IB Mathematics AI HL - Prediction Exams May 2025 - Paper 2

Paper 2 | 7 questions | 120 mins | 110 marks

Question 1

CALCULATOR | Easy

[Maximum mark: 17]

Professor Smith teaches a calculus course to a group of students in a school. He has noticed that the length of the lectures he gives follows a normal distribution with mean m minutes.

There is a 20% chance that any lecture lasts at most 48.6 minutes.

(a) Sketch a diagram to represent this probability. [2]

There is also a 20% chance that any lecture lasts longer than 55.4 minutes.

(b) Show that $m = 52$. [2]

The standard deviation of the number of minutes a lecture lasts is 4 minutes.

(c) Find the probability that a lecture

(i) lasts between 45 and 51 minutes;

(ii) lasts more than 55 minutes. [4]

There is a 70% chance that a lecture lasts less than x minutes.

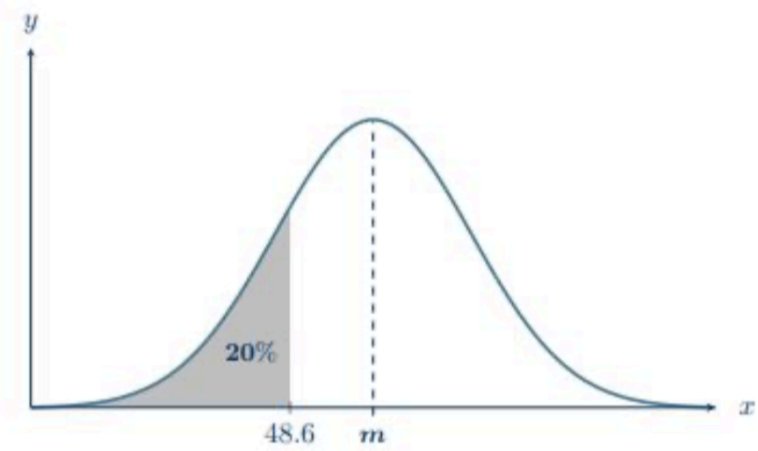
(d) Find the value of x. [2]

Professor Smith delivers one lecture per day to this group of students, Monday to Friday. It can be assumed that the length of each lecture is independent.

(e) Find the probability that the lecture on Monday lasts between 45 and 51 minutes but the lecture on Tuesday lasts more than 55 minutes. [3]

(f) Calculate the probability that during a given week, at least 3 lectures last more than 55 minutes. [4]

(a) We have



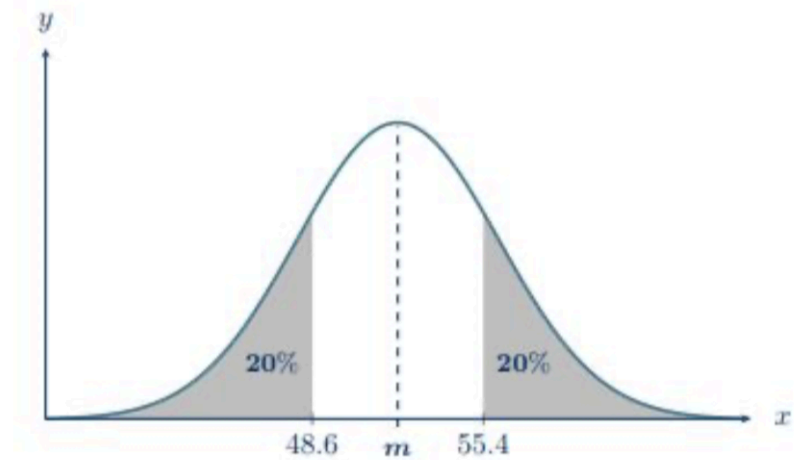
Normal distribution curve, 48.6 labelled to left of mean

A1

Shading to left of mean

A1

(b) If we consider the symmetry of the normal distribution curve, we can see that the values 48.6 and 55.4 must be equidistant from the mean:



Hence, the mean of the distribution, m, will be equal to the mean of 48.6 and 55.4.

(R1)

We have

$$m = \frac{48.6 + 55.4}{2} \quad \textbf{M1}$$

$$\boxed{m = 52} \quad \textbf{AG}$$

(c) (i) Let the continuous random variable X represent the length of a randomly selected lecture.

We have $X \sim \mathrm{N}\left(52, 4^2\right)$. Hence we find

$$\mathrm{P}(45 < X < 51) = \text{normalCdf}(45, 51, 52, 4) \qquad \textbf{(M1)}$$

$$= 0.3612\ldots$$

$$\boxed{= 0.361\,(3\text{ s.f.})} \qquad [\text{using G.D.C.}] \qquad \textbf{A1}$$

