IB Maths DP

5. Calculus

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CONTENTS

5.1 Differentiation

5.1.1 Introduction to Differentiation

Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of x) approaches as x approaches a particular value from either side
    - Limits are of interest when the function is undefined at a particular value
    - For example, the function $f(x) = \frac{x^4 - 1}{x - 1}$ will approach a limit as x approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

What might I be asked about limits?

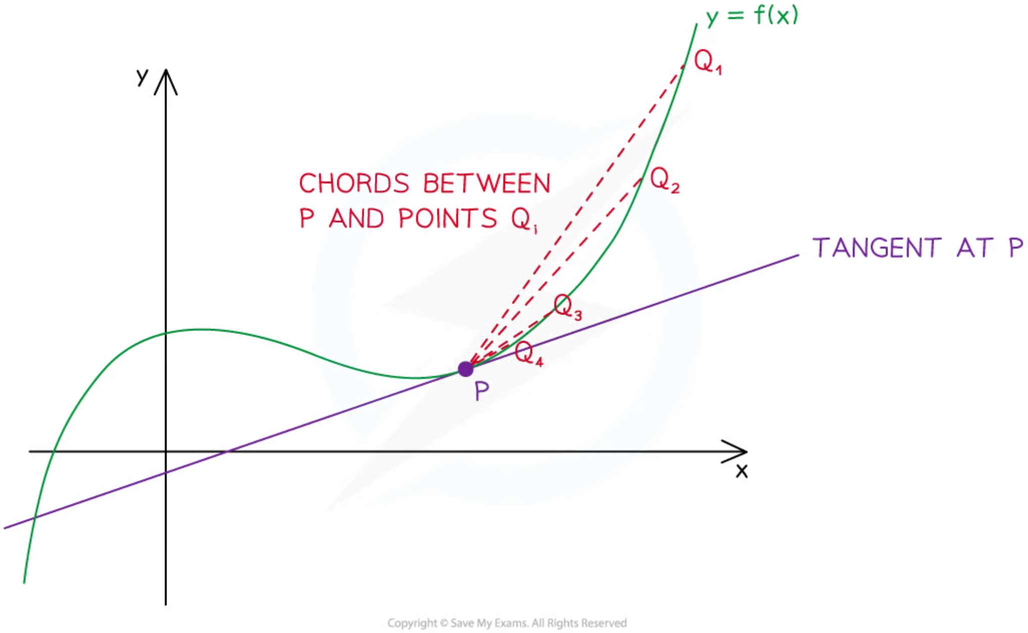
- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = f(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- **Calculus** is about **rates** of **change**
    - the way a car's position on a road changes is its speed (velocity)
    - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the **gradient** to the value of x
- The derivative is also called the **gradient function**

How are limits and derivatives linked?

- Consider the point P on the graph of $y = f(x)$ as shown below
    - $[PQ_i]$ is a series of chords



- The **gradient** of the **function** $f(x)$ at the point P is **equal** to the **gradient** of the **tangent** at point P
- The **gradient** of the **tangent** at point P is the **limit** of the **gradient** of the chords $[PQ_i]$ as point Q 'slides' down the curve and gets ever closer to point P
- The **gradient** of the function changes as x changes
- The **derivative** is the function that calculates the gradient from the value x

What is the notation for derivatives?

- For the function $y = f(x)$, the **derivative**, with respect to x, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

- Different variables may be used
    - e.g. If $V = f(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = f'(s)$

What might I be asked about derivatives?

- You may be asked to use the graphing features of your GDC to find the **gradients** of a **function** at different values of x
- From a **series** of **gradient** values, you may be asked to suggest an **expression** for the **derivative** (gradient function) of a function

Worked Example

The graph of $y = f(x)$ where $f(x) = x^3 - 2$ passes through the points $P(2, 6)$, $A(2.3, 10.167)$, $B(2.1, 7.261)$ and $C(2.05, 6.615125)$.

a)
Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is " $\frac{y_2-y_1}{x_2-x_1}$ "

$[PA]$: $\frac{10.167-6}{2.3-2} = 13.89$

$[PB]$: $\frac{7.261-6}{2.1-2} = 12.61$

$[PC]$: $\frac{6.615125-6}{2.05-2} = 12.3$

Gradient of chords are: [PA] 13.89
[PB] 12.61
[PC] 12.3025

b)
Estimate the gradient of the tangent to the curve at the point P.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x=2$ is 12

c)
Use your GDC to find the gradient of the tangent at the pont P.

Using GDC, plot $y=x^3-2$,
draw a tangent at $x=2$
GDC can tell you either/both of the equation of the tangent and $\frac{dy}{dx}$

GDC gradient is 12

Differentiating Powers of x

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What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- **Powers** of x are **differentiated** according to the following formula:
    - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Q}$
    - This is given in the **formula booklet**
- If the power of x is **multiplied** by a **constant** then the derivative is also multiplied by that constant
    - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Q}$ and a is a constant
- The **alternative notation** (to $f'(x)$) is to use $\frac{\mathrm{d}y}{\mathrm{d}x}$
    - If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$
        - e.g. If $y = -4x^{\frac{1}{2}}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = -2x^{-\frac{1}{2}}$
- Don't forget these **two** special cases:
    - If $f(x) = ax$ then $f'(x) = a$
        - e.g. If $y = 6x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$
    - If $f(x) = a$ then $f'(x) = 0$
        - e.g. If $y = 5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
    - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
    - e.g. If $f(x) = 2\sqrt{x}$ then rewrite as $f(x) = 2x^{\frac{1}{2}}$ and differentiate
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
    - e.g. If $f(x) = \frac{4}{x}$ then rewrite as $f(x) = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of x apply to **all rational** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of x
    - e.g. If $f(x) = 5x^4 - 3x^{\frac{2}{3}} + 4$ then

$$f'(x) = 5 \times 4x^{4-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1} + 0$$

$$f'(x) = 20x^3 - 2x^{-\frac{1}{3}}$$

- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = (2x - 3)(x^2 - 4)$ then expand to $f(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can be differentiated

What might I be asked to do once I've found the derivative (gradient function)?

- The **derivative** can be used to find the **gradient** of a **function** at any point
  - The gradient of a function at a point is equal to the **gradient** of the **tangent** to the curve at that point
  - A question may refer to the gradient of the tangent
  - A GDC can be used to help find gradients or verify answers
- Take extra care when differentiating negative and fractional powers of x
  - $\frac{1}{2} - 1 = -\frac{1}{2}$ but $\frac{1}{3} - 1 = -\frac{2}{3}$

Exam Tip

- A common mistake is not simplifying expressions before differentiating
  - The derivative of $(x^2 + 3)(x^3 - 2x + 1)$ can **not** be found by multiplying the derivatives of $(x^2 + 3)$ and $(x^3 - 2x + 1)$

Worked Example

The function $f(x)$ is given by

$$f(x) = 2x^3 + \frac{4}{\sqrt{x}}$$

where $x > 0$.

a)

Find the derivative of $f(x)$.

Rewrite f(x) so every term is a power of x

$f(x) = 2x^3 + 4x^{-1/2}$

Differentiate by applying the formula

$f'(x) = 6x^2 - 2x^{-3/2}$

anx^{n-1}

take care with negatives

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$\boxed{\therefore\ f'(x) = 6x^2 - 2x^{-3/2}}$

b)

Find the gradient of the tangent to the curve $y = f(x)$ at the point where $x = 4$.

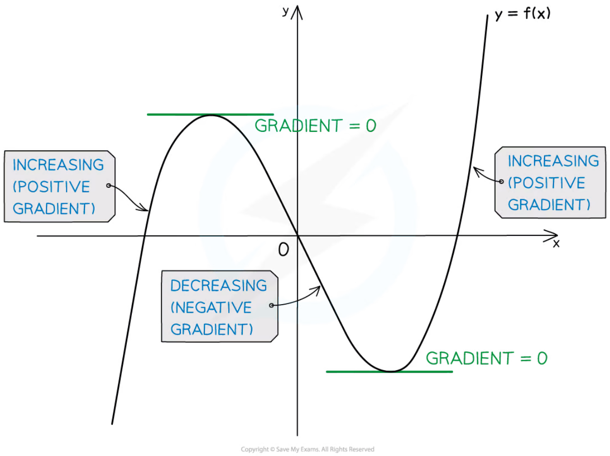
$f'(4) = 6(4)^2 - \frac{2}{(4)^{3/2}} = 96 - \frac{2}{(2)^3} = 96 - \frac{1}{4} = 95.75$

∴ The gradient of the tangent to the curve $y = f(x)$ when $x = 4$ is 95.75

Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, $f(x)$, is **increasing** if $\boldsymbol{f'(x) > 0}$
  - This means the **value** of the **function** ('output') **increases** as x **increases**
- A function, $f(x)$, is **decreasing** if $\boldsymbol{f'(x) < 0}$
  - This means the **value** of the **function** ('output') **decreases** as x **increases**
- A function, $f(x)$, is **stationary** if $\boldsymbol{f'(x) = 0}$



How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

STEP 1
Find the derivative **f'(x)**

STEP 2
Solve the inequalities
$\boldsymbol{f'(x) > 0}$ (for increasing intervals) and/or
$\boldsymbol{f'(x) < 0}$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
  - a range of values of x (**interval**) is given where a function satisfies each condition
  - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
    - $f(x)$ is **decreasing** for $x < 0$
    - $f(x)$ is **stationary** at $x = 0$
    - $f(x)$ is **increasing** for $x > 0$

Worked Example

$f(x) = x^2 - x - 2$

a)
Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ $\therefore$ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 6 > 0$ $\therefore$ increasing

$\therefore$ At $x = 0$, $f(x)$ is decreasing
At $x = 3$, $f(x)$ is increasing

b)
Find the values of x for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

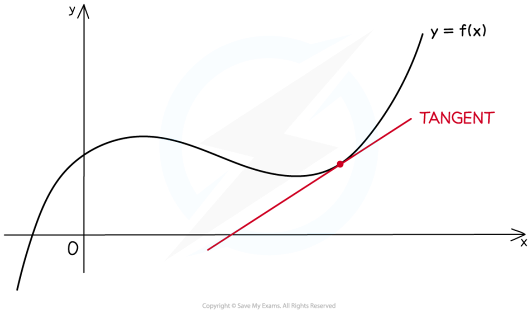
$f'(x) > 0$
$2x - 1 > 0$
$x > \frac{1}{2}$

$\therefore$ $f(x)$ is increasing for $x > \frac{1}{2}$

Tangents & Normals

What is a tangent?

- At any point on a curve (the graph of a (non-linear) **function**), the **tangent** is the straight line that passes through that point and has the same **gradient** as the curve at that point



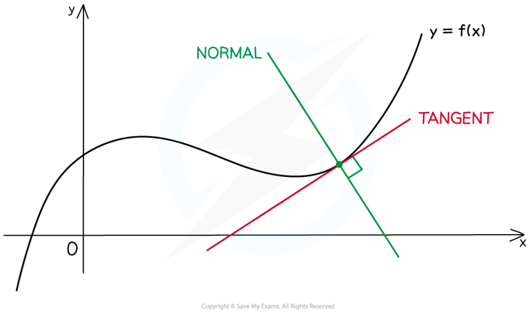
How do I find the equation of a tangent?

- The **equation** of the **tangent** to the function $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = f'(x_1)(x - x_1)$$

What is a normal?

- At any point on a curve (the graph of a (non-linear) **function**), the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent** at that point



How do I find the equation of a normal?

- The **equation** of the **normal** to the function $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

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Exam Tip

- The equations of a tangent and a normal are not in the formula booklet
    - However both can be derived from the equation of a straight line $y - y_1 = m(x - x_1)$
    - This is given in the formula booklet

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Worked Example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a)
Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find f'(x) by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$$

$\therefore \quad y - 5 = 2(x - 1)$

$$\boxed{\text{Tangent at } x = 1, \text{ is } y = 2x + 3}$$

b)
Find an equation for the normal to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

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For a normal, "$y-y_1 = \frac{-1}{f'(a)}(x-x_1)$"

Using results from part (a):

$$y-5 = \frac{-1}{2}(x-1)$$

$$y = -\frac{1}{2}x + \frac{11}{2}$$

$$2y = -x + 11$$

$\therefore$ Equation of normal is $x+2y-11=0$

Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the **lowest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the **greatest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions tend to infinity for large values of x (and/or minus infinity for large negative values of x)
- The **nature** of a stationary point refers to whether it is a local **minimum** or local **maximum** point

How do I find the coordinates and nature of stationary points?

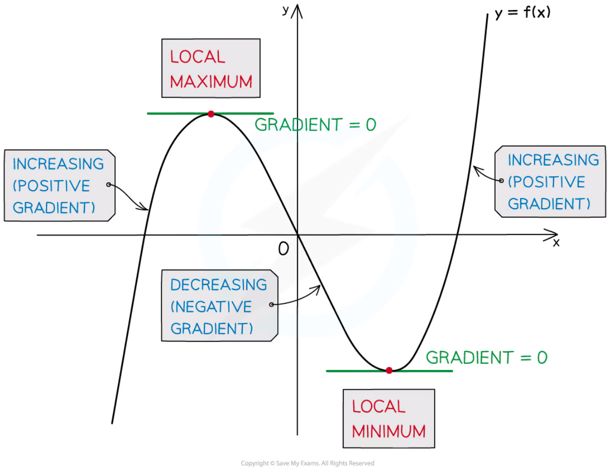
- The instructions below describe how to find **local minimum** and **maximum points** using a **GDC** on the graph of the function $y = f(x)$.

STEP 1
Plot the graph of $y = f(x)$
Sketch the graph as part of the solution

STEP 2
Use the options from the graphing screen to "solve for minimum"
The GDC will display the x and y coordinates of the first minimum point
Scroll onwards to see there are anymore minimum points
Note down the coordinates and the type of stationary point

STEP 3
Repeat **STEP 2** but use "solve for maximum" on your GDC

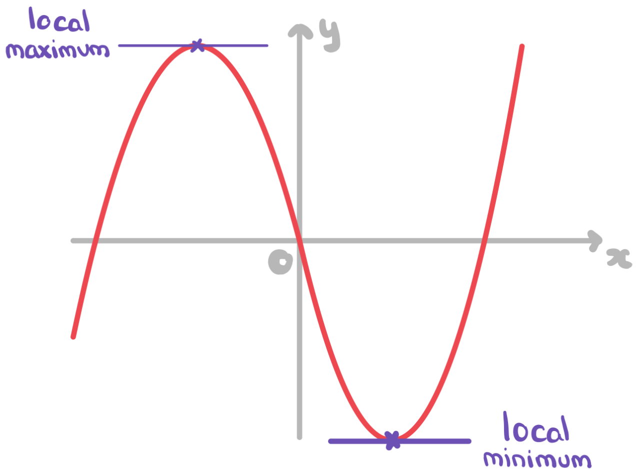
- In **STEP 2** the **nature** of the stationary point should be easy to tell from the graph
  - a local **minimum** changes the function from **decreasing** to **increasing**
    - the gradient changes from **negative** to **positive**
  - a local **maximum** changes the function from **increasing** to **decreasing**
    - the gradient changes from **positive** to **negative**



Worked Example

Find the stationary points of $f(x) = x(x^2 - 27)$, and state their nature.

Plot the graph of $y = x(x^2-27)$ on GDC and sketch here.



∴ Stationary points are

(3, -54) LOCAL MINIMUM POINT
(-3, 54) LOCAL MAXIMUM POINT

5.1.3 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
    - **minimising** the cost of raw materials in manufacturing a product
    - the **maximum** height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being **optimised** needs to be dependent on a **single** variable
    - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
        - minimising the cost of the **main** raw material – timber in manufacturing furniture say – the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
    - Other **modelling assumptions** may have to be made too; for example
        - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In optimisation problems, letters other than x, y and f are often used including capital letters
    - V is often used for volume, S for surface area
    - r for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which variable is independent (x) and which is dependent (y)
    - a GDC may always use x and y but ensure you use the correct variable throughout your working and final answer
- Problems often start by **linking two connected** quantities together – for example **volume** and **surface area**
    - where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of a **single** variable
- Once the quantity of interest is written as a function of a **single** variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

STEP 1
Rewrite the quantity to be optimised as a single variable, using any constraints given in the question

STEP 2
Use your GDC to find the (local) maximum or minimum points as required
Plot the graph of the function and use the graphing features of the GDC to "solve for minimum/maximum" as required

STEP 3

Note down the solution from your GDC and interpret the answer(s) in the context of the question

Exam Tip

- The first part of rewriting a quantity as a single variable is often a "show that" question – this means you may still be able to access later parts of the question even if you can't do this bit

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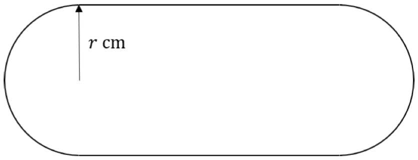
Worked Example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



The total area of the bed is to be 100π m^2.

a)
Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

The width of the rectangle is $2r$ m and its length L m

The AREA of the bed, 100π m^2 is given by

$$\underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} + \underbrace{2rL}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} = \underbrace{100\pi}_{\text{total area (this is the constraint)}}$$

$$\therefore \pi r^2 + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^2$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

Write L in terms of r

The PERIMETER of the bed is

$$P = \underbrace{\pi r + \pi r}_{\text{semi-circular arcs}} + \underbrace{2L}_{\text{two straight lengths}}$$

Use L from the area constraint to write P in terms of r only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$

$$\boxed{\therefore P = \pi\left(r + \frac{100}{r}\right)}$$

b) Find $\frac{\mathrm{d}P}{\mathrm{d}r}$.

