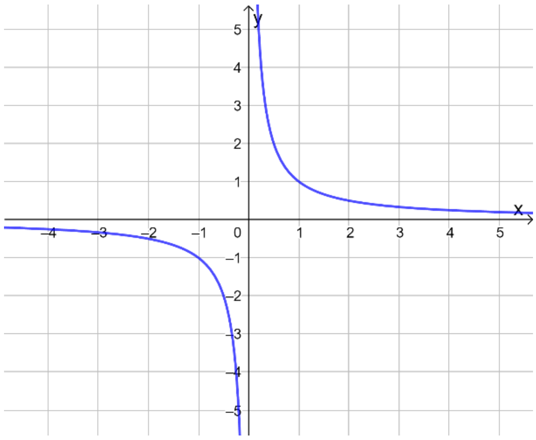
L'Hôpital's Rule

You will be familiar with the idea of limits from other areas of mathematics, for example rational functions:



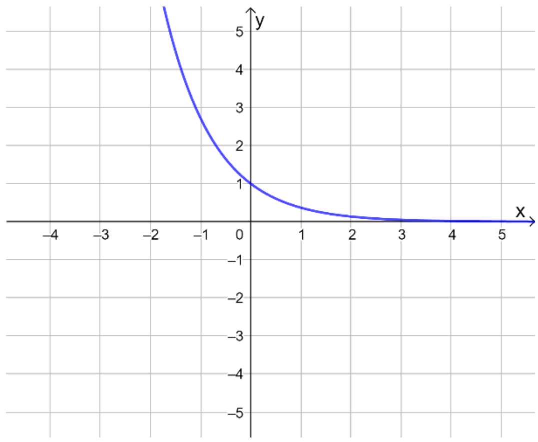
$f(x) = \frac{1}{x}$, $x \neq 0$

You cannot evaluate $f(0)$

You cannot find a value for $f(x) = 0$

However, we can say that as x gets bigger, then $f(x)$ tends towards 0

This is a limit which can be written $\lim\limits_{x \to \infty} \frac{1}{x} = 0$



Similarly, for the function $g(x) = e^{-x}$

$\lim\limits_{x \to \infty} e^{-x} = 0$

Imagine, we wanted to evaluate the following limit

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{e^{-x}}$$

This would give $\frac{0}{0}$ which is an indeterminate form.

L'Hôpital's Rule gives us a method to evaluate limits in this form $\frac{0}{0}$ and also $\frac{\infty}{\infty}$

L'Hôpital's Rule $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

This works provided that

1. $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists
   AND
2. $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$
   OR
3. $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$

Let's try to evaluate $\lim_{x\to\infty} \frac{\frac{1}{x}}{e^{-x}}$

This can be re-written $\lim_{x\to\infty} \frac{e^x}{x}$ which is in the indeterminate form $\frac{\infty}{\infty}$

We also need to consider what we mean by $x \to \infty$: $\quad x \to +\infty \;$ or $\; x \to -\infty$

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x)}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(x) = 1$$

Therefore,

$$\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = \frac{\infty}{1} = \infty$$

And,

$$\lim_{x\to-\infty} \frac{e^x}{x} = \lim_{x\to-\infty} \frac{e^x}{1} = \frac{0}{1} = 0$$

This makes sense when we consider the graph of $y = \frac{e^x}{x}$