a) Given that $z = cos\theta + isin\theta$, show that $z + \frac{1}{z} = 2cos\theta$

b) Hence, show that $z^n + \frac{1}{z^n} = 2cosn\theta$

c) Use the Binomial expansion to expand $\left(z + \frac{1}{z}\right)^5$

d) Hence, show that $cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}cos3\theta + \frac{5}{8}cos\theta$

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a) If $z = cos\theta + isin\theta$

$|z| = 1$

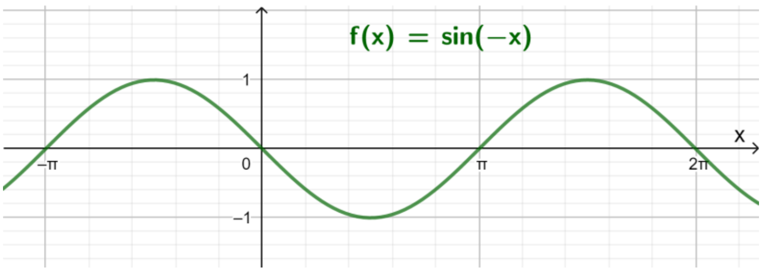
$|z^{-1}| = 1$

$arg(z) = \theta$

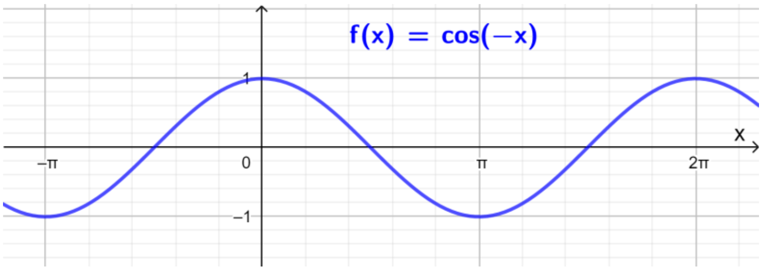
$arg(z^{-1}) = -\theta$

$$\frac{1}{z} = z^{-1} = \cos(-\theta) + isin(-\theta)$$

$sinx$ is an odd function: $\sin(-x) = -sinx$



$cosx$ is an even function: $\cos(-x) = cosx$



Therefore $\frac{1}{z} = \cos(\theta) - isin(\theta)$

$$z + \frac{1}{z} = cos\theta + isin\theta + cos\theta - isin\theta$$

$$z + \frac{1}{z} = 2cos\theta$$

b) $arg(z^n) = n\theta$

so $z^n = \cos(n\theta) + isin(n\theta)$

$arg(z^{-n}) = -n\theta$

and $\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + isin(-n\theta)$

$$z^{-n} = \cos(n\theta) - isin(n\theta)$$

$$z^n + \frac{1}{z^n} = \cos(n\theta) + isin(n\theta) + \cos(n\theta) - isin(n\theta)$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

c) $$\left(z + \frac{1}{z}\right)^5 = z^5 + 5(z^4)\left(\frac{1}{z}\right) + 10(z^3)\left(\frac{1}{z}\right)^2 + 10(z^2)\left(\frac{1}{z}\right)^3 + 5(z)\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$$

$$= z^5 + 5z^3 + 10z + 10\frac{1}{z} + 5\frac{1}{z^3} + \frac{1}{z^5}$$

d) $$\left(z + \frac{1}{z}\right)^5 = z^5 + \frac{1}{z^5} + 5z^3 + 5\frac{1}{z^3} + 10z + 10\frac{1}{z}$$

$$(2cos\theta)^5 = z^5 + \frac{1}{z^5} + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$2^5(cos\theta)^5 = 2\cos 5\theta + 10cos3\theta + 20cos\theta$$

$$32(cos\theta)^5 = 2\cos 5\theta + 10cos3\theta + 20cos\theta$$

$$cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}cos3\theta + \frac{5}{8}cos\theta$$