Equation of Planes - Vector, Normal and Cartesian Form

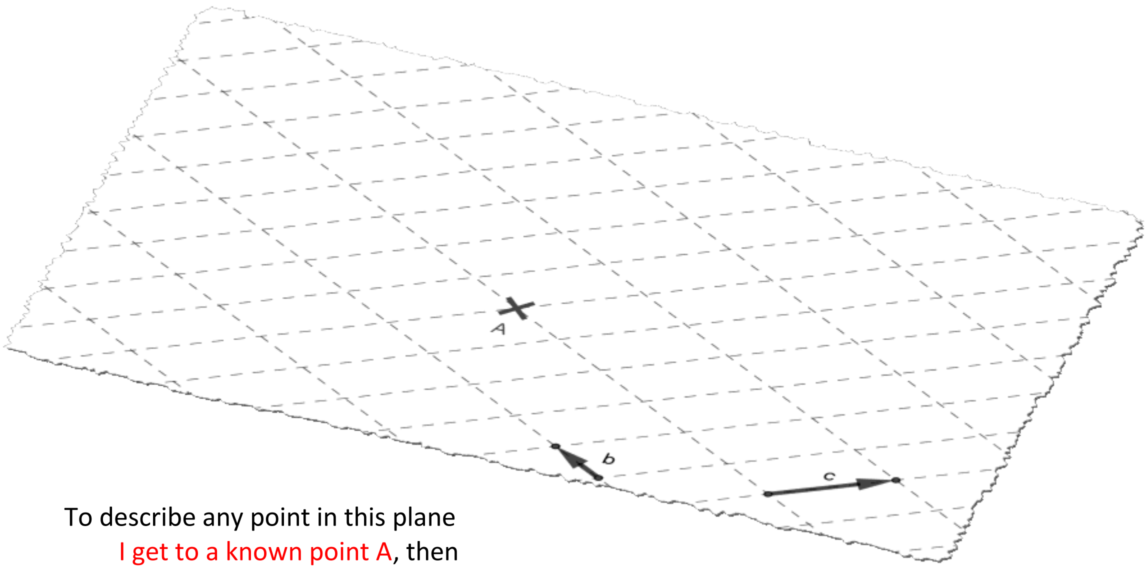
There are 3 forms of the equation of a line, although the last two are pretty much the same

$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ $\qquad$ *Vector form*

$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ $\qquad$ *Normal form*

$ax + by + cz = d$ $\qquad$ *Cartesian form*

$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ $\qquad$ *Vector form*



To describe any point in this plane

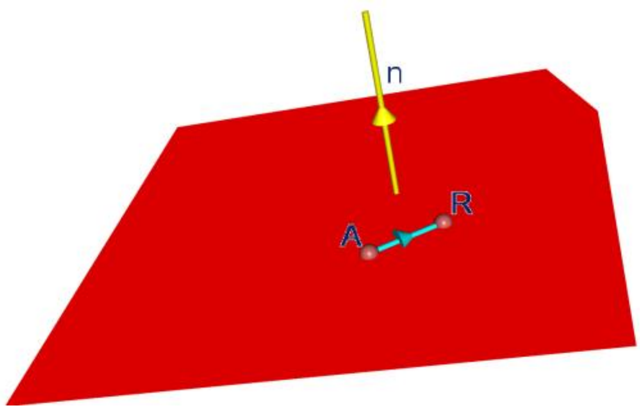
I get to a known point A, then

I add a multiple of a direction vector, **b** then

I add a multiple of a different direction vector **c**

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$
$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$

$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ $\qquad$ *Normal form*

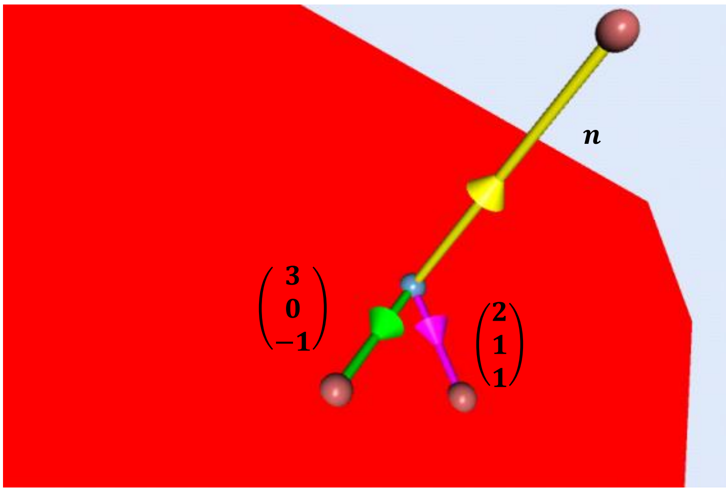


$$\overrightarrow{\boldsymbol{AR}} \cdot \boldsymbol{n} = \boldsymbol{0}$$
$$\left(\overrightarrow{\boldsymbol{OR}} - \overrightarrow{\boldsymbol{OA}}\right) \cdot \boldsymbol{n} = \boldsymbol{0}$$
$$(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = \boldsymbol{0}$$
$$\boldsymbol{r} \cdot \boldsymbol{n} - \boldsymbol{a} \cdot \boldsymbol{n} = \boldsymbol{0}$$
$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Example

Convert the following into normal and Cartesian form

$$\boldsymbol{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$$



The vector product finds a vector perpendicular to 2 vectors

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}3\\0\\-1\end{pmatrix}$$

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}3\\0\\-1\end{pmatrix} = \begin{pmatrix}1\times -1 - 0\times 1\\ -(2\times -1 - 3\times 1)\\ 2\times 0 - 3\times 1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\5\\-3\end{pmatrix}$$

Check this is correct by finding the scalar products

$$\begin{pmatrix}2\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\-3\end{pmatrix} = 0 \qquad \begin{pmatrix}3\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\-3\end{pmatrix} = 0$$

As the scalar products are equal to zero, the vector is perpendicular

Use the formula for the normal form

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\-3\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\-3\end{pmatrix}$$

Find the scalar products to find the cartesian form

$$x \cdot (-1) + y \cdot 5 + z \cdot (-3) = 1 \cdot (-1) + 2 \cdot 5 + 0 \cdot (-3)$$

$$-x + 5y - 3z = -9$$