(ii) Again using the G.D.C., we have

$$\mathrm{P}(X > 55) = \text{normalCdf}(55, \infty, 52, 4) \qquad \textbf{(M1)}$$

$$= 0.2266\ldots$$

$$\boxed{= 0.227\,(3\text{ s.f.})} \qquad \textbf{A1}$$

(d) Using the G.D.C., we get

$$\mathrm{P}(X < x) = 0.7$$

$$x = \text{InvNorm}(0.7, 52, 4) \qquad \textbf{(M1)}$$

$$= 54.09\ldots$$

$$\boxed{= 54.1\,(3\text{ s.f.})} \qquad \textbf{A1}$$

(e) Taking the results in part (c) (i) and (ii), we can use the formula for independent events. We have

$$\mathrm{P}(45 < \mathrm{M} < 51 \cap \mathrm{T} > 55) = \mathrm{P}(45 < \mathrm{M} < 51) \times \mathrm{P}(\mathrm{T} > 55) \qquad \textbf{(M1)}$$

$$= 0.3612\ldots \times 0.2266\ldots \qquad \textbf{(M1)}$$

$$= 0.08186\ldots$$

$$\boxed{= 0.0819\,(3\text{ s.f.})} \qquad \textbf{A1}$$

(f) Let L represent the event that a randomly selected lecture during a given week lasts more than 55 minutes. We can say that L follows a binomial distribution, because

- We have a fixed number of trials ($n = 5$ lectures per week)
- Each trial is independent (given information)
- Each trial has two outcomes (a lecture must last either more than 55 minutes, or less than or equal to 55 minutes)
- We have a constant probability (from (c) (ii), $p = 0.2266...$)

The criteria for a binomial distribution are met, therefore $L \sim \mathrm{B}(5, 0.2266...)$

(R1A1)

We require $\mathrm{P}(L \geq 3)$. We have

$$\mathrm{P}(L \geq 3) = \mathrm{binomCdf}(5, 0.2266..., 3, 5) \qquad \textbf{(M1)}$$

$$= 0.08041...$$

$$= 0.0804 \text{ (3 s.f.)} \qquad \textbf{A1}$$

Question 2

CALCULATOR   Medium

[Maximum mark: 16]

At the start 2001, a team of zoologists introduced a new species of rabbit onto a large island. 1000 rabbits were initially introduced and there were 1728 rabbits at the start of 2004.

The size of the population of the species, N, t years after the start of 2001, can be modelled by the following function

$$N(t) = p \times q^t, \qquad t \geq 0$$

(a) (i) Show that the value of p is 1000.

(ii) Verify that $q = 1.2$. [2]

(b) State the annual growth rate of the population as a percentage. [1]

(c) Use the model to predict the population size at the start of 2011. [2]

In 2012, the team observed that the growth rate of the rabbit species was changing and that the overall population was now in decline. Upon investigation, they found that the rabbits had became the prey of a new type of snake on the island. Zoologists adjusted their model to the following new function, where t is still the number of years after the start of 2001.

$$N(t) = 1380 \times (0.87)^{(t-12)} \qquad t \geq 12$$

(d) Use this model to find the size of the population at the start of 2019. [2]

(e) Find the year in which the population of the species will first drop below 200. [3]

(f) In the period from 2013 onwards, find the number of complete years in which the size of the population of the species was greater than or equal to 800. [3]

At the start of 2020, the remaining population of the species was transferred to another island along with 300 more rabbits. The team of zoologists now expect the population growth will recover and will increase at a rate of 10% per year.

(g) Estimate the size of the population at the start of 2030. [3]

(a) (i) We have been given that $N(0) = 1000$. Therefore

$$N(0) = p \times q^0 \quad \textbf{A1}$$
$$1000 = p$$

We have shown that $p = 1000$.

AG

(ii) Now we know that $N(t) = 1000 \times q^t$ and we have been given that $N(3) = 1728$. We can use the value $q = 1.2$ to verify that $N(3)$ gives the correct result.

$$N(3) = 1000 \times (1.2)^3 \quad \textbf{A1}$$
$$= 1728$$

Using the value $q = 1.2$, we have shown that the model predicts 1728 rabbits at the start of 2004. This verifies that $q = 1.2$.

AG

(b) We have shown that the model is $N(t) = 1000(1.2)^t$.

This tells us that each year, we multiply the existing population by 1.2 to generate the next population value.

Therefore the annual growth rate is 20%

A1

(c) After ten years the population size is

$$N(10) = 1000 \times 1.2^{10} \quad \textbf{(M1)}$$
$$= 6191.7...$$
$$= 6190 \text{ (3 s.f.)} \quad \textbf{A1}$$

(d) We have

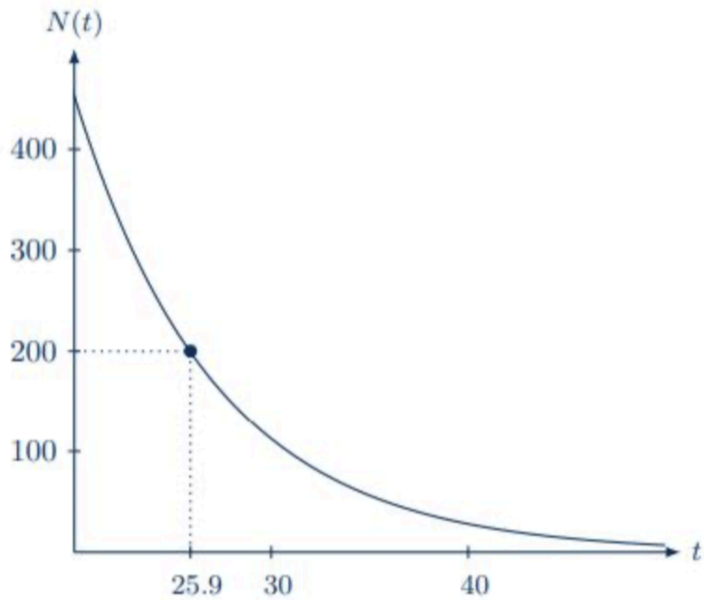
$$N(t) = 1380 \times (0.87)^{(t-12)}$$

The start of 2019 will be when $t = 18$. We have

$$N(18) = 1380 \times (0.87)^{(18-12)} \quad \textbf{(M1)}$$
$$= 598.40...$$
$$= 598 \text{ (3 s.f.)} \quad \textbf{A1}$$

