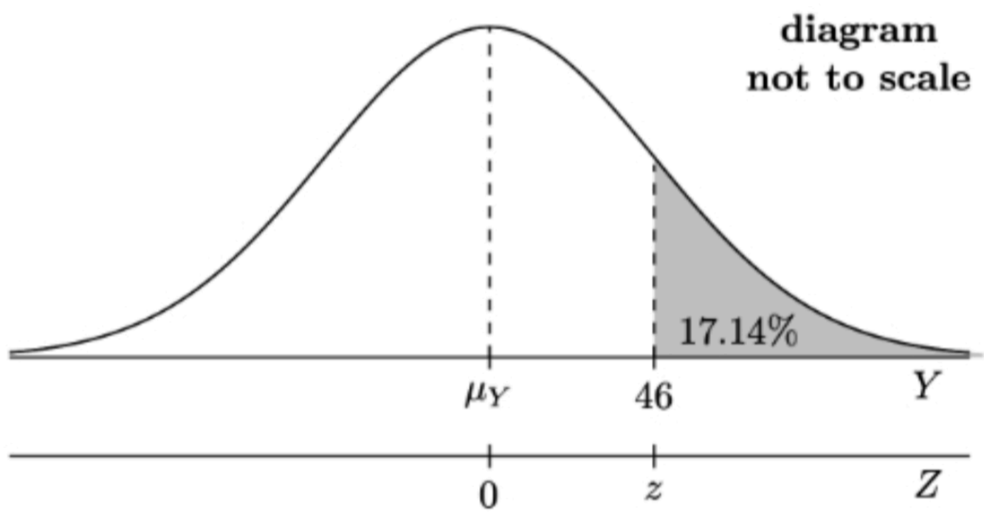
To find the $Z-$value associated with $\mathrm{P}(Z < 0.03061)$ we must use the inverse normal function.

$$z = \mathrm{invNorm}(0.03061, 0, 1)$$
$$= -1.871... \qquad \textbf{(A1)}$$

Substituting z and 35 into the formula

$$-1.8719... = \frac{35 - \mu_Y}{\sigma_Y} \qquad \textbf{M1}$$
$$-1.872\,\sigma_Y = 35 - \mu_Y$$
$$\boxed{\mu_Y = 35 + 1.872\sigma_Y} \qquad \text{...as required.} \qquad \textbf{AG}$$

(c) (i) As in part (b) we calculate the Z-value and use the standardisation formula.



We need to find the associated $Z-$value on the standard normal curve for $P(Z > z) = 0.1714$.

$\text{P}(Z > 0.1714) = \text{P}(Z < 0.8286)$. Hence we can use the calculator to find z.

$$z = \text{invNorm}(0.8286, 0, 1)$$

$$= 0.94864...$$ **A1**

Substituting z and 46 into the formula

$$0.9486 = \frac{46 - \mu_Y}{\sigma_Y}$$

$$0.9486\,\sigma_Y = 46 - \mu_Y$$

$$\mu_Y = 46 - 0.9486\sigma_Y$$ ...as required. **AG**

(ii) Now we have a system of equations that can be solved simultaneously.

$$\mu_Y = 35 + 1.872\sigma_Y$$

$$\mu_Y = 46 - 0.9486\sigma_Y$$

Using the G.D.C.

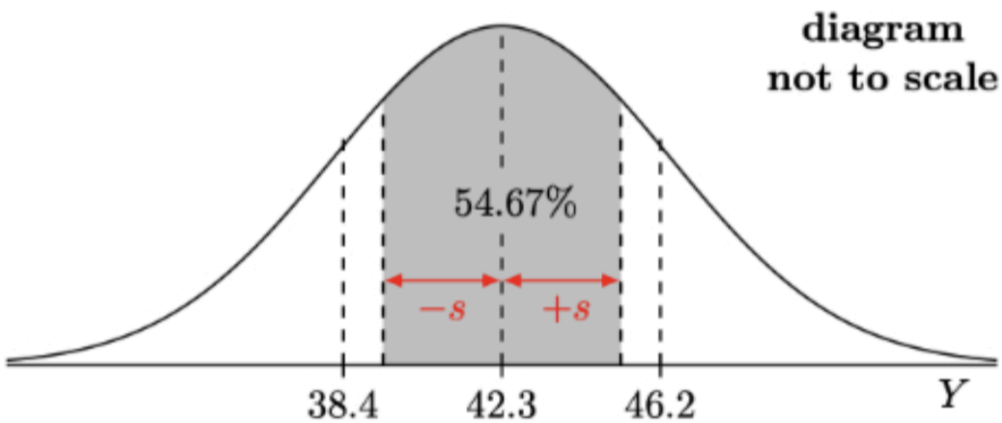
$$\sigma_Y = 3.899... \quad \textbf{A1}$$

