IB Mathematics AI HL - Prediction Exams May 2025 - Paper 3

Paper 3 | 2 questions | 75 mins | 55 marks

Question 1

CALCULATOR   Medium

[Maximum mark: 26]

A health and fitness app allows users to enter data about their regular exercise routine, and a body fat percentage (PBF) is calculated. The app then designs a 3 month fitness programme, and gives a predicted PBF if the user follows the programme for the full 3 months.

Two local gyms, Awesome Fitness (A) and Better Bodies (B), have claimed that their 3 month programmes will produce a lower final PBF than that predicted by the app.

The following table shows the predicted PBF and the final PBF of 10 randomly selected people who have obtained a prediction from the app, and then followed the 3 month programme at gym A.

No.	Gender	Predicted PBF	Final PBF
1	male	18.1	13.9
2	female	23.1	19.5
3	female	21.5	18.1
4	male	27.4	22.5
5	female	26.3	22.0
6	female	29.5	28.1
7	female	33.4	29.9
8	female	22.8	18.4
9	male	26.6	20.7
10	female	28.7	24.2

The difference between the final PBF and the predicted PBF was calculated for each user.

(a) (i) Find the mean PBF difference.

(ii) Find the standard deviation of the differences. [3]

A hypothesis test is carried out by Awesome Fitness to test their claim.

(b) (i) State the name of an appropriate test.

(ii) State an assumption that must be made in order to carry out the hypothesis test.

(iii) State the null and alternate hypotheses.

(iv) Determine the p-value.

(v) Hence state whether there is any evidence at the 5% significance level to support the claim made by Awesome Fitness. Justify your answer. [7]

The following table shows the corresponding differences between the PBF scores of a random sample of app users who have completed 3 months at gym B.

No.	Gender	Difference
1	male	−3.5
2	female	−2.3
3	male	−3.1
4	male	−2.9
5	male	−3.3
6	female	−1.4
7	male	−4.1
8	male	−3.7
9	male	−2.9
10	female	−0.5

(c) Given that the assumption made in part (b) is valid, conduct an appropriate hypothesis test to determine, at the 5% significance level, whether Awesome Fitness can claim to have better results than Better Bodies. Justify your answer. [6]

Awesome Fitness have also tracked their users diets over the 3 month programme, and given each user a diet score out of 10, higher scores indicating better diet.

No.	Gender	Predicted PBF	Final PBF	Diet score
1	male	18.1	13.9	8
2	female	23.1	19.5	7.5
3	female	21.5	18.1	8.1
4	male	27.4	22.5	9.3
5	female	26.3	22.0	6.8
6	female	29.5	28.1	4.5
7	female	33.4	29.9	7.7
8	female	22.8	18.4	8.9
9	male	26.6	20.7	9.4
10	female	28.7	24.2	7.8

A test is conducted to determine if there is a linear correlation between Final PBF and diet score for their members, of which there are several hundred.

(d) (i) State the null and alternate hypotheses for this test.

(ii) Determine the p-value for the test.

(iii) Hence, determine at the 5% significance level whether there is significant evidence of a linear correlation. Justify your answer. [5]

One member of staff noticed that any comparison between the two gyms may not be valid because the sample from gym A contained mainly females while the sample from gym B contained mainly males, and there might be a difference in the change of body fat percentage due to biological reasons.

Hence, another test is conducted over the following table showing how many users obtained a given reduction, r, in their PBF.

	$r < -6$	$-6 < r < -3$	$-3 < r < 0$	$0 < r$
Male	13	9	5	3
Female	7	6	7	10

(e) (i) Use an appropriate test to show, at the 10% significance level, that there is evidence that PBF reduction and gender are not independent.

(ii) Hence, suggest a sampling method that may lead to better data analysis in the previous parts. [5]

(a) Using the G.D.C.,

(M1)

(i) $\bar{x} = -4.01$

A1

(ii) $\sigma = 1.12\,(3\text{ s.f.})$

A1

(b) (i) A paired t-test or a t-test on the differences

A1

(ii) The underlying distribution must be normal.

A1

(iii) Using a t-test on the differences, we have

$H_0\ :\ \mu_D = 0$

$H_1\ :\ \mu_D < 0$

A1

(iv) Using the G.D.C.,

(M1)

$p \approx 9.657 \times 10^{-7}$

A1

(v) The p-value is less than the significance level, therefore we reject H_0.

R1

There is significant evidence at the 5% level that Awesome Fitness's programme will produce a lower final PBF than the app predicts.

A1

(c) In this case, we must perform a 2-sample t-test. It is important not to do a paired t-test, or a t-test on differences, because the data from gym A and gym B are not paired in any way.

(M1)

Our null and alternate hypotheses will be

$$H_0 : \mu_A = \mu_B$$
$$H_1 : \mu_A < \mu_B$$

(A1)

Using the G.D.C., we can calculate a p-value.

(M1)

We find a p-value of 0.01274...

(A1)

The p-value is less than the significance level, therefore we reject the null hypothesis.