(e) The graph of $N(t) = 1380 \times (0.87)^{(t-12)}$ is shown below. We can sketch this on the G.D.C. and use it to determine the exact value of t when $N = 200$, and interpret this value in context.

(M1)



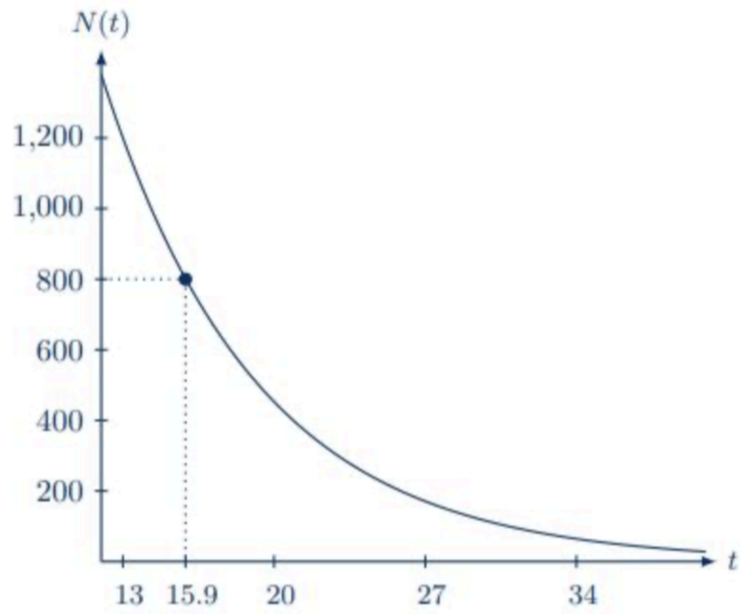
From the graph we have $N(t) < 200$ when $t > 25.9$.

(M1)

The population will first drop below 200 during the 25th year after the start of 2001, which is towards the end of the year 2026.

A1

(f) The graph of $N(t) = 1380 \times (0.87)^{(t-12)}$ is shown below



From the graph we have $N(t) \geq 800$ when $t \leq 15.9$

(M1)

This means we have a population of at least 800 for the full duration of 2013, 2014 and 2015. The value $t = 15.9$ occurs during the year 2016.

(M1)

The population size is greater than or equal to 800 for three complete years (2013 to 2015).

A1

(g) At the start of 2020, $t = 19$. We have

$$N(19) = 1380 \times (0.87)^{(19-12)}$$
$$= 520.61... \quad \textbf{(A1)}$$

We can round this to the nearest whole number, as we require a whole number of rabbits to be transferred to the new island.

The starting population on the new island will be $521 + 300 = 821$

With the predicted growth rate of 10%, the model for population growth on the new island will be $N(t) = 821 \times (1.1)^t$, where t will be time since the start of 2020, therefore we will use $t = 10$.

$$\text{Population at the start of 2030} = 821 \times (1.1)^{10} \quad \textbf{(M1)}$$
$$= 2129.4...$$
$$= 2130 \text{ (3 s.f.)} \quad \textbf{A1}$$

Question 3

CALCULATOR Medium

[Maximum mark: 16]

The cost to manufacture an electronic micro-component at a company can be modelled by the cost function

$$C(x) = x^3 - 3x^2 + 4x$$

where x is in hundreds of micro-components, and $C(x)$ is in hundreds of dollars.

(a) Find $C'(x)$. [2]

The marginal cost of production is the cost of producing one additional unit. This can be approximated by the gradient of the cost function.

(b) Find the marginal cost when 200 micro-components are produced and interpret its meaning in this context. [3]

The revenue from selling the micro-components is given by the function

$$R(x) = 0.6x^3 + x^2 + 10x - 2$$

where x is in hundreds of micro-components and $R(x)$ is in hundreds of dollars.

(c) Given that Profit = Revenue − Cost, determine a function for the profit, $P(x)$, in hundreds of dollars from selling x hundreds of micro-components. [2]

(d) Find $P'(x)$. [2]

(e) Determine the intervals where $P(x)$ is increasing and decreasing. [4]

The derivative $P'(x)$ gives the marginal profit. The production will reach its optimal level when the marginal profit is zero and $P(x)$ is positive.