$$= \boxed{3.90}$$

$$\mu_Y = 42.30... \quad \textbf{A1}$$

$$= \boxed{42.3}$$

(d) (i) The probability of using a sample of *Alloy Y* is 54.67%. Using μ_Y and σ_Y from part (c)(ii) we can represent this on a normal distribution curve.



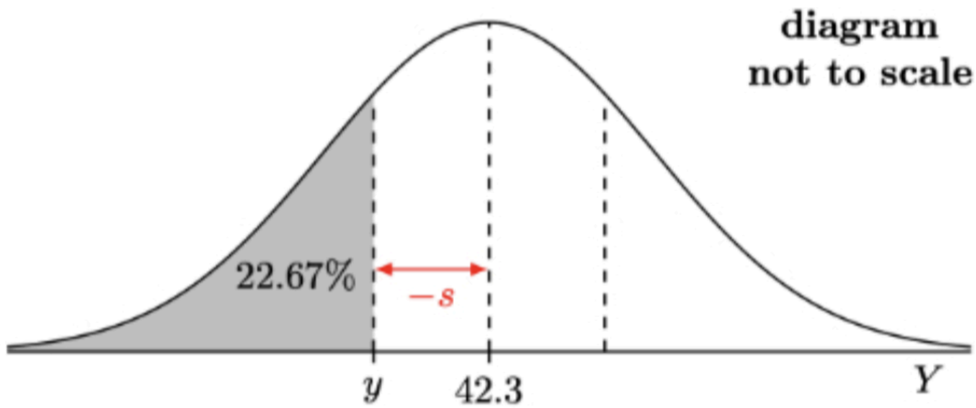
s is the length between the boundaries of the shaded region and the mean.

Notice that the region is centred around $\mu_Y (= 42.3)$. With the lower boundary of the region being $42.3 - s$ and the upper boundary being $42.3 + s$

By subtracting from 1 and dividing by 2 we can find the area of the lower tail of the distribution.

$$1 - 0.5467 = 0.4533$$

$$\frac{0.4533}{2} = 0.22665 \qquad \textbf{(M1)}$$



Now we can use the calculator to find the $Y-$value which is the lower boundary of the shaded region.

$$\begin{aligned} y &= \text{invNorm}(0.22665, 42.30..., 3.899...) \\ &= 39.37... \end{aligned}$$ **(A1)**

We now have the lower boundary of the region, hence

$$\begin{aligned} 42.30... - s &= 39.37... \\ s &= 2.924... \end{aligned}$$ **A1**

k is the number of standard deviations from the mean, hence

$$\begin{aligned} k &= \frac{s}{\sigma_Y} \\ &= \frac{2.924...}{3.899...} \\ &= 0.7499... \\ &= \boxed{0.750} \end{aligned}$$ **A1**

(ii) In this part of the question we can define a new discrete random variable S as the number of samples used in an experiment.

Where $S = \{0, 1..., 8\}$.

Addtionally

- We are given a fixed number of trials $n = 8$.
- he possible outcomes of each trial are the sample is *used* or *not used.* i.e. each trial is a Bernoulli trial.
- The probability of success, $p = 0.5467$, of an individual trial is independent of any other trial - note we can assume this.

Hence the conditions have been fulfilled such that S follows a binomial distribution.

Therefore we can say that

$$S \sim B(n, p)$$
$$S \sim B(8, 0.5467)$$

Where n is the number of trials and p is the probability of success of a single trial.

When we are asked for **more than** 5 selected we can use the G.D.C. to calculate

$$\mathrm{P}(Y > 5) = \mathrm{binomCdf}(8, 0.5467, 6, 8) \qquad \textbf{(M1)}$$

$$= 0.2145...$$

$$= \boxed{0.215} \qquad \textbf{A1}$$

(e) The probability of a usable *Alloy X* sample is 0.8394... from part (a).

The probability of a usable *Alloy Y* sample is 0.5467 from part (d).

