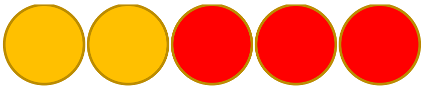
Alphonse and Bettina are playing a game. A bag contains 2 yellow beads and 3 red beads. They take it in turns to pick a bead from the bag at random. Alphonse goes first. If Alphonse picks a yellow bead, he wins and the game stops. If he picks a red bead, he replaces the bead and it is Bettina's turn. If Bettina picks a red bead, she wins and the game stops. If she picks a yellow bead, she replaces the bead and it is Alphonse's turn again.

a) Find the probability that Alphonse wins on his first turn.

b) Show that the probability that Alphonse wins on his second turn is $\frac{12}{125}$

c) The game continues until one of the player wins. What is the probability that Alphonse wins the game?



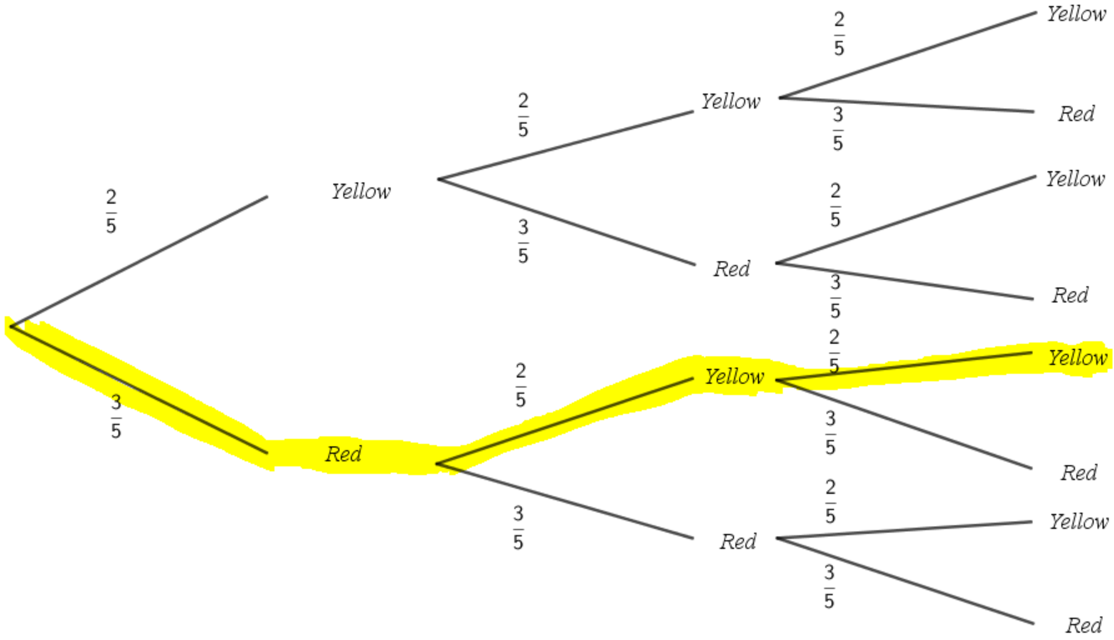
a)

$$P(\text{Alphonse wins on his first turn}) = P(\text{YELLOW})$$
$$= \frac{2}{5}$$

b)

For Alphonse to win on his 2^{nd} turn,

- Alphonse must lose on 1^{st} turn (RED)
- Bettina must lose on her 1^{st} turn (YELLOW)
- Alphonse must win on 2^{nd} turn (YELLOW)



$$P(\text{Alphonse wins on his 2nd turn}) = \frac{3}{5}\times\frac{2}{5}\times\frac{2}{5} = \frac{12}{125}$$

c)

For Alphonse to win:

He wins on his 1st turn
Or
He wins on his 2nd turn
Or
He wins on his 3rd turn
Or
etc…

$$P(\text{Alphonse wins on his 1st turn}) = \frac{2}{5}$$

$$P(\text{Alphonse wins on his 2nd turn}) = \frac{3}{5}\times\frac{2}{5}\times\frac{2}{5}$$

$$P(\text{Alphonse wins on his 3rd turn}) = \frac{3}{5}\times\frac{2}{5}\times\frac{3}{5}\times\frac{2}{5}\times\frac{2}{5}$$

$$P(\text{Alphonse wins}) = \frac{2}{5} + \frac{3}{5}\times\frac{2}{5}\times\frac{2}{5} + \frac{3}{5}\times\frac{2}{5}\times\frac{3}{5}\times\frac{2}{5}\times\frac{2}{5} + \cdots$$

This is an infinite geometric series

$$U_1 = \frac{2}{5}$$

$$r = \frac{3}{5}\times\frac{2}{5} = \frac{6}{25}$$

$$
\begin{aligned}
S_\infty &= \frac{U_1}{1-r} \\
P(\text{Alphonse wins}) &= \frac{\frac{2}{5}}{1-\frac{6}{25}} \\
&= \frac{\frac{2}{5}}{\frac{19}{25}} \\
&= \frac{2}{5} \times \frac{25}{19} \\
&= \frac{10}{19}
\end{aligned}
$$