Rewrite P as powers of r

$P = \pi(r + 100r^{-1})$

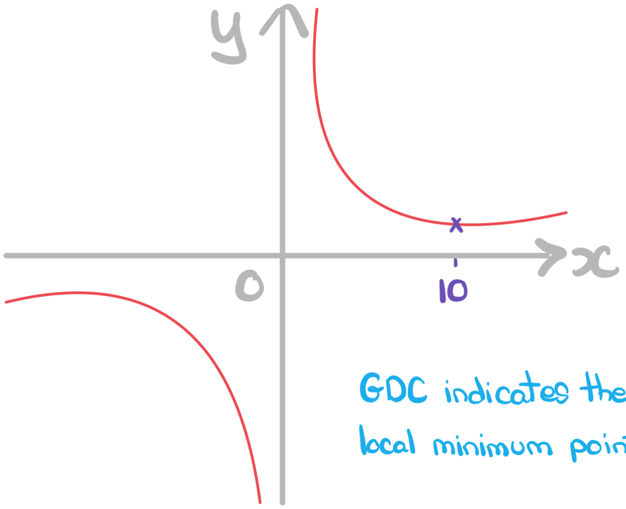
$\frac{dP}{dr} = \pi(1 - 100r^{-2})$

$$\boxed{\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)}$$

c)

Find the value of r that minimises the perimeter.

Use GDC to plot $y = \pi\left(x + \frac{100}{x}\right)$ and sketch the result



GDC indicates the ONLY local minimum point is at $x = 10$

$$\boxed{\therefore \text{The value of } r \text{ that minimises the perimeter is } r = 10}$$

d)

Hence find the minimum perimeter.

The minimum perimeter will be the y-coordinate of the local minimum point found in part (c)

From GDC, $y = 62.831\,853\ldots$ (when $x = 10$)

∴ Minimum perimeter is 62.8 m (3 s.f.)

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5.2 Further Differentiation

5.2.1 Differentiating Special Functions

Differentiating Trig Functions

How do I differentiate sin, cos and tan?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- The derivative of $y = \tan x$ is $\frac{dy}{dx} = \frac{1}{\cos^2 x}$
    - This result can be derived using **quotient rule**
- All three of these derivatives are given in the **formula booklet**
- For the **linear** function $ax + b$, where a and b are constants,
    - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
    - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
    - **the derivative of** $y = \tan(ax + b)$ **is** $\frac{dy}{dx} = \frac{a}{\cos^2(ax + b)}$
- For the **general** function $f(x)$,
    - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
    - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
    - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
    - Ensure you know how to change the angle mode on your GDC

Exam Tip

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode

Worked Example

a)

Find $f'(x)$ for the functions

i. $f(x) = \sin x$
ii. $f(x) = \cos(5x + 1)$

i. $\boxed{f'(x) = \cos x}$

ii. $\boxed{f'(x) = -5\sin(5x+1)}$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = \frac{12x}{\cos^2\left(6x^2 - \frac{\pi}{4}\right)}$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = \frac{12\left(\frac{\sqrt{\pi}}{2}\right)}{\cos^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]}$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$

$\boxed{\therefore \frac{dy}{dx} = 12\sqrt{\pi} \text{ at } x = \frac{\sqrt{\pi}}{2}}$

Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

- The derivative of $y = \mathrm{e}^x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \mathrm{e}^{(ax+b)}$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = a\mathrm{e}^{(ax+b)}$
  - the derivative of $y = \ln(ax+b)$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ (a's cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = \mathrm{e}^{f(x)}$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)\mathrm{e}^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)\ln(f(x))$
- The last two sets of results can be derived using the **chain rule**

Exam Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to x is $\frac{1}{x}$, NOT $\frac{k}{x}$
  - the derivative of e^{kx} with respect to x is $k\mathrm{e}^{kx}$, NOT $kx\mathrm{e}^{kx-1}$

Worked Example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^c$, where $a,\ b$ and c are integers to be found.

$$y = e^{-3x+1} + 2(\ln 5x)$$

$$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

Chain Rule

What is the chain rule?

- The **chain rule** states if $\boldsymbol{y}$ is a function of $\boldsymbol{u}$ and $\boldsymbol{u}$ is a function of $\boldsymbol{x}$ then

$$y = f(u(x))$$

$$\frac{\mathbf{d}\boldsymbol{y}}{\mathbf{d}\boldsymbol{x}} = \frac{\mathbf{d}\boldsymbol{y}}{\mathbf{d}\boldsymbol{u}} \times \frac{\mathbf{d}\boldsymbol{u}}{\mathbf{d}\boldsymbol{x}}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ – **not** a composite function, x 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1
Identify the two functions
Rewrite y as a function of u; $y = f(u)$
Write u as a function of x; $u = g(x)$

STEP 2
Differentiate y with respect to u to get $\frac{\mathrm{d}y}{\mathrm{d}u}$
Differentiate u with respect to x to get $\frac{\mathrm{d}u}{\mathrm{d}x}$

STEP 3
Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ and substitute u back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

Are there any standard results for using chain rule?

- There are **five** general results that can be useful

- If $y = (f(x))^n$ then $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$
- If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
- If $y = \ln(f(x))$ then $\frac{dy}{dx} = f'(x)\ln(f(x))$
- If $y = \sin(f(x))$ then $\frac{dy}{dx} = f'(x)\cos(f(x))$
- If $y = \cos(f(x))$ then $\frac{dy}{dx} = f'(x)\sin(f(x))$

Exam Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
    - every time you use it, say it to yourself in your head *"differentiate the first function ignoring the second, then multiply by the derivative of the second function"*

Worked Example

a)
Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u) = u^7$

$u = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x - 5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$$\frac{dy}{dx} = 7u^6(2x-5)$$

and substitute u back for g(x)

$$\boxed{\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6}$$

b)
Find the derivative of $y = \sin(e^{2x})$.

$y = \sin(e^{2x})$ "... differentiate $\sin \square$, ignore e^{2x}..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of e^{2x}..."

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"
or by applying chain rule again

$$\boxed{\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})}$$

Product Rule

What is the product rule?

- The **product rule** states if y is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x\cos x$ is a product, "sin x times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v'

Exam Tip

- Use u, v, u' and v' for the elements of product rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (u and v', v and u')
- For trickier functions chain rule may be reuqired inside product rule
  - i.e. chain rule may be needed to differentiate u and v

Worked Example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$$\begin{array}{ccc} u = e^x & \times & v = \sin x \\ u' = e^x & & v' = \cos x \end{array}$$

↑ arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$$\boxed{\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)}$$

It is straightforward to take a factor of e^x out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1

$$\begin{array}{ccc} u = 5x^2 & \times & v = \cos 3x^2 \\ u' = 10x & & v' = -\sin 3x^2 \times 6x \\ & & v' = -6x \sin 3x^2 \end{array}$$

chain rule

STEP 2 $y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$$\boxed{\therefore \frac{dy}{dx} = 10x\left(\cos 3x^2 - 3x^2 \sin 3x^2\right)}$$

Quotient Rule

YOUR NOTES
⬇

What is the quotient rule?

- The **quotient rule** states if y is the quotient $\dfrac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathbf{d}y}{\mathbf{d}x} = \frac{v\dfrac{\mathbf{d}u}{\mathbf{d}x} - u\dfrac{\mathbf{d}v}{\mathbf{d}x}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

STEP 1
Identify the two functions, u and v
Differentiate both u and v with respect to x to find u' and v'

STEP 2
Obtain $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by applying the quotient rule formula $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important
Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v',

Exam Tip

- Use u, v, u' and v' for the elements of quotient rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (v and u' , u and v')
- Look out for functions of the form $y = f(x)(g(x))^{-1}$
  - These can be differentiated using a combination of **chain rule** and **product rule**
    (it would be good practice to try!)
  - ... but it can also be seen as a quotient rule question in disguise
  - ... and vice versa!
    - A quotient could be seen as a product by rewriting the denominator as $(g(x))^{-1}$

Worked Example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to x.

STEP 1 Identify u and v, differentiate

$u = \cos 2x \qquad v = 3x+2$

$u' = -2\sin 2x \qquad v' = 3$

↑ chain rule

↑ opposite diagonals match up

STEP 2 Apply quotient rule: '$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$'

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\boxed{\therefore\ f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

Related Rates of Change

What is meant by rates of change?

- A **rate of change** is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are **derivatives**
    - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be tthe rate at which the volume of a sphere cahnges relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
    - A positive rate of change would indicate an increase
        - e.g. the change in volume of water as a bathtub fills
    - A negative rate of change would indicate a decrease
        - e.g. the change in volume of water in a leaking bucket

What is meant by related rates of change?

- **Related rates of change** are connected by a linking variable or parameter
    - this is usually **time**, represented by t
    - **seconds** is the standard unit for time but this will depend on context
- e.g. Water running into a large bowl
    - both the height and volume of water in the bowl change with time
    - time is the linking parameter

How do I solve problems involving related rates of change?

- Use of chain rule

$$y = g(u) \quad u = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- Chain rule is given in the **formula booklet** in the format above
    - **Different letters** may be used relative to the context
        - e.g. V for **volume**, S for **surface area**, h for **height**, r for **radius**
- Problems often involve one quantity being **constant**
    - so another quantity can be expressed in terms of a **single** variable
    - this makes finding a derivative a lot easier
- For **time** problems at least, it is more convenient to use

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}y}{\mathrm{d}x}$$

and if it is more convenient to find $\frac{\mathrm{d}x}{\mathrm{d}y}$ than $\frac{\mathrm{d}y}{\mathrm{d}x}$ then use chain rule in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}y}$$

- **Neither** of these alternative versions of chain rule are in the **formula booklet**

STEP 1
Write down the rate of change given and the rate of change required
(If unsure of the rates of change involved, use the units given as a clue
e.g. $\mathrm{m\,s^{-1}}$ (metres per second) would be the rate of change of length, per time, $\frac{\mathrm{d}l}{\mathrm{d}t}$)

STEP 2
Use chain rule to form an equation connecting these rates of change with a third rate
The third rate of change will come from a related quantity such as volume, surface area, perimeter

STEP 3
Write down the formula for the related quantity (volume, etc) accounting for any fixed quantities
Find the third rate of change of the related quantity (derivative) using differentiation

STEP 4
Substitute the derivative and known rate of cahnge into the equation and solve it

Exam Tip

- If you struggle to determine which rate to use in an exam then you can look at the units to help
  - e.g. A rate of 5 **cm^3** *per* **second** implies **volume** *per* **time** so the rate would be $\frac{\mathrm{d}V}{\mathrm{d}t}$

YOUR NOTES
↓

Worked Example

A cuboid has a square cross-sectional area of side length x cm and a fixed height of 5 cm.
The volume of the cuboid is increasing at a rate of 20 cm^3 s^{-1}.
Find the rate at which the side length is increasing at the point when its side length is 3 cm.

STEP 1: Write down rates of change given and required

$\frac{dV}{dt} = 20$ (Units are cm^3 (volume) s^{-1} (per second))

$\frac{dx}{dt}$ is required

STEP 2: Form equation from chain rule and a third 'connecting' rate

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

STEP 3: Formula for linking quantity, and its derivative

Volume (of a cuboid) is the link

$V = x^2 \times 5 = 5x^2$ (Cross-section is square, height is constant)

Differentiate, $\frac{dV}{dx} = 10x$

STEP 4: Substitute and solve

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

$$20 = \frac{dx}{dt} \times 10(3)$$

x, side length is 3

$$\boxed{\therefore \frac{dx}{dt} = \frac{2}{3} \text{ cm s}^{-1}}$$

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ (First order derivative)
  - $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d}^2$) with respect to x twice (so $\boldsymbol{x}^2$)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
      - test for local minimum and maximum points
      - help determine the nature of stationary points
      - help determine the concavity of a function
      - graph derivatives

How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions**, **roots**, etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

Exam Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term

YOUR NOTES
↓

Worked Example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a)
Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$

DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b)
Evaluate $f''(3)$.
Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$

$x^{\frac{3}{2}} = x\sqrt{x}$ $\quad x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$

RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** exhibits this property, but in addition the **function changes** from **increasing** to **decreasing,** or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection *could*** also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function $y = f(x)$, **stationary points** can be found using the following process

STEP 1

Find the **gradient function,** $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

STEP 2
Solve the equation $f'(x) = 0$ to find the x-coordiante(s) of any stationary points

STEP 3
If the y-coordaintes of the stationary points are also required then substitute the x-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x, f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x, f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of x (and/or **minus infinity** for **large negative** values of x)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of x
  - similar for a **global** maximum point

How do I find the nature of a stationary point?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ ...

STEP 1

Find $f'(x)$ and solve $f'(x) = 0$ to find the x-coordinates of any stationary points

STEP 2 (Second derivative)

Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

STEP 3 (Second derivative)

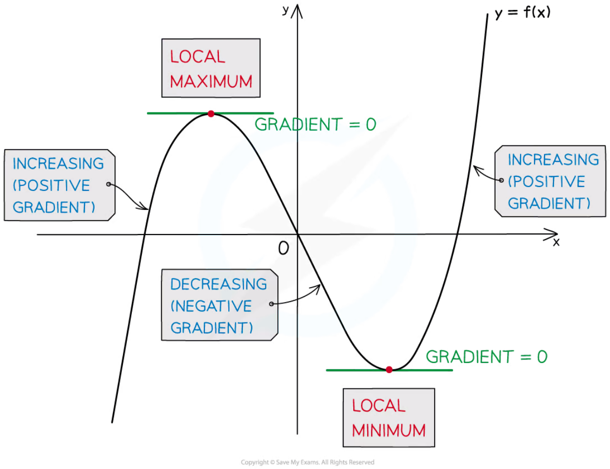
- If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
- If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
- If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

STEP 4 (First derivative)

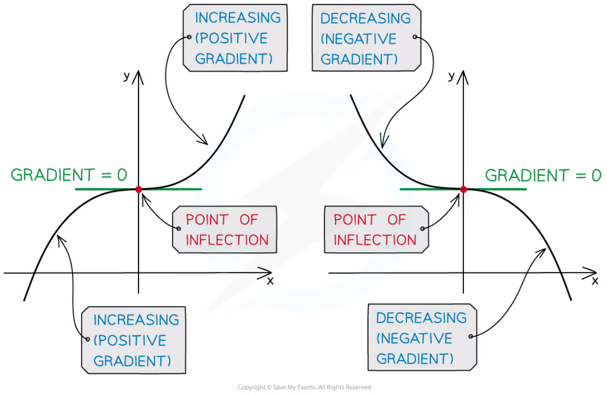
Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small h

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h) < 0,\ f'(x) = 0,\ f'(x+h) > 0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**

    - $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$



- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**



Exam Tip

- Exam questions may use the phrase "classify turning points" instead of "find the nature of turning points"
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says "show that..." or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you're aiming for and to check your work

YOUR NOTES
↓

Worked Example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

At stationary points, $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x-3)(x+2) = 0$

$x = 3,\quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2,\quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(3) = 6(2 \times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:
$(3, -56)$ local minimum point
$(-2, 69)$ local maximum point

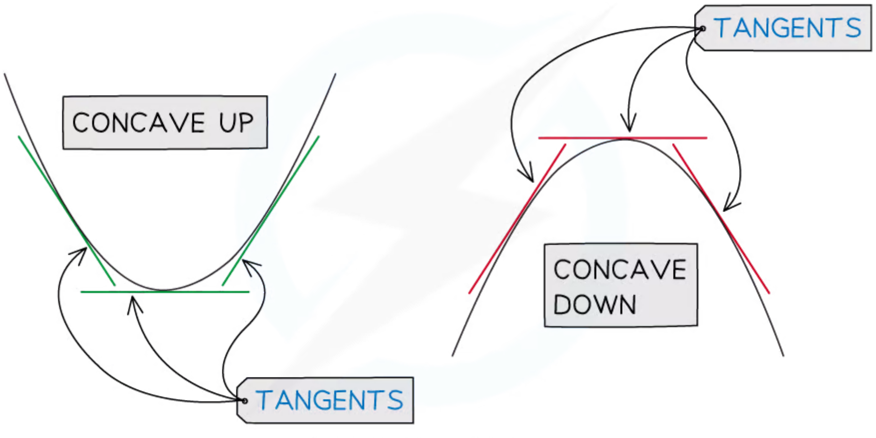
Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

5.2.6 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) \leq 0$ for all values of x in an interval
  - a curve is **CONCAVE UP** if $f''(x) \geq 0$ for all values of x in an interval
- **CONCAVE DOWN** is often called **concave**
- **CONCAVE UP** is often called **convex**



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Exam Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley ☹
  - **Concave up** is the shape of (the mouth of) a happy smiley ☺

YOUR NOTES
↓

Worked Example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a)
Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

At $x = -2$, $y = f(x)$ is concave down
At $x = 2$, $y = f(x)$ is concave up

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b)
Find the values of x for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)
Concave up is $f''(x) > 0$
$6x > 0$ when $x > 0$

$\therefore$ $y = f(x)$ is concave up for $x > 0$

Use your GDC to check your answer

Points of Inflection

What is a point of inflection?