R1

There is significant evidence at the 5% level that Awesome Fitness's programme will produce a lower final PBF than Better Bodies programme.

A1

(d) (i) This is a two-tailed test for ρ, hence we have

$H_0 : \rho = 0$

$H_1 : \rho \neq 0$

A1

(ii) Using the G.D.C.,

(M1)

$p \approx 0.178$

A1

(iii) The p-value is greater than the significance level, therefore we do not reject the null hypothesis.

R1

There is insufficient evidence at the 5% level of a linear correlation between final PBF and diet score.

A1

(e) (i) This will be a χ^2 test for independence.

(A1)

Our null and alternate hypotheses are

$$H_0 : \text{PBF reduction and gender are independent}$$

$$H_1 : \text{PBF reduction and gender are not independent}$$

Using the G.D.C.,

(M1)

We calculate a p-value of $p = 0.08956...$

A1

The p-value is less than the significance level, therefore we reject the null hypothesis.

R1

There is significant evidence at the 10% level that PBF reduction and gender are not independent.

AG

(ii) We would like to have an equal number of females and males, therefore stratified sampling would be appropriate.

A1

Question 2

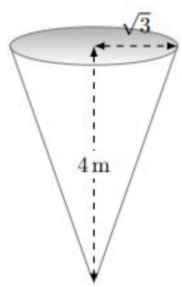
CALCULATOR

Hard ●●●●

[Maximum mark: 29]

This question explores different models for the height of syrup in a cone shaped container.

A candy manufacturer has a syrup mixing machine in the form of an inverted cone with radius $\sqrt{3}$ m and height 4 m as shown in the following diagram.



The specification manual has the following table showing the expected height of the syrup, h, at time t minutes once a valve at the bottom of the container is open.

Time	Height
0	4
16	0.5

Jennifer, who is in charge of the syrup mixer, believes that the rate at which the height of the syrup is decreasing is proportional to the height at any time, as shown by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = kh \quad , \quad k \in \mathbb{R}$$

(a) (i) Find an expression for h in terms of k and t.

(ii) Hence find the value of k and write h only in terms of t. [7]

As Jennifer reads further in the manual, she finds that for the best syrup consistency, the syrup should completely drain from the cone in exactly 20 minutes. That is, when $t = 20$, $h = 0$.

(b) Explain why Jennifer's model would not be suitable for this machine. [2]

Let V be the volume, in cubic metres, of the syrup in the container at time t minutes and R be the radius, in metres, of the circular valve at the bottom of the container. After contacting the machine manufacturer, Jennifer learns that the rate of change of the *volume* is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -180\pi R^2\sqrt{h}$$

Note that h is the height of the syrup in the container at any time t, and r is the radius of the top of the syrup at the same time.

(c) (i) Find an expression for r in terms of h.

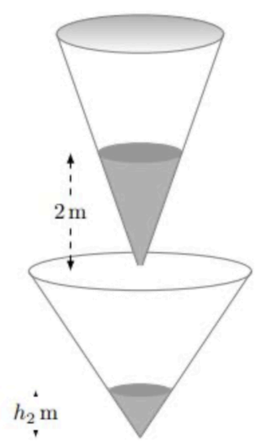
(ii) Hence, show that $V = \frac{\pi}{16}h^3$. [4]

(d) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{960R^2}{\sqrt{h^3}}$. [4]

(e) Using the initial condition from the manual, find an expression for h in terms of t and R. [6]

(f) Given that the container drains in exactly 20 minutes, find the value of R, the radius of the circular valve. [2]

Once the syrup passes through the valve, it settles in a second inverted cone shaped container of the same volume, but with a height of 3 m, and a radius of 2 m.



A disruption at the factory causes production to be shut down.

At this time, the height of the syrup in the first container is 2 m, and the height of the syrup in the second container is h_2 m.

(g) Given that the first container was initially full, find the value of h_2. [4]

(a) (i) To find an expression for k, we must integrate, but first we must separate the variables. We have

$$\int \frac{1}{h}\,\mathrm{d}h = \int k\,\mathrm{d}t \qquad \textbf{(M1)}$$

$$\ln|h| = kt + C$$

$$\ln h = kt + C \qquad [\text{because } h > 0] \qquad \textbf{(A1)}$$

Given that $h = 4$ when $t = 0$, we can evaluate C.

$$\ln 4 = k(0) + C$$

$$C = \ln 4 \qquad \textbf{(A1)}$$

Hence we have

$$\ln h = kt + \ln 4$$

$$h = e^{kt+\ln 4} \qquad \textbf{(M1)}$$

$$\boxed{h = 4e^{kt}} \qquad \textbf{A1}$$

(ii) Given that $h = 0.5$ when $t = 16$, we have

$$0.5 = 4e^{16k} \qquad \textbf{(M1)}$$

$$k = -0.1299... \qquad [\text{Using G.D.C.}] \qquad \textbf{A1}$$

$$\boxed{h = 4e^{-0.13t}}$$

(b) Evaluating $h(20)$ we have

$$h(20) = 4e^{-0.13(20)}$$

$$= 0.2970... \qquad \textbf{A1}$$

At this time, the height should be 0 m, but according to the model it is still approximately 0.3 m.