(f) Find the optimal production level and the expected profit at this level. [3]

(a) Differentiating the cost function, we get

$$C(x) = x^3 - 3x^2 + 4x$$

$$C'(x) = 3x^2 - 3(2)x + 4 \qquad \textbf{(M1)}$$

$$= 3x^2 - 6x + 4 \qquad \textbf{A1}$$

(b) Since x represents hundreds of micro-components, we substitute $x = 2$ into $C'(x)$ and evaluate:

$$C'(2) = 3(2)^2 - 6(2) + 4 \qquad \textbf{(M1)}$$

$$= \$4 \text{ [hundreds of dollars]}$$

$$= \$400 \qquad \textbf{A1}$$

Hence the marginal cost when 200 micro-components are produced is $400.

For every additional 100 micro-components produced, the cost increases by $400.

A1

(c) Given that Profit = Revenue − Cost, we get

$$P(x) = R(x) - C(x)$$

$$= 0.6x^3 + x^2 + 10x - 2 - (x^3 - 3x^2 + 4x) \qquad \textbf{(M1)}$$

$$= -0.4x^3 + 4x^2 + 6x - 2 \qquad \textbf{A1}$$

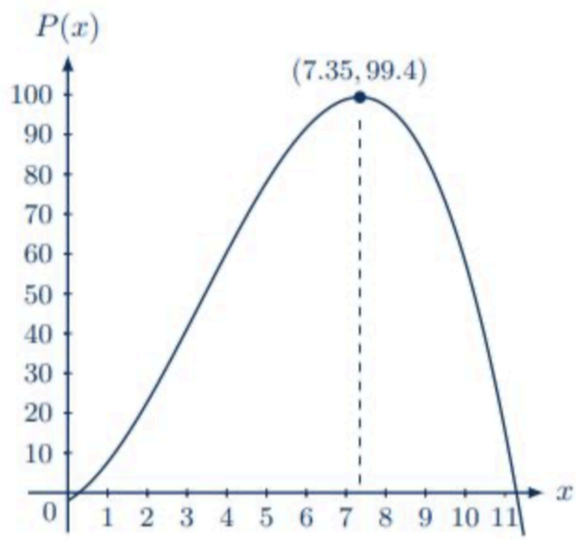
(d) Differentiating $P(x)$, we get

$$P'(x) = -0.4(3)x^2 + 4(2)x + 6(1) \quad \textbf{(M1)}$$

$$= -1.2x^2 + 8x + 6 \quad \textbf{A1}$$

(e) The graph below shows the curve of $P(x)$ with the coordinates of the turning point shown. Note that x (number of micro-components) is restricted to be non-negative.

(M2)



We can see that $P(x)$ is increasing to the left of the turning point (the gradient is positive) and decreasing to the right of the turning point (the gradient is negative).

Hence, $P(x)$ is increasing for $0 \leq x < 7.35$ and decreasing for $x > 7.35$.

A2

(f) From part (e) we see that the marginal profit is zero at $x = 7.35$ (i.e. production of 735 micro-components). We can therefore calculate the expected profit at this production level

$$P(7.35) = -0.4(7.35)^3 + 4(7.35)^2 + 6(7.35) - 2 \quad \textbf{(M1)}$$

$$\approx 99.36 \text{ hundreds of dollars}$$

$$\approx \$9\,936$$

Thus, the optimal production level is 735 micro-components and the expected profit at this level is \$9 936.

A2

Question 4

CALCULATOR

Medium

[Maximum mark: 14]

A submarine is moving in a straight line such that $x = 5 + 12t$ and $y = -10 - 18t$, where t is the travel time, in minutes, after 3:40 pm. x represents displacement due east, and y represents displacement due north, relative to an origin O. Distances are measured in metres.

(a) Find the coordinates of the submarine's position at 4:25 pm. [2]

(b) (i) Write down the velocity vector of the submarine.

(ii) Hence find the speed of the submarine. [3]

An iceberg is located at $P(1500, -1400)$.

(c) Find the distance from the submarine to the iceberg at 4:55 pm. [3]

(d) Find the time, to the nearest minute, when the submarine is closest to the iceberg. [3]

(e) If the submarine changed course at the time found in part (d), to go directly underneath the iceberg, determine the time at which it would be directly underneath. [3]

(a) When $t = 45$, we have

$$
\begin{aligned}
x &= 5 + 12(45) \\
&= 545 \\
y &= -10 - 18(45) \\
&= -820 \qquad \textbf{(M1)}
\end{aligned}
$$

At 4:25 pm, the coordinates of the submarine's position are $A(545, -820)$

A1

(b) (i) We could write a vector equation for the path of the submarine. This would represent the displacement of the submarine from the origin O at any time t. We have

$$\begin{aligned} r &= \begin{pmatrix} 5 + 12t \\ -10 - 18t \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -10 \end{pmatrix} + t \begin{pmatrix} 12 \\ -18 \end{pmatrix} \end{aligned}$$

The velocity vector of the submarine is $\boxed{\mathbf{v} = \begin{pmatrix} 12 \\ -18 \end{pmatrix}}$

A1

(ii) Using the magnitude of a vector formula, we get

$$\begin{aligned} \text{speed} &= |\mathbf{v}| \\ &= \sqrt{12^2 + (-18)^2} && \textbf{(M1)} \\ &= 21.63... \\ &\boxed{= 21.6\ \text{m}\,\text{min}^{-1}\ (3\ \text{s.f.})} && \textbf{A1} \end{aligned}$$

(c) Evaluating $x = 5 + 12t$ for $t = 75$, we obtain

$$\begin{aligned} x &= 5 + 12(75) \\ &= 905 \end{aligned}$$

Evaluating $y = -10 - 18t$ for $t = 75$, we obtain

$$\begin{aligned} y &= -10 - 18(75) \\ &= -1360 \end{aligned}$$

Hence the position of the submarine at 4:55 pm is Q(905, −1 360).

