Intersection of a Line and a Plane

The method
1. Write the equation of the line in parametric form
2. Substitute these values for x, y and z in to the cartesian equation of the plane
3. Solve for the parameter (e.g. λ)
4. Find the point of intersection by substituting this value in to the parametric equation of a line

Example 1 - Point of Intersection

Find the point of intersection of the line $-x = \frac{y-5}{2} = 2z - 8$
with the plane 3x – y + z = 8

1) Write the equation of the line in parametric form

$$-x = \frac{y-5}{2} = 2z - 8 = \boldsymbol{\lambda}$$

$$-x = \boldsymbol{\lambda}$$
$$\frac{y-5}{2} = \boldsymbol{\lambda}$$
$$2z - 8 = \boldsymbol{\lambda}$$

$$x = -\lambda$$
$$\mathrm{y} = 2\lambda + 5$$
$$\mathrm{z} = \frac{\lambda + 8}{2}$$

2) Substitute these in to the equation of the plane

$$3x - y + z = 8$$
$$3(-\lambda) - (2\lambda + 5) + \frac{\lambda+8}{2} = 8$$

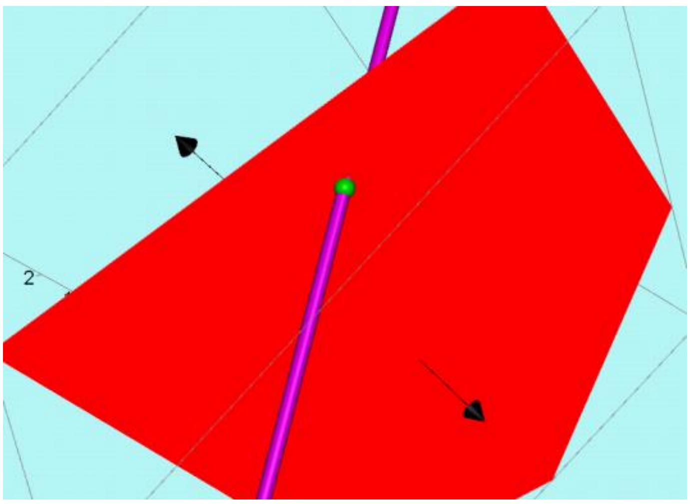
3) Solve for λ

$$6(-\lambda) - 2(2\lambda + 5) + \lambda + 8 = 16$$
$$-6\lambda - 4\lambda - 10 + \lambda + 8 = 16$$
$$-18 = 9\lambda$$
$$-2 = \lambda$$

4) Find the point of intersection by substituting this value in to the parametric equation of a line

$x = 2$
$\mathrm{y} = 1$
$\mathrm{z} = 3$

Point of intersection (2, 1, 3)



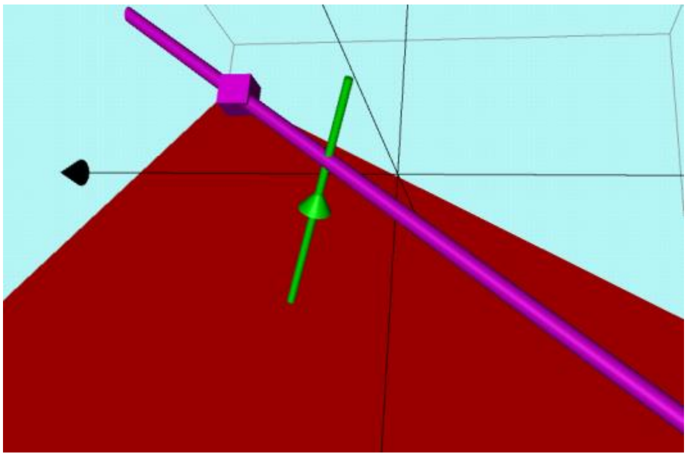
Example 2 - No Intersection

Find the point of intersection of the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

with the plane x - 2y + 2z = 6

$$x = 2 + 4\lambda$$
$$y = 1 + 3\lambda$$
$$z = \lambda$$

$$x \quad - \quad 2y \quad + 2z = 6$$
$$2 + 4\lambda - 2(1 + 3\lambda) + 2\lambda = 6$$
$$2 + 4\lambda - 2 - 6\lambda + 2\lambda = 6$$
$$0 \neq 6$$



The line and the plane do not meet.
The line must be parallel to the plane.

The normal to the plane is perpendicular to the line

Normal = $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Direction of the line = $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

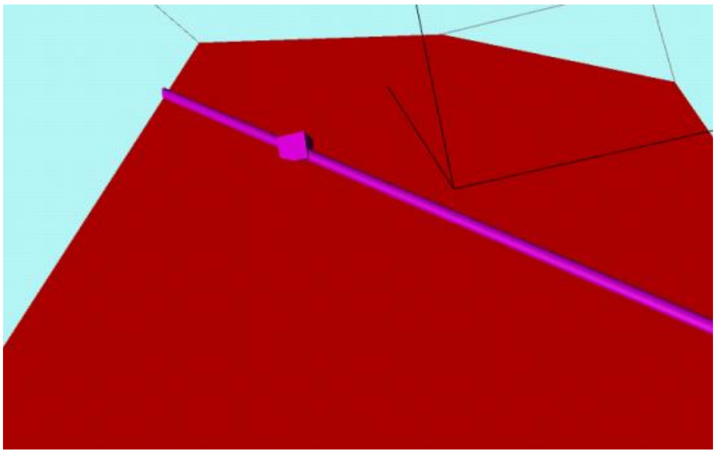
$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = 1 \times 4 - 2 \times 3 + 2 \times 1 = 0$$

The scalar product of the normal and the direction of the line is equal to zero. Hence, the normal and the line are perpendicular. Therefore, the plane and the line are parallel.

Example 3 - Line lies in Plane

Find the point of intersection of the line x = 2 + 4λ , y = 1 + 3λ , z = λ
with the plane x - 2y + 2z = 0

x = 2 + 4λ
y = 1 + 3λ
z = λ



$$x \quad - \quad 2y \quad + \quad 2z = 0$$
$$(2 + 4\lambda) - 2(1 + 3\lambda) + 2\lambda = 0$$
$$2 + 4\lambda - 2 - 6\lambda + 2\lambda = 0$$
$$4\lambda - 6\lambda + 2\lambda = 0$$
$$0\lambda = 0$$
$$0 = 0$$

The line lies in the plane