a) Use a deductive proof to prove that $even \times even = even$

b) Similarly prove that $odd \times odd = odd$

c) Hence, use proof by contradiction to prove that $\log_2 5$ is irrational

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a) Let a be an even number, then

$$a = 2m$$

Let b be an even number, then

$$b = 2n$$

$$even \times even = \quad a \times b = 2m \times 2n = 2(2mn)$$

Which is divisible by 2, hence even

b) Let c be an odd number, then

$$c = 2m + 1$$

Let d be an even number, then

$$d = 2n + 1$$

$$\begin{aligned} odd \times odd = \quad & c \times d = (2m + 1) \times (2n + 1) \\ & = 4mn + 2m + 2n + 1 \\ & = 2(2mn + m + n) + 1 \end{aligned}$$

Which is odd

c) If $\log_2 5$ is irrational, then it cannot be written as a fraction

Assume that $\log_2 5$ is rational

$$\log_2 5 = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$$

$$5 = 2^{\frac{a}{b}}$$

$$5^b = \left(2^{\frac{a}{b}}\right)^b$$

$$5^b = 2^a$$

From b) 5^b is always odd for any integer b

From a) 2^a is always even for any integer a

This is a contradiction, since

$$odd \neq even$$

Therefore, $\log_2 5$ is irrational