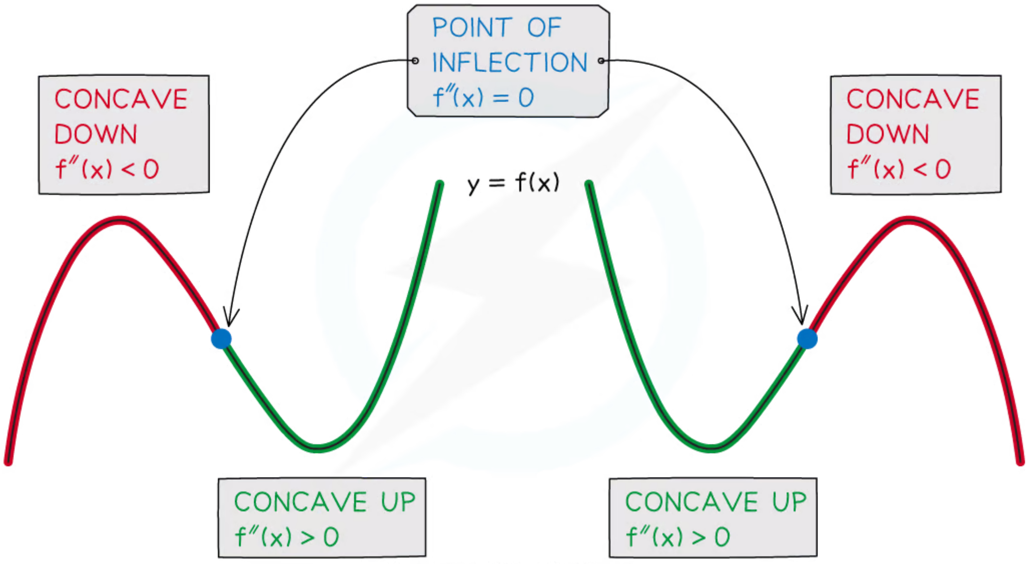
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
    - the **second derivative** is zero
        - $f''(x) = 0$

AND

- the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
    - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
        - the **first derivative** test would be needed
    - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

What about the first derivative, like with turning points?

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ($f'(x) = 0$)
    - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
        - the tangent to the curve at this point would be horizontal

  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

How do I find the coordinates of a point of inflection?

- For the function $f(x)$

STEP 1

Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinates of possible points of inflection

STEP 2

Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

- If $f''(x) < 0$ then $f(x)$ is concave down
- If $f''(x) > 0$ then $f(x)$ is concave up

If concavity changes, $x = a$ is a **point of inflection**

STEP 3

If required, the y-coordinate of a point of inflection can be found by substituting the x-coordinate into $f(x)$

Exam Tip

- You can find the x-coordinates of the point of inflections of $y = f(x)$ by drawing the graph $y = f'(x)$ and finding the x-coordinates of any local maximum or local minimum points
- Another way is to draw the graph $y = f''(x)$ and find the x-coordinates of the points where the graph crosses (not just touches) the x-axis

Worked Example

Find the coordinates of the point of inflection on the graph of
$y = 2x^3 - 18x^2 + 24x + 5$.
Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$
$f'(x) = 6x^2 - 36x + 24$
$f''(x) = 12x - 36$
$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$
$f''(2.9) < 0$ (concave down)
$f''(3.1) > 0$ (concave up)
$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ **AND** the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible

5.3 Integration

5.3.1 Trapezoid Rule: Numerical Integration

Trapezoid Rule: Numerical Integration

What is the trapezoid rule?

- The **trapezoidal rule** is a numerical method used to find the **approximate area** enclosed by a curve, the x-axis and two vertical lines
    - it is also known as '**trapezoid rule**' and '**trapezium rule**'
- The trapezoidal rule finds an **approximation** of the area by **summing of the areas** of trapezoids beneath the curve
    - $y_0 = f(a),\ \ y_1 = f(a+h),\ \ y_2 = f(a+2h)$ etc

$$\int_a^b f(x)\ dx \approx \frac{1}{2}h\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

- Note that there are n trapezoids (also called strips) but $(n+1)$ function values (y_i)
- The trapezoidal rule is given in the **formula booklet**

What else can be asked to do with the trapezoid rule?

- Comparing the **true** answer with the answer from the trapezoid rule
    - This may involve finding the **percentage error** in the approximation
    - The true answer may be given in the question, found from a GDC or from work on **integration**

Exam Tip

- Ensure you are clear about the difference between the number of data points (y values) and the number of strips (number of trapezoids) used in a Trapezoid Rule question
- Although it shouldn't be too much trouble to type the trapezoid rule into your GDC in one go, it may be wise to work parts of it out separately and write these down as part of your working out

Worked Example

a)

Using the trapezoidal rule, find an approximate value for

$$\int_0^4 \frac{6x^2}{x^3+2}\,dx$$

to 3 decimal places, using $n = 4$.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$ USING $\int_{a=0}^{b=4}$ AND $n=4$

STEP 2: USING WIDTH $h = 1$, VALUES ARE 0, 1, 2, 3, 4

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y\,dx \approx \frac{1}{2}h\left(y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$$

SUBSTITUTE ALL VALUES

$$\frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...))$$

$$= \frac{1}{2}(1.454... + 12.524...)$$

$$= 6.9893... = 6.989 \quad (3\text{ dp})$$

b)

Given that the area bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is 6.993 to three decimal places, calculate the percentage error in the trapezoidal rule approximation.

USING $\%\text{ ERROR} = \frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$

b) $\frac{6.989 - 6.993}{6.993} \times 100$

$= -0.\dot{0}5720\dot{0}$

IGNORE MINUS, YOU COULD DO EXACT − ESTIMATE TO GET RID OFF THIS

$= 0.06\% \quad (2\text{ dp})$

Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
  - Integration is referred to as **antidifferentiation**
  - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

What is the notation for integration?

- An **integral** is normally written in the form

$$\int f(x)\ \mathrm{d}x$$

  - the large operator $\int$ means "integrate"
  - "$\mathrm{d}x$" indicates which variable to integrate with respect to
  - $f(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $\mathrm{F}(x)$
  - there's then no need to keep writing the whole integral; refer to it as $\mathrm{F}(x)$
- $\mathrm{F}(x)$ may also be called the **indefinite integral** of $f(x)$

What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
  - If $f(x) = a$ then $f'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
  - a zero term wouldn't be written as part of a function
  - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant** of **integration** and is usually denoted by the letter c
  - it is often referred to as "plus c"
- Without more information it is impossible to deduce the value of this constant
  - there are endless antiderivatives, $\mathrm{F}(x)$, for a function $f(x)$

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formulae:
  - If $f(x) = x^n$ then $\int f(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q},\ n \neq -1$ and c is the **constant** of **integration**
  - This is given in the **formula booklet**
- If the power of x is multiplied by a constant then the integral is also multiplied by that constant
  - If $f(x) = ax^n$ then $\int f(x)\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q},\ n \neq -1$ and a is a constant and c is the **constant** of **integration**
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation can still be used with integration
- Note that the formulae above do not apply when $x = -1$ as this would lead to division by zero
- Remember the special case:
  - $\int a\,\mathrm{d}x = ax + c$
    - e.g. $\int 4\,\mathrm{d}x = 4x + c$
  - This allows **constant** terms to be integrated
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - eg. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of $\boldsymbol{x}$ will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- The formulae for integrating powers of x apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of x
  - e.g. If $f(x) = 8x^3 - 2x + 4$ then $\int f(x)\,\mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$
- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = 8x^2(2x-3)$ then
    $\int f(x)\,\mathrm{d}x = \int (16x^3 - 24x^2)\,\mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$

What might I be asked to do once I've found the anti-derivative (integrated)?

- With more information the **constant of integration**, c, can be found
- The **area under** a **curve** can be found using integration

Exam Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
    - In general you can think of it as 'raising the power by one and dividing by the new power'
    - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

Worked Example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for y in terms of x.

Firstly rewrite all terms as powers of x

$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}$ ← fractional AND negative!

$$y = \int \left(3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}\right) dx$$

$$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

special case

take care with negatives, $-\frac{1}{2} + 1 = \frac{1}{2}$

constant of integration

$$\boxed{\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c}$$

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
  - this constant term, usually called c, is the **constant** of **integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
  - collectively these may be referred to as the **family** of **antiderivatives** or **family** of **curves**
  - the constant of integration is determined by the **exact** location of the curve
    - if a **point** on the **curve** is **known,** the **constant** of **integration** can be found

How do I find the constant of integration?

- For $\mathrm{F}(x) + c = \int f(x)\, \mathrm{d}x$, the **constant** of **integration,** c – and so the particular **antiderivative** - can be found if a point the graph of $y = \mathrm{F}(x) + c$ passes through is known

STEP 1
If need be, rewrite $f(x)$ into an integrable form
Each term needs to be a power of x (or a constant)

STEP 2
Integrate each term of $f'(x)$, remembering the constant of integration, "$+c$"
(Increase power by 1 and divide by new power)

STEP 3
Substitute the x and y coordinates of a given point in to $\mathrm{F}(x) + c$ to form an equation in c
Solve the equation to find c

Exam Tip

- If a constant of integration can be found then the question will need to give you some extra information
  - If this is given then make sure you use it to find the value of c

YOUR NOTES
↓

Worked Example

The graph of $y = f(x)$ passes through the point $(3, -4)$. The gradient function of $f(x)$ is given by $f'(x) = 3x^2 - 4x - 4$.

Find $f(x)$.

STEP 1 f'(x) is already in an integrable form

$f'(x) = 3x^2 - 4x - 4$

STEP 2 Integrate, remembering "+c"

$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$

$f(x) = x^3 - 2x^2 - 4x + c$

STEP 3 Substitute x and y coordinates to find c

$f(3) = -4$

$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$

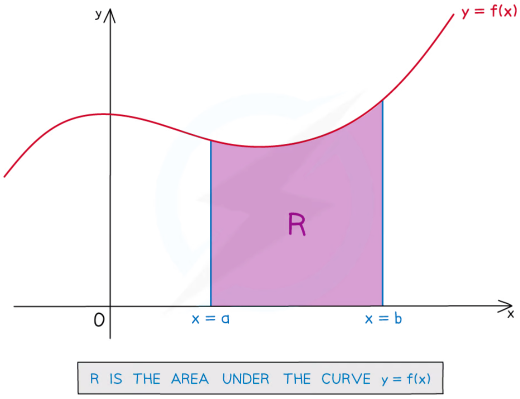
$27 - 18 - 12 + c = -4$

$c = -1$

$\boxed{\therefore f(x) = x^3 - 2x^2 - 4x - 1}$

Area Under a Curve Basics

What is meant by the area under a curve?



- The phrase "**area under a curve**" refers to the area bounded by
  - the graph of $y = f(x)$
  - the x-axis
  - the **vertical** line $x = a$
  - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
  - the use of **integration** described below would still apply
    - but the shape created would be a **trapezoid**
    - so it is easier to use "$A = \frac{1}{2}h(a+b)$"

What is a definite integral?

$$\int_a^b f(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
  - **a** is the **lower** limit
  - **b** is the **upper** limit
- $f(x)$ is the **integrand**
- $\mathrm{F}(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("$+c$") is not needed in **definite integration**
  - "$+c$" would appear alongside both **F(a)** and **F(b)**
  - subtracting means the "$+c$"'s cancel

How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the x-axis should be obvious boundaries for the area so the key here is in finding a and b - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits
You may prefer to plot the graph on your GDC and find the limits from there

STEP 2
Look carefully where the 'left' and 'right' boundaries of the area lie
If the boundaries are vertical lines, the limits will come directly from their equations
Look out for the y-axis being one of the (vertical) boudnaries – in this case the limit (x) will be 0
One, or both, of the limits, could be a root of the equation $f(x) = 0$
    i.e. where the graph of $y = f(x)$ crosses the x-axis
In this case solve the equation $f(x) = 0$ to find the limit(s)
A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3
The definite integral for finding the area can now be set up in the form

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Exam Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)
- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
    - Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
    - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutating definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

How do I use my GDC to find definite integrals?

Without graphing first …

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
    - The function to be integrated (**integrand**) ($f(x)$)
    - The **lower** limit (a from $x = a$)
    - The **upper** limit (b from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

With graphing first …

- Plot the graph of $y = f(x)$
    - You may also wish to plot the vertical lines $x = a$ and $x = b$
        - make sure your GDC is expecting an "$x =$ " style equation
    - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
        - it may appear as the integral symbol (e.g. $\int dx$)
        - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
        - your GDC may allow you to type the exact limits required from the keypad
            - the lower limit would be typed in first
            - read any information that appears on screen carefully to make sure

Exam Tip

- When revising for your exams always use your GDC to check any definite integrals you have carried out by hand
    - This will ensure you are confident using the calculator you plan to take into the exam and should also get you into the habit of using you GDC to check your work, something you should do if possible

Worked Example

a)

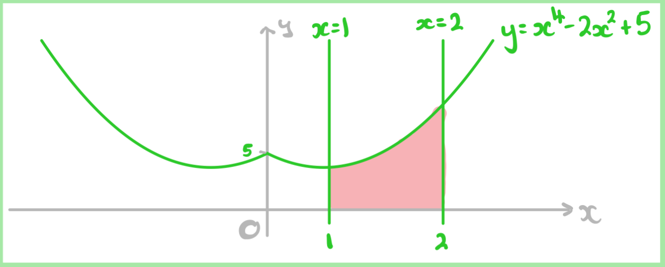
Using your GDC to help, or otherwise, sketch the graphs of

$y = x^4 - 2x^2 + 5$,

$x = 1$ and

$x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
You may then need to change the 'input type' to '$x =$'
to enter $x = 1$ and $x = 2$.
Plot the graph on your GDC and sketch the result, ensuring
to include all the main properties of each graph.



b)

The area enclosed by the three graphs from part (a) and the x-axis is to be found. Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5)\,dx$$

c)

Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5)\,dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral value was given as 6.53333333 – not exact!

5.4 Further Integration

5.4.1 Integrating Special Functions

YOUR NOTES
↓

Integrating Trig Functions

How do I integrate sin, cos and 1/cos^2?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin\ x\ \mathrm{d}x = -\cos\ x + c$$

$$\int \cos\ x\ \mathrm{d}x = \sin\ x + c$$

where c is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \frac{1}{\cos^2\ x}\ \mathrm{d}x = \tan\ x + c$$

- All three of these standard integrals are in the **formula booklet**
- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin\ (ax + b)\ \mathrm{d}x = -\frac{1}{a}\cos\ (ax + b) + c$$

$$\int \cos\ (ax + b)\ \mathrm{d}x = \frac{1}{a}\sin\ (ax + b) + c$$

$$\int \frac{1}{\cos^2\ (ax + b)}\ \mathrm{d}x = \frac{1}{\mathrm{a}}\tan(ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
    - Ensure you know how to change the angle mode on your GDC

Exam Tip

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet
    - However, do be familiar with the **layout** of the formula booklet
        - You'll be able to quickly locate whatever you are after
        - You do not want to be searching every line of every page!
    - For formulae you think you have remembered, use the booklet to double-check

YOUR NOTES
↓

Worked Example

a)
Find, in the form $\mathrm{F}(x) + c$, an expression for each integral

i. $\int \cos x \, \mathrm{d}x$

ii. $\int \frac{1}{\cos^2\left(3x - \frac{\pi}{3}\right)} \mathrm{d}x$

i.

$$\boxed{\int \cos x \, dx = \sin x + c}$$

ii.

$$\boxed{\int \frac{1}{\cos^2\left(3x - \frac{\pi}{3}\right)} dx = \frac{1}{3}\tan\left(3x - \frac{\pi}{3}\right) + c}$$

(Linear function ax+b)

b) A curve has equation $y = \int 2\sin\left(2x + \frac{\pi}{6}\right) \mathrm{d}x$.

The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for y.

$$y = 2\int \sin\left(2x + \frac{\pi}{6}\right) dx$$

$$y = 2\left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{6}\right)\right] + c$$

At $x = \frac{\pi}{3}, y = \sqrt{3}$, $\sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$

$$c = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}$$

$$c = \frac{\sqrt{3}}{2}$$

$$\boxed{\therefore y = \frac{\sqrt{3}}{2} - \cos\left(2x + \frac{\pi}{6}\right)}$$

Integrating e^x & 1/x

YOUR NOTES
↓

How do I integrate exponentials and logarithms?

- The **antiderivatives** involving $\mathbf{e}^{x}$ and $\ln\ x$ are

$$\int \mathrm{e}^{x}\ \mathrm{d}x = \mathrm{e}^{x} + c$$

$$\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax+b)$, where a and b are constants,

$$\int \mathrm{e}^{ax+b}\ \mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + c$$

$$\int \frac{1}{ax+b}\ \mathrm{d}x = \frac{1}{\mathrm{a}}\ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b}\ \mathrm{d}x = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + \ln\ k = \ln\ k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax+b)$ when $b=0$

Exam Tip

- When revising, familiarise yourself with the layout of this section of the formula booklet, make sure you know what is and isn't in there and how to find it very quickly

Worked Example

A curve has the gradient function $f'(x) = \dfrac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x}\right) dx$$

$$f(x) = 3\int \frac{1}{3x+2}\,dx + \int e^{4-x}\,dx$$

$$= 3\left[\frac{1}{3}\ln|3x+2|\right] - e^{4-x} + c$$

$f(1) = \ln 10 - e^3$, $\quad \ln|3\times 1+2| - \cancel{e^{4-1}} + c = \ln 10 - \cancel{e^3}$

$\therefore c = \ln 10 - \ln 5$

$c = \ln\left(\frac{10}{5}\right) = \ln 2$

$$\boxed{\begin{array}{c}\therefore f(x) = \ln|3x+2| - e^{4-x} + \ln 2 \\ (= \ln 2|3x+2| - e^{4-x})\end{array}}$$

Integrating Composite Functions (ax+b)

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $\boldsymbol{ax+b}$

How do I integrate linear (*ax+b*) functions?

