Prove that $\sqrt[3]{5}$ is irrational

Let $r = \sqrt[3]{5}$

Assume that the opposite is true

Assume that r is rational

r can be written as a fraction in its lowest terms

$r = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$

...a and b are coprime

...there is **no** common factor of a and b other than 1

$r^3 = \frac{a^3}{b^3} = 5$

$a^3 = 5b^3$

$5b^3$ is divisible by 5

a^3 is divisible by 5

Using the Fundamental Theorem of Arithmetic and since 5 is prime:

a is divisible by 5

$a = 5n$

$a^3 = 5b^3$

$(5n)^3 = 5b^3$

$125n^3 = 5b^3$

$25n^3 = b^3$

b^3 is divisible by 5

b is divisible by 5

If a and b are both divisible by 5, then $\frac{a}{b}$ has a common factor of 5

a and b are NOT coprime

This is a contradiction

$\sqrt[3]{5}$ is not rational. We have proved that $\sqrt[3]{5}$ is irrational.