(A1)

Using the distance between two points formula, we find

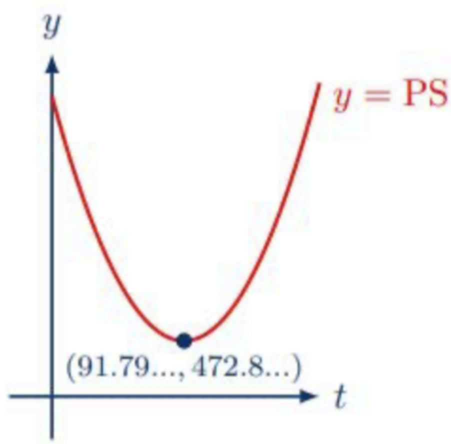
$$\begin{aligned} \text{PQ} &= \sqrt{(1500 - 905)^2 + (-1400 + 1360)^2} && \textbf{(M1)} \\ &= 596.34... \\ &\boxed{= 596\ \text{m}\ (3\ \text{s.f.})} && \textbf{A1} \end{aligned}$$

(d) Similar to part (c), we can note that the position of the submarine at any time t is $S(5+12t, -10-18t)$. We can use the distance formula to obtain an expression for the distance between the submarine and the iceberg at $P(1500, -1400)$. We have

$$\mathrm{PS} = \sqrt{(1500-(5+12t))^2 + (-1400-(-10-18t))^2} \qquad \textbf{(M1)}$$

We could now sketch the graph of $y = \mathrm{PS}$ on the G.D.C. and find the minimum point.

(M1)



In context, The closest the submarine comes to the iceberg is 473 metres, and this occurs 92 minutes after 3:40 pm.

The submarine is closest to the iceberg at 5:12 pm.

A1

(e) We have from part (b) (ii) that the speed of the submarine is 21.63... metres per minute, and from part (d) we have the distance of 472.8... metres. We can use the formula relating speed, distance and time. We have

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{speed}} && \textbf{(M1)} \\ &= \frac{472.8...}{21.63...} \\ &= 21.85... && \textbf{(A1)} \end{aligned}$$

The submarine will pass underneath the iceberg approximately 22 minutes after 5:12 pm (from part (d)).

The submarine will pass underneath the iceberg at approximately 5:34 pm.

A1

Question 5

CALCULATOR   Hard

[Maximum mark: 19]

A proposed differential equation modelling the displacement of a person descending a water slide is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 10 - 0.8\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

where x is the displacement of the person in metres at time t seconds.

It can be assumed that initially, $x = \frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

(a) By using the substitution $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, find an expression for $\frac{\mathrm{d}v}{\mathrm{d}t}$ in terms of v. [2]

(b) Hence use Euler's method for coupled systems, with a step size of 0.5, to approximate a person's velocity after 5 seconds. [3]

(c) Solve the differential equation formed in part (a) to find an expression for v in terms of t. [6]

The $\lim_{t\to\infty} v(t)$ is known as the terminal velocity.

(d) State the value of the terminal velocity. [1]

(e) Determine the time at which $v(t)$ reaches 96% of the terminal velocity. Give your answer to the nearest second. [3]

(f) Calculate the percentage error in the approximation of the velocity found by Euler's method at this time. [2]

(g) Explain why Euler's method has overestimated the velocity at this time. [2]

(a) If $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, then $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

(M1)

Substituting these expressions into our original differential equation, we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.8v$$ **A1**

(b) In order to use euler's method for coupled systems, we need to determine formulae that will generate the next of values, x_{n+1}, v_{n+1} and t_{n+1}, based on the values of x_n, v_n and t_n.

We also have been given $h = 0.5$, $x_0 = 0$, $v_0 = 0$ and $t_0 = 0$.

Our formulae will be

$$x_{n+1} = x_n + h \times f_1(x_n, v_n, t_n)$$

$$v_{n+1} = v_n + h \times f_2(x_n, v_n, t_n)$$

$$t_{n+1} = t_n + h$$

Where $f_1(x_n, v_n, t_n) = \dfrac{\mathrm{d}x_n}{\mathrm{d}t_n}$ and $f_2(x_n, v_n, t_n) = \dfrac{\mathrm{d}v_n}{\mathrm{d}t_n}$

Hence we have

$$x_{n+1} = x_n + 0.5v_n$$

$$v_{n+1} = v_n + 0.5(10 - 0.8v_n)$$

$$t_{n+1} = t_n + 0.5$$

(A1)

Using the G.D.C., we can generate 10 steps of Euler's method, which will take us to $t = 5$, the time at which we need to approximate the velocity.

(M1)

The following table shows these steps, the final step is highlighted in red.

n	t_n	x_n	v_n
0	0	0	0
1	0.5	0	5
2	1	2.5	8
3	1.5	6.5	9.8
4	2	11.4	10.88
5	2.5	16.84	11.52...
6	3	22.60...	11.91...
7	3.5	28.56...	12.15...
8	4	34.63...	12.29...
9	4.5	40.78...	12.37...
10	5	46.96...	12.42...

