Let $y = sinx$

Prove by induction that $\frac{d^n y}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right),\ n \in \mathcal{Z}^+$

Let P(n) be the proposition that $\frac{d^n y}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right), \qquad n \in \mathcal{Z}^+$

Show true for $n = 1$

$$y = sinx$$

$$\frac{dy}{dx} = cosx$$

$$\frac{d^1 y}{dx^1} = \sin\left(x + \frac{\pi}{2}\right)$$

$\sin(A + B) \equiv sinAcosB + cosAsinB$

$$\frac{dy}{dx} = sinxcos\frac{\pi}{2} + cosxsin\frac{\pi}{2}$$

$$cos\frac{\pi}{2} = 0$$

$$sin\frac{\pi}{2} = 1$$

$$\frac{dy}{dx} = sinx \times 0 + cosx \times 1$$

$$\frac{dy}{dx} = cosx$$

Hence true for $n = 1$

We do not have to use the compound angle formula to prove this. There is another way involving a simple trigonometric identity:

$cosx = \sin\left(x + \frac{\pi}{2}\right)$

We will use this identity later on in the proof

Assume true for $n = k$

Assume that $\frac{d^k y}{dx^k} = \sin\left(x + \frac{k\pi}{2}\right)$ is true

Show true for $n = k + 1$

Show that $\frac{d^{k+1} y}{dx^{k+1}} = \sin\left(x + \frac{(k+1)\pi}{2}\right)$ is true

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(\sin\left(x + \frac{k\pi}{2}\right)\right)$$

$$= \cos\left(x + \frac{k\pi}{2}\right)$$

Let's use that identity now

$$cos x \equiv \sin\left(x + \frac{\pi}{2}\right)$$

$$cos\left(x + \frac{k\pi}{2}\right) \equiv \sin\left(\left(x + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right)$$

$$\frac{d^{k+1}y}{dx^{k+1}} = \sin\left(\left(x + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right)$$

$$= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$$

$$= \sin\left(x + \frac{(k+1)\pi}{2}\right)$$

Hence true for $n = k + 1$

Concluding statement

True for $n = 1$
Assuming it is true for $n = k$ then it is true for $n = k + 1$
Therefore it is true for all $n \in \mathbb{Z}^+$