- A **linear function** (of x) is of the form $ax+b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
    - $\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+c$
    - $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b)+c$
    - $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b}+c$
    - $\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b|+c$
- There is one more special case
    - $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1}+c$ where $n \in \mathbb{Q},\ n \neq -1$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
    - However, spotting them can make solutions more efficient

Exam Tip

- Although the specific formulae in this revision note are NOT in the **formula booklet**
    - almost all of the information you will need to apply reverse chain rule is provided
    - make sure you have the formula booklet open at the right page(s) and practice using it

YOUR NOTES
↓

Worked Example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\,\mathrm{d}x$

$$I = \int 3(7-2x)^{5/3}\,dx = 3\int(-2x+7)^{5/3}\,dx$$

Using $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,$

$$I = 3\left[\frac{1}{-2\times\frac{8}{3}}(-2x+7)^{8/3}\right] + c \quad \xleftarrow{\quad} \frac{5}{3}+1$$

$$\boxed{\therefore\ I = -\frac{9}{16}(7-2x)^{8/3} + c}$$

b) $\int \frac{1}{2}\cos(3x-2)\,\mathrm{d}x$

$$I = \int \frac{1}{2}\cos(3x-2)\,dx = \frac{1}{2}\int\cos(3x-2)\,dx$$

Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

$$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$$

$$\boxed{\therefore\ I = \frac{1}{6}\sin(3x-2)+c}$$

Reverse Chain Rule

What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **second function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\,\mathrm{d}x \neq \frac{1}{f'(x)} e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I = \int g'(x)\, f\big(g(x)\big)\,\mathrm{d}x$$

  - this does not have to be strictly true, but '*algebraically*' it should be
    - if **coefficients** do not match '**adjust** and **compensate'** can be used
    - e.g. $5x^2$ is not quite the derivative of $4x^3$
      - the *algebraic* part (x^2) is '*correct*'
      - but the coefficient 5 is '*wrong*'
      - use '**adjust** and **compensate'** to '*correct*' it
- A particularly useful instance of reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**
  - '**adjust** and **compensate**' may need to be used to deal with any coefficients
    - e.g.

$$I = \int \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\ln|x^3+3x| + c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
  - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
  - this is indicated in the steps below by quote marks

STEP 1

Spot the 'main' function

e.g. $I = \int x(5x^2-2)^6\,\mathrm{d}x$

"the main function is $(\,\dots)^6$ which would come from $(\,\dots)^7$"

STEP 2

'**Adjust**' and '**compensate**' any coefficients required in the integral
e.g. " $(\ \dots\ )^7$ would differentiate to $7(\ \dots\ )^6$"
"chain rule says multiply by the derivative of $5x^2-2$, which is $10x$"
"there is no '7' or '10' in the integrand so adjust and compensate"

$$I=\frac{1}{7}\times\frac{1}{10}\times\int 7\times 10\times x(5x^2-2)^6\,\mathrm{d}x$$

STEP 3
Integrate and simplify

e.g. $I=\frac{1}{7}\times\frac{1}{10}\times(5x^2-2)^7+c$

$$I=\frac{1}{70}(5x^2-2)^7+c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
    - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**

Exam Tip

- Before the exam, practice this until you are confident with the pattern and do not need to worry about the formula or steps anymore
    - This will save time in the exam
- You can always check your work by differentiating, if you have time

Worked Example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Given that the curve passes through the point $(0,\ 1)$, find an expression for $f(x)$.

$$f(x) = \int 5x^2\sin(2x^3)\ dx$$

$$f(x) = 5\int x^2\sin(2x^3)\ dx$$ Take 5 out as a factor

This is a product, almost in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to −sin (...)"

"$2x^3$ would differentiate to $6x^2$"

$$f(x) = \underbrace{5 \times - \times \frac{1}{6} \times}_{\text{compensate}} \int \underbrace{- \times 6 \times}_{\text{adjust}} x^2\sin(2x^3)\ dx$$

STEP 3: Integrate and simplify

$$\boxed{f(x) = -\frac{5}{6}\cos(2x^3) + c}$$

Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually u) in terms of the original variable (usually x)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

How do I integrate using substitution?

STEP 1
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

STEP 2
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction
(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

STEP 3
Replace all parts of the integral
All x terms should be replaced with equivalent u terms, including $\mathrm{d}x$
If finding a **definite integral** change the limits from x-values to u-values too

STEP 4
Integrate and either
substitute x back in
or
evaluate the definte integral using the u limits (either using a GDC or manually)

STEP 5
Find c, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

Exam Tip

- Use your GDC to check the value of a definite integral, even in cases where working needs to be shown

YOUR NOTES
↓

Worked Example

a)
Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,\mathrm{d}x$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate u and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore\ du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute x back in

(STEP 5 not needed, evaluating c is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\boxed{\therefore\ I = \frac{-1}{2(3x^2+5x-1)^2} + c}$$

b)
Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,\mathrm{d}x$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3} \, du$

Change limits too, $x=1, \quad u = 3(1)^2 + 5(1) - 1 = 7$

$x=2, \quad u = 3(2)^2 + 5(2) - 1 = 21$

STEP 4: Integrate and evaluate

$$I = \left[-\frac{1}{2}u^{-2}\right]_7^{21} = \left[-\frac{1}{2}(21)^{-2}\right] - \left[-\frac{1}{2}(7)^{-2}\right]$$

$$\boxed{\therefore I = \frac{4}{441}}$$

5.4.3 Further Applications of Integration

YOUR NOTES
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Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
    - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
    - will be **negative** if the area is **fully** under the x-axis
    - possibly **negative** if the area is **partially** under the y-axis
        - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**

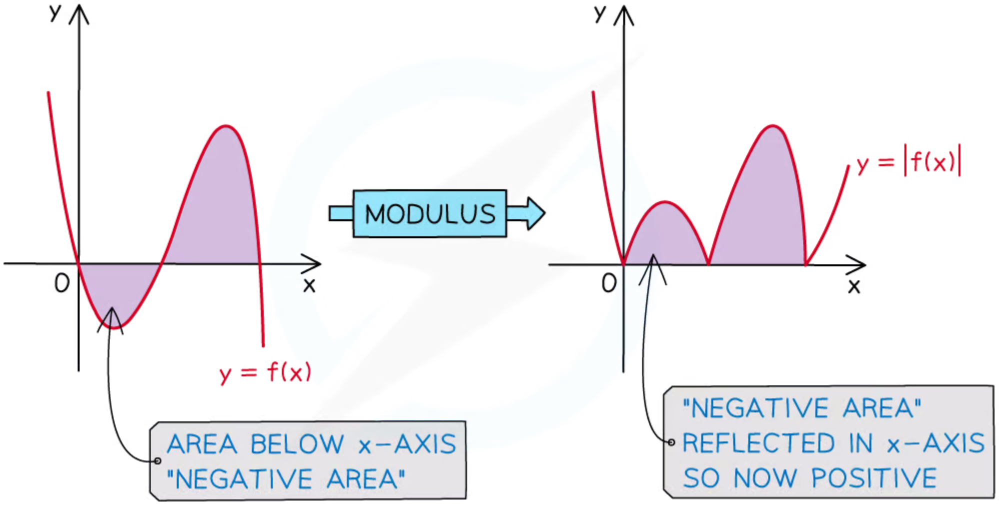
How do I find the area under a curve when the curve is fully under the x-axis?

STEP 1
Write the expression for the definite integral to find the area as usual
This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

STEP 2
The answer to the definite integral will be negative
Area must always be positive so take the modulus (absolute value) of it
e.g. If $I = -36$ then the area would be 36 (square units)

How do I find the area under a curve when all, or some, of the curve is below the x-axis?

- Use the **modulus** function
    - The **modulus** is also called the **absolute value** (Abs)
    - Essentially the modulus function makes **all** function **values positive**
    - Graphically, this means any negative areas are reflected in the x-axis



- A GDC will recognise the modulus function
  - look for a key or on-screen icon that says 'Abs' (absolute value)

$$A = \int_a^b |y| \, \mathrm{d}x$$

  - This is given in the **formula booklet**

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
If not identifiable from the question, use the graph to find the limits a and b

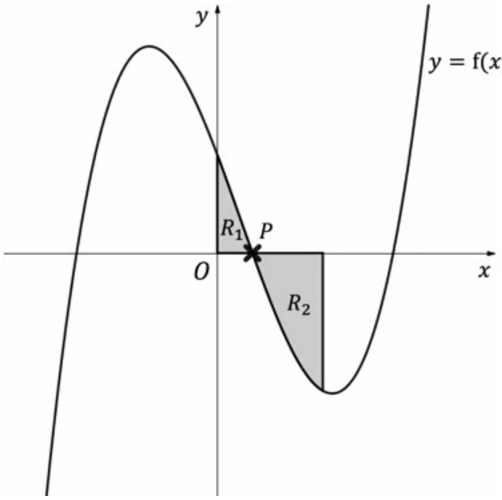
STEP 2
Write down the definite integral needed to find the required area
Remember to include the modulus (| ... |) symbols around the function
Use the GDC to evaluate it

Exam Tip

- If no diagram is provided, quickly sketch one so that you can see where the curve is above and below the x - axis and split up your integrals accordingly
  - You should use your GDC to do this

Worked Example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.



The region R_1 is bounded by the curve $y = f(x)$, the x-axis and the y-axis.

The region R_2 is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

Find the total area of the shaded regions, R_1 and R_2.

STEP 1: Graph given, identify limits

$a = 0$ (y-axis)

$b = 3$ (line $x = 3$)

STEP 2: Write down the integral required and use a GDC to evaluate it

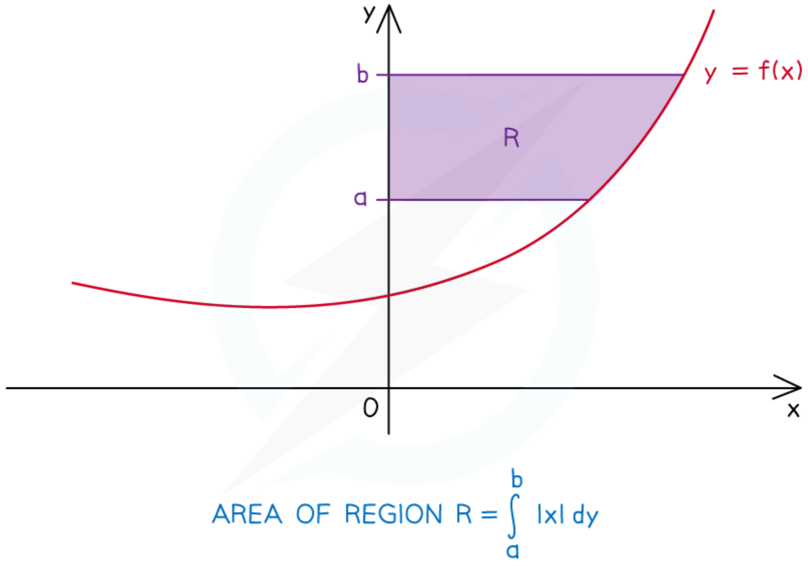
$$A = \int_0^3 \left| (x+4)(x-1)(x-5) \right| \, dx$$

$A = 43.166\,666\ldots$

$$\boxed{\therefore A = 43.2 \text{ square units} \quad (3 \text{ s.f.})}$$

Area Between Curve & y-axis

What is meant by the area between a curve and the y-axis?



- The **area** referred to is the **region bounded** by
  - the graph of $y = f(x)$
  - the y-axis
  - the **horizontal** line $y = a$
  - the **horizontal** line $y = b$
- The **exact area** can be found by evaluating a **definite integral**

How do I find the area between a curve and the y-axis?

- Use the formula

$$A = \int_a^b |x| \, dy$$

  - This is given in the **formula booklet**
  - The function is normally given in the form $y = f(x)$
    - so will need rearranging into the form $x = g(y)$
  - a and b may not be given directly as could involve the the x-axis ($y = 0$) and/or a root of $x = g(y)$
    - use a GDC to plot the curve and find roots as necessary

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
(or $x = g(y)$ if already in that form)
If not identifiable from the question, use the graph to find the limits a and b

STEP 2
If needed, rearrange $y = f(x)$ into the form $x = g(y)$

STEP 3
Write down the definite integral needed to find the required area
Use a GDC to evaluate it
A GDC is likely to require the function written with 'x' as the variable (not 'y')
Remember to include the modulus (| ... |) symbols around the function
Modulus may be called 'Absolute value (Abs)' on some GDCs

- In trickier problems some (or all) of the area may be 'negative'
  - this would be any area that is to the **left** of the y-axis (negative x values)
  - $|x|$ makes such areas 'positive' by **reflecting** them in the y-axis
    - a GDC will apply $|x|$ automatically as long as the modulus (| ... |)symbols are included

Exam Tip

- If no diagram is provided, quickly sketch one so that you can see where the curve is to the left and right of the y - axis and split up your integrals accordingly
  - You should use your GDC to do this

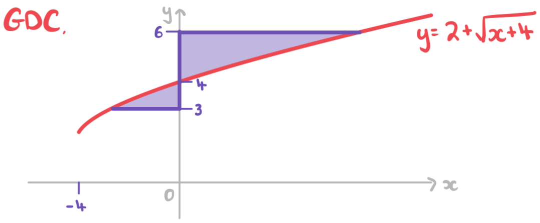
YOUR NOTES
↓

Worked Example

Find the area enclosed by the curve with equation $y = 2 + \sqrt{x+4}$, the y-axis and the horizontal lines with equations $y = 3$ and $y = 6$.

STEP 1: GDC plot shows partially negative area; limits given in question

From GDC.



STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = 2 + \sqrt{x+4}$

$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$

$x = y^2 - 4y$

STEP 3: Write down integral; use GDC to evaluate

$A = \int_3^6 |y^2 - 4y| \, dy$

(Type this as $\int_3^6 |x^2 - 4x| \, dx$ on a GDC)

might be 'Abs' on a GDC

$A = 12.333\ 333\ldots$

$\boxed{\therefore A = 12.3 \text{ square units } (3 \text{ s.f.})}$

The exact answer is $^{37}/_{3}$ but our GDC was not able to recognise this, despite us trying to use the exact-approximate button (S-D). This may vary between makes/models and will be due to the algorithm used to calculate integrals.

Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral
    - using a GDC, one method is not particularly trickier than the other
- The **total area** required could be the **sum** or **difference** of the area under the curve and the area under the line

How do I find the area between a curve and a line?

STEP 1
If a diagram is not given, use a GDC to draw the graphs of the curve and line and identify the area to be found

STEP 2
Use a GDC to find the root(s) of the curve, the root of the line and the x-coordinate of any intersections between the curve and line

STEP 3
Use the graph to determine whether areas will need adding or subtracting
Deduce the limits and so the definite integral(s) to find the area(s) under hte curve and line
Use a GDC to calculate the area under the curve

$$\int_a^b |y| \, \mathrm{d}x$$

Remember to include the modulus (| ... |) symbols around the function

Use a GDC to calculate the area under the line - this could be another definite integral or $\frac{1}{2}bh$ for a triangle or $\frac{1}{2}h(a+b)$ for a trapezium

STEP 4
Add or subtract areas accordingly to obtain a final answer

Exam Tip

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram provided, use your GDC to graph one and if you have time copy the sketch into your working

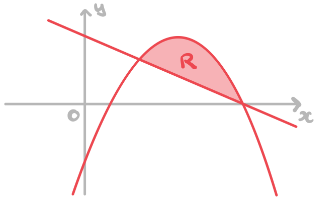
Worked Example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

R lies entirely in the first quadrant.

Find the area of the region R.

STEP 1: Sketch the graph from GDC plot; identify area required



STEP 2: Only intersections are required (use GDC)

Points of intersection are $(3,5)$ and $(8,0)$

STEP 3: Determine +/−, limits, integrals, etc

Area under curve $= \int_3^8 |10x - x^2 - 16| \, dx = \frac{100}{3}$

Area under line $= \frac{1}{2} \times (8-3) \times 5 = \frac{25}{2}$

$\therefore$ Area of R $= \frac{100}{3} - \frac{25}{2} = \frac{125}{6}$

$$\boxed{\text{Area of R} = \frac{125}{6} \text{ square units} \qquad (20.8 \ \ 3 \text{ s.f.})}$$

If finding the area of R directly from your GDC you may find it will not give an exact answer

In this case, an exact answer was not demanded so either $125/6$ or 20.8 (3 s.f.) is acceptable

YOUR NOTES
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5.4.4 Volumes of Revolution

Volumes of Revolution Around x-axis

What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the x-axis
- The **volume** of **revolution** is the volume of this solid

- Be careful – the 'front' and 'back' of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

How do I solve problems involving the volume of revolution around the x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 \, \mathrm{d}x$$

  - This is given in the **formula booklet**
  - y is a function of x
  - $x = a$ and $x = b$ are the equations of the (vertical) lines bounding the area
    - If $x = a$ and $x = b$ are not stated in a question, the boundaries could involve the y-axis $(x = 0)$ and/or a root of $y = f(x)$
    - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
If not identifiable from the question, use the graph to find the limits a and b

STEP 2
Use a GDC and the formula to evaluate the integral
Thus find the volume of revolution

Exam Tip

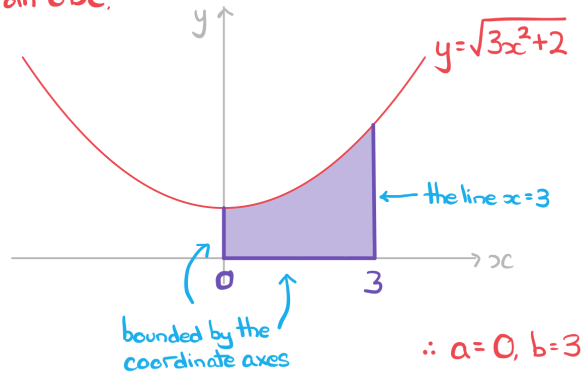
- Functions involved can be quite complicated so type them into your GDC carefully
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems

Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y=\sqrt{3x^2+2}$, the coordinate axes and the line $x=3$ by 2π radians around the x-axis. Give your answer as an exact multiple of π.

STEP 1: Use GDC to plot $y=f(x)$; identify limits

From GDC.



