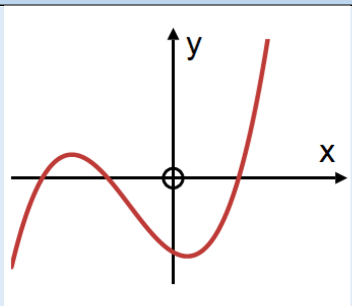
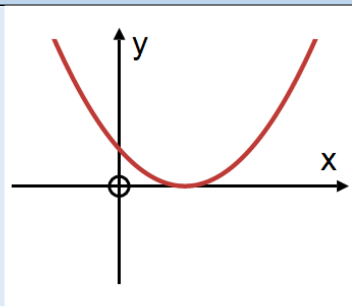
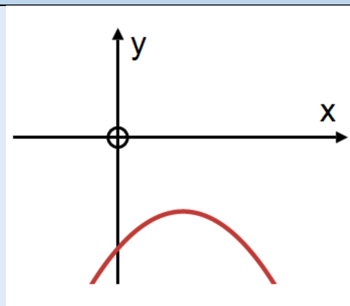
Roots of Polynomials

The **Conjugate Root** Theorem states that if the complex number $\boldsymbol{a} + \boldsymbol{ib}$ is a root of a polynomial in one variable with real coefficients, then the complex **conjugate** $\boldsymbol{a} - \boldsymbol{bi}$ also a root of that polynomial. This is a useful theorem for solving polynomials with real coefficients. Since the coefficients of the polynomial are real numbers, complex roots must always come in pairs and more than that they must be conjugate pairs - this way two complex numbers can multiply to give a real number.

It is worth considering how the number of real and complex roots of a polynomial relates to the shape of the graph

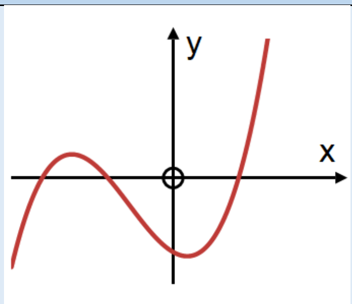
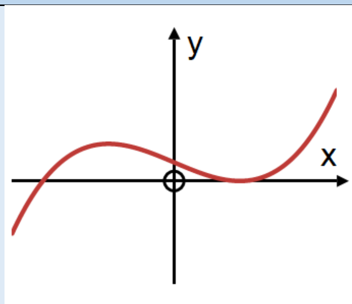
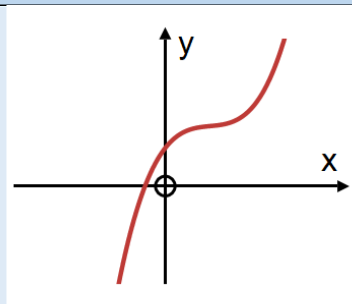
Quadratic

A quadratic equation will have 2 roots

2 distinct real roots	1 repeated real root	2 distinct complex roots
		

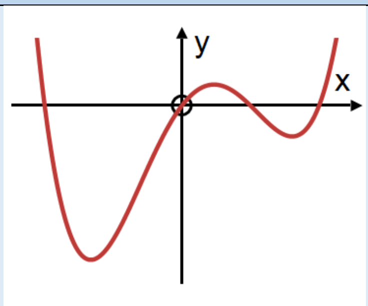
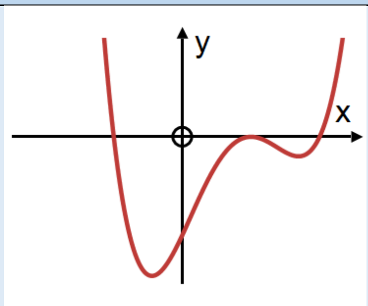
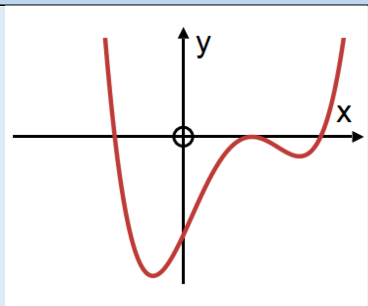
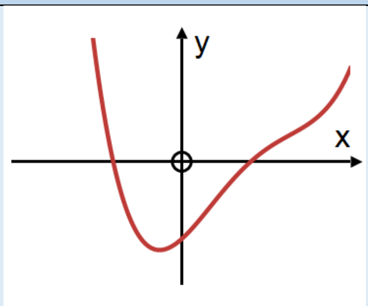
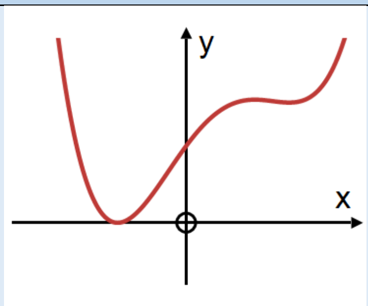
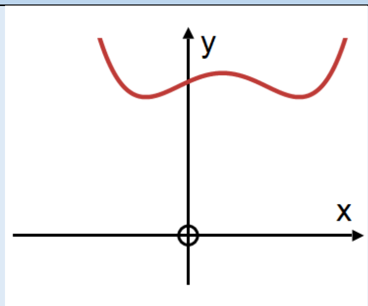
Cubic

A cubic equation will have 3 roots

3 distinct real roots	1 repeated real root & 1 other real root	1 real root & 2 complex roots
		

Quartic

A quartic equation will have 4 roots

4 distinct real roots	2 distinct real roots & 1 repeated real root	2 repeated real roots
		
2 distinct real roots & 2 complex roots	1 repeated real root & 2 complex roots	2 pairs of complex roots
		

Questions often give us a complex root of a polynomial equation and require us to work out the other roots.

Example

One root of the cubic equation $2z^3 + az^2 + bz + 15$ is $1 + 2i$. Find the other roots and a and b.

If $z = 1 + 2i$ is a root to the polynomial

Then $z = 1 - 2i$ is another root

$(z - 1 - 2i)$ is a factor

$(z - 1 + 2i)$ is a factor

We can multiply these factors together to find a quadratic equation

$$(z - 1 - 2i)(z - 1 + 2i) = z(z - 1 + 2i) \\ -1(z - 1 + 2i) \\ -2i(z - 1 + 2i)$$

$$= z^2 - z + 2zi$$
$$-z + 1 - 2i$$
$$-2zi + 2i - 4i^2$$

$$= z^2 - 2z + 5$$

You can find the other factor by dividing the cubic by this quadratic, but it is fairly easy to work out the unknown values by observation

$$2z^3 + az^2 + bz + 15 = (2z + 3)(z^2 - 2z + 5)$$

Now we can work out a and b by expanding the brackets

$$= 2z(z^2 - 2z + 5)$$
$$+3(z^2 - 2z + 5)$$

$$= 2z^3 - z^2 + 4z + 15$$