Therefore after 5 seconds, the approximate velocity is $12.4\,\mathrm{ms}^{-1}$ (3 s.f.)

A1

(c) We have here the differential equation formed in part (a).

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.8v$$

To solve this, we must integrate. But first we must separate the variables, in order to have our expression for v on the left hand side (so we can integrate with respect to v). We have

$$\int \frac{1}{10-0.8v}\,\mathrm{d}v = \int \mathrm{d}t \qquad \textbf{(M1)}$$

$$-\frac{1}{0.8}\ln|10-0.8v| = t + C \qquad \textbf{(A1)}$$

At this point we could note that $-\frac{1}{0.8} = -\frac{5}{4}$. Hence we have

$$-\frac{5}{4}\ln|10-0.8v| = t + C$$

Using our initial condition, $v = 0$ when $t = 0$, we can solve for C.

$$-\frac{5}{4}\ln|10-0.8(0)| = 0 + C \qquad \textbf{(M1)}$$

$$C = -\frac{5}{4}\ln 10$$

Hence we have

$$-\frac{5}{4}\ln|10-0.8v| = t - \frac{5}{4}\ln 10 \qquad \textbf{(A1)}$$

We could now multiply through by $-\frac{4}{5}$ to make things more straightforward:

$$\left(-\frac{4}{5}\right)\left(-\frac{5}{4}\ln|10-0.8v|\right) = \left(-\frac{4}{5}\right)\left(t-\frac{5}{4}\ln 10\right)$$

$$\ln|10-0.8v| = -\frac{4t}{5}+\ln 10$$

$$e^{\ln|10-0.8v|} = e^{-\frac{4t}{5}+\ln 10} \qquad \textbf{(M1)}$$

$$|10-0.8v| = e^{-\frac{4t}{5}}\cdot e^{\ln 10} \qquad [\text{because } a^{m+n}=a^m\cdot a^n]$$

$$10-0.8v = 10\,e^{-\frac{4t}{5}}$$

Note that in the last step we have removed the absolute value signs, because the right hand side is greater than 0 for all values of t.

$$-0.8v = -10+10\,e^{-\frac{4t}{5}}$$

$$v = \frac{-10+10\,e^{-\frac{4t}{5}}}{-0.8}$$

$$v = 12.5-12.5\,e^{-\frac{4t}{5}} \qquad \textbf{A1}$$

Note: there are several correct ways to write the final answer.

(d) As $t\to\infty$, $e^{-\frac{4t}{5}}\to 0$, so the function approaches a value of 12.5 (from below).

The terminal velocity is $12.5\,\text{ms}^{-1}$.

A1

(e) We need to solve the following equation for t:

$$12.5-12.5\,e^{-\frac{4t}{5}} = 0.96(12.5) \qquad \textbf{(M1)}$$

Using the numerical solver on the G.D.C., we find that $t=4.023...$

(A1)

To the nearest second, $v(t)$ reaches 96% of the terminal velocity after 4 seconds.

A1

(f) Using our table from part (b), we see that Euler's method predicted a velocity of $12.29...\text{ms}^{-1}$ after 4 seconds.

Using the percentage error formula with $v_A = 12.29...$ and $v_E = 0.96(12.5) = 12$, we have

$$\begin{aligned}\varepsilon &= \left|\frac{12.29... - 12}{12}\right| \times 100\% \\ &= 2.417...\end{aligned}$$

(M1)

$$\boxed{\varepsilon = 2.42\%\ (3\text{ s.f.})}$$

A1

(g) Our function, $v(t)$, is always concave down.

A1

Euler's method will always overestimate if a function is concave down, and always underestimate if a function is concave up.

R1

Question 6

CALCULATOR | Hard

[Maximum mark: 15]

A water park lends life jackets to its visitors. The park has two stations, A and B, where clients can take a life jacket and they can return the life jacket to either of the two stations. It is found that at the end of a day, 81% of the life jackets are returned to the same station they were taken from, while the other 19% are returned to the other station.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the proportion of the life jackets moving between the two stations. [2]

On February 1st, before opening the park, the staff counted the number of life jackets and found there were 400 life jackets at station A and 500 at station B.

(b) Determine the number of life jackets expected to be at each station when the park opens on February 5th. Give your answer to the nearest whole number. [2]

(c) (i) Show that the eigenvalues of $\boldsymbol{T}$ are $\lambda_1 = 1$ and $\lambda_2 = 0.62$.

(ii) Find a corresponding eigenvector for each eigenvalue from **part (c) (i)**.