$\therefore\ a=0,\ b=3$

STEP 2: Use GDC and formula, find volume

$$V=\pi\int_0^3\left(\sqrt{3x^2+2}\right)^2\,dx = 33\pi$$

$$\boxed{\therefore V = 33\pi \text{ cubic units} \quad (104\ \ 3\text{s.f.})}$$

Depending on make/model of your GDC you may or may not get an exact answer. If you don't, try evaluating the integral without π (but remember to put it back for your written answer!)

Volumes of Revolution Around y-axis

What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid** of **revolution** which is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the y-axis
- The **volume** of **revolution** is the volume of this solid

How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 \, dy$$

  - This is given in the **formula booklet**
  - x is a function of y
    - the function is usually given in the form $y = f(x)$
    - this will need rearranging into the form $x = g(y)$
  - $y = a$ and $y = b$ are the equations of the (horizontal) lines bounding the area
    - If $y = a$ and $y = b$ are not stated in the question, the boundaries could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$
    - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
(or $x = g(y)$ if already in that form)
If not identifiable from the question use the graph to find the limits a and b

STEP 2
If needed, rearrange $y = f(x)$ into the form $x = g(y)$

STEP 3
Use a GDC and the formula to evaluate the integral
A GDC will likely require the function written with 'x' as the variable (not 'y')
Thus find the volume of revolution

Exam Tip

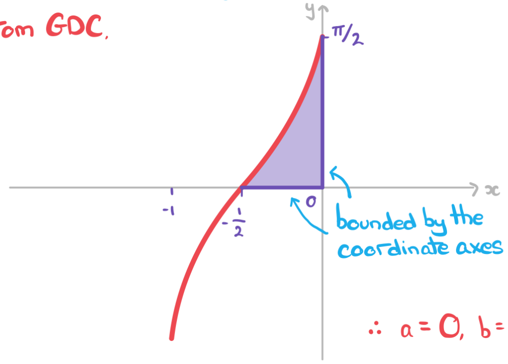
- Functions involved can be quite complicated so type them into your GDC carefully
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems

Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \arcsin(2x+1)$ and the coordinate axes by 2π radians around the y-axis. Give your answer to three significant figures.

STEP 1: Use GDC to plot $y=f(x)$; identify limits

From GDC.



$\therefore\ a = 0,\ b = \pi/2$

STEP 2: Rearrange $y=f(x)$ into $x=g(y)$

$y = \arcsin(2x+1)$

$\sin y = 2x+1$

$x = \frac{1}{2}(\sin y - 1)$

STEP 3: Use GDC and formula, find volume

$V = \pi\int_0^{\pi/2}\left[\frac{1}{2}(\sin y - 1)\right]^2 dy$ (Type as $\left[\frac{1}{2}(\sin x - 1)\right]^2$ on GDC)

$V = 0.279\ 754\ \ldots$

$\boxed{\therefore V = 0.280 \text{ cubic units} \quad (3 \text{ s.f.})}$

5.5 Kinematics

5.5.1 Kinematics Toolkit

Displacement, Velocity & Acceleration

What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects
- Common words such as **distance, speed** and **acceleration** are used in kinematics but are used according to their technical definition

What definitions do I need to be aware of?

- Firstly, only motion of an object in a **straight line** is considered
    - this could be a **horizontal** straight line
        - the **positive** direction would be to the **right**
    - or this could be a **vertical** straight line
        - the **positive** direction would be **upwards**

Particle

- A **particle** is the general term for an **object**
    - some questions may use a **specific** object such as a **car** or a **ball**

Time t seconds

- **Displacement**, **velocity** and **acceleration** are all **functions** of **time** t
- **Initially** time is zero $t = 0$

Displacement s m

- The **displacement** of a particle is its **distance relative** to a **fixed point**
    - the fixed point is often (but not always) the particle's **initial position**
- **Displacement** will be **zero** $s = 0$ if the object is at or has returned to its initial position
- **Displacement** will be negative if its **position relative** to the **fixed point** is in the **negative direction** (left or down)

Distance d m

- Use of the word **distance** needs to be considered carefully and could refer to
    - the distance **travelled** by a particle
    - the **(straight line)** distance the particle is from a **particular point**
- Be careful not to confuse **displacement** with **distance**
    - if a bus route starts and ends at a bus depot, when the bus has returned to the depot, its **displacement** will be **zero** but the distance the bus has travelled will be the length of the route
- **Distance** is always **positive**

Velocity v m s^{-1}

- The **velocity** of a particle is the **rate of change** of its **displacement** at time t

- **Velocity** will be **negative** if the **particle** is moving in the **negative direction**
- A **velocity** of **zero** means the particle is **stationary** $v = 0$

Speed $|v|$ m s^{-1}

- **Speed** is the **magnitude** (a.k.a. absolute value or modulus) of **velocity**
  - as the particle is **moving** in a **straight line, speed** is the **velocity ignoring** the **direction**
    - if $v = 4,\ |v| = 4$
    - if $v = -6,\ |v| = 6$

Acceleration a m s^{-2}

- The **acceleration** of a particle is the **rate of change** of its **velocity** at time t
- Acceleration can be **negative** but this alone cannot fully describe the particle's motion
  - if **velocity** and **acceleration** have the **same** sign the particle is **accelerating** (speeding up)
  - if **velocity** and **acceleration** have **different** signs then the particle is **decelerating** (slowing down)
  - if **acceleration** is **zero** $a = 0$ the particle is moving with **constant** velocity
  - in all cases the **direction** of **motion** is determined by the **sign** of **velocity**

Are there any other words or phrases in kinematics I should know?

- Certain words and phrases can imply values or directions in kinematics
  - a particle described as "at **rest**" means that its velocity is zero, $\boldsymbol{v = 0}$
  - a particle described as moving "**due east**" or "**right**" or would be moving in the **positive horizontal** direction
    - this also means that $\boldsymbol{v > 0}$
  - a particle "**dropped from the top of a cliff**" or "**down**" would be moving in the **negative vertical** direction
    - this also means that $\boldsymbol{v < 0}$

What are the key features of a velocity-time graph?

- The **gradient** of the graph equals the **acceleration** of an object
- A **straight line** shows that the object is **accelerating** at a **constant rate**
- A **horizontal** line shows that the object is moving at a **constant velocity**
- The **area** between graph and the x-axis tells us the **change in displacement** of the object
  - Graph **above** the x-axis means the object is moving **forwards**
  - Graph **below** the x-axis means the object is moving **backwards**

- The **total displacement** of the object from its starting point is the sum of the **areas above** the x-axis **minus** the sum of the **areas below** the x-axis
- The **total distance travelled** by the object is the sum of **all** the **areas**
- If the graph **touches** the **x-axis** then the object is **stationary** at that time
- If the graph is **above** the **x-axis** then the object has positive velocity and is **travelling forwards**
- If the graph is **below** the **x-axis** then the object has negative velocity and is **travelling backwards**

YOUR NOTES
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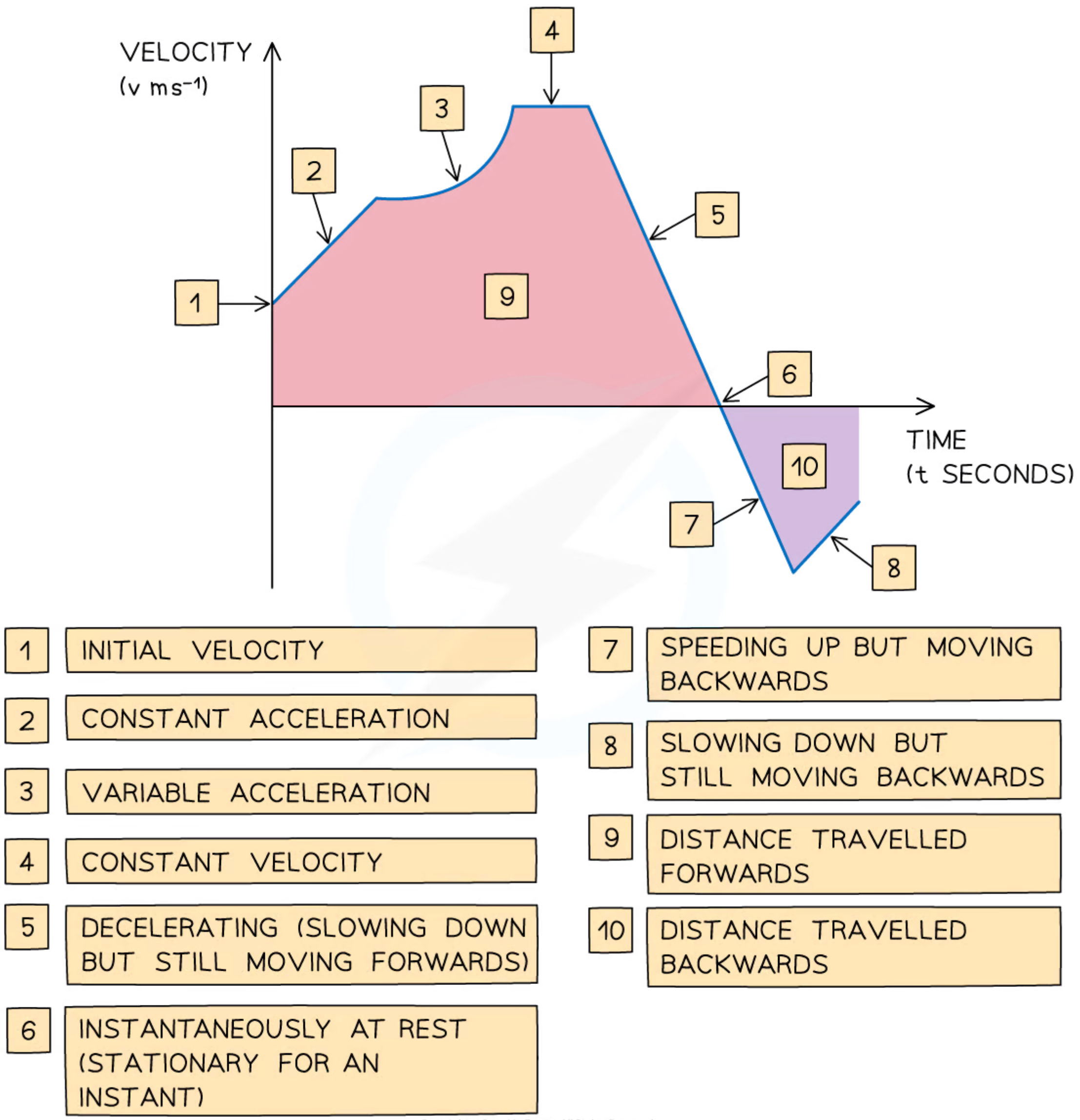

Exam Tip

- In an exam if you are given an expression for the velocity then sketching a velocity-time graph can help visualise the problem

YOUR NOTES
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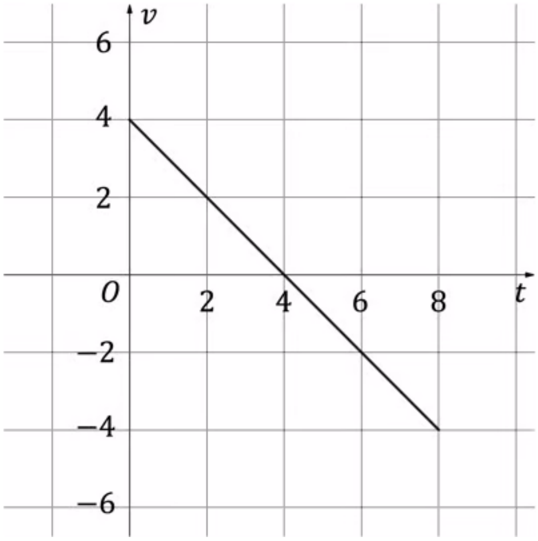
Worked Example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground.

The velocity-time graph below illustrates the motion of the particle for these 8 seconds.



i)
How many seconds does the particle take to reach its maximum height? Give a reason for your answer.
ii)
State, with a reason, whether the particle is accelerating or decelerating at time $t = 3$.

i. At maximum height, velocity is zero

$v = 0$ at $t = 4$

∴ The particle takes 4 seconds to reach its maximum height. This is because its velocity is 0 m s^{-1} at 4 seconds.

ii. At $t=3$, velocity is POSITIVE

Acceleration is the gradient of velocity

At $t=3$, acceleration is NEGATIVE

∴ At 3 seconds the particle is decelerating as its velocity and acceleration have different signs.

Differentiation for Kinematics

How is differentiation used in kinematics?

- **Displacement**, **velocity** and **acceleration** are related by calculus
- In terms of differentiation and derivatives
  - **velocity** is the **rate** of **change** of **displacement**
    - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ or $v(t) = s'(t)$
  - **acceleration** is the **rate** of **change** of **velocity**
    - $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ or $a(t) = v'(t)$
  - so **acceleration** is also the **second derivative** of **displacement**
    - $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ or $a(t) = s''(t)$
  - Sometimes **velocity** may be a **function** of **displacement** rather than time
    - $v(s)$ rather than $v(t)$
    - in such circumstances, **acceleration** is $a = s\frac{\mathrm{d}v}{\mathrm{d}s}$
    - this result is derived from the **chain rule**
  - All acceleration formulae are given in the **formula booklet**
- Even if a motion graph is given, if possible, use your GDC to draw one
  - you can then use your GDC's graphing features to find **gradients**
    - **velocity** is the **gradient** on a **displacement** (-time) graph
    - **acceleration** is the **gradient** on a **velocity** (-time) graph
- **Dot notation** is often used to indicate time derivatives
  - x is sometimes used as displacement (rather than s) in such circumstances
  - $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$, so $\dot{x}$ is **velocity**
  - $\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, so $\ddot{x}$ is **acceleration**

Worked Example

a)

The displacement, x m, of a particle at t seconds, is modelled by the function $x(t) = 2t^3 - 27t^2 + 84t$.

Find expressions for $\dot{x}$ and $\ddot{x}$.

$$x = 2t^3 - 27t^2 + 84t$$

$$\dot{x} = \frac{dx}{dt} \qquad \therefore \dot{x} = 6t^2 - 54t + 84$$

$$\dot{x} = 6(t^2 - 9t + 14)$$

$$\boxed{\dot{x} = 6(t-2)(t-7)}$$

It is not essential to factorise answers

$$\ddot{x} = \frac{d^2x}{dt^2} \qquad \therefore \ddot{x} = 12t - 54$$

$$\boxed{\ddot{x} = 6(2t-9)}$$

b)

The velocity, v m s^{-1}, of a particle is given as $v(s) = 6s - 5s^2 - 4$, where s m is the displacement of the particle.

Find an expression, in terms of s, for the acceleration of the particle.

$$v = 6s - 5s^2 - 4$$

$$a = v\frac{dv}{ds} \qquad \therefore a = \underset{\overset{\uparrow}{v(s)}}{(6s - 5s^2 - 4)}\underset{\overset{\uparrow}{\frac{dv}{ds}}}{(6 - 10s)}$$

$$\boxed{a = 3(2-5s)(6s-5s^2-4)}$$

Integration for Kinematics

How is integration used in kinematics?

- Since **velocity** is the **derivative** of **displacement** ($v = \frac{\mathrm{d}s}{\mathrm{d}t}$) it follows that

$$s = \int v \, \mathrm{d}t$$

- Similarly, **velocity** will be an **antiderivative** of **acceleration**

$$v = \int a \, \mathrm{d}t$$

- You might be given the **acceleration** in terms of the **velocity and/or** the **displacement**
  - In this case you can solve a differential equation to find an **expression for the velocity in terms of the displacement**

$$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$$

How would I find the constant of integration in kinematics problems?

- A **boundary** or **initial** condition would need to be known
  - phrases involving the word **"initial"**, or **"initially"** are referring to **time** being **zero,** i.e. $t = 0$
  - you might also be given information about the object at some other time (this is called a **boundary condition**)
  - **substituting** the values in from the **initial or boundary condition** would allow the **constant** of **integration** to be found

How are definite integrals used in kinematics?

- Definite integrals can be used to find the displacement of a particle between two points in time
  - $\int_{t_1}^{t_2} v(t) \, \mathrm{d}t$ would give the **displacement** of the particle **between** the times $t = t_1$ and $t = t_2$
    - This can be found using a velocity-time graph by **subtracting** the **total area below** the horizontal axis from the **total area above**
  - $\int_{t_1}^{t_2} |v(t)| \, \mathrm{d}t$ gives the **distance** a particle has **travelled** between the times $t = t_1$ and $t = t_2$
    - This can be found using a velocity velocity-time graph by **adding** the **total area below** the horizontal axis to the **total area above**
    - Use a GDC to plot the modulus graph $y = |v(t)|$

Exam Tip

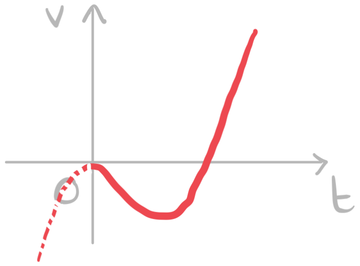
- Sketching the velocity-time graph can help you visualise the distances travelled using areas between the graph and the horizontal axis

Worked Example

A particle moving in a straight horizontal line has velocity ($v \text{ m s}^{-2}$) at time t seconds modelled by $v(t) = 8t^3 - 12t^2 - 2t$.

i. Given that the initial position of the particle is at the origin, find an expression for its displacement from the origin at time t seconds.
ii. Find the displacement of the particle from the origin in the first five seconds of its motion.
iii. Find the distance travelled by the particle in the first five seconds of its motion.

