The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} a & 0 \le x < 2 \\ a\sin\left(\frac{\pi x}{4}\right) & 2 \le x < 4 \\ 0 & otherwise \end{cases}$$

a) Draw a sketch of ***f***
b) Show that the value of $a = \frac{\pi}{2\pi+4}$
c) Find E(x)
d) Find the exact value of the median of X
e) Find $P(X \le 2 | X \le 3)$

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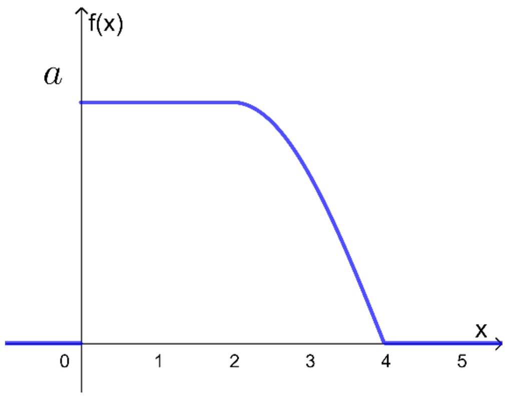
a) It is useful to work out some key points:

$$f(2) = a\sin\left(\frac{\pi \times 2}{4}\right) = a$$

$$f(4) = a\sin\left(\frac{\pi \times 4}{4}\right) = 0$$



b) For continuous random variables, the total area under the function = 1

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

In our case, we can split the area into 2 parts, $0 \le x < 2$ and $2 \le x < 4$

$$2a + \int_2^4 a\sin\left(\frac{\pi x}{4}\right) dx = 1$$

$$2a + a\int_2^4 \sin\left(\frac{\pi x}{4}\right) dx = 1$$

If you work out this integral using your GDC, it will not give you an exact value, so it is better to do it by hand:

$$2a + a\left[-\frac{4}{\pi}cos\left(\frac{\pi x}{4}\right)\right]_2^4 = 1$$

$$2a + a\left[\frac{4}{\pi} + 0\right] = 1$$

$$2a + \frac{4a}{\pi} = 1$$

$$\frac{2a\pi + 4a}{\pi} = 1$$

$$a(2\pi + 4) = \pi$$

$$a = \frac{\pi}{2\pi + 4}$$

c) For continuous random variables,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

In this case, we have to split the integrals into

$$E(X) = \int_0^2 \frac{\pi}{2\pi + 4}xdx + \int_2^4 \frac{\pi}{2\pi + 4}xsin\left(\frac{\pi x}{4}\right)dx$$

$$E(X) = \frac{\pi}{2\pi + 4}\left(\int_0^2 xdx + \int_2^4 xsin\left(\frac{\pi x}{4}\right)\mathrm{dx}\right)$$

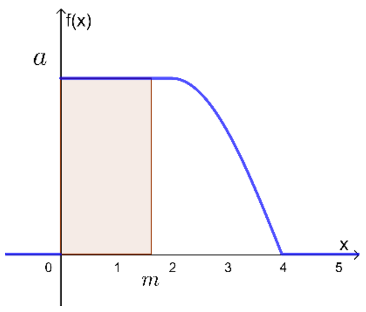
We can use a graphical calculator to work this out

$$E(X) \approx 1.67$$

d) For continuous random variables, median, m

$$\int_{-\infty}^{\mathrm{m}} f(x)dx = 0.5$$

Clearly this happens in the first part of the function

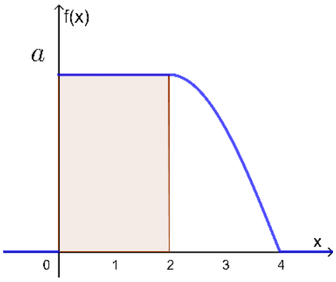


Therefore $\mathrm{m} \times \mathrm{a} = 0.5$

$$m \times \frac{\pi}{2\pi + 4} = 0.5$$

$$\mathrm{m} = \frac{\pi + 2}{\pi}$$

e) $$P(X \leq 2 | X \leq 3) = \frac{P(X \leq 2)}{P(X \leq 3)}$$



$$P(X \leq 2) = 2a$$

$$= \frac{2\pi}{2\pi + 4} = \frac{\pi}{\pi + 2}$$

$$P(X \leq 3) = P(X \leq 2) + P(2 \leq X \leq 3)$$

$$= \frac{\pi}{\pi + 2} + \int_2^3 \frac{\pi}{2\pi + 4} sin\left(\frac{\pi x}{4}\right) dx$$

Using the GDC

$$\approx 0.88607$$

$$P(X \leq 2 | X \leq 3) = \frac{P(X \leq 2)}{P(X \leq 3)} = \frac{\frac{\pi}{\pi + 2}}{0.88607} \approx 0.690$$

INTHINKING