(iii) Hence express $\boldsymbol{T}$ in the form $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [6]

(d) Show that $\boldsymbol{T}^n = \frac{1}{2}\begin{bmatrix} 1+(0.62)^n & 1-(0.62)^n \\ 1-(0.62)^n & 1+(0.62)^n \end{bmatrix}$, where $n \in \mathbb{Z}^+$. [2]

(e) Hence, find an expression for the number of life jackets in station A after n days, where $n \in \mathbb{Z}^+$ [3]

revisionvillage

(a) We have

		FROM	
		Station A	Station B
TO	Station A	0.81	0.19
	Station B	0.19	0.81

Hence the transition matrix is

$$\boldsymbol{T} = \begin{bmatrix} 0.81 & 0.19 \\ 0.19 & 0.81 \end{bmatrix}$$

A2

(b) There are four transitions between February 1st and February 5th. We have

$$\begin{aligned} \boldsymbol{S}_4 &= \boldsymbol{T}^4 \boldsymbol{S}_0 \\ &= \begin{bmatrix} 0.81 & 0.19 \\ 0.19 & 0.81 \end{bmatrix}^4 \begin{bmatrix} 400 \\ 500 \end{bmatrix} \\ &= \begin{bmatrix} 442.61 \\ 457.39 \end{bmatrix} \qquad \text{[by using G.D.C.]} \end{aligned}$$

(M1)

Hence the distribution of the life jackets after four days is

Station A : 443

Station B : 457

A1

(c) (i) If we solve the characteristic equation $\det(\boldsymbol{T} - \lambda \boldsymbol{I}) = 0$ for λ, we get

$$\begin{vmatrix} 0.81-\lambda & 0.19 \\ 0.19 & 0.81-\lambda \end{vmatrix} = 0$$

$$(0.81-\lambda)^2 - (0.19)^2 = 0$$ **A1**

$$\lambda = 0.62,\ 1 \qquad \text{[by using G.D.C.]}$$

Hence the eigenvalues of $\boldsymbol{T}$ are $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 0.62}$

AG

(ii) If we solve the matrix equation $(\boldsymbol{T} - \lambda_1 \boldsymbol{I})\boldsymbol{X}_1 = 0$ for $\boldsymbol{X}_1$, we find

$$(\boldsymbol{T} - \boldsymbol{I})\boldsymbol{X}_1 = 0 \qquad [\text{since } \lambda_1 = 1]$$

$$\begin{bmatrix} -0.19 & 0.19 \\ 0.19 & -0.19 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.19x + 0.19y \\ 0.19x - 0.19y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = y$$

(M1)

$$\boxed{\boldsymbol{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad [\text{choose } x = 1]$$

A1

If we solve the matrix equation $(\boldsymbol{T} - \lambda_2 \boldsymbol{I})\boldsymbol{X}_2 = 0$ for $\boldsymbol{X}_2$, we find

$$(\boldsymbol{T} - 0.62\boldsymbol{I})\boldsymbol{X}_2 = 0 \qquad [\text{since } \lambda_2 = 0.62]$$

$$\begin{bmatrix} 0.19 & 0.19 \\ 0.19 & 0.19 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.19x + 0.19y \\ 0.19x + 0.19y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = -y$$

(M1)

$$\boxed{\boldsymbol{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \qquad [\text{choose } x = 1]$$

A1

(iii) Let

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \end{bmatrix} \qquad \text{[invertible matrix of e-vectors]}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{[diagonal matrix of e-values]}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.62 \end{bmatrix}$$

Using the inverse of a 2×2 matrix formula, we have

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence we obtain

$$\boldsymbol{T} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.62 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A1

(d) We have

$$\begin{aligned}
\boldsymbol{T}^n &= \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} \\
&= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0.62\end{bmatrix}^n\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & (0.62)^n\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} && \textbf{M1} \\
&= \frac{1}{2}\begin{bmatrix}1 & (0.62)^n\\ 1 & -(0.62)^n\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} && \textbf{A1} \\
&= \boxed{\frac{1}{2}\begin{bmatrix}1+(0.62)^n & 1-(0.62)^n\\ 1-(0.62)^n & 1+(0.62)^n\end{bmatrix}} && \textbf{AG}
\end{aligned}$$

(e) We need to determine the nth state matrix. The expression in the top row will represent the number of lifejackets in station A after n days.

$$\boldsymbol{S}_n = \boldsymbol{T}^n\boldsymbol{S}_0$$

Using our expression for $\boldsymbol{T}^n$ from part (d), we have

$$\begin{aligned}
\boldsymbol{S}_n &= \frac{1}{2}\begin{bmatrix}1+(0.62)^n & 1-(0.62)^n\\ 1-(0.62)^n & 1+(0.62)^n\end{bmatrix}\begin{bmatrix}400\\ 500\end{bmatrix} && \textbf{(M1)} \\
&= \begin{bmatrix}1+(0.62)^n & 1-(0.62)^n\\ 1-(0.62)^n & 1+(0.62)^n\end{bmatrix}\begin{bmatrix}200\\ 250\end{bmatrix} \\
&= \begin{bmatrix}200[1+(0.62)^n]+250[1-(0.62)^n]\\ 200[1-(0.62)^n]+250[1+(0.62)^n]\end{bmatrix} && \textbf{(M1)} \\
\boldsymbol{S}_n &= \begin{bmatrix}450-50(0.62)^n\\ 450+50(0.62)^n\end{bmatrix}
\end{aligned}$$

Hence, an expression for the number of life jackets in station A after n days is

$$\boxed{A(n) = 450 - 50(0.62)^n} \qquad \textbf{A1}$$

Question 7

CALCULATOR Hard

[Maximum mark: 13]

The sunset time in Paris can be modelled by the function

$$S(t) = 1.85\sin(0.52t - 1.57) + 19.5$$

where t is time in months, from the beginning of January.