R1

(c) (i) We have been given that the radius of the container is $\sqrt{3}$ and the height is 4. The ratio of container radius to height is $\frac{\sqrt{3}}{4}$, and this will also be true for the syrup at any time, regardless of how much syrup is left in the container. We have

$$\frac{r}{h} = \frac{\sqrt{3}}{4} \qquad \textbf{(M1)}$$

$$\boxed{r = \frac{\sqrt{3}}{4}h} \qquad \textbf{A1}$$

(ii) The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. At any time t, the volume of syrup will be in the shape of a cone. Using our expression for r from part (i), we have

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{\sqrt{3}}{4}h\right)^2 h \qquad \textbf{M1}$$

$$= \frac{1}{3}\pi\left(\frac{3}{16}h^2\right) h \qquad \textbf{M1}$$

$$\boxed{V = \frac{\pi}{16}h^3} \qquad \textbf{AG}$$

(d) We can use related rates to find an expression for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} \qquad \textbf{M1}$$

Differentiating our solution to (c) part (ii), we have $\frac{dV}{dh} = \frac{3\pi h^2}{16}$ and therefore $\frac{dh}{dV} = \frac{16}{3\pi h^2}$.

A1

We have been given that $\frac{dV}{dt} = -180\pi R^2\sqrt{h}$, and we can substitute both of these into our related rates equation:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= -180\pi R^2\sqrt{h} \times \frac{16}{3\pi h^2} \qquad \textbf{M1}$$

$$= \frac{-960R^2h^{1/2}}{h^2}$$

$$= \frac{-960R^2}{h^{3/2}} \qquad \textbf{M1}$$

$$\boxed{\frac{dh}{dt} = -\frac{960R^2}{\sqrt{h^3}}} \qquad \textbf{AG}$$

(e) To find an expression for h, we must integrate. First we will separate the variables.

$$\int h^{3/2}\,\mathrm{d}h = \int -960R^2\,\mathrm{d}t \qquad \textbf{(M1)}$$

$$\frac{h^{5/2}}{5/2} = -960R^2t + C \qquad \textbf{(M1)}$$

$$\frac{2h^{5/2}}{5} = -960R^2t + C$$

Using the initial condition, $h = 4$ when $t = 0$, we have

$$\frac{(2)4^{5/2}}{5} = -960R^2(0) + C \qquad \textbf{(M1)}$$

$$\frac{64}{5} = C \qquad \textbf{(A1)}$$

Hence we have

$$\frac{2h^{5/2}}{5} = -960R^2t + \frac{64}{5}$$

$$h^{5/2} = -2400R^2t + 32 \qquad \textbf{(M1)}$$

$$h = \left(-2400R^2t + 32\right)^{2/5}$$

$$\boxed{h = \sqrt[5]{(32 - 2400R^2t)^2}} \qquad \textbf{A1}$$

(f) Using $h = 0$ when $t = 20$, we have

$$0 = \sqrt[5]{(32 - 2400R^2(20))^2} \qquad \textbf{(M1)}$$

$$R = 0.02581\ldots \qquad \text{[by numerical solver]} \qquad \textbf{A1}$$

The radius of the circular valve is 0.0258 m (3 s.f.)

(g) The total volume of syrup can be found from the dimensions of the first cone. We have

$$V = \frac{1}{3}\pi\left(\sqrt{3}\right)^2(4)$$
$$= 4\pi\,\mathrm{m}^3 \qquad \textbf{(A1)}$$

At the time when the height is 2 metres in the first container, the radius of the syrup will be $\frac{\sqrt{3}}{2}$, from our answer to (c) part (i).

Hence the volume of syrup remaining in the first container will be

$$V = \frac{1}{3}\pi\left(\frac{\sqrt{3}}{2}\right)^2(2)$$
$$= \frac{\pi}{2}\,\mathrm{m}^3$$

This means there is exactly $\frac{7\pi}{2}$ m^3 of syrup in container 2.

(A1)

We have been given the dimensions of container 2, and from this we can determine that the ratio of the radius of the syrup to its height, at any time, is $\frac{2}{3}$, that is, $r = \frac{2}{3}h$.

Hence we can form an equation for the volume of syrup in container 2 at the time when the syrup height in container 1 is 2 metres. We have

$$V = \frac{1}{3}\pi r^2 h_2$$
$$\frac{7\pi}{2} = \frac{1}{3}\pi\left(\frac{2}{3}h_2\right)^2 h_2 \qquad \textbf{(M1)}$$
$$\frac{21}{2} = \frac{4}{9}h_2^3$$
$$h_2^3 = \frac{189}{8}$$
$$h_2 = 2.869\ldots$$

$$\boxed{h_2 = 2.87\,\mathrm{m}\ (3\text{ s.f.})} \qquad \textbf{A1}$$