Use your GDC to sketch a velocity (-time) graph and use it to check to see if your answers are sensible.



i. "initial" – $t=0$, "origin" – $s=0$

$s(t) = \int v(t)\,dt = \int (8t^3 - 12t^2 - 2t)\,dt$

$s(t) = 2t^4 - 4t^3 - t^2 + c$ where c is a constant

at $t=0$, $s=0$, $\therefore c = 0$

$\boxed{\therefore s(t) = 2t^4 - 4t^3 - t^2}$

ii. "first five seconds" – $t_1 = 0$, $t_2 = 5$

Using a GDC this would be

$s = \int_0^5 (8t^3 - 12t^2 - 2t)\,dt$

$\boxed{s = 725 \text{ m}}$

iii. Using a GDC this would be

$d = \int_0^5 |8t^3 - 12t^2 - 2t|\,dt$ d for distance

$d = 736.734\,020\ldots$

$\boxed{\therefore d = 737 \text{ m (3 s.f.)}}$

5.6 Differential Equations

5.6.1 Modelling with Differential Equations

Modelling with Differential Equations

Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
    - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time t

How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
        - For example, the rate of change of a population of bacteria, P, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of…', 'rate of growth of…', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time t
        - So with the bacteria example above, the equation will involve the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$
- Recall the basic equation of proportionality
    - If y is proportional to x, then $y = kx$ for some **constant of proportionality** k
        - So for the bacteria example above the differential equation needed would be $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
    - The precise value of k will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
    - It can often be useful to assume that $k > 0$ when setting up your equation
        - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
        - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{\mathrm{d}P}{\mathrm{d}t} = -kP$

Worked Example

a)

In a particular pond, the rate of change of the area covered by algae, *A*, at any time *t* is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$$\boxed{\frac{dA}{dt} = k\sqrt{A} \quad \text{(where } k \text{ is a constant of proportionality)}}$$

b)

Newton's Law of Cooling states that the rate of change of the temperature of an object, *T*, at any time *t* is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, T_a, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\boxed{\frac{dT}{dt} = -k(T - T_a) \quad \text{(where } k>0 \text{ is a constant of proportionality)}}$$

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k>0$ assures that $\frac{dT}{dt}$ is negative.

5.6.2 Separation of Variables

Separation of Variables

What is separation of variables?

- **Separation of variables** can be used to solve certain types of first order differential equations
- Look out for equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$
  - i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a function of x multiplied by a function of y
  - be careful – the 'function of x' $g(x)$ may just be a constant!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x} = 6y$, $g(x) = 6$ and $h(y) = y$
- If the equation is in that form you can use separation of variables to try to solve it
- If the equation is not in that form you will need to use another solution method

How do I solve a differential equation using separation of variables?

- STEP 1: Rearrange the equation into the form $\left(\frac{1}{h(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$
- STEP 2: Take the integral of both sides to change the equation into the form

$$\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

  - You can think of this step as 'multiplying the $\mathrm{d}x$ across and integrating both sides'
    - Mathematically that's not *quite* what is actually happening, but it will get you the right answer here!
- STEP 3: Work out the integrals on both sides of the equation to find the **general solution** to the differential equation
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
- STEP 4: Use any boundary or initial conditions in the question to work out the value of the integration constant
- STEP 5: If necessary, rearrange the solution into the form required by the question

Exam Tip

- Be careful with letters – the equation on an exam may not use x and y as the variables
- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Worked Example

For each of the following differential equations, either (i) solve the equation by using separation of variables giving your answer in the form $y = f(x)$, or (ii) state why the equation may not be solved using separation of variables.

a) $\dfrac{dy}{dx} = \dfrac{e^x + 4x}{3y^2}$.

STEP 1 : $3y^2 \frac{dy}{dx} = e^x + 4x$ $\quad g(x) = e^x + 4x \quad h(y) = \frac{1}{3y^2}$

STEP 2 : $\int 3y^2 \, dy = \int (e^x + 4x) \, dx$

STEP 3 : $y^3 = e^x + 2x^2 + c$ ← Don't forget constant of integration

STEP 4 : No boundary conditions given, so skip step

STEP 5 : $\boxed{y = \sqrt[3]{e^x + 2x^2 + c}}$ $\quad y = f(x)$

b) $\dfrac{dy}{dx} = 4xy - 2\ln x$.

$4xy - 2\ln x$ is not of the form $g(x)h(y)$, so it may not be solved using separation of variables.

c) $\dfrac{dy}{dx} = 3y$, given that $y = 2$ when $x = 0$.

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STEP 1 : $\frac{1}{y}\frac{dy}{dx} = 3$ $g(x) = 3$ $h(y) = y$

STEP 2 : $\int \frac{1}{y}\,dy = \int 3\,dx$

STEP 3 : $\ln|y| = 3x + c$ ← Don't forget constant of integration

STEP 4 : $\ln|2| = 3(0) + c \implies c = \ln 2$ ← y = 2 when x = 0

STEP 5 : For the boundary condition $y = 2$, $y > 0$.
Therefore we can drop the modulus sign from $|y|$.

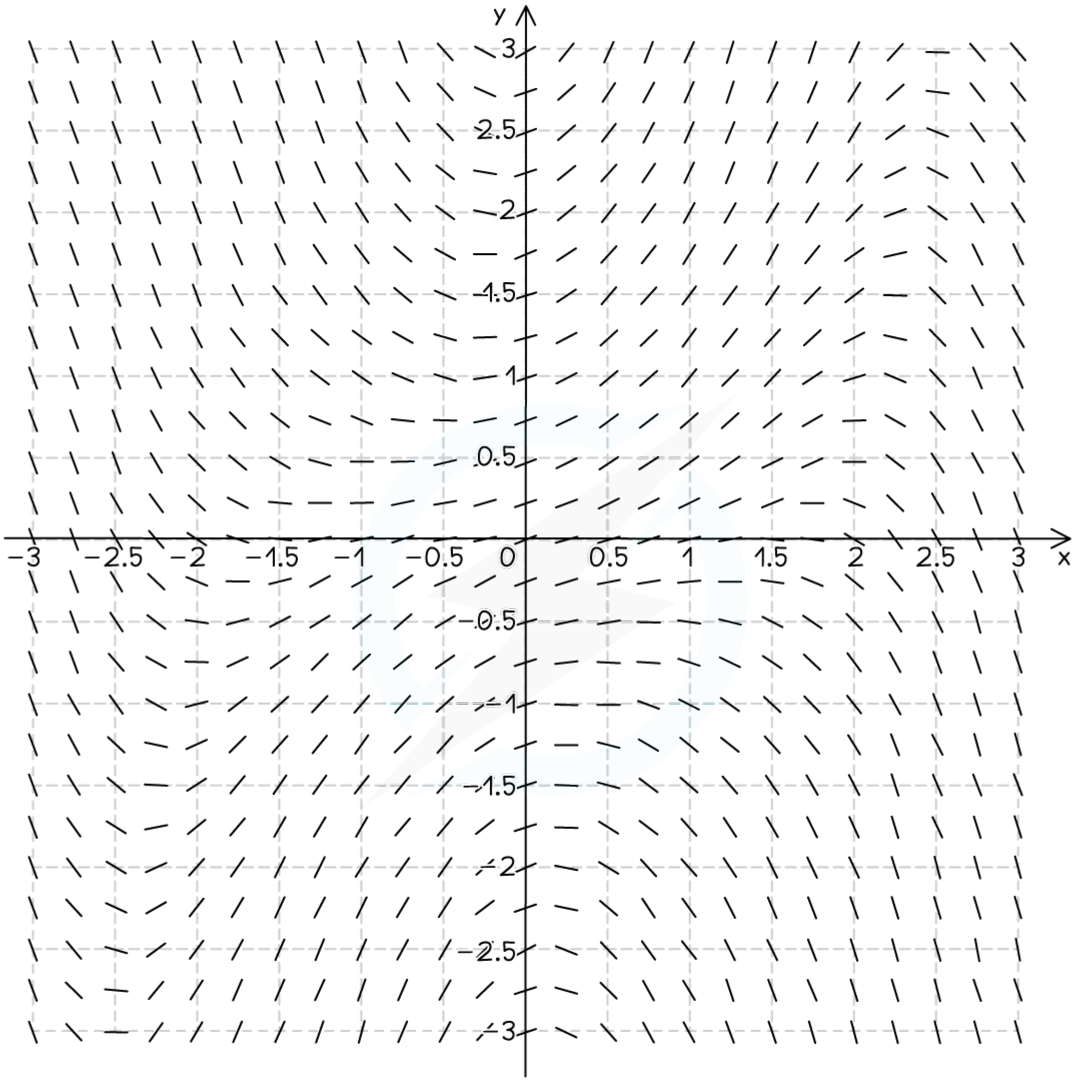
$y = e^{3x + \ln 2} = (e^{3x})(e^{\ln 2})$

$\Longrightarrow \boxed{y = 2e^{3x}}$ $y = f(x)$

Slope Fields

What are slope fields?

- We are considering here a differential equation involving two variables of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x, y)$
    - I.e., the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is equal to some function of *x* and *y*
- In some cases it may be possible to solve the differential equation analytically, while in other cases this is not possible
- Whether or not the equation can be analytically solved, however, it is always possible to calculate the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at any point (*x*, *y*) by putting the *x* and *y* values into $g(x, y)$
    - This means that we can calculate the gradient of the solution curve at any point that the solution might go through
- A **slope field** for a differential equation is a diagram with short tangent lines drawn at a number of points
    - The gradient of the tangent line drawn at any given point will be equal to the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at that point
    - Normally the tangent lines will be drawn for points that form a regularly-spaced grid of *x* and *y* values

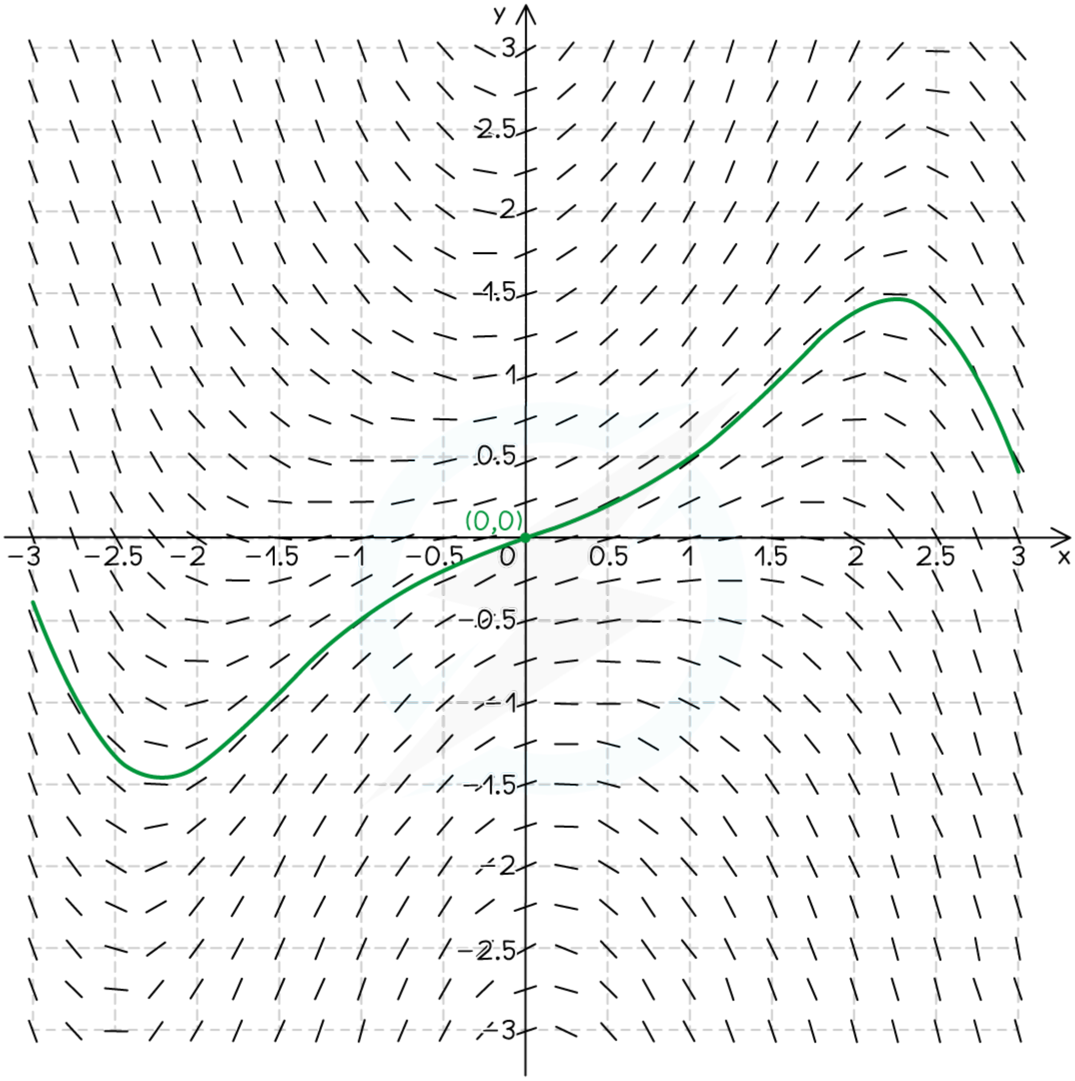


SLOPE FIELD FOR $\frac{dy}{dx} = y \sin x - e^{-\cos x} \cos x$

How can I use slope fields to study the solutions of a differential equation?

- Looking at the tangent lines in a slope field diagram will give you a general sense for what the solution curves to the differential equation will look like
    - Remember that the solution to a given differential equation is actually a family of solutions
    - We need appropriate boundary conditions or initial conditions to determine which of that family of solutions is the precise solution in a particular situation
- You can think of the tangent lines in a slope diagram as 'flow lines'
    - From a given point the solution curve through that point will 'flow' away from the point in the direction of the tangent line
- For a given point, you can use a slope field to sketch the general shape of the solution curve that goes through that point

- The given point here serves as a boundary condition, letting you know which of all the possible solution curves is the one you want to sketch
- The sketch should go through the given point, and follow the general 'flow' of the tangent lines through the rest of the slope field diagram
- In general, the sketched solution curve should **not** attempt to connect together a number of different tangent lines in the diagram
    - There is no guarantee that the solution curve will go through any exact point in the 'grid' of points at which tangent lines have been drawn
    - The only tangent line that your solution curve should definitely go through is one at the given 'boundary condition' point
- The sketched solution curve may go along some of the tangent lines, but it should not should not cut across any of them



SOLUTION CURVE THROUGH THE ORIGIN

- Look out for places where the tangent lines are horizontal

- At such points $\frac{\mathrm{d}y}{\mathrm{d}x}=0$
- Therefore such points may indicate local minimum or maximum points for a solution curve
    - Be careful – not every point where $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ is a local minimum or maximum
    - But every local minimum or maximum will be at a point where $\frac{\mathrm{d}y}{\mathrm{d}x}=0$
- Don't forget that you can also solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=g(x,\ y)=0$ directly to identify points where the gradient is zero
    - For example if $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin(x-y)$, then the gradient will be zero anywhere where $x-y=0, \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$
    - This is another way to identify possible local minimum and maximum points for the solution curves
    - If such a point falls between the 'grid points' at which the tangent lines have been drawn, this may be the only way to identify such a point exactly

Worked Example

Consider the differential equation

$$\frac{dy}{dx} = -0.4(y-2)^{\frac{1}{3}}(x-1)e^{-\frac{(x-1)^2}{25}}.$$

a)
Using the equation, determine the set of points for which the solutions to the differential equation will have horizontal tangents.

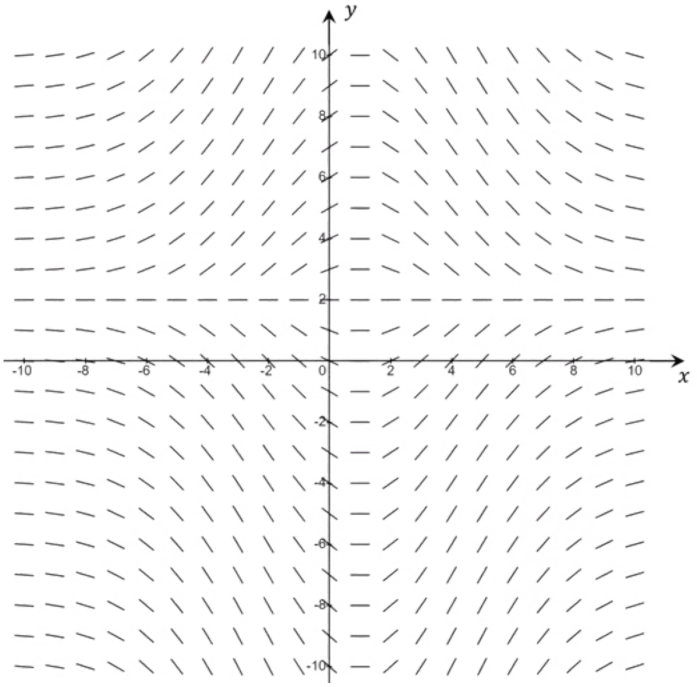
The solution will have horizontal tangents wherever $\frac{dy}{dx} = 0$.

The exponential function is never equal to zero.

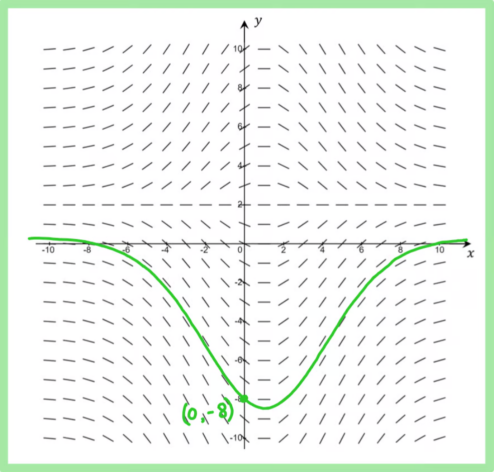
$\frac{dy}{dx} = 0$ when $y-2=0$ or $x-1=0$.