The sunrise time can similarly be modelled by

$$R(t) = 1.35\sin(0.52t - 4.7) + 7.5$$

Both $S(t)$ and $R(t)$ are measured in hours since midnight.

$S(t)$ can be expressed as $\text{im}(z_1) + 19.5$ and $R(t)$ can be expressed as $\text{im}(z_2) + 7.5$, where $z_1, z_2 \in \mathbb{C}$.

(a) Write down expressions for z_1 and z_2. Give your answers in the form $z_{1,2} = r\,e^{\theta i}$. [2]

The number of daylight hours in Paris, $D(t)$, is given by $D(t) = S(t) - R(t)$.

(b) Hence show that the number of hours of daylight in Paris can be modelled by the function

$$D(t) = 3.20\sin(0.52t - 1.57) + 12$$

[6]

(c) According to the model, find:

(i) the highest number of hours of daylight in a year;

(ii) The approximate number of months per year in which there is at least 14 hours of daylight. [5]

(a) Equating our function $S(t)$ to $\text{im}(z_1) + 19.5$, we must have

$$\text{im}(z_1) = 1.85\sin(0.52t - 1.57)$$

This is the imaginary part of the complex number

$$z_1 = 1.85\cos(0.52t - 1.57) + i\,(1.85\sin(0.52t - 1.57))$$

In the required form, we have $z_1 = 1.85e^{(0.52t-1.57)i}$

A1

Using similar reasoning with $R(t)$, we have $z_2 = 1.35e^{(0.52t-4.7)i}$

A1

(b) We have

$$\begin{aligned} \mathrm{D}(t) &= \mathrm{S}(t) - \mathrm{R}(t) \\ &= \mathrm{im}(z_1) + 19.5 - (\mathrm{im}(z_2) + 7.5) \\ &= \mathrm{im}(z_1) - \mathrm{im}(z_2) + 12 \\ &= \mathrm{im}(z_1 - z_2) + 12 \end{aligned}$$ **(A1)**

Let's find $z_1 - z_2$ so that we can identify the imaginary part.

$$z_1 - z_2 = 1.85e^{(0.52t-1.57)i} - 1.35e^{(0.52t-4.7)i}$$

$$= 1.85e^{0.52ti-1.57i} - 1.35e^{0.52ti-4.7i}$$ **(M1)**

$$= 1.85\left(e^{0.52ti}\right)\left(e^{-1.57i}\right) - 1.35\left(e^{0.52ti}\right)\left(e^{-4.7i}\right)$$

$$= e^{0.52ti}\left(1.85e^{-1.57i} - 1.35e^{-4.7i}\right)$$ **(M1)**

We have now factorised the variable t out of the brackets, allowing us to compute a single complex number inside the brackets.

$$= e^{0.52ti}\left(3.199...e^{-1.565...i}\right) \qquad \text{[by G.D.C.]}$$ **(M1)**

$$= 3.199...e^{0.52ti-1.565...i}$$

$$= 3.199...e^{(0.52t-1.565...)i}$$

$$z_1 - z_2 = 3.20e^{(0.52t-1.57)i} \text{ (3 s.f.)}$$ **(A1)**

Hence we get

$$\mathrm{D}(t) = \mathrm{im}\left[3.20e^{(0.52t-1.57)i}\right] + 12$$

$$= 3.20\sin(0.52t - 1.57) + 12$$ **A1**

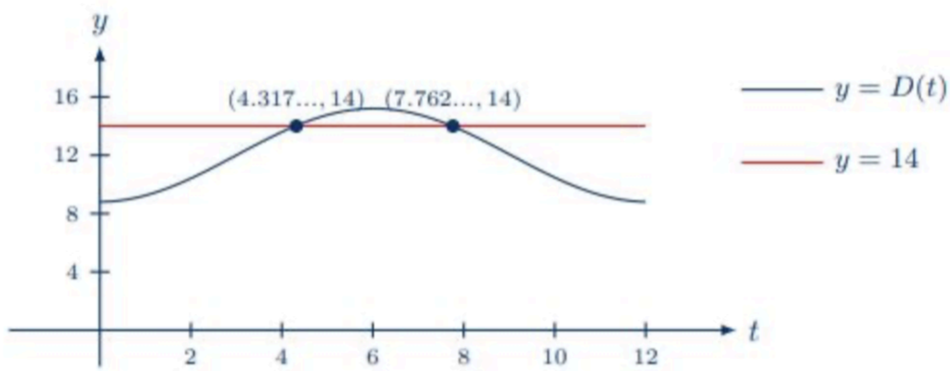
(c) (i) According to the model, the highest number of hours of daylight is

$$D_{\text{max}} = 3.20 + 12$$ **M1**

$$= 15.2 \text{ hours}$$ **A1**

(ii) If we sketch the graph of $y = \mathrm{D}(t)$, and also the graph of $y = 14$ [by using G.D.C.], we can find the points of intersection. Our solution is the domain for which $\mathrm{D}(t) > 14$.

(M1)



We have

$$7.762... - 4.317... = 3.444...$$ **(M1)**

There are at least 14 hours of daylight for approximately 3.44 months of the year.

A1

Note: An answer of 3 months is acceptable.