We can use more of *Alloy X* than *Alloy Y* in the following ways

2 *Alloy X* and 0 *Alloy Y*

2 *Alloy X* and 1 *Alloy Y*

1 *Alloy X* and 0 *Alloy Y* **(M1)**

Let's calculate the probability of each of these outcomes.

There is only 1 way to choose 2 samples of X (or Y) or 0 samples of X (or Y).

Hence for 2 *Alloy X* and 0 *Alloy Y* we have

$$(0.8394...)^2 \times (1 - 0.5467)^2 = 0.1447... \quad \textbf{(A1)}$$

However there are 2 ways to choose 1 sample of Y. We can either choose the first and reject the second or reject the first and choose the second. Therefore we need to multiply the probability by 2.

Hence for 2 *Alloy X* and 1 *Alloy Y* we have

$$(0.8394)^2 \times 2(1 - 0.5467)(0.5467) = 0.3492... \quad \textbf{(A1)}$$

and similarly for 1 *Alloy X* and 0 *Alloy Y* we have

$$2(0.8394...)(1 - 0.8394) \times (1 - 0.5467)^2 = 0.05540... \quad \textbf{(A1)}$$

Adding them all together we get

$$\begin{aligned} P(\text{More samples of } \textit{Alloy X} \text{ than } \textit{Alloy Y}) &= 0.1447... + 0.3492... + 0.05540... \\ &= 0.5494... \\ &= \boxed{0.549} \end{aligned}$$

A1

Question 9

CALCULATOR

Hard

[Maximum mark: 17]

Juanita wants to borrow some money to buy an apartment.

She finds an offer allowing her to borrow $480,000 over 10 years with an interest rate of $r\%$ P.A. compounded monthly. She repays the loan with a fixed amount p every month.

Juanita takes the loan out at the beginning of the month. At the end of the month, the interest is added **and then** she makes the monthly payment of p.

This continues until after 10 complete years, she has repaid the loan in its entirety.

Juanita wants to analyse three different scenarios in which she could repay the loan.

(a) In the first scenario her monthly payment is $p = \$5\,000$.

If $k = 1 + \frac{r}{1200}$

(i) Write down the number of payments that will be made over the entire 10 year term of the loan.

(ii) Show that

$$96k^{120} = \frac{k^{120} - 1}{k - 1}$$

(iii) Hence, or otherwise, find r. [6]

(b) In the second scenario Juanita uses the same values for p and r as part (a). She makes the monthly payments of p for 7 years and 4 months.

She then makes a final payment to clear the remaining balance of the loan.

(i) Find the number of payments she makes **before** the final payment.

(ii) Hence, find the final payment required to clear the remaining balance to 4 significant figures. [3]

(c) In the third scenario Juanita pays p per month for 5 complete years and then she increases her monthly loan repayment to $2p$ for the remaining 5 years.

Find the value of p, to the nearest dollar, for the third scenario. [8]

(a) (i) There are 12 payments per year for 10 years

$$= 10 \times 12$$

$$= 120$$ **A1**

(ii) Note: Multiplying by k represents an increase of $r\%$.

We can build up an expression which represents the remaining value, FV, of the loan. After the first month, this remaining value is increased by $r\%$ and then the payment \$5 000 is subtracted

$$\begin{aligned} FV &= 480\,000 \times k - 5\,000 \\ &= 480\,000k - 5\,000 \end{aligned}$$

We then continue into the second month, this process is repeated. The remaining value is increased by $r\%$ and the payment \$5 000 is subtracted

$$\begin{aligned} FV &= (480\,000k - 5\,000) \times k - 5\,000 \\ &= 480\,000k^2 - 5\,000k - 5\,000 \end{aligned}$$

Similarly for the third month

(M1)

$$\begin{aligned} FV &= (480\,000k^2 - 5\,000k - 5\,000) \times k - 5\,000 \\ &= 480\,000k^3 - 5\,000k^2 - 5\,000k - 5\,000 \end{aligned}$$

This continues for all 120 payments and can be shown using the following series

$$FV = 480\,000k^{120} - 5\,000k^{119} - 5\,000k^{118} - 5\,000k^{117} - \dots - 5\,000k - 5\,000$$

(A1)

As the loan has been repaid in full we can replace FV with 0.