The solutions will have horizontal tangents at any point where $y=2$ or $x=1$.

The diagram below shows the slope field for the differential equation, for $-10 \le x \le 10$ and $-10 \le y \le 10$.



b)
Sketch the solution curve for the solution to the differential equation that passes through the point $(0, -8)$.


y
x
10
8
6
4
2
0
-2
-4
-6
-8
-10
(0, -8)

Euler's Method: First Order

What is Euler's method?

- **Euler's method** is a numerical method for finding approximate solutions to differential equations
- It treats the derivatives in the equation as being constant over short 'steps'
- The accuracy of the Euler's Method approximation can be improved by making the step sizes smaller

How do I use Euler's method with a first order differential equation?

- STEP 1: Make sure your differential equation is in $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ form
- STEP 2: Write down the recursion equations using the formulae $y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ from the exam formula booklet
  - *h* in those equations is the **step size**
  - the exam question will usually tell you the correct value of *h* to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for x_0 and y_0 will come from the boundary conditions given in the question

Exam Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x* and *y*
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

YOUR NOTES
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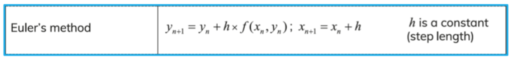
Worked Example

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x + 1$ with the boundary condition $y(0) = 0.5$.

a)
Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 1$.



Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)

} from formula booklet

STEP 1: $\frac{dy}{dx} = \underbrace{x - y + 1}_{f(x,y)}$

STEP 2: $y_{n+1} = y_n + \underbrace{0.2}_{h \text{ (from question)}} \times \underbrace{(x_n - y_n + 1)}_{f(x_n, y_n)}$ $\quad x_{n+1} = x_n + 0.2$

STEP 3: We need to get x from 0 to 1, so we will need $\frac{1-0}{0.2} = 5$ steps.

$y(0) = 0.5$

n	x_n	y_n
0	0	0.5
1	0.2	0.6
2	0.4	0.72
3	0.6	0.856
4	0.8	1.0048
5	1	1.16384

} from GDC

$\boxed{y(1) = 1.16 \text{ (3 s.f.)}}$

b)
Explain how the accuracy of the approximation in part (a) could be improved.

Make the step size smaller.

Euler's Method: Coupled Systems

How do I use Euler's method with coupled first order differential equations?

- STEP 1: Make sure your coupled differential equations are in $\frac{dx}{dt} = f_1(x,y,t)$ and $\frac{dy}{dt} = f_2(x,y,t)$ form
- STEP 2: Write down the recursion equations using the formulae $x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$, $y_{n+1} = y_n + h \times f_1(x_n, y_n, t_n)$ and $t_{n+1} = t_n + h$ from the exam formula booklet
    - *h* in those equations is the **step size**
    - the exam question will usually tell you the correct value of *h* to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
    - the values for x_0, y_0 and t_0 will come from the boundary conditions given in the question
    - frequently you will be given an initial condition
        - look out for terms like 'initially' or 'at the start'
        - in this case t_0

Exam Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x*, *y* and *t*.
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

YOUR NOTES
↓

Worked Example

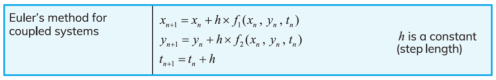
Consider the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - 3y + 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y + \frac{1}{t+1}$$

Initially $x = 10$ and $y = 2$.

Use Euler's method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$.



Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

} from formula booklet

STEP 1: Equations are already in the proper form

STEP 2: $x_{n+1} = x_n + 0.1 \times \overbrace{(2x_n - 3y_n + 1)}^{f_1(x_n,\, y_n,\, t_n)}$

$y_{n+1} = y_n + \underbrace{0.1}_{h \text{ (from question)}} \times \underbrace{\left(x_n + y_n + \frac{1}{t_n+1}\right)}_{f_2(x_n,\, y_n,\, t_n)}$ $\quad t_{n+1} = t_n + 0.1$

STEP 3: To get t from 0 to 0.5 we need $\frac{0.5-0}{0.1} = 5$ steps.

Initially $x = 10$ and $y = 2$

n	t_n	x_n	y_n
0	0	10	2
1	0.1	11.5	3.3
2	0.2	12.91	4.8709
3	0.3	14.13	6.7323
4	0.4	15.037	8.8955
5	0.5	15.475	11.36

} from GDC

$\boxed{x(0.5) = 15.5 \text{ (3 s.f.)} \qquad y(0.5) = 11.4 \text{ (3 s.f.)}}$

5.7 Further Differential Equations

5.7.1 Coupled Differential Equations

Solving Coupled Differential Equations

How do I write a system of coupled differential equations in matrix form?

- The coupled differential equations considered in this part of the course will be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

  - $a, b, c, d \in \mathbb{R}$ are constants whose precise value will depend on the situation being modelled
    - In an exam question the values of the constants will generally be given to you
- This system of equations can also be represented in matrix form:

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- It is usually more convenient, however, to use the 'dot notation' for the derivatives:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- This can be written even more succinctly as $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$
  - Here $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

How do I find the exact solution for a system of coupled differential equations?

- The exact solution of the coupled system $\dot{\mathbf{x}} = \boldsymbol{Mx}$ depends on the **eigenvalues** and **eigenvectors** of the matrix of coefficients $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  - The eigenvalues and/or eigenvectors may be given to you in an exam question
  - If they are not then you will need to calculate them using the methods learned in the matrices section of the course
- On the exam you will only be asked to find exact solutions for cases where the two eigenvalues of the matrix are real, distinct, and non-zero
  - Similar solution methods exist for non-real, non-distinct and/or non-zero eigenvalues, but you don't need to know them as part of the IB AI HL course

- Let the eigenvalues and corresponding eigenvectors of matrix $\boldsymbol{M}$ be λ_1 and λ_2, and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively
  - Remember from the definition of eigenvalues and eigenvectors that this means that $\boldsymbol{M}\boldsymbol{p}_1 = \lambda_1\boldsymbol{p}_1$ and $\boldsymbol{M}\boldsymbol{p}_2 = \lambda_2\boldsymbol{p}_2$
- The exact solution to the system of coupled differential equations is then

$$\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$$

  - This solution formula is in the exam formula booklet
  - $A,\ B \in \mathbb{R}$ are constants (they are essentially constants of integration of the sort you have when solving other forms of differential equation)
- If initial or boundary conditions have been provided you can use these to find the precise values of the constants A and B
  - Finding the values of A and B will generally involve solving a set of simultaneous linear equations (see the worked example below)

Worked Example

The rates of change of two variables, x and y, are described by the following system of coupled differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

Initially $x = 2$ and $y = 1$.

Given that the matrix $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ has eigenvalues of 3 and 2 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the exact solution to the system of coupled differential equations.

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

} From exam formula booklet

$$\underline{x} = Ae^{3t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}1\\2\end{pmatrix}$$

Put eigenvalues and eigenvectors into the solution formula

At $t = 0$, $\underline{x} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

Use initial condition to find values of A and B

So $Ae^0\begin{pmatrix}1\\1\end{pmatrix} + Be^0\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$$\begin{pmatrix}A + B\\A + 2B\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$ Simultaneous equations

$\Rightarrow A = 3, \quad B = -1$

$$\boxed{\underline{x} = 3e^{3t}\begin{pmatrix}1\\1\end{pmatrix} - e^{2t}\begin{pmatrix}1\\2\end{pmatrix}}$$

I.e., $x = 3e^{3t} - e^{2t}$

$y = 3e^{3t} - 2e^{2t}$

Phase Portraits

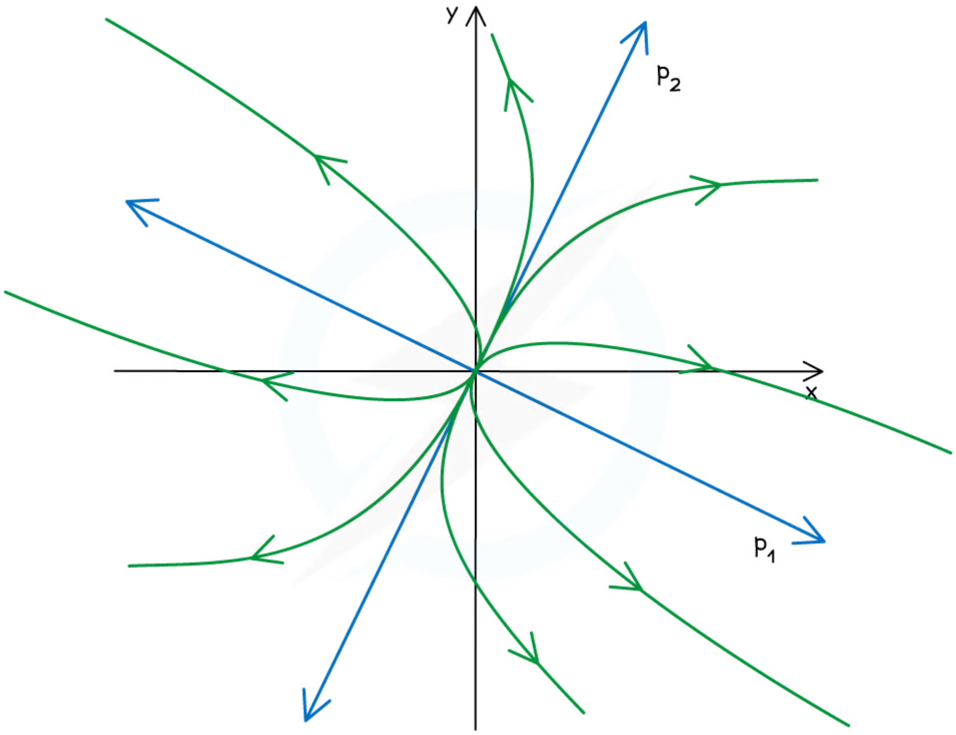
What is a phase portrait for a system of coupled differential equations?

- Here we are again considering systems of coupled equations that can be represented in the matrix form $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$, where $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$
- A **phase portrait** is a diagram showing how the values of *x* and *y* change over time
  - On a phase portrait we will usually sketch several typical solution trajectories
  - The precise trajectory that the solution for a particular system will travel along is determined by the initial conditions for the system
- Let λ_1 and λ_2 be the eigenvalues of the matrix $\boldsymbol{M}$
  - The overall nature of the phase portrait depends in large part on the values of λ_1 and λ_2

What does the phase portrait look like when λ_1 and λ_2 are real numbers?

- Recall that for real distinct eigenvalues the solution to a system of the above form is $\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$, where λ_1 and λ_2 are the eigenvalues of $\boldsymbol{M}$ and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding eigenvectors
  - AI HL only considers cases where λ_1 and λ_2 are distinct (i.e., $\lambda_1 \neq \lambda_2$) and non-zero
- A phase portrait will always include two ‘eigenvector lines’ through the origin, each one parallel to one of the eigenvectors
    - So if $\boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{p}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, for example, then these lines through the origin will have equations $y = 2x$ and $y = -\frac{4}{3}x$, respectively
  - These lines will define two sets of solution trajectories
  - If the eigenvalue corresponding to a line’s eigenvector is **positive**, then there will be solution trajectories along the line **away from** the origin in both directions as *t* increases
  - If the eigenvalue corresponding to a line’s eigenvector is **negative**, then there will be solution trajectories along the line **towards** the origin in both directions as *t* increases
  - No solution trajectory will ever cross an eigenvector line
- If **both eigenvalues are positive** then all solution trajectories will be directed **away from** the origin as *t* increases
  - In between the ‘eigenvector lines’ the trajectories as they move away from the origin will all curve to become approximately parallel to the line whose eigenvector corresponds to the larger eigenvalue

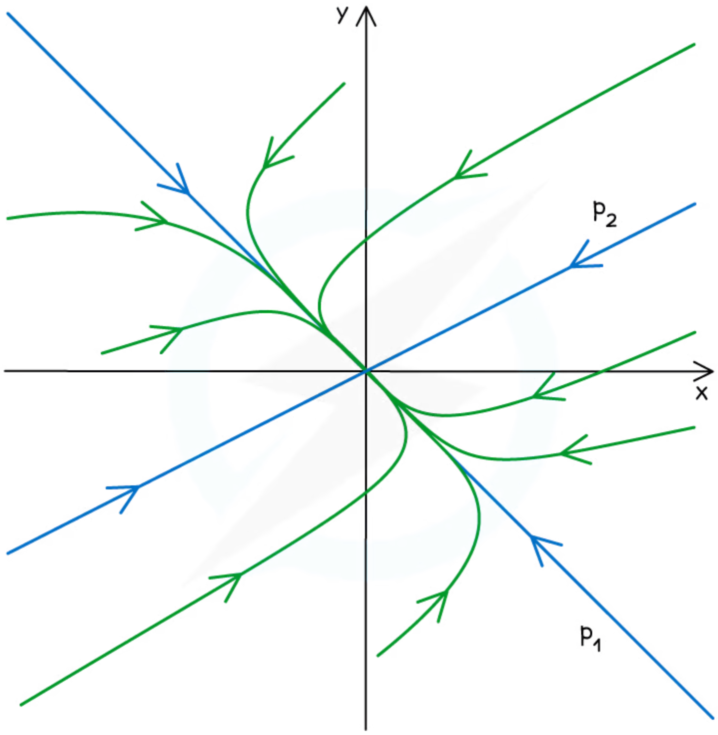
BOTH EIGENVALUES REAL AND POSITIVE



$\lambda_1 > \lambda_2 > 0$, SO SOLUTION TRAJECTORIES BECOME PARALLEL TO p_1 AS THEY MOVE AWAY FROM THE ORIGIN

- If **both eigenvalues are negative** then all solution trajectories will be directed **towards** the origin as *t* increases
  - In between the 'eigenvector lines' the trajectories will all curve so that at points further away from the origin they are approximately parallel to the line whose eigenvector corresponds to the more negative eigenvalue
    - They will then converge on the other eigenvalue line as they move in towards the origin

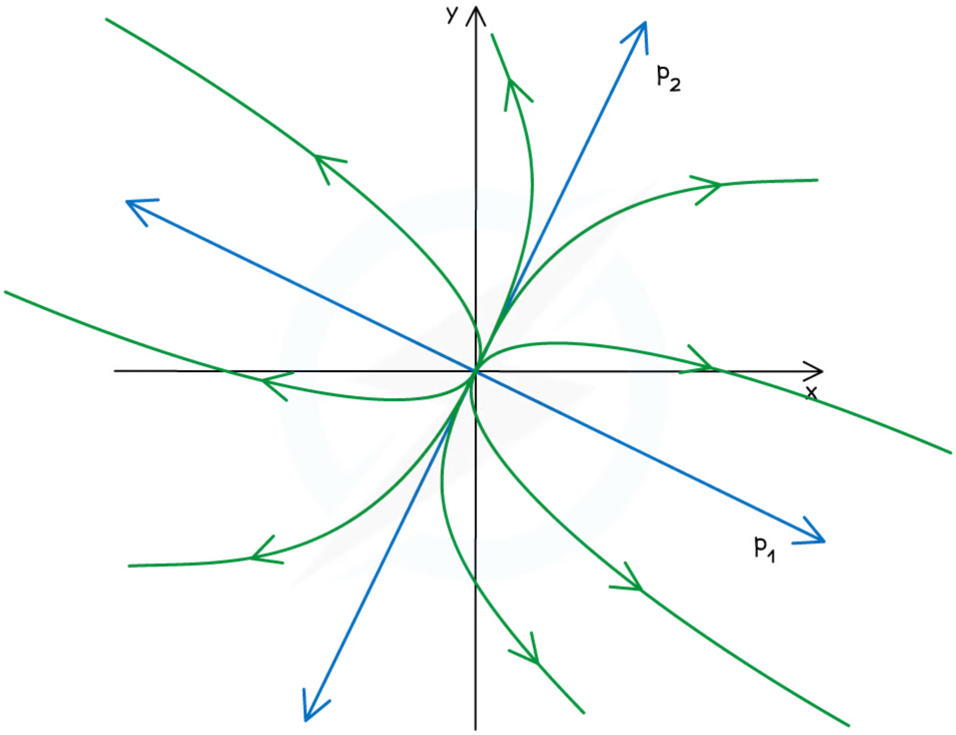
BOTH EIGENVALUES REAL AND NEGATIVE



$0 > \lambda_1 > \lambda_2$, SO SOLUTION TRAJECTORIES START PARALLEL TO p_2, AND CONVERGE ON p_1 AS THEY APPROACH THE ORIGIN

- If **one eigenvalue is positive and one eigenvalue is negative** then solution trajectories will generally start by heading in towards the origin before curving to head out away again from the origin as *t* increases
    - In between the 'eigenvector lines' the solution trajectories will all move in towards the origin along the direction of the eigenvector line that corresponds to the negative eigenvalue, before curving away and converging on the eigenvector line that corresponds to the positive eigenvalue as they head away from the origin

BOTH EIGENVALUES REAL AND POSITIVE

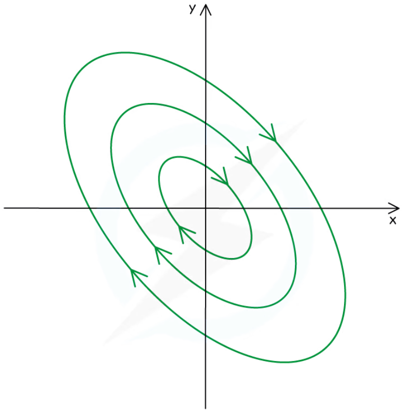


$\lambda_1 > \lambda_2 > 0$, SO SOLUTION TRAJECTORIES BECOME PARALLEL TO p_1 AS THEY MOVE AWAY FROM THE ORIGIN

What does the phase portrait look like when λ_1 and λ_2 are imaginary numbers?

- Here the solution trajectories will all be either circles or ellipses with their centres at the origin

PURELY IMAGINARY EIGENVALUES

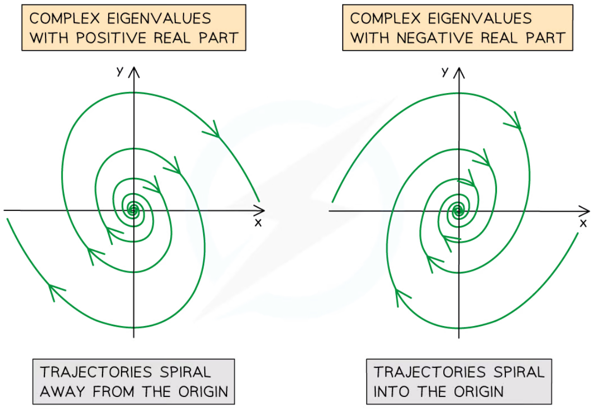


TRAJECTORIES 'ORBIT' AROUND THE ORIGIN. THEY MAY BE ELLIPTICAL OR CIRCULAR, AND MAY BE CLOCKWISE OR ANTICLOCKWISE (BUT ALL WILL EITHER BE ONE OR THE OTHER FOR ANY PARTICULAR DIFFERENTIAL EQUATION).

- You can determine the direction (clockwise or anticlockwise) and the shape (circular or elliptical) of the trajectories by considering the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ for points on the coordinate axes
  - For example, consider the system $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\boldsymbol{x}$
    - The eigenvalues of $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ are i and $-\mathrm{i}$, so the trajectories will be elliptical or circular
  - When $x = 1$ and $y = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(1) - 2(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1(1) - 1(0) = 1$
    - This shows that from a point on the positive x-axis the solution trajectory will be moving 'to the right and up' in the direction of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  - When $x = 0$ and $y = 1$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(0) - 2(1) = -2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1(0) - 1(1) = -1$
    - This shows that from a point on the positive y-axis the solution trajectory will be moving 'to the left and down' in the direction of the vector $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$
  - The directions of the trajectories at those points tell us that the directions of the trajectories will be anticlockwise
  - They also tell us that the trajectories will be ellipses
    - For circular trajectories, the direction of the trajectories when they cross a coordinate axis will be perpendicular to that coordinate axis

What does the phase portrait look like when λ_1 and λ_2 are complex numbers?

- In this case λ_1 and λ_2 will be complex conjugates of the form $a \pm bi$, where a and b are non-zero real numbers
  - If $a = 0$, $b \neq 0$, then we have the imaginary eigenvalues case above
- Here the solution trajectories will all be spirals
  - If the real part of the eigenvalues is **positive** (i.e., if $a > 0$), then the trajectories will spiral **away from** the origin
  - If the real part of the eigenvalues is **negative** (i.e., if $a < 0$), then the trajectories will spiral **towards** the origin



- You can determine the direction (clockwise or anticlockwise) of the trajectories by considering the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ for points on the coordinate axes
  - For example, consider the system $\dot{\mathbf{x}} = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}\mathbf{x}$
    - The eigenvalues of $\begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}$ are $2 + 3\mathrm{i}$ and $2 - 3\mathrm{i}$, so the trajectories will be spirals
    - Because the real part of the eigenvalues (2) is positive, the trajectories will spiral away from the origin
  - When $x = 1$ and $y = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(1) + 5(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -2(1) + 3(0) = -2$
    - This shows that from a point on the positive x-axis the solution trajectory will be moving 'to the right and down' in the direction of the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
  - When $x = 0$ and $y = 1$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(0) + 5(1) = 5$ and $\frac{\mathrm{d}t}{\mathrm{d}t} = -2(0) + 3(1) = 3$
    - This shows that from a point on the positive y-axis the solution trajectory will be moving 'to the right and up' in the direction of the vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$
  - The directions of the trajectories at those points tell us that the directions of the trajectory spirals will be clockwise

Worked Example

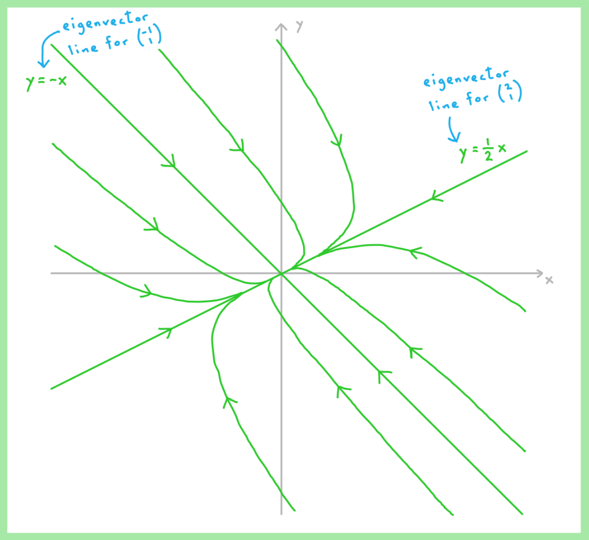
Consider the system of coupled differential equations

$$\frac{dx}{dt} = -2x + 2y$$

$$\frac{dy}{dt} = x - 3y$$

Given that -1 and -4 are the eigenvalues of the matrix $\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, draw a phase portrait for the solutions of the system.

Both eigenvalues are negative, so all trajectories will converge on the origin. -4 is more negative than -1, so away from the origin the trajectories will curve towards the $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector line.



Equilibrium Points

What is an equilibrium point?

- For a system of coupled differential equations, an **equilibrium point** is a point (x, y) at which both $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$
    - Because both derivatives are zero, the rates of change of both x and y are zero
    - This means that x and y will not change, and therefore that if the system is ever at the point (x, y) then it will remain at that point (x, y) forever
- An equilibrium point can be **stable** or **unstable**
    - An equilibrium point is stable if for **all** points close to the equilibrium point the solution trajectories move back towards the equilibrium point
        - This means that if the system is perturbed away from the equilibrium point, it will tend to move back towards the state of equilibrium
    - If an equilibrium point is not stable, then it is unstable
        - If a system is perturbed away from an unstable equilibrium point, it will tend to continue moving further and further away from the state of equilibrium
- For a system that can be represented in the matrix form $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$, where $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the origin $(0, 0)$ is always an equilibrium point
    - Considering the nature of the **phase portrait** for a particular system will tell us what sort of equilibrium point the origin is
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real and negative**, then the origin is a **stable** equilibrium point
        - This sort of equilibrium point is sometimes known as a **sink**
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real and positive**, then the origin is an **unstable** equilibrium point
        - This sort of equilibrium point is sometimes known as a **source**
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real**, with **one positive and one negative**, then the origin is an **unstable** equilibrium point
        - This sort of equilibrium point is known as a **saddle point** (you will be expected to identify saddle points if they occur in an AI HL exam question)
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **imaginary**, then the origin is an **unstable** equilibrium point
        - Recall that for all points other than the origin, the solution trajectories here all 'orbit' around the origin along circular or elliptical paths
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **complex with a negative real part**, then the origin is an **stable** equilibrium point
        - All solution trajectories here spiral in towards the origin
    - If both eigenvalues of the matrix $\boldsymbol{M}$ are **complex with a positive real part**, then the origin is an **unstable** equilibrium point
        - All solution trajectories here spiral away from the origin

Worked Example

a)

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - 3y + 6$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y - 7$$

Show that $(3, 4)$ is an equilibrium point for the system.

When $x = 3$ and $y = 4$,

$$\frac{dx}{dt} = 2(3) - 3(4) + 6 = 0$$

$$\frac{dy}{dt} = 3 + 4 - 7 = 0$$

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both zero at $(3, 4)$, therefore $(3, 4)$ is an equilibrium point for the system.

b)

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 3y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + 2y$$

Given that 4 and -1 are the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, determine the coordinates and nature of the equilibrium point for the system.

YOUR NOTES
↓

When $x = 0$ and $y = 0$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

Therefore the origin $(0,0)$ is the equilibrium point for the system.

One of eigenvalues is positive and one is negative.

Therefore the origin is a saddle point, which is an unstable equilibrium point.

Sketching Solution Trajectories

How do I sketch a solution trajectory for a system of coupled differential equations?

- A **phase portrait** shows typical trajectories representing all the possible solutions to a system of coupled differential equations
- For a given set of initial conditions, however, the solution will only have one specific trajectory
- **Sketching a particular solution trajectory** will generally involve the following:
    - Make sure you know what the 'typical' solutions for the system look like
        - You don't need to sketch a complete phase portrait unless asked, but you should know what the phase portrait for your system would look like
        - If the phase portrait includes 'eigenvector lines', however, it is worth including these in your sketch to serve as guidelines
    - Mark the starting point for your solution trajectory
        - The coordinates of the starting point will be the x and y values when $t = 0$
        - Usually these are given in the question as the initial conditions for the system
    - Determine the initial direction of the solution trajectory
        - To do this find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $t = 0$
        - This will tell you the directions in which x and y are changing initially
        - For example if $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 3$ when $t = 0$, then the trajectory from the starting point will initially be 'to the left and up', parallel to the vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
    - Use the above considerations to create your sketch
        - The trajectory should begin at the starting point (be sure to mark and label the starting point on your sketch!)
        - It should move away from the starting point in the correct initial direction
        - As it moves further away from the starting point, the trajectory should conform to the nature of a 'typical solution' for the system

YOUR NOTES
⬇

Worked Example

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 5y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x + 3y$$

The initial conditions of the system are such that the exact solution is given by

$$\boldsymbol{x} = \mathrm{e}^{6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - 2\mathrm{e}^{-2t}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

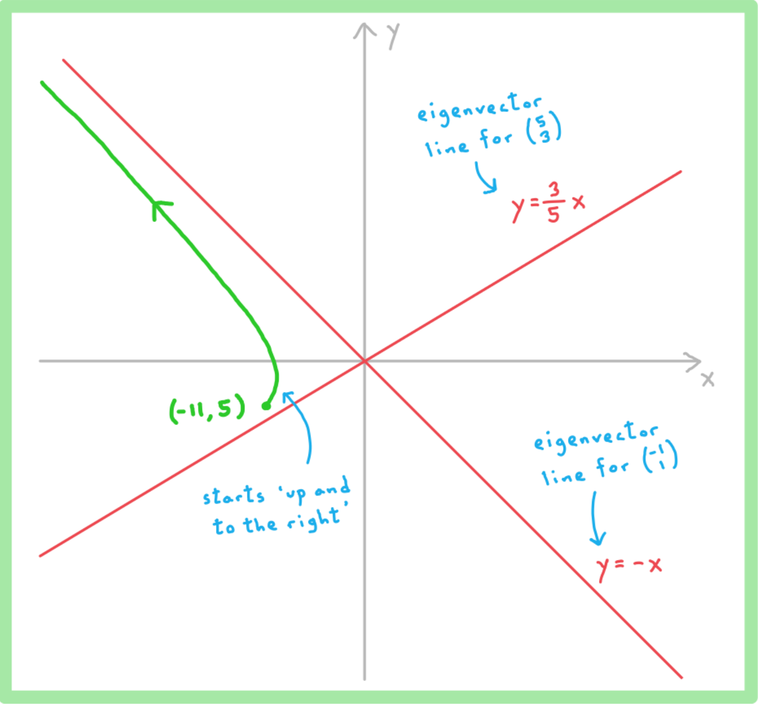
Sketch the trajectory of the solution, showing the relationship between x and y as t increases from zero.

The eigenvalues have different signs, so the trajectory will become approximately parallel to the positive eigenvalue's eigenvector line as it moves away from the origin.

When $t = 0$, $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix}$ starting point

$\frac{dx}{dt} = (-11) - 5(-5) = 14$ $\frac{dy}{dt} = -3(-11) + 3(-5) = 18$

So the initial trajectory will be 'up and to the right' in the direction of the vector $\begin{pmatrix} 18 \\ 14 \end{pmatrix}$.



Euler's Method: Second Order

How do I apply Euler's method to second order differential equations?

- A **second order differential equation** is a differential equation containing one or more second derivatives
- In this section of the course we consider second order differential equations of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=f\left(x,\frac{\mathrm{d}x}{\mathrm{d}t},t\right)$$

  - You may need to rearrange the differential equation given to get it in this form
- In order to apply Euler's method, use the substitution $y=\frac{\mathrm{d}x}{\mathrm{d}t}$ to turn the second order differential equation into a pair of coupled first order differential equations
  - If $y=\frac{\mathrm{d}x}{\mathrm{d}t}$, then $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$
  - This changes the second order differential equation into the coupled system
- Approximate solutions to this coupled system can then be found using the standard Euler's method for coupled systems
  - See the notes on this method in the revision note 5.6.4 Approximate Solutions to Differential Equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=f(x,y,t))$$

Worked Example

Consider the second order differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 50\cos t$.

a)
Show that the equation above can be rewritten as a system of coupled first order differential equations.

$$\frac{d^2x}{dt^2} = -x - 2\frac{dx}{dt} + 50\cos t \qquad \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$$

Let $y = \frac{dx}{dt}$. Substitution

Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, so the equation becomes

$$\frac{dy}{dt} = -x - 2y + 50\cos t$$

This gives the coupled system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x - 2y + 50\cos t$$

b)

Initially $x = 2$ and $\frac{dx}{dt} = -1$. By applying Euler's method with a step size of 0.1, find approximations for the values of x and $\frac{dx}{dt}$ when $t = 0.5$.

Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

From exam formula booklet

$$x_{n+1} = x_n + 0.1 y_n \qquad t_{n+1} = t_n + 0.1$$

$$y_{n+1} = y_n + 0.1\left(-x_n - 2y_n + 50 \cos t_n\right)$$

Initially $x = 2$ and $y = \frac{dx}{dt} = -1$

n	t_n	x_n	y_n
0	0	2	-1
1	0.1	1.9	4
2	0.2	2.3	7.985
3	0.3	3.0985	11.058
4	0.4	4.2043	13.313
5	0.5	5.5356	14.835

from GDC

At $t = 0.5$,

$x = 5.54$ (3 s.f.)

$\frac{dx}{dt} = 14.8$ (3 s.f.)

Exact Solutions & Phase Portraits: Second Order

How can I find the exact solution for a second order differential equation?

- In some cases we can apply methods we already know to find the exact solutions for second order differential equations
- In this section of the course we consider second order differential equations of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=0$$

  - are constants
- Use the substitution $y=\frac{\mathrm{d}x}{\mathrm{d}t}$ to turn the second order differential equation into a pair of coupled first order differential equations
  - If $y=\frac{\mathrm{d}x}{\mathrm{d}t}$, then $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$
  - This changes the second order differential equation into the coupled system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t}=-bx-ay$$

  - The coupled system may also be represented in matrix form as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

    - In the 'dot notation' here and
  - That can be written even more succinctly as $\dot{\boldsymbol{x}}=\boldsymbol{M}\boldsymbol{x}$
    - Here $\dot{\boldsymbol{x}}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix}x\\ y\end{pmatrix}$, and $\boldsymbol{M}=\begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}$
- Once the original equation has been rewritten in matrix form, the standard method for finding exact solutions of systems of coupled differential equations may be used
  - The solutions will depend on the eigenvalues and eigenvectors of the matrix **M**
  - For the details of the solution method see the revision note 5.7.1 Coupled Differential Equations
  - Remember that exam questions will only ask for exact solutions for cases where the eigenvalues of **M** are real and distinct

How can I use phase portraits to investigate the solutions to second order differential equations?

- Here we are again considering second order differential equations of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=0$$

  - *a* & *b* are real constants
- As shown above, the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ can be used to convert this second order differential equation into a system of coupled first order differential equations of the form $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$
  - Here $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$
- Once the equation has been rewritten in this form, you may use the standard methods to construct a phase portrait or sketch a solution trajectory for the equation
  - For the details of the phase portrait and solution trajectory methods see the revision note 5.7.1 Coupled Differential Equations
  - When interpreting a phase portrait or solution trajectory sketch, don't forget that $y = \frac{\mathrm{d}x}{\mathrm{d}t}$
    - So if *x* represents the displacement of a particle, for example, then $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ will represent the particle's velocity

YOUR NOTES
↓

Worked Example

Consider the second order differential equation $\frac{d^2x}{dt^2}+3\frac{dx}{dt}-4x=0$. Initially $x=3$ and $\frac{dx}{dt}=-2$.

a)
Show that the equation above can be rewritten as a system of coupled first order differential equations.

$$\frac{d^2x}{dt^2} = 4x - 3\frac{dx}{dt} \qquad \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$$

Let $y = \frac{dx}{dt}$. Substitution

Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, so the equation becomes

$$\frac{dy}{dt} = 4x - 3y$$

This gives the coupled system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = 4x - 3y$$

b)
Given that the matrix $\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$ has eigenvalues of 1 and -4 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find the exact solution to the second order differential equation.

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

} From exam formula booklet

We have $\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} \underline{x}$, so

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-4t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

At $t = 0$, $x = 3$ and $y = \frac{dx}{dt} = -2$, so

$$\begin{pmatrix} A - B \\ A + 4B \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \Longrightarrow A = 2, \ B = -1$$

$$\Longrightarrow \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = 2e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-4t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\boxed{x = 2e^{t} + e^{-4t}}$$

c)
Sketch the trajectory of the solution to the equation on a phase diagram, showing the relationship between x and $\frac{\mathrm{d}x}{\mathrm{d}t}$.

At $t = 0$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 4(3) - 3(-2) = 18$.

So initially the solution trajectory is to the left and up.