(M1)

$$0 = 480\,000k^{120} - 5\,000k^{119} - 5\,000k^{118} - 5\,000k^{117} - \dots - 5\,000k - 5\,000$$

We can then rearrange the equation above to get

$$480\,000k^{120} = 5\,000k^{119} + 5\,000k^{118} + 5\,000k^{117} + \dots + 5\,000k + 5\,000$$

Recognise that the R.H.S. of the above is a geometric series with a first term of 5 000 and a common ratio of k. Hence we can use the sum of a geometric sequence formula to write

$$480\,000k^{120} = 5\,000\left(\frac{k^{120} - 1}{k - 1}\right)$$ **A1**

$$\boxed{96k^{120} = \frac{k^{120} - 1}{k - 1}}$$ ... as required.

(iii) Using a G.D.C. we can solve the equation found above.

$$96k^{120} = \frac{k^{120} - 1}{k - 1}$$

$$k = 1.0038412\dots$$ [by using G.D.C.]

Hence

$$1.0038412\dots = 1 + \frac{r}{1200}$$

$$r = 4.609419\dots$$ **A1**

$$= \boxed{4.61\%}$$ [by using G.D.C.]

(b) (i) She makes 7 full years of payments plus an additional 4 months, which gives

$$= 7 \times 12 + 4$$
$$= 88$$

She makes 88 payments before the final payment.

A1

(ii) You could use the Financial Solver on the G.D.C. with the following values

N	I	PV	Pmt	FV	PpY	CpY
88	4.609...%	\$480 000	−\$5 000		12	12

(A1)

Solving for FV gives a remaining value of 150 286.301....

Hence the final payment is \$150 300.

A1

(c) Juanita pays p per month for 60 months, she then pays $2p$ per month for the remaining 60 months.

(A1)

Lets consider the first 60 payments of p using a similar method to part (a) where $k = 1 + \frac{4.61}{1200} = 1.0038...$

$$FV = ((480\,000 \times k - p) \times k - p)...$$

$$FV = 480\,000k^{60} - pk^{59} - pk^{58}... - pk - p \qquad \textbf{A1}$$

Now let's consider the next 60 payments of $2p$ which will clear the remaining balance, which means the future value is \$0

(M1)

$$0 = ((480\,000k^{60} - pk^{59} - pk^{58}... - p) \times k - 2p) \times k - 2p)...$$

$$0 = 480\,000k^{120} - pk^{119}... - pk^{60} - 2pk^{59} - 2pk^{58} - ... - 2pk - 2p$$

$$480\,000k^{120} = pk^{119}... + pk^{60} + 2pk^{59} + 2pk^{58} + ... + 2pk + 2p \qquad \textbf{A1}$$

This equation has a single unknown (p). However in its current form we cannot enter it into the calculator as there are too many terms!

We need to find a way to rewrite the sum in a simpler form.

We can do this by summing the RHS of the equation in two parts. We will let the first 60 terms be part A and the last 60 terms be part B

To find the sum of part A we can use the formula directly, pasrt A has parameters $u_1 = 2p$, $r = 1.0038...$ and $n = 60$, therefore

$$S_A = 2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1} \qquad \textbf{A1}$$

Let's reconsider the original equation and replace part A with the simpler form found above

$$480\,000k^{120} = \underbrace{pk^{119}... + pk^{60}}_{\text{B}} + \underbrace{2pk^{59} + 2pk^{58} + ... + 2pk + 2p}_{\text{A}}$$

$$480\,000k^{120} = \underbrace{pk^{119}... + pk^{60}}_{\text{B}} + \underbrace{2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{A}}$$

Part B is more complicated. We can't simply sum 120 terms as we only want to sum the last 60 terms of the sequence.

Lets find the sum of the full 120 terms and then subtract the sum of the first 60 terms.

Part B has parameters $u_1 = p$, $r = 1.0038...$ and $n = 120$, therefore

$$S_B = p\frac{(1.0038...)^{120} - 1}{(1.0038...) - 1} - p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}$$ **M1A1**

Let's now rewrite the original equation and replace S_B with the formula above

$$480\,000(1.0038...)^{120} = \underbrace{p\frac{(1.0038...)^{120} - 1}{(1.0038...) - 1} - p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{B}} + \underbrace{2p\frac{(1.0038...)^{60} - 1}{(1.0038...) - 1}}_{\text{A}}$$

We can now solve this on the G.D.C.

$$p = 3465.61...$$

$$p = \boxed{\$3466}$$ **A1**